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Differential Geometry

Structures on Manifolds and Submanifolds

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Mohammad Hasan Shahid

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Differential Geometry: Structures on Manifolds and Submanifolds

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Contents

About the Editor

Mohammad Hasan Shahid

Mohammad Hasan Shahid (Professor) obtained his PhD in Mathematics from Aligarh Muslim University, India, in 1988. His main research interest is differential geometry and its applications. He has published more than 100 research papers in several renowned international journals and conference proceedings, additionally writing and editing books in this area. He was awarded the Sultana Nahar Distinguished Teacher award in 2017-2018 for his outstanding contribution to this field of research. He has conducted research and delivered talks at various universities of the world, such as the following: the University of Leeds, UK; the University of Montpellier, France; the University of Sevilla, Spain; Hokkaido University, Japan; Chua University, Japan; and Manisa Celal Bayar University, Turkey.

Preface

The theory of structures on manifolds and their submanifolds is a fruitful area of research of modern differential geometry.

The objective of this special issue titled "Differential Geometry: Structures on manifolds and submanifolds" was to invite high-quality and interesting papers on the topic and to collect papers from different areas of differential geometry. Many papers were submitted and 19 papers were accepted after going through a careful review process conducted by leading experts. The topics covered included twister product Hermitian manifolds, Lorentzian manifolds, framed natural mate, framed curve Ricci solitons, spacelike hypersurface, nearly Sasakian manifolds, cosymplectic manifolds, heat equattions, Li–Yau-type gradient, Ricci–Yamabe solitons, quasi-Einstein spacetime, quarter symmetric metric connection, etc. I hope that these research papers are of great interest and encourage other researchers to conduct further studies.

I, as a Guest Editor of this special issue, am grateful to all authors for their contributions and also thankful to all the reviewers for their valuable comments and suggestions.

I would like to thank the MDPI publishing editorial team who gave me the opportunity of being the Guest Editor of this special issue, especially Ms. Ashley Chang, the Section Managing Editor, for her support in helping me process all the manuscripts.

Mohammad Hasan Shahid
Editor

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Article

Gradient Ricci Solitons on Spacelike Hypersurfaces of Lorentzian Manifolds Admitting a Closed Conformal Timelike Vector Field

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Abstract: In this article, we investigate Ricci solitons occurring on spacelike hypersurfaces of Einstein Lorentzian manifolds. We give the necessary and sufficient conditions for a spacelike hypersurface of a Lorentzian manifold, equipped with a closed conformal timelike vector field $\bar{\xi}$, to be a gradient Ricci soliton having its potential function as the inner product of $\bar{\xi}$ and the timelike unit normal vector field to the hypersurface. Moreover, when the ambient manifold is Einstein and the hypersurface is compact, we establish that, under certain straightforward conditions, the hypersurface is an extrinsic sphere, that is, a totally umbilical hypersurface with a non-zero constant mean curvature. In particular, if the ambient Lorentzian manifold has a constant sectional curvature, we show that the compact spacelike hypersurface is essentially a round sphere.

Keywords: gradient Ricci soliton; Einstein manifold; conformal vector field; spacelike hypersurfaces with constant mean curvature

MSC: 53A10; 53C40; 53C42; 53C50; 53C65

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1. Introduction

It is a well-established fact that Ricci solitons are closely linked with Ricci flows, as outlined in [1]. Essentially, a pseudo-Riemannian metric g defined on M provides a Ricci soliton on a smooth manifold M if and only if there exists a positive function $\sigma(t)$ and a one-parameter family $\psi(t)$ of diffeomorphisms of M such that the one-parameter family of metrics $g(t) = \sigma(t)\psi(t)^* g$ satisfies the Ricci flow equation:

$$\frac{\partial}{\partial t} g(t) = -2Ric_g(t),$$

with the initial condition $g = g(0)$. Here, $\psi(t)^*$ denotes the pullback along the diffeomorphism $\psi(t)$, and $Ric_g(t)$ represents the Ricci curvature of $g(t)$.

A pseudo-Riemannian manifold (M, g) is called a Ricci soliton if there exists a nonzero smooth vector field X and a constant λ satisfying

$$\frac{1}{2} L_X g + Ric = \lambda g, \tag{1}$$

where L_X is the Lie derivative with respect to X and Ric is the Ricci tensor with g. We denote a Ricci soliton by (M, g, X, λ). The concept of a Ricci soliton was first introduced by Hamilton [2,3].

Ricci solitons are a type of manifold in differential geometry that generalize the concept of Einstein metrics, that is, $Ric = cg$ for some constant c. The Ricci soliton is classified as shrinking, steady, or expanding based on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. The vector field X is referred to as the potential field of (M, g, X, λ). If the potential field X is the gradient of some smooth function f on M, that is $X = \nabla f$, then (M, g, f, λ) will denote the gradient Ricci soliton $(M, g, \nabla f, \lambda)$. In this case, Equation (1) takes the form

$$Ric + Hess(f) = \lambda g, \tag{2}$$

where $Hess(f)$ is the Hessian of the function f. The function f is called a potential function of the Ricci soliton (M, g, f, λ).

Additionally, if $L_X g = 0$, the Ricci soliton is considered trivial, and from Equation (1), M is an Einstein manifold.

One of the significant areas of focus in differential geometry and mathematical physics is the theory of submanifolds, which presents challenging topics related to submanifold geometry. In many research endeavors, the Gauss, Codazzi, and Ricci Equations for submanifolds play a crucial role as they can be formulated in a manageable manner. The exploration of Ricci solitons on hypersurfaces has gained traction, particularly in understanding the conditions under which hypersurfaces within Riemannian manifolds can exhibit Ricci soliton structures. While Ricci solitons on hypersurfaces in Riemannian manifolds have been extensively investigated, there is a relative scarcity of studies focusing on Ricci solitons in a Lorentzian manifold ambient space, despite their significance in terms of geometry and applications in theoretical physics. These circumstances have motivated our investigation into Ricci solitons on Riemannian hypersurfaces within Lorentzian manifolds.

The fascination with the geometry of Ricci solitons stems from its diverse applications in various disciplines, particularly in the context of hypersurfaces in Riemannian manifolds, as exemplified in [4–19]. This paper directs its attention to spacelike hypersurfaces in Lorentzian manifolds, which, to the best of our knowledge, represent an underexplored area in the existing literature. More specifically, our investigation centers on the analysis of gradient Ricci solitons on spacelike hypersurfaces of Lorentzian manifolds. These hypersurfaces are characterized by the presence of a closed conformal vector field of the ambient manifold, with the potential function denoted as θ, that is, the inner product between the closed conformal vector field and the timelike unit normal vector field to the hypersurface

This paper is organized as follows: the second section revisits essential concepts and formulas concerning spacelike hypersurfaces in Lorentzian manifolds. Section 3 presents the main results, focusing on characterizing conditions under which a spacelike hypersurface in a Lorentzian manifold, endowed with a closed conformal vector field, displays a gradient Ricci soliton structure with θ as the potential function.

The examination then focuses on compact gradient Ricci solitons, particularly when the ambient manifold is Einstein. We provide sufficient conditions to characterize spacelike hypersurfaces as extrinsic spheres, that is, totally umbilical hypersurfaces with a nonzero constant mean curvature. In the special case where the ambient manifold has a constant sectional curvature, it is deduced that the hypersurface is a round sphere. In the future, we look forward to generalizing this research in the case where the ambient manifold is a generalized Robertson Walker (GRW) spacetime.

2. Preliminaries

Let (M, g) be a hypersurface in an orientable Lorentzian manifold $(\overline{M}, \overline{g})$ of dimension (n + 1). Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\overline{M}$, respectively. Two fundamental equations apply to all vector fields X and Y that are tangential to M.

$$\overline{\nabla}_X Y = \nabla_X Y - h(X, Y), \tag{3}$$

$$\overline{\nabla}_X N = -A(X). \tag{4}$$

Formula (3) is called Gauss' formula and Formula (4) is called Weingarten' formula, where h is the second fundamental form, and A is the shape operator of M derived from a normal vector field N to $\overline{M}$.

There is a relationship between the second fundamental form h and the shape operator A of M.

$$\overline{g}(A(X), Y) = \overline{g}(h(X, Y), N). \tag{5}$$

The Codazzi equation describes the normal part of the curvature $\overline{R}(X, Y)Z$ as follows:

$$(\overline{R}(X, Y)Z)^{\perp} = (\nabla_Y h)(X, Z) - (\nabla_X h)(Y, Z), \tag{6}$$

where X, Y, and Z are tangent to M, while N is normal to M, and $\overline{R}$ is the curvature tensor of $\overline{M}$, defined as follows:

$$\overline{R}(X, Y)Z = \overline{\nabla}_{[X,Y]}Z - [\overline{\nabla}_X, \overline{\nabla}_Y]Z.$$

The covariant derivative of h is denoted as ∇h, and it is defined as follows:

$$(\nabla_X h)(Y, Z) = \overline{\nabla}_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The Gauss–Codazzi equation is a mathematical formula that is widely known and used.

$$R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + \overline{g}(h(X, Z), h(Y, W)) - \overline{g}(h(X, W), h(Y, Z)), \tag{7}$$

for all X, Y, Z and W tangent to M, where R and $\overline{R}$ are the curvature tensors of M and $\overline{M}$, respectively.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame of a pseudo-Riemannian manifold (M, g). Then, the Ricci curvature tensor on M is a symmetric tensor given by

$$Ric(X, Y) = \sum_{i=1}^{n} \epsilon_i R(X, e_i, Y, e_i),$$

where X and Y are tangent to M, and the scalar curvature S of M is defined by

$$S = \sum_{i=1}^{n} \epsilon_i Ric(e_i, e_i).$$

The divergence of the vector field X of M is defined by

$$div(X) = \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} X, e_i), \tag{8}$$

where $\epsilon_i = g(e_i, e_i)$. The trace of the curvature tensor is the Ricci curvature, and the trace of the Ricci is the scalar curvature.

The mean curvature H of a spacelike hypersurface M in a Lorentzian manifold $(\overline{M}, \overline{g})$ is define by

$$H = -\frac{1}{n} tr(A),$$

where $tr(A)$ is the trace of the shape operator A of M derived from a normal vector field N to $\overline{M}$.

Equation (7) results in a relationship between the Ricci curvatures Ric and $\overline{Ric}$ of M and $\overline{M}$, respectively. Furthermore, it can be expressed as follows:

$$Ric(X, Y) = \overline{Ric}(X, Y) + \overline{g}(\overline{R}(N, X)Y, N) + g(A(X), nHY + A(Y)). \tag{9}$$

The Hessian $Hess(f)$ of a smooth function f on a pseudo-Riemannian manifold (M, g) is a symmetric tensor defined by

$$Hess(f)(X, Y) = g(S_f(X), Y),$$

where S_f is the Hessian operator defined by $S_f = \nabla_X \nabla f$ and ∇f is the gradient of the function f.

A point p of a pseudo-Riemannian hypersurface M of $\overline{M}$ is called an umbilical point if the shape operator A at p, $A_p = \Phi I$, where Φ is scalar. M is called totally umbilical if every point of M is umbilical. In particular, M is called totally geodesic if $A = 0$.

A hypersurface M of a pseudo-Riemannian $(\overline{M}, \overline{g})$ is called an extrinsic sphere if it is a totally umbilical sphere with a non-zero constant mean curvature.

3. Ricci Solitons on Spacelike Hypersurfaces in Einstein Lorentzian Manifolds

Let (M, g) be an orientable spacelike hypersurface of a Lorentz manifold $(\overline{M}, \overline{g})$ of dimension $(n+1)$, and let $\bar{\xi}$ be a timelike closed conformal vector field on $\overline{M}$ which means

$$\overline{\nabla}_X \bar{\xi} = \psi X,$$

for all $X \in \mathfrak{X}(\overline{M})$ ($\mathfrak{X}(\overline{M})$ is the set of all vector fields on $\overline{M}$) and ψ is called the conformal function, a smooth function on $\overline{M}$. The restriction of $\bar{\xi}$ to M is denoted by ξ. Let N be a unit timeline normal vector field on M, which can be chosen so that $\theta = \overline{g}(\xi, N) < 0$. Then, we can write

$$\xi = \xi^T - \theta N, \tag{10}$$

where ξ^T is the tangential component of ξ. By using Gauss and Weingarten formulas, it yields

$$\nabla_X \xi^T = \psi X - \theta A(X), \tag{11}$$

and

$$A(\xi^T) = -\nabla \theta. \tag{12}$$

From (8), it is straightforward to derive

$$div \xi^T = n(\psi + \theta H). \tag{13}$$

Let Q and $\overline{Q}$ be Ricci operators on M and $\overline{M}$, respectively, where Q and $\overline{Q}$ satisfy $Ric(X, Y) = g(QX, Y)$ and $\overline{Ric}(X, Y) = \overline{g}(\overline{Q}X, Y)$.

Some of the notation are reviewed, which are needed in our results. It is easy to see that $\overline{R}(N, X)N$ is tangent to M for all $X \in \mathfrak{X}(M)$, and, thus, we can define the normal Jacobi operator $R_N : TM \to TM$ by

$$R_N(X) = \overline{R}(N, X)N.$$

Define the operator $(\nabla A)\xi^T$ on M by

$$(\nabla A)\xi^T(X) = \nabla A(X, \xi^T) = (\nabla_X A)(\xi^T).$$

The following lemma is crucial for proving the main results.

Lemma 1.

$$tr((\nabla A)\xi^T) = -\overline{Ric}(\xi^T, N) - n\xi^T(H).$$

Proof. Let $\{e_1, \ldots, e_n\}$ denote a local orthonormal frame on M that can be taken as parallel. By using the Codazzi Equation (6), it yields

$$\begin{aligned}
tr((\nabla A)\xi^T) &= \sum_{i=1}^{n} g((\nabla_{e_i} A)(\xi^T), e_i) \\
&= -\sum_{i=1}^{n} \overline{g}(\overline{R}(\xi^T, e_i)e_i, N) - \sum_{i=1}^{n} g((\nabla_{\xi^T} A)(e_i), e_i) \\
&= -\sum_{i=1}^{n} \overline{g}(\overline{R}(\xi^T, e_i)e_i, N) - \sum_{i=1}^{n} g(\nabla_{\xi^T}(A(e_i)), e_i) + \sum_{i=1}^{n} g(A(\nabla_{\xi^T})e_i), e_i) \\
&= -\sum_{i=1}^{n} \overline{g}(\overline{R}(\xi^T, e_i)e_i, N) + \overline{g}(\overline{R}(N, N)\xi^T, N) - \xi^T \sum_{i=1}^{n} g(A(e_i), e_i) \\
&= -\overline{Ric}(\xi^T, N) - n\xi^T(H).
\end{aligned}$$

□

Our first result presents the conditions that a spacelike hypersurface must satisfy to be identified as a gradient Ricci soliton of the particular type (M, g, θ, λ).

Theorem 1. *Let $(\overline{M}, \overline{g})$ be an (n+1)-dimensional Lorentzian manifold endowed with a timelike closed conformal vector field $\bar{\xi}$. Let (M, g) be a spacelike hypersurface of $(\overline{M}, \overline{g})$, and let ξ, ξ^T, and θ be the same as above. (M, g, θ, λ) is a Ricci soliton if and only if the following equation is satisfied:*

$$\overline{Q} + R_N - (\nabla A)(\xi^T) + (nH - \psi)A + (1 + \theta)A^2 = \lambda I. \tag{14}$$

Proof. By using Equations (3) and (4), it follows that

$$\begin{aligned}
Hess(\theta)(X, Y) &= g(\nabla_X \nabla\theta, Y) \\
&= -g(\nabla_X(A(\xi^T)), Y) \\
&= -g((\nabla_X A)(\xi^T), Y) - g(A(\nabla_X \xi^T), Y) \\
&= -g((\nabla_X A)(\xi^T), Y) - g(\psi A(X) - \theta A^2(X), Y).
\end{aligned}$$

From this last expression, we have

$$S_\theta = -(\nabla A)\xi^T - \psi A + \theta A^2.$$

By using Equation (2), it yields

$$Q = (\nabla A)\xi^T + \psi A - \theta A^2 + \lambda I. \tag{15}$$

By substituting (15) into (9), we obtain (14). □

The next result outlines a practical condition applicable to a spacelike hypersurface, establishing its characterization as a gradient Ricci soliton of the type (M, g, θ, λ).

Theorem 2. *Let $(\overline{M}, \overline{g})$ be an (n+1)-dimensional Lorentzian manifold with a timelike closed conformal vector field $\bar{\xi}$ on $\overline{M}$. Let (M, g) be a spacelike hypersurface of $(\overline{M}, \overline{g})$, and let ξ, ξ^T, and θ be the same as above. If (M, g, θ, λ) is a Ricci soliton, then*

$$\bar{S} + 2\overline{Ric}(N, N) + (\psi - nH)nH + (1 + \theta)|A|^2 + \overline{Ric}(\xi^T, N) + n\xi^T(H) = n\lambda. \tag{16}$$

Proof. Formula (16) is obtained just by tracing Equation (14) and using Lemma 1. □

In the case of an Einstein ambient manifold, Theorem 2 yields the following implication.

Theorem 3. *Let $(\overline{M}, \overline{g})$ be an (n+1)-dimensional Einstein Lorentzian manifold with $\overline{Ric} = n\bar{c}\overline{g}$, where $\bar{c}$ is a constant. Let $\bar{\xi}$ be a timelike closed conformal vector field on $\overline{M}$. Let (M, g) be a spacelike hypersurface of $(\overline{M}, \overline{g})$, and let ξ, ξ^T, and θ be the same as above. If (M, g, θ, λ) is a Ricci soliton, then*

$$(1+\theta)(|A|^2 - nH^2) + n(n-1)(\bar{c} - H^2) + ndiv(H\xi^T) = n\lambda. \tag{17}$$

Proof. Using (13) and (16), it follows that

$$\begin{aligned} \bar{S} + 2\overline{Ric}(N,N) + (1+\theta)(|A|^2 - nH^2) - n(n-1)H^2 + nHdiv(\xi^T) \\ +\overline{Ric}(\xi^T, N) + n\xi^T(H) = n\lambda. \end{aligned} \tag{18}$$

Since $\overline{M}$ is Einstein, then $\overline{Ric}(N,N) = -n\bar{c}$, $\overline{Ric}(\xi^T, N) = 0$ and $\bar{S} = n(n+1)\bar{c}$. It follows that Equation (18) becomes

$$(1+\theta)(|A|^2 - nH^2) + n(n-1)(\bar{c} - H^2) + nHdiv(\xi^T) + n\xi^T(H) = n\lambda. \tag{19}$$

Using $div(H\xi^T) = Hdiv(\xi^T) + \xi^T(H)$ yields (17). □

A simple consequence of the last theorem is the following result.

Theorem 4. *Let $(\overline{M}, \overline{g})$ be an (n+1)-dimensional Einstein Lorentzian manifold with $\overline{Ric} = n\bar{c}\overline{g}$, where $\bar{c}$ is a constant, and let $\bar{\xi}$ be a timelike closed conformal vector field on $\overline{M}$. Let (M, g) be a compact spacelike hypersurface of $(\overline{M}, \overline{g})$, and let ξ, ξ^T, and θ be the same as above. If (M, g, θ, λ) is a Ricci soliton, then*

$$\int_M (1+\theta)(|A|^2 - nH^2)dV = n\int_M (\lambda - (n-1)(\bar{c} - H^2))dV. \tag{20}$$

There are interesting results after imposing certain assumptions on the function θ.

Theorem 5. *Consider the manifolds (M, g) and $(\overline{M}, \overline{g})$ as defined in Theorem 4, with the additional assumption that $\theta < -1$ (resp. $-1 < \theta < 0$) everywhere. If (M, g, θ, λ) is a non-trivial Ricci soliton, then $\lambda \leq (n-1)(\bar{c} - H^2)$ (resp. $\lambda \geq (n-1)(\bar{c} - H^2)$), with equality holds if and only if M is an extrinsic sphere. In particular, if $(\overline{M}, \overline{g})$ has a constant sectional curvature, then M is necessarily a sphere with a constant sectional curvature $c = \bar{c} - H^2 > 0$. In this case, the Ricci soliton is shrinking.*

Proof. Applying Schwartz's inequality leads to the conclusion that $\lambda \leq (n-1)(\bar{c} - H^2)$. It is a well-established fact that when equality is achieved, it indicates that M is totally umbilical. In [20], Lemma 35 on page 116 implies that M has a constant sectional curvature $c = \bar{c} - H^2$. As M is compact, it must be that M is a sphere with a constant positive curvature $c = \bar{c} - H^2$. This implies that $\lambda > 0$. Consequently, the Ricci soliton is shrinking. □

Remark 1. *In Theorem 5, assuming that $\overline{M}$ is a space form implies that it is isometric to the de Sitter Space $S_1^{n+1}(\bar{c})$, where $\bar{c} > 0$.*

The consequences derived from Equation (20) in Theorem 4 also lead to the following result.

Theorem 6. *Let $(\overline{M}, \overline{g})$ be an (n+1)-dimensional Einstein Lorentzian manifold with $\overline{Ric} = n\bar{c}\overline{g}$, where $\bar{c}$ is a constant, and let $\bar{\xi}$ be a timelike closed conformal vector field on $\overline{M}$. Let (M, g) be a compact spacelike hypersurface of $(\overline{M}, \overline{g})$, and let ξ, ξ^T, and θ be the same as above. If (M, g, θ, λ) is a Ricci soliton, such that either $\theta < -1$ and $\lambda \geq (n-1)(\bar{c} - H^2)$, or $-1 < \theta < 0$ and $\lambda \leq (n-1)(\bar{c} - H^2)$, then M is totally umbilical, H is a constant, and M is an extrinsic sphere.*

Proof. Clearly, Equation (20) implies $|A|^2 - nH^2 = 0$ if $\theta < -1$ and $\lambda \geq (n-1)(\bar{c} - H^2)$ or $-1 < \theta < 0$ and $\lambda \leq (n-1)(\bar{c} - H^2)$. It is concluded that M is totally umbilical with a constant mean curvature H, since $\lambda = (n-1)(\bar{c} - H^2)$. It must be that $H \neq 0$ because otherwise, M will be totally geodesic, a contradiction of the compactness of M. □

4. Conclusions

Gradient Ricci solitons have been extensively studied in Riemannian manifolds, as discussed in the introduction. However, this concept has received limited attention in the Lorentzian context, with only a few papers published on the topic. In this paper, we investigate gradient Ricci solitons on spacelike hypersurfaces of Lorentzian manifolds, marking the first attempt to do so. We believe that our research offers several advantages and potential impacts compared to the existing literature, thereby advancing knowledge in Lorentzian geometry. By studying gradient Ricci solitons in Lorentzian manifolds, particularly on spacelike hypersurfaces, we aim to gain a deeper understanding of their properties and behavior. This understanding could have significant implications across various branches of physics, particularly in general relativity, where Lorentzian manifolds are fundamental. Our research also aims to contribute to the development of a more comprehensive theoretical framework applicable to diverse mathematical and physical fields. Additionally, we hope that our findings will inspire further research and potentially lead to practical applications in fields such as cosmology, gravitational physics, and geometric analysis. In summary, we think that our work represents a significant step forward in understanding gradient Ricci solitons on spacelike hypersurfaces of Lorentzian manifolds, with implications for both theoretical mathematics and applied physics.

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Chern Flat and Chern Ricci-Flat Twisted Product Hermitian Manifolds

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Abstract: Let (M_1, g) and (M_2, h) be two Hermitian manifolds. The twisted product Hermitian manifold $(M_1 \times {}_f M_2, G)$ is the product manifold $M_1 \times M_2$ endowed with the Hermitian metric $G = g + f^2 h$, where f is a positive smooth function on $M_1 \times M_2$. In this paper, the Chern curvature, Chern Ricci curvature, Chern Ricci scalar curvature and holomorphic sectional curvature of the twisted product Hermitian manifold are derived. The necessary and sufficient conditions for the compact twisted product Hermitian manifold to have constant holomorphic sectional curvature are obtained. Under the condition that the logarithm of the twisted function is pluriharmonic, it is proved that the twisted product Hermitian manifold is Chern flat or Chern Ricci-flat, if and only if (M_1, g) and (M_2, h) are Chern flat or Chern Ricci-flat, respectively.

Keywords: Hermitian manifold; twisted product; holomorphic sectional curvature; Chern flat; Chern Ricci-flat

MSC: 53C55

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1. Introduction

Warped product and twisted product are important methods used to construct new classes of geometric spaces, and these models are widely applied in theoretical physics. In 1969, warped product was firstly introduced by O′Neill and Bishop to construct Riemannian manifolds with negative sectional curvature [1]. In 2001, Kozma, Peter and Varga [2] extended the warped product to real Finsler manifolds. Asanov [3,4] obtained some models of relativity theory by studying the warped product Finsler metric. In 2018, the notion of warped product was extended to Hermitian geometry by the work of He and Zhang [5], and they obtained the necessary and sufficient conditions for the compact nontrivial doubly warped product (abbreviated as DWP) Hermitian manifold to have constant holomorphic sectional curvature.

The notion of twisted product, as a generalization of warped product, was first introduced by Chen [6]. In 1993, Ponge and Reckziegel [7] extended twisted product to pseudo-Riemannian manifolds. Then, Fernández-López showed that a mixed Ricci-flat twisted product semi-Riemannian manifold can be expressed as a warped product semi-Riemannian manifold [8]. In 2017, Kazan and Sahin [9] deeply investigated the twisted product and multiply twisted product semi-Riemannian manifolds, which further promoted the development of twisted product in Riemannian geometry. Kozma, Peter and Shimada [10] extended the twisted product to real Finsler manifolds and studied some geometric properties relating to Cartan connection, geodesic and completeness. Recently, Xiao and He [11] extended the twisted product to complex Finsler manifolds and gave the formulae of holomorphic curvature and Ricci scalar curvature of the doubly twisted product (abbreviated as DTP) complex Finsler manifold. In light of the above results, we shall extend the twisted product to Hermitian manifold, and attempt to derive the Chern curvature, Chern Ricci curvature, Chern Ricci scalar curvature and holomorphic sectional curvature of the twisted product Hermitian manifold. In addition, we intend to find the necessary and sufficient conditions for the compact Hermitian manifold to have constant holomorphic sectional curvature.

One of the most important problems in geometry is to characterize Chern flat or Chern Ricci-flat manifolds. In 1967, Tani [12] firstly gave the definition of Ricci-flat space in Riemannian geometry. Later, Bando and Kobayashi [13] constructed Ricci-flat metrics on Einstein–Kähler manifolds. Liu and Yang [14] obtained the sufficient and necessary conditions for the Hopf manifold to be Levi-Civita Ricci-flat. Recently, Ni and He [15] gave the necessary and sufficient conditions for DWP-Hermitian manifold to be Levi-Civita Ricci-flat. In 2012, Di Scala [16] showed that quasi-Kähler Chern flat almost Hermitian structures on compact manifolds correspond to complex parallelizable Hermitian structures satisfying the second Gray identity. Wu and Zheng [17] proved that the compact Hermitian manifold with complex dimension 3, having vanishing real bisectional curvature, must be Chern flat. Based on the above mentioned studies, we are interested in the condition under which the twisted product Hermitian manifold is Chern flat or Chern Ricci-flat.

The structure of this paper is as follows. In Section 2, we briefly recall some basic concepts of Hermitian geometry and related symbolic conventions. In Section 3, we shall extend the concept of twisted product to Hermitian geometry, and derive the Chern connection coefficients of a twisted product Hermitian manifold. In Section 4, we shall give the formulae of Chern curvature, Chern Ricci curvature and Chern Ricci scalar curvature of the twisted product Hermitian manifold. In Section 5, we focus on investigating the twisted product Hermitian manifold with constant holomorphic sectional curvature. In Section 6, under the condition that the logarithm of the twisted function is pluriharmonic, we shall show that the twisted product Hermitian manifold is Chern flat or Chern Ricci-flat if and only if (M_1, g) and (M_2, h) are Chern flat or Chern Ricci-flat, respectively.

2. Preliminary

In this section, we briefly introduce the definitions and notations which we need in this paper.

Let (M, J, G) be a n-dimensional Hermitian manifold with complex structure J and Hermitian metric G. Let $T^{\mathbb{C}}M$ denote the complexified tangent bundle of M, which can be decomposed as

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M$ and $T^{0,1}M$ are eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Let $z = (z^1, \cdots, z^n)$ denote the local holomorphic coordinates on M, then vector fields $\{\partial_\alpha\}$ and $\{\partial_{\bar{\alpha}}\}$ form the basis of $T^{1,0}M$ and $T^{0,1}M$, respectively, where $\partial_\alpha = \frac{\partial}{\partial z^\alpha}, \partial_{\bar{\alpha}} = \frac{\partial}{\partial \bar{z}^\alpha}$. On the Hermitian holomorphic tangent bundle $T^{1,0}M$, the coefficients of Chern connection ∇ are [18]

$$\Gamma^\beta_{\gamma\alpha} = G^{\beta\bar{\sigma}} \partial_\gamma G_{\alpha\bar{\sigma}}, \tag{1}$$

and their complex conjugate.

Definition 1 ([18])**.** *Let ∇ be the Chern connection, its Chern curvature tensor K on the Hermitian manifold (M, J, G) is defined by*

$$K = K_{\alpha\bar{\beta}\gamma\bar{\sigma}} dz^\alpha \otimes d\bar{z}^\beta \otimes dz^\gamma \otimes d\bar{z}^\sigma, \tag{2}$$

where

$$K_{\alpha\bar{\beta}\gamma\bar{\sigma}} = -G_{\epsilon\bar{\beta}} K^\epsilon_{\alpha\gamma\bar{\sigma}}, \tag{3}$$

$$K^\epsilon_{\alpha\gamma\bar{\sigma}} = -\partial_{\bar{\sigma}} \Gamma^\epsilon_{\gamma\alpha}. \tag{4}$$

Definition 2 ([14]). *The first and the second Chern Ricci curvature on the Hermitian manifold (M, J, G) are defined by*

$$K^{(1)} = -\sqrt{-1}K^{(1)}_{\alpha\overline{\beta}}dz^{\alpha} \wedge d\overline{z}^{\beta}, \tag{5}$$

$$K^{(2)} = -\sqrt{-1}K^{(2)}_{\alpha\overline{\beta}}dz^{\alpha} \wedge d\overline{z}^{\beta},$$

respectively, where

$$K^{(1)}_{\alpha\overline{\beta}} = G^{\gamma\overline{\sigma}}K_{\alpha\overline{\beta}\gamma\overline{\sigma}}, \tag{6}$$

$$K^{(2)}_{\alpha\overline{\beta}} = G^{\gamma\overline{\sigma}}K_{\gamma\overline{\sigma}\alpha\overline{\beta}}. \tag{7}$$

Definition 3 ([14]). *The Chern Ricci scalar curvature on the Hermitian manifold (M, J, G) is defined by*

$$S_G = G^{\alpha\overline{\beta}}K^{(1)}_{\alpha\overline{\beta}} = G^{\alpha\overline{\beta}}K^{(2)}_{\alpha\overline{\beta}}. \tag{8}$$

For research purposes, we introduce the following two definitions.

Definition 4 ([19]). *Let D be open in C^n. A function $f \in C^2(D)$ is said to be pluriharmonic if it satisfies the differential equations*

$$\frac{\partial^2 f}{\partial z^{\alpha}\partial\overline{z}^{\beta}} = 0. \tag{9}$$

Definition 5 ([20]). *The complex Laplace operator*

$$L = G^{\overline{\beta}\alpha}\frac{\partial^2}{\partial z^{\alpha}\partial\overline{z}^{\beta}} \tag{10}$$

is a second-order elliptic partial differential operator with smooth coefficients.

Clearly, if f is a pluriharmonic function, then $L(f) = 0$.

3. Twisted Product Hermitian Manifold

Let (M_1, g) and (M_2, h) be two Hermitian manifolds with $dim_{\mathbb{C}}M_1 = m$ and $dim_{\mathbb{C}}M_2 = n$, respectively, then $M = M_1 \times M_2$ is a Hermitian manifold with $dim_{\mathbb{C}}M = m + n$.

We denote $z_1 = (z^1, \ldots, z^m) \in M_1$ and $z_2 = (z^{m+1}, \ldots, z^{m+n}) \in M_2$, so $z = (z_1, z_2) \in M$. Let $\pi_1 : M_1 \times M_2 \to M_1, \pi_2 : M_1 \times M_2 \to M_2$ be the natural projection maps, then $\pi_1(z) = z_1, \pi_2(z) = z_2$.

Let $T^{1,0}M_1$ and $T^{1,0}M_2$ be the holomorphic tangent bundle of M_1 and M_2, respectively. Denote $v_1 = (v^1, \cdots, v^m) \in T^{1,0}M_1$ and $v_2 = (v^{m+1}, \cdots, v^{m+n}) \in T^{1,0}M_2$, then $v = (v_1, v_2) \in T^{1,0}M$. Let $d\pi_1 : T^{1,0}(M_1 \times M_2) \to T^{1,0}M_1, d\pi_2 : T^{1,0}(M_1 \times M_2) \to T^{1,0}M_2$ be the holomorphic tangent maps induced by π_1 and π_2, then $d\pi_1(z, v) = (z_1, v_1)$, $d\pi_2(z, v) = (z_2, v_2)$, where z is called the base coordinates (or points) on M and v is called the fiber coordinates (or tangent directions).

For the reader′s convenience, the lowercase Greek indices like $\alpha, \beta, \gamma, \cdots$ run from 1 to $m + n$, the lowercase Latin indices like $i, j, k, s, t, \cdots$ run from 1 to m, while the lowercase Latin indices with a prime like $i', j', k', s', t', \cdots$ run from $m + 1$ to $m + n$. Quantities associated with (M_1, g) and (M_2, h) are denoted with upper indices 1 and 2, respectively; for example, $\overset{1}{\Gamma}{}^{i}_{jk'}$, $\overset{2}{\Gamma}{}^{i'}_{j'k'}$ are Chern connection coefficients of (M_1, g) and (M_2, h), respectively. In the following, we use the Einstein summation convention.

Definition 6. *Let (M_1, g) and (M_2, h) be two Hermitian manifolds. Let $f : M_1 \times M_2 \to (0, +\infty)$ be a positive smooth function. The twisted product Hermitian manifold $(M_1 \times {}_f M_2, G)$ is the product manifold $M = M_1 \times M_2$ endowed with the Hermitian metric $G : TM \to (0, +\infty)$:*

$$G(z, v) = g(\pi_1(z), d\pi_1(v)) + f^2 h(\pi_2(z), d\pi_2(v)), \tag{11}$$

for $z = (z_1, z_2) \in M$ and $v = (v_1, v_2) \in T^{1,0}M$. The function f is called the twisted function and G is called the twisted product Hermitian metric for simplicity.

In particular, if f only depends on M_1, then $(M_1 \times {}_f M_2, G)$ is a warped product Hermitian manifold. If f only depends on M_2, then $(M_1 \times {}_f M_2, G)$ is the product Hermitian manifold.

Denote

$$g_{i\bar{j}} = \frac{\partial^2 g}{\partial v^i \partial \overline{v}^j}, \quad h_{i'\overline{j'}} = \frac{\partial^2 h}{\partial v^{i'} \partial \overline{v}^{j'}}. \tag{12}$$

Then, the fundamental tensor matrix $(G_{\alpha\overline{\beta}})$ of G has the following forms

$$(G_{\alpha\overline{\beta}}) = \begin{pmatrix} g_{i\bar{j}} & 0 \\ 0 & f^2 h_{i'\overline{j'}} \end{pmatrix}, \tag{13}$$

its inverse matrix $(G^{\bar{\beta}\alpha})$ is also given by

$$(G^{\overline{\beta}\alpha}) = \begin{pmatrix} g^{\bar{j}i} & 0 \\ 0 & f^{-2} h^{\overline{j'}i'} \end{pmatrix}. \tag{14}$$

Proposition 1. *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Then, the Chern connection coefficients associated with G are given by*

$$\Gamma^i_{jk} = \overset{1}{\Gamma}{}^i_{jk}, \quad \Gamma^{i'}_{j'k'} = 2f^{-1}\delta^{i'}_{k'}\partial_{j'} f + \overset{2}{\Gamma}{}^{i'}_{j'k'}, \tag{15}$$

$$\Gamma^{i'}_{jk'} = 2f^{-1}\delta^{i'}_{k'}\partial_j f, \quad \Gamma^i_{j'k} = \Gamma^i_{jk'} = \Gamma^i_{j'k'} = \Gamma^{i'}_{jk} = \Gamma^{i'}_{j'k} = 0. \tag{16}$$

Proof. By putting $\alpha = k', \beta = i', \gamma = j'$ in (1), we have

$$\Gamma^{i'}_{j'k'} = G^{i'\overline{\sigma}}\partial_{j'} G_{k'\overline{\sigma}} = G^{i'\overline{s}}\partial_{j'} G_{k'\overline{s}} + G^{i'\overline{s'}}\partial_{j'} G_{k'\overline{s'}}. \tag{17}$$

Plunging (13) and (14) into (17), we can obtain

$$\begin{aligned} \Gamma^{i'}_{j'k'} &= f^{-2} h^{\overline{s'}i'} \partial_{j'}(f^2 h_{k'\overline{s'}}) \\ &= 2f^{-1}\delta^{i'}_{k'}\partial_{j'} f + h^{\overline{s'}i'}\partial_{j'} h_{k'\overline{s'}} \\ &= 2f^{-1}\delta^{i'}_{k'}\partial_{j'} f + \overset{2}{\Gamma}{}^{i'}_{j'k'}. \end{aligned}$$

Similarly, the other equalities of Proposition 1 can be deduced. □

4. Curvatures of Twisted Product Hermitian Manifold

In this section, we shall derive the Chern curvature, Chern Ricci curvature and Chern Ricci scalar curvature of the twisted product Hermitian manifold.

Proposition 2. *Let* $(M_1 \times {}_f M_2, G)$ *be a twisted product Hermitian manifold. Then, the coefficients of Chern curvature tensor* $K^{\epsilon}_{\alpha\gamma\overline{\sigma}}$ *are given by*

$$K^t_{kj\overline{s}} = \overset{1}{K}{}^t_{kj\overline{s}}, \quad K^{t'}_{k'j'\overline{s'}} = -2\frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{s'}} \delta^{t'}_{k'} + \overset{2}{K}{}^{t'}_{k'j'\overline{s'}}, \tag{18}$$

$$K^{t'}_{k'j\overline{s}} = -2\frac{\partial^2 \ln f}{\partial z^{j} \partial \overline{z}^{s}} \delta^{t'}_{k'}, \quad K^{t'}_{k'j'\overline{s}} = -2\frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{s}} \delta^{t'}_{k'}, \quad K^{t'}_{k'j\overline{s'}} = -2\frac{\partial^2 \ln f}{\partial z^{j} \partial \overline{z}^{s'}} \delta^{t'}_{k'}, \tag{19}$$

$$K^t_{k'j\overline{s}} = K^t_{kj'\overline{s}} = K^t_{kj\overline{s'}} = K^t_{k'j'\overline{s}} = K^t_{k'j\overline{s'}} = K^t_{kj'\overline{s'}} = K^t_{k'j'\overline{s'}} = 0, \tag{20}$$

$$K^{t'}_{kj\overline{s}} = K^{t'}_{kj'\overline{s}} = K^{t'}_{kj\overline{s'}} = K^{t'}_{kj'\overline{s'}} = 0. \tag{21}$$

Proof. By putting $\alpha = k', \gamma = j', \sigma = s', \epsilon = t'$ in (4), we have

$$K^{t'}_{k'j'\overline{s'}} = -\partial_{\overline{s'}} \Gamma^{t'}_{j'k'}. \tag{22}$$

Substituting the second equality of (15) into (22), and using (4), we have

$$\begin{aligned} K^{t'}_{k'j'\overline{s'}} &= -\partial_{\overline{s'}}(2f^{-1}\delta^{t'}_{k'}\partial_{j'} f + \overset{2}{\Gamma}{}^{t'}_{j'k'}) \\ &= 2f^{-2}\delta^{t'}_{k'}(\partial_{\overline{s'}} f)(\partial_{j'} f) - 2f^{-1}\frac{\partial^2 f}{\partial z^{j'} \partial \overline{z}^{s'}} \delta^{t'}_{k'} - \partial_{\overline{s'}} \overset{2}{\Gamma}{}^{t'}_{j'k'} \\ &= -2\frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{s'}} \delta^{t'}_{k'} + \overset{2}{K}{}^{t'}_{k'j'\overline{s'}}. \end{aligned}$$

Similarly, we can obtain other equalities of Proposition 2. □

Proposition 3. *Let* $(M_1 \times {}_f M_2, G)$ *be a twisted product Hermitian manifold. Then,*

$$K_{k\overline{i}j\overline{s}} = \overset{1}{K}_{k\overline{i}j\overline{s}}, \tag{23}$$

$$K_{k'\overline{i'}j\overline{s}} = 2f^2 \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^s} h_{k'\overline{i'}}, \tag{24}$$

$$K_{k'\overline{i'}j'\overline{s}} = 2f^2 \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^s} h_{k'\overline{i'}}, \tag{25}$$

$$K_{k'\overline{i'}j\overline{s'}} = 2f^2 \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^{s'}} h_{k'\overline{i'}}, \tag{26}$$

$$K_{k'\overline{i'}j'\overline{s'}} = 2f^2 \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{s'}} h_{k'\overline{i'}} + f^2 \overset{2}{K}_{k'\overline{i'}j'\overline{s'}}, \tag{27}$$

$$K_{k'\overline{i}j\overline{s}} = K_{k\overline{i'}j\overline{s}} = K_{k\overline{i}j'\overline{s}} = K_{k\overline{i}j\overline{s'}} = K_{k'\overline{i}j'\overline{s'}} = K_{k\overline{i'}j'\overline{s'}} = 0, \tag{28}$$

$$K_{k'\overline{i}j'\overline{s}} = K_{k'\overline{i}j\overline{s'}} = K_{k\overline{i'}j'\overline{s}} = K_{k\overline{i'}j\overline{s'}} = K_{k\overline{i}j'\overline{s'}} = 0. \tag{29}$$

Proof. By putting $\alpha = k', \beta = i', \gamma = j', \sigma = s'$ in (3), we have

$$K_{k'\overline{i'}j'\overline{s'}} = -G_{\epsilon\overline{i'}} K^{\epsilon}_{k'j'\overline{s'}} = -G_{t\overline{i'}} K^t_{k'j'\overline{s'}} - G_{t'\overline{i'}} K^{t'}_{k'j'\overline{s'}}. \tag{30}$$

Plunging (13) and the second equality of (18) into (30), a trivial caculation yields

$$
\begin{aligned}
K_{k'\overline{i'}j'\overline{s'}} &= -f^2 h_{t'\overline{i'}}\Big(-2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \overline{z}^{s'}}\delta^{t'}_{k'} + \overset{2}{K}{}^{t'}_{k'j'\overline{s'}}\Big) \\
&= 2f^2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \overline{z}^{s'}} h_{k'\overline{i'}} - f^2 h_{t'\overline{i'}}\overset{2}{K}{}^{t'}_{k'j'\overline{s'}} \\
&= 2f^2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \overline{z}^{s'}} h_{k'\overline{i'}} + f^2 \overset{2}{K}_{k'\overline{i'}j'\overline{s'}}.
\end{aligned}
$$

Similarly, we can obtain other equalities of Proposition 3. □

Proposition 4. *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Then, the coefficients of the first and the second Chern Ricci curvature tensor are given by*

$$
K^{(1)}_{k\overline{i}} = \overset{1}{K}{}^{(1)}_{k\overline{i}}, \qquad K^{(1)}_{k'\overline{i}} = K^{(1)}_{k\overline{i'}} = 0, \tag{31}
$$

$$
K^{(1)}_{k'\overline{i'}} = 2f^2 \overset{1}{L}(\ln f) h_{k'\overline{i'}} + 2\overset{2}{L}(\ln f) h_{k'\overline{i'}} + \overset{2}{K}{}^{(1)}_{k'\overline{i'}}, \tag{32}
$$

and

$$
K^{(2)}_{k\overline{i}} = \overset{1}{K}{}^{(2)}_{k\overline{i}} + 2\frac{\partial^2 \ln f}{\partial z^k \partial \overline{z}^i}, \quad K^{(2)}_{k'\overline{i}} = 2\frac{\partial^2 \ln f}{\partial z^{k'} \partial \overline{z}^i}, \tag{33}
$$

$$
K^{(2)}_{k'\overline{i'}} = 2\frac{\partial^2 \ln f}{\partial z^{k'} \partial \overline{z}^{i'}} + \overset{2}{K}{}^{(2)}_{k'\overline{i'}}, \quad K^{(2)}_{k\overline{i'}} = 2\frac{\partial^2 \ln f}{\partial z^{k} \partial \overline{z}^{i'}}. \tag{34}
$$

Proof. Letting $\alpha = k', \beta = i'$ in (6), we have

$$
K^{(1)}_{k'\overline{i'}} = G^{\gamma\overline{\sigma}} K_{k'\overline{i'}\gamma\overline{\sigma}} = G^{j\overline{s}} K_{k'\overline{i'}j\overline{s}} + G^{j'\overline{s}} K_{k'\overline{i'}j'\overline{s}} + G^{j\overline{s'}} K_{k'\overline{i'}j\overline{s'}} + G^{j'\overline{s'}} K_{k'\overline{i'}j'\overline{s'}}. \tag{35}
$$

Substituting (14), (24) and (27) into (35), and noticing that (10), we can obtain

$$
\begin{aligned}
K^{(1)}_{k'\overline{i'}} &= 2f^2 g^{j\overline{s}}\frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^s} h_{k'\overline{i'}} + f^{-2} h^{j'\overline{s'}}\Big(2f^2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \overline{z}^{s'}} h_{k'\overline{i'}} + f^2 \overset{2}{K}_{k'\overline{i'}j'\overline{s'}}\Big) \\
&= 2f^2 \overset{1}{L}(\ln f) h_{k'\overline{i'}} + 2\overset{2}{L}(\ln f) h_{k'\overline{i'}} + h^{j'\overline{s'}} \overset{2}{K}_{k'\overline{i'}j'\overline{s'}} \\
&= 2f^2 \overset{1}{L}(\ln f) h_{k'\overline{i'}} + 2\overset{2}{L}(\ln f) h_{k'\overline{i'}} + \overset{2}{K}{}^{(1)}_{k'\overline{i'}}.
\end{aligned}
$$

Similarly, we can obtain other equalities of Proposition 4. □

Theorem 1. *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Then, the Chern Ricci scalar curvature of G along a nonzero vector $v = \left(v^i, v^{i'}\right) \in T^{1,0}M$ is given by*

$$
S_G(v) = S_g(v_1) + f^{-2} S_h(v_2) + 2\overset{1}{L}(\ln f) + 2f^{-2}\overset{2}{L}(\ln f). \tag{36}
$$

Proof. According to (8), we have

$$
S_G(v) = G^{k\overline{i}} K^{(1)}_{k\overline{i}} + G^{k'\overline{i}} K^{(1)}_{k'\overline{i}} + G^{k\overline{i'}} K^{(1)}_{k\overline{i'}} + G^{k'\overline{i'}} K^{(1)}_{k'\overline{i'}}. \tag{37}
$$

Substituting (14), (31) and (32) into (37), after a straightfoward computation, we see that

$$
\begin{aligned}
S_G(v) &= g^{k\overline{i}} \overset{1}{K}{}^{(1)}_{k\overline{i}} + f^{-2} h^{k'\overline{i'}}\big[2f^2 \overset{1}{L}(\ln f) h_{k'\overline{i'}} + 2\overset{2}{L}(\ln f) h_{k'\overline{i'}} + \overset{2}{K}{}^{(1)}_{k'\overline{i'}}\big] \\
&= S_g(v_1) + f^{-2} S_h(v_2) + 2\overset{1}{L}(\ln f) + 2f^{-2}\overset{2}{L}(\ln f).
\end{aligned}
$$

Thus, we complete the proof. □

According to Definitions 4 and 5, we can obtain the following.

Corollary 1. *Let* $(M_1 \times {}_f M_2, G)$ *be a twisted product Hermitian manifold. Suppose* $\ln f$ *is a pluriharmonic function, then* $S_G(v) = S_g(v_1) + f^{-2} S_h(v_2)$.

5. Holomorphic Sectional Curvature of Twisted Product Hermitian Manifold

In this section, we would like to derive the holomorphic sectional curvature of the twisted product Hermitian manifold, and give the necessary and sufficient conditions for the compact twisted product Hermitian manifold to have constant holomorphic sectional curvature.

Definition 7 ([21])**.** *Let* (M, G) *be a Hermitian manifold. Then, the holomorphic sectional curvature of G along a nonzero vector* $v = (v^i, v^{i'}) \in T^{1,0}M$ *is defined by*

$$K_G(v) = -\frac{1}{G^2(v,\overline{v})} K_{\alpha\overline{\beta}\gamma\overline{\sigma}} v^\alpha \overline{v}^\beta v^\gamma \overline{v}^\sigma. \tag{38}$$

Theorem 2. *Let* $(M_1 \times {}_f M_2, G)$ *be a twisted product Hermitian manifold. Then, the holomorphic sectional curvature of G along a nonzero vector* $v = (v^i, v^{i'}) \in T^{1,0}M$ *is given by*

$$\begin{aligned} K_G(v) = \frac{1}{G^2(v,\overline{v})}[g^2 K_g(v_1) + f^2 h^2 K_h(v_2) - 2f^2 h(\frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^s} v^j \overline{v}^s \\ + \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^s} v^{j'} \overline{v}^s + \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^{s'}} v^j \overline{v}^{s'} + \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{s'}} v^{j'} \overline{v}^{s'})]. \end{aligned} \tag{39}$$

Proof. According to (28), (29) and (38), we have

$$\begin{aligned} K_G(v) = -\frac{1}{G^2(v,\overline{v})}(K_{k\bar{i}j\bar{s}} v^k \overline{v}^i v^j \overline{v}^s + K_{k'\overline{i'}j\overline{s}} v^{k'} \overline{v}^{i'} v^j \overline{v}^s + K_{k'\overline{i'}j'\overline{s}} v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^s \\ + K_{k'\overline{i'}j\overline{s'}} v^{k'} \overline{v}^{i'} v^j \overline{v}^{s'} + K_{k'\overline{i'}j'\overline{s'}} v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^{s'}). \end{aligned} \tag{40}$$

Using (27) and noting that $h_{k'\overline{i'}} v^{k'} \overline{v}^{i'} = h$, we have

$$\begin{aligned} K_{k'\overline{i'}j'\overline{s'}} v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^{s'} &= (2f^2 \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{s'}} h_{k'\overline{i'}} + f^2 \overset{2}{K}_{k'\overline{i'}j'\overline{s'}}) v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^{s'} \\ &= 2f^2 h \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{s'}} v^{j'} \overline{v}^{s'} - f^2 h^2 K_h(v_2). \end{aligned} \tag{41}$$

Similarly, we can obtain

$$K_{k\bar{i}j\bar{s}} v^k \overline{v}^i v^j \overline{v}^s = \overset{1}{K}_{k\bar{i}j\bar{s}} v^k \overline{v}^i v^j \overline{v}^s = -g^2 K_g(v_1), \tag{42}$$

$$K_{k'\overline{i'}j\overline{s}} v^{k'} \overline{v}^{i'} v^j \overline{v}^s = 2f^2 \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^s} h_{k'\overline{i'}} v^{k'} \overline{v}^{i'} v^j \overline{v}^s = 2f^2 h \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^s} v^j \overline{v}^s, \tag{43}$$

$$K_{k'\overline{i'}j'\overline{s}} v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^s = 2f^2 \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^s} h_{k'\overline{i'}} v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^s = 2f^2 h \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^s} v^{j'} \overline{v}^s, \tag{44}$$

$$K_{k'\overline{i'}j\overline{s'}} v^{k'} \overline{v}^{i'} v^j \overline{v}^{s'} = 2f^2 \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^{s'}} h_{k'\overline{i'}} v^{k'} \overline{v}^{i'} v^j \overline{v}^{s'} = 2f^2 h \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^{s'}} v^j \overline{v}^{s'}. \tag{45}$$

Plunging (41)–(45) into (40), we can obtain (39). □

According to Definition 4, we can easily obtain

Corollary 2. *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Suppose $\ln f$ is a pluriharmonic function, then*

$$K_G(v) = \frac{1}{G^2(v,\bar{v})}(g^2 K_g(v_1) + f^2 h^2 K_h(v_2)).$$

Theorem 3 ([21])**.** *Let (M, G) be a compact Hermitian manifold. Then, M has constant holomorphic sectional curvature κ if and only if, at every point of M,*

$$\Theta_{\alpha\bar{\beta}\gamma\bar{\sigma}} = -\frac{1}{2}\kappa(G_{\alpha\bar{\beta}}G_{\gamma\bar{\sigma}} + G_{\alpha\bar{\sigma}}G_{\gamma\bar{\beta}}), \tag{46}$$

where

$$\Theta_{\alpha\bar{\beta}\gamma\bar{\sigma}} = \frac{1}{4}(K_{\alpha\bar{\beta}\gamma\bar{\sigma}} + K_{\gamma\bar{\sigma}\alpha\bar{\beta}} + K_{\alpha\bar{\sigma}\gamma\bar{\beta}} + K_{\gamma\bar{\beta}\alpha\bar{\sigma}}). \tag{47}$$

Proposition 5. *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Then,*

$$\Theta_{k\bar{i}j\bar{s}} = \overset{1}{\Theta}_{k\bar{i}j\bar{s}}, \tag{48}$$

$$\Theta_{k'\bar{i}'j\bar{s}} = \Theta_{k'\bar{s}j\bar{i}'} = \Theta_{j\bar{i}'k'\bar{s}} = \Theta_{j\bar{s}k'\bar{i}'} = \frac{1}{2}f^2\frac{\partial^2 \ln f}{\partial z^j \partial \bar{z}^s}h_{k'\bar{i}'}, \tag{49}$$

$$\Theta_{k'\bar{i}'j'\bar{s}} = \Theta_{k'\bar{s}j'\bar{i}'} = \frac{1}{2}f^2(\frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^s}h_{j'\bar{i}'} + \frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^s}h_{k'\bar{i}'}), \tag{50}$$

$$\Theta_{k'\bar{i}'j\bar{s}'} = \Theta_{j\bar{i}'k'\bar{s}'} = \frac{1}{2}f^2(\frac{\partial^2 \ln f}{\partial z^j \partial \bar{z}^{i'}}h_{k'\bar{s}'} + \frac{\partial^2 \ln f}{\partial z^j \partial \bar{z}^{s'}}h_{k'\bar{i}'}), \tag{51}$$

$$\begin{aligned}\Theta_{k'\bar{i}'j'\bar{s}'} = \frac{1}{2}f^2(&\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{s'}}h_{k'\bar{i}'} + \frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{i'}}h_{j'\bar{s}'} + \frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{i'}}h_{k'\bar{s}'} \\ &+ \frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{s'}}h_{j'\bar{i}'}) + f^2\overset{2}{\Theta}_{k'\bar{i}'j'\bar{s}'},\end{aligned} \tag{52}$$

$$\Theta_{k'\bar{i}j\bar{s}} = \Theta_{k\bar{i}'j\bar{s}} = \Theta_{j\bar{i}k'\bar{s}} = \Theta_{k\bar{s}j\bar{i}'} = \Theta_{k'\bar{i}j'\bar{s}} = \Theta_{k\bar{i}'j\bar{s}'} = 0. \tag{53}$$

Proof. By putting $\alpha = k', \beta = i', \gamma = j', \sigma = s'$ in (47), we have

$$\Theta_{k'\bar{i}'j'\bar{s}'} = \frac{1}{4}(K_{k'\bar{i}'j'\bar{s}'} + K_{j'\bar{s}'k'\bar{i}'} + K_{k'\bar{s}'j'\bar{i}'} + K_{j'\bar{i}'k'\bar{s}'}). \tag{54}$$

By using (27), we obtain

$$\begin{aligned}\Theta_{k'\bar{i}'j'\bar{s}'} &= \frac{1}{4}(2f^2\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{s'}}h_{k'\bar{i}'} + f^2\overset{2}{K}_{k'\bar{i}'j'\bar{s}'} + 2f^2\frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{i'}}h_{j'\bar{s}'} + f^2\overset{2}{K}_{j'\bar{s}'k'\bar{i}'} \\ &\quad + 2f^2\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{i'}}h_{k'\bar{s}'} + f^2\overset{2}{K}_{k'\bar{s}'j'\bar{i}'} + 2f^2\frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{s'}}h_{j'\bar{i}'} + f^2\overset{2}{K}_{j'\bar{i}'k'\bar{s}'}) \\ &= \frac{1}{2}f^2(\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{s'}}h_{k'\bar{i}'} + \frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{i'}}h_{j'\bar{s}'} + \frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{i'}}h_{k'\bar{s}'} + \frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{s'}}h_{j'\bar{i}'}) + f^2\overset{2}{\Theta}_{k'\bar{i}'j'\bar{s}'}.\end{aligned}$$

Similar calculations give the rest of the equalities of Proposition 5. □

Theorem 4. *Let $(M_1 \times {}_f M_2, G)$ be a compact twisted product Hermitian manifold. Then, G has constant holomorphic sectional curvature κ if and only if $\kappa = 0$ and the following equalities hold*

$$\begin{cases} \overset{1}{\Theta}_{k\bar{i}j\bar{s}} = 0, & (55a) \\ \overset{1}{L}(\ln f) = 0, & (55b) \\ \dfrac{\partial^2 lnf}{\partial z^{j'}\partial \overline{z}^{s}} = 0, & (55c) \\ \dfrac{\partial^2 lnf}{\partial z^{j}\partial \overline{z}^{s'}} = 0, & (55d) \\ 2\overset{2}{L}(\ln f) + h^{\overline{s'}j'}h^{\overline{i'}k'}\overset{2}{\Theta}_{k'\overline{i'}j'\overline{s'}} = 0. & (55e) \end{cases}$$

Proof. According to Theorem 3, (13) and (53), $(M_1 \times {}_f M_2, G)$ has constant holomorphic sectional curvature if and only if

$$\begin{cases} \Theta_{k\bar{i}j\bar{s}} = -\dfrac{1}{2}\kappa(G_{k\bar{i}}G_{j\bar{s}} + G_{k\bar{s}}G_{j\bar{i}}), & (56a) \\ \Theta_{k'\overline{i'}j\bar{s}} = -\dfrac{1}{2}\kappa G_{k'\overline{i'}}G_{j\bar{s}}, & (56b) \\ \Theta_{k'\overline{i'}j'\bar{s}} = 0, & (56c) \\ \Theta_{k'\overline{i'}j\overline{s'}} = 0, & (56d) \\ \Theta_{k'\overline{i'}j'\overline{s'}} = -\dfrac{1}{2}\kappa(G_{k'\overline{i'}}G_{j'\overline{s'}} + G_{k'\overline{s'}}G_{j'\overline{i'}}). & (56e) \end{cases}$$

Substituting (13) and (48)–(52) into (56a)–(56e), and noticing that $f^2 \neq 0$, (56a)–(56e) are thus equivalent to the following equalities

$$\begin{cases} \overset{1}{\Theta}_{k\bar{i}j\bar{s}} = -\dfrac{1}{2}\kappa(g_{k\bar{i}}g_{j\bar{s}} + g_{k\bar{s}}g_{j\bar{i}}), \\ \dfrac{\partial^2 lnf}{\partial z^{j}\partial \overline{z}^{s}}h_{k'\overline{i'}} = -\kappa h_{k'\overline{i'}}g_{j\bar{s}}, \\ \dfrac{\partial^2 \ln f}{\partial z^{k'}\partial \overline{z}^{s}}h_{j'\overline{i'}} + \dfrac{\partial^2 lnf}{\partial z^{j'}\partial \overline{z}^{s}}h_{k'\overline{i'}} = 0, \\ \dfrac{\partial^2 \ln f}{\partial z^{j}\partial \overline{z}^{i'}}h_{k'\overline{s'}} + \dfrac{\partial^2 lnf}{\partial z^{j}\partial \overline{z}^{s'}}h_{k'\overline{i'}} = 0, \\ \dfrac{\partial^2 \ln f}{\partial z^{j'}\partial \overline{z}^{s'}}h_{k'\overline{i'}} + \dfrac{\partial^2 \ln f}{\partial z^{k'}\partial \overline{z}^{i'}}h_{j'\overline{s'}} + \dfrac{\partial^2 \ln f}{\partial z^{j'}\partial \overline{z}^{i'}}h_{k'\overline{s'}} + \dfrac{\partial^2 \ln f}{\partial z^{k'}\partial \overline{z}^{s'}}h_{j} \\ +2\overset{2}{\Theta}_{k'\overline{i'}j'\overline{s'}} = -\kappa f^2(h_{k'\overline{i'}}h_{j'\overline{s'}} + h_{k'\overline{s'}}h_{j'\overline{i'}}). \end{cases}$$

The above equalities are equivalent to

$$\begin{cases} \overset{1}{\Theta}_{k\bar{i}j\bar{s}} = -\dfrac{1}{2}\kappa(g_{k\bar{i}}g_{j\bar{s}} + g_{k\bar{s}}g_{j\bar{i}}), & (58a) \\ \overset{1}{L}(\ln f) = -\kappa, & (58b) \\ \dfrac{\partial^2 lnf}{\partial z^{j'}\partial \overline{z}^{s}} = 0, & (58c) \\ \dfrac{\partial^2 lnf}{\partial z^{j}\partial \overline{z}^{s'}} = 0, & (58d) \\ 2\overset{2}{L}(\ln f) + h^{\overline{s'}j'}h^{\overline{i'}k'}\overset{2}{\Theta}_{k'\overline{i'}j'\overline{s'}} = -\kappa f^2. & (58e) \end{cases}$$

In fact, contracting (57b) with $h^{\overline{i'}k'}$ and $g^{\bar{s}j}$ successively, and noticing that $\overset{1}{L} = g^{\bar{i}k}\frac{\partial^2}{\partial z^k \partial \overline{z}^i}$, we can obtain (58b). Contracting (57c) and (57d) with $h^{\overline{i'}k'}$, respectively, we can obtain (58c) and

(58d). Contracting (57e) with $h^{\bar{i}'k'}$ and $h^{\bar{s}'j'}$ successively, and noticing that $\overset{2}{L} = h^{\bar{i}'k'} \frac{\partial^2}{\partial z^{k'} \partial \bar{z}^{i'}}$, we can obtain (58e).

Proof of the necessity.

Let us suppose that $\kappa \neq 0$, combining (58a) and (58b), we have

$$2\overset{1}{\Theta}_{k\bar{i}j\bar{s}} = \overset{1}{L}(\ln f)(g_{k\bar{i}}g_{j\bar{s}} + g_{k\bar{s}}g_{j\bar{i}}), \tag{59}$$

since $\overset{1}{\Theta}_{k\bar{i}j\bar{s}}$, $g_{k\bar{i}}$ depend only on z_1, which says that f only depends on M_1. These are contradicted by the fact that $(M_1 \times {}_f M_2, G)$ is a twisted product Hermitian manifold. Thus,

$$\kappa = 0. \tag{60}$$

Plunging (60) into (58a), (58b) and (58e), we can check that (58a)–(58e) can be simplified as (55a)–(55e).

Next, we prove the sufficiency.

Suppose that $\kappa = 0$ and (55a)–(55e) hold; this immediately confirms that (57a)–(57e) hold, i.e., $(M_1 \times {}_f M_2, G)$ has constant holomorphic sectional curvature κ. Thus, we complete the proof. □

6. Chern Flat and Chern Ricci-Flat Twisted Product Hermitian Manifolds

Let (M_1, g) and (M_2, h) be two Chern flat or Chern Ricci-flat Hermitian manifolds, respectively. We would like to know under what conditions the twisted product Hermitian manifold $(M_1 \times {}_f M_2, G)$ is Chern flat or Chern Ricci-flat.

Definition 8 ([22])**.** *A Hermitian manifold (M, G) is called Chern flat if*

$$K = 0,$$

where K is the Chern curvature tensor.

Definition 9 ([22])**.** *A Hermitian manifold (M, G) is called Chern Ricci-flat if*

$$K^{(1)} = 0,$$

where $K^{(1)}$ is the first Chern Ricci curvature tensor.

Theorem 5. *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Suppose $\ln f$ is pluriharmonic, then $(M_1 \times {}_f M_2, G)$ is Chern flat if and only if (M_1, g) and (M_2, h) are Chern flat.*

Proof. Since $\ln f$ is pluriharmonic, then

$$\frac{\partial^2 \ln f}{\partial z^j \partial \bar{z}^s} = 0, \tag{61}$$

$$\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^s} = 0, \tag{62}$$

$$\frac{\partial^2 \ln f}{\partial z^j \partial \bar{z}^{s'}} = 0, \tag{63}$$

$$\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{s'}} = 0. \tag{64}$$

According to Definition 8 and (2), $(M_1 \times {}_f M_2, G)$ is Chern flat if and only if

$$K_{\alpha\bar{\beta}\gamma\bar{\sigma}} = 0. \tag{65}$$

Using Proposition 3 and (61)–(64), and noticing that $f^2 \neq 0$, (65) is equivalent to following equalities

$$\begin{cases} \overset{1}{K}_{k\bar{i}j\bar{s}} = 0, \\ \overset{2}{K}_{k'\bar{i'}j'\bar{s'}} = 0. \end{cases} \tag{66}$$

which means that (M_1, g) and (M_2, h) are Chern flat. □

Theorem 6. *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Suppose $\ln f$ is pluriharmonic, then $(M_1 \times {}_f M_2, G)$ is Chern Ricci-flat if and only if (M_1, g) and (M_2, h) are Chern Ricci-flat.*

Proof. Suppose that $\ln f$ is pluriharmonic, then

$$\overset{1}{L}(\ln f) = \overset{2}{L}(\ln f) = 0. \tag{67}$$

By Definition 9 and (5), $(M_1 \times {}_f M_2, G)$ is Chern Ricci flat if and only if

$$K^{(1)}_{\alpha\bar{\beta}} = 0. \tag{68}$$

Using (31), (32) and (67), (68) is equivalent to the following equalities

$$\begin{cases} \overset{1}{K}{}^{(1)}_{k\bar{i}} = 0, \\ \overset{2}{K}{}^{(1)}_{k'\bar{i'}} = 0. \end{cases} \tag{69}$$

Which means that (M_1, g) and (M_2, h) are Chern Ricci flat. □

7. Conclusions

In this paper, we extended the twisted product to Hermitian manifold. Based on this, we confirmed that the compact twisted product Hermitian manifold has constant holomorphic sectional curvature if and only if $\kappa = 0$ and a system of differential equations holds. Under the condition that the logarithm of the twisted function is pluriharmonic, we obtained the necessary and sufficient conditions for the twisted product Hermitian manifold to be Chern flat or Chern Ricci-flat, respectively, so then we gave an effective way to construct Chern flat or Chern Ricci-flat Hermitian manifolds.

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Article

An Invariant of Riemannian Type for Legendrian Warped Product Submanifolds of Sasakian Space Forms

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Abstract: In the present paper, we investigate the geometry and topology of warped product Legendrian submanifolds in Sasakian space forms $\mathbb{D}^{2n+1}(\epsilon)$ and obtain the first Chen inequality that involves extrinsic invariants like the mean curvature and the length of the warping functions. This inequality also involves intrinsic invariants (δ-invariant and sectional curvature). In addition, an integral bound is provided for the Bochner operator formula of compact warped product submanifolds in terms of the gradient Ricci curvature. Some new results on mean curvature vanishing are presented as a partial solution to the well-known problem given by S.S. Chern.

Keywords: warped products; Legendrian; Sasakian space form; Ricci curvature; ordinary differential equations; Riemannian invariants; Bochner operator formula; eigenvalues

MSC: 53C20; 53C21; 53C40; 53C42; 53C80; 53Z05

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1. Introduction and Main Motivations

The geometry of warped product manifolds is rich and varied, and their properties depend crucially on the choice of the warping function. Understanding the behavior of this function is therefore of fundamental importance in the study of these objects. In recent years, there has been a surge of interest in the study of warped product manifolds, driven in part by their wide-ranging applications and their connections to other areas of mathematics. Therefore, the study of warped product manifolds has many important applications in geometry and physics. For example, in general relativity, warped product manifolds are used to model certain types of black hole spacetimes. In algebraic geometry, they arise in studying moduli spaces of vector bundles on algebraic varieties. In topology, they have been used to construct examples of exotic manifolds that do not admit a smooth structure [1–3]. On the other hand, the Chen delta invariant is a numerical invariant in algebraic topology that measures the extent to which a loop in space fails to be a boundary of a surface. More precisely, if a loop is the boundary of a surface, then the Chen delta invariant is zero. Otherwise, it gives a measure of how "far" the loop is from being a boundary. Applications of the delta-invariant can be found in various areas of mathematics, including topology, geometry, and algebraic geometry. For example, it has been used to study the topology of moduli spaces of algebraic curves, the geometry of the Kähler–Einstein metric on a complex manifold, and the topology of configuration spaces of particles in a Euclidean space. It has also found applications in physics, particularly in the study of topological field theories [4–6]. Numerous mathematicians have also investigated product manifolds and related submanifolds. To address the issues, new forms of Riemannian invariants, distinct from classical invariants, must be introduced. Furthermore, general optimum links between the essential extrinsic invariants and the new intrinsic invariants for submanifolds must be established. This was the reason for Chen [7] to introduce a notion that delta-invariants on Riemannian manifolds and discussed in detail [4,8]. More specifically, they introduced a novel family of curvature functions on submanifolds in the 1990s. A good isometric immersion that creates the least amount of tension from the surrounding space at each point roughly describes the ideal immersion of a Riemannian manifold into a real space form [9]. Chen proposed that the submanifold satisfying the equality condition is known as the ideal submanifold and developed numerous inequalities in terms of invariants. Chen's submanifolds are a substitute for these submanifolds in [4]. Chen has described the ideal submanifolds in real space forms and complex space forms [6,7,9–11]. In addition, Dillen, Petrovic, Verstraelen, Mihai, and Tripathi investigated conformally flat, semisymmetric, and Ricci-semisymmetric submanifolds obeying Chen's inequality in real space forms [12–18] and also (see [10] and references therein) for more information about ideal submanifolds.

It should be noted that there are few studies on the δ-invariant for warped product structures other than the Chen-derived optimal inequality for CR-warped products in complex space form [19]. Recently, Mustafa et al. [20] constructed the first Chen invariant for warped product submanifolds in real space forms and discussed the minimality conditions on submanifolds. From this point of view, by using the Gauss equation instead of the Codazzi equation in the sense of [13], in the first part of this paper, we provide a sharp estimate of the squared norm of the mean curvature in terms of a warping function and the constant holomorphic sectional curvature in the spirit of [21–33], motivated by the historical development on the study of a warping function of a warped product submanifold [34]. As the main objective of our study, we present a novel method for establishing inequalities for δ-invariant curvature inequalities for warped product Legendrian submanifolds isometrically immersed in Sasakian space. This has been discussed in [20,21,35]. As a consequence of the main results discussed in this paper, we generalize a number of inequalities for areas on Euclidean spheres and Euclidean spaces. There is another significant group of Riemannian products in this family.

2. Preliminaries

A $(2m+1)$-dimensional manifold $\widetilde{\mathcal{D}}^{2m+1}$ endowed with an almost-contact structure (φ, ξ, η, g) is called an almost-contact metric manifold when it satisfies the following properties:

$$\varphi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \varphi(\xi) = 0, \;\; \eta \circ \varphi = 0, \tag{1}$$

$$g(\varphi \mathcal{X}_1, \varphi \mathcal{X}_2) = g(\mathcal{X}_1, \mathcal{X}_2) - \eta(\mathcal{X}_1)\eta(\mathcal{X}_2), \;\; and \;\; \eta(\mathcal{X}_1) = g(\mathcal{X}_1, \xi), \tag{2}$$

for any $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}(T\widetilde{\mathcal{D}}^{2m+1})$, where the Lie algebra of vector fields is on a manifold $\widetilde{\mathcal{D}}^{2m+1}$. In this case, φ, g, ξ, and η are called $(1,1)$-tensor fields, a structure vector field, and dual 1-form, respectively. Furthermore, an almost-contact metric manifold is known to be a *Sasakian manifold* (cf. [22,36,37]) if

$$(\widetilde{\nabla}_{\mathcal{X}_1} \varphi)\mathcal{X}_2 = g(\mathcal{X}_1, \mathcal{X}_2)\xi - \eta(\mathcal{X}_2)\mathcal{X}_1, \quad \widetilde{\nabla}_{\mathcal{X}_1}\xi = -\varphi \mathcal{X}_1, \tag{3}$$

for any vector fields $\mathcal{X}_1, \mathcal{X}_2$ on $\widetilde{\mathcal{D}}^{2m+1}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g. An n-dimensional Riemannian submanifold $\mathbb{D}^n$ of $\widetilde{\mathcal{D}}^{2m+1}$ is referred to as totally real if the standard almost-contact structure φ of $\widetilde{\mathcal{D}}^{2m+1}$ maps any tangent space of $\mathbb{D}^n$ into its corresponding normal space (see [22,35,38,39]). Now, let $\mathbb{D}^n$ be an isometric immersed submanifold of dimension n in $\widetilde{\mathcal{D}}^{2m+1}$, then $\mathbb{D}^n$ is referred to as a Legendrian submanifold if ξ is a normal vector field on $\mathbb{D}^n$ (i.e., $\mathbb{D}^n$ is a C-totally real submanifold) and $m = n$ [22,35,38]. Legendrian submanifolds play a substantial role in contact geometry. From the Riemannian geometric perspective, studying the Legendrian submanifolds of Sasakian manifolds was initiated in the 1970s.

Let $\mathbb{D}$ be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian $\widetilde{\mathcal{D}}^{2m+1}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are induced connections on the tangent bundle TM and normal bundle $T^{\perp}\mathbb{D}$ of $\mathbb{D}^n$, respectively. Then, the Gauss and Weingarten formulas are given by

$$(i)\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_2 = \nabla_{\mathcal{X}_1}\mathcal{X}_2 + \zeta(\mathcal{X}_1,\mathcal{X}_2), (ii)\widetilde{\nabla}_{\mathcal{X}_1}N = -A_N\mathcal{X}_1 + \nabla^{\perp}_{\mathcal{X}_1}N, \tag{4}$$

for each $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}(T\mathbb{D})$ and $N \in \mathfrak{X}(T^{\perp}\mathbb{D})$, where ζ and A_N are the second fundamental form and shape operator (corresponding to the normal vector field N), respectively, for the immersion of $\mathbb{D}^n$ into $\widetilde{\mathcal{D}}^{2m+1}$, and they are related as follows:

$$g(\zeta(\mathcal{X}_1,\mathcal{X}_2),N) = g(A_N\mathcal{X}_1,\mathcal{X}_2). \tag{5}$$

Similarly, the equations of Gauss and Codazzi are, respectively, given by

$$(i)R(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\mathcal{X}_4) = \widetilde{R}(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\mathcal{X}_4) + g\big(\zeta(\mathcal{X}_1,\mathcal{X}_4),\zeta(\mathcal{X}_2,\mathcal{X}_3)\big) - g\big(\zeta(\mathcal{X}_1,\mathcal{X}_3),\zeta(\mathcal{X}_2,\mathcal{X}_4)\big). \tag{6}$$

$$(ii)\big(\widetilde{R}(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3\big)^{\perp} = (\widetilde{\nabla}_{\mathcal{X}_1}\zeta)(\mathcal{X}_2,\mathcal{X}_3) - (\widetilde{\nabla}_{\mathcal{X}_2}\zeta)(\mathcal{X}_1,\mathcal{X}_3). \tag{7}$$

For all $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\mathcal{X}_4 \in \mathfrak{X}(T\widetilde{M})$, $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are the curvature tensor of $\widetilde{\mathcal{D}}^{2n+1}$ and $\mathbb{D}^n$, respectively. The mean curvature $\mathbb{H}$ of Riemannian submanifold $\mathbb{D}^n$ is given by

$$\mathbb{H} = \frac{1}{n}trace(\zeta). \tag{8}$$

A submanifold $\mathbb{D}^n$ of Riemannian manifold $\widetilde{\mathcal{D}}^{2n+1}$ is said to be a *totally umbilical* if

$$\zeta(\mathcal{X}_1,\mathcal{X}_2) = g(\mathcal{X}_1,\mathcal{X}_1)\mathbb{H},$$

and *totally geodesic* if

$$\zeta(\mathcal{X}_1,\mathcal{X}_2) = 0,$$

for any $\mathcal{X}_1,\mathcal{X}_2 \in \mathfrak{X}(TM)$, respectively, where $\mathbb{H}$ is the mean curvature vector of $\mathbb{D}^n$. Furthermore, if $\mathbb{H} = 0$, then $\mathbb{D}^n$ is *minimal* in $\widetilde{\mathbb{D}}^{2m+1}$. Moreover, the related null space or kernel of the second fundamental form of $\mathbb{D}^n$ at x is defined by

$$\mathbb{D}_x = \big\{\mathcal{X}_1 \in T_x\mathbb{D} : \zeta(\mathcal{X}_1,\mathcal{X}_2) = 0, forall\mathcal{X}_2 \in T_x\mathbb{D}\big\}. \tag{9}$$

In this context, we shall define another important Riemannian intrinsic invariant called the scalar curvature of $\widetilde{\mathcal{D}}^{2m+1}$, denoted at $\widetilde{\tau}(T_x\widetilde{\mathcal{D}}^{2m+1})$, which, at some x in $\widetilde{\mathcal{D}}^{2m+1}$, is given as

$$\widetilde{\tau}(T_x\widetilde{\mathcal{D}}^{2m+1}) = \sum_{1\le i<j\le 2m+1}\widetilde{\mathcal{K}}_{ij}, \tag{10}$$

where $\widetilde{\mathcal{K}}_{ij} = \widetilde{\mathcal{K}}\big(e_i \wedge e_j\big)$. It is clear that Equality (10) is congruent to the following equation, which will be frequently used in a subsequent proof:

$$2\widetilde{\tau}(T_x\widetilde{\mathcal{D}}^{2m+1}) = \sum_{1\le i<j\le 2m+1}\widetilde{\mathcal{K}}_{ij}, \tag{11}$$

Similarly, scalar curvature $\widetilde{\tau}(L_x)$ of $L-$plan is given by

$$\widetilde{\tau}(L_x) = \sum_{1\le i<j\le m}\widetilde{\mathcal{K}}_{ij}, \tag{12}$$

Let $\{e_1, \cdots e_n\}$ be an orthonormal basis of the tangent space $T_x\mathbb{D}$ and $e_r = (e_{n+1}, \cdots e_{2m+1})$ belonging to an orthonormal basis of the normal space $T^{\perp}\mathbb{D}$, then we have

$$\zeta^r_{ij} = g(\zeta(e_i, e_j), e_r) \ \ and \ \ ||\zeta||^2 = \sum_{i,j=1}^{n} g\big(\zeta(e_i, e_j), \zeta(e_i, e_j)\big). \tag{13}$$

Let $\mathcal{K}_{ij}$ and $\widetilde{\mathcal{K}}_{ij}$ denote the sectional curvature of the plane section spanned and e_i at x in the submanifold $\mathbb{D}^n$ and in the Riemannian space form $\widetilde{\mathcal{D}}^{2n+1}(c)$, respectively. Thus, $\mathcal{K}_{ij}$ and $\widetilde{\mathcal{K}}_{ij}$ are the intrinsic and extrinsic sectional curvatures of the span $\{e_i, e_j\}$ at x, thus from Gauss Equation (6)(i), we have

$$2\widetilde{\tau}(T_x\widetilde{\mathbb{D}}^n) = \mathcal{K}_{ij} = 2\widetilde{\tau}(T_x\widetilde{\mathcal{D}}^{2m+1}) = \widetilde{\mathcal{K}}_{ij} + \sum_{r=n+1}^{2m+1}\Big(\zeta^r_{ii}\zeta^r_{jj} - (\zeta^r_{ij})^2\Big). \tag{14}$$

The second invariant is called the *Chen first invariant*, which is defined as

$$\delta_{\widetilde{\mathcal{D}}^{2m+1}}(x) = \widetilde{\tau}(T_x\widetilde{\mathcal{D}}^{2m+1}) - inf\Big\{\widetilde{\mathcal{K}}(\pi) : \pi \subset T_x\widetilde{\mathcal{D}}^{2m+1}, x \in \widetilde{\mathcal{D}}^{2m+1}, \dim\zeta = 2\Big\} \tag{15}$$

Assume that $\mathbb{D}_1^{s_1}$ and $\mathbb{D}_2^{s_2}$ are two Riemannian manifolds with their Riemannian metrices g_1 and g_2, respectively. Let $f > 0$ be a smooth function defined on $\mathbb{D}_1^{s_1}$. Then, warped product manifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is the manifold $\mathbb{D}_1^{s_1} \times \mathbb{D}_2^{s_2}$ furnished by the Riemannian metric $g = g_1 + f^2 g_2$ [1]. Assume that $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is a warped product manifold, then for any $\mathcal{X}_1 \in \Gamma(T\mathbb{D}_1^{s_1})$ and $\mathcal{X}_3 \in \Gamma(T\mathbb{D}_2^{s_2})$, we find that

$$\nabla_{\mathcal{X}_3}\mathcal{X}_1 = \nabla_{\mathcal{X}_1}\mathcal{X}_3 = (\mathcal{X}_1 \ln f)\mathcal{X}_3. \tag{16}$$

Similarly, from unit vector fields, $\mathcal{X}_1$ and $\mathcal{X}_3$ are tangent to $\mathbb{D}_1^{s_1}$ and $\mathbb{D}_2^{s_2}$, respectively, thus deriving

$$\begin{aligned}\mathcal{K}(\mathcal{X}_1 \wedge \mathcal{X}_3) =& g(R(\mathcal{X}_1, \mathcal{X}_3)\mathcal{X}_1, \mathcal{X}_3)\\ =&(\nabla_{\mathcal{X}_1}\mathcal{X}_1)\ln f g(\mathcal{X}_3, \mathcal{X}_3) - g\big(\nabla_{\mathcal{X}_1}((\mathcal{X}_1 \ln f)\mathcal{X}_3), \mathcal{X}_3\big)\\ =& (\nabla_{\mathcal{X}_1}\mathcal{X}_1)\ln f g(\mathcal{X}_3, \mathcal{X}_3) - g\big(\nabla_{\mathcal{X}_1}(\mathcal{X}_1 \ln f)\mathcal{X}_3 + (\mathcal{X}_1 \ln f)g(\nabla_{\mathcal{X}_1}\mathcal{X}_3, \mathcal{X}_3\big)\\ =& (\nabla_{\mathcal{X}_1}\mathcal{X}_1)\ln f g(\mathcal{X}_3, \mathcal{X}_3) - (\mathcal{X}_1 \ln f)^2 - \mathcal{X}_1(\mathcal{X}_1 \ln f).\end{aligned} \tag{17}$$

Suppose that $\{e_1, \cdots e_n\}$ is an orthonormal frame for $\mathbb{D}^n$, then sum up over the vector fields such that

$$\sum_{i=1}^{s_1}\sum_{j=1}^{s_2}\mathcal{K}(e_i \wedge e_j) = \sum_{i=1}^{s_1}\sum_{j=1}^{s_2}\Bigg((\nabla_{e_i}e_i)\ln f - e_i(e_i \ln f) - (e_i \ln f)^2\Bigg),$$

which implies that

$$\sum_{i=1}^{s_1}\sum_{j=1}^{s_2}\mathcal{K}(e_i \wedge e_j) = s_2\Big(\Delta(\ln f) - ||\nabla(\ln f)||^2\Big). \tag{18}$$

But, it was proved [9] that for arbitrary warped product submanifolds,

$$\sum_{i=1}^{s_1}\sum_{j=1}^{s_2}\mathcal{K}(e_i \wedge e_j) = \frac{s_2\Delta f}{f}. \tag{19}$$

Thus, from (18) and (19), we obtain

$$\frac{\Delta f}{f} = \Delta(\ln f) - ||\nabla(\ln f)||^2. \tag{20}$$

The following remarks are consequences of warped product submanifolds:

Remark 1. *A warped product manifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is said to be trivial if the warping function f is constant or simply a Riemannian product manifold.*

Remark 2. *If $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is a warped product manifold, then $\mathbb{D}_1$ is a totally geodesic and $\mathbb{D}_2$ is a totally umbilical submanifold of $\mathbb{D}^n$, respectively.*

A Sasakian manifold is said to be Sasakian space form with a constant φ-sectional curvature ϵ if and only if the Riemannian curvature tensor $\widetilde{R}$ can be written as (see [22,38]):

$$\begin{aligned}\widetilde{R}(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4) = \left(\frac{\epsilon+3}{4}\right)&\Big\{g(\mathcal{X}_2, \mathcal{X}_3)g(\mathcal{X}_1, \mathcal{X}_4) - g(\mathcal{X}_1, \mathcal{X}_3)g(\mathcal{X}_2, \mathcal{X}_4)\Big\} \\ + \left(\frac{\epsilon-1}{4}\right)&\Big\{\eta(\mathcal{X}_1)\eta(\mathcal{X}_3)g(\mathcal{X}_2, \mathcal{X}_4) + \eta(\mathcal{X}_4)\eta(\mathcal{X}_2)g(\mathcal{X}_1, \mathcal{X}_3) \\ &- \eta(\mathcal{X}_2)\eta(\mathcal{X}_3)g(\mathcal{X}_1, \mathcal{X}_4) - \eta(\mathcal{X}_1)g(\mathcal{X}_2, \mathcal{X}_3)\eta(\mathcal{X}_4) \\ &+ g(\varphi\mathcal{X}_2, \mathcal{X}_3)g(\varphi\mathcal{X}_1, \mathcal{X}_4) - g(\varphi\mathcal{X}_1, \mathcal{X}_3)g(\varphi\mathcal{X}_2, \mathcal{X}_4) \\ &+ 2g(\mathcal{X}_1, \varphi\mathcal{X}_2)g(\varphi\mathcal{X}_3, \mathcal{X}_4)\Big\},\end{aligned} \tag{21}$$

where $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4 \in \mathfrak{X}(T\widetilde{\mathcal{D}}^{2m+1})$. Moreover, $\mathbb{R}^{2m+1}$ and $\mathbb{S}^{2m+1}$ with standard Sasakian structures can be given as typical examples of Sasakian space forms. Many geometers have drawn significant attention to minimal Legendrian submanifolds in particular.
We recall the following important algebraic lemma.

Lemma 1. *Let $t_1, t_2 \cdots t_n, s$ $(n+1)(n \geq 2)$ be a real number such that*

$$\sum_{i=1}^{n}(t_i)^2 = (n-1)\Big(\sum_{i=1}^{n} t_1^2 + s\Big). \tag{22}$$

Then, $2t_1t_2 \geq s$ with an equality holds if and only if $t_1 + t_2 = t_3 = \cdots t_n$.

Theorem 1. *Let $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ be an isometric immersion of a warped product Legendrian submanifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ into a Sasakian space form $\widetilde{\mathcal{D}}^{2n+1}(\epsilon)$. Then, for each point $x \in \mathbb{D}^n$ and each plane section $\pi_i \subset T_x\mathbb{D}_i^{n_i}$, for $i = 1, 2$, we obtain*

(1) Let $\pi_1 \subset T_x\mathbb{D}_1^{s_1}$, then

$$\begin{aligned}\delta_{\widetilde{\mathbb{D}}^{s_1}}(x) \leq \frac{n^2}{2}||\mathbb{H}||^2 + s_2||\nabla(\ln f)||^2 - s_2\Delta(\ln f) \\ + \left\{\frac{s_1}{2}\Big(s_1 + 2s_2 - 1\Big) - 1\right\}\left(\frac{\epsilon+3}{4}\right).\end{aligned} \tag{23}$$

The equality of the above inequality holds at $x \in \mathbb{D}^n$ if and only if there exists an orthonormal basis $\{e_1 \cdots e_n\}$ of $T_x\mathbb{D}^n$ and orthonormal basis $\{e_n + 1 \cdots e_{2n+1}\}$ of $T_x^{\perp}$ such that (a) $\pi = Span\{e_1, e_2\}$ and (b) shape operators take the following form

$$(i) A_{e_{n+1}} = \left(\begin{array}{ccccc|ccc} \mu_1 & \zeta_{12}^{n+1} & 0 & \cdots & 0_{1s_1} & 0_{1s_1+1} & \cdots & 0_{1n} \\ \zeta_{12}^{n+1} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{s_1 1} & 0 & 0 & \cdots & \mu & 0_{s_1 s_1+1} & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+11} & \cdots & \cdots & \cdots & 0_{s_1+1s_1} & \zeta_{s_1+1s_1+1}^{n+1} & \cdots & \zeta_{s_1+1n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \cdots & \zeta_{nn}^{n+1} \end{array}\right),$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \cdots m\}$, then we have the matric

$$(ii) A_{e_r} = \left(\begin{array}{ccccc|ccc} \zeta_{11}^{r} & \zeta_{12}^{r} & 0 & \cdots & 0_{1s_1} & 0_{1s_1+1} & \cdots & 0_{1n} \\ \zeta_{21}^{r} & -\zeta_{11}^{r} & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots\vdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{s_1 1} & 0 & 0 & \cdots & 0_{s_1 s_1} & 0_{s_1 s_1+1} & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+11} & \cdots & \cdots & \cdots & 0_{s_1+1s_1} & \zeta_{s_1+1s_1+1}^{n+1} & \cdots & \zeta_{s_1+1n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \cdots & \zeta_{nn}^{n+1} \end{array}\right),$$

(2) *If $\pi_2 \subset T_x\mathbb{D}_2^{s_2}$, then*

$$\delta_{\widetilde{\mathbb{D}}^{s_2}}(x) \leq \frac{n^2}{2}||\mathbb{H}||^2 + s_2||\nabla(\ln f)||^2 - s_2\Delta(\ln f) + \left\{\frac{s_2}{2}\Big(s_2 + 2s_1 - 1\Big) - 1\right\}\left(\frac{\epsilon+3}{4}\right). \tag{24}$$

Equalities of the above equation hold if and only if

$$(iii) A_{e_{n+1}} = \left(\begin{array}{cccc|ccccc} \zeta_{11}^{n+1} & \cdots & \cdots & \zeta_{1s_1}^{n+1} & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \zeta_{s_1 11}^{n+1} & \cdots & \cdots & \zeta_{s_1 s_1}^{n+1} & 0_{s_1 s_1+1} & \cdots & \cdots & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+11} & \cdots & \cdots & 0_{s_1+1s_1} & \mu_1 & \zeta_{s_1+1s_1+2}^{n+1} & 0 & \cdots & 0_{s_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \zeta_{s_1+2s_1+1}^{n+1} & \mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & \mu \end{array}\right),$$

where $\mu = \mu_1 + \mu_2$*. If* $r \in \{n+2, \cdots, m\}$*, thus we have*

$$(iv) A_{e_r} = \left(\begin{array}{cccc|cccccc} \zeta^r_{11} & \cdots & \cdots & \zeta^r_{1s_1} & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \zeta^r_{s_1 11} & \cdots & \cdots & \zeta^r_{s_1 s_1} & 0_{s_1 s_1+1} & \cdots & \cdots & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+11} & \cdots & \cdots & 0_{s_1+1s_1} & \zeta^r_{s_1+1s_1+1} & \zeta^{n+1}_{s_1+1s_1+2} & 0 & \cdots & 0_{s_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \zeta^{n+1}_{s_1+2s_1+1} & -\zeta^r_{s_1+1s_1+1} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & 0 \end{array}\right),$$

(v) If the equality holds in (1) or (2), then $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ *is mixed totally geodesic in* $\mathbb{D}_\epsilon^{2n+1}$*. Moreover,* $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ *is both* $\mathbb{D}_1^{s_1}$*-minimal and* $\mathbb{D}_2^{s_2}$*-minimal. Thus,* $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ *is a minimal warped product submanifold in* $\mathbb{D}_\epsilon^{2n+1}$*.*

Proof. Let $\pi_1 \subset T_x\mathbb{D}_1$ be 2-plane for $x \in \mathbb{D}^n$, then we consider the orthonormal basis $\{e_1 \cdots e_{s_1}, e_{s_1+1}, \cdots e_n\}$ of $T_x\mathbb{D}^n$ such that $\{e_1, \cdots e_{s_1}\}$ is an orthonomal basis for $T_x\mathbb{D}_1$ and $\{e_{s_1+1}, \cdots e_n\}$ is for $T_x\mathbb{D}_2$. Similarly, $\{e_{n+1}, \cdots e_{2n+1}\}$ is an orthonormal basis for $T_x^\perp\mathbb{D}^n$. Assume that $\pi = Span\{e_1, e_2\}$ such that the normal vector e_{n+1} is in the direction of mean curvature vector $\mathbb{H}$, thus from (21) and Gauss Equation (6), we obtain

$$n^2\|\mathbb{H}\|^2 = 2\tau(T_x\mathbb{D}^n) + ||\zeta||^2 - n(n-1)\left(\frac{\epsilon+3}{4}\right). \tag{25}$$

which implies that

$$\Big(\sum_{i=1}^{s_1} \zeta_{ii}^{n+1}\Big)^2 = 2\tau(T_x\mathbb{D}^n) + ||\zeta||^2 - n(n-1)\left(\frac{\epsilon+3}{4}\right) - \Big(\sum_{j=s_1+1}^{n} \zeta_{jj}^{n+1}\Big)^2 - 2\sum_{A=s_1+1}^{s_1}\sum_{B=s_1+1}^{n} \zeta_{AA}^{n+1}\zeta_{BB}^{n+1}. \tag{26}$$

Let us consider the following:

$$\Omega = 2\tau(T_x\mathbb{D}^n) - n(n-1)\left(\frac{\epsilon+3}{4}\right) - \frac{(s_1-2)}{(s_1-1)}\Big(\sum_{i=1}^{s_1} \zeta_{ii}^{n+1}\Big)^2 - \Big(\sum_{j=s_1+1}^{n} \zeta_{jj}^{n+1}\Big)^2 - 2\sum_{A=s_1+1}^{s_1}\sum_{B=s_1+1}^{n} \zeta_{AA}^{n+1}\zeta_{BB}^{n+1}. \tag{27}$$

It follows from (26) and (27), and we find that

$$\Big(\sum_{i=1}^{s_1} \zeta_{ii}^{n+1}\Big)^2 = (s_1-1)\Big(\Omega + ||\zeta||^2\Big). \tag{28}$$

The above equation can be expressed as

$$\left(\sum_{i=1}^{s_1} \zeta_{ii}^{n+1}\right)^2 = (s_1 - 1)\Bigg\{\Omega + \sum_{i=1}^{s_1} \left(\zeta_{ii}^{n+1}\right)^2 + \sum_{j=s_1+1}^{n} \left(\zeta_{jj}^{n+1}\right)^2 + \sum_{i\neq j=1}^{n} \left(\zeta_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=1}^{n} \left(\zeta_{jj}^{n+1}\right)^2\Bigg\} \tag{29}$$

Therefore, we shall apply Lemma 1 on the above equation, i.e.,

$$t_\alpha = \zeta_{\alpha\alpha}^{n+1}, \forall t_\alpha \in \{1 \cdots, s_1\}$$
$$and$$
$$s = \Omega + \sum_{j=s_1+1}^{n} \left(\zeta_{jj}^{n+1}\right)^2 + \sum_{i\neq j=1}^{n} \left(\zeta_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=1}^{n} \left(\zeta_{jj}^{n+1}\right)^2$$

Thus, we obtain that

$$\zeta_{11}^{n+1}\zeta_{22}^{n+1} \geq \frac{1}{2}\Bigg\{\Omega + \sum_{j=s_1+1}^{n} \left(\zeta_{jj}^{n+1}\right)^2 + \sum_{i\neq j=1}^{n} \left(\zeta_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=1}^{n} \left(\zeta_{jj}^{n+1}\right)^2\Bigg\} \tag{30}$$

Then, from (21) and (14), we derive

$$K(\pi_1) = \left(\frac{\epsilon+3}{4}\right) + \sum_{r=n+1}^{2n+1} \left(\zeta_{11}^r \zeta_{22}^r - (\zeta_{12}^r)^2\right). \tag{31}$$

If we combine Equations (30) and (31), we obtain

$$K(\pi_1) \geq \left(\frac{\epsilon+3}{4}\right) + \frac{1}{2}\Omega + \frac{1}{2}\sum_{j=s_1+1}^{n} \left(\zeta_{jj}^{n+1}\right)^2 + \sum_{r=n+1}^{2n+1} \left(\zeta_{11}^r \zeta_{22}^r - (\zeta_{12}^r)^2\right) + \frac{1}{2}\sum_{i\neq j=1}^{n} \left(\zeta_{ij}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+2}^{2n+1} \sum_{i,j=1}^{n} \left(\zeta_{jj}^{n+1}\right)^2. \tag{32}$$

We choose the last two terms of the above equation, and we derive

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(\zeta_{ij}^{n+1}\right)^2 = \sum_{\substack{i,j=3\\ i\neq j}}^{n} \left(\zeta_{ij}^{n+1}\right)^2 + 2\sum_{j=3}^{n} \left(\zeta_{1j}^{n+1}\right)^2 + 2\left(\zeta_{12}^{n+1}\right)^2 + 2\sum_{j=3}^{n} \left(\zeta_{2j}^{n+1}\right)^2. \tag{33}$$

Moreover, for the last term, we obtain

$$\sum_{r=n+2}^{2n+1} \sum_{i,j=1}^{n} \left(\zeta_{jj}^{n+1}\right)^2 = \sum_{r=n+2}^{2n+1} \sum_{i,j=3}^{n} \left(\zeta_{jj}^{n+1}\right)^2 + 2\sum_{r=n+2}^{2n+1} \sum_{j=3}^{n} \left(\zeta_{1j}^{n+1}\right)^2 + 2\sum_{r=n+2}^{2n+1} \sum_{j=3}^{n} \left(\zeta_{2j}^{n+1}\right)^2 + 2\left(\zeta_{12}^{n+1}\right)^2 + \sum_{r=n+2}^{2n+1} \Big((\zeta_{11})^2 + (\zeta_{22})^2\Big).$$

Furthermore, we have

$$\sum_{r=n+2}^{2n+1} \zeta_{11}^{r}\zeta_{22}^{2} + \frac{1}{2}\sum_{r=n+2}^{2n+1}\left((\zeta_{11}^{r})^2 + (\zeta_{22}^{r})^2\right) = \frac{1}{2}\sum_{r=n+2}^{2n+1}\left(\zeta_{11}^{r} + \zeta_{22}^{r}\right)^2, \tag{34}$$

$$\sum_{j=3}^{n}\left((\zeta_{1j}^{n+1})^2 + (\zeta_{2j}^{n+1})^2\right) + \sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}(\zeta_{1j}^{n+1})^2 + \sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}(\zeta_{2j}^{n+1})^2$$
$$= \sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}\left\{(\zeta_{2j}^{n+1})^2 + (\zeta_{2j}^{n+1})^2\right\} \tag{35}$$

After adding (33) and (59), then using (34) and (35), and taking into account that

$$(\zeta_{12}^{n+1})^2 + \sum_{r=n+2}^{2n+1}(\zeta_{12}^{n+1})^2 = \sum_{r=n+1}^{2n+1}(\zeta_{12}^{n+1})^2.$$

We obtain

$$\begin{aligned}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1}\sum_{i,j=1}^{n}(\zeta_{jj}^{n+1})^2 =& 2\sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}\left\{(\zeta_{2j}^{n+1})^2 + (\zeta_{2j}^{n+1})^2\right\}\\ &+ \sum_{\substack{i,j=3\\ i\neq j}}^{n}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}(\zeta_{jj}^{n+1})^2\\ &- 2\sum_{r=n+2}^{2n+1}\left\{\zeta_{11}^{r}\zeta_{22}^{r} - (\zeta_{12}^{r})^2\right\}\\ &+ \sum_{r=n+2}^{2n+1}\left(\zeta_{11}^{r} + \zeta_{22}^{2}\right)^2.\end{aligned} \tag{36}$$

It follows from (32) and (36) that one derives

$$\begin{aligned}K(\pi_1) \geq& \left(\frac{\epsilon+3}{4}\right) + \frac{1}{2}\Omega + \frac{1}{2}\sum_{\beta=s_1+1}^{n}(\zeta_{\beta\beta}^{n+1})^2\\ &+ \sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}\left\{(\zeta_{2j}^{n+1})^2 + (\zeta_{2j}^{n+1})^2\right\}\\ &+ \frac{1}{2}\left\{\sum_{\substack{i,j=3\\ i\neq j}}^{n}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}(\zeta_{jj}^{n+1})^2\right\} + \frac{1}{2}\sum_{r=n+2}^{2n+1}\left(\zeta_{11}^{r} + \zeta_{22}^{2}\right)^2,\end{aligned}$$

which implies that

$$\begin{aligned}K(\pi_1) \geq& \left(\frac{\epsilon+3}{4}\right)\\ &+ \frac{1}{2}\left\{\Omega + \sum_{\substack{i,j=3\\ i\neq j}}^{n}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}(\zeta_{jj}^{n+1})^2 + \sum_{\beta=s_1+1}^{n}(\zeta_{\beta\beta}^{n+1})^2\right\}.\end{aligned}$$

From (27), we arrive at

$$K(\pi_1) \geq \left(\frac{\epsilon+3}{4}\right) + \tau(T_x\mathbb{D}^n) + \frac{1}{2(s_1-1)}\Big(\sum_{\alpha=s_1+1}^{n}\zeta_{\alpha\alpha}^{n+1}\Big)^2$$

$$
\begin{aligned}
&-\frac{n^2}{2}||\mathbb{H}||^2-\frac{n(n-1)}{2}\left(\frac{\epsilon+3}{4}\right)\\
&+\frac{1}{2}\left\{\sum_{\substack{i,j=3\\i\neq j}}^{n}\left(\zeta_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\zeta_{jj}^{n+1}\right)^2+\sum_{\beta=s_1+1}^{n}\left(\zeta_{\beta\beta}^{n+1}\right)^2\right\}.
\end{aligned}
\tag{37}
$$

Using (11) and (19) together in (37), we obtain

$$
\begin{aligned}
K(\pi_1)\geq&\tau(T_x\mathbb{D}_1^{s_1})+\tau(T_x\mathbb{D}_2^{s_2})+\frac{s_2\nabla f}{f}-\frac{n^2}{2}||\mathbb{H}||^2\\
&+\left(1-\frac{n(n-1)}{2}\right)\left(\frac{\epsilon+3}{4}\right)\\
&+\frac{1}{2}\left\{\sum_{\substack{i,j=3\\i\neq j}}^{n}\left(\zeta_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\zeta_{ij}^{n+1}\right)^2+\sum_{\beta=s_1+1}^{n}\left(\zeta_{\beta\beta}^{n+1}\right)^2\right\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\tau(T_x\mathbb{D}_1^{s_1})-K(\pi_1)\leq&\frac{n^2}{2}||\mathbb{H}||^2-\tau(T_x\mathbb{D}_2^{s_2})-\frac{s_2\nabla f}{f}\\
&+\left(\frac{n(n-1)}{2}-1\right)\left(\frac{\epsilon+3}{4}\right)\\
&-\frac{1}{2}\left\{\sum_{\substack{i,j=3\\i\neq j}}^{n}\left(\zeta_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\zeta_{ij}^{n+1}\right)^2+\sum_{\beta=s_1+1}^{n}\left(\zeta_{\beta\beta}^{n+1}\right)^2\right\}.
\end{aligned}
\tag{38}
$$

The Gauss Equation (6)(i) for $\tau(T_x\mathbb{D}_2^{s_2})$ gives us

$$
\begin{aligned}
\tau(T_x\mathbb{D}_2^{s_2})=&\frac{s_2(s_2-1)}{2}\left(\frac{\epsilon+3}{4}\right)\\
&-\frac{1}{2}\sum_{r=n+1}^{2n+1}\sum_{A,B=s_1+1}^{n}\left(\zeta_{AB}^{n+1}\right)^2\\
&-\frac{1}{2}\sum_{r=n+1}^{2n+1}\Big(\zeta_{s_1+1s_1+1}+\cdots+\zeta_{nn}^{r}\Big).
\end{aligned}
\tag{39}
$$

In view of Equations (38) and (39), we find that

$$
\begin{aligned}
\tau(T_x\mathbb{D}_1^{s_1})-K(\pi_1)\leq&\frac{n^2}{2}||\mathbb{H}||^2-\frac{s_2(s_2-1)}{2}\left(\frac{\epsilon+3}{4}\right)\\
&-\frac{1}{2}\left\{\sum_{\substack{i,j=3\\i\neq j}}^{n}\left(\zeta_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{i,j=3}^{n}\left(\zeta_{ij}^{r}\right)^2\right.\\
&\left.+\sum_{\beta=s_1+1}^{n}\left(\zeta_{\beta\beta}^{n+1}\right)^2-\sum_{r=n+1}^{2n+1}\sum_{A,B=s_1+1}^{n}\left(\zeta_{AB}^{n+1}\right)^2\right\}\\
&+\left(\frac{n(n-1)}{2}-1\right)\left(\frac{\epsilon+3}{4}\right)-\frac{s_2\nabla f}{f}.
\end{aligned}
\tag{40}
$$

Then, the last relation turns into

$$
\begin{aligned}
\tau(T_x\mathbb{D}_1^{s_1}) - K(\pi_1) \leq & \frac{n^2}{2}||\mathbb{H}||^2 - \frac{s_2(s_2-1)}{2}\left(\frac{\epsilon+3}{4}\right) + \left(\frac{n(n-1)}{2} - 1\right)\left(\frac{\epsilon+3}{4}\right) \\
& -\frac{1}{2}\Bigg\{ \sum_{\substack{k,l=3\\k\neq l}}^{s_1} (\zeta_{kl}^{n+1})^2 + 2\sum_{k=3}^{2n+1}\sum_{l=s_1+1}^{n} (\zeta_{kl}^{n+1})^2 + \sum_{\substack{A,B=s_1+1\\A\neq B}}^{s_1} (\zeta_{kl}^{n+1})^2 \\
& + \sum_{r=n+2}^{2n+1}\sum_{k,l=3}^{s_1} (\zeta_{kl}^{n+1})^2 + 2\sum_{r=n+2}^{2n+1}\sum_{k=3}^{s_1}\sum_{A=s_1+1}^{n} (\zeta_{kl}^{r})^2 \\
& + \sum_{r=n+2}^{2n+1}\sum_{A,B=s_1+1}^{n} (\zeta_{AB}^{r})^2 + \sum_{\beta=s_1+1}^{n} (\zeta_{\beta\beta}^{n+1})^2 \\
& - \sum_{r=n+1}^{2n+1}\sum_{A,B=s_1+1}^{n} (\zeta_{AB}^{r})^2 \Bigg\} - \frac{s_2\nabla f}{f}.
\end{aligned} \tag{41}
$$

With the preceding above equation and the help of the following two relations:

$$
\sum_{A=s_1+1}^{n} (\zeta_{AA}^2)^2 + \sum_{\substack{A,B=3\\A\neq B}}^{n} (\zeta_{AB}^{n+1})^2 = \sum_{A,B=s_1+1}^{n} (\zeta_{AB}^{n+1})^2.
$$

and

$$
\sum_{A,B=s_1+1}^{n} (\zeta_{AB}^{n+1})^2 + \sum_{r=n+2}^{2n+1}\sum_{A,B=s_1+1}^{n} (\zeta_{AB}^{r})^2 = \sum_{r=n+1}^{2n+1}\sum_{A,B=s_1+1}^{n} (\zeta_{AB}^{r})^2.
$$

Assertion (41) is as follows:

$$
\begin{aligned}
\tau(T_x\mathbb{D}_1^{s_1}) - K(\pi_1) \leq & \left\{ \frac{s_1}{2}\Big(s_1 + 2s_2 - 1\Big) - 1 \right\}\left(\frac{\epsilon+3}{4}\right) \\
& -\frac{1}{2}\Bigg\{ \sum_{\substack{k,l=3\\k\neq l}}^{s_1} (\zeta_{kl}^{n+1})^2 + \sum_{r=n+2}^{2n+1}\sum_{k,l=3}^{s_1} (\zeta_{kl}^{n+1})^2 \\
& + 2\sum_{\alpha=3}^{s_1}\sum_{\beta=s_1+1}^{n} (\zeta_{\alpha\beta}^{n+1})^2 + 2\sum_{r=n+2}^{2n+1}\sum_{A=3}^{s_1}\sum_{B=s_1+1}^{n} (\zeta_{AB}^{r})^2 \Bigg\} \\
& + \frac{n^2}{2}||\mathbb{H}||^2 - \frac{s_2\nabla f}{f}.
\end{aligned} \tag{42}
$$

The first inequality of Theorem 1 holds from the above equation and (15). For the second case, if $\pi \subset T_x\mathbb{D}_2^{s_2}$, we consider $\pi_2 = Span\{e_{s_1+1}, e_{s_1+1}\}$, following same methodology as first case as:

$$
\begin{aligned}
\Big(\sum_{\alpha=s_1+1}^{n} \zeta_{\alpha\alpha}^{n+1}\Big)^2 =& 2\tau(T_x\mathbb{D}^n) + ||\zeta||^2 - n(n-1)\left(\frac{\epsilon+3}{4}\right) - \Big(\sum_{\beta=s_1+1}^{n} \zeta_{\beta\beta}^{s_1}\Big)^2 \\
& - 2\sum_{\alpha=1}^{s_1}\sum_{\beta=s_1+1}^{n} \zeta_{\alpha\alpha}^{n+1}\zeta_{\beta\beta}^{n+1}.
\end{aligned}
$$

Considering the following:

$$
\Psi = 2\tau(T_x\mathbb{D}^n) - n(n-1)\left(\frac{\epsilon+3}{4}\right)
$$

$$- \frac{(s_1-2)}{(s_1-1)}\Big(\sum_{\alpha=s_1+1}^{s_1} \zeta_{\alpha\alpha}^{n+1}\Big)^2 - \Big(\sum_{\beta=s_1+1}^{n} \zeta_{\beta\beta}^{s_1}\Big)^2$$
$$-2\sum_{\alpha=1}^{s_1}\sum_{\beta=s_1+1}^{n} \zeta_{\alpha\alpha}^{n+1}\zeta_{\beta\beta}^{n+1}.$$

The last two equation implies that

$$\Big(\sum_{\alpha=s_1+1}^{n} \zeta_{\alpha\alpha}^{n+1}\Big)^2 = (s_2-1)\Big(\Psi + ||\zeta||^2\Big),$$

which implies that

$$\Big(\sum_{\alpha=s_1+1}^{n} \zeta_{\alpha\alpha}^{n+1}\Big)^2 = (s_2-1)\Bigg\{\Psi + \Big(\sum_{\alpha=1}^{s_1} \zeta_{\alpha\alpha}^{n+1}\Big)^2 + \Big(\sum_{\beta=s_1+1}^{n} \zeta_{\beta\beta}^{n+1}\Big)^2 + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n} \big(\zeta_{\alpha\beta}^{n+1}\big)^2 + \sum_{r=n+2}^{2n+1}\sum_{\alpha=,\beta=1}^{n} \big(\zeta_{\alpha\beta}^{r}\big)^2\Bigg\}. \tag{43}$$

Similarly, applying Lemma 1 in the above equation, we obtain

$$\zeta_{s_1+1s_1+1}^{n+1}\zeta_{s_1+2s_1+2}^{n+1} \geq \frac{1}{2}\Bigg\{\Psi + \Big(\sum_{\alpha=1}^{s_1} \zeta_{\alpha\alpha}^{n+1}\Big)^2 + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n} \big(\zeta_{\alpha\beta}^{n+1}\big)^2 + \sum_{r=n+2}^{2n+1}\sum_{\alpha=,\beta=1}^{n} \big(\zeta_{\alpha\beta}^{r}\big)^2\Bigg\} \tag{44}$$

From (21) and (14), we find that

$$K(\pi_2) = \left(\frac{\epsilon+3}{4}\right) + \sum_{r=n+1}^{2n+1}\left(\zeta_{s_1+1s_1+1}^{r}\zeta_{s_1+2s_1+2}^{r} - \big(\zeta_{s_1+1s_1+2}^{r}\big)^2\right) \tag{45}$$

Equations (44) and (45) are implied such that

$$K(\pi_2) \geq \left(\frac{\epsilon+3}{4}\right) + \sum_{r=n+1}^{2n+1}\left(\zeta_{s_1+1s_1+1}^{r}\zeta_{s_1+2s_1+2}^{r} - \big(\zeta_{s_1+1s_1+2}^{r}\big)^2\right)$$
$$\frac{1}{2}\Bigg\{\Psi + \Big(\sum_{\alpha=1}^{s_1} \zeta_{\alpha\alpha}^{n+1}\Big)^2 + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n} \big(\zeta_{\alpha\beta}^{n+1}\big)^2 + \sum_{r=n+2}^{2n+1}\sum_{\alpha=,\beta=1}^{n} \big(\zeta_{\alpha\beta}^{r}\big)^2\Bigg\}. \tag{46}$$

Following the method from (27) and (42), we obtain the second inequality of Theorem 1. On the other hand, for the equality condition, we define two different cases whether the 2-plane π_i is tangent to the first factor or to the second factor. In the first case, we consider $\pi_1 \subset T_x\mathbb{D}_1^{s_1}$, then the equality holds if and only if equalities hold in (30), (32), (38), (39) and (42), and we obtain the following condition:

$$\zeta_{11}^{n+1} + \zeta_{22}^{n+1} = \zeta_{33}^{n+1} = \cdots = \zeta_{s_1s_1}^{n+1} \tag{47}$$

$$\sum_{r=n+2}^{2n+1}\sum_{j=3}^{n}\left(\big(\zeta_{2j}^{n+1}\big)^2 + \big(\zeta_{2j}^{n+1}\big)^2\right) + \sum_{r=n+2}^{2n+1}\Big(\zeta_{11}^{r} + \zeta_{22}^{2}\Big)^2 = 0, \tag{48}$$

$$\sum_{r=n+1}^{2n+1}\Big(\zeta_{s_1+1s_1+1} + \cdots + \zeta_{nn}^{r}\Big) = \Big(\sum_{\alpha=1}^{s_1} \zeta_{\alpha\alpha}^{n+1}\Big)^2 = 0, \tag{49}$$

$$\sum_{\substack{k,l=3\\k\neq l}}^{s_1}\left(\zeta_{kl}^{n+1}\right)^2+\sum_{r=n+2}^{2n+1}\sum_{k,l=3}^{s_1}\left(\zeta_{kl}^{n+1}\right)^2+\sum_{\alpha=3}^{s_1}\sum_{\beta=s_1+1}^{n}\left(\zeta_{\alpha\beta}^{n+1}\right)^2$$
$$+\sum_{r=n+2}^{2n+1}\sum_{A=3}^{s_1}\sum_{B=s_1+1}^{n}\left(\zeta_{AB}^{r}\right)^2=0. \quad (50)$$

Equation (49) clearly indicates that the warped product $\mathbb{D}_1^{s_1}\times_f\mathbb{D}_2^{s_1}$ is both a $\mathbb{D}_1^{s_1}$-minimal and $\mathbb{D}_2^{s_2}$-minimal warped product Legendrian submanifold in $\mathbb{D}_\epsilon^{2n+1}$. It can be concluded that the warped product Legendrian submanifold $\mathbb{D}_1^{s_1}\times_f\mathbb{D}_2^{s_1}$ is minimal in $\mathbb{D}_\epsilon^{2n+1}$. Moreover, we shall classify the other case in two techniques, as they depend on the vector fields r. Assuming that $r=n+1$, we define the following:

$$\zeta_{11}^{n+1}+\zeta_{22}^{n+1}=\zeta_{33}^{n+1}=\cdots=\zeta_{s_1s_1}^{n+1}$$
$$\text{and}$$
$$\sum_{j=3}^{n}\zeta_{1j}^{n+1}=\sum_{j=3}^{n}\zeta_{2j}^{n+1}=\sum_{\substack{k,l=3\\k\neq l}}^{s_1}\left(\zeta_{kl}^{n+1}\right)^2=\sum_{\alpha=3}^{s_1}\sum_{\beta=s_1+1}^{n}\left(\zeta_{\alpha\beta}^{n+1}\right)^2=0.$$

Thus, the above condition is equivalent to the following matrices:

$$(i)A_{e_{n+1}}=\left(\begin{array}{ccccc|ccc}
\mu_1 & \zeta_{12}^{n+1} & 0 & \cdots & 0_{1s_1} & 0_{1s_1+1} & \cdots & 0_{1n}\\
\zeta_{12}^{n+1} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots\\
0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots\\
0_{s_11} & 0 & 0 & \cdots & \mu & 0_{s_1s_1+1} & \cdots & 0_{s_1n}\\
\hline
0_{s_1+11} & \cdots & \cdots & \cdots & 0_{s_1+1s_1} & \zeta_{s_1+1s_1+1}^{n+1} & \cdots & \zeta_{s_1+1n}^{n+1}\\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0_{n1} & \cdots & \cdots & \cdots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \cdots & \zeta_{nn}^{n+1}
\end{array}\right),$$

where $\mu=\mu_1+\mu_2$ gives the (i) theorem. Similarly, if $r\in\{n+2,\cdots,m\}$, then the above condition implies that

$$\zeta_{11}^{n+1}+\zeta_{22}^{n+1}=\sum_{j=3}^{n}\zeta_{1j}^{n+1}=\sum_{j=3}^{n}\zeta_{2j}^{n+1}=\sum_{\substack{k,l=3\\k\neq l}}^{s_1}\left(\zeta_{kl}^{n+1}\right)^2=\sum_{\alpha=3}^{s_1}\sum_{\beta=s_1+1}^{n}\left(\zeta_{\alpha\beta}^{n+1}\right)^2=0.$$

This is equivalent to the second metric:

$$(ii)A_{e_r}=\left(\begin{array}{ccccc|ccc}
\zeta_{11}^{r} & \zeta_{12}^{r} & 0 & \cdots & 0_{1s_1} & 0_{1s_1+1} & \cdots & 0_{1n}\\
\zeta_{21}^{r} & -\zeta_{11}^{r} & 0 & \cdots & \vdots & \vdots & \cdots & \vdots\\
0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots\vdots & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots\\
0_{s_11} & 0 & 0 & \cdots & 0_{s_1s_1} & 0_{s_1s_1+1} & \cdots & 0_{s_1n}\\
\hline
0_{s_1+11} & \cdots & \cdots & \cdots & 0_{s_1+1s_1} & \zeta_{s_1+1s_1+1}^{n+1} & \cdots & \zeta_{s_1+1n}^{n+1}\\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0_{n1} & \cdots & \cdots & \cdots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \cdots & \zeta_{nn}^{n+1}
\end{array}\right),$$

It is clear that the above two conditions show that $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ is a mixed totally geodesic warped product Legendrian submanifold in $\mathbb{D}_\epsilon^{2n+1}$. Furthermore, the equality sign in (ii) holds if and only if the following two matrices are satisfied:

$$(iii) A_{e_{n+1}} = \left(\begin{array}{cccc|ccccc}
\zeta_{11}^{n+1} & \cdots & \cdots & \zeta_{1s_1}^{n+1} & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\zeta_{s_1 11}^{n+1} & \cdots & \cdots & \zeta_{s_1 s_1}^{n+1} & 0_{s_1 s_1+1} & \cdots & \cdots & \cdots & 0_{s_1 n} \\
\hline
0_{s_1+11} & \cdots & \cdots & 0_{s_1+1s_1} & \mu_1 & \zeta_{s_1+1s_1+2}^{n+1} & 0 & \cdots & 0_{s_1+1n} \\
\vdots & \ddots & \ddots & \vdots & \zeta_{s_1+2s_1+1}^{n+1} & \mu_2 & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\
0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & \mu
\end{array}\right),$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \cdots, 2n+1\}$, thus we have

$$(iv) A_{e_r} = \left(\begin{array}{cccc|ccccc}
\zeta_{11}^{r} & \cdots & \cdots & \zeta_{1s_1}^{r} & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\zeta_{s_1 11}^{r} & \cdots & \cdots & \zeta_{s_1 s_1}^{r} & 0_{s_1 s_1+1} & \cdots & \cdots & \cdots & 0_{s_1 n} \\
\hline
0_{s_1+11} & \cdots & \cdots & 0_{s_1+1s_1} & \zeta_{s_1+1s_1+1}^{r} & \zeta_{s_1+1s_1+2}^{n+1} & 0 & \cdots & 0_{s_1+1n} \\
\vdots & \ddots & \ddots & \vdots & \zeta_{s_1+2s_1+1}^{n+1} & -\zeta_{s_1+1s_1+1}^{r} & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\
0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & 0
\end{array}\right),$$

From the above, it is also clear that $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ is both a $\mathbb{D}_1^{s_1}$-minimal and $\mathbb{D}_2^{s_2}$-minimal warped product Legendrian submanifold in $\mathbb{D}_\epsilon^{2n+1} \times \mathbb{R}$, which implies the minimilty of the warped product Legendrian submanifold $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ in $\mathbb{D}_\epsilon^{2n+1}$. This completes the proof of the theorem. □

Warped product manifolds have studied themselves to be a profitable ambient space to obtain a wide range of distinct geometrical properties for immersion. We now find the inequalities for the Riemannian manifold that has constant sectional curvature $\epsilon \in \{1, -3\}$ and can be expressed as a product manifold of $\mathbb{D}_\epsilon^{2n+1}$. We find the following result as follows.

2.1. An Application for Warped Product Legendrian Submanifold in $\mathbb{S}^{2n+1}$ with $\epsilon = 1$

Theorem 2. *Assume that $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is an isometric immersion of a warped product submanifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ into a Euclidean sphere $\mathbb{S}^{2n+1}$. Then, for each point $x \in \mathbb{D}^n$ and each plane section $\pi_i \subset T_x\mathbb{D}_i^{n_i}$, for $i = 1, 2$, we obtain the following for*

(a) $\pi_1 \subset T_x\mathbb{D}_1^{s_1}$

$$\delta_{\widetilde{\mathbb{D}}^{s_1}}(x) \leq \frac{n^2}{2}||\mathbb{H}||^2 + s_2||\nabla(\ln f)||^2 - s_2\Delta(\ln f) + \left\{\frac{s_1}{2}\Big(s_1 + 2s_2 - 1\Big) - 1\right\}.$$

The equality of the above inequality holds at $x \in \mathbb{D}^n$ if and only if there exists an orthonormal basis $\{e_1 \cdots e_n\}$ of $T_x\mathbb{D}^n$ and orthonormal basis $\{e_{n+1} \cdots e_{2n+1}\}$ of $T_x^\perp$ such that (a) $\pi = Span\{e_1, e_2\}$ and (b) shape operators take the following form

$$(i)A_{e_{n+1}} = \left(\begin{array}{ccccc|ccc} \mu_1 & \zeta_{12}^{n+1} & 0 & \cdots & 0_{1s_1} & 0_{1s_1+1} & \cdots & 0_{1n} \\ \zeta_{12}^{n+1} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{s_11} & 0 & 0 & \cdots & \mu & 0_{s_1s_1+1} & \cdots & 0_{s_1n} \\ \hline 0_{s_1+11} & \cdots & \cdots & \cdots & 0_{s_1+1s_1} & \zeta_{s_1+1s_1+1}^{n+1} & \cdots & \zeta_{s_1+1n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \cdots & \zeta_{nn}^{n+1} \end{array}\right),$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \cdots m\}$, then we have the matric

$$(ii)A_{e_r} = \left(\begin{array}{ccccc|ccc} \zeta_{11}^r & \zeta_{12}^r & 0 & \cdots & 0_{1s_1} & 0_{1s_1+1} & \cdots & 0_{1n} \\ \zeta_{21}^r & -\zeta_{11}^r & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots \vdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{s_11} & 0 & 0 & \cdots & 0_{s_1s_1} & 0_{s_1s_1+1} & \cdots & 0_{s_1n} \\ \hline 0_{s_1+11} & \cdots & \cdots & \cdots & 0_{s_1+1s_1} & \zeta_{s_1+1s_1+1}^{n+1} & \cdots & \zeta_{s_1+1n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \cdots & \zeta_{nn}^{n+1} \end{array}\right),$$

(b) *for $\pi_2 \subset T_x\mathbb{D}_2^{s_2}$*

$$\delta_{\widetilde{\mathbb{D}}^{s_2}}(x) \leq \frac{n^2}{2}||\mathbb{H}||^2 + s_2||\nabla(\ln f)||^2 - s_2\Delta(\ln f) + \left\{\frac{s_2}{2}\Big(s_2 + 2s_1 - 1\Big) - 1\right\}.$$

The equality of the above equation hold if and only if

$$(iii)A_{e_{n+1}} = \left(\begin{array}{cccc|ccccc} \zeta_{11}^{n+1} & \cdots & \cdots & \zeta_{1s_1}^{n+1} & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \zeta_{s_111}^{n+1} & \cdots & \cdots & \zeta_{s_1s_1}^{n+1} & 0_{s_1s_1+1} & \cdots & \cdots & \cdots & 0_{s_1n} \\ \hline 0_{s_1+11} & \cdots & \cdots & 0_{s_1+1s_1} & \mu_1 & \zeta_{s_1+1s_1+2}^{n+1} & 0 & \cdots & 0_{s_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \zeta_{s_1+2s_1+1}^{n+1} & \mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & \mu \end{array}\right),$$

where $\mu = \mu_1 + \mu_2$. *If* $r \in \{n+2, \cdots, 2n+1\}$, *thus we have*

$$(iv) A_{e_r} = \left(\begin{array}{cccc|ccccc}
\zeta^r_{11} & \cdots & \cdots & \zeta^r_{1s_1} & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\zeta^r_{s_1 11} & \cdots & \cdots & \zeta^r_{s_1 s_1} & 0_{s_1 s_1+1} & \cdots & \cdots & \cdots & 0_{s_1 n} \\
\hline
0_{s_1+11} & \cdots & \cdots & 0_{s_1+1s_1} & \zeta^r_{s_1+1s_1+1} & \zeta^{n+1}_{s_1+1s_1+2} & 0 & \cdots & 0_{s_1+1n} \\
\vdots & \ddots & \ddots & \vdots & \zeta^{n+1}_{s_1+2s_1+1} & -\zeta^r_{s_1+1s_1+1} & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\
0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & 0
\end{array}\right),$$

(v) If the equality holds in (1) or (2), then $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ *is mixed totally geodesic in space form* $\mathbb{D}_\epsilon^{2n+1}$. *Moreover,* $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ *is both* $\mathbb{D}_1^{s_1}$*-minimal and* $\mathbb{D}_2^{s_2}$*-minimal. Thus,* $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ *is a minimal warped product submanifold in Sasakian space form* $\mathbb{D}_\epsilon^{2n+1}$.

Proof. Now we consider the constant sectional curvature $\epsilon = 1$ and $\mathbb{D}_\epsilon^{2n+1} = \mathbb{S}^{2n+1}$ for the product manifold $\mathbb{S}^{2n+1}$. Then, inserting the proceeding value in (23) and (24), we obtain the required result. □

2.2. An Application for Warped Product Submanifold in $\mathbb{R}^{2n+1}$ *with* $\epsilon = -3$

Theorem 3. *Assume that* $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ *is an isometric immersion of a warped product Legendrian submanifold* $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ *into a Euclidean spaces* $\mathbb{R}^{2n+1}$. *Then, for each point* $x \in \mathbb{D}^n$ *and each plane section* $\pi_i \subset T_x\mathbb{D}_i^{n_i}$, *for* $i = 1, 2$, *we obtain the following for*

(a) $\pi_1 \subset T_x\mathbb{D}_1^{s_1}$ *or* $\pi_2 \subset T_x\mathbb{D}_2^{s_2}$

$$\delta_{\widetilde{\mathbb{D}}^{s_1}}(x) \leq \frac{n^2}{2}\|\mathbb{H}\|^2 + s_2||\nabla(\ln f)||^2 - s_2\Delta(\ln f).$$

(b) *for* $\pi_2 \subset T_x\mathbb{D}_2^{s_2}$

$$\delta_{\widetilde{\mathbb{D}}^{s_2}}(x) \leq \frac{n^2}{2}\|\mathbb{H}\|^2 + s_2||\nabla(\ln f)||^2 - s_2\Delta(\ln f).$$

The equality of the above inequality holds as in Theorem 1.

Proof. Now we assume that $\mathbb{D}_\epsilon^{2n+1} = \mathbb{R}^{2n+1}$ and constant sectional curvature $\epsilon = -3$ for the Euclidean spaces $\mathbb{R}^{2n+1}$. Then, using these values in (23) and (24), we obtain the required result. □

Remark 3. *It should be noticed that Theorem 2 coincides with Theorem 4.1 in [20]. If* $f = 1$, *then Theorem 2 is generalized the result in [4]. Therefore, our result is a generalization of [4,20].*

2.3. Some Applications to Obtain Dirichlet Eigenvalue Inequalities

Now, if the first eigenvalue of the Dirichlet boundary condition is denoted by $v_1(\Sigma) > 0$ on a complete noncompact Riemannian manifold $\mathbb{D}^n$ with the compact domain Σ in $\mathbb{D}^n$, then we have

$$\Delta\sigma + v\sigma = 0, \text{ on } \Sigma \text{ and } \quad \sigma = 0 \text{ on } \partial\Sigma, \tag{51}$$

where Δ is the Laplacian on $\mathbb{D}^n$, and σ is a non-zero function defined on $\mathbb{D}^n$. Then, $v_1(\mathbb{D}^n)$ is expressed as $inf_\Sigma v_1(\Sigma)$.

From the above motivation, assume that f is the non-constant warping function on compact warped product submanifold $\mathbb{D}^n$. Then, the minimum principle on v_1 leads to (see, for instance, [1,9])

$$\int_{\mathbb{D}^n} ||\nabla\sigma||^2 dV \geq v_1 \int_{\mathbb{D}^n} (\sigma)^2 dV \tag{52}$$

and the equality is satisfied if and only if

$$\Delta\sigma = v_1\sigma. \tag{53}$$

Implementing the integration along the base manifold $\mathbb{D}^{s_1}$ in Equations (23) and (24), we obtain the following result.

Theorem 4. *Assume that $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is a compact warped product Legendrian submanifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ into a Sasakian space form $\widetilde{\mathcal{D}}^{2n+1}(\epsilon)$. If v_1 is an eigenvalue of the eigenfunction $\sigma = \ln f$ satisfies* (53)*, then we have*

$$\begin{aligned}\int_{\mathbb{D}_1\times\{s_2\}} \delta_{\mathbb{D}^{s_1}}(x) dV \leq \frac{n^2}{2}\int_{\mathbb{D}_1\times\{s_2\}} \|\mathbb{H}\|^2 dV + s_2 v_1 \int_{\mathbb{D}_1\times\{s_2\}} (\ln f)^2 dV \\ + \int_{\mathbb{D}_1\times\{s_2\}} \left\{\left(\frac{s_1}{2}\Big(s_1 + 2s_2 - 1\Big) - 1\right)\left(\frac{\epsilon+3}{4}\right)\right\} dV,\end{aligned} \tag{54}$$

for $\pi_1 \subset T\mathbb{D}_1$. Moreover, we have

$$\begin{aligned}\int_{\mathbb{D}_1\times\{s_2\}} \delta_{\mathbb{D}^{s_2}}(x) dV \leq \frac{n^2}{2}\int_{\mathbb{D}_1\times\{s_2\}} \|\mathbb{H}\|^2 dV + s_2 v_1 \int_{\mathbb{D}_1\times\{s_2\}} (\ln f)^2 dV \\ + \int_{\mathbb{D}_1\times\{s_2\}} \left\{\left(\frac{s_2}{2}\Big(s_2 + 2s_1 - 1\Big) - 1\right)\left(\frac{\epsilon+3}{4}\right)\right\} dV,\end{aligned} \tag{55}$$

for $\pi_2 \subset T\mathbb{D}_2$.

Proof. As we know from the Stokes theorem, $\int \Delta\sigma dV = 0$ for a compact support. Then, we use the proceeding condition in (23) and (24) by replacing $\sigma = \ln f$, and we easily obtain the result. □

2.4. An Applications for Brochler Formulas

Theorem 5. *Assume that $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is a compact warped product Legendrian submanifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ into a Sasakian space form $\widetilde{\mathcal{D}}^{2n+1}(\epsilon)$. If v_1 is an eigenvalue of the eigenfunction $\sigma = \ln f$ satisfies* (53)*, then we have*

$$\begin{aligned}\int_{\mathbb{D}_1\times\{s_2\}} Ric(\nabla \ln f, \nabla \ln f) dV \geq \frac{v_1}{s_2}\int_{\mathbb{D}_1\times\{s_2\}} \delta_{\widetilde{\mathbb{D}}^{s_1}}(x) dV - \frac{n^2 v_1}{2s_2}\int_{\mathbb{D}_1\times\{s_2\}} \|\mathbb{H}\|^2 dV \\ + \frac{v_1}{s_2}\int_{\mathbb{D}_1\times\{s_2\}} \left\{1 - \left(\frac{s_1}{2}(s_1 + 2s_2 - 1)\right)\right\}\left(\frac{\epsilon+3}{4}\right) dV \\ - \int_{\mathbb{D}_1\times\{s_2\}} \|\nabla^2 \ln f\|^2 dV,\end{aligned} \tag{56}$$

for $\pi_1 \subset T\mathbb{D}_1$. Moreover, we have

$$\int_{\mathbb{D}_1\times\{s_2\}} Ric(\nabla \ln f, \nabla \ln f) dV \geq \frac{v_1}{s_2}\int_{\mathbb{D}_1\times\{s_2\}} \delta_{\mathbb{D}^{s_2}}(x) dV - \frac{n^2 v_1}{2s_2}\int_{\mathbb{D}_1\times\{s_2\}} \|\mathbb{H}\|^2 dV$$

$$+\frac{v_1}{s_2}\int_{\mathbb{D}_1\times\{s_2\}}\left\{1-\left(\frac{s_2}{2}\big(s_2+2s_1-1\big)\right)\right\}\left(\frac{\epsilon+3}{4}\right)dV$$
$$-\int_{\mathbb{D}_1\times\{s_2\}}\|\nabla^2\ln f\|^2 dV, \tag{57}$$

for $\pi_2 \subset T\mathbb{D}_2$.

Proof. If σ is the first eigenfunction of the Laplacian $\Delta\sigma = div(\nabla\sigma)$ for $\mathbb{D}^n$ connected to the first non-zero eigenvalue v_1, such that, $\Delta\sigma = -v_1\sigma$, then recalling the Bochner formula (see [40]) that gives the following relation of the differentiable function σ denoted at the Riemannian manifold as:

$$\frac{1}{2}\Delta\|\nabla\sigma\|^2 = \|\nabla^2\sigma\|^2 + Ric(\nabla\sigma,\nabla\sigma) + g\big(\nabla\sigma,\nabla(\Delta\sigma)\big).$$

By the integration of the previous equation, using the Stokes theorem, we have

$$\int_{\mathbb{D}_1\times\{s_2\}}\|\nabla^2\sigma\|^2 dV + \int_{\mathbb{D}_1\times\{s_2\}} Ric(\nabla\sigma,\nabla\sigma)dV + \int_{\mathbb{D}_1\times\{s_2\}} g\big(\nabla\sigma,\nabla(\Delta\sigma)\big)dV = 0. \tag{58}$$

Now, using $\Delta\sigma = v_1\sigma$ and making some rearrangement in Equation (58), we derive

$$\int_{\mathbb{D}_1\times\{s_2\}}\|\nabla\sigma\|^2 dV = \frac{1}{v_1}\left(\int_{\mathbb{D}_1\times\{s_2\}}\|\nabla^2\sigma\|^2 dV + \int_{\mathbb{D}_1\times\{s_2\}} Ric(\nabla\sigma,\nabla\sigma)dV\right). \tag{59}$$

Taking the integration in (23) and (24) and inserting the above equation, we obtain the desired results. □

3. Chern's Problem: Finding the Conditions under Which Warped Products Must Be Minimal

In this section, we provide the partial answer to the Chern problem [41], that is, the necessary condition for a warped product Legendrian submanifold to be a minimal in Sasakian space form $\widetilde{\mathcal{D}}^{2n+1}(\epsilon)$.

Corollary 1. *Let* $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ *be an isometric immersion of a warped product Legendrian submanifold* $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ *into a Sasakian space form* $\widetilde{\mathcal{D}}^{2n+1}(\epsilon)$. *Then, for each point* $x \in \mathbb{D}^n$ *and each* $\pi_1 \subset T_x\mathbb{D}_1^{n_1}$, *we have*

$$\delta_{\widetilde{\mathbb{D}}^{s_1}}(x) + s_2\Delta(\ln f) \leq \left\{\frac{s_1}{2}\Big(s_1+2s_2-1\Big)-1\right\}\left(\frac{\epsilon+3}{4}\right) + s_2||\nabla(\ln f)||^2. \tag{60}$$

and if the equality satisfies, then ϕ *is minimal.*

The second result is:

Corollary 2. *Let* $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ *be an isometric immersion of a warped product Legendrian submanifold* $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ *into a Sasakian space form* $\widetilde{\mathcal{D}}^{2n+1}(\epsilon)$. *Then, for each point* $x \in \mathbb{D}^n$ *and each* $\pi_2 \subset T_x\mathbb{D}_2^{n_2}$, *we have*

$$\delta_{\widetilde{\mathbb{D}}^{s_2}}(x) + s_2\Delta(\ln f) \leq \left\{\frac{s_2}{2}\Big(s_2+2s_1-1\Big)-1\right\}\left(\frac{\epsilon+3}{4}\right) + s_2||\nabla(\ln f)||^2. \tag{61}$$

and if the equality satisfies, then ϕ *is minimal.*

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Metallic Structures for Tangent Bundles over Almost Quadratic ϕ-Manifolds

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Abstract: This paper aims to explore the metallic structure $J^2 = pJ + qI$, where p and q are natural numbers, using complete and horizontal lifts on the tangent bundle TM over almost quadratic ϕ-structures (briefly, (ϕ, ξ, η)). Tensor fields $\widetilde{F}$ and F^* are defined on TM, and it is shown that they are metallic structures over (ϕ, ξ, η). Next, the fundamental 2-form Ω and its derivative $d\Omega$, with the help of complete lift on TM over (ϕ, ξ, η), are evaluated. Furthermore, the integrability conditions and expressions of the Lie derivative of metallic structures $\widetilde{F}$ and F^* are determined using complete and horizontal lifts on TM over (ϕ, ξ, η), respectively. Finally, we prove the existence of almost quadratic ϕ-structures on TM with non-trivial examples.

Keywords: metallic structure; tangent bundle; partial differential equations; nijenhuis tensor; mathematical operators; lie derivatives

MSC: 53D15; 58D17; 53C15

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1. Introduction

Many ancient societies have made extensive use of the golden mean as a foundation for proportions, whether for creating music, sculptures, paintings, or buildings, such as temples and palaces [1]. Fractal geometry has been explained using the silver mean [2]. Some uses of a class of polynomial structures have been constructed on Riemannian manifolds for the metallic means family (a generalization of the golden mean) and generalized Fibonacci sequences in differential geometry. The geometric properties (such as totally geodesic, totally umbilical hypersurfaces, etc.) in metallic Riemannian manifolds have been explored in [3]. This manuscript is focused on studying the properties of metallic structures for tangent bundles over a class of metallic Riemannian manifolds.

A quadratic equation of type

$$x^2 = px + q,$$

where p and q are natural numbers, whose positive solutions are given by

$$\sigma_p^q = \frac{p + \sqrt{p^2 + 4q}}{2}$$

is known as a metallic means family [4]. The most notable member is the well-known "Golden Mean" for $p = q = 1$. The metallic means family includes the silver mean for $p = 2, q = 1$, the bronze mean for $p = 3, q = 1$, the copper mean for $p = 1, q = 2$, and many others.

Let M be an n-dimensional differentiable manifold and TM be its tangent bundle. Let $\Im(M)$ and $\Im(TM)$ be the algebra of tensor fields of M and TM, respectively. The differential geometry of tangent bundle has been broadly studied by Davis [5], Sasaki [6], Tachibana and Okumura [7], Yano and Ishihara [8], and others. Yano and Kabayashi [9] defined the natural mapping (say complete lift) of $\Im(M)$ into $\Im(TM)$ and studied complete lifts of an almost complex structure and the symplectic structure on TM. Tanno [10] studied complete and vertical lifts of an almost contact structure on TM and defined a tensor field $\tilde{J}$ of type (1,1) and proved that it is an almost complex structure on TM. Numerous investigators have studied various geometric structures on TM—an almost complex structure by Yano [11], paracomplex structures by Tekkoyun [12], almost r-contact structures by Das and Khan [13], and many others [14–19].

In [20], Azami explored complete and horizontal lifts of metallic structures and analyzed the geometric properties of these structures. Salimov et al. [19] studied complete lifts of symplectic vector fields on tangent and cotangent bundles. Recently, Khan [21] introduced a new tensor field J of type (1,1) and demonstrated that J is a metallic structure (MS) on the frame bundle FM. Furthermore, the derivative and the coderivative of fundamental 2-form and the Nijenhuis tensor of J on FM are discussed.

On the other hand, Sasaki [6] defined a structure named as an almost contact structure and demonstrated its basic algebraic properties such as a Riemannian metric, the fundamental 2-form, etc., on M. Later on, Sato [22] defined the notion of an almost paracontact structure and analyzed its geometrical properties.

Debnath et al. [23] defined the notion of a (ϕ, ξ, η) on a differentiable manifold M and established its existence. Later on, Gonul et al. [24] developed a relation between MS and (ϕ, ξ, η). They proved that the warped product manifold has structure (ϕ, ξ, η). Most recently, Gök et al. [25] introduced the notion of $f_{(a,b)}(3,2,1)$-structures and investigated a necessary condition for these structures to be a (ϕ, ξ, η).

The main aim of this paper is summarized as:

- Tensor fields $\tilde{F}$ and F^* are defined on TM over the structure (ϕ, ξ, η) and we prove that they are metallic structures, which generalize the notion of almost complex structure $\tilde{J}$ introduced by Tanno [10].
- The basic geometrical properties of fundamental 2-Form and its derivative on TM over the structure (ϕ, ξ, η) are studied.
- The integrability conditions and expressions of the Lie derivative of metallic structures $\widetilde{F}$ and F^* with the help of complete and horizontal lifts, respectively, on TM over the structure (ϕ, ξ, η) are investigated.
- The existence of almost quadratic ϕ-manifolds on TM with non-trivial examples are proved.

2. Preliminaries

Let M be an n-dimensional differentiable manifold of class C^∞ and TM be the tangent bundle over a manifold M such that $TM = \bigcup_{x \in M} T_x M$ with the projection map $\pi : TM \to M$, where $T_x M$ represents the tangent space at a point x of M. Let (U, x^h) be a local chart in M and (x^h, y^h) be a local coordinate in $\pi^{-1}(U) \subset TM$ and be called the induced coordinate in $\pi^{-1}(U)$.

Let f, η, Υ_1, and F be a function, a 1-form, a vector field, and a tensor field of type (1,1) of M, respectively. The vertical lifts $f^V, \eta^V, {\Upsilon_1}^V$, and F^V on TM in terms of partial differential equations are given by [8,25]

$$\begin{aligned} f^V &= f \circ \pi, \\ \eta^V &= (\eta_i)^V (dx^i)^V, \\ {\Upsilon_1}^V &= x^h \frac{\partial}{\partial y^h}, \\ F^V &= F_i^h \frac{\partial}{\partial y^h} \otimes dx^i, \end{aligned}$$

where η_i, x^h, and F_i^h, $i, h = 1, 2, \ldots, n$ are local components of η, Υ_1, and F on M, respectively.

The complete lifts $f^C, \eta^C, {\Upsilon_1}^C$, and F^C on TM in the term of partial differential equations are given by

$$\begin{aligned} f^C &= y^i \partial_i f = \partial f, \\ \eta^C &= y^i \partial_i \eta, \\ {\Upsilon_1}^C &= x^h \frac{\partial}{\partial x^h} + \frac{\partial x^h}{\partial x^i} y^i \frac{\partial}{\partial y^h}, \\ F^C &= (F_i^h)^C \frac{\partial}{\partial y^h} \otimes dx^i + (F_i^h)^V \frac{\partial}{\partial x^h} \otimes dx^i + (F_i^h)^V \frac{\partial}{\partial y^h} \otimes dy^h. \end{aligned}$$

By the definition of the lift, we have

$$\begin{aligned} (i)\ \ {\Upsilon_1}^V f^V &= 0,\ \ {\Upsilon_1}^V f^C = (\Upsilon_1 f)^V, \\ (ii)\ \ {\Upsilon_1}^C f^V &= (\Upsilon_1 f)^V,\ \ {\Upsilon_1}^C f^C = (\Upsilon_1 f)^C, \\ (iii)\ \ \eta^V({\Upsilon_1}^V) &= 0,\ \ \eta^V({\Upsilon_1}^C) = (\eta \Upsilon_1)^V, \\ (iv)\ \ \eta^C({\Upsilon_1}^V) &= (\eta \Upsilon_1)^V,\ \ \eta^C({\Upsilon_1}^C) = (\eta \Upsilon_1)^C, \\ (v)\ \ F^V {\Upsilon_1}^V &= 0,\ \ F^V {\Upsilon_1}^C = (F \Upsilon_1)^V, \\ (vi)\ \ F^C {\Upsilon_1}^V &= (F \Upsilon_1)^V,\ \ F^C {\Upsilon_1}^C = (F \Upsilon_1)^C. \end{aligned} \tag{1}$$

By the definition of the Lie product of the lift, we have

$$[{\Upsilon_1}^V, {\Upsilon_2}^V] = 0,\ \ [{\Upsilon_1}^V, {\Upsilon_2}^C] = [\Upsilon_1, \Upsilon_2]^V,\ \ [{\Upsilon_1}^C, {\Upsilon_2}^C] = [\Upsilon_1, \Upsilon_2]^C. \tag{2}$$

Let f be a function and ∇ is an affine connection on M. The horizontal lift is

$$f^H = f^C - \nabla_\gamma f,$$

where ∇f is a gradient of f on M, γ is an operator, and $\nabla_\gamma f = \gamma(\nabla f)$ is in $\pi^{-1}(U)$ (see [8], p. 86).

Let Υ_1, η, and S be a vector field, a 1-form, and a tensor field of arbitrary type on M, respectively. The horizontal lifts ${\Upsilon_1}^H, \eta^H$, and S^H on TM are given by

$${\Upsilon_1}^H = {\Upsilon_1}^C - \nabla_\gamma \Upsilon_1, \tag{3}$$

$$\eta^H = \eta^C - \nabla_\gamma \eta, \tag{4}$$

$$S^H = S^C - \nabla_\gamma S. \tag{5}$$

By the definitions of the lifts, we have

$${\Upsilon_1}^H f^V = (\Upsilon_1 f)^V,\ \ F^V {\Upsilon_1}^H = (F \Upsilon_1)^V,\ \ \eta^V({\Upsilon_1}^H) = (\eta(\Upsilon_1))^V. \tag{6}$$

By the definitions of the Lie product of the lifts, we have

$$\begin{aligned} [{\Upsilon_1}^V, {\Upsilon_2}^H] &= [\Upsilon_1, \Upsilon_2]^V - (\nabla_{\Upsilon_1} \Upsilon_2)^V = -(\hat{\nabla}_{\Upsilon_2} \Upsilon_1)^V, \\ [{\Upsilon_1}^C, {\Upsilon_2}^H] &= [\Upsilon_1, \Upsilon_2]^H - \gamma £_{\Upsilon_1} \Upsilon_2, \\ [{\Upsilon_1}^H, {\Upsilon_2}^H] &= [\Upsilon_1, \Upsilon_2]^H - \gamma \hat{R}(\Upsilon_1, \Upsilon_2), \end{aligned} \tag{7}$$

where $£_{\Upsilon_1}$ represents the Lie derivative with respect to Υ_1 and $\hat{R}$ represents the curvature tensor of $\hat{\nabla}$ given by $\hat{\nabla}_{\Upsilon_1} \Upsilon_2 = \nabla_{\Upsilon_2} \Upsilon_1 + [\Upsilon_1, \Upsilon_2]$.

In addition, let P and Q be arbitrary elements of $\Im(M)$, then

$$\begin{aligned} (P \otimes Q)^V &= P^V \otimes Q^V, \\ (P \otimes Q)^C &= P^C \otimes Q^V + P^V \otimes Q^C, \\ (P \otimes Q)^H &= P^H \otimes Q^V + P^V \otimes Q^H. \end{aligned}$$

Let g^C be the complete lift on TM of a Riemannian metric g on M. Then [20]

$$\begin{aligned} g^C({\rm Y}_1{}^C, {\rm Y}_2{}^V) &= g^C({\rm Y}_1{}^V, {\rm Y}_2{}^C) = (g({\rm Y}_1, {\rm Y}_2))^V, \\ g^C({\rm Y}_1{}^V, {\rm Y}_2{}^V) &= 0, \\ g^C({\rm Y}_1{}^C, {\rm Y}_2{}^C) &= (g({\rm Y}_1, {\rm Y}_2))^C, \end{aligned}$$

where ${\rm Y}_1$ and ${\rm Y}_2$ are vector fields on M.

2.1. Metallic Structure

The quadratic structure J on M satisfying

$$J^2 = pJ + qI, \tag{8}$$

where J denotes a tensor field of type (1,1), I is the identity vector field, and p, q are natural numbers, named as a metallic structure. The structure (M, J) is called a metallic manifold [26–31].

Let g be a Riemannian metric on M such that

$$g(J{\rm Y}_1, {\rm Y}_2) = g({\rm Y}_1, J{\rm Y}_2)$$

or equally,

$$g(J{\rm Y}_1, J{\rm Y}_2) = pg(J{\rm Y}_1, {\rm Y}_2) + qg({\rm Y}_1, {\rm Y}_2),$$

where ${\rm Y}_1$ and ${\rm Y}_2$ are vector fields on M. The structure (M, J, g) is said to be a metallic Riemannian manifold [32,33].

The Nijenhuis tensor of J is denoted by N_J and given by

$$N_J({\rm Y}_1, {\rm Y}_2) = [J{\rm Y}_1, J{\rm Y}_2] - J[J{\rm Y}_1, {\rm Y}_2] - J[{\rm Y}_1, J{\rm Y}_2] + J^2[{\rm Y}_1, {\rm Y}_2],$$

J is integrable if $N_J({\rm Y}_1, {\rm Y}_2) = 0$.

2.2. Almost Quadratic ϕ-Structure

Debnath et al. [23] introduced the notion of structure (ϕ, ξ, η) and discussed some geometric properties of such structures. Next, Gonul et al. [24] investigated the connection between MS and almost quadratic ϕ-structures. Consider a non-null tensor fields ϕ of type (1,1), a 1-form η and a vector field ξ on M satisfying

$$\phi^2 = p\phi + qI - q\eta \otimes \xi, \quad p^2 + 4q \neq 0, q \neq 0 \tag{9}$$

$$\eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi(\xi) = 0, \tag{10}$$

where p and q are constants and I is the identity vector field. The structure (ϕ, ξ, η) is called an almost quadratic ϕ-structure on M and the manifold (M, ϕ, ξ, η) is called an almost quadratic ϕ-manifold [23,24,34].

Furthermore,

$$g(\phi{\rm Y}_1, {\rm Y}_2) = g({\rm Y}_1, \phi{\rm Y}_2)$$

or equally,

$$g(\phi{\rm Y}_1, \phi{\rm Y}_2) = pg(\phi{\rm Y}_1, {\rm Y}_2) + qg({\rm Y}_1, {\rm Y}_2) - q\eta({\rm Y}_1)\eta({\rm Y}_2).$$

The structure (ϕ, ξ, η, g) is termed as an almost quadratic metric ϕ-structure and the manifold (M, ϕ, ξ, η, g) is called an almost quadratic metric ϕ-manifold.

In addition, the 1-form η associated with g such that

$$g(Y_1, \xi) = \eta(Y_1)$$

and the 2-Form Φ is given by [35]

$$\Phi(Y_1, Y_2) = g(Y_1, \phi Y_2) \tag{11}$$

is said to be the fundamental form of an almost quadratic metric ϕ-manifold.

The Nijenhuis tensor of (ϕ, ξ, η) is denoted by N_ϕ and given by

$$N_\phi(Y_1, Y_2) = [\phi Y_1, \phi Y_2] - \phi[\phi Y_1, Y_2] - \phi[Y_1, \phi Y_2] + \phi^2[Y_1, Y_2],$$

where Y_1 and Y_2 are vector fields on M.

Proposition 1 ([24])**.** *Let $(M, \phi, \xi, \eta, g, \nabla)$ be a (β, ϕ)-Kenmotsu quadratic metric manifold such that $(\nabla_{Y_1}\phi)Y_2 = \beta g(Y_1, \phi Y_2)\xi + \beta\eta(Y_2)\phi Y_1$. Then the structure (ϕ, ξ, η) is integrable; that is, the Nijenhuis tensor $N_\phi = 0$, where ∇ is the Levi-Civita connection.*

3. Proposed Theorems for the Complete Lifts of Metallic Structures on the Tangent Bundle Over (ϕ, ξ, η)

In this section, we study the structure (ϕ, ξ, η) geometrically using complete lift on TM. A tensor field $\widetilde{F}$ on the tangent bundle is defined and show that it is an MS by using the complete lift on TM over (ϕ, ξ, η). Next, mathematical operators, namely fundamental 2-Form Ω and the derivative $d\Omega$ using the complete lift on TM over (ϕ, ξ, η), are calculated. Furthermore, the integrability condition and the Lie derivative of an $MS(\widetilde{F})$ by using the complete lift on TM over (ϕ, ξ, η) are established.

Let M be an n dimensional differentiable manifold and ϕ, η, and ξ be a tensor field of type (1,1), a 1-form and a vector field on M, respectively.

Applying complete lifts on (9), (10) and using (1), we obtain

$$\begin{aligned}
(\phi^C)^2 &= p\phi^C + qI - q(\eta^V \otimes \xi^C + \eta^C \otimes \xi^V), \\
\eta^C(\xi^C) &= \eta^V(\xi^V) = 0, \;\; \eta^V(\xi^C) = \eta^C(\xi^V) = 1, \\
\eta^C \circ \phi^C &= \eta^V \circ \phi^C = \eta^C \circ \phi^V = \eta^V \circ \phi^V = 0, \\
\phi^C(\xi^V) &= \phi^V(\xi^C) = \phi^C(\xi^C) = \phi^V(\xi^V) = 0,
\end{aligned}$$

where $\phi^C, \eta^C, \xi^C, \phi^V, \eta^V$, and ξ^V are complete and vertical lifts of ϕ, η, and ξ, respectively, on TM. Azami [20] defined a tensor field J of type (1,1) on TM with an almost paracontact structure (ϕ, η, ξ, g) as

$$J = \frac{p}{2}I - \left(\frac{2\sigma_p^q - p}{2}\right)\left(\phi^C + \eta^V \otimes \xi^V + \eta^C \otimes \xi^C\right)$$

and proved that it is an MS on TM.

Recently, Khan [21] introduced a tensor $\widetilde{J}$ on FM immersed with an almost contact structure (ϕ, η, ξ, g) as

$$\begin{aligned}
\widetilde{J} &= \frac{p}{2}I - \left(\frac{2\sigma_p^q - p}{2}\right)[\phi^H + \sum_{\alpha=1}^{n} \eta^{H_\alpha} \otimes \xi^{(\alpha+n)} \\
&- \sum_{\alpha=1}^{n} \eta^{H_{\alpha+n}} \otimes \xi^{(\alpha)} + \eta^V \otimes \xi^{(2n+1)} - \eta^{H_{2n+1}} \otimes \xi^H],
\end{aligned}$$

where $\phi^H, \eta^{H_\alpha}, \alpha = 1, 2, \ldots, n$ and ξ^H are horizontal lifts of a tensor field ϕ of type (1,1), a 1-form η and a vector field ξ, respectively, and $\xi^{(\alpha)}$ is α-vertical lift of ξ on FM.

From Azami [20] and Khan [21], let us introduce a new $(1,1)$-type tensor field $\widetilde{F}$ on TM as

$$\widetilde{F} = \frac{p}{2}I - A\Big[\phi^C + \sqrt{q}\Big(\eta^V \otimes \xi^V + \eta^C \otimes \xi^C\Big)\Big], \tag{12}$$

where $A = \frac{2\sigma_p^q - p}{2\sqrt{p\phi^C + q}}$. Since p, q are natural numbers and ϕ is non-singular, therefore $p\phi^C + q > 0$ and $A \neq 0$.

Theorem 1. *Let TM be a tangent bundle of M immersed with structure (ϕ, ξ, η). Then $\widetilde{F}$ given by (12) is a metallic structure on TM.*

Proof. Let Y_1 be a vector field on M and ${\mathrm{Y}_1}^C$ and ${\mathrm{Y}_1}^V$ be complete and vertical lifts of Y_1, respectively, on TM. Applying ξ^V, ξ^C, and ϕ^C on (12), we obtain

$$\widetilde{F}(\xi^V) = \frac{p}{2}\xi^V - A\sqrt{q}\xi^C, \tag{13}$$

$$\widetilde{F}(\xi^C) = \frac{p}{2}\xi^C - A\sqrt{q}\xi^V, \tag{14}$$

$$\widetilde{F}(\phi^C\widetilde{\mathrm{Y}}_1) = \frac{p}{2}\phi^C\widetilde{\mathrm{Y}}_1 - A[p\phi^C\widetilde{\mathrm{Y}}_1 + q\widetilde{\mathrm{Y}}_1 - q(\eta^V(\widetilde{\mathrm{Y}}_1)\xi^C + \eta^C(\widetilde{\mathrm{Y}}_1)\xi^V)], \tag{15}$$

where $\widetilde{\mathrm{Y}}_1$ is a vector field on TM.

In the view of (12)–(15), we obtain

$$\begin{aligned} \widetilde{F}^2(\widetilde{\mathrm{Y}}_1) &= \frac{p}{2}\widetilde{F}(\widetilde{\mathrm{Y}}_1) - A\Big[\varphi^C\widetilde{\mathrm{Y}}_1 + \sqrt{q}(\eta^V(\widetilde{\mathrm{Y}}_1)\widetilde{F}(\xi^V) + \eta^C(\widetilde{\mathrm{Y}}_1)\widetilde{F}(\xi^C))\Big], \\ &= p\widetilde{F}(\widetilde{\mathrm{Y}}_1) + q(\widetilde{\mathrm{Y}}_1). \end{aligned}$$

This shows that $\widetilde{F}$ is an MS on TM. □

Corollary 1. *Let Y_1 and Y_2 be vector fields on M and $\widetilde{F}$ be an MS on TM given by (12) such that $\eta(\mathrm{Y}_1) = 0$, then*

$$\begin{aligned} \widetilde{F}{\mathrm{Y}_1}^V &= \frac{p}{2}{\mathrm{Y}_1}^V - A\Big[(\phi\mathrm{Y}_1)^V + \sqrt{q}(\eta(\mathrm{Y}_1))^V\xi^C\Big], \\ \widetilde{F}\mathrm{Y}_1^C &= \frac{p}{2}\mathrm{Y}_1^C - A\Big[(\phi\mathrm{Y}_1)^C + \sqrt{q}(\eta(\mathrm{Y}_1))^V\xi^V + (\eta(\mathrm{Y}_1))^C\xi^C\Big]. \end{aligned}$$

If $\eta(\mathrm{Y}_1) = 0$, then

$$\widetilde{F}\mathrm{Y}_1^V = \frac{p}{2}\mathrm{Y}_1^V - A(\phi\mathrm{Y}_1)^V, \tag{16}$$

$$\widetilde{F}\mathrm{Y}_1^C = \frac{p}{2}\mathrm{Y}_1^C - A(\phi\mathrm{Y}_1)^C. \tag{17}$$

Proof. The proof is obtained by applying Y_1^C and Y_1^V on $\widetilde{F}$ given by (12) and using $\eta(\mathrm{Y}_1) = 0$.

Let g^C be the complete lift of the metric g on TM. The 2-form on TM defined by

$$\Omega(\widetilde{\mathrm{Y}}_1, \widetilde{\mathrm{Y}_2}) = g^C(\widetilde{\mathrm{Y}}_1, \widetilde{\mathrm{Y}}_2), \tag{18}$$

where $\widetilde{\mathrm{Y}}_1$ and $\widetilde{\mathrm{Y}_2}$ are vector fields and $\widetilde{F}$ is an MS given by (12) on TM. □

Theorem 2. *Let TM be the tangent bundle of M, g^C be the complete lift of g and $\widetilde{F}$ be an MS given by (12) on TM, then the 2-form Ω is given by*

$$\begin{aligned}
(i)\ \Omega(Y_1^C, Y_2{}^C) &= \frac{p}{2}(g(Y_1, Y_2))^C - A\sqrt{q}\{\eta(Y_1)^V\eta(Y_2)^V + \eta(Y_1)^C\eta(Y_2)^C\} \\
&- A(g(Y_1, \phi Y_2))^C, \\
(ii)\ \Omega(Y_1^C, Y_2{}^V) &= \frac{p}{2}(g(Y_1, Y_2))^V - A\sqrt{q}\eta(Y_1)^C\eta(Y_2)^V - A(g(Y_1, \phi Y_2))^V, \\
(iii)\ \Omega(Y_1^V, Y_2{}^C) &= \frac{p}{2}(g(Y_1, Y_2))^V - A\sqrt{q}\eta(Y_1)^V\eta(Y_2)^C - A(g(Y_1, \phi Y_2))^V, \\
(iv)\ \Omega(Y_1^V, Y_2{}^V) &= -A\sqrt{q}\eta(Y_1)^V\eta(Y_2)^V,
\end{aligned}$$

where $\widetilde{Y}_1$ and $\widetilde{Y}_2$ are vector fields on TM.

Proof. (i) Let $\widetilde{Y}_1 = Y_1^C$ and $\widetilde{Y}_2 = Y_2^C$ in (18) and using (1) and (12), we have

$$\begin{aligned}
\Omega(Y_1^C, Y_2{}^C) &= g^C(Y_1^C, \widetilde{F}Y_2{}^C) \\
&= g^C\Big(Y_1^C, \frac{p}{2}Y_2{}^C - A\Big[(\phi Y_2)^C + \sqrt{q}(\eta(Y_2))^V\xi^V + (\eta(Y_2))^C\xi^C\Big]\Big) \\
&= \frac{p}{2}(g(Y_1, Y_2))^C - A\sqrt{q}[(\eta(Y_1))^V(\eta(Y_2))^V + (\eta(Y_1))^C(\eta(Y_2))^C] \\
&- A(g(Y_1, \phi Y_2))^C.
\end{aligned}$$

(ii) Let $\widetilde{Y}_1 = Y_1^C$ and $\widetilde{Y}_2 = Y_2^V$ in (18) and using (1) and (12), we have

$$\begin{aligned}
\Omega(Y_1^C, Y_2{}^V) &= g^C(Y_1^C, \widetilde{F}Y_2{}^V) \\
&= g^C\Big(Y_1^C, \frac{p}{2}Y_2{}^V - A\Big[(\phi Y_2)^V + \sqrt{q}(\eta(Y_2))^V\xi^C\Big]\Big) \\
&= \frac{p}{2}(g(Y_1, Y_2))^V - A\sqrt{q}\eta(Y_1)^C\eta(Y_2)^V - A(g(Y_1, \phi Y_2))^V.
\end{aligned}$$

(iii) Let $\widetilde{Y}_1 = Y_1^V$ and $\widetilde{Y}_2 = Y_2{}^C$ in (18) and using (1) and (12), we have

$$\begin{aligned}
\Omega(Y_1^V, Y_2{}^C) &= g^C(Y_1^V, \widetilde{F}Y_2{}^C) \\
&= g^C\Big(Y_1^V, \frac{p}{2}Y_2{}^C - A\Big[(\phi Y_2)^C + \sqrt{q}(\eta(Y_2))^V\xi^V + \eta(Y_2))^C\xi^C\Big]\Big) \\
&= \frac{p}{2}(g(Y_1, Y_2))^V - A\sqrt{q}\eta(Y_1)^V\eta(Y_2)^C - A(g(Y_1, \phi Y_2))^V.
\end{aligned}$$

(iv) Let $\widetilde{Y}_1 = Y_1^V$ and $\widetilde{Y}_2 = Y_2{}^V$ in (18) and using (1) and (12), we have

$$\begin{aligned}
\Omega(Y_1^V, Y_2{}^V) &= g^C(Y_1^V, \widetilde{F}Y_2{}^V) \\
&= g^C\Big(Y_1^V, \frac{p}{2}Y_2{}^V - A\Big[(\phi Y_2)^V + \sqrt{q}(\eta(Y_2))^V\xi^C\Big]\Big) \\
&= -A\sqrt{q}\eta(Y_1)^V\eta(Y_2)^V.
\end{aligned}$$

□

Theorem 3. *Let TM be the tangent bundle of M, g^C be the complete lift of g, and $\widetilde{F}$ be an MS given by (12), then the derivative $d\Omega$ is given by*

(i) $$\begin{aligned}
3d\Omega(Y_1^C, Y_2{}^C, Y_3{}^V) &= \tfrac{p}{2}((Y_1 g(Y_2, Y_3))^V - (Y_2 g(Y_1, Y_3))^V + (Y_3 g(Y_1, Y_2))^V) \\
&- A((Y_1 g(Y_2, \phi Y_3))^V - (Y_2 g(Y_1, \phi Y_3))^V + (Y_3 g(Y_1, \phi Y_2))^V) \\
&- A\sqrt{q}(Y_1^C\eta(Y_2)^V\eta(Y_3)^C - Y_2{}^C(\eta(Y_3))^V(\eta(Y_1))^C \\
&+ Y_3{}^C\eta(Y_2)^V\eta(Y_1)^C + Y_3{}^V\eta(Y_2)^C\eta(Y_1)^C) - \tfrac{p}{2}((g([Y_1, Y_2], Y_3))^V \\
&- (g([Y_1, Y_3], Y_2))^V + (g([Y_2, Y_3], Y_1))^V) + A((g([Y_1, Y_2], \phi Y_3))^V
\end{aligned}$$

$$- (g([Y_1, Y_3], \phi Y_2))^V + (g([Y_2, Y_3], \phi Y_1))^V) + A\sqrt{q}((\eta(Y_3))^V \eta([Y_1, Y_2])^C - \eta(Y_2)^C \eta([Y_1, Y_3])^V + \eta(Y_1)^C \eta([Y_2, Y_3])^V).$$

(ii) $$3d\Omega(Y_1^C, Y_2{}^V, Y_3{}^V) = \frac{p}{2}(((Y_3 g(Y_1, Y_2))^V - (Y_2 g(Y_1, Y_3))^V) + A((Y_3 g(Y_1, \phi Y_2))^V - (Y_2 g(Y_1, \phi Y_3))^V) - A\sqrt{q}(Y_1^C \eta(Y_2)^V \eta(Y_3)^V + Y_2{}^V \eta(Y_3)^V \eta(Y_1)^V + Y_3{}^V \eta(Y_1)^C \eta(Y_2)^V).$$

(iii) $$3d\Omega(Y_1^V, Y_2{}^V, Y_3{}^V) = -A\sqrt{q}(Y_1^V \eta(Y_2)^V \eta(Y_3)^V + Y_2{}^V \eta(Y_3)^V \eta(Y_1)^V - Y_3{}^V \eta(Y_1)^V \eta(Y_2)^V).$$

(iv) $$3d\Omega(Y_1^C, Y_2{}^C, Y_3{}^C) = \frac{p}{2}((Y_1 g(Y_2, Y_3))^C - (Y_2 g(Y_1, Y_3))^C + (Y_3 g(Y_1, Y_2))^C) - A((Y_1 g(Y_2, \phi Y_3))^C - (Y_2 g(Y_1, \phi Y_3))^C + (Y_3 g(Y_1, \phi Y_2))^C) - A\sqrt{q}(Y_1^C \eta(Y_2)^V \eta(Y_3)^V + Y_1^C \eta(Y_2)^C \eta(Y_3)^C - Y_2{}^C \eta(Y_3)^V \eta(Y_1)^C - Y_2{}^C \eta(Y_3)^C \eta(Y_1)^C + Y_3{}^C \eta(Y_2)^V \eta(Y_1)^V + Y_3{}^C \eta(Y_2)^C \eta(Y_1)^C) - \frac{p}{2}((g([Y_1, Y_2], Y_3))^C - (g([Y_1, Y_3], Y_2))^C + (g([Y_2, Y_3], Y_1))^C) + A((g([Y_1, Y_2], \phi Y_3))^C - (g([Y_1, Y_3], \phi Y_2))^C + (g([Y_2, Y_3], \phi Y_1))^C) + A\sqrt{q}(\eta(Y_3)^V \eta([Y_1, Y_2])^V - \eta(Y_2)^C \eta([Y_1, Y_3])^V + \eta(Y_1)^C \eta([Y_2, Y_3])^V) + A\sqrt{q}(\eta(Y_3)^C \eta([Y_1, Y_2])^C - \eta(Y_2)^C \eta([Y_1, Y_3])^C + \eta(Y_1)^C \eta([Y_2, Y_3])^C).$$

Proof. We have

$$\begin{aligned} 3d\Omega(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3) &= \{\tilde{Y}_1(\Omega(\tilde{Y}_2, \tilde{Y}_3)) - \tilde{Y}_2(\Omega(\tilde{Y}_1, \tilde{Y}_3)) + \tilde{Y}_3(\Omega(\tilde{Y}_1, \tilde{Y}_2)) \\ &- \Omega([\tilde{Y}_1, \tilde{Y}_2], \tilde{Y}_3) + \Omega([\tilde{Y}_1, \tilde{Y}_3], \tilde{Y}_2) - \Omega([\tilde{Y}_2, \tilde{Y}_3], \tilde{Y}_1)\}, \end{aligned}$$

called coboundary formula [35]. Here $\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3$ are arbitrary vector fields on TM.

Applying (1)–(7), (12)–(15), Theorem 2 and using $\eta(Y_1) = \eta(Y_2) = 0$, we have

(i) $$\begin{aligned} 3d\Omega(Y_1^C, Y_2{}^C, Y_3{}^V) &= Y_1^C(\Omega(Y_2{}^C, Y_3{}^V)) - Y_2{}^C(\Omega(Y_1^C, Y_3{}^V)) \\ &+ Y_3{}^V(\Omega(Y_1^C, Y_2{}^C)) - \Omega([Y_1^C, Y_2{}^C], Y_3{}^V) + \Omega([Y_1^C, Y_3{}^V], Y_2{}^C) \\ &- \Omega([Y_2{}^C, Y_3{}^C], Y_1^V) = Y_1^C(\frac{p}{2}(g(Y_2, Y_3))^V - A(g(Y_2, \phi Y_3))^V \\ &- Y_2{}^C(\frac{p}{2}(g(Y_1, Y_3))^V - A(g(Y_1, \phi Y_3))^V - A\sqrt{q}\eta(Y_1)^C \eta(Y_3)^V) \\ &+ Y_3{}^C(\frac{p}{2}(g(Y_1, Y_2))^V - A(g(Y_1, \phi Y_2))^V - A\sqrt{q}\eta(Y_1)^C \eta(Y_2)^V) \\ &- (\frac{p}{2}(g([Y_1, Y_2], Y_3))^V - A(g([Y_1, Y_2], \phi Y_3))^V - A\sqrt{q}\eta([Y_1, Y_2])^C \eta(Y_3)^V) \\ &+ (\frac{p}{2}(g([Y_1, Y_3], Y_2))^V - A(g([Y_1, Y_3], \phi Y_2))^V - A\sqrt{q}\eta([Y_1, Y_3])^C \eta(Y_2)^V) \\ &- (\frac{p}{2}(g([Y_2, Y_3], Y_1))^V - A(g([Y_2, Y_3], \phi Y_1))^V - A\sqrt{q}\eta([Y_2, Y_3])^C \eta(Y_1)^V) \\ &= \frac{p}{2}((Y_1 g(Y_2, Y_3))^V - (Y_2 g(Y_1, Y_3))^V + (Y_3 g(Y_1, Y_2))^V) \\ &- A((Y_1 g(Y_2, \phi Y_3))^V - (Y_2 g(Y_1, \phi Y_3))^V + (Y_3 g(Y_1, \phi Y_2))^V) \\ &- A\sqrt{q}(Y_1^C \eta(Y_2)^V \eta(Y_3)^C - Y_2{}^C \eta(Y_3)^V \eta(Y_1)^C + Y_3{}^C \eta(Y_2)^V \eta(Y_1)^C \\ &+ Y_3{}^V \eta(Y_2)^C \eta(Y_1)^C) - \frac{p}{2}((g([Y_1, Y_2], Y_3))^V - (g([Y_1, Y_3], Y_2))^V \\ &+ (g([Y_2, Y_3], Y_1))^V) + A((g([Y_1, Y_2], \phi Y_3))^V - (g([Y_1, Y_3], \phi Y_2))^V \\ &+ (g([Y_2, Y_3], \phi Y_1))^V) + A\sqrt{q}(\eta(Y_3)^V \eta([Y_1, Y_2])^C \\ &- \eta(Y_2)^C \eta([Y_1, Y_3])^V + \eta(Y_1)^C \eta([Y_2, Y_3])^V). \end{aligned}$$

Other results were obtained by using similar devices. □

Theorem 4. *A metallic structure $\widetilde{F}$, defined by (12), is integrable on TM over (ϕ,ξ,η) if and only if $N_{\widetilde{F}}=0$, which is equivalent to the conditions*

$$\eta([\phi\mathrm{Y}_1,\mathrm{Y}_2])=0,\ \ \eta([\phi\mathrm{Y}_1,\xi])=0,\ \ \eta([\mathrm{Y}_1,\mathrm{Y}_2])=0,\ \ \eta([\mathrm{Y}_1,\xi])=0,$$

and (ϕ,ξ,η) is integrable i.e. $N_\phi=0$.

Proof. Let $N_{\widetilde{F}}$ stand for the Nijenhuis tensor of $\widetilde{F}$. Then

$$N_{\widetilde{F}}(\widetilde{\mathrm{Y}}_1,\widetilde{\mathrm{Y}}_2)=[\widetilde{F}\widetilde{\mathrm{Y}}_1,\widetilde{F}\widetilde{\mathrm{Y}}_2]-\widetilde{F}[\widetilde{F}\widetilde{\mathrm{Y}}_1,\widetilde{\mathrm{Y}}_2]-\widetilde{F}[\widetilde{\mathrm{Y}}_1,\widetilde{F}\widetilde{\mathrm{Y}}_2]+\widetilde{F}^2[\widetilde{\mathrm{Y}}_1,\widetilde{\mathrm{Y}}_2], \tag{19}$$

where $\widetilde{\mathrm{Y}}_1$ and $\widetilde{\mathrm{Y}}_2$ are vector fields on TM.

Applying (1)–(7), (12)–(15) on (19), and using $\eta(\mathrm{Y}_1)=\eta(\mathrm{Y}_2)=0$, we have

$$\begin{aligned}
N_{\widetilde{F}}(\mathrm{Y}_1^V,\mathrm{Y}_2^V) &= 0,\\
N_{\widetilde{F}}(\mathrm{Y}_1^V,\mathrm{Y}_2^C) &= A^2(N_\phi(\mathrm{Y}_1,\mathrm{Y}_2))^V+A^2\sqrt{q}\eta([\phi\mathrm{Y}_1,\mathrm{Y}_2])^V\xi^C\\
&+ A\sqrt{q}\eta([\mathrm{Y}_1,\phi\mathrm{Y}_2])^V\xi^C+q\eta([\mathrm{Y}_1,\mathrm{Y}_2])^V\xi^V\\
&+ q\eta([\mathrm{Y}_1,\mathrm{Y}_2])^V\xi^C,\\
N_{\widetilde{F}}(\mathrm{Y}_1^C,\mathrm{Y}_2^C) &= A^2(N_\phi(\mathrm{Y}_1,\mathrm{Y}_2))^C-A^2\sqrt{q}\eta([\phi\mathrm{Y}_1,\mathrm{Y}_2])^V\xi^V\\
&- A^2\sqrt{q}\eta([\mathrm{Y}_1,\phi\mathrm{Y}_2])^V\xi^V\\
&- A^2\sqrt{q}\eta([\phi\mathrm{Y}_1,\mathrm{Y}_2])^C\xi^C-A^2\sqrt{q}\eta([\mathrm{Y}_1,\phi\mathrm{Y}_2])^C\xi^C,\\
N_{\widetilde{F}}(\mathrm{Y}_1^V,\xi^V) &= 0,\\
N_{\widetilde{F}}(\mathrm{Y}_1^V,\xi^C) &= A^2(\phi^2[\mathrm{Y}_1,\xi])^V+A^2q(\eta[\mathrm{Y}_1,\mathrm{Y}_2])\xi^V\\
&- A^2(\phi[\phi\mathrm{Y}_1,\xi])^V-A^2\sqrt{q}([\phi\mathrm{Y}_1,\xi])^V\xi^C,\\
N_{\widetilde{F}}(\mathrm{Y}_1^C,\xi^V) &= A^2(\phi^2[\mathrm{Y}_1,\xi])^V+A^2\sqrt{q}([\phi\mathrm{Y}_1,\xi])^C-A^2(\phi[\phi\mathrm{Y}_1,\xi])^V\\
&- A^2\sqrt{q}\eta([\phi\mathrm{Y}_1,\xi])^V\xi^C+-A\sqrt{q}(\phi[\mathrm{Y}_1,\xi])^C-\frac{p^2}{2}[\mathrm{Y}_1,\xi]^V\\
&- A^2q\eta([\mathrm{Y}_1,\xi])^C\xi^C+Ap(\phi[\mathrm{Y}_1,\xi])^V+pA\sqrt{q}\eta([\mathrm{Y}_1,\xi])^V\xi^C,\\
N_{\widetilde{F}}(\mathrm{Y}_1^C,\xi^C) &= A^2(\phi^2[\mathrm{Y}_1,\xi])^C+A^2q(\eta[\mathrm{Y}_1,\xi]\xi)^C+A^2\sqrt{q}[\phi\mathrm{Y}_1,\xi]^V\\
&- A^2(\phi[\phi\mathrm{Y}_1,\xi])^C+A^2\sqrt{q}(\eta[\phi\mathrm{Y}_1,\mathrm{Y}_2])^V\xi^V+A^2\sqrt{q}(\eta[\phi\mathrm{Y}_1,\mathrm{Y}_2])^C\xi^C\\
&- A^2\sqrt{q}(\phi[\mathrm{Y}_1,\xi])^V-A^2q(\eta[\mathrm{Y}_1,\xi])^V\xi^C,\\
N_{\widetilde{F}}(\xi^V,\xi^C) &= 0.
\end{aligned}$$

Let $\widetilde{\mathrm{Y}}_1$ and $\widetilde{F}$ be a vector field and an MS, respectively, on TM. The Lie derivative of $\widetilde{F}$ with respect to $\widetilde{\mathrm{Y}}_1$ is given by ([8], p. 113)

$$\left(\pounds_{\widetilde{\mathrm{Y}}_1}\widetilde{F}\right)\widetilde{\mathrm{Y}_2}=[\widetilde{\mathrm{Y}}_1,\widetilde{F}\widetilde{\mathrm{Y}}_2]-\widetilde{F}[\widetilde{\mathrm{Y}}_1,\widetilde{\mathrm{Y}}_2], \tag{20}$$

where $\widetilde{\mathrm{Y}_2}$ is a vector field on TM. □

Theorem 5. *Let $\widetilde{F}$ be an MS on TM given by (12) and Υ_1 and Υ_2 be vector fields on M such that $\eta(\Upsilon_1) = \eta(\Upsilon_2) = 0$, then*

$$\begin{aligned}
(i) \;\; (\pounds_{\Upsilon_2^V}\widetilde{F})\Upsilon_1^V &= 0,\\
(ii) \;\; (\pounds_{\Upsilon_2^V}\widetilde{F})\Upsilon_1^C &= A\Big((\phi[\Upsilon_2,\Upsilon_1])^V - [\Upsilon_2,\phi\Upsilon_1]^V + \sqrt{q}(\eta[\Upsilon_2,\Upsilon_1])^V\Big),\\
(iii) \;\; (\pounds_{\Upsilon_2^V}\widetilde{F})\xi^V &= -Aq[\Upsilon_2,\xi]^V,\\
(iv) \;\; (\pounds_{\Upsilon_2^V}\widetilde{F})\xi^C &= A\Big(\phi[\Upsilon_2,\xi]^V - \sqrt{q}(\eta[\Upsilon_2,\xi])^V\xi^C\Big),\\
(v) \;\; (\pounds_{\Upsilon_2^C}\widetilde{F})\Upsilon_1^V &= A\Big(\phi[\Upsilon_2,\Upsilon_1]^V - [\Upsilon_2,\phi\Upsilon_1]^V - \sqrt{q}(\eta([\Upsilon_2,\Upsilon_1])^V\xi^C\Big),\\
(vi) \;\; (\pounds_{\Upsilon_2^C}\widetilde{F})\Upsilon_1^C &= A\Big((\phi[\Upsilon_2,\Upsilon_1])^C - [\Upsilon_2,\phi\Upsilon_1]^C\Big)\\
&- A\sqrt{q}\Big((\eta(\Upsilon_2,\Upsilon_1))^V\xi^V - (\eta(\Upsilon_2,\Upsilon_1))^C\xi^C\Big),\\
(vii) \;\; (\pounds_{\Upsilon_2^C}\widetilde{F})\xi^V &= A\Big((\phi[\Upsilon_2,\xi])^V + \sqrt{q}(\eta[\Upsilon_2,\xi])^V\xi^C - \sqrt{q}[\Upsilon_2,\xi]^C\Big),\\
(viii) \;\; (\pounds_{\Upsilon_2^C}\widetilde{F})\xi^C &= A\Big(\phi[\Upsilon_2,\xi])^C - \sqrt{q}[\Upsilon_2,\xi]^V\Big)\\
&+ A\sqrt{q}\Big((\eta[\Upsilon_2,\xi]^V\xi^V + (\eta[\Upsilon_2,\xi])^C\xi^C\Big),
\end{aligned}$$

Proof. Applying (1)–(7), (12)–(15), and (20), and using $\eta(\Upsilon_1) = \eta(\Upsilon_2) = 0$.

$$\begin{aligned}
(i) \;\; \pounds_{\Upsilon_2^V}\widetilde{F}\Upsilon_1^V &= \pounds_{\Upsilon_2^V}(\frac{p}{2}\Upsilon_1{}^V - A\Big[(\phi\Upsilon_1)^V + \sqrt{q}(\eta(\Upsilon_1))^V\xi^C)\Big]\\
(\pounds_{\Upsilon_2^V}\widetilde{F})\Upsilon_1^V + \widetilde{F}\pounds_{\Upsilon_2^V}\Upsilon_1^V &= \frac{p}{2}\pounds_{\Upsilon_2^V}\Upsilon_1{}^V\\
&- A\pounds_{\Upsilon_2^V}(\phi\Upsilon_1)^V - A\sqrt{q}(\eta(\Upsilon_1))^V\pounds_{\Upsilon_2^V}\xi^C\\
&= 0.
\end{aligned}$$

Others results are obtained by using similar devices. □

4. Proposed Theorems for the Horizontal Lift of Metallic Structures on the Tangent Bundle Over (ϕ, ξ, η)

In this section, we study (ϕ, ξ, η) geometrically using a horizontal lift on TM. A tensor field F^* on the tangent bundle is defined and shows that it is an *MS* by using the horizontal lift on TM over (ϕ, ξ, η). Furthermore, the integrability condition and Lie derivative of an *MS* F^* by using the horizontal lift on TM over (ϕ, ξ, η) are established.

Let M be an n dimensional differentiable manifold and ϕ, η, and ξ be the tensor field of type (1,1), a 1-form, and a vector field on M. Let ϕ^H, η^H, and ξ^H be horizontal lifts of ϕ, η, and ξ, respectively, on TM. Applying horizontal lifts on (9), (10), and using (1), we obtain

$$\begin{aligned}
(\phi^H)^2 &= p\phi^H + qI - q(\eta^V \otimes \xi^H + \eta^H \otimes \xi^V),\\
\eta^H(\xi^H) &= \eta^V(\xi^V) = 0,\;\; \eta^V(\xi^H) = \eta^H(\xi^V) = 1,\\
\eta^H \circ \phi^H &= \eta^V \circ \phi^H = \eta^H \circ \phi^V = \eta^V \circ \phi^V = 0,\\
\phi^H(\xi^V) &= \phi^V(\xi^H) = \phi^H(\xi^V) = \phi^V(\xi^V) = 0.
\end{aligned}$$

From Azami [20] and Khan [21], let us introduced a new tensor field F^* of type (1,1) on TM as

$$F^* = \frac{p}{2}I - B\Big[\phi^H + \sqrt{q}(\eta^V \otimes \xi^V + \eta^H \otimes \xi^H)\Big], \tag{21}$$

where $B = \frac{2\sigma_p^q - p}{2\sqrt{p\phi^H + q}}$. Since p, q are natural numbers and ϕ is non-singular, therefore $p\phi^H + q > 0$ and $A \neq 0$.

Theorem 6. *Let the tangent bundle TM of M be immersed with (ϕ, ξ, η). Then the metallic structure F^*, given by (21), is an MS on TM.*

Proof. Let Y_1 be a vector field on M and Y_1^H, Y_1^C, and Y_1^V be horizontal, complete, and vertical lifts of Y_1, respectively, on TM. Applying ξ^H, ξ^V, ξ^C, and ϕ^H on (21), we obtain

$$\begin{aligned}
(i)\quad F^*(\xi^H) &= \frac{p}{2}\xi^H - B\sqrt{q}\xi^V, \\
(ii)\quad F^*(\xi^V) &= \frac{p}{2}\xi^V - B\sqrt{q}\xi^H, \\
(iii)\quad F^*(\xi^C) &= \frac{p}{2}\xi^C - B[\phi^H(\nabla_\gamma Y_1) + \sqrt{q}\phi^C(\nabla_\gamma Y_1)\xi^H], \\
(iv)\ F^*(\phi^H\widetilde{Y}_1) &= \frac{p}{2}\phi^H(\widetilde{Y}_1) - B[p\phi^H\widetilde{Y}_1 + q\widetilde{Y}_1 - q(\eta^V(\widetilde{Y}_1)\xi^H + \eta^H(\widetilde{Y}_1)\xi^V)].
\end{aligned} \tag{22}$$

In the view of (21) and (22), we obtain

$$\begin{aligned}
(F^*)^2(\widetilde{Y}_1) &= \frac{p}{2}F^*(\widetilde{Y}_1) - B\Big[F^*\varphi^H\widetilde{Y}_1 + \sqrt{q}(\eta^V(\widetilde{Y}_1)F^*(\xi^V) + \eta^H(\widetilde{Y}_1)F^*(\xi^H))\Big], \\
&= pF^*(\widetilde{Y}_1) + q(\widetilde{Y}_1).
\end{aligned}$$

This shows that F^* is an MS. □

Corollary 2. *Let Y_1 and Y_2 be the vector fields on M and F^* be an MS on TM given by (21) such that $\eta(Y_1) = 0$. Then*

$$\begin{aligned}
(i)\quad F^*Y_1^V &= \frac{p}{2}Y_1^V - B\Big[(\phi Y_1)^V + \sqrt{q}(\eta(Y_1))^V\xi^H\Big], \\
(ii)\quad F^*Y_1^H &= \frac{p}{2}Y_1^H - B\Big[(\phi Y_1)^H + \sqrt{q}(\eta(Y_1))^V\xi^V\Big], \\
(iii)\quad F^*Y_1^C &= \frac{p}{2}Y_1^C - B[(\phi Y_1)^H + \phi^H(\nabla_\gamma Y_1) \\
&+ \sqrt{q}((\eta(Y_1))^V\xi^V + \eta^C(\nabla_\gamma Y_1)\xi^H)], \\
(iv)\quad F^*Y_1^C &= \frac{p}{2}Y_1^C - B[(\phi Y_1)^H + \phi^C(\nabla_\gamma Y_1) \\
&+ \sqrt{q}((\eta(Y_1))^V\xi^V + \eta^C(\nabla_\gamma Y_1)\xi^H)].
\end{aligned} \tag{23}$$

If $\eta(Y_1) = 0$, then

$$\begin{aligned}
(i)\quad F^*Y_1^H &= \frac{p}{2}Y_1^H - B(\phi Y_1)^H, \\
(ii)\quad F^*Y_1^V &= \frac{p}{2}Y_1^V - B(\phi Y_1)^V, \\
(iii)\quad F^*Y_1^C &= \frac{p}{2}Y_1^C - B[(\phi Y_1)^H + \phi^H(\nabla_\gamma Y_1) \\
&+ \sqrt{q}\eta^C(\nabla_\gamma Y_1)\xi^H].
\end{aligned} \tag{24}$$

Proof. The proof is obtained by applying Y_1^C and Y_1^V on F^* given by (21) and using $\eta(Y_1) = 0$. □

Theorem 7. *The metallic structure F^* given by (21) is integrable on TM over (ϕ, ξ, η) if and only if $N_{F^*} = 0$, which is equivalent to the conditions*

$$\eta([\phi Y_1, Y_2]) = 0, \ \ \eta([\phi Y_1, \xi]) = 0, \ \ \eta([Y_1, Y_2]) = 0,$$

$$\eta([Y_1, \xi]) = 0, \ \ \nabla Y_1 = 0, \ \ \hat{R} = 0,$$

and (ϕ, ξ, η) is integrable, i.e., $N_\phi = 0$.

Proof. Let N_{F^*} be the Nijenhuis tensor of the metallic structure F^*, then

$$N_{F^*}(\widetilde{Y}_1, \widetilde{Y}_2) = [F^*\widetilde{Y}_1, F^*\widetilde{Y}_2] - F^*[F^*\widetilde{Y}_1, \widetilde{Y}_2] - F^*[\widetilde{Y}_1, F^*\widetilde{Y}_2] + (F^*)^2[\widetilde{Y}_1, \widetilde{Y}_2], \tag{25}$$

where $\widetilde{Y}_1$ and $\widetilde{Y}_2$ are vector fields on TM.
Applying (3)–(7), (21), (23), and (16) on (25), and using $\eta(Y_1) = \eta(Y_2) = 0$.

$$\begin{aligned}
N_{F^*}({Y_1}^V, {Y_2}^V) &= -B^2\Big((\phi\nabla_{Y_2}\xi)^V + \sqrt{q}(\eta(\nabla_{Y_2}\xi))^V\xi^H\Big) + Bq(\eta[Y_1,\xi])^V\xi^H. \\
N_{F^*}({Y_1}^V, {Y_2}^H) &= B^2\big(N_\phi(Y_1,Y_2)\big)^V + Bp\Big((\phi[Y_1,Y_2])^V + \sqrt{q}\eta([Y_1,Y_2])^V\xi^H\Big) \\
&- Bp\Big((\phi\nabla_{Y_1}Y_2)^V + \sqrt{q}\eta(\nabla_{Y_1}Y_2)^V\xi^H\Big) - B^2(\nabla_{\phi Y_1}\phi Y_2) \\
&- B^2\sqrt{q}\eta([\phi Y_1, Y_2])^V\xi^H - B^2\sqrt{q}\eta([Y_1,\phi Y_2])^V\xi^H \\
&+ \Big((\phi(\nabla_{\phi Y_1}Y_2)^V + \sqrt{q}\eta(\nabla_{\phi Y_1}Y_2)^V\xi^H\Big) - \frac{p^2}{4}(\nabla_{Y_1}Y_2)^V - q(\nabla_{Y_1}Y_2)^V \\
&+ B^2\Big((\phi(\nabla_{Y_1}\phi Y_2))^V + \sqrt{q}\eta(\nabla_{Y_1}\phi Y_2)^V\xi^H\Big) - Bp((\phi[Y_1,Y_2])^V \\
&+ \sqrt{q}\eta([Y_1,Y_2])^V\xi^H) + B\Big((\phi(\nabla_{Y_1}Y_2))^V + \sqrt{q}\eta(\nabla_{Y_1}Y_2)^V\xi^H)\Big). \\
N_{F^*}(Y_1^H, {Y_2}^H) &= B^2(N_\phi(Y_1,Y_2))^H + B^2q\Big((\eta[Y_1,Y_2]\xi)^H + (\eta[Y_1,Y_2]\xi)^V\Big) \\
&- B^2(p\phi^H + q)\gamma\hat{R}(Y_1,Y_2) + \frac{p}{2}B\gamma\hat{R}(Y_1,\phi Y_2) + \frac{p}{2}B\hat{R}(\phi Y_1, Y_2) \\
&- B^2\gamma\hat{R}(\phi Y_1,\phi Y_2) - B\sqrt{q}(\eta[\phi Y_1,Y_2])^V\xi^V - BF^*\gamma\hat{R}(\phi Y_1,Y_2) \\
&- B^2\sqrt{q}(\eta[Y_1,\phi Y_2])^V\xi^V - BF^*\gamma\hat{R}(Y_1,\phi Y_2). \\
N_{F^*}(Y_1^V, \xi^V) &= B^2\sqrt{q}\Big([\phi Y_1,\xi]^V - \phi[Y_1,\xi]^V - (\nabla_{\phi Y_1}\xi)^V - (\phi\nabla_X\xi)^V\Big) \\
&+ B^2q\Big(\eta(\nabla_{Y_1}\xi)^V\xi^H - (\eta[Y_1,\xi])^V\xi^H\Big). \\
N_{F^*}({Y_1}^V, \xi^H) &= B^2\Big((\phi^2[Y_1,\xi])^V - q(\nabla_{Y_1}\xi)^V\Big). \\
N_{F^*}(Y_1^H, \xi^V) &= -B^2((p\phi+q)[\xi,Y_1])^V + B^2\big((p\phi+q)(\nabla_\xi Y_1)\big)^V \\
&+ \frac{p}{2}B\sqrt{q}\gamma\hat{R}(Y_1,\xi) + B^2\sqrt{q}\Big([\phi Y_1,\xi]^H - \gamma\hat{R}(\phi Y_1,\xi)\Big) \\
&+ B^2(\phi[\xi,\phi Y_1])^V + B^2\sqrt{q}(\eta[\xi,\phi Y_1])^V\xi^H - B^2(\phi(\nabla_\xi\phi Y_1))^V \\
&- B^2\sqrt{q}(\eta(\nabla_\xi\phi Y_1))^V\xi^H - B^2\sqrt{q}\phi[Y_1,\xi]^H \\
&- B^2q(\eta[Y_1,\xi])^V\xi^V - B\sqrt{q}F^*\gamma\hat{R}(Y_1,\xi). \\
N_{F^*}(Y_1^H, \xi^H) &= B^2\phi^2[Y_1,\xi]^H + B^2\eta[Y_1,\xi]^H + B^2(\eta[Y_1,\xi]\xi)^V - \frac{p^2+4q}{4}(\gamma\hat{R}(Y_1,\xi) \\
&+ pB\sqrt{q}[\xi,Y_1]^V + \frac{p}{2}B\gamma\hat{R}(\phi Y_1,\xi) - B^2\phi[\phi Y_1,\xi] - B^2\sqrt{q}(\eta[\phi Y_1,\xi])^V\xi^V \\
&- B^2\sqrt{q}(\phi[\xi,Y_1])^V - B^2q(\eta[\xi,Y_1])^V\xi^H + B^2\sqrt{q}(\phi\nabla_\xi Y_1)^V \\
&+ B^2q(\eta(\nabla_\xi Y_1))^V\xi^H - Bp\sqrt{q}(\eta[Y_1,\xi])^V\xi^V. \\
N_{F^*}(\xi^V, \xi^H) &= -(\frac{p^2}{2} + q)(\nabla_\xi\xi)^V + B^2q(\nabla_\xi\xi)^V + \frac{p}{2}B\sqrt{q}(\nabla_\xi\xi)^H.
\end{aligned}$$

□

Theorem 8. *Let F^* be a MS in TM given by (21) and Υ_1 and Υ_2 be vector fields on M such that $\eta(\Upsilon_1) = \eta(\Upsilon_2) = 0$, then*

$$\begin{aligned}
(i)\quad (\pounds_{\Upsilon_2{}^H} F^*)\Upsilon_1^H &= B\Big([\phi\Upsilon_1, \Upsilon_2]^H - \gamma\hat{R}(\phi\Upsilon_1, \Upsilon_2)\Big) + \frac{p}{2}\gamma\hat{R}(\Upsilon_1, \Upsilon_2)\\
&- B\Big((\phi[\Upsilon_1, \Upsilon_2])^H - \sqrt{q}\eta([\Upsilon_1, \Upsilon_2])^V\xi^V\Big) - F^*\gamma\hat{R}(\Upsilon_1, \Upsilon_2),\\
(ii)\quad (\pounds_{\Upsilon_2{}^V} F^*)\Upsilon_1^H &= B\Big((\phi[\Upsilon_2, \Upsilon_1])^V + \sqrt{q}(\eta([\Upsilon_2, \Upsilon_1]))^V\xi^H - [\Upsilon_2, \phi\Upsilon_1]^V\Big)\\
&- B\Big((\phi(\nabla_{\Upsilon_2}\Upsilon_1))^V + \sqrt{q}(\eta(\nabla_{\Upsilon_2}\Upsilon_1)^V\xi^H - \nabla_{\Upsilon_2}\phi\Upsilon_1)^V\Big),\\
(iii)\quad (\pounds_{\Upsilon_2{}^V} F^*)\Upsilon_1^V &= 0,\\
(iv)\quad (\pounds_{\Upsilon_2{}^H} F^*)\Upsilon_1{}^V &= B\Big((\phi\nabla_{\Upsilon_1}\Upsilon_2)^V + \sqrt{q}\eta(\nabla_{\Upsilon_1}\Upsilon_2)^V\xi^H) - [\phi\Upsilon_1, \Upsilon_2]^V\Big)\\
&- B\Big((\phi[\Upsilon_1, \Upsilon_2])^V = \sqrt{q}\eta([\Upsilon_1, \Upsilon_2])^V\xi^H - \nabla_{\phi\Upsilon_1}\Upsilon_2\Big).
\end{aligned}$$

Proof. Applying (21), (23), (16), and (20), and using $\eta(\Upsilon_1) = \eta(\Upsilon_2) = 0$.

$$\begin{aligned}
(i)\quad \pounds_{\Upsilon_2{}^V}(F^*\Upsilon_1^H) &= \frac{p}{2}(\pounds_{\Upsilon_2{}^V}\Upsilon_1^H - B\pounds_{\Upsilon_2{}^V}(\phi\Upsilon_1)^H - B\sqrt{q}(\eta\Upsilon_1)^V\pounds_{\Upsilon_2{}^V}\xi^V)\\
(\pounds^V_{\Upsilon_2}F^*)\Upsilon_1^H + F^*(\pounds_{\Upsilon_2{}^V}X^H) &= \frac{p}{2}[\Upsilon_2^H, \Upsilon_1^H] - B[\Upsilon_2^H, (\phi\Upsilon_1)^H]\\
&- B\sqrt{q}(\eta\Upsilon_1)^V[\Upsilon_2{}^H, \xi^H]\\
(\pounds_{\Upsilon_2{}^H}F^*)\Upsilon_1^H &= B\Big([\phi\Upsilon_1, \Upsilon_2]^H - \gamma\hat{R}(\phi\Upsilon_1, \Upsilon_2)\Big) + \frac{p}{2}\gamma\hat{R}(\Upsilon_1, \Upsilon_2)\\
&- B\Big((\phi[\Upsilon_1, \Upsilon_2])^H - \sqrt{q}\eta([\Upsilon_1, \Upsilon_2])^V\xi^V\Big) - F^*\gamma\hat{R}(\Upsilon_1, \Upsilon_2),
\end{aligned}$$

Others results are obtained by using similar devices. □

Example 1. *Setting $p = q = 1$ in (8), then $F^2 - F - I = 0$ is obtained and named as the Golden Structure. Also, from (21), we have*

$$F^* = \frac{1}{2} - B\Big[\phi^H + \eta^V \otimes \xi^V + \eta^H \otimes \xi^H\Big]. \tag{26}$$

Using (22), we infer

$$\begin{aligned}
(i)\quad F^*(\xi^H) &= \frac{1}{2}\xi^H - B\xi^V,\\
(ii)\quad F^*(\xi^V) &= \frac{1}{2}\xi^V - B\xi^H,\\
(iii)\quad F^*(\xi^C) &= \frac{1}{2}\xi^C - B[\phi^H(\nabla_\gamma\Upsilon_1) + \phi^C(\nabla_\gamma\Upsilon_1)\xi^H],\\
(iv)\ F^*(\phi^H\widetilde{\Upsilon}_1) &= \frac{1}{2}\phi^H(\widetilde{\Upsilon}_1) - B[\phi^H\widetilde{\Upsilon}_1 + \widetilde{\Upsilon}_1 - (\eta^V(\widetilde{\Upsilon}_1)\xi^H + \eta^H(\widetilde{\Upsilon}_1)\xi^V)].
\end{aligned} \tag{27}$$

Apply $F^*(\widetilde{\Upsilon}_1)$ in (26), we infer

$$\begin{aligned}
(F^*)^2(\widetilde{\Upsilon}_1) &= \frac{1}{2}F^*(\widetilde{\Upsilon}_1) - B\Big[F^*\varphi^H\widetilde{\Upsilon}_1 + \eta^V(\widetilde{\Upsilon}_1)F^*(\xi^V) + \eta^H(\widetilde{\Upsilon}_1)F^*(\xi^H)\Big],\\
&= F^*(\widetilde{\Upsilon}_1) + (\widetilde{\Upsilon}_1).
\end{aligned}$$

This shows that F^* is a golden structure.

5. Examples of Almost Quadratic ϕ-Manifolds

In this section, we prove the existence of almost quadratic ϕ-manifolds on the tangent bundle with non-trivial examples.

Example 2. *Let $M = \{(x, y, z) : x, y, z \in \Re, z \neq 0\}$ be a differentiable manifold of dimension 3, $\Re$ is a set of real numbers. We suppose that e_i^C and $e_i^V; i = 1, 2, 3$ be complete and vertical lifts on TM of independent vector fields $e_i; i = 1, 2, 3$ on M, then they form a basis $\{e_i^C, e_i^V; i = 1, 2, 3\}$ for TM of M. Let g^C be the complete lift of a Riemannian metric g such that $g_{ij} = \delta_{ij}$, where δ_{ij} is Kronecker delta. That is,*

$$\begin{aligned} g^C(Y_1^V, e_3^C) &= (g(Y_1, e_3))^V = (\eta(e_3))^V, \\ g^C(Y_1^C, e_3^C) &= (g(Y_1, e_3))^C = (\eta(e_3))^C, \\ g^C(e_3^C, e_3^C) &= 1, \ g^V(Y_1^V, e_3^C) = 0, \ g^V(e_3^V, e_3^V) = 0, \end{aligned}$$

where Y_1 is a vector field on M. If ϕ represents the (1,1) symmetric tensor on M such that

$$\phi^V(e_i^V) = (1 + \sqrt{2})e^z e_i^V, i = 1, 2,$$

$$\phi^C(e_i^C) = (1 + \sqrt{2})e^z e_i^C, i = 1, 2,$$

$$\phi^V(e_3^V) = \phi^C(e_3^C) = 0,$$

Then we can easily verify that

$$(\phi^C)^2 = p\phi^C + qI - q(\eta^V \otimes \xi^C + \eta^C \otimes \xi^V),$$

where $p = 2e^z$, $q = e^{2z} \Rightarrow p^2 + 4q = 8e^{2z} \neq 0$. This shows that M is an almost quadratic ϕ-manifold and the structure (ϕ, ξ, η) is an almost quadratic ϕ-structure on M.

Again, from the straightforward calculations, we prove that

$$g^C((\phi e_i)^C, e_j^C) = g^C(e_i^C, (\phi e_j)^C),$$

$$g^C((\phi e_i)^V, e_j^C) = g^C(e_i^V, (\phi e_j)^C),$$

and

$$\begin{aligned} g^C((\phi e_i)^C, (\phi e_j)^C) &= p g^C((\phi e_i)^C, e_j^C) + q(g^C(e_i^C, e_j^C) \\ &- (\eta(e_i))^C(\eta(e_j))^V - (\eta(e_i))^V(\eta(e_j))^C), \ \forall \, i, j = 1, 2, 3. \end{aligned}$$

The manifold M is an almost quadratic metric ϕ-manifold and the structure (ϕ, ξ, η, g) is an almost quadratic metric ϕ-structure on M.

Example 3. *A paracontact structure (ϕ, η, ξ) on M such that [22]*

$$\phi^2 = I - \eta \otimes \xi$$

is an almost quadratic ϕ-structure when $p = 0, q = 1$ in (9). The new tensor $\widetilde{F}$ of type (1,1) given by (12) becomes

$$\widetilde{F} = -\phi^C + \eta^V \otimes \xi^V + \eta^C \otimes \xi^C.$$

It can be easily proved that $\widetilde{F}$ is almost a product structure.

Remark 1. *For the horizontal lift, we can obtain the similar examples of almost quadratic ϕ-manifolds.*

6. Conclusions

In this work, we have characterized a metallic structure by using the complete and horizontal lifts over an almost quadratic ϕ-structure (ϕ, ξ, η). Tensor fields $\tilde{F}$ and F^* are defined on TM over the structure (ϕ, ξ, η) and we proved that they are metallic structures, which generalizes the notion of an almost complex structure $\tilde{J}$ introduced by Tanno [10]. The fundamental geometrical properties of fundamental 2-Form and its derivative on TM over the structure (ϕ, ξ, η) were calculated. The integrability conditions and expressions of the Lie derivative of metallic structures $\widetilde{F}$ and F^* on TM over the structure (ϕ, ξ, η) were determined. Finally, we demonstrated that almost quadratic ϕ-manifolds exist on TM with non-trivial examples. Future studies could fruitfully explore this issue further by considering the polynomial structure $Q(F) = F_n + a_n F_{n-1} + \cdots + a_2 F + a_1 I$, where F is the tensor field of type (1,1).

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Article

Tangent Bundles Endowed with Quarter-Symmetric Non-Metric Connection (QSNMC) in a Lorentzian Para-Sasakian Manifold

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Abstract: The purpose of the present paper is to study the complete lifts of a QSNMC from an LP-Sasakian manifold to its tangent bundle. The lifts of the curvature tensor, Ricci tensor, projective Ricci tensor, and lifts of Einstein manifold endowed with QSNMC in an LP-Sasakian manifold to its tangent bundle are investigated. Necessary and sufficient conditions for the lifts of the Ricci tensor to be symmetric and skew-symmetric and the lifts of the projective Ricci tensor to be skew-symmetric in the tangent bundle are given. An example of complete lifts of four-dimensional LP-Sasakian manifolds in the tangent bundle is shown.

Keywords: Lorentzian para-Sasakian manifolds; complete lifts; tangent bundle; quarter-symmetric non-metric connection; partial differential equations; mathematical operators; curvature tensor; projective Ricci tensor; Einstein manifold

MSC: 53C05; 53C07; 53C25; 58A30

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1. Introduction

Tangent bundle geometry has long been a source of interest in differential geometry. Tangent bundle investigation introduces several novel challenges to the study of modern differential geometry. Using the lift function, it is convenient to generalize differentiable structures on any manifold M to its tangent bundle. The theory of vertical, complete, and horizontal lifts of geometrical structures and connections from a manifold to its tangent bundle was developed by Yano and Ishihara [1]. Numerous researchers have examined various connections and geometric structures on the tangent bundle like Yano and Kobayashi [2], Tani [3], Pandey and Chaturvedi [4], and Khan [5,6]. Different lifts of metallic structures to tangent bundles have been studied in [7–9]. Tangent bundles immersed with quarter-symmetric non-metric connections, semi-symmetric P-connections, and semi-symmetric non-metric connections on almost Hermitian manifolds, Kähler manifolds, Kenmotsu manifolds, Sasakian manifolds, para-Sasakian manifolds, Riemannian manifolds and their submanifolds, and statistical manifolds and their submanifolds have been studied in [5,10–18]. Recently, Khan et al. [19] studied the tangent bundle of P-Sasakian manifolds endowed with a quarter-symmetric metric connection (QSMC).

On the other hand, the notion of quarter-symmetric connection in a Riemannian manifold with affine connection was introduced by Golab in 1975 [20]. This was further developed by many geometers like Yano and Imai [21], Rastogi [22,23], Mishra and Pandey [24], Mukhopadhyay et al. [25], Biswas and De [26], Sengupta and Biswas [27], Singh and Pandey [28], and others.

Let ∇ be a linear connection on an n-dimensional differentiable manifold M^n of class C^∞. If the torsion tensor T of ∇ defined by

$$T(X_0, Y_0) = \nabla_{X_0} Y_0 - \nabla_{Y_0} X_0 - [X_0, Y_0], \tag{1}$$

satisfies

$$T(X_0, Y_0) = \lambda_0(Y_0)\phi_0 X_0 - \lambda_0(X_0)\phi_0 Y_0, \tag{2}$$

where λ_0 is a 1-form and ϕ_0 is a $(1,1)$ tensor field, then the connection ∇ is called a quarter-symmetric connection [21,29,30]. Also, if ∇ satisfies

$$(\nabla_{X_0} g)(Y_0, Z_0) \neq 0, \tag{3}$$

for all $X_0, Y_0, Z_0 \in \mathfrak{X}(M^n)$, the set of all vector fields on M^n, then ∇ is called a quarter-symmetric non-metric connection (QSNMC).

We start this paper with Section 1. Section 2 is devoted to preliminaries. In Section 3, a QSNMC in an LP-Sasakian manifold is studied. The complete lifts of LP-Sasakian manifolds and QSNMC in an LP-Sasakian manifold to its tangent bundle are investigated in Sections 4 and 5. In Sections 6 and 7, the complete lifts of the curvature tensor and symmetric and skew-symmetric condition of the Ricci tensor in an LP-Sasakian manifold endowed with QSNMC to its tangent bundle are investigated. The skew-symmetric properties of the projective Ricci tensor and Einstein manifold endowed with QSNMC in an LP-Sasakian manifold to its tangent bundle are studied in Sections 8 and 9. Lastly, an example of the lift of four-dimensional LP-Sasakian manifolds to its tangent bundle is shown in Section 9, followed by a conclusion section.

2. Preliminaries

Let M^n be a differentiable manifold and $T_0M^n = \bigcup_{p\in M^n} T_{0p}M^n$ be the tangent bundle, where $T_{0p}M^n$ is the tangent space at a point $p \in M^n$ and $\pi : T_0M^n \to M^n$ is the natural bundle structure of T_0M^n over M^n. For any co-ordinate system (Q, x^h) in M^n, where (x^h) is a local co-ordinate system in the neighborhood Q, then $(\pi^{-1}(Q), x^h, y^h)$ is a co-ordinate system in T_0M^n, where (x^h, y^h) is an induced co-ordinate system in $\pi^{-1}(Q)$ from (x^h) [1].

2.1. Vertical and Complete Lifts

Let us define a vector field X_0, a tensor field F_0 of type $(1,1)$, a function f_0, a 1-form ω_0, and an affine connection ∇ in M^n; their vertical and complete lifts are denoted by f_0^v, X_0^v, ω_0^v, F_0^v, ∇^v, and f_0^c, X_0^c, ω_0^c, F_0^c, ∇^c, respectively. The following formulas of complete and vertical lifts are defined by [1,5]

$$(f_0X_0)^v = f_0^v X_0^v, \ \ (f_0X_0)^c = f_0^c X_0^v + f_0^v X_0^c, \tag{4}$$

$$X_0^v f_0^v = 0, \ \ X_0^v f_0^c = X_0^c f_0^v = (X_0 f_0)^v, \ \ X_0^c f_0^c = (X_0 f_0)^c, \tag{5}$$

$$\omega_0(f_0^v) = 0, \ \ \omega_0^v(X_0^c) = \omega_0^c(X_0^v) = \omega_0(X_0)^v, \ \ \omega_0^c(X_0^c) = \omega_0(X_0)^c, \tag{6}$$

$$F_0^v X_0^c = (F_0X_0)^v, \ \ F_0^c X_0^c = (F_0 X_0)^c, \tag{7}$$

$$[X_0, Y_0]^v = [X_0^c, Y_0^v] = [X_0^v, Y_0^c], \ \ [X_0, Y_0]^c = [X_0^c, Y_0^c], \tag{8}$$

$$\nabla^c{}_{X_0^c} Y_0^c = (\nabla_{X_0} Y_0)^c, \ \ \nabla^c{}_{X_0^c} Y_0^v = (\nabla_{X_0} Y_0)^v. \tag{9}$$

Suppose T_0M is the tangent bundle and let $X_0 = X_0^i \frac{\partial}{\partial x^i}$ be a local vector field on M, then its vertical and complete lifts in the term of partial differential equations are

$$X_0^v = X_0^i \frac{\partial}{\partial y^i} \qquad \text{and} \qquad X_0^c = X_0^i \frac{\partial}{\partial x^i} + \frac{\partial X_0^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}.$$

2.2. LP-Sasakian Manifolds

An n-dimensional differentiable manifold M^n is called a Lorentzian para-Sasakian (briefly LP-Sasakian) [31] of dimension n if it admits a $(1,1)$- tensor field ϕ_0, a contravariant vector field ξ_0, a 1-form η_0, and a Lorentzian metric g which satisfy

$$\phi_0{}^2(X_0) = X_0 + \eta_0(X_0)\xi_0, \tag{10}$$
$$\eta_0(\xi_0) = -1, \tag{11}$$
$$g(\phi_0 X_0, \phi_0 Y_0) = g(X_0, Y_0) + \eta_0(X_0)\eta_0(Y_0), \tag{12}$$
$$g(X_0, \xi_0) = \eta_0(X_0), \tag{13}$$
$$(\nabla_{X_0}\phi_0)(Y_0) = g(X_0, Y_0)\xi_0 + \eta_0(Y_0)X_0 + 2\eta_0(X_0)\eta_0(Y_0)\xi_0, \tag{14}$$
$$\nabla_{X_0}\xi_0 = \phi_0 X_0. \tag{15}$$

In an LP-Sasakian manifold, the following relations also hold:

$$\phi_0\xi_0 = 0, \ \ \eta_0 \circ \phi_0 = 0, \tag{16}$$
$$rank\ \phi_0 = n - 1. \tag{17}$$

If we take a tensor field $\Phi_0(X_0, Y_0)$ as

$$\Phi_0(X_0, Y_0) = g(X_0, \phi_0 Y_0), \tag{18}$$

for any vector fields X_0 and Y_0, then the tensor field $\Phi_0(X_0, Y_0)$ is a symmetric $(0,2)$ tensor field [31]. Since the 1-form η_0 is closed in an LP-Sasakian manifold, we have [31,32]

$$(\nabla_{X_0}\eta_0)(Y_0) = \Phi_0(X_0, Y_0), \quad \Phi_0(X_0, \xi_0) = 0, \tag{19}$$

for all $X_0, Y_0 \in M^n$. In an LP-Sasakian manifold, the following relations hold [32,33]:

$$g(R_0(X_0, Y_0)Z_0, \xi_0) = g(Y_0, Z_0)\eta_0(X_0) - g(X_0, Z_0)\eta_0(Y_0), \tag{20}$$
$$R_0(\xi_0, X_0)Y_0 = g(X_0, Y_0)\xi_0 - \eta_0(Y_0)X_0, \tag{21}$$
$$R_0(X_0, Y_0)\xi_0 = \eta_0(Y_0)X_0 - \eta_0(X_0)Y_0, \tag{22}$$
$$R_0(\xi_0, X_0)\xi_0 = X_0 + \eta_0(X_0)\xi_0, \tag{23}$$
$$S_0(X_0, \xi_0) = (n-1)\eta_0(X_0), \tag{24}$$
$$S_0(\phi_0 X_0, \phi_0 Y_0) = S_0(X_0, Y_0) + (n-1)\eta_0(X_0)\eta_0(Y_0), \tag{25}$$

where R_0 is the Riemannian curvature tensor and S_0 is the Ricci tensor of the manifold.

3. QSNMC

In an LP-Sasakian manifold (M^n, g), the linear connection $\ddot{\nabla}$ on M^n is given by [29]

$$\ddot{\nabla}_{X_0} Y_0 = \nabla_{X_0} Y_0 + \eta_0(Y_0)\phi_0 X_0 + a_0(X_0)\phi_0 Y_0, \tag{26}$$

where η_0 and a_0 are 1-form associated with vector field ξ_0 and A_0 on M^n is given by

$$\eta_0(X_0) = g(X_0, \xi_0), \tag{27}$$
$$a_0(X_0) = g(X_0, A_0), \tag{28}$$

for all vector fields $X_0 \in \mathfrak{X}_0(M^n)$, where $\mathfrak{X}_0(M^n)$ is the set of all differentiable vector fields on M^n and the torsion tensor is given by

$$\ddot{T}(X_0, Y_0) = \eta_0(Y_0)\phi_0 X_0 - \eta_0(X_0)\phi_0 Y_0 + a_0(X_0)\phi_0 Y_0 - a_0(Y_0)\phi_0 X_0. \tag{29}$$

A linear connection satisfying (29) is called a quarter-symmetric connection. Further, by using (26), we have

$$(\ddot{\nabla}_{X_0}g)(Y_0, Z_0) = -\eta_0(Y_0)g(\phi_0 X_0, Z_0) - \eta_0(Z_0)g(\phi_0 X_0, Y_0) - 2a_0(X_0)g(\phi_0 Y_0, Z_0). \quad (30)$$

A linear connection $\ddot{\nabla}$ defined by (26) which satisfies (29) and (30) is called QSNMC.

4. Complete Lifts from an LP-Sasakian Manifold to Its Tangent Bundle

Let the tangent bundle be denoted by T_0M^n in an LP-Sasakian manifold (M^n, g). Taking complete lifts by mathematical operators on (10)–(16) and (18)–(25), we obtain

$$(\phi_0^2(X_0))^c = X_0^c + \eta_0^c(X_0^c)\xi_0^v + \eta_0^v(X_0^c)\xi_0^c, \quad (31)$$

$$\eta_0^c(\xi_0^c) = \eta_0^v(\xi_0^v) = 0, \ \ \eta_0^c(\xi_0^v) = \eta_0^v(\xi_0^c) = -1, \quad (32)$$

$$g^c\Big((\phi_0 X_0)^c, (\phi_0 Y_0)^c\Big) = g^c(X_0^c, Y_0^c) + \eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X^c)\eta_0^c(Y_0^c), \quad (33)$$

$$g^c(X_0^c, \xi_0^c) = \eta_0^c(X_0^c), \quad (34)$$

$$\begin{aligned}(\nabla^c_{X_0^c}\phi_0^c)Y_0^c = {} & g^c(X_0^c, Y_0^c)\xi_0^v + g^c(X_0^v, Y_0^c)\xi_0^c + \eta_0^c(Y_0^c)X_0^v + \eta_0^v(Y_0^c)X_0^c \\ & + 2\Big\{\eta_0^c(X_0^c)\eta_0^c(Y_0^c)\xi_0^v + \eta_0^c(X_0^c)\eta_0^v(Y_0^c)\xi_0^c + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\xi_0^c\Big\},\end{aligned} \quad (35)$$

$$\nabla^c_{X_0^c}\xi_0^c = (\phi_0 X_0)^c, \quad (36)$$

$$\phi_0^c\xi_0^c = \phi_0^v\xi_0^v = \phi_0^c\xi_0^v = \phi_0^v\xi_0^c = 0, \quad (37)$$

$$\eta_0^c \circ \phi_0^c = \eta_0^v \circ \phi_0^v = \eta_0^c \circ \phi_0^v = \eta_0^v \circ \phi_0^c = 0, \quad (38)$$

$$\Phi_0^c(X_0^c, Y_0^c) = g^c(X_0^c, \phi_0^c Y_0^c), \quad (39)$$

$$(\nabla^c_{X_0^c}\eta_0^c)Y_0^c = \Phi_0^c(X_0^c, Y_0^c), \quad (40)$$

$$\Phi_0^c(X_0^c, \xi_0^c) = 0, \quad (41)$$

$$\begin{aligned}g^c(R^c(X_0^c, Y_0^c)Z_0^c, \xi_0^c) = {} & g^c(Y_0^c, Z_0^c)\eta_0^v(X_0^c) + g^c(Y_0^v, Z_0^c)\eta_0^c(X_0^c) \\ & - g^c(X_0^c, Z_0^c)\eta_0^v(Y_0^c) - g^c(X_0^v, Z_0^c)\eta_0^c(Y_0^c),\end{aligned} \quad (42)$$

$$R^c(\xi^c, X_0^c)Y_0^c = g^c(X_0^c, Y_0^c)\xi_0^v + g^c(X_0^v, Y_0^c)\xi_0^c - \eta_0^c(Y_0^c)X_0^v - \eta_0^v(Y_0^c)X_0^c, \quad (43)$$

$$R^c(X_0^c, Y_0^c)\xi_0^c = \eta_0^c(Y_0^c)X_0^v + \eta_0^v(Y_0^c)X_0^c - \eta_0^c(X_0^c)Y_0^v - \eta_0^v(X_0^c)Y_0^c, \quad (44)$$

$$R^c(\xi_0^c, X_0^c)\xi_0^c = X_0^c + \eta_0^c(X_0^c)\xi_0^v + \eta_0^v(X_0^c)\xi_0^c, \quad (45)$$

$$S^c(X_0^c, \xi_0^c) = (n-1)\eta_0^c(X_0^c), \quad (46)$$

$$S^c(\phi_0^c X_0^c, \phi_0^c Y_0^c) = S^c(X_0^c, Y_0^c) + (n-1)\Big\{\eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big\}. \quad (47)$$

5. Complete Lifts of QSNMC of an LP-Sasakian Manifold in the Tangent Bundle

In an LP-Sasakian manifold (M^n, g) and its tangent bundle T_0M^n, let us take complete lifts by mathematical operators on Equations (26)–(30), and we have

$$\begin{aligned}\ddot{\nabla}^c_{X_0^c}Y_0^c = {} & \nabla^c_{X_0^c}Y_0^c + \eta_0^c(Y_0^c)(\phi_0 X_0)^v + \eta_0^v(Y_0^c)(\phi_0 X_0)^c + a^c(X_0^c)(\phi_0 Y_0)^v \\ & + a^v(X_0^c)(\phi_0 Y_0)^c,\end{aligned} \quad (48)$$

$$\ddot{T}^c(X_0^c, Y_0^c) = \eta_0^c(Y_0^c)(\phi_0 X_0)^v + \eta_0^v(Y_0^c)(\phi_0 X_0)^c - \eta_0^c(X_0^c)(\phi_0 Y_0)^v - \eta_0^v(X_0^c)(\phi_0 Y_0)^c + a_0^c(X_0^c)(\phi_0 Y_0)^v + a_0^v(X_0^c)(\phi_0 Y_0)^c - a_0^c(Y_0^c)(\phi_0 X_0)^v - a_0^v(Y_0^c)(\phi_0 X_0)^c, \tag{49}$$

$$\eta_0^c(X_0^c) = g^c(X_0^c, \xi_0^c), \tag{50}$$

$$a^c(X_0^c) = g^c(X_0^c, A_0^c), \tag{51}$$

$$(\ddot{\nabla}^c_{X_0^c} g^c)(Y_0^c, Z_0^c) = -\eta_0^c(Y_0^c) g^c\Big((\phi_0 X_0)^v, Z_0^c\Big) - \eta_0^v(Y_0^c) g^c\Big((\phi_0 X_0)^c, Z_0^c\Big) - \eta_0^c(Z_0^c) g^c\Big((\phi_0 X_0)^v, Y_0^c\Big) - \eta_0^v(Z_0^c) g^c\Big((\phi_0 X_0)^c, Y_0^c\Big) - 2a^c(X_0^c) g^c\Big((\phi_0 Y_0)^v, Z_0^c\Big) - 2a^v(X_0^c) g^c\Big((\phi_0 Y_0)^c, Z_0^c\Big). \tag{52}$$

The connection given by Equation (48) is said to be a QSNMC on an LP-Sasakian manifold in its tangent bundle if the torsion tensor $\ddot{T}^c$ of $T_0 M^n$ endowed with $\ddot{\nabla}^c$ satisfies Equation (49) and the complete lifts of Lorentzian metric g^c fulfill the relation (52).

Theorem 1. *If an LP-Sasakian manifold (M^n, g) with an almost Lorentzian para-contact metric structure $(\phi_0, \xi_0, \eta_0, g)$ admitting a QSNMC $\ddot{\nabla}$ which satisfies (49) and (52), then the QSNMC in the tangent bundle is given by*

$$\ddot{\nabla}^c_{X_0^c} Y_0^c = \nabla^c_{X_0^c} Y_0 + \eta_0^c(Y_0^c)(\phi_0 X_0)^v + \eta_0^v(Y_0^c)(\phi_0 X_0)^c + a^c(X_0^c)(\phi_0 Y_0)^v + a^v(X_0^c)(\phi_0 Y_0)^c.$$

Proof. Let $\ddot{\nabla}^c$ be the complete lifts of a linear connection in M^n given by

$$\ddot{\nabla}^c_{X_0^c} Y_0^c = \nabla^c_{X_0^c} Y_0^c + H_0^c(X_0^c, Y_0^c). \tag{53}$$

Now, we shall determine the complete lifts of the tensor field H_0^c such that $\ddot{\nabla}^c$ satisfies (49) and (52). From (53), we have

$$\ddot{T}^c(X_0^c, Y_0^c) = H_0^c(X_0^c, Y_0^c) - H_0^c(Y_0^c, X_0^c). \tag{54}$$

We denote

$$G_0^c(X_0^c, Y_0^c, Z_0^c) = (\ddot{\nabla}^c_{X_0^c} g^c)(Y_0^c, Z_0^c). \tag{55}$$

From (53) and (55), we have

$$g^c\Big(H_0^c(X_0^c, Y_0^c), Z_0^c\Big) + g^c\Big(H_0^c(X_0^c, Z_0^c), Y_0^c\Big) = -G_0^c(X_0^c, Y_0^c, Z_0^c). \tag{56}$$

Using (52), (53), (55), and (56) we have

$$
\begin{aligned}
&g^c\left(\ddot{T}^c(X_0^c, Y_0^c), Z_0^c\right) + g^c\left(\ddot{T}^c(Z_0^c, X_0^c), Y_0^c\right) + g^c\left(\ddot{T}^c(Z_0^c, Y_0^c), X_0^c\right) \\
&= g^c\left(H_0^c(X_0^c, Y_0^c), Z_0^c\right) - g^c\left(H_0^c(Y_0^c, X_0^c), Z_0^c\right) + g^c\left(H_0^c(Z_0^c, X_0^c), Y_0^c\right) \\
&- g^c\left(H_0^c(X_0^c, Z_0^c), Y_0^c\right) + g^c\left(H_0^c(Z_0^c, Y_0^c), X_0^c\right) - g^c\left(H_0^c(Y_0^c, Z_0^c), X_0^c\right) \\
&= g^c\left(H_0^c(X_0^c, Y_0^c), Z_0^c\right) - g^c\left(H_0^c(X_0^c, Z_0^c), Y_0^c\right) - G_0^c(Z_0^c, X_0^c, Y_0^c) + G_0^c(Y_0^c, X_0^c, Z_0^c) \\
&= 2g^c\left(H_0^c(X_0^c, Y_0^c), Z_0^c\right) + G_0^c(X_0^c, Y_0^c, Z_0^c) + G_0^c(Y_0^c, X_0^c, Z_0^c) - G_0^c(Z_0^c, X_0^c, Y_0^c) \\
&= 2g^c\left(H_0^c(X_0^c, Y_0^c), Z_0^c\right) - 2\left\{\eta_0^c(Z_0^c)g^c\left((\phi_0 X_0)^v, Y_0^c\right) + \eta_0^v(Z_0^c)g^c\left((\phi_0 X_0)^c, Y_0^c\right)\right\} \\
&- 2\left\{a_0^c(X_0^c)g^c\left((\phi_0 Y_0)^v, Z_0^c\right) + a_0^v(X_0^c)g^c\left((\phi_0 Y_0)^c, Z_0^c\right)\right\} - 2\left\{a_0^c(Y_0^c)g^c\left((\phi_0 X_0)^v, Z_0^c\right)\right. \\
&\left. + a_0^v(Y_0^c)g^c\left((\phi_0 X_0)^c, Z_0^c\right)\right\} + 2\left\{a_0^c(Z_0^c)g^c\left((\phi_0 X_0)^v, Y_0^c\right) + a_0^v(Z_0^c)g^c\left((\phi_0 X_0)^c, Y_0^c\right)\right\},
\end{aligned}
$$

or,

$$
\begin{aligned}
H_0^c(X_0^c, Y_0^c) = \frac{1}{2}&\left\{\ddot{T}^c(X_0^c, Y_0^c) + {}'\ddot{T}^c(X_0^c, Y_0^c) + {}'\ddot{T}^c(Y_0^c, X_0^c)\right\} + a_0^c(X_0^c)(\phi_0 Y_0)^v \\
&+ a_0^v(X_0^c)(\phi_0 Y_0)^c + a_0^c(Y_0^c)(\phi_0 X_0)^v + a_0^v(Y_0^c)(\phi_0 X_0)^c \\
&+ g^c\left(\phi_0 X_0)^c, Y_0^c\right)\xi_0^v + g^c\left(\phi_0 X_0)^v, Y_0^c\right)\xi_0^c \\
&- g^c\left(\phi_0 X_0)^c, Y_0^c\right)A_0^v - g^c\left(\phi_0 X_0)^v, Y_0^c\right)A_0^c,
\end{aligned}
$$

where ${}'\ddot{T}^c$ is a tensor field of type $(1,2)$ defined by

$$
g^c\left({}'\ddot{T}^c(X_0^c, Y_0^c), Z_0^c\right) = g^c\left(\ddot{T}^c(Z_0^c, X_0^c), Y_0^c\right),
$$

or,

$$
H_0^c(X_0^c, Y_0^c) = \eta_0^c(Y_0^c)(\phi_0 X_0)^v + \eta_0^v(Y_0^c)(\phi_0 X_0)^c + a^c(X_0^c)(\phi_0 Y_0)^v + a^v(X_0^c)(\phi_0 Y_0)^c,
$$

which gives

$$
\ddot{\nabla}^c_{X_0^c} Y_0^c = \nabla^c_{X_0^c} Y_0 + \eta_0^c(Y_0^c)(\phi_0 X_0)^v + \eta_0^v(Y_0^c)(\phi_0 X_0)^c + a^c(X_0^c)(\phi_0 Y_0)^v + a^v(X_0^c)(\phi_0 Y_0)^c.
$$

□

6. Curvature Tensor of LP-Sasakian Manifolds Endowed with QSNMC to Tangent Bundle

Let $\ddot{R}_0^c$ and R_0^c be the curvature tensors of the connections $\ddot{\nabla}^c$ and ∇^c to tangent bundle T_0M^n, respectively.

$$
\ddot{R}_0^c(X_0^c, Y_0^c)Z_0^c = \ddot{\nabla}^c_{X_0^c}\ddot{\nabla}^c_{Y_0^c}Z_0^c - \ddot{\nabla}^c_{Y_0^c}\ddot{\nabla}^c_{X_0^c}Z_0^c - \ddot{\nabla}^c_{[X_0^c, Y_0^c]}Z_0^c. \tag{57}
$$

Using (48) in (57), we have

$$\begin{aligned}\ddot{R}_0^c(X_0^c, Y_0^c)Z_0^c = {} & R_0^c(X_0^c, Y_0^c)Z_0^c + g^c\Big((\phi_0 X_0)^c, Z_0^c\Big)(\phi_0 Y_0)^v \\ & + g^c\Big((\phi_0 X_0)^v, Z_0^c\Big)(\phi_0 Y_0)^c - g^c\Big((\phi_0 Y_0)^c, Z_0^c\Big)(\phi_0 X_0)^v \\ & - g^c\Big((\phi_0 Y_0)^v, Z_0^c\Big)(\phi_0 X_0)^c + \eta_0^c(Y_0^c)\eta_0^c(Z_0^c)X_0^v \\ & + \eta_0^c(Y_0^c)\eta_0^v(Z_0^c)X_0^c + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)X_0^c \\ & - \eta_0^c(X_0^c)\eta_0^c(Z_0^c)Y_0^v - \eta_0^c(X_0^c)\eta_0^v(Z_0^c)Y_0^c \\ & - \eta_0^v(X_0^c)\eta_0^c(Z_0^c)Y_0^c + a_0^c(Y_0^c)g^c(X_0^c, Z_0^c)\xi_0^v \\ & + a_0^c(Y_0^c)g^c(X_0^v, Z_0^c)\xi_0^c + a_0^v(Y_0^c)g^c(X_0^c, Z_0^c)\xi_0^c \\ & - a_0^c(X_0^c)g^c(Y_0^c, Z_0^c)\xi_0^v - a_0^c(X_0^c)g^c(Y_0^v, Z_0^c)\xi_0^c \\ & - a_0^v(X_0^c)g^c(Y_0^c, Z_0^c)\xi_0^c + a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\xi_0^v \\ & + a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^v(Z_0^c)\xi_0^c + a_0^c(Y_0^c)\eta_0^v(X_0^c)\eta_0^c(Z_0^c)\xi_0^c \\ & + a_0^v(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\xi_0^c - a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\xi_0^v \\ & - a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^v(Z_0^c)\xi_0^c - a_0^c(X_0^c)\eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\xi_0^c \\ & - a_0^v(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\xi_0^c + da_0^c(X_0^c, Y_0^c)(\phi_0 Z_0)^v \\ & + da_0^v(X_0^c, Y_0^c)(\phi_0 Z_0)^c,\end{aligned} \tag{58}$$

where

$$R_0^c(X_0^c, Y_0^c)Z_0^c = \nabla^c_{X_0^c}\nabla^c_{Y_0^c}Z_0 - \nabla^c_{Y_0^c}\nabla^c_{X_0^c}Z_0 - \nabla^c_{[X_0^c, Y_0^c]}Z_0^c, \tag{59}$$

is the curvature tensor of ∇^c with respect to the Riemannian connection. Contracting (58), we obtain

$$\begin{aligned}\ddot{S}_0^c(Y_0^c, Z_0^c) = {} & S_0^c(Y_0^c, Z_0^c) - \gamma^c g^c\Big((\phi_0 Y_0)^c, Z_0^c\Big) + \Big[1 - a_0^c(\xi_0^c)\Big]g^c(Y_0^c, Z_0^c) \\ & + \Big[\eta_0^c - a_0^c(\xi_0^c)\Big]\Big[\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\Big] \\ & + da_0^c\Big((\phi_0 Z_0)^c, Y_0^c\Big),\end{aligned} \tag{60}$$

and

$$\ddot{r_0}^c = r_0^c - (n-1)a_0^c(\xi_0^c) + \lambda_0^c - \gamma^{c^2}, \tag{61}$$

where $\ddot{S}_0^c$ and $\ddot{r_0}^c$ are the Ricci tensor and scalar curvature with respect to $\ddot{\nabla}^c$.

$$\lambda_0^c = trace\ da_0^c\Big((\phi_0 Z_0)^c, Y_0^c\Big) \text{ and } \gamma^c = trace\ \phi_0^c. \tag{62}$$

Theorem 2. *In an LP-Sasakian manifold* (M^n, g) *with tangent bundle* T_0M^n *admitting QSNMC, we have the following:*

1. *The complete lifts of curvature tensor* $\ddot{R}_0^c$ *are given by Equation (58).*
2. *The complete lifts of Ricci tensor* $\ddot{S}_0^c$ *are given by Equation (60).*
3. *The complete lifts of scalar curvature* $\ddot{r}_0$ *are given by Equation (61).*

Let us consider that $\ddot{R}_0^c(X_0^c, Y_0^c) = 0$ in (58), and by contracting it we also obtain

$$\begin{aligned}S_0^c(Y_0^c, Z_0^c) = {} & \gamma^c g^c\Big((\phi_0 Y_0)^c, Z_0^c\Big) - \Big[1 - a_0^c(\xi_0^c)\Big]g^c(Y_0^c, Z_0^c) \\ & - \Big[\eta_0^c - a_0^c(\xi_0^c)\Big]\Big[\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\Big] \\ & - da_0^c\Big((\phi_0 Z_0)^c, Y_0^c\Big),\end{aligned} \tag{63}$$

which gives

$$r_0^c = (n-1)a_0^c(\xi_0^c) - \lambda_0^c + \gamma^{c^2}. \tag{64}$$

Theorem 3. *In an LP-Sasakian manifold, (M^n, g), with tangent bundle T_0M^n endowed with QSNMC whose curvature tensor vanishes, then the complete lift of r_0^c is given by (64).*

From (58), it follows that

$$'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) + {}'\ddot{R}_0^c(Y_0^c, X_0^c, Z_0^c, W_0^c) = 0, \tag{65}$$

$$\begin{aligned}
&'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) + {}'\ddot{R}_0^c(X_0^c, Y_0^c, W_0^c, Z_0^c)\\
&= \eta_0^c(Y_0^c)\eta_0^v(Z_0^c)g^c(X_0^c, W_0^c) + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)g^c(X_0^c, W_0^c)\\
&- \eta_0^c(X_0^c)\eta_0^v(Z_0^c)g^c(Y_0^c, W_0^c) - \eta_0^v(X_0^c)\eta_0^c(Z_0^c)g^c(Y_0^c, W_0^c)\\
&+ \eta_0^c(Y_0^c)\eta_0^v(W_0^c)g^c(X_0^c, Z_0^c) + \eta_0^v(Y_0^c)\eta_0^c(W_0^c)g^c(X_0^c, Z_0^c)\\
&- \eta_0^c(X_0^c)\eta_0^v(W_0^c)g^c(Y_0^c, Z_0^c) - \eta_0^v(X_0^c)\eta_0^c(W_0^c)g^c(Y_0^c, Z_0^c)\\
&+ a_0^c(Y_0^c)\eta_0^v(W_0^c)g^c(X_0^c, Z_0^c) + a_0^v(Y_0^c)\eta_0^c(W_0^c)g^c(X_0^c, Z_0^c)\\
&- a_0^c(X_0^c)\eta_0^v(W_0^c)g^c(Y_0^c, Z_0^c) - a_0^v(X_0^c)\eta_0^c(W_0^c)g^c(Y_0^c, Z_0^c)\\
&+ a_0^c(Y_0^c)\eta_0^v(Z_0^c)g^c(X_0^c, W_0^c) + a_0^v(Y_0^c)\eta_0^c(Z_0^c)g^c(X_0^c, W_0^c)\\
&- a_0^c(X_0^c)\eta_0^v(Z_0^c)g^c(Y_0^c, W_0^c) - a_0^v(X_0^c)\eta_0^c(Z_0^c)g^c(Y_0^c, W_0^c)\\
&+ 2\Big[a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\eta_0^v(W_0^c) + a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^v(Z_0^c)\eta_0^c(W_0^c)\\
&+ a_0^c(Y_0^c)\eta_0^v(X_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c) + a_0^v(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c)\Big]\\
&- 2\Big[a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\eta_0^v(W_0^c) + a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^v(Z_0^c)\eta_0^c(W_0^c)\\
&+ a_0^c(X_0^c)\eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c) + a_0^v(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c)\Big]\\
&+ 2da_0^c(X_0^c, Y_0^c)g^c\Big((\phi_0 Z_0)^c, W_0^c\Big).
\end{aligned} \tag{66}$$

$$\begin{aligned}
&'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) - {}'\ddot{R}_0^c(Z_0^c, W_0^c, X_0^c, Y_0^c)\\
&= \eta_0^c(Y_0^c)\eta_0^v(Z_0^c)g^c(X_0^c, W_0^c) + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)g^c(X_0^c, W_0^c)\\
&- \eta_0^c(X_0^c)\eta_0^v(W_0^c)g^c(Y_0^c, Z_0^c) - \eta_0^v(X_0^c)\eta_0^c(W_0^c)g^c(Y_0^c, Z_0^c)\\
&+ a_0^c(Y_0^c)\eta_0^v(W_0^c)g^c(X_0^c, Z_0^c) + a_0^v(Y_0^c)\eta_0^c(W_0^c)g^c(X_0^c, Z_0^c)\\
&- a_0^c(X_0^c)\eta_0^v(W_0^c)g^c(Y_0^c, Z_0^c) - a_0^v(X_0^c)\eta_0^c(W_0^c)g^c(Y_0^c, Z_0^c)\\
&+ a_0^c(Z_0^c)\eta_0^v(Y_0^c)g^c(X_0^c, W_0^c) + a_0^v(Z_0^c)\eta_0^c(Y_0^c)g^c(X_0^c, W_0^c)\\
&- a_0^c(W_0^c)\eta_0^v(Y_0^c)g^c(X_0^c, Z_0^c) - a_0^v(W_0^c)\eta_0^c(Y_0^c)g^c(X_0^c, Z_0^c)\\
&+ a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\eta_0^v(W_0^c) + a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^v(Z_0^c)\eta_0^c(W_0^c)\\
&+ a_0^c(Y_0^c)\eta_0^v(X_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c) + a_0^v(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c)\\
&- a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\eta_0^v(W_0^c) - a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^v(Z_0^c)\eta_0^c(W_0^c)\\
&- a_0^c(X_0^c)\eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c) - a_0^v(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c)\\
&+ a_0^c(Z_0^c)\eta_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^v(W_0^c) + a_0^c(Z_0^c)\eta_0^c(X_0^c)\eta_0^v(Y_0^c)\eta_0^c(W_0^c)\\
&+ a_0^c(Z_0^c)\eta_0^v(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(W_0^c) + a_0^v(Z_0^c)\eta_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(W_0^c)\\
&- a_0^c(W_0^c)\eta_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) - a_0^c(W_0^c)\eta_0^c(X_0^c)\eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\\
&- a_0^c(W_0^c)\eta_0^v(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c) - a_0^v(W_0^c)\eta_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\\
&+ da_0^c(X_0^c, Y_0^c)g^c\Big((\phi_0 Z_0)^c, W_0^c\Big) - da_0^c(Z_0^c, W_0^c)g^c\Big((\phi_0 X_0)^c, Y_0^c\Big),
\end{aligned} \tag{67}$$

and

$$\begin{aligned} &{}'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) + {}'\ddot{R}_0^c(Y_0^c, Z_0^c, X_0^c, W_0^c) + {}'\ddot{R}_0^c(Z_0^c, X_0^c, Y_0^c, W_0^c) \\ &\quad = da_0^c(X_0^c, Y_0^c) g^c\Big((\phi_0 Z_0)^c, W_0^c\Big) + da_0^c(Y_0^c, Z_0^c) g^c\Big((\phi_0 X_0)^c, W_0^c\Big) \\ &\quad + da_0^c(Z_0^c, X_0^c) g^c\Big((\phi_0 Y_0)^c, W_0^c\Big). \end{aligned} \tag{68}$$

If the 1-form a_0^c is closed, then from (68) we have

$$\begin{aligned} {}'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) + {}'\ddot{R}_0^c(Y_0^c, Z_0^c, X_0^c, W_0^c) \\ + {}'\ddot{R}_0^c(Z_0^c, X_0^c, Y_0^c, W_0^c) = 0, \end{aligned} \tag{69}$$

where

$$\begin{aligned} {}'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) &= g^c\Big(\ddot{R}_0^c(X_0^c, Y_0^c) Z_0^c, W_0^c\Big) \\ \text{and } {}'R_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) &= g^c\Big(R_0^c(X_0^c, Y_0^c) Z_0^c, W_0^c\Big). \end{aligned}$$

Theorem 4. *In an LP-Sasakian manifold, (M^n, g) with tangent bundle T_0M^n endowed with a QSNMC, the curvature tensor satisfies relations (65)–(68). In particular, if the complete lift of 1-form a_0^c is closed, then*

$${}'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) + {}'\ddot{R}_0^c(Y_0^c, Z_0^c, X_0^c, W_0^c) + {}'\ddot{R}_0^c(Z_0^c, X_0^c, Y_0^c, W_0^c) = 0.$$

7. Symmetric and Skew-Symmetric Condition of the Ricci Tensor of $\ddot{\nabla}^c$ in an LP-Sasakian Manifold Endowed with a QSNMC to Tangent Bundle

From Equation (60), we have

$$\begin{aligned} \ddot{S}_0^c(Z_0^c, Y_0^c) &= S_0^c(Z_0^c, Y_0^c) - \gamma^c g^c\Big((\phi_0 Z_0)^c, Y_0^c\Big) \\ &+ \Big[1 - a_0^c(\xi_0^c)\Big] g^c(Y_0^c, Z_0^c) + \Big[\eta_0^c - a_0^c(\xi_0^c)\Big]\Big[\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) \\ &+ \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\Big] + da_0^c\Big((\phi_0 Y_0)^c, Z_0^c\Big). \end{aligned} \tag{70}$$

From (60) and (70), we have

$$\ddot{S}_0^c(Y_0^c, Z_0^c) - \ddot{S}_0^c(Z_0^c, Y_0^c) = da_0^c\Big((\phi_0 Z_0)^c, Y_0^c\Big) - da_0^c\Big((\phi_0 Y_0)^c, Z_0^c\Big). \tag{71}$$

If $\ddot{S}_0^c(Y_0^c, Z_0^c)$ is symmetric, then the left-hand side of (71) vanishes, and then

$$da_0^c\Big((\phi_0 Z_0)^c, Y_0^c\Big) = da_0^c\Big((\phi_0 Y_0)^c, Z_0^c\Big). \tag{72}$$

Moreover, if Equation (72) holds, then from (71), $\ddot{S}_0^c(Y_0^c, Z_0^c)$ is symmetric.

Theorem 5. *In an LP-Sasakian manifold (M^n, g) with tangent bundle T_0M^n endowed with QSNMC $\ddot{\nabla}^c$, the Ricci tensor $\ddot{S}_0^c(Y_0^c, Z_0^c)$ with respect to QSNMC is symmetric if and only if relation (72) holds.*

From (60) and (70), we have

$$\begin{aligned}\ddot{S}_0^c(Y_0^c, Z_0^c) + \ddot{S}_0^c(Z_0^c, Y_0^c) = {} & 2S_0^c(Y_0^c, Z_0^c) - 2\gamma^c g^c\Big((\phi_0 Y_0)^c, Z_0^c\Big) \\ & + 2\Big[1 - a_0^c(\xi_0^c)\Big] g^c(Y_0^c, Z_0^c) \\ & + 2\Big[n - a_0^c(\xi_0^c)\Big]\Big[\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) \\ & + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\Big] + da_0^c\Big((\phi_0 Y_0)^c, Z_0^c\Big) \\ & + da_0^c\Big((\phi_0 Z_0)^c, Y_0^c\Big).\end{aligned} \tag{73}$$

By taking the skew-symmetry of $\ddot{S}_0^c(Y_0^c, Z_0^c)$, the left-hand side of (73) will vanish and we have

$$\begin{aligned}S_0^c(Y_0^c, Z_0^c) = {} & \gamma^c g^c\Big((\phi_0 Y_0)^c, Z_0^c\Big) \\ & - \Big[1 - a_0^c(\xi_0^c)\Big] g^c(Y_0^c, Z_0^c) - \Big[n - a_0^c(\xi_0^c)\Big] \\ & \Big[\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) - \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\Big] \\ & - \frac{1}{2}\Big[da_0^c\Big((\phi_0 Y_0)^c, Z_0^c\Big) + da_0^c\Big((\phi_0 Z_0)^c, Y_0^c\Big)\Big].\end{aligned} \tag{74}$$

Moreover if $S_0^c(Y_0^c, Z_0^c)$ is given by (74), then from (73), we have

$$S_0^c(Y_0^c, Z_0^c) + S_0^c(Z_0^c, Y_0^c) = 0.$$

Theorem 6. *The necessary and sufficient condition for the Ricci tensor of $\ddot{\nabla}^c$ in an LP-Sasakian manifold (M^n, g) endowed with QSNMC $\ddot{\nabla}^c$ in the tangent bundle T_0M^n to be skew-symmetric is that the Ricci tensor of the Levi-Civita connection ∇^c is given by (74).*

8. Skew-Symmetric Properties of the Projective Ricci Tensor in an LP-Sasakian Manifold Endowed with QSNMC $\ddot{\nabla}^c$ in the Tangent Bundle

Chaki and Saha defined the projective Ricci tensor in a Riemannian manifold as [34]

$$P_0(X_0, Y_0) = \frac{n}{n-1}\Big[S_0(X_0, Y_0) - \frac{r_0}{n} g(X_0, Y_0)\Big]. \tag{75}$$

So, the projective Ricci tensor with respect to QSNMC $\ddot{\nabla}$ is defined as

$$\ddot{P}_0(X_0, Y_0) = \frac{n}{n-1}\Big[\ddot{S}_0(X_0, Y_0) - \frac{\ddot{r}_0}{n} g(X_0, Y_0)\Big]. \tag{76}$$

Taking a complete lift by mathematical operators on (76), we have

$$\ddot{P}_0^c(X_0^c, Y_0^c) = \frac{n}{n-1}\Big[\ddot{S}_0^c(X_0^c, Y_0^c) - \frac{\ddot{r}_0^c}{n} g^c(X_0^c, Y_0^c)\Big]. \tag{77}$$

Using (60) and (61) in (77), we have

$$\begin{aligned}\ddot{P}_0^c(X_0^c, Y_0^c) = {} & \frac{n}{n-1}\Big[S_0^c(X_0^c, Y_0^c) - \gamma^c g^c\Big((\phi_0 X_0)^c, Y_0^c\Big) \\ & + \Big(1 - a_0^c(\xi_0^c)\Big) g^c(X_0^c, Y_0^c) + \Big(n - a_0^c(\xi_0^c)\Big)\Big(\eta_0^c(X_0^c)\eta_0^v(Y_0^c) \\ & + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big) + da_0^c\Big((\phi_0 Y_0)^c, X_0^c\Big) \\ & - \frac{1}{n}\Big(r_0^c - (n-1)a_0^c(\xi_0^c) + \lambda_0^c - \gamma^{c^2}\Big) g^c(X_0^c, Y_0^c)\Big].\end{aligned} \tag{78}$$

Similarly, we have

$$\begin{aligned}\ddot{P}_0^c(Y_0^c, X_0^c) = \frac{n}{n-1}\Big[&S_0^c(Y_0^c, X_0^c) - \gamma^c g^c\Big((\phi_0 Y_0)^c, X_0^c\Big)\\ &+ \Big(1 - a_0^c(\xi_0^c)\Big)g^c(Y_0^c, X_0^c) + \Big(n - a_0^c(\xi_0^c)\Big)\Big(\eta_0^c(X_0^c)\eta_0^v(Y_0^c)\\ &+ \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big) + da_0^c\Big((\phi_0 X_0)^c, Y_0^c\Big)\\ &- \frac{1}{n}\Big(r_0^c - (n-1)a_0^c(\xi_0^c) + \lambda_0^c - \gamma^{c^2}\Big)g^c(Y_0^c, X_0^c)\Big].\end{aligned} \tag{79}$$

From (78) and (79), we have

$$\begin{aligned}&\ddot{P}_0^c(X_0^c, Y_0^c) + \ddot{P}_0^c(Y_0^c, X_0^c)\\ &= \frac{n}{n-1}\Big[2S_0^c(X_0^c, Y_0^c) - 2\gamma^c g^c\Big((\phi_0 X_0)^c, Y_0^c\Big)\\ &+ 2\Big(1 - a_0^c(\xi_0^c)\Big)g^c(X_0^c, Y_0^c) + 2\Big(n - a_0^c(\xi_0^c)\Big)\\ &\Big(\eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big)\\ &- \frac{2}{n}\Big(r_0^c - (n-1)a_0^c(\xi_0^c) + \lambda_0^c - \gamma^{c^2}\Big)g^c(X_0^c, Y_0^c)\\ &+ da_0^c\Big((\phi_0 X_0)^c, Y_0^c\Big) + da_0^c\Big((\phi_0 Y_0)^c, X_0^c\Big)\Big].\end{aligned} \tag{80}$$

If $\ddot{P}_0^c(X_0^c, Y_0^c)$ is skew-symmetric, then the left-hand side of (80) vanishes and we have

$$\begin{aligned}S_0^c(X_0^c, Y_0^c) = \Big[&\gamma^c g^c\Big((\phi_0 X_0)^c, Y_0^c\Big) - \Big(1 - a_0^c(\xi_0^c)\Big)g^c(X_0^c, Y_0^c)\\ &- \Big(n - a_0^c(\xi_0^c)\Big)\Big(\eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big)\\ &+ \frac{1}{n}\Big(r_0^c - (n-1)a_0^c(\xi_0^c) + \lambda_0^c - \gamma^{c^2}\Big)g^c(X_0^c, Y_0^c)\\ &- \frac{1}{2}\Big(da_0^c\Big((\phi_0 X_0)^c, Y_0^c\Big) + da_0^c\Big((\phi_0 Y_0)^c, X_0^c\Big)\Big)\Big].\end{aligned} \tag{81}$$

Moreover, if $S_0^c(X_0^c, Y_0^c)$ is given by (81), then from (80) we obtain

$$\ddot{P}_0^c(X_0^c, Y_0^c) + \ddot{P}_0^c(Y_0^c, X_0^c) = 0 \;\text{ s.t }\; \ddot{P}_0^c(X_0^c, Y_0^c) = -\ddot{P}_0^c(Y_0^c, X_0^c). \tag{82}$$

which gives a skew-symmetric condition of the projective Ricci tensor of $\ddot{\nabla}^c$.

Theorem 7. *The necessary and sufficient condition for the projective Ricci tensor of $\ddot{\nabla}^c$ in an LP-Sasakian manifold (M^n, g) endowed with QSNMC $\ddot{\nabla}^c$ in the tangent bundle T_0M^n to be skew-symmetric is that the Ricci tensor of the Levi-Civita connection $\ddot{\nabla}^c$ is given by (81).*

9. Lifts of Einstein Manifold Endowed with QSNMC $\ddot{\nabla}^c$ in an LP-Sasakian Manifold to the Tangent Bundle

A Riemannian manifold (M^n, g) is called an Einstein manifold with respect to Riemannian connection if

$$S_0^c(X_0^c, Y_0^c) = \frac{r_0^c}{n} g^c(X_0^c, Y_0^c). \tag{83}$$

Then, the Einstein manifold with respect to QSNMC $\ddot{\nabla}^c$ is given by

$$\ddot{S}_0^c(X_0^c, Y_0^c) = \frac{\ddot{r}_0^c}{n} g^c(X_0^c, Y_0^c). \tag{84}$$

Using (60) and (61) in (84), we have

$$\begin{aligned}
&\ddot{S}_0^c(X_0^c, Y_0^c) - \frac{\ddot{r}_0^c}{n} g^c(X_0^c, Y_0^c) \\
&= S_0^c(X_0^c, Y_0^c) - \frac{r_0^c}{n} g^c(X_0^c, Y_0^c) - \gamma^c g^c\Big((\phi_0 X_0)^c, Y_0^c\Big) \\
&+ da_0^c\Big((\phi_0 Y_0)^c, X_0^c\Big) + \frac{1}{n}\Big[n + \gamma^{c^2} - \lambda_0^c - a_0^c(\xi_0^c)\Big] g^c(X_0^c, Y_0^c) \\
&+ \Big(n - a_0^c(\xi_0^c)\Big)\Big[\eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big].
\end{aligned} \tag{85}$$

If

$$\begin{aligned}
&\gamma^c g^c\Big((\phi_0 X_0)^c, Y_0^c\Big) + da_0^c\Big(X_0^c, (\phi_0 Y_0)^c\Big) \\
&= \frac{1}{n}\Big[n + \gamma^{c^2} - \lambda_0^c - a_0^c(\xi_0^c)\Big] g^c(X_0^c, Y_0^c) \\
&+ \Big(n - a_0^c(\xi_0^c)\Big)\Big[\eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big],
\end{aligned} \tag{86}$$

then from (85), we have

$$\ddot{S}_0^c(X_0^c, Y_0^c) - \frac{\ddot{r}_0^c}{n} g^c(X_0^c, Y_0^c) = S_0^c(X_0^c, Y_0^c) - \frac{r_0^c}{n} g^c(X_0^c, Y_0^c). \tag{87}$$

Theorem 8. *In an LP-Sasakian manifold (M^n, g) with tangent bundle T_0M^n admitting QSNMC if Equation (86) holds, then the manifold reduces to an Einstein manifold for the Riemannian connection if and only if it is an Einstein manifold for the connection $\ddot{\nabla}^c$.*

10. Example

Let M be a four-dimensional manifold defined as

$$M = \Big\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_4 \neq 0\Big\}, \tag{88}$$

where $\mathbb{R}$ is the set of real numbers. Let x_1, x_2, x_3, x_4 be given by

$$e_1 = \frac{x_1}{x_4}\frac{\partial}{\partial x_1}, \quad e_2 = \frac{x_2}{x_4}\frac{\partial}{\partial x_2}, \quad e_3 = \frac{x_3}{x_4}\frac{\partial}{\partial x_3}, \quad e_4 = x_4\frac{\partial}{\partial x_4},$$

where $\{e_1, e_2, e_3, e_4\}$ are a linearly independent global frame on M. Let the 1-form η_0 be given by

$$\eta_0(X_0) = g(X_0, e_4).$$

The Lorentzian metric g is defined by

$$g(e_i, e_j) = \begin{cases} -1, & i = j = 4 \\ 1, & i = j = 1, 2, 3 \\ 0, & \textit{otherwise}. \end{cases}$$

Let ϕ_0 be the tensor field defined by

$$\phi_0 e_i = \begin{cases} 0, & i = 4 \\ e_i, & i = 1, 2, 3. \end{cases}$$

Using the linearity of ϕ_0 and g, we acquire $\eta_0(e_4) = -1$, $\phi_0^2 X_0 = -X_0 + \eta_0(X_0)e_4$ and $g(\phi_0 X_0, \phi_0 Y_0) = g(X_0, Y_0) + \eta_0(X_0)\eta_0(Y_0)$. Thus, for $e_4 = \xi_0$, then the structure (ϕ, ξ_0, η_0, g) is an almost para-contact metric structure on M and M is called an almost para-contact metric manifold. In addition, M satisfies

$$(\nabla_{X_0}\phi_0)Y_0 = g(X_0, Y_0)e_4 + \eta_0(Y_0)X_0 + 2\eta_0(X_0)\eta_0(Y_0)e_4.$$

Here, for $e_4 = \xi_0$, M is an LP-Sasakian manifold. In tangent bundle T_0M, let the complete and vertical lifts of e_1, e_2, e_3, e_4 be $e_1^c, e_2^c, e_3^c, e_4^c$ and $e_1^v, e_2^v, e_3^v, e_4^v$ on M and let g^c be the complete lift of the Lorentzian metric g on T_0M such that

$$g^c(X_0^v, e_4^c) = \Big(g^c(X_0, e_4)\Big)^v = \Big(\eta_0(X_0)\Big)^v \tag{89}$$

$$g^c(X_0^c, e_4^c) = \Big(g^c(X_0, e_4)\Big)^c = \Big(\eta_0(X_0)\Big)^c \tag{90}$$

$$g^c(e_4^c, e_4^c) = -1, \quad g^v(X_0^v, e_4^c) = 0, \quad g^v(e_4^v, e_4^c) = 0, \tag{91}$$

and so on. Let ϕ_0^c and ϕ_0^v be the complete and vertical lifts of the $(1,1)$ tensor field ϕ_0 defined by

$$\phi_0^v(e_4^v) = \phi_0^c(e_4^c) = 0, \tag{92}$$

$$\phi_0^v(e_1^v) = e_1^v, \quad \phi_0^c(e_1^c) = e_1^c, \tag{93}$$

$$\phi_0^v(e_2^v) = e_2^v, \quad \phi_0^c(e_2^c) = e_2^c, \tag{94}$$

$$\phi_0^v(e_3^v) = e_3^v, \quad \phi_0^c(e_3^c) = e_3^c. \tag{95}$$

Using the linearity of ϕ_0 and g, we infer that

$$(\phi_0^2 X_0)^c = X_0^c + \eta_0^c(X_0)e_4^v + \eta_0^v(X_0)e_4^c, \tag{96}$$

$$g^c\Big((\phi_0 e_4)^c, (\phi_0 e_3)^c\Big) = g^c(e_4^c, e_3^c) + \eta_0^c(e_4^c)\eta_0^v(e_3^c) + \eta_0^v(e_4^c)\eta_0^c(e_3^c). \tag{97}$$

Thus, for $e_4 = \xi_0$ in (89)–(91) and (96), the structure $(\phi_0^c, \xi_0^c, \eta_0^c, g^c)$ is an almost para-contact metric structure on T_0M and satisfies the relation

$$\begin{aligned}(\nabla^c_{e_4^c}\phi_0^c)e_3^c = {} & g^c(e_4^c, e_3^c)\xi_0^v + g^c(e_4^v, e_3^c)\xi_0^c + \eta_0^c(e_3^c)e_4^v + \eta_0^v(e_3^c)e_4^c \\ & + 2\Big\{\eta_0^c(e_4^c)\eta_0^c(e_3^c)\xi_0^v + \eta_0^c(e_4^c)\eta_0^v(e_3^c)\xi_0^c + \eta_0^v(e_4^c)\eta_0^c(e_3^c)\xi_0^c\Big\}.\end{aligned}$$

Thus, $(\phi_0^c, \xi_0^c, \eta_0^c, g^c, T_0M)$ is an LP-Sasakian manifold.

11. Conclusions

The current work investigates the lifts of a QSNMC and LP-Sasakian manifold to the tangent bundle. First, the LP-Sasakian manifold lifts to the tangent bundle are presented. The relationship between the Riemannian connection and the QSNMC from an LP-Sasakian manifold to the tangent bundle is established. An expression of the curvature tensor of the lifts of an LP-Sasakian manifold associated with QSNMC to its tangent bundle is given. The Ricci tensor and the scalar curvature lifts to the tangent bundle are provided. Some theorems regarding the properties of the lifts of the curvature tensor of an LP-Sasakian manifold endowed with QSNMC in an LP-Sasakian manifold to the tangent bundle are given.

Necessary and sufficient conditions for the symmetric and skew-symmetric properties of the lifts of the Ricci tensor are investigated. Sufficient conditions for the skew-symmetric property of the lifts of the projective Ricci tensor in the tangent bundle are provided. The lifts of the Einstein manifold associated with QSNMC on an LP-Sasakian manifold to the tangent bundle are also established. An example of the lifts of LP-Sasakian manifolds in the tangent bundle is constructed.

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mathematics

Article

Framed Natural Mates of Framed Curves in Euclidean 3-Space

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Abstract: In this study, we consider framed curves as regular or singular space curves with an adapted frame in Euclidean 3-space. We define framed natural mates of a framed curve that are tangent to the generalized principal normal of the framed curve. Subsequently, we present the relationships between a framed curve and its framed natural mates. In particular, we establish some necessary and sufficient conditions for the framed natural mates of specific framed curves, such as framed spherical curves, framed helices, framed slant helices, and framed rectifying curves. Finally, we support the concept with some examples.

Keywords: framed curve; spherical curve; helix; slant helix; rectifying curve

MSC: 53A04; 58K05

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1. Introduction

A regular space curve has no singular point in Euclidean space. In this case, the curvature and torsion functions of a regular space curve are well defined at every point. However, this situation is not applicable to all space curves, as some may have singular points. Therefore, the Frenet–Serret frame fails at singular points. Honda and Takahashi [1] introduced a framed curve, which is a regular curve or singular space curve with a moving frame in Euclidean space. Similar to curvature functions of a regular curve, they also defined the framed curvature functions, which are well defined even at singular points. Also, Fukunaga and Takahashi [2] studied the existence conditions of framed curves. Additionally, Wang et al. [3] proposed an adapted frame as an alternative to the moving frame of a framed curve in Euclidean space, with its elements referred to as the generalized tangent vector, generalized principal normal vector, and generalized binormal vector, respectively.

Naturally, the theory of framed curves, which includes regular curves as well, has captured the interest of researchers. As a result, concepts traditionally belonging to the category of regular curves (e.g., helix [4,5], slant helix [6,7], rectifying curve [8], Salkowski curve [9], etc.) have now been extended to the theory of framed curves. In this regard, recently, the concepts of framed helix [10], framed slant helix [11], framed clad helix [12], framed rectifying curve [3,13], framed normal curve [14], and framed Bertrand and Mannheim curves [15] have been introduced. References [16–18] are additional noteworthy studies that contribute to the theory of framed curves. Furthermore, a group of researchers, known as Li et al. and referenced in [19–24], conducted theoretical research and development on submanifold theory, soliton theory, etc. We can find more motivations from some papers [25–51]. Their work has contributed to the advancement of related research topics.

Moreover, Legendre curves are a special case of framed curves. Therefore, References [52–65] are other notable studies that contribute to the field of framed curves, specifically in the category of frontal or front curves.

Additionally, in the category of curves associated with the Frenet–Serret elements of regular curves, the concept of the principal direction (binormal direction) curve was introduced. It is defined as the integral curve of the principal normal vector (binormal vector) of a regular Frenet curve by Choi et al. [66]. Moreover, the natural mate (resp. conjugate mate) is a regular curve that is tangent to the principal normal (resp. binormal) vector of the base regular curve. These curves were introduced as partner curves of any regular curve by Deshmukh et al. [67].

On the other hand, natural and conjugate mates correspond to principal direction and binormal direction curves from the algebraic viewpoint, respectively. But, since the integral curve is defined only for vector fields on a region that contains a curve (i.e., not along a curve), it is more suitable to use the terminology of natural and conjugate mate from a geometric viewpoint. In this sense, as a generalization of the concept of natural mates of a regular space curve, we introduce framed natural mates of the framed curve in Euclidean space by using the adapted frame in [3]. After, we give the necessary and sufficient conditions for framed natural mates of a framed curve when the frame curve is a framed helix, framed slant helix, framed rectifying curve, or framed spherical curve. Finally, the concept of framed natural mate with some examples is enriched.

2. Preliminary

Let $\mathbb{R}^3$ denote the Euclidean 3-space, that is, the 3-dimensional real vector space endowed with the standard inner product $\langle x, y\rangle = \sum_{i=1}^{3} x_i y_i$, for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The norm of a vector $x \in \mathbb{R}^3$ is defined by $\|x\| = \sqrt{\langle x, x\rangle}$. Also, the cross-product of vectors x and y is given by $x \wedge y = (x_2y_3 - x_3y_2, -x_1y_3 + x_3y_1, x_1y_2 - x_2y_1)$.

Framed Curves in Euclidean 3-Space

Let γ: $I \to \mathbb{R}^3$ be a space curve. If $\dot{\gamma}(t_0) = \frac{d\gamma}{dt}(t_0) = 0$ at $t_0 \in I$, then t_0 is called a singular point of γ. It is easy to see that the Frenet frame of any space curve is not well defined at any singular of the curve. Now, let us give the following concept about framed curves, which is a regular curve with linear independent condition or singular space curve in $\mathbb{R}^3$ (see [1–3,10] for more detail and background).

Let us take the set $\Delta_2 = \left\{ \boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2) \in \mathbb{S}^2 \times \mathbb{S}^2 | \langle \boldsymbol{u}_1, \boldsymbol{u}_2\rangle = 0 \right\}$ as a 3-dimensional manifold.

Definition 1. *$(\gamma, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$: $I \to \mathbb{R}^3 \times \Delta_2 \subset \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{S}^2$ is called a framed curve, if $\langle \dot{\gamma}(t), \boldsymbol{\mu}_i(t)\rangle = 0$ for all $t \in I$. $\gamma : I \to \mathbb{R}^3$ is also called a framed curve (or framed base curve) if there exists $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$: $I \to \Delta_2$ such that $(\gamma, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$ is a framed curve [1].*

Now, unlike the Frenet frame, a well-defined moving frame can be constructed along the framed curve γ, which may have singular points. Let $(\gamma, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$ be a framed curve, and let $\boldsymbol{\vartheta} : I \to \mathbb{S}^2$ be a regular spherical curve such that $\boldsymbol{\vartheta}(t) = \boldsymbol{\mu}_1(t) \wedge \boldsymbol{\mu}_2(t)$ for all $t \in I$. Hence, $\{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\vartheta}\}$ is an orthonormal frame, which is a moving frame along the framed curve γ in $\mathbb{R}^3$. Then, the Frenet–Serret-type formulas are given by:

$$\begin{aligned} \dot{\boldsymbol{\mu}}_1(t) &= \mathfrak{l}(t)\boldsymbol{\mu}_2(t) + \mathfrak{m}(t)\boldsymbol{\vartheta}(t), \\ \dot{\boldsymbol{\mu}}_2(t) &= -\mathfrak{l}(t)\boldsymbol{\mu}_1(t) + \mathfrak{n}(t)\boldsymbol{\vartheta}(t), \\ \dot{\boldsymbol{\vartheta}}(t) &= -\mathfrak{m}(t)\boldsymbol{\mu}_1(t) - \mathfrak{n}(t)\boldsymbol{\mu}_2(t), \end{aligned}$$

and there exists a smooth function $\mathfrak{a} : I \to \mathbb{R}$ such that

$$\dot{\gamma}(t) = \mathfrak{a}(t)\boldsymbol{\vartheta}(t). \tag{1}$$

Here, the quadruple smooth functions $(\mathfrak{l}, \mathfrak{m}, \mathfrak{n}, \mathfrak{a}) = (\langle \dot{\boldsymbol{\mu}}_1, \boldsymbol{\mu}_2\rangle, \langle \dot{\boldsymbol{\mu}}_1, \boldsymbol{\vartheta}\rangle, \langle \dot{\boldsymbol{\mu}}_2, \boldsymbol{\vartheta}\rangle, \langle \dot{\gamma}, \boldsymbol{\vartheta}\rangle)$ are called the curvature of the framed curve γ.

Remark 1. *It is clear that if* $t_0 \in I$ *is a singular point of* γ, *then* $\mathfrak{a}(t_0) = 0$. *Moreover, since we suppose that* $\boldsymbol{\vartheta}$ *is a regular spherical curve (i.e., that* $\dot{\boldsymbol{\vartheta}}(t) \neq 0$*), then* $(\mathfrak{m}(t), \mathfrak{n}(t)) \neq (0,0)$ *for all* $t \in I$.

Similar to Bishop frame [68] of regular curves, Wang et al. [3] give the following adapted frame, which is an alternative to the moving frame of the framed curve:

Let $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in \Delta_2$ and $\theta : I \to \mathbb{R}$ be a smooth function such that

$$\begin{pmatrix} \boldsymbol{\eta}_1(t) \\ \boldsymbol{\eta}_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}_1(t) \\ \boldsymbol{\mu}_2(t) \end{pmatrix}.$$

It is easy to see that $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$: $I \to \mathbb{R}^3 \times \Delta_2$ is also a framed curve and $\boldsymbol{\vartheta} = \boldsymbol{\mu}_1 \wedge \boldsymbol{\mu}_2 = \boldsymbol{\eta}_1 \wedge \boldsymbol{\eta}_2$. Now, we assume that $\mathfrak{m}(t) = -\mathfrak{p}(t)\cos\theta(t)$ and $\mathfrak{n}(t) = \mathfrak{p}(t)\sin\theta(t)$ such that $\mathfrak{m}(t)\sin\theta(t) + \mathfrak{n}(t)\cos\theta(t) = 0$, then we have an adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$ along the framed curve γ and the following Frenet–Serret-type formulas:

$$\dot{\boldsymbol{\vartheta}}(t) = \mathfrak{p}(t)\boldsymbol{\eta}_1(t), \quad \dot{\boldsymbol{\eta}}_1(t) = -\mathfrak{p}(t)\boldsymbol{\vartheta}(t) + \mathfrak{q}(t)\boldsymbol{\eta}_2(t), \quad \dot{\boldsymbol{\eta}}_2(t) = -\mathfrak{q}(t)\boldsymbol{\eta}_1(t), \tag{2}$$

where

$$\begin{cases} \mathfrak{p} = \langle \dot{\boldsymbol{\vartheta}}, \boldsymbol{\eta}_1 \rangle = \|\dot{\boldsymbol{\vartheta}}\| = \sqrt{\mathfrak{m}^2 + \mathfrak{n}^2} > 0 \\ \mathfrak{q} = \langle \dot{\boldsymbol{\eta}}_1, \boldsymbol{\eta}_2 \rangle = \mathfrak{l} - \dot{\theta} = \mathfrak{l} + \left(\dfrac{\mathfrak{m}^2}{\mathfrak{m}^2 + \mathfrak{n}^2} \right) \left(\dfrac{\mathfrak{n}}{\mathfrak{m}} \right)^{\cdot} \\ \mathfrak{a} = \langle \dot{\gamma}, \boldsymbol{\vartheta} \rangle \end{cases} \tag{3}$$

The triple smooth functions $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$ are called framed curvature with respect to the adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$ along the framed curve γ. Moreover, the vectors $\boldsymbol{\vartheta}(t), \boldsymbol{\eta}_1(t), \boldsymbol{\eta}_2(t)$ are called the generalized tangent vector, the generalized principal normal vector, and the generalized binormal vector of the framed curve, respectively.

Proposition 1. *Let* $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$: $I \to \mathbb{R}^3 \times \Delta_2$ *be a framed curve with framed curvature* $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. *If the framed curve* γ *is a regular curve with curvature* κ *and torsion* τ, *then we have* $\kappa = \frac{\mathfrak{p}}{|\mathfrak{a}|}$ *and* $\tau = \frac{\mathfrak{q}}{\mathfrak{a}}$ *[3].*

Now, we introduce the framed Darboux vector (framed centrode) and the framed co-Darboux vector (framed co-centrode) with respect to the adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$ of framed curve γ, respectively.

Definition 2. *Let* $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ *be a framed curve in* $\mathbb{R}^3 \times \Delta_2$ *with adapted frame apparatus* $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\}$. *Then, the framed Darboux vector of the framed curve* γ *is defined by* $\boldsymbol{\Omega}(t) = \mathfrak{q}(t)\boldsymbol{\vartheta}(t) + \mathfrak{p}(t)\boldsymbol{\eta}_2(t)$, *which satisfies the following equations:*

$$\dot{\boldsymbol{\vartheta}}(t) = \boldsymbol{\Omega}(t) \wedge \boldsymbol{\vartheta}(t), \quad \dot{\boldsymbol{\eta}}_1(t) = \boldsymbol{\Omega}(t) \wedge \boldsymbol{\eta}_1(t), \quad \dot{\boldsymbol{\eta}}_2(t) = \boldsymbol{\Omega}(t) \wedge \boldsymbol{\eta}_2(t).$$

Moreover, we call that

$$\boldsymbol{\Omega}_0(t) = \frac{\mathfrak{q}(t)\boldsymbol{\vartheta}(t) + \mathfrak{p}(t)\boldsymbol{\eta}_2(t)}{\sqrt{\mathfrak{p}^2(t) + \mathfrak{q}^2(t)}} \tag{4}$$

is the unit framed Darboux vector of γ.

Definition 3. *Let* $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ *be a framed curve in* $\mathbb{R}^3 \times \Delta_2$ *with adapted frame apparatus* $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\}$. *Then, the framed co-Darboux vector of the framed curve* γ *is defined by* $\widehat{\boldsymbol{\Omega}}(t) = -\mathfrak{p}(t)\boldsymbol{\vartheta}(t) + \mathfrak{q}(t)\boldsymbol{\eta}_2(t)$. *Moreover, we call that*

$$\widehat{\mathbf{\Omega}}_{\mathbf{0}}(t)=\frac{-\mathfrak{p}(t)\boldsymbol{\vartheta}(t)+\mathfrak{q}(t)\boldsymbol{\eta}_2(t)}{\sqrt{\mathfrak{p}^2(t)+\mathfrak{q}^2(t)}} \tag{5}$$

is the unit framed co-Darboux vector of γ*. Also, it is easy to see that* $\widehat{\mathbf{\Omega}}_{\mathbf{0}}=\frac{\dot{\boldsymbol{\eta}}_1}{\|\dot{\boldsymbol{\eta}}_1\|}$.

Now, we give the following framed versions (i.e., that generalized versions) of well-known definitions and characterizations for regular space curves.

Definition 4. *Let* $(\gamma,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ *be a framed curve in* $\mathbb{R}^3\times\Delta_2$*. Then,* γ *is called a framed planar curve if it lies on a plane in* $\mathbb{R}^3$ *[1].*

By using Proposition 3.3 in [1] with Equation (3), we give the following characterization of framed planar curves with respect to the adapted curvature.

Theorem 1. *Let* $(\gamma,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ *be a framed curve in* $\mathbb{R}^3\times\Delta_2$ *with framed curvature* $(\mathfrak{p},\mathfrak{q},\mathfrak{a})$*. Then,* γ *is a framed planar curve if and only if* $\mathfrak{q}=0$*.*

Definition 5. *Let* $(\gamma,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ *be a framed curve in* $\mathbb{R}^3\times\Delta_2$*. Then,* γ *is called a framed spherical curve if it lies on a sphere with a radius* r *in* $\mathbb{R}^3$ *[3].*

We give Theorem 2 and Corollary 1 by using Proposition 2 and Corollary 1 in [3] with Equation (3).

Theorem 2. *Let* $(\gamma,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ *be a framed curve in* $\mathbb{R}^3\times\Delta_2$ *with framed curvature* $(\mathfrak{p},\mathfrak{q}=0,\mathfrak{a})$*. Then,* γ *is a framed spherical curve, which is a circle in* $\mathbb{R}^3$ *if and only if* $\mathfrak{q}=0$ *and* $\frac{\mathfrak{p}}{|\mathfrak{a}|}$ *is a constant such that* $\mathfrak{a}\neq 0$*.*

Corollary 1. *Let* $(\gamma,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ *be a framed curve in* $\mathbb{R}^3\times\Delta_2$ *with framed curvature* $(\mathfrak{p},\mathfrak{q}=0,\mathfrak{a})$*. Then,* γ *is a framed spherical curve, which is a great circle in* $\mathbb{S}^2(r)$ *if and only if* $\mathfrak{q}=0$ *and* $\frac{\mathfrak{p}}{|\mathfrak{a}|}=\frac{1}{r}$ *such that* $\mathfrak{a}\neq 0$*.*

Theorem 3. *Let* $(\gamma,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ *be a framed curve in* $\mathbb{R}^3\times\Delta_2$ *with framed curvature* $(\mathfrak{p},\mathfrak{q}\neq 0,\mathfrak{a})$*. Then,* γ *is a framed spherical curve in* $\mathbb{S}^2(r)$ *if and only if*

$$\left(\frac{1}{\mathfrak{q}}\left(\frac{\mathfrak{a}}{\mathfrak{p}}\right)^{\cdot}\right)^2+\left(\frac{\mathfrak{a}}{\mathfrak{p}}\right)^2=r^2, \tag{6}$$

or equivalently,

$$\left(\frac{1}{\mathfrak{q}}\left(\frac{\mathfrak{a}}{\mathfrak{p}}\right)^{\cdot}\right)^{\cdot}+\frac{\mathfrak{a}\,\mathfrak{q}}{\mathfrak{p}}=0.$$

Ref. [14].

Definition 6. *Let* $(\gamma,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ *be a framed curve in* $\mathbb{R}^3\times\Delta_2$ *with framed curvature* $(\mathfrak{p},\mathfrak{q},\mathfrak{a})$*. Then, the framed harmonic curvature of* γ *is given by* $\mathfrak{h}=\frac{\mathfrak{q}}{\mathfrak{p}}$ *[11].*

Definition 7. *Let* $(\gamma,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ *be a framed curve in* $\mathbb{R}^3\times\Delta_2$ *with adapted frame* $\{\boldsymbol{\vartheta},\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\}$*. Then,* γ *is called a framed helix if its generalized tangent vector* $\boldsymbol{\nu}$ *makes a constant angle with a fixed unit vector* $\boldsymbol{\zeta}$ *[3,10].*

Theorem 4. *Let* $(\gamma,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ *be a framed curve in* $\mathbb{R}^3\times\Delta_2$ *with framed curvature* $(\mathfrak{p},\mathfrak{q},\mathfrak{a})$*. Then,* γ *is a framed helix if and only if* $\mathfrak{h}=\cot\phi$ *such that* ϕ *is a constant angle [3].*

Definition 8. *Let $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$. Then, γ is called a framed slant helix if its generalized principal normal vector $\boldsymbol{\eta}_1$ makes a constant angle with a fixed unit vector $\boldsymbol{\zeta}$. That is, $\langle \boldsymbol{\eta}_1, \boldsymbol{\zeta} \rangle = \cos\phi$, where $\phi \neq \pi/2$ is a constant angle [11].*

Theorem 5. *Let $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. Then, γ is a framed slant helix if and only if*

$$\sigma = \frac{\mathfrak{p}^2 \left(\frac{\mathfrak{q}}{\mathfrak{p}}\right)^{\cdot}}{(\mathfrak{p}^2 + \mathfrak{q}^2)^{3/2}} = \frac{\dot{\mathfrak{h}}}{\mathfrak{p}(1 + \mathfrak{h}^2)^{3/2}}$$

is a constant function [11].

Definition 9. *Let $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$. Then, γ is called a framed rectifying curve if its position vector satisfies*

$$\gamma(t) = \lambda_1(t)\boldsymbol{\vartheta}(t) + \lambda_2(t)\boldsymbol{\eta}_2(t)$$

for some smooth functions $\lambda_1(t), \lambda_2(t)$ [3].

Theorem 6. *Let $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. Then, γ is a framed rectifying curve if and only if its framed harmonic curvature is given by*

$$\mathfrak{h}(t) = c_1 \int \mathfrak{a}(t)dt + c_2$$

for some constants $c_1 \neq 0$, and c_2 [3].

3. Framed Natural Mates

In this section, we give the concept of framed natural mates of a framed curve as a regular or singular space curve. This concept is more general than the concept of a natural mate of a Frenet curve in [67].

Definition 10. *Let $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$: $I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve with an adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$. Then, a framed curve $(\gamma_\star, \boldsymbol{\eta}_{1\star}, \boldsymbol{\eta}_{2\star})$: $I \to \mathbb{R}^3 \times \Delta_2$ with an adapted frame $\{\boldsymbol{\vartheta}_\star, \boldsymbol{\eta}_{1\star}, \boldsymbol{\eta}_{2\star}\}$ is called a framed natural mate of $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$, if the generalized tangent vector $\boldsymbol{\vartheta}_\star$ of $\gamma_\star$ is tangent to the generalized principal normal vector $\boldsymbol{\eta}_1$ of the framed curve γ (i.e., that $\boldsymbol{\vartheta}_\star(t) = \boldsymbol{\eta}_1(t)$ for all $t \in I$).*

From now on, we call that the framed curve $\gamma_\star$ is a framed natural mate of the framed curve γ, if $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ and $(\gamma_\star, \boldsymbol{\eta}_{1\star}, \boldsymbol{\eta}_{2\star})$ are framed natural mates.

Theorem 7. *Let $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ and $(\gamma_\star, \boldsymbol{\eta}_{1\star}, \boldsymbol{\eta}_{2\star})$ be framed curves. Then, a framed natural mate $\gamma_\star$ of the framed curve γ is given by*

$$\gamma_\star(t) = \int \mathfrak{a}_\star(t)\boldsymbol{\eta}_1(t)dt \tag{7}$$

with the following adapted frame apparatus

$$\boldsymbol{\vartheta}_\star = \boldsymbol{\eta}_1, \quad \boldsymbol{\eta}_{1\star} = \widehat{\boldsymbol{\Omega}}_0, \quad \boldsymbol{\eta}_{2\star} = \boldsymbol{\Omega}_0, \quad \mathfrak{p}_\star = \sqrt{\mathfrak{p}^2 + \mathfrak{q}^2}, \quad \mathfrak{q}_\star = \frac{\dot{\mathfrak{h}}}{1 + \mathfrak{h}^2} \tag{8}$$

where $\mathfrak{a}_\star : I \to \mathbb{R}$ is a smooth function.

Proof. Let $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with adapted frame apparatus $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\}$. Then, we see that $\left\{\boldsymbol{\eta}_1, \widehat{\boldsymbol{\Omega}}_0, \boldsymbol{\Omega}_0\right\}$ is an orthonormal basis along the framed curve γ

in $\mathbb{R}^3$, where $\boldsymbol{\Omega_0}, \widehat{\boldsymbol{\Omega}}_{\mathbf{0}}$ are given by (4) and (5), respectively. Moreover, by using Equation (2), we have the following equations:

$$\dot{\boldsymbol{\eta}}_1 = \mathfrak{p}_\star \widehat{\boldsymbol{\Omega}}_{\mathbf{0}}, \quad \dot{\widehat{\boldsymbol{\Omega}}}_0 = -\mathfrak{p}_\star \boldsymbol{\eta}_1 + \mathfrak{q}_\star \boldsymbol{\Omega_0}, \quad \dot{\boldsymbol{\Omega}}_0 = -\mathfrak{q}_\star \widehat{\boldsymbol{\Omega}}_{\mathbf{0}},$$

such that

$$\mathfrak{p}_\star = \sqrt{\mathfrak{p}^2 + \mathfrak{q}^2}, \quad \mathfrak{q}_\star = \frac{\dot{\mathfrak{h}}}{1 + \mathfrak{h}^2}$$

where $\mathfrak{h}$ is the framed harmonic curvature of γ. In that case, from the Existence and Uniqueness Theorems of framed curves in [1], there exists a framed curve $(\gamma_\star, \boldsymbol{\eta}_{1\star}, \boldsymbol{\eta}_{2\star})$ in $\mathbb{R}^3 \times \Delta_2$ with the adapted frame apparatus $\{\boldsymbol{\vartheta}_\star, \boldsymbol{\eta}_{1\star}, \boldsymbol{\eta}_{2\star}, (\mathfrak{p}_\star, \mathfrak{q}_\star, \mathfrak{a}_\star)\}$ whose elements are determined by the Equation (8). Also, by using (1), we have $\dot{\boldsymbol{\gamma}}_\star = \mathfrak{a}_\star \boldsymbol{\vartheta}_\star$. This equality leads to the framed curve $\boldsymbol{\gamma}_\star$ being given by (7) such that $\mathfrak{a}_\star \colon I \to \mathbb{R}$ is a smooth function. Finally, by Definition 10, it is nothing but a framed natural mate of $\boldsymbol{\gamma}$. □

Remark 2. *By Theorem 7, there exists a smooth function* $\mathfrak{a}_\star \colon I \to \mathbb{R}$ *such that a framed natural mate of* γ *is given by* (7). *Hence, we see that each smooth function* $\mathfrak{a}_\star$ *generates a different framed natural mate of* γ*, and so a framed natural mate of a framed curve is not unique. Moreover, by Definition 1, it is easy to see that the framed natural mate of* $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ *is given by* $\left(\gamma_\star, \widehat{\boldsymbol{\Omega}}_{\mathbf{0}}, \boldsymbol{\Omega_0}\right)$*, which is also a framed curve in* $\mathbb{R}^3 \times \Delta_2$*.*

Remark 3. *Particularly, when framed curves* $\gamma, \gamma_\star$ *are regular curves and* $\mathfrak{a}(t), \mathfrak{a}_\star(t)$ *are their speed functions, which are equal to 1, then the concept of framed natural mate coincides with the concept of Frenet natural mate [67] (also, the concept of principal normal direction curve of* γ *[66]). So, the concept of framed natural mate is a generalized version of [67].*

Now, when γ is a framed helix or slant helix, it is easy to see that the following results by using Theorems 1, 4, and 5.

Corollary 2. *Let* $\gamma_\star$ *be a framed natural mate of the framed curve* γ *in* $\mathbb{R}^3$*. Then,* γ *is a framed helix if and only if* $\gamma_\star$ *is a framed planar curve.*

Corollary 3. *Let* $\gamma_\star$ *be a framed natural mate of the framed curve* γ *in* $\mathbb{R}^3$*. Then,* γ *is a framed slant helix if and only if* $\gamma_\star$ *is a framed helix.*

Now, we give the following relationship between framed curvatures of a framed rectifying curve and its framed natural mate.

Corollary 4. *Let* γ *and* $\gamma_\star$ *be framed natural mates in* $\mathbb{R}^3$ *with framed curvatures* $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$ *and* $(\mathfrak{p}_\star, \mathfrak{q}_\star, \mathfrak{a}_\star)$*, respectively. Then,* γ *is a framed rectifying curve if and only if the following equation holds:*

$$\lambda\, \mathfrak{a}\, \mathfrak{p}^2 = \mathfrak{p}_\star^2\, \mathfrak{q}_\star \tag{9}$$

where λ *is a nonzero constant.*

Proof. Assume that γ is a framed rectifying curve in $\mathbb{R}^3$ with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$ and $\mathfrak{h}$ is its framed harmonic curvature. Then, by using Theorem 6, $\mathfrak{h}(t) = c_1 \int \mathfrak{a}(t)dt + c_2$ for some constants $c_1 \neq 0$ and c_2. Also, let $\gamma_\star$ be a framed natural mate of γ with a framed curvature $(\mathfrak{p}_\star, \mathfrak{q}_\star, \mathfrak{a}_\star)$. Then, by using (8), we have

$$\mathfrak{p}_\star^2 = \mathfrak{p}^2(1 + \mathfrak{h}^2), \quad \mathfrak{q}_\star = \frac{c_1 \mathfrak{a}}{1 + \mathfrak{h}^2}.$$

Thus, the last equations lead to easily (9). Conversely, let γ and $\gamma_\star$ be framed natural mates, which satisfy the Equation (9). Then, if (8) is taken into account with (9), $\dot{\mathfrak{h}}(t) = c_1\mathfrak{a}(t)$, where c_1 is a nonzero constant. Consequently, γ is a framed rectifying curve in $\mathbb{R}^3$ with respect to Theorem 6, $\square$

Now, we give a relation with respect to the framed curvatures of a framed spherical curve and its framed natural mate.

Corollary 5. *Let γ and $\gamma_\star$ be framed natural mates in $\mathbb{R}^3$ with framed curvatures $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$ and $(\mathfrak{p}_\star, \mathfrak{q}_\star, \mathfrak{a}_\star)$, respectively. Then, γ is a framed spherical curve in $\mathbb{S}^2(r)$ if and only if the following equation holds:*

$$\frac{\dot{\mathfrak{p}}_\star}{\mathfrak{p}_\star} = \mathfrak{q}_\star\mathfrak{h} + \frac{\dot{\mathfrak{a}}}{\mathfrak{a}} \mp \frac{\mathfrak{q}}{\mathfrak{a}}\sqrt{r^2\mathfrak{p}^2 - \mathfrak{a}^2}. \tag{10}$$

Proof. Assume that γ is a framed spherical curve in $\mathbb{S}^2(r)$ with framed curvatures $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. Then, by using Theorem 3, we have

$$\left(\frac{\mathfrak{a}}{\mathfrak{p}}\right)^{\cdot} = \pm\frac{\mathfrak{q}}{\mathfrak{p}}\sqrt{r^2\mathfrak{p}^2 - \mathfrak{a}^2}$$

and so,

$$\frac{\dot{\mathfrak{p}}}{\mathfrak{p}} = \frac{\dot{\mathfrak{a}}}{\mathfrak{a}} \mp \frac{\mathfrak{q}}{\mathfrak{a}}\sqrt{r^2\mathfrak{p}^2 - \mathfrak{a}^2}.$$

Moreover, after using Equation (8) and accordingly, the ratio of $\dot{\mathfrak{p}}_\star$ over $\mathfrak{p}_\star$ is

$$\frac{\dot{\mathfrak{p}}_\star}{\mathfrak{p}_\star} = \frac{\dot{\mathfrak{p}}}{\mathfrak{p}} + \frac{\mathfrak{h}\dot{\mathfrak{h}}}{1+\mathfrak{h}^2} = \frac{\dot{\mathfrak{p}}}{\mathfrak{p}} + \mathfrak{q}_\star\mathfrak{h}$$

Thus, it is easy to see that Equation (10) from the last two equations. Conversely, let γ and $\gamma_\star$ be framed natural mates, which satisfy the Equation (10). Then, by taking into account the Equation (8), we obtain

$$\frac{\dot{\mathfrak{p}}}{\mathfrak{p}} = \frac{\dot{\mathfrak{p}}_\star}{\mathfrak{p}_\star} - \mathfrak{q}_\star\mathfrak{h} = \frac{\dot{\mathfrak{a}}}{\mathfrak{a}} \mp \frac{\mathfrak{q}}{\mathfrak{a}}\sqrt{r^2\mathfrak{p}^2 - \mathfrak{a}^2}.$$

This leads to the following equation

$$\dot{\mathfrak{p}}\mathfrak{a} - \mathfrak{p}\dot{\mathfrak{a}} = \mp\mathfrak{p}\mathfrak{q}\sqrt{r^2\mathfrak{p}^2 - \mathfrak{a}^2},$$

and after suitable settings, we obtain

$$\left(\frac{\mathfrak{a}}{\mathfrak{p}}\right)^{\cdot} = \pm\frac{\mathfrak{q}}{\mathfrak{p}}\sqrt{r^2\mathfrak{p}^2 - \mathfrak{a}^2} \tag{11}$$

As the first case, if $\mathfrak{q} = 0$ in (11), then the proof is clear by Theorem 2. In the other case, if $\mathfrak{q} \neq 0$ in (11), then we reach

$$\left(\frac{1}{\mathfrak{q}}\left(\frac{\mathfrak{a}}{\mathfrak{p}}\right)^{\cdot}\right)^2 = r^2 - \left(\frac{\mathfrak{a}}{\mathfrak{p}}\right)^2.$$

Finally, the desired result is obtained by using Theorem 3. $\square$

After that, let us concentrate on the results of some special framed natural mates of γ.

Theorem 8. *Let γ and $\gamma_\star$ be framed natural mates in $\mathbb{R}^3$ with framed curvatures $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$ and $(\mathfrak{p}_\star, \mathfrak{q}_\star, \mathfrak{a}_\star)$, respectively. If γ is a framed curve with framed curvature $(r, \mathfrak{q}, \mathfrak{a})$ such that r is a* *positive constant, then its framed natural mate $\gamma_\star$ is a framed spherical curve in $\mathbb{S}^2(\frac{\mathfrak{a}_\star}{r})$ such that $\mathfrak{a}_\star$ is a positive constant. The converse is true only when $\gamma_\star$ is a framed spherical curve that is not a circle (i.e., that $\mathfrak{q}_\star \neq 0$) and $\mathfrak{a}_\star$ is a positive constant.*

Proof. Suppose that the framed curvature of γ is $(r, \mathfrak{q}, \mathfrak{a})$ such that r is a positive constant. Then, for framed curvature functions of a framed natural mate $\gamma_\star$, we have

$$\mathfrak{p}_\star = \sqrt{r^2 + \mathfrak{q}^2}, \quad \mathfrak{q}_\star = \frac{r\,\dot{\mathfrak{q}}}{r^2 + \mathfrak{q}^2}. \tag{12}$$

As the first case, if $\mathfrak{q} = 0$ in (12), then it is clear that $\gamma_\star$ is a framed circle with a radius $1/r$ by Theorem 2. In the other case, we suppose that $\mathfrak{q} \neq 0$ in (12) and $\mathfrak{a}_\star = a_0$ is a positive constant, then by using (12), we obtain

$$\left(\frac{1}{\mathfrak{q}_\star}\left(\frac{\mathfrak{a}_\star}{\mathfrak{p}_\star}\right)^{\cdot}\right)^2 + \left(\frac{\mathfrak{a}_\star}{\mathfrak{p}_\star}\right)^2 = \frac{{a_0}^2}{r^2}.$$

Thus, by using Theorem 3, $\gamma_\star$ is a framed spherical curve in $\mathbb{S}^2(\frac{\mathfrak{a}_0}{r})$.

Conversely, we assume that $\gamma_\star$ is a framed spherical curve in $\mathbb{S}^2(\frac{\mathfrak{a}_0}{r})$ such that $\mathfrak{q}_\star \neq 0$, and $\mathfrak{a}_\star = a_0$ is a positive constant. Then, by using Equation (6),

$$\frac{{a_0}^2(\dot{\mathfrak{p}}_\star)^2}{\mathfrak{q}_\star^2\,\mathfrak{p}_\star^4} + \frac{{a_0}^2}{\mathfrak{p}_\star^2} = \frac{{a_0}^2}{r^2},$$

and after suitable settings and integration, without loss of generality, we obtain

$$\mathfrak{p}_\star = r\sec\left(\int \mathfrak{q}_\star dt\right). \tag{13}$$

Now, if we choose as the framed harmonic curvature $\mathfrak{h} = \tan\varphi$ such that φ is a smooth function, then this choice leads to $\mathfrak{q}_\star = \dot{\varphi}$ by applying Equation (8). Thus, by taking into account (13), we obtain $\mathfrak{p}_\star = r\sec\varphi$. Moreover, by applying Equation (8), $\mathfrak{p}_\star = \mathfrak{p}\sqrt{1+\mathfrak{h}^2} = \mathfrak{p}\sec\varphi$. Hence, we conclude that $\mathfrak{p} = r$ by the last two equations of $\mathfrak{p}_\star$. □

Let γ be a framed curve with framed curvature $(\mathfrak{p} = \lambda\cos\phi, \mathfrak{q} = \lambda\sin\phi, \mathfrak{a})$ such that λ is a positive constant and ϕ is a smooth function. Then, by using Equation (8), we see that its framed natural mate $\gamma_\star$ has the framed curvature $(\mathfrak{p}_\star = \lambda, \mathfrak{q}_\star, \mathfrak{a}_\star)$. Now, let us give the following theorem for the converse of this statement.

Theorem 9. *Let $\gamma_\star$ be a framed natural mate of the framed curve γ in $\mathbb{R}^3$. If $\gamma_\star$ has the framed curvature $(\mathfrak{p}_\star = \lambda, \mathfrak{q}_\star, \mathfrak{a}_\star)$ such that λ is a positive constant, then the framed curvature of γ is given by:*

$$\left(\mathfrak{p} = \lambda\cos\left(\int \mathfrak{q}_\star dt\right), \mathfrak{q} = \lambda\sin\left(\int \mathfrak{q}_\star dt\right), \mathfrak{a}\right).$$

Proof. Assume that the framed curvature of the framed natural mate $\gamma_\star$ is $(\mathfrak{p}_\star = \lambda, \mathfrak{q}_\star, \mathfrak{a}_\star)$ such that λ is a positive constant. Then, by using (8), we have

$$\mathfrak{p} = \sqrt{\lambda^2 - \mathfrak{q}^2}.$$

This leads to $\mathfrak{h} = \mathfrak{q}/\sqrt{\lambda^2 - \mathfrak{q}^2}$ and again, by taking into account (8), we obtain

$$\mathfrak{q}_\star = \frac{\left(\frac{\mathfrak{q}}{\sqrt{\lambda^2-\mathfrak{q}^2}}\right)^{\cdot}}{1+\frac{\mathfrak{q}^2}{\lambda^2-\mathfrak{q}^2}} = \frac{\dot{\mathfrak{q}}}{\sqrt{\lambda^2-\mathfrak{q}^2}} = \frac{\left(\frac{\mathfrak{q}}{\lambda}\right)^{\cdot}}{\sqrt{1-\left(\frac{\mathfrak{q}}{\lambda}\right)^2}}.$$

After integration, we obtain $\mathfrak{q} = \lambda\sin(\int \mathfrak{q}_\star dt)$ and so; it leads to the conclusion that $\mathfrak{p} = \lambda\cos(\int \mathfrak{q}_\star dt)$. $\square$

By Theorem 9, we obtain the following results, which are answer to the question: "When does the framed curve γ become a framed spherical curve for its framed natural mate $\gamma_\star$ with a framed curvature $(\mathfrak{p}_\star = \lambda, \mathfrak{q}_\star, \mathfrak{a}_\star)$."

Theorem 10. *Let γ and $\gamma_\star$ be framed natural mates with framed curvatures $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a} = a_0)$ and $(\mathfrak{p}_\star = \lambda, \mathfrak{q}_\star, \mathfrak{a}_\star)$ such that a_0 and λ are some positive constants, respectively. Then, γ is a framed spherical curve in $\mathbb{S}^2(r)$ if and only if $\gamma_\star$ has the framed curvature $\left(\lambda, \pm\frac{\lambda^2\sqrt{\rho^2-\lambda^2}\cos(\lambda t)}{\lambda^2+(\rho^2-\lambda^2)\sin^2(\lambda t)}, \mathfrak{a}_\star\right)$ for a positive constant $\rho \geq \lambda$.*

Proof. Suppose that γ is a framed spherical curve with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a} = a_0)$ in $\mathbb{S}^2(r)$ such that a_0 is a positive constant, and its framed natural mate $\gamma_\star$ has framed curvature $(\mathfrak{p}_\star = \lambda, \mathfrak{q}_\star, \mathfrak{a}_\star)$ such that λ is a positive constant. Then, by using Theorems 3 and 9, we have

$$\frac{{a_0}^2}{\lambda^4}\sec^2 f\left(\lambda^2 + \dot{f}^2\sec^2 f\right) = r^2$$

where $f = \int \mathfrak{q}_\star dt$, and so

$$\frac{\lambda^4 r^2}{{a_0}^2}\cos^2 f - \lambda^2 = \dot{f}^2\sec^2 f.$$

We see that there exists a positive constant $\rho = \frac{\lambda^2 r}{a_0}$ such that $\rho \geq \lambda$ by the last equation. Accordingly, after suitable settings, we have

$$\frac{\lambda\dot{f}\sec^2 f}{\sqrt{\rho^2 - \lambda^2\sec^2 f}} = \pm\lambda$$

and next step, by applying integration, we obtain

$$\arcsin\left(\frac{\lambda\tan f}{\sqrt{\rho^2-\lambda^2}}\right) = \pm\lambda t.$$

This equation leads to the following equation

$$f = \arctan\left(\pm\frac{\sqrt{\rho^2-\lambda^2}}{\lambda}\sin(\lambda t)\right) \tag{14}$$

Finally, the desired result is obtained by $\mathfrak{q}_\star = \dot{f}$.

Conversely, we suppose that γ has framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a} = a_0)$ such that a_0 is a positive constant, and $\gamma_\star$ has framed curvature

$$\left(\mathfrak{p}_\star = \lambda, \mathfrak{q}_\star = \pm\frac{\lambda^2\sqrt{\rho^2-\lambda^2}\cos(\lambda t)}{\lambda^2+(\rho^2-\lambda^2)\sin^2(\lambda t)}, \mathfrak{a}_\star\right)$$

such that λ is a positive constant. Now, if we take as $f = \int \mathfrak{q}_\star dt$, then by using Theorem 9 and hypothesis, we have

$$\mathfrak{p} = \lambda\cos f, \quad \mathfrak{q} = \lambda\sin f, \quad \mathfrak{a} = a_0. \tag{15}$$

Moreover, by using hypothesis, f is given by (14). Now, we must check the Equation (6). After applying Equations (14) and (15) in Equation (6), we obtain the positive constant $\frac{a_0{}^2\rho^2}{\lambda^4}$. Consequently, γ is a framed spherical curve in $\mathbb{S}^2(\frac{a_0\rho}{\lambda^2})$ by Theorem 3. $\square$

Corollary 6. *Let γ be a framed curve and $\gamma_\star$ be its framed natural mate with framed curvature $(\mathfrak{p}_\star = \lambda, \mathfrak{q}_\star, \mathfrak{a}_\star)$ such that λ is a positive constant. Then, γ is a framed spherical curve, which is not a circle in $\mathbb{S}^2(r)$ if and only if $\gamma_\star$ has the framed curvature $\left(\mathfrak{p}_\star = \lambda,\ \mathfrak{q}_\star = \frac{-\dot{\mathfrak{a}}\pm\mathfrak{q}\sqrt{r^2\mathfrak{p}^2-\mathfrak{a}^2}}{\mathfrak{a}\,\mathfrak{h}},\ \mathfrak{a}_\star\right)$ such that $\mathfrak{q} \neq 0$.*

Proof. The proof is clear by using Corollary 5 and Theorem 10. $\square$

4. Some Examples of Framed Natural Mates

By using the following Frenet-type method (cf. [10]), we can uniquely determine the adapted frame apparatus of any framed curves as regular or singular space curves in $\mathbb{R}^3$.

Let $\boldsymbol{\vartheta} : I \to \mathbb{S}^2$ be a regular spherical curve, and $\mathfrak{a} : I \to \mathbb{R}$ be a smooth function. Then, there exists a framed curve $\gamma : I \to \mathbb{R}^3$ with the adapted frame

$$\left\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1 = \frac{\dot{\boldsymbol{\vartheta}}}{\|\dot{\boldsymbol{\vartheta}}\|}, \boldsymbol{\eta}_2 = \boldsymbol{\vartheta} \wedge \boldsymbol{\eta}_1\right\} \tag{16}$$

such that $\dot{\gamma} = \mathfrak{a}\,\boldsymbol{\vartheta}$. Thus, the smooth function $\mathfrak{a}$ corresponds to the speed function of γ.

Example 1. *Let $\boldsymbol{\vartheta}$ be a small circle in $\mathbb{S}^2$ given by:*

$$\boldsymbol{\vartheta}(t) = \left(\frac{2\sqrt{2}}{3}\cos t, \frac{2\sqrt{2}}{3}\sin t, \frac{1}{3}\right).$$

Then, by integrating (1) *for any smooth function $\mathfrak{a}$, we obtain a family of framed helices γ with framed curvature $(\mathfrak{p}(t), \mathfrak{q}(t), \mathfrak{a}(t)) = \left(\frac{2\sqrt{2}}{3}, \frac{1}{3}, \mathfrak{a}(t)\right)$, which are generated by $\mathfrak{a}$ and $\boldsymbol{\vartheta}$. Moreover, by using* (7) *and* (16)*, framed natural mates $\gamma_\star$ of γ are a family of framed planar curves with framed curvature $(\mathfrak{p}_\star, \mathfrak{q}_\star, \mathfrak{a}_\star) = (1, 0, \mathfrak{a}_\star)$ for any smooth function $\mathfrak{a}_\star$. For example, if $\mathfrak{a}(t) = \cos(3t)$, then the parametrization of framed helix γ is given by*

$$\gamma(t) = \left(\frac{1}{12}\left(2\sqrt{2}\sin(2t) + \sqrt{2}\sin(4t)\right), \frac{1}{12}\left(2\sqrt{2}\cos(2t) - \sqrt{2}\cos(4t) + 2\right), \frac{1}{9}\sin(3t)\right)$$

(see Figure 1e) and the framed natural mate $\gamma_\star$, which is a framed planar curve, is given by

$$\gamma_\star(t) = \left(\frac{1}{8}(-2\cos(2t) + \cos(4t) - 2), \frac{1}{8}(2\sin(2t) + \sin(4t)), 0\right)$$

such that $\mathfrak{a}_\star(t) = \cos(3t)$ (see Figure 2e).

Example 2. *Let $\boldsymbol{\vartheta}$ be a unit speed spherical helix in $\mathbb{S}^2$ given by*

$$\boldsymbol{\vartheta}(t) = \left(\frac{32t^7 - 2352t^5 + 51450t^3 - 300125t}{51450\sqrt{35}}, \frac{\left(16t^4 - 336t^2 + 735\right)\left(35 - t^2\right)^{3/2}}{25725\sqrt{35}}, \frac{t}{6}\right)$$

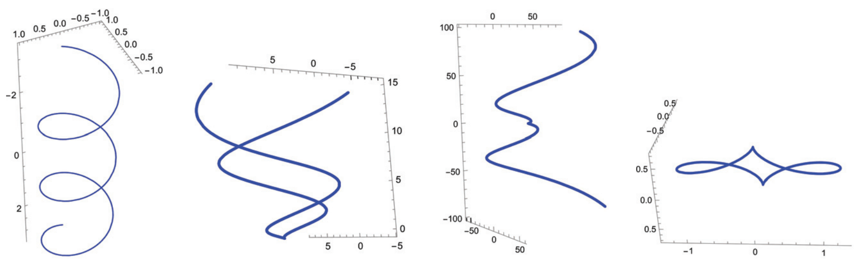
and $\mathfrak{a}(t) = t$, then the parametrization of framed slant helix γ is given by

$$\gamma(t) = \left(\frac{t^3\left(32t^6 - 3024t^4 + 92610t^2 - 900375\right)}{463050\sqrt{35}}, \frac{\left(35 - t^2\right)^{5/2}\left(-16t^4 + 112t^2 + 245\right)}{231525\sqrt{35}}, \frac{t^3}{18}\right)$$

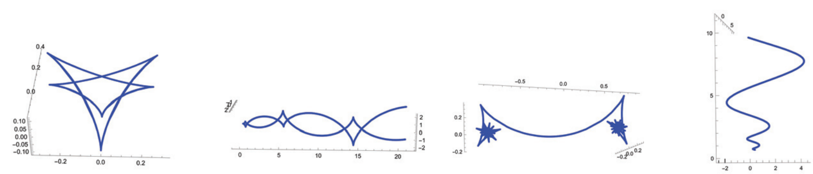
with framed curvature $(\mathfrak{p}(t), \mathfrak{q}(t), \mathfrak{a}(t)) = \left(1, \frac{t}{\sqrt{35-t^2}}, t\right)$ *and* $\sigma = \frac{1}{\sqrt{35}}$ *(see Figure 3a). Thus, for the choice* $\mathfrak{a}_\star(t) = t$*, a framed natural mate, which is a framed helix of* γ *is given by*

$$\gamma_\star(t) = \left(\frac{t^2(t^2-35)(8t^4-280t^2+1225)}{14700\sqrt{35}}, \frac{t^3(2t^2-35)(35-t^2)^{3/2}}{3675\sqrt{35}}, \frac{t^2}{12}\right)$$

with framed curvature $(\mathfrak{p}_\star, \mathfrak{q}_\star, \mathfrak{a}_\star) = \left(\frac{\sqrt{35}}{\sqrt{35-t^2}}, \frac{1}{\sqrt{35-t^2}}, t\right)$ *and* $\mathfrak{h}_\star(t) = \frac{1}{\sqrt{35}}$ *(see Figure 3b).*

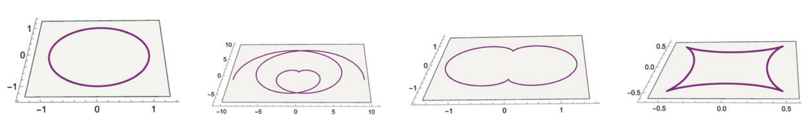


(**a**) $\mathfrak{a}(t) = 1$ (**b**) $\mathfrak{a}(t) = t$ (**c**) $\mathfrak{a}(t) = t^2$ (**d**) $\mathfrak{a}(t) = \cos\frac{t}{2}$

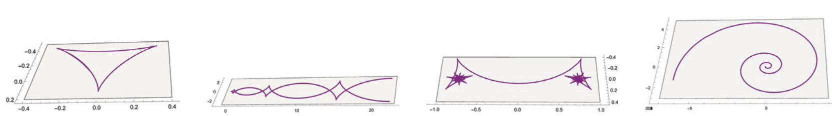


(**e**) $\mathfrak{a}(t) = \cos 3t$ (**f**) $\mathfrak{a}(t) = t\cos t$ (**g**) $\mathfrak{a}(t) = \cos t^2$ (**h**) $\mathfrak{a}(t) = e^{t/5}$

Figure 1. Some framed helices with framed curvature $\left(\frac{2\sqrt{2}}{3}, \frac{1}{3}, \mathfrak{a}(t)\right)$.

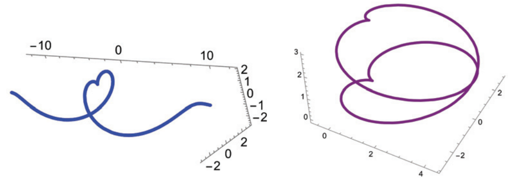


(**a**) $\mathfrak{a}_\star(t) = 1$ (**b**) $\mathfrak{a}_\star(t) = t$ (**c**) $\mathfrak{a}_\star(t) = \cos\frac{t}{2}$ (**d**) $\mathfrak{a}_\star(t) = \cos 2t$



(**e**) $\mathfrak{a}_\star(t) = \cos 3t$ (**f**) $\mathfrak{a}_\star(t) = t\cos t$ (**g**) $\mathfrak{a}_\star(t) = \cos t^2$ (**h**) $\mathfrak{a}_\star(t) = e^{t/5}$

Figure 2. Some framed planar curves with framed curvature $(1, 0, \mathfrak{a}_\star(t))$, which are framed natural mates of framed helices in Figure 1.

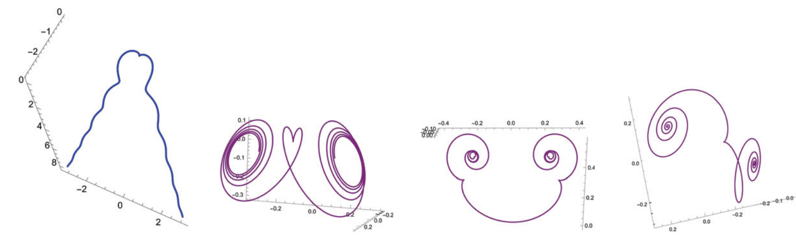


(**a**) $\mathfrak{a}(t) = t$ (**b**) $\mathfrak{a}_\star(t) = t$

Figure 3. (**a**) Framed slant helix γ; (**b**) its framed natural mate $\gamma_\star$, which is a framed helix.

Example 3. *Let* $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\}$ *and* $\{\boldsymbol{\vartheta}_\star, \boldsymbol{\eta}_{1\star}, \boldsymbol{\eta}_{2\star}, (\mathfrak{p}_\star, \mathfrak{q}_\star, \mathfrak{a}_\star)\}$ *be adapted frame apparatus of* γ *and* $\gamma_\star$*, respectively. According to the Existence and Uniqueness Theorems of framed curves in [1], if the framed curvature of a framed curve is given, then we can draw a congruent graphic to the framed curve by applying numerical solution method to Frenet-type differential Equations* (1)–(3) *with the initial conditions. Also, the framed curvature of its framed natural mate is determined from by Theorem 7. Thus, the following graphics of the framed curve and its framed natural mate are obtained by using the "NDSolve" command in Mathematica [69].*

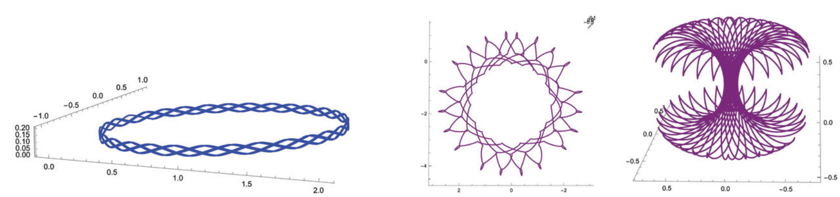
Let γ *be a framed curve with framed curvature* $(\mathfrak{p}(t), \mathfrak{q}(t), \mathfrak{a}(t)) = (3, 3t^2, 2t)$*. By using Theorem 6,* γ *is a framed rectifying curve (see Figure 4a). Thus, we obtain its framed natural mates, which has the framed curvature* $(\mathfrak{p}_\star(t), \mathfrak{q}_\star(t), \mathfrak{a}_\star(t)) = \left(3\sqrt{1+t^4}, \frac{2t}{1+t^4}, \mathfrak{a}_\star(t)\right)$ *(see Figure 4b–d).*



(**a**) $\mathfrak{a}(t) = t$ (**b**) $\mathfrak{a}_\star(t) = t$ (**c**) $\mathfrak{a}_\star(t) = \cos 2t$ (**d**) $\mathfrak{a}_\star(t) = \sin t$

Figure 4. (**a**) Framed rectifying curve γ; (**b**–**d**) its framed natural mates $\gamma_\star$, which are a spiral-type framed curve.

Finally, as an application of by Theorem 10, if we choose $\lambda = 2, r = 1, \mathfrak{a}(t) = \frac{1}{5}$ *such that* $\rho = 20$*. Then,* γ *is a framed spherical curve in* $\mathbb{S}^2$ *with framed curvature* $(\mathfrak{p}(t), \mathfrak{q}(t), \mathfrak{a}(t)) = \left(\frac{2}{\sqrt{1+99(\sin 2t)^2}}, \frac{6\sqrt{11}\sin 2t}{\sqrt{1+99(\sin 2t)^2}}, \frac{1}{5}\right)$ *(see Figure 5a). Thus, we obtain its framed natural mates, which has the framed curvature* $(\mathfrak{p}_\star(t), \mathfrak{q}_\star(t), \mathfrak{a}_\star(t)) = \left(2, \frac{6\sqrt{11}\sin 2t}{\sqrt{1+99(\sin 2t)^2}}, \mathfrak{a}_\star(t)\right)$ *(see Figure 5b,c).*



(**a**) $\mathfrak{a}(t) = t$ (**b**) $\mathfrak{a}_\star(t) = \cos\frac{t}{3}$ (**c**) $\mathfrak{a}_\star(t) = \cos t$

Figure 5. (**a**) Framed spherical curve γ; (**b**,**c**) its framed natural mates $\gamma_\star$.

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mathematics

Article

Surface Pencil Pair Interpolating Bertrand Pair as Common Asymptotic Curves in Euclidean 3-Space

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Abstract: In this paper, we obtain the necessary and sufficient conditions of a surface pencil pair interpolating a Bertrand pair as common asymptotic curves in Euclidean 3-space $\mathbb{E}^3$. Afterwards, the conclusion to the ruled surface pencil pair is also obtained. Meanwhile, the epitomes are stated to emphasize that the proposed methods are effective in product manufacturing by adjusting the shapes of the surface pencil pair.

Keywords: asymptotic curve; Bertrand mate; ruled surface

MSC: 53A04; 53A05; 53A17

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1. Introduction

An asymptotic curve on a surface is an essential geometrical characteristic that plays a major role in a variety of implementations, such as the design of hulls, car shells, knife rests, cloths, etc. In the context of geometry, an asymptotic curve is a curve constantly tangent to an asymptotic trend (direction) of the surface. It is occasionally named an asymptotic line, although it is not required to be a line. An asymptotic trend is one for which the normal curvature is identically zero. This means that, for a point on an asymptotic curve, we take the plane that affords both the surface's normal and the curve's tangent at this point. The intersection curve of the plane and the surface will have zero curvature at this point. Asymptotic trends can only arise if the Gaussian curvature is negative (or zero). There will be two asymptotic trends if every point has a negative Gaussian curvature; these trends are halved by the principal lines [1,2]. In practical applications, essential work has focused on the reverse problem or backward analysis: given a 3D curve, how can we define those surfaces that possess this curve as a special curve, rather than finding and assorting curves on analytical curved surfaces? Wang et al. [3] considered the issue of constructing a surface pencil from a specified spatial geodesic curve, through which each surface can be a candidate for style design. They proved the necessary and sufficient conditions for the coefficients to be content with both the geodesic and the isoparametric requirements. This scheme has been utilized by numerous researchers (see, for example, [4–16]).

In the context of the theory of special curves, the consistency relationship among the curves is a fascinating issue. The Bertrand curve is one of the best-known special curves. Two curves are called a Bertrand pair if there exists a consistency relationship among their principal normals at the analogical points [1,2]. The Bertrand curve can be evaluated as the popularization of the helix. The helix, as a certain type of curve, has attracted the attention of mathematicians as well as scientists because of its diverse implementations; for instance, the Bertrand curves represent special models of offset curves, which are used in computer-aided manufacturing (CAM) and computer-aided design (CAD) (see [17–19]). However, to our knowledge, there is no work that designs a surface pencil pair interpolating a Bertrand pair to be asymptotic curves in Euclidean 3-space $\mathbb{E}^3$. This paper is intended to satisfy such a requirement; we evaluate a Bertrand pair as asymptotic curves to model a surface pencil pair in $\mathbb{E}^3$. Moreover, a ruled surface is indispensable for various areas of CAGD; regarding the aim of this work, the conclusion to the ruled surface pencil pair is also outlined. Meanwhile, some examples are shown to depict the surface pencil and ruled surface pencil with common Bertrand asymptotic curves.

2. Preliminaries

In this section, we list the most important notations that we use in this paper [1,2]. A curve is regular if it possesses a tangent line at each point of the curve. In the following, all curves are supposed to be regular. Given a spatial curve $\boldsymbol{\alpha}(s)$, it is expressed by arc length parameter s. We assume $\ddot{\boldsymbol{\alpha}}(s) \neq 0$ for all $s \in [0, L]$, since this would give a straight line. In this paper, $\dot{\boldsymbol{\alpha}}(s)$ and $\boldsymbol{\alpha}^{'}(v)$ indicate the derivatives of $\boldsymbol{\alpha}$ with respect to arc length parameter s and arbitrary parameter v, respectively. For each point of $\boldsymbol{\alpha}(s)$, the set $\{\chi_1(s), \chi_2(s), \chi_3(s)\}$ is named the Serret–Frenet frame on $\boldsymbol{\alpha}(s)$, where $\chi_1(s) = \dot{\boldsymbol{\alpha}}(s)$, $\chi_2(s) = \ddot{\boldsymbol{\alpha}}(s)/\left\|\ddot{\boldsymbol{\alpha}}(s)\right\|$ and $\chi_3(s) = \chi_1(s) \times \chi_2(s)$ are the unit tangent, principal normal, and binormal vectors of the curve at the point $\boldsymbol{\alpha}(s)$, respectively. The arc length derivative of the Serret–Frenet frame is [1,2]

$$\begin{pmatrix} \dot{\chi}_1 \\ \dot{\chi}_2 \\ \dot{\chi}_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix}, \quad (1)$$

where the curvature $\kappa(s)$ and torsion $\tau(s)$ are specified by

$$\kappa(s) = \left\|\ddot{\boldsymbol{\alpha}}(s)\right\|, \ \tau(s) = \frac{\det(\dot{\boldsymbol{\alpha}}(s), \ddot{\boldsymbol{\alpha}}(s), \dddot{\boldsymbol{\alpha}}(s))}{\left\|\ddot{\boldsymbol{\alpha}}(s)\right\|^2}.$$

In spite of the fact that the arc length parameter is simple to analyze, in the majority of feasible situations, the parameter of a specified curve is commonly not in arc length parametrization. We can symbolize the specified curve by employing arc length parametrization. Given the curve

$$\boldsymbol{\alpha}(v) = (\alpha_1(v), \alpha_2(v), \alpha_3(v)), 0 \leq v \leq H,$$

where the parameter v is not the arc length. The synthesis of the Serret–Frenet frame is specified by [1,2]

$$\chi_1(v) = \frac{\boldsymbol{\alpha}^{'}(v)}{\left\|\boldsymbol{\alpha}^{'}(v)\right\|}, \ \chi_3(v) = \frac{\boldsymbol{\alpha}^{'}(v) \times \boldsymbol{\alpha}^{''}(v)}{\left\|\boldsymbol{\alpha}^{'}(v) \times \boldsymbol{\alpha}^{''}(v)\right\|}, \ \chi_2(v) = \chi_3(v) \times \chi_1(v), \ ({}^{'} \overset{def}{=} \frac{d}{dv}), \quad (2)$$

and the Serret–Frenet formula is

$$\begin{pmatrix} \chi_1^{'}(v) \\ \chi_2^{'}(v) \\ \chi_3^{'}(v) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(v)\left\|\boldsymbol{\alpha}^{'}(v)\right\| & 0 \\ -\kappa(v)\left\|\boldsymbol{\alpha}^{'}(v)\right\| & 0 & \tau(v)\left\|\boldsymbol{\alpha}^{'}(v)\right\| \\ 0 & -\tau(v)\left\|\boldsymbol{\alpha}^{'}(v)\right\| & 0 \end{pmatrix} \begin{pmatrix} \chi_1(v) \\ \chi_2(v) \\ \chi_3(v) \end{pmatrix}. \quad (3)$$

We utilize basic notation for the Bertrand pair from [1,2]: Let $\boldsymbol{\alpha}(s)$ and $\widehat{\boldsymbol{\alpha}}(s)$ be two curves in $\mathbb{E}^3$; $\chi_2(s)$ and $\widehat{\chi}_2(s)$ are the principal normal vectors of them, respectively; the pair $\{\boldsymbol{\alpha}(s), \widehat{\boldsymbol{\alpha}}(s)\}$ is named a Bertrand pair if $\chi_2(s)$ and $\widehat{\chi}_2(s)$ are linearly dependent at the congruent points. $\boldsymbol{\alpha}(s)$ is named the Bertrand mate of $\widehat{\boldsymbol{\alpha}}(\widehat{s})$, and

$$\widehat{\boldsymbol{\alpha}}(s) = \boldsymbol{\alpha}(s) + f\chi_2(s). \quad (4)$$

where f is steady. Therefore, the associations of the Serret–Frenet frame of $\boldsymbol{\alpha}(s)$ with that of $\widehat{\boldsymbol{\alpha}}(s)$ are

$$\begin{pmatrix} \widehat{\chi}_1 \\ \widehat{\chi}_2 \\ \widehat{\chi}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix}, \tag{5}$$

where φ is a constant angle.

We represent a surface M by

$$M : \boldsymbol{y}(s,t) = (y_1(s,t), y_2(s,t), y_3(s,t)), \quad (s,t) \in \mathbb{D} \subseteq \mathbb{R}^2. \tag{6}$$

If $\boldsymbol{y}_j(s,t) = \frac{\partial \boldsymbol{y}}{\partial j}$, the surface normal is

$$\boldsymbol{\zeta}(s,t) = \boldsymbol{y}_s \wedge \boldsymbol{y}_t, \tag{7}$$

which is perpendicular to the vectors $\boldsymbol{y}_s$ and $\boldsymbol{y}_t$.

Definition 1 ([1,2])**.** *A curve on a surface is asymptotic if and only if the binormal vector of the curve is everywhere parallel to the surface normal.*

A curve $\boldsymbol{\alpha}(s)$ on a surface $\boldsymbol{y}(s,t)$ is an isoparametric curve if it has a stationary s or t-parameter value. In other words, there exists a parameter t_0 such that $\boldsymbol{\alpha}(s) = \boldsymbol{y}(s,t_0)$ or $\boldsymbol{\alpha}(t) = \boldsymbol{y}(s_0,t)$. Given a parametric curve $\boldsymbol{\alpha}(s)$, we call it an isoasymptotic curve of the surface $\boldsymbol{y}(s,t)$ if it is both an asymptotic and a parameter curve on $\boldsymbol{y}(s,t)$.

3. Main Results

This section presents a new approach to constructing a surface pencil pair interpolating a Bertrand pair as common asymptotic curves in $\mathbb{E}^3$. To do this, we take into consideration a Bertrand pair, such that the surfaces' tangent planes are concomitant with the curves' osculating planes.

Let $\boldsymbol{\alpha}(s)$ be a curve with $\left\|\ddot{\boldsymbol{\alpha}}(s)\right\| \neq 0$; $\widehat{\boldsymbol{\alpha}}(s)$ is the Bertrand mate of $\boldsymbol{\alpha}(s)$, and $\{\widehat{\kappa}(s), \widehat{\tau}(s), \widehat{\chi}_1(s), \widehat{\chi}_2(s), \widehat{\chi}_3(s)\}$ is the Frenet–Serret apparatus of $\widehat{\boldsymbol{\alpha}}(s)$ as in Equation (1). The surface pencil M interpolating $\boldsymbol{\alpha}(s)$ can be denoted by

$$M : \boldsymbol{y}(s,t) = \boldsymbol{\alpha}(s) + a(s,t)\chi_1(s) + b(s,t)\chi_2(s); \ \ 0 \le t \le T. \tag{8}$$

Similarly, the surface pencil $\widehat{M}$ interpolating $\widehat{\boldsymbol{\alpha}}(s)$ is denoted by

$$\widehat{M} : \widehat{\boldsymbol{y}}(s,t) = \widehat{\boldsymbol{\alpha}}(s) + a(s,t)\widehat{\chi}_1(s) + b(s,t)\widehat{\chi}_2(\widehat{s}); \ \ 0 \le t \le T. \tag{9}$$

Here, $a(s,t), b(s,t) \in C^1$ are named marching-scale functions, and $b(s,t) \neq 0$.

In order to obtain the $\widehat{M}$ interpolating $\widehat{\boldsymbol{\alpha}}(s)$ as a mutual asymptotic curve, according to Equations (8) and (9), we examine what the marching-scale functions should fulfill. To do this, we have

$$\left.\begin{aligned} \widehat{\boldsymbol{y}}_s(s,t) &= (1 + a_s - b\widehat{\kappa})\widehat{\chi}_1 + (b_s + a\widehat{\kappa})\widehat{\chi}_2 + b\widehat{\tau}\widehat{\chi}_3, \\ \widehat{\boldsymbol{y}}_t(s,t) &= a_t\widehat{\chi}_1 + b_t\widehat{\chi}_2, \end{aligned}\right\} \tag{10}$$

and

$$\widehat{\boldsymbol{\zeta}}(s,t) := \widehat{\boldsymbol{y}}_s \times \widehat{\boldsymbol{y}}_t = b\widehat{\tau}(b_t\widehat{\chi}_1 + a_t\widehat{\chi}_2) + [(1 + a_s - b\widehat{\kappa})b_t - (a\widehat{\kappa} + b_s)a_t]\widehat{\chi}_3(s). \tag{11}$$

Since $\widehat{\boldsymbol{\alpha}}(s)$ is isoparametric on $\widehat{M}$, there exists a value $t = t_0 \in [0,T]$ such that $\widehat{\boldsymbol{y}}(s,t_0) = \widehat{\boldsymbol{\alpha}}(s)$; in other words,

$$a(s,t_0) = b(s,t_0) = 0, \ a_s(s,t_0) = b_s(s,t_0) = 0. \tag{12}$$

Thus, as $t = t_0$, i.e., over $\widehat{\boldsymbol{\alpha}}(s)$, we have

$$\widehat{\boldsymbol{\zeta}}(s,t_0) = b_t\widehat{\chi}_3(s). \tag{13}$$

The coincidence of the binormal $\widehat{\chi}_3(s)$ with surface normal $\widehat{\boldsymbol{\zeta}}(s,t_0)$ identifies $\widehat{\boldsymbol{\alpha}}(s)$ as an asymptotic curve. We utilize $\{M,\ \widehat{M}\}$ to indicate the surface pencil pair. Then, from Equations (9)–(13), we derive the following theorem.

Theorem 1. *$\{M,\ \widehat{M}\}$ interpolate the Bertrand pair $\{\boldsymbol{\alpha}(s),\ \widehat{\boldsymbol{\alpha}}(s)\}$ as common asymptotic curves if and only if*

$$\left.\begin{array}{l} a(s,t_0)=b(s,t_0)=0, \\ b_t(s,t_0)\neq 0,\ 0\leq t_0\leq T,\ \ 0\leq s\leq L. \end{array}\right\} \tag{14}$$

We call M and $\widehat{M}$, expressed by by Equations (8) and (9) and satisfying the conditions (14), an asymptotic surface pencil pair interpolating a Bertrand pair, since the common asymptotic curves are also a Bertrand pair. Any $\{M,\ \widehat{M}\}$ satisfying the conditions of Equation (14) is a member of this surface pencil pair. As reported in [3], for ease of interpretation, the marching-scale functions $a(s,t)$ and $b(s,t)$ can be displayed as two factors:

$$\begin{array}{c} a(s,t)=l(s)A(t), \\ b(s,t)=m(s)B(t). \end{array} \tag{15}$$

$l(\widehat{s})$, $m(\widehat{s})$, $A(t)$ and $B(t)$ are C^1 functions that do not identically vanish. Then, we can obtain the below corollary.

Corollary 1. *$\{M,\ \widehat{M}\}$ interpolate the Bertrand pair $\{\boldsymbol{\alpha}(s),\ \widehat{\boldsymbol{\alpha}}(s)\}$ as common asymptotic curves if and only if*

$$\left.\begin{array}{l} A(t_0)=B(t_0)=0,\ \ l(s)=const.\neq 0,\ m(s)=const.\neq 0, \\ \frac{dB(t_0)}{dt}=const.\neq 0,\ 0\leq t_0\leq T,\ \ 0\leq s\leq L. \end{array}\right\} \tag{16}$$

To confirm that $\{M,\ \widehat{M}\}$ interpolate the Bertrand pair $\{\boldsymbol{\alpha}(s),\ \widehat{\boldsymbol{\alpha}}(s)\}$, we can first design the marching-scale functions in Equation (16), and then use them in Equations (8) and (9) to specify the parameterization. For suitability in practice, the functions $a(s,t)$ and $b(s,t)$ can be moreover constrained to be in extra limited forms and still possess sufficient degrees of freedom to specify a large pencil pair interpolating the Bertrand pair $\{\boldsymbol{\alpha}(s),\ \widehat{\boldsymbol{\alpha}}(s)\}$ as common asymptotic curves. Therefore, let us assume that $a(s,t)$ and $b(s,t)$ can be displayed as follows.

(1) If

$$\left\{\begin{array}{l} a(s,t)=\sum\limits_{k=1}^{p} a_{1k}l(s)^k A(t)^k, \\ b(s,t)=\sum\limits_{k=1}^{p} b_{1k}m(s)^k B(t)^k, \end{array}\right. \tag{17}$$

then,

$$\left\{\begin{array}{c} A(t_0)=B(t_0)=0, \\ b_{11}\neq 0,\ m(s)\neq 0,\text{ and } \frac{dB(t_0)}{dt}=const.\neq 0, \end{array}\right. \tag{18}$$

where $l(s)$, $m(s)$, $A(t), B(t)\in C^1$, a_{ij}, $B_{ij}\in\mathbb{R}$ $(i=1,2; j=1,2,...,p)$ and $l(s)$, and $m(s)$ are not identically zero.

(2) If

$$\left\{\begin{array}{l} a(s,t)=f(\sum\limits_{k=1}^{p} a_{1k}l^k(s)A^k(t)), \\ b(s,t)=g(\sum\limits_{k=1}^{p} b_{1k}m^k(s)B^k(t)), \end{array}\right. \tag{19}$$

then

$$\left\{\begin{array}{c} A(t_0)=B(t_0)=f(0)=g(0)=0, \\ b_{11}\neq 0,\ \frac{dB(t_0)}{dt}=const\neq 0,\ m(s)\neq 0,\ g^{'}(0)\neq 0, \end{array}\right. \tag{20}$$

where $l(s)$, $m(s)$, $A(t)$, $B(t) \in C^1$, a_{ij}, $b_{ij} \in \mathbb{R}$ $(i = 1,2; j = 1,2,...,p)$ and $l(s)$, and $m(s)$ are not identically zero.

Since the parameters a_{ij}, $b_{ij} \in \mathbb{R}$ $(i = 1,2; j = 1,2,...,p)$ control the shape of $\{M, \widehat{M}\}$, we can adjust these parameters to output $\{M, \widehat{M}\}$, which represent definite restrictions, such as conditions on the boundary, curvature, etc. The marching-scale functions in Equations (15), (17) and (19) are general for $\{M, \widehat{M}\}$ interpolating the given Bertrand curves as common asymptotic curves. Furthermore, since there are no constraints related to the Bertrand curves in Equations (16), (18) or (20), the surface pencil pair interpolating the given Bertrand curves, acting as both isoparametric curves and asymptotic curves, can always be found by choosing suitable marching-scale functions. Furthermore, some more conditions for various types of $\{M, \widehat{M}\}$ interpolating the given Bertrand curves can be obtained from $0 \le \varphi \le \frac{\pi}{2}$; in the special cases, if $\varphi = 0$ $(\varphi = \pi/2)$, then the pair $\{M, \widehat{M}\}$ are named the oriented pair and right pair, respectively.

Example 1. *If* $\boldsymbol{a}_0 = (0,0,0)$, $\boldsymbol{a}_1 = (0,1,1)$ *and* $\boldsymbol{a}_2 = (1,2,0)$ *are points in the Euclidean 3-space* $\mathbb{E}^3$, *then the quadratic Bézier curve can be specified as*

$$\boldsymbol{\alpha}(v) = b_0(v)\boldsymbol{a}_0 + b_1(v)\boldsymbol{a}_1 + b_2(v)\boldsymbol{a}_2,\ 0 \le v \le 1.$$

where

$$b_0(v) = (1-v)^2,\ b_1(v) = 2v(1-v),\ b_2(v) = v^2,$$

are the blending functions of the curve $\boldsymbol{\alpha}(v)$. *It is easy to show that*

$$\kappa(v) = \frac{1}{2}\sqrt{\frac{6}{5v^2 - 4v + 2}},\ \tau(v) = 0.$$

After simple computation, we obtain

$$\chi_1(v) = \frac{(v, 1, 1-2v)}{\rho},\ \chi_2(v) = \frac{(2(1-v), 2-5v, -(2+v))}{\sqrt{6}\rho},\ \chi_3(v) = (-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}),$$

where $\rho(v) = \sqrt{5v^2 - 4v + 2}$. *Selecting* $a(v,t) = -4vt$, $b(v,t) = -t$, $\gamma \neq 0$, *and* $t_0 = 0$. *Clearly, Equation* (16) *is satisfied, and the parametric surface specified by Equation* (8) *is*

$$M : \boldsymbol{y}(v,t) = \left(v^2, 2v, 2v - 2v^2\right) + t(-4v, -1, 0)\begin{pmatrix} \frac{v}{\rho} & \frac{1}{\rho} & \frac{1-2v}{\rho} \\ \frac{2(1-v)}{\sqrt{6}\rho} & \frac{2-5v}{\sqrt{6}\rho} & \frac{-(2+v)}{\sqrt{6}\rho} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix}.$$

Let $f = \sqrt{6}$ *in Equation* (7), *and we obtain*

$$\widehat{\boldsymbol{\alpha}}(v) = (v^2 - \frac{2v}{\rho}, 2v - \frac{(2-5v)}{\rho}, 2v(1-v) - \frac{(2+v)}{\rho}).$$

Via Equation (5), *we find*

$$\widehat{\chi}_1 = \begin{pmatrix} \chi_{11} \\ \chi_{12} \\ \chi_{13} \end{pmatrix} = \begin{pmatrix} \frac{v}{\rho}\cos\varphi - \frac{2}{\sqrt{6}}\sin\varphi \\ \frac{1}{\rho}\cos\varphi + \frac{1}{\sqrt{6}}\sin\varphi \\ \frac{1-2v}{\rho}\cos\varphi + \frac{1}{\sqrt{6}}\sin\varphi \end{pmatrix},$$

$$\widehat{\chi}_3 = \begin{pmatrix} \chi_{31} \\ \chi_{32} \\ \chi_{33} \end{pmatrix} = \begin{pmatrix} -\frac{v}{\rho}\sin\varphi - \frac{2}{\sqrt{6}}\cos\varphi \\ -\frac{1}{\rho}\sin\varphi + \frac{1}{\sqrt{6}}\cos\varphi \\ -\frac{(1-2v)}{\rho}\sin\varphi + \frac{1}{\sqrt{6}}\cos\varphi \end{pmatrix}.$$

Then, we have

$$\widehat{M}:\widehat{\boldsymbol{y}}(v,t)=(v^2-\frac{2v}{\rho},2v+\frac{2-3v}{\rho},2v-2v^2-\frac{(2+v)}{\rho})+t(-4v,-1,0)\begin{pmatrix}\chi_{11} & \chi_{12} & \chi_{13}\\ 0 & 1 & 0\\ \chi_{31} & \chi_{32} & \chi_{33}\end{pmatrix}.$$

For $\beta=\gamma=-1$, the oriented pair and the right pair, respectively, are shown in Figures 1 and 2, where $0\le r\le 1$, and $-15\le t\le 15$.

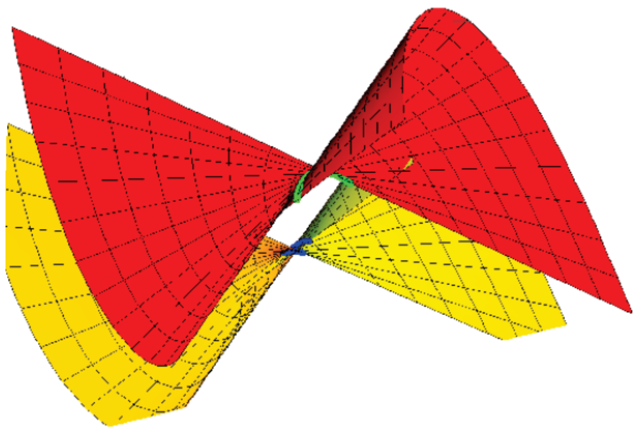

Figure 1. Oriented pair $\{M,\widehat{M}\}$ with $\widehat{\boldsymbol{\alpha}}(v)$ blue and $\boldsymbol{\alpha}(v)$ green.

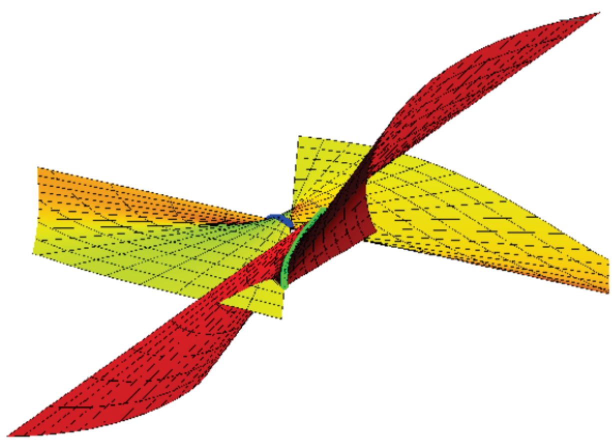

Figure 2. Right pair $\{M,\widehat{M}\}$ with $\widehat{\boldsymbol{\alpha}}(v)$ blue and $\boldsymbol{\alpha}(v)$ green.

Example 2. *Given a helix*

$$\boldsymbol{\alpha}(s)=\frac{1}{\sqrt{2}}(\cos s,\sin s,s),\ 0\le s\le 2\pi.$$

The Serret–Frenet frame is

$$\chi_1(s)=\frac{1}{\sqrt{2}}(-\sin s,\cos s,1),\ \chi_2(s)=(-\cos s,-\sin s,0),\ \widehat{\chi}_3(s)=\frac{1}{\sqrt{2}}(\sin s,-\cos s,1).$$

Then, the parametric surface defined by Equation (8) *is*

$$M : \boldsymbol{y}(s,t) = \frac{1}{\sqrt{2}}(\cos s, \sin s, s) + (a(s,t), b(s,t), 0)\begin{pmatrix} \frac{-\sin s}{\sqrt{2}} & \frac{\cos s}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos s & -\sin s & 0 \\ \frac{\sin s}{\sqrt{2}} & \frac{-\cos s}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Let $f = \sqrt{2}$ *in Equation* (7)*, and we have*

$$\widehat{\boldsymbol{\alpha}}(s) = \frac{1}{\sqrt{2}}(-\cos s, -\sin s, s),\ 0 \leq s \leq 2\pi.$$

Via Equation (5)*, we find*

$$\widehat{\chi}_1 = \begin{pmatrix} \chi_{11} \\ \chi_{12} \\ \chi_{13} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(-\cos\varphi + \sin\varphi)\sin s \\ \frac{1}{\sqrt{2}}(\cos\varphi - \sin\varphi)\cos s \\ \frac{1}{\sqrt{2}}(\cos\varphi + \sin\varphi) \end{pmatrix},$$

$$\widehat{\chi}_3 = \begin{pmatrix} \chi_{31} \\ \chi_{32} \\ \chi_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\sin\varphi + \cos\varphi)\sin s \\ \frac{1}{\sqrt{2}}(-\sin\varphi - \cos\varphi)\cos s \\ \frac{1}{\sqrt{2}}(\cos\varphi - \sin\varphi) \end{pmatrix}.$$

Then, we have

$$\widehat{M} : \widehat{\boldsymbol{y}}(s,t) = \frac{1}{\sqrt{2}}(-\cos s, -\sin s, s) + (a(s,t), b(s,t), 0)\begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ -\cos s & -\sin s & 0 \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix}.$$

(1) If we take $a(s,t) = 0$, $b(s,t) = 1 - \cosh t + \sum_{k=2}^{4} b_{2k}(1-\cosh t)^k$, *where* $t_0 = 0$ *and* $b_{2k} \in \mathbb{R}$, *then Equation* (18) *is satisfied. The oriented pair and the right pair are identical; where* b_{2k} *approaches zero,* $0 \leq t \leq 0.2$, *and* $0 \leq s \leq 2\pi$ *(Figure 3).*

(2) If we take $a(s,t) = \sin(\sum_{k=1}^{4} t^k s^k)$, $b(s,t) = \sum_{k=1}^{4} t^k s^k$, *and* $t_0 = 0$, *then Equation* (20) *is satisfied. The oriented pair and the right pair, respectively, are shown in Figures 4 and 5, where* $0 \leq t \leq 0.1$, *and* $0 \leq s \leq 2\pi$.

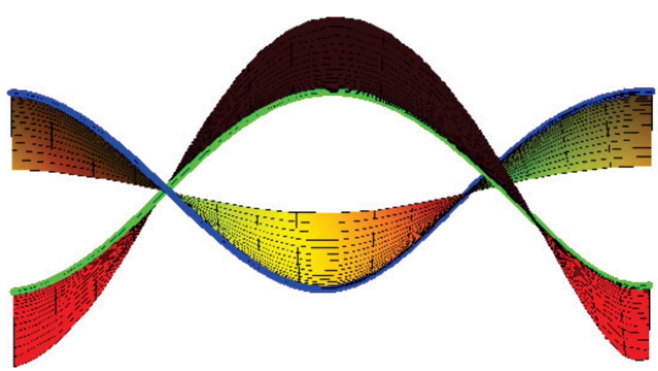

Figure 3. Oriented right pair $\{M, \widehat{M}\}$ with $\widehat{\boldsymbol{\alpha}}(s)$ blue and $\boldsymbol{\alpha}(s)$ green.

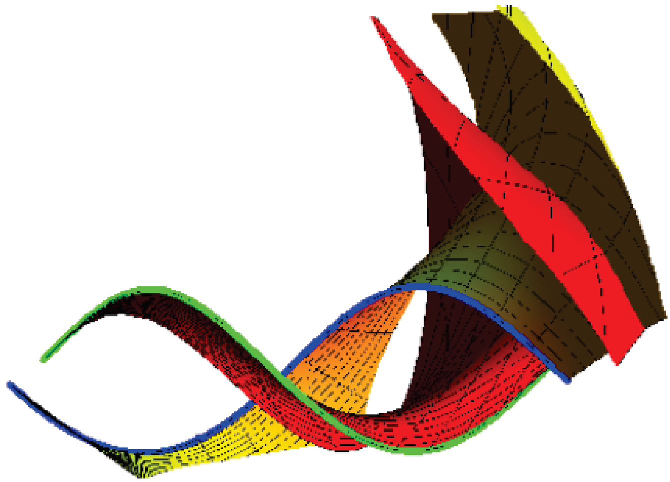

Figure 4. Oriented pair $\{M, \widehat{M}\}$ with $\widehat{\boldsymbol{\alpha}}(s)$ blue and $\boldsymbol{\alpha}(s)$ green.

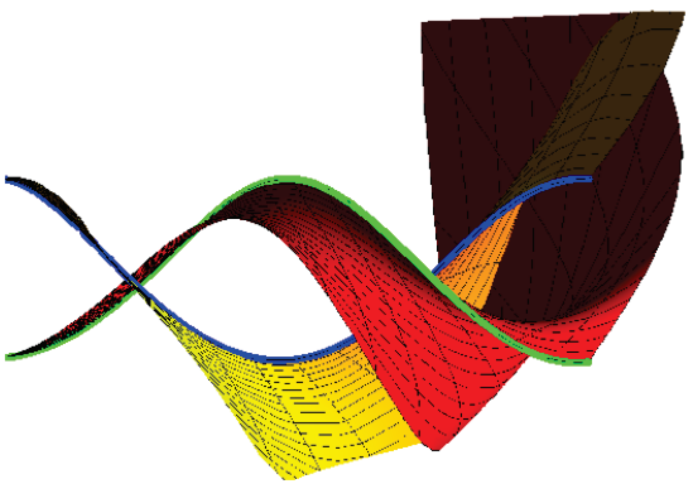

Figure 5. Right pair $\{M, \widehat{M}\}$ with $\widehat{\boldsymbol{\alpha}}(s)$ blue and $\boldsymbol{\alpha}(s)$ green.

Ruled Surface Family Pair with Bertrand Pair as Mutual Asymptotic Curves

Ruled surfaces are simple and common surfaces in geometric designs. Suppose that $\widehat{\boldsymbol{y}}(s,t)$ is a ruled surface with the base $\widehat{\boldsymbol{\alpha}}(s)$ and $\widehat{\boldsymbol{\alpha}}(s)$ is also an isoparametric curve of $\widehat{\boldsymbol{y}}(s,t)$; then, there exists t_0 such that $\widehat{\boldsymbol{y}}(s,t_0) = \widehat{\boldsymbol{\alpha}}(s)$. It follows that the surface can be represented as

$$\widehat{M} : \widehat{\boldsymbol{y}}(s,t) - \widehat{\boldsymbol{y}}(s,t_0) = (t - t_0)\widehat{\boldsymbol{e}}(s), 0 \leq s \leq L, \text{ with } t,\ t_0 \in [0,T], \tag{21}$$

where $\widehat{\boldsymbol{e}}(s)$ defines the direction of the rulings. In view of Equation (9), we have

$$(t - t_0)\widehat{\boldsymbol{e}}(s) = a(s,t)\widehat{\boldsymbol{\chi}}_1(s) + v(s,t)\widehat{\boldsymbol{\chi}}_2(s),\ 0 \leq s \leq L, \text{ with } t,\ t_0 \in [0,T].$$

For $a(s,t)$ and $b(s,t)$, we have

$$\begin{aligned} a(s,t) &= (t-t_0) < \widehat{e}(s), \widehat{\chi}_1(s) >, \\ b(s,t) &= (t-t_0) < \widehat{e}(s), \widehat{\chi}_2(s) > . \end{aligned} \tag{22}$$

The above equations are simply the necessary and sufficient conditions for which $\widehat{y}(s,t)$ is a ruled surface with a base $\widehat{\alpha}(s)$.

Now, we examine whether $\widehat{\alpha}(s)$ is also asymptotic on $\widehat{M}$ by employing Theorem 1. It is apparent that, in this case, it follows that

$$< \widehat{e}(s), \widehat{\chi}_2(s) >= \det(\widehat{e}, \widehat{\chi}_3, \widehat{\chi}_1) \neq 0. \tag{23}$$

Then, at any point on $\widehat{\alpha}(s)$, the $\widehat{e}$ should be in the osculating plane. Moreover, the $\widehat{e}$ and $\widehat{t}$ must not be parallel. It follows that

$$\widehat{e}(s) = x(s)\widehat{\chi}_1(s) + y(s)\widehat{\chi}_2(s), 0 \leq s \leq L. \tag{24}$$

Substituting Equation (24) into Equation (22), we obtain

$$tx(s) = a(s,t), \text{ and } ty(s) = b(s,t), \text{ with } y(s) \neq 0. \tag{25}$$

Then, the ruled surface family with the mutual geodesic $\widehat{\alpha}(s)$ can be specified as

$$\widehat{M} : \widehat{y}(s,t) = \widehat{\alpha}(s) + t(x(s)\widehat{\chi}_1(s) + y(s)\widehat{\chi}_2(s)),\ 0 \leq s \leq L,\ 0 \leq t \leq T, \tag{26}$$

where $x(s), y(s) \neq 0,\ 0 \leq s \leq L$, and $0 \leq t \leq T$. However, the normal vector to $\widehat{M}$ along the curve $\widehat{\alpha}(s) \mid$ is

$$\widehat{\zeta}(s,t_0) = y(s)\widehat{\chi}_3(s), \tag{27}$$

which shows that $\widehat{\alpha}(s)$ is an asymptotic curve on $\widehat{M}$. Then, the following theorem can be stated.

Theorem 2. *The ruled surface family pair* {M, $\widehat{M}$} *interpolate the Bertrand pair* {$\alpha(s)$, $\widehat{\alpha}(s)$} *as common asymptotic curves if and only if there exists a parameter $t_0 \in [0,T]$, and the functions $x(s), y(s) \neq 0$, so that $\widehat{M}$ and M are, respectively, parameterized by Equation* (26), *and*

$$M : y(s,t) = \alpha(s) + t(x(s)\chi_1(s) + y(s)\chi_2(s)),\ 0 \leq s \leq L,\ 0 \leq t \leq T. \tag{28}$$

It must be pointed out that in Equations (26) and (28), there exist two asymptotic curves crossing every point on the curves $\widehat{\alpha}(s)(\alpha(s))$. One is $\widehat{\alpha}$ itself and the other is a line in the orientation $\widehat{e}(s)$ as given in Equation (24). Every constituent of the isoparametric ruled surface family with the mutual asymptotic $\widehat{\alpha}$ is established by two set functions $x(s)$, $y(s) \neq 0$.

Example 3. *In view of Example 1, for $x(v) = y(v) = -1$, the ruled oriented pair* {M, $\widehat{M}$} *and the ruled right pair* {M, $\widehat{M}$}*, respectively, are shown in Figures 6 and 7, where $0 \leq v \leq 1$, and $-4 \leq t \leq 4$.*

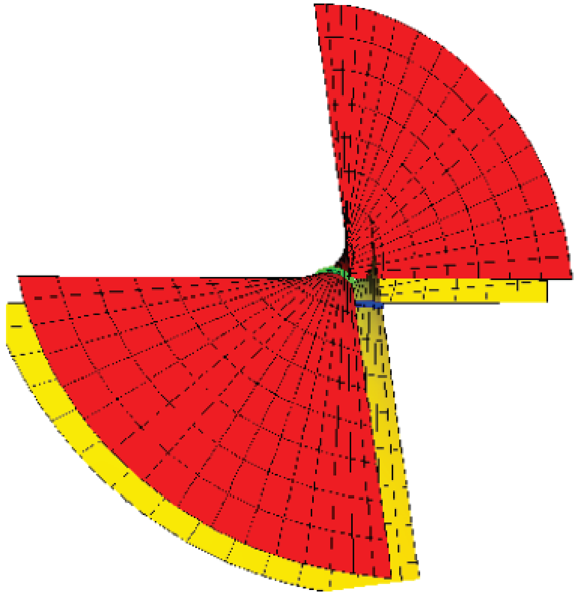

Figure 6. Ruled oriented pair $\{M, \widehat{M}\}$ with $\widehat{\boldsymbol{\alpha}}(v)$ blue and $\boldsymbol{\alpha}(v)$ green.

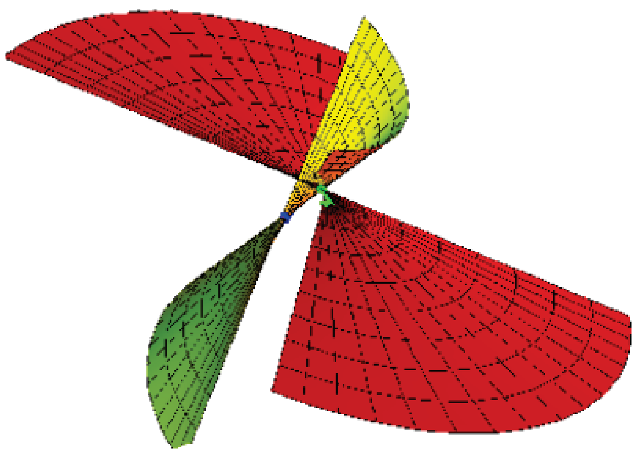

Figure 7. Ruled right pair $\{M, \widehat{M}\}$ with $\widehat{\boldsymbol{\alpha}}(v)$ blue and $\boldsymbol{\alpha}(v)$ green.

Example 4. *In view of Example 2, for $x(s) = y(s) = 1$, the ruled oriented pair $\{M, \widehat{M}\}$, and the ruled right pair $\{M, \widehat{M}\}$, respectively, are shown in Figures 8 and 9, where $-2 \leq t \leq 2$, and $0 \leq s \leq 2\pi$ (Figure 8).*

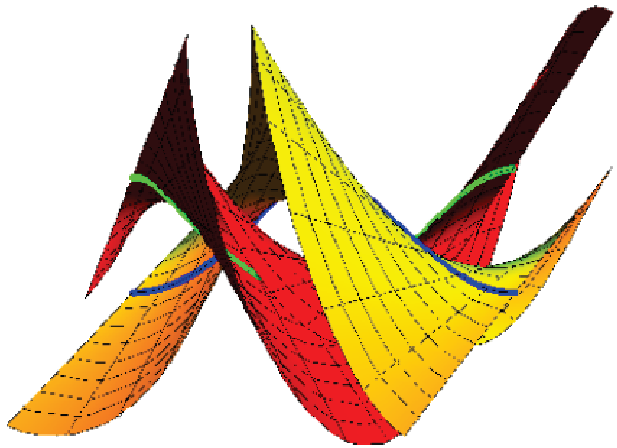

Figure 8. Ruled oriented pair $\{M, \widehat{M}\}$ with $\widehat{\boldsymbol{\alpha}}(s)$ blue and $\boldsymbol{\alpha}(s)$ green.

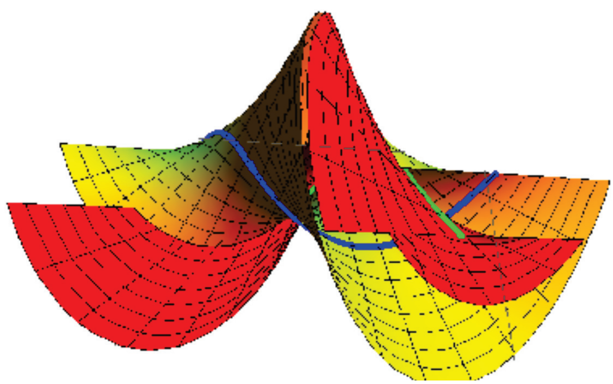

Figure 9. Ruled right pair $\{M, \widehat{M}\}$ with $\widehat{\boldsymbol{\alpha}}(s)$ blue and $\boldsymbol{\alpha}(s)$ green.

4. Conclusions

In this work, we constructed the surface pencil pair and ruled surface pencil pair interpolating a Bertrand pair as common asymptotic curves in Euclidean 3-space $\mathbb{E}^3$. Moreover, some curves were selected to organize the surface pencil pair and ruled surface pencil pair that have the Bertrand pair $\{\widehat{\boldsymbol{\alpha}}(s), \boldsymbol{\alpha}(s)\}$ as common asymptotic curves. Hopefully, these results will be advantageous for work in computer-aided manufacturing and to those exploring manufacturing. Our results, presented in this paper, can contribute to the field of CAGD and have practical applications in CAM. The authors plan to register the study in different spaces and examine the classification of singularities as reported in [20–22].

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Article

The Homology of Warped Product Submanifolds of Spheres and Their Applications

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Abstract: The aim of the current article is to formulate sufficient conditions for the Laplacian and a gradient of the warping function of a compact warped product submanifold $\Sigma^{\beta_1+\beta_2}$ in a unit sphere $\mathbb{S}^d$ that provides trivial homology and fundamental groups. We also validate the instability of current flows in $\pi_1(\Sigma^{\beta_1+\beta_2})$. The constraints are also applied to the warped function eigenvalues and integral Ricci curvatures.

Keywords: warped product submanifolds; standard sphere; homology groups; fundamental group; stable currents

MSC: 53C40; 58C35; 53C55; 58Z05; 58J60

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1. Introduction and Main Results

An algebraic description of a manifold can be found in its homology groups, which are significant topological invariants. This theory has various applications since these groups carry extensive topological information about the connected components, holes, tunnels, and dimensions of the manifold. In fact, homology theory has applications in root construction, protein docking, image segmentation, and gene expression data [1]. It is generally acknowledged that any non-trivial integral homology class in $H_{\beta_2}(\Sigma^n, \mathbb{Z})$ is associated with the topological properties of submanifolds in different ambient spaces. The authors of [2], Federer and Fleming, were the first to demonstrate this idea using the method of variational calculus to represent the idea of geometric measure spaces. Later on, Lawson-Simons [3] constructed an escalation for the second fundamental form, which enforces the vanishing of the homology in a region of intermediate dimensions and the non-existence of stable current flows in the submanifold of the simply connected space form, and discovered the following idea, the main motivation for this work:

Theorem 1 ([3,4]). *If the following optimization inequality holds for a compact m-dimensional submanifold in a space form $\widetilde{M}(c)$ such that the curvature $c \geq 0$ and β_1 is an integer satisfying $0 < \beta_1 < m$,*

$$\sum_{b_1=\beta_1+1}^{m} \sum_{b_2=1}^{\beta_1} \left(2||\mathbf{A}(\mathcal{E}_{b_1}, \mathcal{E}_{b_2})||^2 - g\big(\mathbf{A}(\mathcal{E}_{b_1}, \mathcal{E}_{b_1}), \mathbf{A}(\mathcal{E}_{b_2}, \mathcal{E}_{b_2})\big) \right) < \beta_1(m-\beta_1)c, \quad (1)$$

then no stable β_1-currents flow in Σ^m and

$$H_{\beta_1}(\Sigma^m, \mathbb{Z}) = H_{\beta_2}(\Sigma^m, \mathbb{Z}) = 0$$

for any integer $\beta_2 = m - \beta_1$, and $H_i(\Sigma^m, \mathbb{Z})$ is the i-th homology group of Σ^m with integer coefficients.

The first study of vanishing homology groups on warped product submanifold theory can be found in [5]. By placing appropriate limits on the Laplacian and the gradient of the warping function, Sahin [5] was able to confirm some conclusions regarding the nonexistence of stable current and vanishing homology groups into the contact CR-warped product that was immersed in a sphere with odd dimensions with the implementation of Theorem 1. In recent years, a significant number of studies have been conducted on the geometric structure and topological characteristics of submanifolds in various ambient spaces. These structures and characteristics have been demonstrated in numerous papers covering numerous applications, such as Euclidean spaces [6], complex projective spaces [7], CR-warped product submanifolds in Sasakian space forms [5], CR-warped product submanifolds in Euclidean spaces [8], CR-warped products in nearly Kaeher manifolds [9,10], CR-warped products in hyperbolic spaces [11], and many others (see [12,13]). There is a closed relationship between the absence of stable currents and the vanishing homology groups of submanifolds in various ambient manifold classes. These conclusions were reached by applying pinching conditions to the second fundamental form. Many writers have examined a variety of topological features in response to the lack of stable currents or stable submanifolds [13,14] motivated by Theorem [3] in the references. In the above literature, we observed that the topological and geometrical approaches have lately become useful ideas in machine learning theory due to the need for deep learning models in curved spaces. The significance of submanifold theory was once again demonstrated by the notion that the data might be viewed as a submanifold of Euclidean space. It is obvious that the theory of submanifolds will continue to be studied in light of this new area of applications.

2. Preliminaries

Let a sphere with a constant sectional curvature, c, be represented by $\mathbb{S}^d$, $c = 1 > 0$, and the d-dimension. Given that $\mathbb{S}^d$ accepts a canonical isometric immersion in $\mathbb{R}^{d+1}$, we use this as our main argument:

$$\mathbb{S}^d = \{V_2 \in \mathbb{R}^{d+1} \ : ||V_2||^2 = 1\}.$$

The Riemannian curvature tensor $\widetilde{R}$ of the sphere $\mathbb{S}^d$ satisfies

$$\widetilde{R}(V_2, V_3, V_4, V_5) = g(V_2, V_5)g(V_3, V_4) - g(V_3, V_5)g(V_2, V_4), \tag{2}$$

for any $V_3, V_2, V_4, V_5 \in \Gamma(T\widetilde{M})$, where $T\widetilde{M}$ is the tangent bundle of $\mathbb{S}^d$, and g is the Riemannian metric. In other words, the unit sphere $\mathbb{S}^d$ is a manifold with a constant sectional curvature equal to 1.

Let us assume that Σ^m is an m-dimensional Riemannian submanifold of a Riemannian manifold $\widetilde{M}$. Let us denote $\Gamma(T\Sigma)$ for the section of the tangent bundle of Σ and $\Gamma(T\Sigma^{\perp})$ for the set of all normal vector fields of Σ, respectively. Let us also denote ∇ for the Levi–Civita connection on tangent bundle $T\Sigma$ and $\nabla^{\perp}$ for the Levi–Civita connection on the normal bundle $T\Sigma$. If $\widetilde{R}$ and R are represented as the Riemannian curvature tensors on the Riemannian manifold $\widetilde{M}$ and submanifold Σ^m, respectively, then the Gauss equation is given by

$$\begin{aligned}\widetilde{R}(V_2, V_3, V_4, V_5) =& R(V_2, V_3, V_4, V_5) + g\big(\mathbf{A}(V_2, V_4), \mathbf{A}(V_3, V_5)\big) \\ &- g\big(\mathbf{A}(V_2, V_5), \mathbf{A}(V_3, V_4)\big),\end{aligned} \tag{3}$$

for any $V_2, V_3, V_4, V_5 \in \Gamma(T\widetilde{M})$, and $\mathbf{A}$ is the second fundamental form of Σ^m. A local orthonormal frame's $\{\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_m\}$ and the mean curvature vector $\mathcal{H}$ on Σ is defined by

$$||\mathcal{H}||^2 = \frac{1}{m^2} \sum_{r=m+1}^{d} \left(\sum_{b_1=1}^{m} \mathbf{A}_{b_1 b_1} \right)^2. \tag{4}$$

The scalar curvature of submanifold Σ^m, denoted by $\tau(T_x\Sigma^m)$, is formulated as follows:

$$\tau(T_x\Sigma^m) = \sum_{1 \leq b_1 < b_2 \leq m} K_{b_1 b_2} \tag{5}$$

where $K_{b_1 b_2} = K(\mathcal{E}_{b_1} \wedge \mathcal{E}_{b_2})$ is the sectional curvature of Σ^m. The first Equality (5) is proportionate to the following equation, which will be used often in later proofs:

$$2\tau(T_x\Sigma^m) = \sum_{1 \leq b_1 \neq b_2 \leq m} K_{b_1 b_2}. \tag{6}$$

Similarly, the scalar curvature $\tau(\prod_x)$ of a $\prod -$plane is given by

$$\tau(\prod_x) = \sum_{1 \leq b_1 < b_2 \leq m} K_{b_1 b_2}. \tag{7}$$

If the plane sections are spanned by $\mathcal{E}_{b_1}$ and $\mathcal{E}_{b_2}$ at x, then the sectional curvatures of the submanifold Σ^m and the Riemannian manifold $\widetilde{M}^d$ are denoted by $K_{b_1 b_2}$ and $\widetilde{K}_{b_1 b_2}$, respectively. If $K_{b_1 b_2}$ and $\widetilde{K}_{b_1 b_2}$ are spanned by $\{\mathcal{E}_{b_1} and \mathcal{E}_{b_2}\}$ at x, respectively, then using the Gauss Equations (3) and (5), we have

$$K(\mathcal{E}_{b_1} \wedge \mathcal{E}_{b_2}) = \widetilde{K}(\mathcal{E}_{b_1} \wedge \mathcal{E}_{b_2}) + \sum_{r=m+1}^{d} \left(\mathbf{A}^r_{b_1 b_1} \mathbf{A}^r_{b_2 b_2} - (\mathbf{A}^r_{b_1 b_2})^2 \right). \tag{8}$$

For more details regarding the above definitions, see [15–17].

3. Warped Product Manifolds

A definition of warped product manifolds was given by Bishop and O'Neill [18] by taking the negative curvature of the manifold. The product manifold $\Sigma^n = N_1 \times N_2$ of two Riemannian manifolds, N_1 and N_2, with matrices g_1 and g_2, respectively, is defined as a warped product as $\Sigma^n = N_1 \times_\mu N_2$ if the metric of Σ^n satisfies $g = g_1 + \mu^2 g_2$, where μ stands for the warping function defined on the base N_1. Of course, in this case, μ is constant, and Σ^n is a usual Riemannian product. Some important formulas were given by Bishop and O'Neill [18], including the following equations:

$$\nabla_{V_1} U_1 = \nabla_{U_1} V_1 = \frac{(U_1 \mu)}{\mu} V_1 \tag{9}$$

$$\mathcal{R}(U_1, U_2)V_1 = \frac{\mathcal{H}^\mu(U_1, V_1)}{\mu} U_2, \tag{10}$$

for any $U_1, U_2 \in \Gamma(TN_1)$ and $V_1 \in \Gamma(TN_2)$, where $\mathcal{H}^\mu$ is a Hessian tensor of μ such that

$$\mathcal{H}^\mu(V_2, V_3) = g(\nabla_{V_2} \nabla \mu, V_3).$$

We also have another interesting relationship regarding the connection ∇ on Σ^n that will be very useful for our proof in the main results.

$$g(\nabla \ln f, V_1) = V_1(\ln f). \tag{11}$$

The following remarks are consequences of the definition of warped products:

Remark 1. *A warped product manifold* $\Sigma^n = N_1 \times_f N_2$ *is said to be trivial or simply a Riemannian product manifold if the warping function* f *is a constant function along* N_1.

Remark 2. *If* $\Sigma^n = N_1 \times_f N_2$ *is a warped product manifold, then* N_1 *is totally geodesic, and* N_2 *is a totally umbilical submanifold of* Σ^n.

The $||\nabla\mu||^2$ gradient of the positive differential function μ for an orthonormal frame $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$ is then defined as

$$||\nabla\mu||^2 = \sum_{i=1}^{m} \big(\mathcal{E}_i(\mu)\big)^2. \tag{12}$$

The gradient $\vec{\nabla}\mu$ in [19] is given by

$$g(\vec{\nabla}\mu, V_2) = V_2\mu, \quad and \ \ \vec{\nabla}\mu = \sum_{i=1}^{m} \mathcal{E}_i(\mu)\mathcal{E}_i, \tag{13}$$

and the Laplacian $\Delta\mu$ of μ is defined as

$$\Delta\mu = -\sum_{i=1}^{m} \{(\nabla_{\mathcal{E}_i}\mathcal{E}_i)\mu - \mathcal{E}_i(\mathcal{E}_i(\mu))\} = \sum_{i=1}^{m} g(\nabla_{\mathcal{E}_i}\nabla\mu, \mathcal{E}_i) = trHess(\mu). \tag{14}$$

Remark 3. *It should be emphasized that we take into account Chen's opposite sign of [19] of the function's Laplacian* μ*, that is,* $\Delta\mu = div(\nabla\mu)$*. The sign convention for the Laplacian* Δ *adapted by the authors is* $\Delta = \frac{\partial^2}{\partial t^2}$ *on the real line.*

In addition, as the vector fields V_2 and V_4 are tangent to $N_1^{\beta_1}$ and $N_2^{\beta_2}$, respectively, we obtain

$$\begin{aligned} K(V_2 \wedge V_4) = g(R(V_2, V_4)V_2, V_4) &= (\nabla_{V_2}V_2)\ln hg(V_4, V_4) - g\big(\nabla_{V_2}((V_2 \ln\mu)V_5), V_4\big) \\ &= (\nabla_{V_2}V_2)\ln hg(V_4, V_4) - g\big(\nabla_{V_2}(V_2 \ln\mu)V_4 + (V_2 \ln\mu)\nabla_{V_2}V_4, V_4\big) \\ &= (\nabla_{V_2}V_2)\ln \mu g(V_4, V_4) - (V_2 \ln\mu)^2 - V_2(V_2 \ln\mu). \end{aligned} \tag{15}$$

By summing the vector fields with respect to the orthonormal frame's $\{\mathcal{E}_1, \cdots \mathcal{E}_n\}$, one obtains

$$\sum_{i=1}^{\beta_1}\sum_{j=1}^{\beta_2} K(\mathcal{E}_i \wedge \mathcal{E}_j) = \sum_{i=1}^{\beta_1}\sum_{j=1}^{\beta_2} \left((\nabla_{\mathcal{E}_i}\mathcal{E}_i)\ln\mu - \mathcal{E}_i(\mathcal{E}_i \ln\mu) - (\mathcal{E}_i \ln\mu)^2\right),$$

which implies that

$$\sum_{i=1}^{\beta_1}\sum_{j=1}^{\beta_2} K(\mathcal{E}_i \wedge \mathcal{E}_j) = -\frac{\beta_2\Delta\mu}{\mu}. \tag{16}$$

4. Main Results

We must also use a technique that is an invaluable tool for verifying our results. In the first case, assuming that the warped product submanifold is embedded in $\mathbb{S}^d$, and utilizing Theorem 1, we intend to obtain some identical conclusions regarding the warped product submanifold hypothesis, where pinching criteria on the second fundamental form shall be replaced by the warping function.

Using Theorem 1, the first significant outcome of this paper is now provided.

Theorem 2. *Let $\Psi : \Sigma^{\beta_1+\beta_2} = N_1^{\beta_1} \times_\mu N_2^{\beta_2} \longrightarrow \mathbb{S}^d$ be an $N_1^{\beta_1}$-minimal isometric embedding from a compact warped product submanifold $\Sigma^{\beta_1+\beta_2}$ into an d-dimensional sphere $\mathbb{S}^d$. If the following inequality satisfies*

$$3\mu\Delta\mu < 2\Big(\beta_2\|\nabla\mu\|^2 + \beta_1\mu^2\Big) \tag{17}$$

where $\Delta\mu$ and $\nabla\mu$ are the Laplacian and gradient of the warping function, respectively, then the following are true:

(a) There is no stable integral β_1-current flow in a warped product submanifold $\Sigma^{\beta_1+\beta_2}$.

(b) The i-th integral homology groups of $\Sigma^{\beta_1+\beta_2}$ vanish, which means

$$\mathbb{H}_{\beta_1}(\Sigma^{\beta_1+\beta_2}, \mathbb{Z}) = \mathbb{H}_{\beta_2}(\Sigma^{\beta_1+\beta_2}, \mathbb{Z}) = 0.$$

(c) If $\beta_1 = 1$, then the fundamental group $\pi_1(\Sigma)$ is null, i.e., $\pi_1(\Sigma) = 0$. Moreover, $\Sigma^{\beta_1+\beta_2}$ is a simply connected warped product manifold.

Proof. Suppose $\dim(N_1) = \beta_1$ and $\dim(N_2) = \beta_2$. Let $\{\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_{\beta_1}\}$, and $\{\mathcal{E}^*_{\beta_1+1}, \cdots, \mathcal{E}^*_m\}$ be orthonormal frames of TN_1 and TN_2, respectively. From the Gauss Equation (3) for the standard unit sphere $\mathbb{S}^d$, we then have

$$\begin{aligned}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}&\Big\{2||\mathbf{A}(\mathcal{E}_{b_1}, \mathcal{E}_{b_2})||^2 - g\big(\mathbf{A}(\mathcal{E}_{b_2}, \mathcal{E}_{b_2}), \mathbf{A}(\mathcal{E}_{b_1}, \mathcal{E}_{b_1})\big)\Big\}\\ &= \sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2} g\big(R(\mathcal{E}_{b_1}, \mathcal{E}_{b_2})\mathcal{E}_{b_1}, \mathcal{E}_{b_2}\big)\\ &- \sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2} g\big(\tilde{R}(\mathcal{E}_{b_1}, \mathcal{E}_{b_2})\mathcal{E}_{b_1}, \mathcal{E}_{b_2}\big) + \sum_{r=1}^{d}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}(\mathbf{A}^r_{b_1b_2})^2.\end{aligned} \tag{18}$$

From $R(\mathcal{E}_{b_1}, \mathcal{E}_{b_2})\mathcal{E}_{b_1} = \frac{\mathcal{H}^\mu(\mathcal{E}_{b_1}, \mathcal{E}_{b_1})}{\mu}\mathcal{E}_{b_2}$ in (10), by taking the trace over $N_1^{\beta_1}$ and $N_2^{\beta_2}$, we derive

$$\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2} g\big(R(\mathcal{E}_{b_1}, \mathcal{E}_{b_2})\mathcal{E}_{b_1}, \mathcal{E}_{b_2}\big) = \frac{\beta_2}{\mu}\sum_{b_1=1}^{\beta_1} g\big(\nabla_{\mathcal{E}_{b_1}}\nabla\mu, \mathcal{E}_{b_1}\big). \tag{19}$$

Thus, inserting (19) into the first term of the right-hand side of Equation (18), we derive

$$\begin{aligned}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}&\Big\{2||\mathbf{A}(\mathcal{E}_{b_1}, \mathcal{E}_{b_2})||^2 - g\big(\mathbf{A}(\mathcal{E}_{b_2}, \mathcal{E}_{b_2}), \mathbf{A}(\mathcal{E}_{b_1}, \mathcal{E}_{b_1})\big)\Big\}\\ &= \frac{\beta_2}{\mu}\sum_{b_1=1}^{\beta_1} g\big(\nabla_{\mathcal{E}_{b_1}}\nabla\mu, \mathcal{E}_{b_1}\big) + \sum_{r=1}^{d}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}(\mathbf{A}^r_{b_1b_2})^2\\ &- \sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2} g\big(\tilde{R}(\mathcal{E}_{b_1}, \mathcal{E}_{b_2})\mathcal{E}_{b_1}, \mathcal{E}_{b_2}\big).\end{aligned} \tag{20}$$

By calculating the Laplacian $\Delta\mu$ on $\Sigma^{\beta_1+\beta_2}$, one obtains

$$\Delta\mu = \sum_{i=1}^{m} g\big(\nabla_{\mathcal{E}_i}\nabla\mu, \mathcal{E}_i\big) = \sum_{b_1=1}^{\beta_1} g\big(\nabla_{\mathcal{E}_{b_1}}\nabla\mu, \mathcal{E}_{b_1}\big) + \sum_{b_2=1}^{\beta_2} g\big(\nabla_{\mathcal{E}_{b_2}}\nabla\mu, \mathcal{E}_{b_2}\big).$$

We know this from the warped product submanifold. From the hypothesis, $N_1^{\beta_1}$ is a geodesic submanifold in Σ^m. This implies that $\nabla\mu \in \mathfrak{X}(TN_1)$, and from the description of the warped product, we obtain

$$\Delta\mu = \frac{1}{\mu}\sum_{b_2=1}^{\beta_2} g(\mathcal{E}_{b_2},\mathcal{E}_{b_2})||\nabla\mu||^2 + \sum_{b_1=1}^{\beta_1} g\big(\nabla_{\mathcal{E}_{b_1}}\nabla\mu,\mathcal{E}_{b_1}\big).$$

By multiplying the above equation by $\frac{1}{\mu}$, we obtain

$$\frac{\Delta\mu}{\mu} = \frac{1}{\mu}\sum_{b_1=1}^{\beta_1} g\big(\nabla_{\mathcal{E}_{b_1}}\nabla\mu,\mathcal{E}_{b_1}\big) + \beta_2||\nabla(\ln\mu)||^2.$$

We rewrite the above equations as follows:

$$\frac{\beta_2}{\mu}\sum_{b_1=1}^{\beta_1} g\big(\nabla_{\mathcal{E}_{b_1}}\nabla\mu,\mathcal{E}_{b_1}\big) = \frac{\beta_2\Delta\mu}{\mu} - \beta_2^2||\nabla\ln\mu||^2. \tag{21}$$

Thus, from (20) and (21), one obtains

$$\begin{aligned}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}&\Big\{2||\mathbf{A}(\mathcal{E}_{b_1},\mathcal{E}_{b_2})||^2 - g\big(\mathbf{A}(\mathcal{E}_{b_2},\mathcal{E}_{b_2}),\mathbf{A}(\mathcal{E}_{b_1},\mathcal{E}_{b_1})\big)\Big\}\\ &= \sum_{r=m+1}^{d}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}(\mathbf{A}^r_{b_1b_2})^2 + \frac{\beta_2\Delta\mu}{\mu} - \beta_2^2||\nabla\ln\mu||^2\\ &\quad - \sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2} g\big(\tilde{R}(\mathcal{E}_{b_1},\mathcal{E}_{b_2})\mathcal{E}_{b_1},\mathcal{E}_{b_2}\big).\end{aligned}$$

Next, using the Gauss Equations (3) and (5) for the unit sphere $\mathbb{S}^d$, we find that

$$m^2||\mathcal{H}||^2 + m(m-1) = ||\mathbf{A}||^2 + \sum_{1\le C<B\le m} K(\mathcal{E}_C\wedge\mathcal{E}_B). \tag{22}$$

The warped product manifold $\Sigma^{\beta_1+\beta_2}$ can be expressed using the preceding equation, and using (4) for the mean curvature definition and (8), we obtain

$$\begin{aligned}\sum_{r=m+1}^{d}\left(\sum_{C=1}^{m}\mathbf{A}^r_{CC}\right)^2 &= \sum_{r=m+1}^{d}\sum_{i,j=1}^{\beta_1}(\mathbf{A}^r_{ij})^2 + \sum_{r=m+1}^{d}\sum_{a,b=1}^{\beta_2}(\mathbf{A}^r_{ab})^2 + 2\sum_{r=m+1}^{d}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}(\mathbf{A}^r_{b_1b_2})^2\\ &\quad + \sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}K(\mathcal{E}_{b_1}\wedge\mathcal{E}_{b_2}) + \sum_{1\le i<j\le\beta_1}K(\mathcal{E}_i\wedge\mathcal{E}_j) + \sum_{1\le a<b\le\beta_2}K(\mathcal{E}_a\wedge\mathcal{E}_b).\end{aligned} \tag{23}$$

Using (8) in the fourth term and (16) in the last two terms of the above equation, we derive

$$\begin{aligned}\sum_{r=m+1}^{d}\left(\sum_{C=1}^{m}\mathbf{A}^r_{CC}\right)^2 &= \sum_{r=m+1}^{d}\sum_{i,j=1}^{\beta_1}(\mathbf{A}^r_{ij})^2 + \sum_{r=m+1}^{d}\sum_{a,b=1}^{\beta_2}(\mathbf{A}^r_{ab})^2\\ &\quad + 2\sum_{r=m+1}^{d}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}(\mathbf{A}^r_{b_1b_2})^2 - \frac{\beta_2\Delta\mu}{\mu} - m(m-1)\\ &\quad + \sum_{1\le i<j\le\beta_1}\widetilde{K}(\mathcal{E}_i\wedge\mathcal{E}_j) + \sum_{1\le a<b\le\beta_2}\widetilde{K}(\mathcal{E}_a\wedge\mathcal{E}_b)\\ &\quad + \sum_{r=m+1}^{d}\sum_{1\le i<j\le\beta_1}\Big(\mathbf{A}^r_{ii}\mathbf{A}^r_{jj} - (\mathbf{A}^r_{ij})^2\Big) + \sum_{r=m+1}^{d}\sum_{1\le a<b\le\beta_2}\Big(\mathbf{A}^r_{aa}\mathbf{A}^r_{bb} - (\mathbf{A}^r_{ab})^2\Big).\end{aligned} \tag{24}$$

Thus, by modifying the previous equation and applying the sphere's curvature equation $\mathbb{S}^d$, one can obtain

$$\begin{aligned}\sum_{r=m+1}^{d}\left(\sum_{A=1}^{m}\mathbf{A}^r_{AA}\right)^2 =& \sum_{r=m+1}^{d}\sum_{i,j=1}^{\beta_1}(\mathbf{A}^r_{ij})^2+\sum_{r=m+1}^{d}\sum_{a,b=1}^{\beta_2}(\mathbf{A}^r_{ab})^2+2\sum_{r=m+1}^{d}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}(\mathbf{A}^r_{b_1b_2})^2\\
&-\frac{\beta_2\Delta\mu}{\mu}-\sum_{r=m+1}^{d}\sum_{1\le i<j\le\beta_1}(\mathbf{A}^r_{ij})^2+\sum_{r=m+1}^{d}\sum_{1\le i<j\le\beta_1}\mathbf{A}^r_{ii}\mathbf{A}^r_{jj}\\
&+\sum_{r=m+1}^{d}\left((\mathbf{A}^r_{11})^2+\cdots+(\mathbf{A}_{\beta_1\beta_2})^2\right)-\sum_{r=m+1}^{d}\left((\mathbf{A}^r_{11})^2+\cdots+(\mathbf{A}_{\beta_1\beta_2})^2\right)\\
&+\sum_{r=m+1}^{d}\sum_{1\le a<b\le\beta_2}\mathbf{A}^r_{aa}\mathbf{A}^r_{bb}-\sum_{r=m+1}^{d}\sum_{1\le a<b\le\beta_2}(\mathbf{A}^r_{ab})^2\\
&+\sum_{r=m+1}^{d}\left((\mathbf{A}^r_{\beta_1+1\beta_2+1})^2+\cdots+(\mathbf{A}_{mm})^2\right)\\
&-\sum_{r=m+1}^{d}\left((\mathbf{A}^r_{\beta_1+1\beta_2+1})^2+\cdots+(\mathbf{A}_{mm})^2\right)\\
&+\beta_1(\beta_1-1)+\beta_2(\beta_2-1)-m(m-1).\end{aligned}\tag{25}$$

The result of rearranging the preceding equation is

$$\begin{aligned}\sum_{r=m+1}^{d}\left(\sum_{A=1}^{m}\mathbf{A}^r_{AA}\right)^2 =& \sum_{r=m+1}^{d}\sum_{i,j=1}^{\beta_1}(\mathbf{A}^r_{ij})^2+\sum_{r=m+1}^{d}\sum_{a,b=1}^{\beta_2}(\mathbf{A}^r_{ab})^2+2\sum_{r=m+1}^{d}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}(\mathbf{A}^r_{b_1b_2})^2\\
&+\sum_{r=m+1}^{d}\left\{\sum_{1\le i<j\le\beta_1}\mathbf{A}^r_{ii}\mathbf{A}^r_{jj}+(\mathbf{A}^r_{11})^2+\cdots+(\mathbf{A}_{\beta_1\beta_1})^2\right\}\\
&-\sum_{r=m+1}^{d}\left\{\sum_{1\le i<j\le\beta_1}(\mathbf{A}^r_{ij})^2+(\mathbf{A}^r_{11})^2+\cdots+(\mathbf{A}_{\beta_1\beta_1})^2\right\}\\
&+\sum_{r=m+1}^{d}\left\{\sum_{1\le a<b\le\beta_2}\mathbf{A}^r_{aa}\mathbf{A}^r_{bb}+(\mathbf{A}^r_{\beta_1+1\beta_1+1})^2+\cdots+(\mathbf{A}_{mm})^2\right\}\\
&-\sum_{r=m+1}^{d}\left\{\sum_{1\le a<b\le\beta_2}(\mathbf{A}^r_{ab})^2+(\mathbf{A}^r_{\beta_1+1\beta_1+1})^2+\cdots+(\mathbf{A}_{mm})^2\right\}\\
&-\frac{\beta_2\Delta\mu}{\mu}+2\beta_1\beta_2.\end{aligned}$$

Verifying this using the binomial theorem is straightforward, and it is clear that the base manifold $N_1^{\beta_1}$ of a warped product manifold $N_1^{\beta_1}\times_\mu N_2^{\beta_2}$ is minimal. Therefore, we have

$$\begin{aligned}\sum_{r=m+1}^{d}\left(\sum_{A=p+1}^{m}\mathbf{A}^r_{AA}\right)^2 =& 2\beta_1\beta_2+\sum_{r=m+1}^{d}\sum_{i,j=1}^{\beta_1}(\mathbf{A}^r_{ij})^2+\sum_{r=m+1}^{d}\sum_{a,b=1}^{\beta_2}(\mathbf{A}^r_{ab})^2+2\sum_{r=m+1}^{d}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}(\mathbf{A}^r_{b_1b_2})^2\\
&+\sum_{r=m+1}^{d}\left((\mathbf{A}^r_{11})^2+\cdots+(\mathbf{A}_{\beta_1\beta_1})^2\right)-\sum_{r=m+1}^{d}\sum_{i,j=1}^{\beta_1}(\mathbf{A}^r_{ij})^2-\sum_{r=m+1}^{d}\sum_{a,b=1}^{\beta_2}(\mathbf{A}^r_{ab})^2\\
&+\sum_{r=m+1}^{d}\left((\mathbf{A}^r_{\beta_1+1\beta_1+1})^2+\cdots+(\mathbf{A}_{mm})^2\right)-\frac{\beta_2\Delta\mu}{\mu}.\end{aligned}\tag{26}$$

Since the base manifold $N_1^{\beta_1}$ of the warped product submanifold $N_1^{\beta_1} \times_\mu N_2^{\beta_2}$ is known to be minimal according to the theorem's hypothesis, that is, the partial mean curvature H_1 on $N_1^{\beta_1}$ vanishes, we can use this knowledge to determine that the V^{th} term on the right-hand side of Equation (26) is equal to zero and that the VII^{th} term is equal to the first term on the left side. Thus,

$$2\sum_{r=m+1}^{d}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}(\mathbf{A}^r_{b_1b_2})^2 = \frac{\beta_2\Delta\mu}{\mu} - 2\beta_1\beta_2. \tag{27}$$

From (22) and (27), we have

$$\begin{aligned}\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}&\Big\{2||\mathbf{A}(\mathcal{E}_{b_1},\mathcal{E}_{b_2})||^2 - g\big(\mathbf{A}(\mathcal{E}_{b_2},\mathcal{E}_{b_2}),\mathbf{A}(\mathcal{E}_{b_1},\mathcal{E}_{b_1})\big)\Big\}\\ &= \frac{\beta_2\Delta\mu}{\mu} - \beta_2^2||\nabla\ln\mu||^2 + \frac{\beta_2\Delta\mu}{2\mu} - \beta_1\beta_2\\ &- \sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2} g\big(\tilde{R}(\mathcal{E}_{b_1},\mathcal{E}_{b_2})\mathcal{E}_{b_1},\mathcal{E}_{b_2}\big).\end{aligned}$$

From Equation (2), one then obtains

$$\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2} g\big(\tilde{R}(\mathcal{E}_{b_1},\mathcal{E}_{b_2})\mathcal{E}_{b_1},\mathcal{E}_{b_2}\big) = -\beta_1\beta_2. \tag{28}$$

From this, we obtain

$$\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}\Big\{2||\mathbf{A}(\mathcal{E}_{b_1},\mathcal{E}_{b_2})||^2 - g\big(\mathbf{A}(\mathcal{E}_{b_2},\mathcal{E}_{b_2}),\mathbf{A}(\mathcal{E}_{b_1},\mathcal{E}_{b_1})\big)\Big\} = \frac{3\beta_2\Delta\mu}{2\mu} - \frac{\beta_2^2}{\mu^2}||\nabla\mu||^2. \tag{29}$$

Assuming (17) and (29), we obtain

$$\sum_{b_1=1}^{\beta_1}\sum_{b_2=1}^{\beta_2}\Big\{2||\mathbf{A}(\mathcal{E}_{b_1},\mathcal{E}_{b_2})||^2 - g\big(\mathbf{A}(\mathcal{E}_{b_2},\mathcal{E}_{b_2}),\mathbf{A}(\mathcal{E}_{b_1},\mathcal{E}_{b_1})\big)\Big\} < \beta_1\beta_2. \tag{30}$$

By applying Theorem 1 for a constant curvature $c = 1$, we find that there are no stable β_1-currents in $\Sigma^{\beta_1+\beta_2}$, and $H_{\beta_1}(\Sigma^{\beta_1+\beta_2},\mathbb{Z}) = H_{\beta_2}(\Sigma^{\beta_1+\beta_2},\mathbb{Z}) = 0$, which satisfies Proofs (a) and (b) of the theorem. On the other hand, if in (29) we make the substitution $\beta_1 = 1$, then we obtain

$$\sum_{b_2=2}^{m}\Big\{2||\mathbf{A}(\mathcal{E}_1,\mathcal{E}_{b_2})||^2 - g\big(\mathbf{A}(\mathcal{E}_{b_2},\mathcal{E}_{b_2}),\mathbf{A}(\mathcal{E}_1,\mathcal{E}_1)\big)\Big\} = \frac{3\beta_2\Delta\mu}{2\mu} - \frac{\beta_2}{\mu^2}||\nabla\mu||^2 \tag{31}$$

If the pinching condition (17) for $\beta_1 = 1$ and $\beta_2 = m-1$ holds, then we obtain

$$\sum_{b_2=2}^{m}\Big\{2||\mathbf{A}(\mathcal{E}_1,\mathcal{E}_{b_2})||^2 - g\big(\mathbf{A}(\mathcal{E}_{b_2},\mathcal{E}_{b_2}),\mathbf{A}(\mathcal{E}_1,\mathcal{E}_1)\big)\Big\} < (m-1). \tag{32}$$

There are no stable 1-currents in $\Sigma^{1+\beta_2}$ and $H_1(\Sigma^{1+\beta_2},\mathbb{Z} = H_{n-1}(\Sigma^{1+\beta_2},\mathbb{Z}) = 0$. Let us assume that $\pi_1(\Sigma)$ does not equal 0. The traditional theorem, which uses the findings of Cartan and Hadamard, claims that there is a minimal closed geodesic in any non-trivial homotopy class in $\pi_1(\Sigma)$, which contradicts itself when applied to the compactness of $\Sigma^{1+\beta_2}$. Consequently, $\pi_1(\Sigma) = 0$. The theorem's third component can be expressed as follows. This Riemannian manifold is simply connected if the finite basic group for any Riemannian manifold is null. Therefore, $\Sigma^{\beta_1+\beta_2}$ is simply connected. □

Inspired by geometric rigidity, the second mission of this study is to show a novel vanishing result for compact warped product submanifolds utilizing the Ricci curvature and the eigenvalue of the warping function's Laplacian. The following theorem is detailed below.

Theorem 3. *If the warping function μ is an eigenfunction of the Laplacian of $\Sigma^{\beta_1+\beta_2}$ associated with the first positive eigenvalue λ_1 under the same language of Theorem 2, together they satisfy the following inequality:*

$$\|\nabla^2\mu\|^2 + Ric(\nabla\mu, \nabla\mu) + \frac{\lambda_1(3\lambda_1 + 2\beta_1)\mu^2}{2\beta_2} > 0. \tag{33}$$

Thus,

(a) There is no stable integral β_1-current flow in a warped product submanifold $\Sigma^{\beta_1+\beta_2}$.
(b) The i-th integral homology groups of $\Sigma^{\beta_1+\beta_2}$ with integer coefficients vanish; i.e.,

$$\mathbb{H}_{\beta_1}(\Sigma^{\beta_1+\beta_2}, \mathbb{Z}) = \mathbb{H}_{\beta_2}(\Sigma^{\beta_1+\beta_2}, \mathbb{Z}) = 0.$$

(c) The fundamental group $\pi_1(\Sigma)$ is null, i.e., $\pi_1(\Sigma) = 0$. Furthermore, $\Sigma^{\beta_1+\beta_2}$ is a simply connected warped product submanifold.

Proof. If μ is the first eigenfunction of the Laplacian $\Delta\mu = div(\nabla\mu)$ of $\Sigma^{\beta_1+\beta_2}$ associated with the first non-zero eigenvalue λ_1, that is, $\Delta\mu = -\lambda_1\mu$, then we recall the Bochner formula (see, e.g., [20]), which declares that the next connection is true for a differentiable function μ that is defined on a Riemannian manifold:

$$\frac{1}{2}\Delta\|\nabla\mu\|^2 = \|\nabla^2\mu\|^2 + Ric(\nabla\mu, \nabla\mu) + g\big(\nabla\mu, \nabla(\Delta\mu)\big).$$

Using the Stokes theorem to integrate the preceding equation, we arrive at

$$\begin{aligned}\int_{\Sigma^{\beta_1+\beta_2}} \|\nabla^2\mu\|^2 dV + &\int_{\Sigma^{\beta_1+\beta_2}} Ric(\nabla\mu, \nabla\mu) dV \\ &= -\int_{\Sigma^{\beta_1+\beta_2}} g\big(\nabla\mu, \nabla(\Delta\mu)\big) dV\end{aligned} \tag{34}$$

Now, using $\Delta\mu = -\lambda_1\mu$ and making a change in Equation (34), we derive

$$\int_{\Sigma^{\beta_1+\beta_2}} \|\nabla\mu\|^2 dV = \frac{1}{\lambda_1}\bigg(\int_{\Sigma^{\beta_1+\beta_2}} \|\nabla^2\mu\|^2 dV + \int_{\Sigma^{\beta_1+\beta_2}} Ric(\nabla\mu, \nabla\mu) dV\bigg). \tag{35}$$

If (33) holds, then one obtains

$$\int_{\Sigma^{\beta_1+\beta_2}} \Big\{\|\nabla^2\mu\|^2 + Ric(\nabla\mu, \nabla\mu)\Big\} dV + \frac{\lambda_1(3\lambda_1 + 2\beta_1)}{2\beta_2}\int_{\Sigma^{\beta_1+\beta_2}} \mu^2 dV > 0. \tag{36}$$

By substituting Equation (36) in (35), we obtain

$$-\frac{\lambda_1(3\lambda_1 + 2\beta_1)}{2\beta_2}\int_{\Sigma^{\beta_1+\beta_2}} \mu^2 dV < \lambda_1 \int_{\Sigma^{\beta_1+\beta_2}} \|\nabla\mu\|^2 dV,$$

which implies that

$$-3\lambda_1 \int_{\Sigma^{\beta_1+\beta_2}} \mu^2 dV < 2\beta_1 \int_{\Sigma^{\beta_1+\beta_2}} \mu^2 dV + 2\beta_2 \int_{\Sigma^{\beta_1+\beta_2}} \|\nabla\mu\|^2 dV. \tag{37}$$

Now, using $\Delta = -\lambda_1 \mu$ on the left-hand side of Equation (37), we arrive at

$$\int_{\Sigma^{\beta_1+\beta_2}} \left\{ 3h\Delta\mu - 2\beta_2 \|\nabla\mu\|^2 - 2\beta_1\mu^2 \right\} dV < 0. \tag{38}$$

One then obtains

$$3\mu\Delta\mu < 2\beta_2 \|\nabla\mu\|^2 + 2\beta_1\mu^2. \tag{39}$$

Finally, we arrive at the conclusion of our theorem using the preceding equation as well as Theorem 2. The theorem's proof is now complete. □

Riemannian manifolds with vanishing Ricci curvatures are known as Ricci-flat manifolds. In contrast to Einstein manifolds, Ricci-flat manifolds do not require the cosmological constant to disappear. For Riemannian manifolds of any dimension, with a vanishing cosmological constant, Ricci-flat manifolds are vacuum solutions to the physics equivalents of Einstein's equations. Hence, we regard the warped product submanifold's base as Ricci-flat. We give the following result from Theorem 3.

Theorem 4. *If the warping function μ is an eigenfunction of the Laplacian of $\Sigma^{\beta_1+\beta_2}$ associated with the first positive eigenvalue λ_1 under the same statement of Theorem 2 with assumptions that base manifold is Ricci-flat, then the subsequent stringent inequality holds.*

$$\|\nabla^2\mu\|^2 + \frac{\lambda_1(3\lambda_1 + 2\beta_1)\mu^2}{2\beta_2} > 0. \tag{40}$$

Then, Statements (a), (b), and (c) in Theorem 2 are satisfied.

Proof. As we know that the base manifold $N_1^{\beta_1}$ is Ricci-flat, we then have

$$Ric(\nabla\mu, \nabla\mu) = 0. \tag{41}$$

Thus, by inserting the above-mentioned condition in (33), we obtain the desired result. □

As a quick implementation of Theorem 3, we can provide the following.

Theorem 5. *Let us assume that $\Psi : \Sigma^{\beta_1+\beta_2} = N_1^{\beta_1} \times_\mu N_2^{\beta_2} \longrightarrow \mathbb{S}^{\beta_1+\beta_2+k_1}$ is an $N_1^{\beta_1}$-minimal isometric embedding from a compact warped product submanifold $\Sigma^{\beta_1+\beta_2}$ into an $(\beta_1+\beta_2+k_1)$-dimensional sphere $\mathbb{S}^{\beta_1+\beta_2+k_1}$ that satisfies the following inequality:*

$$\begin{aligned} \int_{\Sigma^{\beta_1+\beta_2}} \|\nabla^2\mu\|^2 dV < \int_{\Sigma^{\beta_1+\beta_2}} \sum_{i=1}^{\beta_1} \|\mathbf{A}(\nabla\mu, \mathcal{E}_i)\|^2 dV \\ + \frac{(\beta_1 - 1 - \lambda_1)(3\lambda_1 + 2\beta_1)}{2\beta_2} \int_{\Sigma^{\beta_1+\beta_2}} \mu^2 dV. \end{aligned} \tag{42}$$

Statements (a), (b), and (c) in Theorem 2 are satisfied. Moreover, $\{\mathcal{E}_i\}$ are orthonormal frames for the base $N_1^{\beta_1}$.

Proof. As we are aware, $\Sigma^{\beta_1+\beta_2}$ is an $N_1^{\beta_1}$-minimal compact warped product submanifold. Then, from the Gauss equation, one obtains

$$R^i_{jkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_{r=1}^{k} \left(\mathbf{A}^r_{ik}\mathbf{A}^r_{jl} - \mathbf{A}^r_{il}\mathbf{A}^r_{jk} \right),$$

which implies the following:

$$R^i_{jij} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} + \sum_{r=1}^{k}\left(\mathbf{A}^r_{ii}\mathbf{A}^r_{jj} - \mathbf{A}^r_{ij}\mathbf{A}^r_{ji}\right). \tag{43}$$

Taking into account that $N_1^{\beta_1}$ is a minimal submanifold and using the argument of the Ricci curvature for a unit sphere, we obtain

$$Ric(\mathcal{E}_i, \mathcal{E}_j) = (\beta_1 - 1)\delta_{ij} - \sum_{r=1}^{k}\sum_{l=1}^{\beta_1}\mathbf{A}^r_{il}\mathbf{A}^r_{jl}$$

The above equation yields that

$$Ric(\mu_i\mathcal{E}_i, \mathcal{E}_j\mu_j) = (\beta_1 - 1)\delta_{ij}\mu_i\mu_j - \sum_{r=1}^{k}\sum_{l=1}^{\beta_1}\mathbf{A}^r_{il}\mathbf{A}^r_{jl}\mu_i\mu_j. \tag{44}$$

Using Equation (44), we obtain

$$Ric(\nabla\mu, \nabla\mu) = (\beta_1 - 1)\|\nabla\mu\|^2 - \sum_{i=1}^{\beta_1}\|\mathbf{A}(\nabla\mu, \mathcal{E}_i)\|^2.$$

Putting the preceding equation into practice in (35), we obtain

$$\int_{\Sigma^{\beta_1+\beta_2}}\sum_{i=1}^{\beta_1}\|\mathbf{A}(\nabla\mu, \mathcal{E}_i)\|^2 dV = \int_{\Sigma^{\beta_1+\beta_2}}\|\nabla^2\mu\|^2 dV + (\beta_1 - 1 - \lambda_1)\int_{\Sigma^{\beta_1+\beta_2}}\|\nabla\mu\|^2 dV. \tag{45}$$

If our assumption in (42) is satisfied, then

$$\int_{\Sigma^{\beta_1+\beta_2}}\|\nabla^2\mu\|^2 dV < \int_{\Sigma^{\beta_1+\beta_2}}\sum_{i=1}^{\beta_1}\|\mathbf{A}(\nabla\mu, \mathcal{E}_i)\|^2 dV + \frac{(\beta_1 - 1 - \lambda_1)(3\lambda_1 + 2\beta_1)}{2\beta_2}\int_{\Sigma^{\beta_1+\beta_2}}\mu^2 dV.$$

The following form can be used to express the previously mentioned equation using $\Delta\mu = -\lambda_1\mu$:

$$\begin{aligned}\frac{3(\beta_1 - 1 - \lambda_1)}{2\beta_2}\int_{\Sigma^{\beta_1+\beta_2}}\mu\Delta\mu dV + \int_{\Sigma^{\beta_1+\beta_2}}\|\nabla^2\mu\|^2 dV \\ < \int_{\Sigma^{\beta_1+\beta_2}}\sum_{i=1}^{\beta_1}\|\mathbf{A}(\nabla\mu, \mathcal{E}_i)\|^2 dV \\ + \frac{\beta_1(\beta_1 - 1 - \lambda_1)}{\beta_2}\int_{\Sigma^{\beta_1+\beta_2}}\mu^2 dV.\end{aligned}$$

Including the previous equation in (45), we derive that

$$\begin{aligned}\frac{3(\beta_1 - 1 - \lambda_1)}{2\beta_2}\int_{\Sigma^{\beta_1+\beta_2}}\mu\Delta\mu dV < (\beta_1 - 1 - \lambda_1)\int_{\Sigma^{\beta_1+\beta_2}}\|\nabla\mu\|^2 dV \\ + \frac{\beta_1(\beta_1 - 1 - \lambda_1)}{\beta_2}\int_{\Sigma^{\beta_1+\beta_2}}\mu^2 dV,\end{aligned}$$

which implies the following from the above equation:

$$3\mu\Delta\mu < 2\beta_2\|\nabla\mu\|^2 + 2\beta_1\mu^2. \tag{46}$$

Consequently, it is the same as Equation (17), and we achieve the desired outcome, i.e., (42). This completes the proof of the corollary. □

Another intriguing outcome that can be attained as a consequence of Theorem 5 is the following:

Corollary 1. *Under the same assumption as Theorem 5, if* $\nabla \mu \in Ker\mathbf{A}$ *holds with*

$$\int_{\Sigma^{\beta_1+\beta_2}} \|\nabla^2 \mu\|^2 dV < \frac{(\beta_1 - 1 - \lambda_1)(3\lambda_1 + 2\beta_1)}{2\beta_2} \int_{\Sigma^{\beta_1+\beta_2}} \mu^2 dV, \tag{47}$$

then the same statements as (a), (b), and (c) in Theorem 2 are satisfied.

Proof. Using the hypothesis of the corollary $\nabla \mu \in Ker\mathbf{A}$, we obtain $\mathbf{A}(\nabla \mu, \mathcal{E}_i) = 0$. Using this condition in (42), we can easily obtain the desired outcome. □

5. Conclusions Remarks

In the present paper, we have found sufficient conditions that have given us information regarding vanishing homology groups and fundamental groups. The homology groups were initially defined in algebraic topology and are a general way to associate a sequence of algebraic objects, such as Abelian groups or modules. If the two shapes are distinguished by examining their holes, this idea can force the definition of the homology groups and homology; it was originally a rigorous mathematical method for defining and categorizing holes in a manifold. The most constructive topological invariants for providing the algebraic summary of the manifold are the homology groups of a manifold. These homologies have many applications and are helpful in finding deep topological information regarding the connected components, holes, tunnels, and dimensions of the manifold. Indeed, homology theory has its applications in gene expression data, protein docking, image segmentation, and root architecture; see [21–23]. Furthermore, they can provide some significant examples of singularity structures in liquid crystals, systems in low-dimensional statistical mechanics, and physical phase transitions [24]. Moreover, the concept of space–time in general relativity uses warped product manifolds. There are two well-known product spaces with warped products: standard static space–times and the generalization of Robertson–Walker space–times [25]. Particularly in mathematical physics, general relativity relies extensively on differential topological approaches [26]. specifically how quantum gravity uses space–time homology [27–29]. The results of this work can be used in physical applications because they are related to the warped product manifold and homotopy–homology theory. We can extend the above work where the curvature is positive or zero to generalized spherical structures.

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mathematics

Article

Conformal Transformations on General (α, β)-Spaces

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Abstract: In this paper, we study conformal transformations between two almost regular general (α, β)-metrics. By using the method of special coordinate system, the necessary and sufficient conditions for conformal transformations preserving the mean Landsberg curvature are obtained. Further, a rigidity theorem for regular general (α, β)-metrics is proved.

Keywords: general (α, β)-spaces; mean Landsberg curvature; conformal transformations

MSC: 53C30; 53C60

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1. Introduction

In Finsler geometry, the Weyl theorem states that the projective and conformal properties of a Finsler space determine the metric properties uniquely [1]. Therefore, the conformal properties of Finsler metrics deserve extra attention. Let F and $\tilde{F}$ be two Finsler metrics on a manifold M. A conformal transformation between F and $\tilde{F}$ is defined by L: $F \to \tilde{F}$, $\tilde{F} = e^{\sigma(x)} F$, where the conformal factor $\sigma := \sigma(x)$ is a scalar function on M. The metrics F and $\tilde{F}$ are conformally related. If σ is a constant, then the conformal transformation is called a homothetic transformation.

A natural problem is knowing how to determine, given a Finsler metric with some properties on a manifold M, all conformally related Finsler metrics with the given properties. B*á*cs*ó*-Cheng [2] characterized conformal transformations that preserve the Riemann curvature, the Ricci curvature, the (mean) Landsberg curvature, or the **S**-curvature, respectively. Chen-Cheng-Zou [3] proved that if both conformally related (α, β)-metrics are of the Douglas type or of isotropic **S**-curvature, then the conformal transformations between them are homothetic. Later, Chen-Liu [4] characterized conformal transformations between two almost regular (α, β)-metrics that preserve the mean Landsberg curvature. Furthermore, they proved that conformal transformations between non-Riemannian regular (α, β)-metrics, which preserve the mean Landsberg curvature, must be homothetic.

Li-Shen [5] studied (α, β)-metrics with the mean Landsberg curvature and obtained its characterizing equation. Cheng–Wang–Wang [6] characterized (α, β)-metrics with the relative isotropic mean Landsberg curvature. Zou-Cheng [7] explored an (α, β)-metric whose $\phi(s)$ is a polynomial about s, and they proved that it has vanishing mean Landsberg curvature if and only if it is a Berwald metric. Under the condition that the 1-form β is a conformal field of the Riemannian metric α, Behzadi–Rafiei [8] proved that the general (α, β)-metric has vanishing mean Landsberg curvature if and only if it is of the Landsberg type. Najafi–Saberali [9] explored the special (α, β)-metric and obtained that it has an isotropic mean Landsberg curvature that is equivalent to that its isotropic Landsberg curvature.

The general (α, β)-metric was first introduced by Yu-Zhu [10] in the following form:

$$F = \alpha\phi(b^2, s), \quad s = \frac{\beta}{\alpha}, \tag{1}$$

where $\phi = \phi(b^2, s)$ is a $\mathbb{C}^\infty$ positive function, $\alpha = \alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = \beta(x, y) = b_i(x)y^i$ is a 1-form, and $b := \|\beta\|_\alpha$. It is known that $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is a Finsler metric with $b < b_0$ if and only if $\phi(b^2, s)$ is a positive $\mathbb{C}^\infty$ function satisfying [10]

$$\phi - s\phi_s > 0, \quad \phi - s\phi_s + (b^2 - s^2)\phi_{ss} > 0$$

for $n \geq 3$. If we consider a 1-form β with $b \leq b_0$ where $b_0 := \sup_{x \in M} b$, then $F = \alpha\phi(b^2, s)$ might be singular in the two extremal directions $y \in T_x M$ with $\beta(x, y) = \pm b_0 \alpha(x, y)$. Such metrics are called almost regular general (α, β)-metrics. In particular, when $\phi = \phi(s)$ in (1), the Finsler metric $F = \alpha\phi(s)$ is called an (α, β)-metric.

Note that $(.)_s$ denotes the partial derivative of the quantity $(.)$ with respect to s.

In this paper, we mainly study conformal transformations preserving the mean Landsberg curvature. It is known that a homothetic transformation must preserve the mean Landsberg curvature. Thus, we focus on non-homothetic conformal transformations.

Theorem 1. *Let F be an almost regular general (α, β)-metric on an $n(\geq 3)$-dimensional manifold M. Assume that F and $\tilde{F}$ are two conformally related metrics with the conformal factor $\sigma = \sigma(x)$. Then, F and $\tilde{F}$ have the same mean Landsberg curvature if and only if one of the following cases holds:*

(1) ϕ must be

$$\phi = k_2 s^{1 - \frac{1}{b^2 k_1}} (b^2 - s^2)^{\frac{1}{2b^2 k_1}},$$

where $k_1 = k_1(b^2)$ and $k_2 = k_2(b^2) > 0$ are arbitrary differential functions. In this case, the conformal factor is arbitrary;

(2) The conformal factor σ satisfies $\sigma_i(x) = \frac{\sigma_b}{b^2} b_i(x)$, and ϕ satisfies

$$\Phi = \frac{k_3 \Delta^{\frac{3}{2}}}{\sqrt{b^2 - s^2}}, \tag{2}$$

where $\sigma_i := \frac{\partial \sigma}{\partial x^i}$, $\sigma_b := \sigma_i b^i$, $b^2 := a^{ij} b_i b_j$, $Q := \frac{\phi_s}{\phi - s\phi_s}$, $\Delta := 1 + sQ + (b^2 - s^2)Q_s$, $\Phi := (1 + sQ)(b^2 - s^2)Q_{ss} + (n\Delta + 1 + sQ)(Q - sQ_s)$, and $k_3 = k_3(b^2)$ is a differential positive function.

This theorem generalizes the results obtained by Chen-Liu about conformal transformations preserving the mean Landsberg curvature on (α, β)-spaces [4].

Based on Theorem 1, we obtain a rigidity theorem for regular general (α, β)-metrics as follows.

Theorem 2. *Let F be a regular general (α, β)-metric on an $n(\geq 3)$-dimensional manifold M. Assume that F and $\tilde{F}$ are two conformally related metrics. Then, F and $\tilde{F}$ have the same mean Landsberg curvature if and only if F is Riemannian.*

2. Preliminaries

Let F be a non-Riemannian Finsler metric on a manifold M of dimension $n(\geq 3)$. Its spray coefficients G^i are defined by

$$G^i := \frac{1}{4} g^{ij} \Big\{ \big[F^2\big]_{x^k y^j} y^k - \big[F^2\big]_{x^j} \Big\},$$

where $g_{ij} := \frac{1}{2}(F^2)_{y^i y^j}$ and $(g^{ij}) := (g_{ij})^{-1}$.

The Cartan tensor is defined by $\mathbf{C} := C_{ijk} dx^i \otimes dx^j \otimes dx^k$, where

$$C_{ijk} := \frac{1}{4} \frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

The mean Cartan torsion $\mathbf{I} := I_i dx^i : T_xM \to \mathbf{R}$ is defined by

$$I_i := g^{jk} C_{ijk}.$$

Deicke's theorem shows that $\mathbf{I} = 0$ if and only if F is Riemannian.

The Landsberg curvature $\mathbf{L} = L_{ijk} dx^i \otimes dx^j \otimes dx^k$ is a horizontal tensor on $TM \backslash \{0\}$, defined by

$$L_{ijk} := -\frac{1}{2} F F_{y^l} \frac{\partial^3 G^l}{\partial y^i \partial y^j \partial y^k}.$$

The mean Landsberg curvature is defined by

$$\mathbf{J} := J_i dx^i, \quad J_i := g^{jk} L_{ijk}.$$

Lemma 1 ([10])**.** *For a general* (α, β)*-metric* $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$*, the coefficients of the fundamental tensor are*

$$g_{ij} = \rho a_{ij} + \bar{\rho} b_i b_j + \frac{1}{\alpha} \tilde{\rho} (b_i y_j + b_j y_i) - \frac{1}{\alpha^2} s \tilde{\rho} y_i y_j,$$

where

$$y_i := a_{ij} y^j, \quad \rho := \phi(\phi - s\phi_s), \quad \bar{\rho} := \phi\phi_{ss} + \phi_s\phi_s, \quad \tilde{\rho} := (\phi - s\phi_s)\phi_s - s\phi\phi_{ss}.$$

And $(g^{ij}) := (g_{ij})^{-1}$ *are as follows:*

$$g^{ij} = \rho^{-1} [a^{ij} + \eta b^i b^j + \bar{\eta} \alpha^{-1} (b^i y^j + b^j y^i) + \tilde{\eta} \alpha^{-2} y^i y^j],$$

where $b^i := a^{ij} b_j$*,*

$$\eta := -\frac{\phi}{\phi - s\phi_s + (b^2 - s^2)\phi_{ss}}, \quad \bar{\eta} := -\frac{(\phi - s\phi_s)\phi_s - s\phi\phi_{ss}}{\phi[\phi - s\phi_s + (b^2 - s^2)\phi_{ss}]},$$
$$\tilde{\eta} := \frac{[s\phi + (b^2 - s^2)\phi_s][(\phi - s\phi_s)\phi_s - s\phi\phi_{ss}]}{\phi^2[\phi - s\phi_s + (b^2 - s^2)\phi_{ss}]}.$$

Lemma 2 ([11])**.** *For a general* (α, β)*-metric* $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$*, the coefficients of the Cartan tensor and the mean Cartan torsion are as follows:*

$$\begin{aligned}
C_{ijk} = & \frac{1}{2} [\alpha^{-1} \tilde{\rho} (a_{ij} b_k + a_{ik} b_j + a_{jk} b_i) - \alpha^{-2} s \tilde{\rho} (a_{ij} y_k + a_{ik} y_j + a_{jk} y_i) \\
& + \alpha^{-4} (3s\tilde{\rho} - s^3 \bar{\rho}_s) y_i y_j y_k + \alpha^{-3} (s^2 \bar{\rho}_s - \tilde{\rho}) (b_i y_j y_k + b_j y_i y_k + b_k y_i y_j) \\
& - \alpha^{-2} s \bar{\rho}_s (b_i b_j y_k + b_j b_k y_i + b_k b_i y_j) + \alpha^{-1} \bar{\rho}_s b_i b_j b_k], \\
I_i = & \frac{1}{2\alpha^2 \rho} A (\alpha b_i - s y_i),
\end{aligned}$$

where $A := \frac{\phi^2 \Phi}{\Delta (1 + sQ)^2}$.

By Deicke's theorem, a general (α, β)-metric is Riemannian if and only if $A = 0$.

3. The Proof of Main Theorems

Before proving Theorem 1, we need following Lemmas.

Lemma 3 ([2])**.** *Let F be a Finsler metric on a manifold M. Assume that F and* $\tilde{F}$ *are two conformally related metrics with the conformal factor* $\sigma = \sigma(x)$*. Then, their mean Landsberg curvature must satisfy*

$$\tilde{J}_i = J_i + \sigma_0 I_i + F^2 \sigma^j I_{i.j} + (\sigma^j I_j) y_i^F - F^2 \sigma^j I_k C^k_{ij},$$

where $\sigma_0 := \sigma_k y^k$, $\sigma^i := g^{ik}\sigma_k$, $I_{i.j} := \frac{\partial I_i}{\partial y^j}$, $y_i^F := g_{ij}y^j$, $C_{ij}^k := g^{kl}C_{ijl}$.

Based on Lemma 3, F and $\tilde{F} = e^{\sigma(x)}F$ have the same mean Landsberg curvature if and only if the following holds:

$$\sigma_0 I_i + F^2\sigma^j I_{i.j} + (\sigma^j I_j)y_i^F - F^2\sigma^j I_k C_{ij}^k = 0.$$

Assume that F is a general (α, β)-metric. By direct computations, the above equation is equivalent to

$$(T_1\sigma_0 + \alpha T_2\sigma_b)y_i + \alpha(T_3\sigma_0 + \alpha T_4\sigma_b)b_i + \alpha^2 T_5\sigma_i = 0, \tag{3}$$

where

$$\begin{aligned}
T_1 &= \left[\frac{(s+b^2Q)(1+sQ+2s^2Q_s)}{(1+sQ)\Delta} + \frac{3s(1+b^2Q_s)}{\Delta} - \frac{2s}{1+sQ} - \frac{s(s+b^2Q)\Delta_s}{\Delta^2}\right]\phi^2 A \\
&\quad + \frac{2s(b^2Q+s)}{\Delta}\phi^2 A_s, \\
T_2 &= \frac{[(1+sQ)s\Delta_s - \Delta(sQ+5s^2Q_s)]}{\Delta^2}\phi^2 A - \frac{2s(1+sQ)}{\Delta}\phi^2 A_s, \\
T_3 &= \left[-\frac{2Q_s(2s^2+b^2sQ-b^2)}{(1+sQ)\Delta} - \frac{2b^2Q_s+s^2Q_s-sQ}{\Delta} + \frac{(s+b^2Q)\Delta_s}{\Delta^2}\right]\phi^2 A \\
&\quad - \frac{2(b^2Q+s)}{\Delta}\phi^2 A_s, \\
T_4 &= -\frac{[(1+sQ)\Delta_s - \Delta(Q+5sQ_s)]}{\Delta^2}\phi^2 A + \frac{2(1+sQ)}{\Delta}\phi^2 A_s, \\
T_5 &= -\frac{s\Delta + (s+b^2Q)}{\Delta}\phi^2 A.
\end{aligned}$$

Note that $T_1 + sT_3 + T_5 = 0$ and $T_2 + sT_4 = 0$ hold.

Lemma 4. *Let a positive C^∞ function $\phi = \phi(b^2, s)$ satisfy $s\Delta + s + b^2Q = 0$. Then,*

$$\phi = k_2 s^{1-\frac{1}{b^2k_1}}(b^2 - s^2)^{\frac{1}{2b^2k_1}},$$

where $k_1 = k_1(b^2)$ and $k_2 = k_2(b^2) > 0$ are arbitrary differential functions.

Proof. $s\Delta + s + b^2Q = 0$ is equivalent to

$$(b^2 - s^2)(Q + sQ_s) + 2s(1+sQ) = 0.$$

It can be rewritten as

$$\left(\frac{b^2 - s^2}{1+sQ}\right)_s = 0.$$

Integrating this equation with respect to s yields

$$Q = \frac{k_1(b^2 - s^2) - 1}{s}, \tag{4}$$

where $k_1 = k_1(b^2)$ is an arbitrary differential function. Since $Q := \frac{\phi_s}{\phi - s\phi_s}$, the above equation leads to

$$\frac{\phi_s}{\phi} = \frac{Q}{1+sQ}.$$

It is equivalent to

$$(\ln \phi)_s = \frac{Q}{1+sQ}.$$

Integrating the above equation with respect to s yields

$$\phi = k_2 \exp\left(\int \frac{Q}{1+sQ} ds\right),$$

where $k_2 = k_2(b^2) > 0$ is a C^∞ function.

Substituting (4) into the above equation yields

$$\phi = k_2 s^{1-\frac{1}{b^2 k_1}} (b^2 - s^2)^{\frac{1}{2b^2 k_1}}.$$

This completes the proof of Lemma 4. □

Lemma 5. *Let a positive C^∞ function $\phi = \phi(b^2, s)$ satisfy $sT_1 + b^2 T_2 = 0$. Then,*

$$\Phi = \frac{k_3 \Delta^{\frac{3}{2}}}{\sqrt{b^2 - s^2}},$$

where $k_3 = k_3(b^2)$ is an arbitrary positive function.

Proof. The direct computation yields

$$sT_1 + b^2 T_2 = \frac{2s^2\Delta[1 + sQ - 2(b^2 - s^2)Q_s] + s(1+sQ)(b^2 - s^2)\Delta_s}{(1+sQ)\Delta^2}\phi^2 A - \frac{2s(b^2 - s^2)}{\Delta}\phi^2 A_s.$$

Thus, $sT_1 + b^2 T_2 = 0$ is equivalent to

$$\{2s\Delta[1 + sQ - 2(b^2 - s^2)Q_s] + (1+sQ)(b^2 - s^2)\Delta_s\}A - 2(b^2 - s^2)(1+sQ)\Delta A_s = 0.$$

It can be rewritten as

$$\frac{A_s}{A} = \frac{Q - sQ_s}{1+sQ} - \frac{Q}{1+sQ} - \frac{sQ_s}{1+sQ} + \frac{2s\Delta + (b^2 - s^2)\Delta_s}{2(b^2 - s^2)\Delta},$$

i.e.,

$$\frac{A_s}{A} = \frac{\rho_s}{\rho} - \frac{\phi_s}{\phi} - \frac{s\phi_{ss}}{\phi - s\phi_s} + \frac{2s\Delta + (b^2 - s^2)\Delta_s}{2(b^2 - s^2)\Delta}.$$

That means

$$(\ln A)_s = (\ln \rho)_s - (\ln \phi)_s + [\ln(\phi - s\phi_s)]_s + \frac{1}{2}\left(\ln \frac{\Delta}{b^2 - s^2}\right)_s.$$

Integrating it with respect to s yields

$$A = k_3 \frac{\phi^2}{(1+sQ)^2}\sqrt{\frac{\Delta}{(b^2 - s^2)}},$$

where $k_3 = k_3(b^2)$ is an arbitrary positive function. This equation is equivalent to

$$\Phi = \frac{k_3 \Delta^{\frac{3}{2}}}{\sqrt{b^2 - s^2}}.$$

This completes the proof of Lemma 5. □

Using above Lemmas, we can prove Theorem 1.

Proof of Theorem 1. "Sufficiency". Assume that $\phi = k_2 s^{1-\frac{1}{b^2k_1}}(b^2-s^2)^{\frac{1}{2b^2k_1}}$ and the conformal factor σ is arbitrary. Note that $0 = T_1 = T_2 = T_3 = T_4 = T_5$ and (3) hold. Thus, the conclusion is obvious. Assume that $\sigma_i = \frac{\sigma_b}{b^2} b_i$ and ϕ satisfies (2). Then, (3) holds. Hence, F and $\tilde{F}$ have the same mean Landsberg curvature.

"Necessity". In general, it is impossible to solve (3) if $\phi = \phi(s)$ is an unknown function. To overcome this difficulty, we choose a special coordinate system at a point x as in [12]. First, we assume that

$$\alpha_x = \sqrt{\sum_{i=1}^{n}(y^i)^2}, \quad \beta_x = by^1.$$

Then, we take another special coordinates: $(s, y^a) \to (y^i)$ given by

$$y^1 = \frac{s}{\sqrt{b^2-s^2}}\bar{\alpha}, \quad y^a = y^a,$$

where

$$\bar{\alpha} = \sqrt{\sum_{a=2}^{n}(y^a)^2}.$$

We make the following agreement

$$1 \le i, j, k, \cdots \le n, \quad 2 \le a, b, c, \cdots \le n.$$

We have

$$\alpha = \frac{b}{\sqrt{b^2-s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2-s^2}}\bar{\alpha}.$$

When we take $i = 1$ in (3), by the rational and irrational terms of y, (3) is equal to

$$(sT_1 + b^2T_3)\bar{\sigma}_0 = 0 \tag{5}$$

and

$$(sT_1 + b^2T_2)\sigma_1 = 0. \tag{6}$$

Similarly, when we take $i = a$ in (3), (3) leads to

$$(sT_1 + b^2T_2)\sigma_1 = 0$$

and

$$\bar{\sigma}_0 y_a T_1 + \frac{b^2}{b^2-s^2}\bar{\alpha}^2\sigma_a T_5 = 0. \tag{7}$$

We divide the problem into two cases:

Case 1: $s\Delta + s + b^2Q = 0$. By Lemma 4, we have

$$\phi = k_2 s^{1-\frac{1}{b^2k_1}}(b^2-s^2)^{\frac{1}{2b^2k_1}},$$

where $k_1 = k_1(b^2)$ and $k_2 = k_2(b^2) \ge 0$ are any differentiable functions. By direct calculations, we have T_1, T_2, T_3, T_4, T_5 all equal to zero. In this case, (3) holds for any conformal factor σ.

Case 2: $s\Delta + s + b^2Q \ne 0$. It implies that $T_5 \ne 0$.

Case 2-1: $\bar{\sigma}_0 \ne 0$. Differentiating (7) with respect to y^b and y^c yields

$$(\sigma_b\delta_{ac} + \sigma_c\delta_{ab})T_1 + 2\frac{b^2}{b^2-s^2}\sigma_a\delta_{bc}T_5 = 0. \tag{8}$$

Contracting it with δ^{bc} yields

$$[T_1 + (n-1)\frac{b^2}{b^2 - s^2}T_5]\sigma_a = 0. \tag{9}$$

On the other hand, contracting (8) with δ^{ac} yields

$$\left(nT_1 + \frac{b^2}{b^2 - s^2}T_5\right)\sigma_b = 0. \tag{10}$$

For $\bar{\sigma}_0 \neq 0$, by (9) and (10), we obtain $T_1 = 0$ and $T_5 = 0$. This contradicts $T_5 \neq 0$. Thus, it is discarded.

Case 2-2: $\bar{\sigma}_0 = 0$, $\sigma_1 \neq 0$. In this case, (5) and (7) hold constantly. By (6), we can obtain $sT_1 + b^2T_2 = 0$. Then, by Lemma 5, ϕ satisfies

$$\Phi = \frac{k_3\Delta^{\frac{3}{2}}}{\sqrt{b^2 - s^2}}, \tag{11}$$

where $k_3 = k_3(b^2)$ is a differentiable positive function.

Substituting (11) into (3) yields

$$[(\alpha s\sigma_b - b^2\sigma_0)(sy_i - \alpha b_i) + \alpha(b^2 - s^2)(\alpha s\sigma_i - \sigma_0 b_i)]\frac{s\Delta + s + b^2Q}{\Delta(b^2 - s^2)} = 0.$$

Since $s\Delta + s + b^2Q \neq 0$,

$$(\alpha s\sigma_b - b^2\sigma_0)(sy_i - \alpha b_i) + \alpha(b^2 - s^2)(\alpha s\sigma_i - \sigma_0 b_i) = 0.$$

Differentiating the above formula with respect to y^j yields

$$2y_j(b^2\sigma_i - \sigma_b b_i) - 2\alpha s\sigma_i b_j + b_j(\sigma_0 b_i + \sigma_b y_i) + \alpha s(\sigma_j b_i + \sigma_b a_{ij}) - b^2(\sigma_j y_i + \sigma_0 a_{ij}) = 0.$$

Contracting it with a^{ij} yields

$$(n-2)(b^2\sigma_0 - \sigma_b\beta) = 0.$$

Because $n \geq 3$, it means that $\sigma_i(x)$ is proportional to $b_i(x)$, i.e., $\sigma_i = \frac{\sigma_b}{b^2}b_i$.

Case 2-3: $\bar{\sigma}_0 = 0$, $\sigma_1 = 0$. Then, conformal transformations between F and $\bar{F}$ are homothetic. □

Remark 1. *Note that $\phi = 1 + s$ or $\phi = 1 + s^2$ does not satisfy* (2)*. Thus, by the definition of general (α, β)-metrics and Theorem 1, conformal transformations that preserve the mean Landsberg curvature of Randers metrics $F = \alpha + \beta$ or square metrics $F = \frac{(\alpha+\beta)^2}{\alpha}$ are homothetic.*

Remark 2. *Let $Q = \sum_{i=0}^{k} f_i(b^2)s^i + g_1(b^2)(b^2 - s^2)^{\frac{1}{2}} + g_2(b^2)(b^2 - s^2)^{\frac{2m+1}{2}}$, where $k(\geq 0)$ and $m(\geq 1)$ are integers. If Q satisfies* (2)*, then $Q = f_1(b^2)s + g_1(b^2)(b^2 - s^2)^{\frac{1}{2}}$. If $\sigma_i = \frac{\sigma_b}{b^2}b_i$, then F and $\tilde{F} = e^{\sigma(x)}F$ have the same mean Landsberg curvature by Theorem 1.*

Before proving Theorem 2, we need the following Lemma.

Lemma 6 ([4])**.** *Let the (α, β)-metric $F = \alpha\phi(s)$ be a regular Finsler metric on an $n(\geq 3)$-dimensional manifold M. If ϕ satisfies*

$$\Phi = \frac{\lambda\Delta^{\frac{3}{2}}}{\sqrt{b^2 - s^2}},$$

where λ is a constant, then F is Riemannian.

Remark 3. *When $\lambda = \lambda(b^2)$, the conclusion is still right.*

Based on Lemma 6, we now give the proof of Theorem 2.

The Proof of Theorem 2. By Theorem 1, we divide the problem into two cases. If $\phi = k_2 s^{1-\frac{1}{b^2k_1}}(b^2 - s^2)^{\frac{1}{2b^2k_1}}$, the general (α,β)-metric $F = \alpha\phi(b^2, s)$ constructed by ϕ, is non-regular. We do not consider this case. If ϕ satisfies $\Phi = \frac{k_3(b^2)\Delta^{\frac{3}{2}}}{\sqrt{b^2-s^2}}$, then $F = \alpha\phi(b^2, s)$ is Riemannian by Lemma 6. □

4. Conclusions

In this paper, we study conformal transformations of general (α,β)-metrics preserving the mean Landsberg curvature. We obtain the necessity and sufficiency conditions for the mean Landsberg curvature and a rigidity theorem for the regular general (α,β)-metric case. The characterization equations for the general (α,β)-metrics with the mean Landsberg curvature are not yet completely solved, and only formal solutions are obtained.

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Article

A Note on Nearly Sasakian Manifolds

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Abstract: A class of nearly Sasakian manifolds is considered in this paper. We discuss the geometric effects of some symmetries on such manifolds and show, under a certain condition, that the class of Ricci semi-symmetric nearly Sasakian manifolds is a subclass of Einstein manifolds. We prove that a Codazzi-type Ricci nearly Sasakian space form is either a Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} = 1$ or a 5-dimensional proper nearly Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} > 1$. We also prove that the spectrum of the operator H^2 generated by the nearly Sasakian space form is a set of a simple eigenvalue of 0 and an eigenvalue of multiplicity 4, and we induce that the underlying space form carries a Sasaki–Einstein structure. We show that there exist integrable distributions with totally geodesic leaves on the same manifolds, and we prove that there are no proper nearly Sasakian space forms with constant sectional curvature.

Keywords: nearly Sasakian space forms; locally symmetric manifold; k-nullity distribution; semi-symmetric manifold; Ricci-symmetric manifold

MSC: 53C15; 53C25

1. Introduction

Blair, Yano, and Showers introduced in [1] the concept of nearly Sasakian structures as an odd-dimensional counterpart of nearly Kähler structures. They proved that a normal nearly Sasakian structure is Sasakian, and, hence, is contact in particular. Also, in the same paper, it was shown that a hypersurface of a nearly Kähler manifold is nearly Sasakian if and only if it is quasi-umbilical with respect to the (almost) contact form. This result was supported by an example stating that S^5 properly imbedded in S^6 inherits a nearly Sasakian structure, which is not a Sasakian structure. That is why nearly Sasakian manifolds may also be considered as an odd-dimensional analogue of nearly Kähler manifolds. However, it is very difficult to find relationships between the two structures, such as for the duo Sasakian and Kähler structures (see [2] for details).

Nearly Sasakian structures can also be seen as the vanishing of the symmetric part of Sasakian structures. Several authors have studied these structures in [2–5] and the references therein. For instance, Olszak in [4,5] gave a good number of properties for nearly Sasakian structures. He proved that if nearly Sasakian manifolds are not Sasakian, they are of dimension 5 and of a constant curvature. Olszak also proved some equivalent conditions for non-Sasakian nearly Sasakian manifolds to be of dimension 5 and showed that such manifolds are Einstein manifolds.

In Ref. [2], among other results, the authors proved that there are two types of integrable distributions with totally geodesic leaves in a nearly Sasakian manifold, which are Sasakian and 5-dimensional nearly Sasakian manifolds. Note that a $(2n+1)$-dimensional nearly Sasakian with $n \geq 3$ is a Sasakian manifold ([3], Theorem 4.9).

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In this paper, we consider the same nearly Sasakian structures by paying attention to certain foliations and curvature properties. We prove that some of these foliations are naturally generated by the symmetry properties on curvature and Ricci tensors.

The study of locally symmetric Riemannian manifolds has a long history, and several authors have worked in this direction. In Ref. [6] and the references therein, a series of results is presented regarding locally symmetric contact manifolds derived under some restrictions. In a direct way, Boeckx and Cho, in [7], proved that a locally symmetric contact manifold is either a Sasakian manifold with a constant sectional curvature 1 or is locally isometric to a unit tangent sphere bundle of a Euclidean space endowed with its standard contact metric structure.

A smooth manifold M is locally symmetric if its Riemannian curvature tensor R is parallel, i.e., $\nabla R = 0$, where ∇ is the Levi–Civita connection on M extended to act on tensors as a derivation. This class of manifolds contains manifolds of a constant curvature. The integrability condition of $\nabla R = 0$ is $R \cdot R = 0$, where again R is extended to act on tensors as a derivation. Manifolds that satisfy the latter condition are called semi-symmetric (see [8,9], for more details). A smooth manifold is said to be Ricci semi-symmetric, if $R \cdot \mathrm{Ric} = 0$. The set of all manifolds that are Ricci semi-symmetric contains the set of manifolds that are semi-symmetric. This means that semi-symmetric conditions imply Ricci semi-symmetric conditions, but the converse is not true, in general.

The present paper studies the two foliations stated by Olszak in papers [4,5]. He proved that, if a proper nearly Sasakian manifold is locally symmetric, then it is of a constant curvature and of dimension 5. These foliations were also investigated by Cappelletti-Montano et al. in [2,3].

The organization of the paper is as follows. Section 2 deals with a definition and properties of a nearly Sasakian manifold and some identity formulas of the underlying tensors, which are supported by two examples. In Section 3, we discuss the two foliations as stated in [2,4]. We establish the geometric effects of semi-symmetry and Ricci semi-symmetry on nearly Sasakian manifolds. Under a certain condition, we show that the class of Ricci-symmetric nearly Sasakian manifolds is a subclass of Einstein manifolds. We prove that these foliations exist canonically in a locally symmetric nearly Sasakian manifold of a constant sectional curvature and k space. Some examples are also established. In Section 4, we derive some algebraic formulas of the curvature tensor for nearly Sasakian manifolds (Proposition 3). We prove that a Codazzi-type Ricci nearly Sasakian space form is either Sasakian with a constant ϕ-holomorphic sectional curvature $\mathcal{H} = 1$ or a 5-dimensional proper nearly Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} > 1$. In the same settings, we also prove that the spectrum of the operator H^2 has a simple eigenvalue of 0 and an eigenvalue of multiplicity 4, which therefore induces that such a Codazzi-type Ricci nearly Sasakian space form carries a Sasaki–Einstein structure. We show that there exist integrable distributions with totally geodesic leaves (Theorems 9 and 10). Contrary to ([4], Theorem 6.1), we prove that there are no proper nearly Sasakian space forms with a constant sectional curvature (Theorem 12).

2. Preliminaries

Let M be a $(2n+1)$-dimensional manifold equipped with an almost contact structure (ϕ, ξ, η), that is, ϕ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying [6]

$$\phi^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

This implies that $\phi\xi = 0$, $\eta \circ \phi = 0$, and $\mathrm{rank}(\phi) = 2n$. In this case, (ϕ, ξ, η, g) is called an almost contact metric structure on M if (ϕ, ξ, η) is an almost contact structure of M and g is a Riemannian metric of M such that [6]

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for any vector field X, Y of M. It is easy to see the $(1,1)$-tensor field ϕ is skew-symmetric, and so $\eta(X) = g(\xi, X)$.

If, moreover,

$$(\nabla_X\phi)Y + (\nabla_Y\phi)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X, \tag{3}$$

where ∇ is the Levi–Civita connection for the Riemannian metric g, then M is called a nearly Sasakian manifold. From (3), one has

$$\nabla_X\xi = -\phi X - HX, \tag{4}$$

where

$$HX = \phi\big(\nabla_\xi\phi\big)X. \tag{5}$$

This operator is skew-symmetric and also anti-commutes with ϕ. The tensor field H is of type $(1,1)$ and satisfies $H\xi = 0$, $\eta \circ H = 0$, and

$$\nabla_\xi H = -\nabla_\xi\phi = \phi H = -\frac{1}{3}\mathcal{L}_\xi\phi, \tag{6}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ. If H vanishes, then a nearly Sasakian manifold is Sasakian (see [10] and the references therein).

It is easy to see that

$$H^2X = \big(\nabla_\xi\phi\big)^2X. \tag{7}$$

The divergence of ξ is given by

$$\operatorname{div}\xi = 0. \tag{8}$$

Example 1. *Let M be a 5-dimensional smooth manifold defined as $M = \{(x_1, x_2, \cdots, x_5) \in \mathbb{R}^5 : x_2 \neq 0,\ x_5 \neq 0\}$ with standard coordinates $(x_1, x_2, \cdots, x_5)$. The vector fields*

$$X_1 = 2\left(x_2\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_1}\right), \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \xi = -\frac{\partial}{\partial x_3},$$
$$X_4 = 2\left(x_5\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right), \quad X_5 = \frac{\partial}{\partial x_5},$$

are linearly independent at each point of M. Denote g to be the Riemannian metric of M, defined as $g(X_i, X_j) = \delta_{ij}$, for any $i,\ j = 1, 2, \cdots, 4$, where δ_{ij} is the Kronecker symbol, and $g(\xi, \xi) = 1$. Locally, the metric g takes the form

$$g = (\frac{1}{4} - x_2^2)dx_1^2 + dx_2^2 + dx_3^2 + (\frac{1}{4} - x_5^2)dx_4^2 + dx_5^2.$$

We define the 1-form η and $(1,1)$-tensor field ϕ, respectively, by, $\eta = -dx_3$ and $\phi X_1 = X_2$, $\phi X_2 = -X_1$, $\phi X_3 = 0$, $\phi X_4 = X_5$, and $\phi X_5 = -X_4$. The relations (1) and (2) are satisfied for $\mathbb{R}^5$ by the linearity of ϕ and g. Thus, the structure (ϕ, ξ, η, g) defines an almost contact metric structure for $\mathbb{R}^5$. Let ∇ be the Levi–Civita connection compatible with the metric g. Then, the non-vanishing Lie brackets are $[X_1, X_2] = [X_4, X_5] = 2\xi$. These lead to the following non-vanishing components of the covariant derivative

$$\nabla_{X_1}X_2 = \xi,\ \ \nabla_{X_1}\xi = -X_2,\ \ \nabla_{X_2}X_1 = -\xi,\ \ \nabla_{X_2}\xi = X_1,$$
$$\nabla_\xi X_1 = -X_2,\ \ \nabla_\xi X_2 = X_1,\ \ \nabla_\xi X_4 = -X_5,\ \ \nabla_\xi X_5 = X_4,$$
$$\nabla_{X_4}\xi = -X_5,\ \ \nabla_{X_4}X_5 = \xi,\ \ \nabla_{X_5}\xi = X_4,\ \ \nabla_{X_5}X_4 = -\xi.$$

Using these covariant derivatives, it is easy to see that relation (3) is satisfied, and, therefore, (ϕ, ξ, η, g) is a nearly Sasakian structure.

Throughout this note, manifolds are assumed to be of class C^∞ and connected, and all tensor fields are of class C^∞. We will denote the $\mathcal{F}(M)$ module of smooth sections of a vector bundle E with $\Gamma(E)$.

A vector field V on M is said to be an affine Killing vector field if it satisfies (see [11], p. 51)

$$\mathcal{L}_V \nabla = 0. \tag{9}$$

Relation (9) reduces to

$$R(V,X)Y + \nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V = 0, \tag{10}$$

where R is the Riemannian curvature tensor R of M defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad \forall\, X,\ Y,\ Z \in \Gamma(TM). \tag{11}$$

Relation (9) is the integrability condition for the Killing vector field V (see [11], for more details). If M is nearly Sasakian, then by using (4), it is easy to see that ξ is a Killing vector. Hence, the vector field ξ is an affine Killing vector field. The converse is not true, in general. In [11], it was proven that the converse holds when the underlying manifold is compact and without a boundary.

Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional nearly Sasakian manifold. Through (10), we obtain

$$R(X,\xi)Y = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi. \tag{12}$$

Therefore, we have [10]:

$$\begin{aligned} R(\xi,X)Y &= (\nabla_X \phi)Y + (\nabla_X H)Y \\ &= g(X - H^2X, Y)\xi - \eta(Y)(X - H^2X) \\ &= \{g(X,Y) - g(H^2X,Y)\}\xi - \eta(Y)X + \eta(Y)H^2X, \end{aligned} \tag{13}$$

$$\left(\nabla_X H^2\right)Y = \eta(Y)(\phi + H)H^2X + g((\phi+H)H^2X, Y)\xi, \tag{14}$$

$$g((\nabla_X \phi)Y, HZ) = -\eta(Y)g(H^2X, \phi Z) + \eta(X)g(H^2Y, \phi Z) + \eta(Y)g(HX, Z) \tag{15}$$

for any $X, Y, Z \in \Gamma(TM)$.

As proven in [10] and using the relations (13)–(15), we have

$$(\nabla_X \phi)Y = -\eta(X)\phi HY - \eta(Y)(X - \phi HX) + g(X - \phi HX, Y)\xi, \tag{16}$$

$$(\nabla_X H)Y = \eta(X)\phi HY + \eta(Y)(H^2X - \phi HX) - g(H^2X - \phi HX, Y)\xi, \tag{17}$$

$$\begin{aligned} (\nabla_X \phi H)Y &= \eta(Y)(\phi H^2X + HX) - \eta(X)(\phi H^2Y + HY) \\ &\quad - g(HX + \phi H^2X, Y)\xi. \end{aligned} \tag{18}$$

Now, for any vector fields X and Y of M,

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y + \eta(X)H^2Y - \eta(Y)H^2X. \tag{19}$$

Then,

$$\mathrm{Ric}(X,\xi) = (2n - \operatorname{trace} H^2)\eta(X), \quad \forall\, X \in \Gamma(TM). \tag{20}$$

By (13), we have, for any $X, Y \in \Gamma(TM)$,

$$R(X,\xi)Y = -g(X,Y)\xi + \eta(Y)X - \eta(Y)H^2X + g(H^2Y, X)\xi. \tag{21}$$

3. Foliations of a Nearly Sasakian Manifold

In Refs. [2,4], for instance, the authors showed that there are two foliations in any nearly Sasakian manifold with leaves that are Sasakian or 5-dimensional nearly Sasakian non-Sasakian manifolds. This fact is led by the square of a skew-symmetric operator H, i.e., H^2. The latter plays an important role, as well as its spectrum.

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional nearly Sasakian manifold. Olszak, in [4], showed that, if M satisfies the condition

$$H^2 = \alpha\{\mathbb{I} - \eta \otimes \xi\} \tag{22}$$

for a real number α, then $\dim M = 5$. The converse is true if the real number α is non-zero (see [5], Theorem 4.1 for more details).

We say that M is a *proper nearly Sasakian manifold* if it is a nearly Sasakian non-Sasakian manifold.

Let $D := \ker \eta$ denote the contact distribution, and let $D^\perp$ denote the one spanned structure vector field ξ. Then, the tangent space TM is decomposed as

$$TM = D \oplus D^\perp, \tag{23}$$

where $\oplus$ is the orthogonal direct sum. Through (23), any $X \in \Gamma(TM)$ can be rewritten as

$$X = QX + Q^\perp X, \tag{24}$$

where Q and $Q^\perp$ are the projection morphisms of TM onto D and $D^\perp$, respectively. Then, for any vector field $X \in \Gamma(TM)$, $Q^\perp X = \eta(X)\xi$, and $X = QX + \eta(X)\xi$.

If (22) is satisfied, then, for any non-zero vector field $X \in \Gamma(D)$,

$$-g(HX, HX) = \alpha g(X, X), \;\text{ i.e., }\; \alpha = -\frac{g(HX, HX)}{g(X, X)}. \tag{25}$$

This means that there is $\lambda \in \mathbb{R}$ such that $\alpha = -\lambda^2 \leq 0$, and, therefore, (22) becomes

$$H^2 = -\lambda^2\{\mathbb{I} - \eta \otimes \xi\}. \tag{26}$$

As examples for both Sasakian and proper nearly Sasakian manifolds, we have the following.

Example 2. *Let us recall the 5-dimensional manifold M considered in Example 1. Then, the components of the tensor field H of the type $(1,1)$ are given*

$$\begin{aligned}
H\xi &= \phi\big(\nabla_\xi \phi\big)\xi = 0,\\
HX_1 &= \phi\nabla_\xi X_2 - \phi^2\nabla_\xi X_1 = \phi X_1 + \phi^2 X_2 = X_2 - X_2 = 0,\\
HX_2 &= -\phi\nabla_\xi X_1 - \phi^2\nabla_\xi X_2 = -X_1 + X_1 = 0,\\
HX_4 &= \phi\nabla_\xi X_5 - \phi^2\nabla_\xi X_4 = X_5 - X_5 = 0\\
HX_5 &= -\phi\nabla_\xi X_4 - \phi^2\nabla_\xi X_5 = -X_4 + X_4 = 0.
\end{aligned}$$

This means that H vanishes everywhere. Therefore, in this case, the structure in (3) reduces to $(\nabla_X\phi)Y = g(X,Y)\xi - \eta(X)Y$, $\forall X,\ Y \in \Gamma(TM)$, which shows that M is a Sasakian manifold.

In [1], the authors showed how to induce a nearly Sasakian structure for S^5. In order to do so, they looked at S^5 as a hypersurface in S^6 equipped with its nearly Kähler structure.

Example 3. *We recall an example of 5-dimensional nearly Sasakian manifolds as detailed in [1,2,6]. Let S^6 be the unit sphere in $\mathbb{R}^7$ with its cross product $\times$ induced by Cayley algebra. Let $\mathcal{N} = \sum_{i=1}^{7} x_i \frac{\partial}{\partial x_i}$ denote the unit outer normal. We define an almost complex structure J for S^6 as $JX = \mathcal{N} \times X$, which implies,*

$$J^2 = \mathcal{N} \times (\mathcal{N} \times X) = -X, \;\; \forall\, X \in \Gamma(TS^6).$$

It is easy to see that J is almost complex structure and is also nearly Kähler (but non-Kähler) when associated with the induced Riemannian metric. As detailed in [2], now we consider S^5 *as a totally umbilical hypersurface of* S^6 *defined by* $x_7 = \frac{\sqrt{2}}{2}$*, with unit normal at each point x, which is given by* $\omega = x - \sqrt{2}\frac{\partial}{\partial x_7} = \sum_{i=1}^{6} x_i \frac{\partial}{\partial x_i} - \frac{\sqrt{2}}{2}\frac{\partial}{\partial x_7}$ *and the shape operator is* $A = -\mathbb{I}$*. Let* (ϕ, ξ, η, g) *be the almost induced contact metric structure with*

$$\xi = -J\omega = \sqrt{2}\left(x_1\frac{\partial}{\partial x_6} - x_2\frac{\partial}{\partial x_5} - x_3\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_3} + x_5\frac{\partial}{\partial x_2} - x_6\frac{\partial}{\partial x_1}\right),$$

and η *is given by the restriction of* $\sqrt{2}(x_1dx_6 - x_6dx_1 + x_5dx_2 - x_2dx_5 + x_4dx_3 - x_3dx_4)$ *to* S^5*. This is a nearly Sasakian non-Sasakian structure with a constant sectional curvature of* 2*. The latter means that*

$$R(X,Y)\xi = 2\{\eta(Y)X - \eta(X)Y\}, \ \forall X, \ Y \in \Gamma(TS^5),$$

which implies that $-\phi^2X - H^2X = 2\{X - \eta(X)\xi\}$*; that is,* $H^2X = -\{X - \eta(X)\xi\}$ *with* $\lambda^2 = 1$*.*

Next, we present some classes of nearly Sasakian manifolds in which condition (26) is satisfied.

Suppose M is a semi-symmetric nearly Sasakian manifold. Then, the curvature tensor R of M satisfies, for any vector fields X and Y of M, $R(X,Y)\cdot R = 0$, where $R(X,Y)$ operates on R as a derivation of the tensor algebra at each point (see [8,9] for more details). Now, let X and Y be vector fields in D such that $g(X,Y) = 0$. Then, using (19) and (21), we have,

$$\begin{aligned}(R(X,\xi)\cdot R)(X,Y)Y &= R(X,\xi)R(X,Y)Y - R(X,Y)R(X,\xi)Y - R(R(X,\xi)X,Y)Y \\ &\quad - R(X,R(X,\xi)Y)Y \\ &= -g(X,R(X,Y)Y)\xi + \eta(R(X,Y)Y)X - \eta(R(X,Y)Y)H^2X \\ &\quad + g(H^2X,R(X,Y)Y)\xi + \{g(X,X) - g(H^2X,X)\}\{g(Y,Y) - g(H^2Y,Y)\}\xi \\ &\quad - g(X,H^2Y)g(X,H^2Y)\xi.\end{aligned} \tag{27}$$

Hence,

$$\begin{aligned}&- g(X,R(X,Y)Y)\xi + \eta(R(X,Y)Y)X - \eta(R(X,Y)Y)H^2X \\ &+ g(H^2X,R(X,Y)Y)\xi + \{g(X,X) - g(H^2X,X)\}\{g(Y,Y) - g(H^2Y,Y)\}\xi \\ &- g(X,H^2Y)g(X,H^2Y)\xi = 0.\end{aligned} \tag{28}$$

Thus, considering the ξ-component of (28), we obtain

$$\begin{aligned}g(R(X,Y)Y,X) &= g(H^2X,R(X,Y)Y) + g(X,X)g(Y,Y) + g(X,X)g(HY,HY) \\ &\quad + g(Y,Y)g(HX,HX) + g(HX,HX)g(HY,HY) \\ &\quad - g(HX,HY)g(HX,HY).\end{aligned} \tag{29}$$

If condition (26) is satisfied, then, from relation (29), one obtains,

$$(1+\lambda^2)g(R(X,Y)Y,X) = (1+2\lambda^2+\lambda^4)g(X,X)g(Y,Y). \tag{30}$$

That is,

$$g(R(X,Y)Y,X) = (1+\lambda^2)g(X,X)g(Y,Y). \tag{31}$$

Therefore, we have

Theorem 1. *Let (M, ϕ, ξ, η, g) be a nearly Sasakian manifold satisfying the Nomizu's condition, i.e., $R(X,Y) \cdot R = 0$ for any vector fields X and Y of M. If*

$$H^2 = -\lambda^2\{\mathbb{I} - \eta \otimes \xi\}$$

for some real number λ, then M is of a constant curvature $1+\lambda^2$. Moreover, M is either a Sasakian manifold or a 5-dimensional proper nearly Sasakian manifold.

Let κ be a real constant. Denote $N(\kappa)$ as the κ-nullity distribution of M. Then, $N(\kappa)$ is seen as the function $p \longmapsto N_p(\kappa)$ with $p \in M$, where $N_p(\kappa)$ is the κ-nullity space at p given by (see [12,13] for more details and reference therein)

$$N_p(\kappa) = \big\{Z \in T_pM : R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y), \ \ \forall X, Y \in T_pM\big\},$$

where T_pM is the tangent space at p. If the vector field ξ on the nearly Sasakian manifold M belongs to $N(\kappa)$, then M is called κ *space*.

Therefore, we have this result.

Theorem 2. *Let (M, ϕ, ξ, η, g) be a nearly Sasakian manifold. Then, M satisfies the condition (26) if and only if M is a $(1+\lambda^2)$ space.*

Proof. If condition (26) is satisfied, then, for any vector vector fields X and Y of M,

$$\begin{aligned} R(X,Y)\xi &= \eta(Y)X - \eta(X)Y - \lambda^2\eta(X)\{Y - \eta(Y)\xi\} + \lambda^2\eta(Y)\{X - \eta(X)\xi\} \\ &= (1+\lambda^2)\{\eta(Y)X - \eta(X)Y\}. \end{aligned}$$

The converse is straightforward and this completes the proof. □

If a nearly Sasakian manifold M is Ricci semi-symmetric, then

$$\begin{aligned} (R(X,Y) \cdot \mathrm{Ric})(Z,W) &= -\mathrm{Ric}(R(X,Y)Z, W) - \mathrm{Ric}(Z, R(X,Y)W) \\ &= 0, \quad \forall\, X,\ \ Y,\ \ Z,\ \ W \in \Gamma(TM). \end{aligned} \tag{32}$$

Using (19) and (20), one has

$$\begin{aligned} (R(X,Y) \cdot \mathrm{Ric})(\xi, Z) &= -\mathrm{Ric}(R(X,Y)\xi, Z) - \mathrm{Ric}(\xi, R(X,Y)Z) \\ &= -\eta(Y)\mathrm{Ric}(X,Z) + \eta(X)\mathrm{Ric}(Y,Z) - \eta(X)\mathrm{Ric}(H^2Y, Z) \\ &+ \eta(Y)\mathrm{Ric}(H^2X, Z) - (2n - \mathrm{trace}\ H^2)\eta(R(X,Y)Z). \end{aligned} \tag{33}$$

Now, through relation (20), we obtain

$$\begin{aligned} (R(\xi, X) \cdot \mathrm{Ric})(Y, \xi) &= -\mathrm{Ric}(R(\xi, X)Y, \xi) - \mathrm{Ric}(Y, R(\xi, X)\xi) \\ &= -(2n - \mathrm{trace}\, H^2)g(X,Y) + (2n - \mathrm{trace}\, H^2)g(H^2X, Y) \\ &+ \mathrm{Ric}(X,Y) - \mathrm{Ric}(H^2X, Y). \end{aligned} \tag{34}$$

If condition (26) is satisfied for M, then (34) becomes

$$(R(\xi, X) \cdot \mathrm{Ric})(Y, \xi) = -2n(1+\lambda^2)^2 g(X,Y) + (1+\lambda^2)\mathrm{Ric}(X,Y). \tag{35}$$

Therefore, we obtain this result.

Theorem 3. *A Ricci semi-symmetric nearly Sasakian manifold satisfying (26) is Einstein.*

Proof. If M is a Ricci semi-symmetric nearly Sasakian manifold satisfying (26), then, using (35), the Ricci tensor is given by $\mathrm{Ric}(X,Y) = 2n(1+\lambda^2)g(X,Y)$ for any vector fields X and Y of M, and the proof is completed. □

In Ref. [4], Olszak proved that if a nearly Sasakian non-Sasakian manifold is locally symmetric, then it is of a constant curvature and of dimension 5. If we assume that the nearly Sasakian manifold M is of a constant sectional curvature κ, then the curvature tensor R of M satisfies the equation in [14,15]:

$$R(X,Y)Z = \kappa\{g(Y,Z)X - g(X,Z)Y\}, \ \ \forall\, X,\ Y \in \Gamma(TM). \tag{36}$$

Then, by putting $Z = \xi$ into (36) and using (19), we obtain

$$\eta(Y)\{(\kappa-1)X + H^2X\} = \eta(X)\{(\kappa-1)Y + H^2Y\}. \tag{37}$$

This implies that

$$H^2X = -(\kappa-1)\{X - \eta(X)\xi\}. \tag{38}$$

Therefore, we obtain:

Theorem 4. *Let $(M,\phi,\xi,\eta,\ g)$ be a nearly Sasakian manifold. If M is of a constant sectional curvature κ, then M is either Sasakian or satisfies condition (26), with $\kappa = 1+\lambda^2$, $\lambda \neq 0$, and a $(1+\lambda^2)$ space.*

A nearly Sasakian manifold M is locally symmetric if

$$(\nabla_W R)(X,Y)Z = 0,, \ \ \forall\, X,\ Y,\ Z,\ W \in \Gamma(TM).$$

We know that the covariant derivative of R, namely, $\nabla\, R$, is defined as

$$\begin{aligned}(\nabla_Z R)(X,Y,W) = {} & \nabla_Z R(X,Y)W - R(\nabla_Z X,Y)W - R(X,\nabla_Z Y)W \\ & - R(X,Y)\nabla_Z W.\end{aligned} \tag{39}$$

By putting $W = \xi$ into (39), one has

$$\begin{aligned}(\nabla_Z R)(X,Y,\xi) = {} & \{g(\phi Z,X) + g(HZ,X)\}Y - \{g(\phi Z,Y) + g(HZ,Y)\}X - \{g(\phi Z,X) \\ & + g(HZ,X)\}H^2Y + \{g(\phi Z,Y) + g(HZ,Y)\}H^2X + \eta(X)(\nabla_Z H^2)Y \\ & - \eta(Y)(\nabla_Z H^2)X + R(X,Y)\phi Z + R(X,Y)HZ.\end{aligned} \tag{40}$$

By using (14), the term $\eta(X)(\nabla_Z H^2)Y - \eta(Y)(\nabla_Z H^2)X$ becomes

$$\begin{aligned}\eta(X)(\nabla_Z H^2)Y - \eta(Y)(\nabla_Z H^2)X = {} & \eta(X)g(\phi H^2Z,Y)\xi + \eta(X)g(H^3Z,Y)\xi \\ & - \eta(Y)g(\phi H^2Z,X)\xi - \eta(Y)g(H^3Z,X)\xi.\end{aligned} \tag{41}$$

Therefore,

$$\begin{aligned}(\nabla_Z R)(X,Y,\xi) = {} & \{g(\phi Z,X) + g(HZ,X)\}Y - \{g(\phi Z,Y) + g(HZ,Y)\}X \\ & - \{g(\phi Z,X) + g(HZ,X)\}H^2Y + \{g(\phi Z,Y) + g(HZ,Y)\}H^2X \\ & + \eta(X)g(\phi H^2Z,Y)\xi + \eta(X)g(H^3Z,Y)\xi - \eta(Y)g(\phi H^2Z,X) \\ & - \eta(Y)g(H^3Z,X)\xi + R(X,Y)\phi Z + R(X,Y)HZ.\end{aligned} \tag{42}$$

If a nearly Sasakian manifold M is locally symmetric, then (42) leads to

$$
\begin{aligned}
0 = &\{g(\phi Z, X) + g(HZ, X)\}g(Y, W) - \{g(\phi Z, Y) + g(HZ, Y)\}g(X, W) \\
&- \{g(\phi Z, X) + g(HZ, X)\}g(H^2 Y, W) + \{g(\phi Z, Y) + g(HZ, Y)\}g(H^2 X, W) \\
&+ \eta(X)g(\phi H^2 Z, Y)\eta(W) + \eta(X)g(H^3 Z, Y)\eta(W) - \eta(Y)g(\phi H^2 Z, X)\eta(W) \\
&- \eta(Y)g(H^3 Z, X)\eta(W) + g(R(X, Y)\phi Z, W) + g(R(X, Y)HZ, W)
\end{aligned} \tag{43}
$$

for any vector field X, Y, Z, and W of M. As a result,

$$
\begin{aligned}
g(R(X, Y)\phi Z, W) + g(R(X, Y)HZ, W) &= -g(R(X, Y)W, \phi Z) - g(R(X, Y)W, HZ) \\
&= -g(R(X, Y)W, \phi Z + HZ).
\end{aligned} \tag{44}
$$

Relation (43) becomes

$$
\begin{aligned}
0 = &g(Y, W)g(\phi Z + HZ, X) - g(X, W)g(\phi Z + HZ, Y) \\
&- g(H^2 Y, W)g(\phi Z + HZ, X) + g(H^2 X, W)g(\phi Z + HZ, Y) \\
&+ \eta(X)\eta(W)g(\phi Z + HZ, H^2 Y) - \eta(Y)\eta(W)g(\phi Z + HZ, H^2 X) \\
&- g(R(X, Y)W, \phi Z + HZ).
\end{aligned} \tag{45}
$$

Thus,

$$
\begin{aligned}
R(X, Y)W = &g(Y, W)X - g(X, W)Y - g(H^2 Y, W)X + g(H^2 X, W)Y \\
&+ \eta(X)\eta(W)H^2 Y - \eta(Y)\eta(W)H^2 X \\
= &\{g(Y, W) - g(H^2 Y, W)\}X - \{g(X, W) - g(H^2 X, W)\}Y \\
&+ \eta(W)\{\eta(X)H^2 Y - \eta(Y)H^2 X\}.
\end{aligned} \tag{46}
$$

Therefore, we have the following.

Theorem 5. *Let (M, ϕ, ξ, η, g) be a nearly Sasakian manifold. If M is locally symmetric, then the curvature tensor R of M is given by, for any vector fields X, Y, and Z of M,*

$$
\begin{aligned}
R(X, Y)Z = &g(Y - H^2 Y, Z)X - g(X - H^2 X, Z)Y \\
&+ \eta(Z)\{\eta(X)H^2 Y - \eta(Y)H^2 X\}.
\end{aligned} \tag{47}
$$

Moreover, the Ricci tensor Ric *and scalar curvature* Scal *are given, respectively, by*

$$
\mathrm{Ric}(X, Y) = 2ng(X - H^2 X, Y) - \eta(X)\eta(Y)\,\mathrm{trace}\, H^2, \tag{48}
$$

$$
\textit{and} \quad \mathrm{Scal} = (2n + 1)\Big\{2n - \mathrm{trace}\, H^2\Big\}. \tag{49}
$$

Proof. Let $\{E_i\}_{1 \le i \le 2n+1}$ be an orthonormal frame with respect to g. Then, the scalar curvature is given by

$$
\mathrm{Scal} = \sum_{i=1}^{2n+1} \mathrm{Ric}(E_i, E_i) = (2n + 1)\Big\{2n - \mathrm{trace}\, H^2\Big\},
$$

which completes the proof. □

Note that the geometric information of relations (47)–(49) depends on the information of the operator H^2. Let M be a locally symmetric nearly Sasakian manifold. Then, the curvature tensor R of M satisfies Equation (47). In addition, if M is of a constant curvature κ, then, by comparing both (36) and (47), one has,

$$(\kappa - 1)\{g(Y,Z)X - g(X,Z)Y\} = -g(H^2Y,Z)X + g(H^2X,Z)Y + \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X\}.$$

By letting $Y = Z = \xi$, this equation reduces to $H^2X = -(\kappa - 1)\{X - \eta(X)\xi\}$. This means that M is either Sasakian (when $\kappa = 1$) or non-Sasakian (when $\kappa \neq 1$), thus satisfying $H^2X = -\lambda^2\{X - \eta(X)\xi\}$, with $\kappa = 1 + \lambda^2$ and $\lambda \neq 0$. The converse is straightforward; that is, if $H^2X = -(\kappa - 1)\{X - \eta(X)\xi\}$, then, using (47), the curvature tensor R satisfies

$$R(X,Y)Z = \kappa\{g(Y,Z)X - g(X,Z)Y\};$$

that is, M is of a constant curvature κ. Thus, according to [5], Theorem 4.1 we have the following.

Theorem 6. *Let (M, ϕ, ξ, η, g) be a locally symmetric nearly Sasakian manifold. Then, M is of a constant curvature κ if and only if M is either Sasakian or is a 5-dimensional proper nearly Sasakian manifold.*

As a consequence to this theorem, we remark the following.

Corollary 1. *There exist no locally symmetric nearly Sasakian manifolds of constant sectional curvature such that, for some real number λ,*

$$H^2 \neq -\lambda^2\{\mathbb{I} - \eta \otimes \xi\}.$$

4. Curvature Tensor Properties

First of all, we shall prove the following propositions.

Proposition 1. *Let (M, ϕ, ξ, η, g) be a nearly Sasakian manifold and R be the Riemannian curvature tensor of M. Then,*

$$\begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z &= 2\{g(\phi X,Y) + g(HX,Y)\}\phi HZ - \eta(Z)\{\eta(X)(\phi H^2Y + HY) \\ &- \eta(Y)(\phi H^2X + HX)\} - g(Y - \phi HY, Z)\{\phi X + HX\} + g(X - \phi HX, Z)\{\phi Y + HY\} \\ &- \{g(\phi Y,Z) + g(HY,Z)\}\{X - \phi HX\} + \{g(\phi X,Z) + g(HX,Z)\}\{Y - \phi HY\} \\ &+ \{\eta(X)g(HY + \phi H^2Y, Z) - \eta(Y)g(HX + \phi H^2X, Z)\}\xi, \end{aligned} \tag{50}$$

for any vector fields X, Y and Z on M.

Proof. The proof follows from straightforward calculations. □

From (2), one obtains the following

$$\begin{aligned} g(R(X,Y)\phi Z, \phi W) - g(R(X,Y)Z,W) &= -\eta(W)g(R(Z,\xi)X,Y) + 2g(\phi X,Y)g(HZ,W) \\ &+ 2g(HX,Y)g(HZ,W) - \eta(X)\eta(Z)g(H^2Y,W) - \eta(X)\eta(Z)g(HY,\phi W) \\ &+ \eta(Y)\eta(Z)g(H^2X,W) + \eta(Y)\eta(Z)g(HX,\phi W) - g(\phi Y,Z)g(X,\phi W) \\ &+ g(\phi Y,Z)g(HX,W) - g(HY,Z)g(X,\phi W) + g(HY,Z)g(HX,W) + g(\phi X,Z)g(Y,\phi W) \\ &- g(\phi X,Z)g(HY,W) + g(HX,Z)g(Y,\phi W) - g(HX,Z)g(HY,W) - g(Y,Z)g(X,W) \\ &+ \eta(X)\eta(W)g(Y,Z) - g(Y,Z)g(HX,\phi W) + g(\phi HY,Z)g(X,W) - \eta(X)\eta(W)g(\phi HY,Z) \\ &+ g(\phi HY,Z)g(HX,\phi W) + g(X,Z)g(Y,W) - \eta(Y)\eta(W)g(X,Z) + g(X,Z)g(HY,\phi W) \\ &- g(\phi HX,Z)g(Y,W) + \eta(Y)\eta(W)g(\phi HX,Z) - g(\phi HX,Z)g(HY,\phi W). \end{aligned} \tag{51}$$

By using the equality, $g(R(X,Y)\phi Z, \phi W) = g(R(\phi Z, \phi W)X,Y)$, the relation (51) reduces to

$$
\begin{aligned}
&g(R(\phi Z,\phi W)X,Y)-g(R(Z,W)X,Y)=-\eta(W)g(R(Z,\xi)X,Y)+2g(\phi X,Y)g(HZ,W)\\
&+2g(HX,Y)g(HZ,W)-\eta(X)\eta(Z)g(H^2Y,W)-\eta(X)\eta(Z)g(HY,\phi W)\\
&+\eta(Y)\eta(Z)g(H^2X,W)+\eta(Y)\eta(Z)g(HX,\phi W)-g(\phi Y,Z)g(X,\phi W)\\
&+g(\phi Y,Z)g(HX,W)-g(HY,Z)g(X,\phi W)+g(HY,Z)g(HX,W)+g(\phi X,Z)g(Y,\phi W)\\
&-g(\phi X,Z)g(HY,W)+g(HX,Z)g(Y,\phi W)-g(HX,Z)g(HY,W)-g(Y,Z)g(X,W)\\
&+\eta(X)\eta(W)g(Y,Z)-g(Y,Z)g(HX,\phi W)+g(\phi HY,Z)g(X,W)-\eta(X)\eta(W)g(\phi HY,Z)\\
&+g(\phi HY,Z)g(HX,\phi W)+g(X,Z)g(Y,W)-\eta(Y)\eta(W)g(X,Z)+g(X,Z)g(HY,\phi W)\\
&-g(\phi HX,Z)g(Y,W)+\eta(Y)\eta(W)g(\phi HX,Z)-g(\phi HX,Z)g(HY,\phi W).
\end{aligned} \tag{52}
$$

Therefore, we have the following.

Proposition 2. *Let (M,ϕ,ξ,η,g) be a nearly Sasakian manifold and R be the Riemannian curvature tensor of M. Then,*

$$
\begin{aligned}
&R(\phi X,\phi Y)Z-R(X,Y)Z=-\eta(Y)g(H^2X,Z)\xi+\eta(Y)\eta(Z)H^2X+2g(HX,Y)\phi Z\\
&+2g(HX,Y)HZ-\eta(Z)\eta(X)H^2Y-\eta(Z)\eta(X)\phi HY+\eta(X)g(H^2Z,Y)\xi\\
&+\eta(X)g(HZ,\phi Y)\xi+g(Z,\phi Y)\phi X-g(HZ,Y)\phi X+g(Z,\phi Y)HX-g(HZ,Y)HX\\
&+g(\phi Z,X)\phi Y+g(\phi Z,Y)HY+g(HZ,X)\phi Y+g(HZ,X)HY-g(Z,Y)X\\
&-g(HZ,\phi Y)X-g(Z,Y)\phi HX+\eta(Z)\eta(Y)\phi HX-g(HZ,\phi Y)\phi HX+g(Z,X)Y\\
&+g(Z,X)\phi HY-g(\phi HZ,X)Y+\eta(Y)g(\phi HZ,X)\xi-g(\phi HZ,X)\phi HY,
\end{aligned} \tag{53}
$$

for any vector fields X, Y, and Z of M.

Next, we deal with the ϕ-holomorphic sectional curvature on a nearly Sasakian manifold. A plane section σ in T_pM of a nearly Sasakian manifold M is called a ϕ section if there exists a vector field X for M that is orthogonal to ξ such that the basis $\{X,\phi X\}$ spans σ. The sectional curvature $K(X,\phi X)$ of a ϕ section is called the *ϕ-sectional curvature*, and it is denoted by $\mathcal{H}$. If M has a pointwise constant ϕ-holomorphic sectional curvature $\mathcal{H}=\mathcal{H}(p)$, $p\in M$, then, for any vector fields X and $Y\in D=\ker\eta$, we have

$$g(R(X,\phi X)X,\phi X)=-\mathcal{H}g(X,X)^2. \tag{54}$$

By taking the g-dot with ϕW of (2) and for any X, Y, and Z of D, we have

$$
\begin{aligned}
g(R(X,Y)\phi Z,\phi W)=\;&g(R(X,Y)Z,W)+2\{g(\phi X,Y)+g(HX,Y)\}g(HZ,W)\\
&-\{g(\phi Y,Z)+g(HY,Z)\}g(X-\phi HX,\phi W)+\{g(\phi X,Z)\\
&+g(HX,Z)\}g(Y-\phi HY,\phi W)-g(Y-\phi HY,Z)g(\phi X+HX,\phi W)\\
&+g(X-\phi HX,Z)g(\phi Y+HY,\phi W).
\end{aligned} \tag{55}
$$

By putting the vector fields $Y=\phi Y$, $Z=\phi X$, and $W=Y$ into (55), one obtains

$$
\begin{aligned}
g(R(X,\phi Y)X,\phi Y)=\;&g(R(X,\phi Y)Y,\phi X)+g(X,Y)^2-g(HX,\phi Y)^2+g(X,\phi Y)^2\\
&-g(HX,Y)^2-g(X,X)g(Y,Y).
\end{aligned} \tag{56}
$$

Likewise, for any $X,Y\in\Gamma(D)$, we have,

$$g(R(X,\phi X)Y,\phi X)=g(R(X,\phi X)X,\phi Y). \tag{57}$$

By substituting $X+Y$ in (54), and by using (57), the left-hand side of relation (54) becomes

$$
\begin{aligned}
& g(R(X+Y,\phi X+\phi Y)(X+Y),\phi X+\phi Y) = g(R(X,\phi X)X,\phi X) + g(R(Y,\phi Y)Y,\phi Y) \\
& + g(R(Y,\phi X)X,\phi X) + g(R(X,\phi Y)X,\phi X) + g(R(Y,\phi Y)X,\phi X) + g(R(X,\phi X)Y,\phi X) \\
& + g(R(Y,\phi X)Y,\phi X) + g(R(X,\phi Y)Y,\phi X) + g(R(Y,\phi Y)Y,\phi X) + g(R(X,\phi X)X,\phi Y) \\
& + g(R(Y,\phi X)X,\phi Y) + g(R(X,\phi Y)X,\phi Y) + g(R(Y,\phi Y)X,\phi Y) + g(R(X,\phi X)Y,\phi Y) \\
& + g(R(Y,\phi X)Y,\phi Y) + g(R(X,\phi Y)Y,\phi Y).
\end{aligned} \tag{58}
$$

By using (56) and (57), the Bianchi Identity, i.e., $g(R(Y,\phi Y)X,\phi X) = g(R(X,\phi Y)Y,\phi X) + g(R(\phi X,\phi Y)X,Y)$ and $g(R(X,\phi X)Y,\phi Y) = g(R(X,\phi Y)Y,\phi X) + g(R(\phi X,\phi Y)X,Y)$, one has

$$
\begin{aligned}
g(R(X+Y,\phi X+\phi Y)(X+Y),\phi X+\phi Y) & \\
= 4g(R(X,\phi Y)Y,\phi X) &+ 4g(R(Y,\phi Y)Y,\phi X) + 4g(R(X,\phi X)X,\phi Y) \\
+ 2g(R(\phi X,\phi Y)X,Y) &+ g(R(Y,\phi X)Y,\phi X) + g(R(X,\phi Y)X,\phi Y) \\
+ g(R(X,\phi X)X,\phi X) &+ g(R(Y,\phi Y)Y,\phi Y).
\end{aligned} \tag{59}
$$

Now, by using (54), the relation (59) becomes, for any X and Y of D,

$$
\begin{aligned}
g(R(X+Y,\phi X+\phi Y)(X+Y),\phi X+\phi Y) & \\
= 4g(R(X,\phi Y)Y,\phi X) &+ 4g(R(Y,\phi Y)Y,\phi X) + 4g(R(X,\phi X)X,\phi Y) \\
+ 2g(R(\phi X,\phi Y)X,Y) &+ g(R(Y,\phi X)Y,\phi X) + g(R(X,\phi Y)X,\phi Y) \\
- \mathcal{H}g(X,X)^2 &- \mathcal{H}g(Y,Y)^2.
\end{aligned} \tag{60}
$$

By substituting $X+Y$ in (54), the right-hand side of relation (54) yields

$$
\begin{aligned}
-\mathcal{H}g(X,X)^2 = -\mathcal{H}\{ & g(X,X)^2 + 4g(X,X)g(X,Y) + 2g(X,X)g(Y,Y) \\
& + 4g(X,Y)^2 + 4g(X,Y)g(Y,Y) + g(Y,Y)^2\}.
\end{aligned} \tag{61}
$$

By calculating equality of (60) and (61), we obtain

$$
\begin{aligned}
& \frac{1}{2}\{4g(R(X,\phi Y)Y,\phi X) + 4g(R(Y,\phi Y)Y,\phi X) + 4g(R(X,\phi X)X,\phi Y) \\
& + 2g(R(\phi X,\phi Y)X,Y) + g(R(Y,\phi X)Y,\phi X) + g(R(X,\phi Y)X,\phi Y)\} \\
& = -\mathcal{H}\{2g(X,Y)^2 + 2g(X,X)g(X,Y) + 2g(X,Y)g(Y,Y) \\
& + g(X,X)g(Y,Y)\}.
\end{aligned} \tag{62}
$$

By putting $X = \phi Y$, $Y = X$, and $Z = Y$ into (2), we have

$$
\begin{aligned}
-g(R(Y,\phi X)Y,\phi X) - g(R(\phi Y,X)Y,\phi X) = {} & g(HY,\phi X)^2 - g(Y,\phi X)^2 \\
& + g(HY,X)^2 + g(Y,Y)g(X,X) - g(X,Y)^2.
\end{aligned} \tag{63}
$$

Therefore, we have

$$
\begin{aligned}
g(R(Y,\phi X)Y,\phi X) = {} & g(R(X,\phi Y)Y,\phi X) + g(X,Y)^2 - g(HY,\phi X)^2 \\
& + g(Y,\phi X)^2 - g(HY,X)^2 - g(X,X)g(Y,Y).
\end{aligned} \tag{64}
$$

By adding (56) and (64), one obtains

$$
\begin{aligned}
g(R(X,\phi Y)X,\phi Y) + g(R(Y,\phi X)Y,\phi X) = {} & 2g(R(X,\phi Y)Y,\phi X) \\
& + 2g(X,Y)^2 - 2g(HX,\phi Y)^2 + 2g(X,\phi Y)^2 - 2g(HX,Y)^2 \\
& - 2g(X,X)g(Y,Y).
\end{aligned} \tag{65}
$$

By putting (65) into (62), we have

$$\begin{aligned} 3g(R(X,\phi Y)Y,\phi X) + 2g(R(Y,\phi Y)Y,\phi X) + 2g(R(X,\phi X)X,\phi Y) \\ + g(R(\phi X,\phi Y)X,Y) + g(X,Y)^2 - g(HX,\phi Y)^2 + g(X,\phi Y)^2 \\ - g(HX,Y)^2 - g(X,X)g(Y,Y) \\ = -\mathcal{H}\{2g(X,Y)^2 + 2g(X,X)g(X,Y) + 2g(X,Y)g(Y,Y) \\ + g(X,X)g(Y,Y)\}. \end{aligned} \tag{66}$$

Since

$$\begin{aligned} g(R(\phi X,\phi Y)X,Y) = g(R(X,Y)X,Y) + g(HX,Y)^2 - g(X,\phi Y)^2 \\ - g(X,Y)^2 + g(HX,\phi Y)^2 + g(X,X)g(Y,Y), \end{aligned} \tag{67}$$

the relation (66) becomes

$$\begin{aligned} 3g(R(X,\phi Y)Y,\phi X) + 2g(R(Y,\phi Y)Y,\phi X) + 2g(R(X,\phi X)X,\phi Y) + g(R(X,Y)X,Y) \\ = -\mathcal{H}\{2g(X,Y)^2 + 2g(X,X)g(X,Y) + 2g(X,Y)g(Y,Y) \\ + g(X,X)g(Y,Y)\}. \end{aligned} \tag{68}$$

By replacing Y with $-Y$ in (68), one obtains

$$\begin{aligned} 3g(R(X,\phi Y)Y,\phi X) - 2g(R(Y,\phi Y)Y,\phi X) - 2g(R(X,\phi X)X,\phi Y) \\ + g(R(X,Y)X,Y) = -\mathcal{H}\{2g(X,Y)^2 - 2g(X,X)g(X,Y) \\ - 2g(X,Y)g(Y,Y) + g(X,X)g(Y,Y)\}. \end{aligned} \tag{69}$$

By summing the relations (68) and (69), we have

$$3g(R(X,\phi Y)Y,\phi X) + g(R(X,Y)X,Y) = -\mathcal{H}\{2g(X,Y)^2 + g(X,X)g(Y,Y)\}. \tag{70}$$

By replacing Y with ϕY in (70) and using curvature identities, we obtain,

$$3g(R(\phi Y,\phi X)Y,X) + g(R(X,\phi Y)X,\phi Y) = -\mathcal{H}\{2g(X,\phi Y)^2 + g(X,X)g(Y,Y)\}. \tag{71}$$

Through the relations (56) and (67), the left-hand side of relation (71) becomes

$$\begin{aligned} 3g(R(\phi Y,\phi X)Y,X) + g(R(X,\phi Y)X,\phi Y) = 3g(R(X,Y)X,Y) + g(R(X,\phi Y)Y,\phi X) \\ + 2g(HX,Y)^2 - 2g(X,\phi Y)^2 - 2g(X,Y)^2 + 2g(HX,\phi Y)^2 \\ + 2g(X,X)g(Y,Y). \end{aligned} \tag{72}$$

By putting the pieces (71) and (72) together, we have

$$\begin{aligned} g(R(X,\phi Y)Y,\phi X) = -3g(R(X,Y)X,Y) - \mathcal{H}\{2g(X,\phi Y)^2 + g(X,X)g(Y,Y)\} \\ - 2g(HX,Y)^2 + 2g(X,\phi Y)^2 + 2g(X,Y)^2 - 2g(HX,\phi Y)^2 \\ - 2g(X,X)g(Y,Y). \end{aligned} \tag{73}$$

Substituting (73) into (70) leads to

$$\begin{aligned} -8g(R(X,Y)X,Y) - 3\mathcal{H}\{2g(X,\phi Y)^2 + g(X,X)g(Y,Y)\} - 6g(HX,Y)^2 \\ + 6g(X,\phi Y)^2 + 6g(X,Y)^2 - 6g(HX,\phi Y)^2 \\ - 6g(X,X)g(Y,Y) = -\mathcal{H}\{2g(X,Y)^2 + g(X,X)g(Y,Y)\}. \end{aligned} \tag{74}$$

Therefore, we have

$$4g(R(X,Y)X,Y) = (\mathcal{H}+3)\{g(X,Y)^2 - g(X,X)g(Y,Y)\} \\ -3(\mathcal{H}-1)g(X,\phi Y)^2 - 3g(HX,Y)^2 - 3g(HX,\phi Y)^2. \tag{75}$$

By replacing X and Y with $X+Z$ and $Y+W$, respectively, in both sides of (75), one has,

$$\begin{aligned} 8g(R(X,Y)Z,W) + 8g(R(Z,Y)X,W) = & -4(\mathcal{H}+3)g(X,Z)g(Y,W) \\ & + 2(\mathcal{H}+3)\{g(X,Y)g(Z,W) + g(X,W)g(Z,Y)\} \\ & - 6(\mathcal{H}-1)\{g(X,\phi Y)g(Z,\phi W) + g(X,\phi W)g(Z,\phi Y)\} \\ & - 6\{g(HX,Y)g(HZ,W) + g(HX,W)g(HZ,Y) \\ & + g(HX,\phi Y)g(HZ,\phi W) + g(HX,\phi W)g(HZ,\phi Y)\}. \end{aligned} \tag{76}$$

In addition, by replacing Y with Z and Z with Y in (76) and then multiplying both sides by -1, we have

$$\begin{aligned} & -8g(R(X,Z)Y,W) - 8g(R(Y,Z)X,W) = 4(\mathcal{H}+3)g(X,Y)g(Z,W) \\ & -2(\mathcal{H}+3)\{g(X,Z)g(Y,W) + g(X,W)g(Z,Y)\} + 6(\mathcal{H}-1)\{g(X,\phi Z)g(Y,\phi W) \\ & + g(X,\phi W)g(Y,\phi Z)\} + 6\{g(HX,Z)g(HY,W) + g(HX,W)g(HY,Z) \\ & + g(HX,\phi Z)g(HY,\phi W) + g(HX,\phi W)g(HY,\phi Z)\}. \end{aligned} \tag{77}$$

By adding (76) and (77), we have

$$\begin{aligned} & 8g(R(X,Y)Z,W) + 16g(R(Z,Y)X,W) - 8g(R(X,Z)Y,W) \\ & = 6(\mathcal{H}+3)\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\} - 6(\mathcal{H}-1)\{g(X,\phi Y)g(Z,\phi W) \\ & - g(X,\phi Z)g(Y,\phi W) + 2g(X,\phi W)g(Z,\phi Y)\} - 6\{g(HX,Y)g(HZ,W) \\ & - g(HX,Z)g(HY,W) + 2g(HX,W)g(HZ,Y) + g(HX,\phi Y)g(HZ,\phi W) \\ & - g(HX,\phi Z)g(HY,\phi W) + 2g(HX,\phi W)g(HZ,\phi Y)\}. \end{aligned} \tag{78}$$

By using the Bianchi identity, that is, $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ and $g(R(Z,Y)X,W) = g(R(X,W)Z,Y)$, the relation ((78) becomes

$$\begin{aligned} & 24g(R(X,W)Z,Y) = 6(\mathcal{H}+3)\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\} \\ & - 6(\mathcal{H}-1)\{g(X,\phi Y)g(Z,\phi W) - g(X,\phi Z)g(Y,\phi W) + 2g(X,\phi W)g(Z,\phi Y)\} \\ & - 6\{g(HX,Y)g(HZ,W) - g(HX,Z)g(HY,W) + 2g(HX,W)g(HZ,Y) \\ & + g(HX,\phi Y)g(HZ,\phi W) - g(HX,\phi Z)g(HY,\phi W) + 2g(HX,\phi W)g(HZ,\phi Y)\}. \end{aligned} \tag{79}$$

By exchanging W and Y in (79), we obtain

$$\begin{aligned} & 24g(R(X,Y)Z,W) = 6(\mathcal{H}+3)\{g(X,W)g(Z,Y) - g(X,Z)g(Y,W)\} \\ & - 6(\mathcal{H}-1)\{g(X,\phi W)g(Z,\phi Y) - g(X,\phi Z)g(W,\phi Y) + 2g(X,\phi Y)g(Z,\phi W)\} \\ & - 6\{g(HX,W)g(HZ,Y) - g(HX,Z)g(HW,Y) + 2g(HX,Y)g(HZ,W) \\ & + g(HX,\phi W)g(HZ,\phi Y) - g(HX,\phi Z)g(HW,\phi Y) + 2g(HX,\phi Y)g(HZ,\phi W)\} \end{aligned} \tag{80}$$

for any X, Y, Z, and $W \in \Gamma(D)$. Now, by considering a vector field X of M as $X = QX + \eta(X)\xi$, where Q is the projection onto D, one has, for any X, Y , Z, and $W \in \Gamma(TM)$,

$$\begin{aligned} & g(R(QX,QY)QZ,QW) = g(R(X,Y)Z,W) - \eta(X)\eta(W)\{g(Y,Z) - g(H^2Z,Y)\} \\ & + \eta(X)\eta(Z)\{g(W,Y) - g(H^2W,Y)\} - \eta(Y)\eta(Z)\{g(W,X) - g(H^2W,X)\} \\ & + \eta(Y)\eta(W)\{g(Z,X) - g(H^2Z,X)\}. \end{aligned} \tag{81}$$

From (80), and by using (81), we have the following

$$\begin{aligned}
24g(R(X,Y)Z,W) &= 6(\mathcal{H}+3)\{g(X,W)g(Z,Y)-\eta(Z)\eta(Y)g(X,W)\\
&-\eta(X)\eta(W)g(Z,Y)-g(X,Z)g(Y,W)+\eta(Y)\eta(W)g(X,Z)\\
&+\eta(X)\eta(Z)g(Y,W)\}-6(\mathcal{H}-1)\{g(X,\phi W)g(Z,\phi Y)-g(X,\phi Z)g(W,\phi Y)\\
&+2g(X,\phi Y)g(Z,\phi W)\}-6\{g(HX,W)g(HZ,Y)-g(HX,Z)g(HW,Y)\\
&+2g(HX,Y)g(HZ,W)+g(HX,\phi W)g(HZ,\phi Y)-g(HX,\phi Z)g(HW,\phi Y)\\
&+2g(HX,\phi Y)g(HZ,\phi W)\}+24\eta(X)\eta(W)\{g(Y,Z)-g(H^2Z,Y)\}\\
&-24\eta(X)\eta(Z)\{g(W,Y)-g(H^2W,Y)\}+24\eta(Y)\eta(Z)\{g(W,X)-g(H^2W,X)\}\\
&-24\eta(Y)\eta(W)\{g(Z,X)-g(H^2Z,X)\}.
\end{aligned} \tag{82}$$

Therefore, one has the following.

Proposition 3. *Let (M,ϕ,ξ,η,g) be a nearly Sasakian manifold. Then, the necessary and sufficient condition for M to have a pointwise constant ϕ-holomorphic sectional curvature $\mathcal{H}$ is*

$$\begin{aligned}
R(X,Y)Z &= \frac{\mathcal{H}+3}{4}\{g(Z,Y)X-g(X,Z)Y\}+\frac{\mathcal{H}-1}{4}\{\eta(X)\eta(Z)Y\\
&-\eta(Z)\eta(Y)X+\eta(Y)g(X,Z)\xi-\eta(X)g(Z,Y)\xi+g(Z,\phi Y)\phi X\\
&+g(X,\phi Z)\phi Y+2g(X,\phi Y)\phi Z\}-\frac{1}{4}\{g(HZ,Y)HX+g(HX,Z)HY\\
&+2g(HX,Y)HZ-g(HZ,\phi Y)\phi HX-g(HX,\phi Z)\phi HY\\
&-2g(HX,\phi Y)\phi HZ\}+\eta(Z)\{\eta(X)H^2Y-\eta(Y)H^2X\}\\
&+\{\eta(Y)g(H^2Z,X)-\eta(X)g(H^2Z,Y)\}\xi
\end{aligned} \tag{83}$$

for all vector fields X, Y, and Z of M.

From relation (83), the Ricci tensor Ric associated with the Riemannian metric g yields

$$\begin{aligned}
\mathrm{Ric}(X,Y) &= \frac{n(\mathcal{H}+3)+\mathcal{H}-1}{2}g(X,Y)-\frac{(n+1)(\mathcal{H}-1)}{2}\eta(X)\eta(Y)\\
&-\frac{5}{2}g(X,H^2Y)-\eta(X)\eta(Y)\operatorname{trace}H^2.
\end{aligned} \tag{84}$$

Moreover, we have the identity for the Ricci curvature:

$$\mathrm{Ric}(\phi X,\phi Y)=\mathrm{Ric}(X,Y)-(2n-\operatorname{trace}H^2)\eta(X)\eta(Y). \tag{85}$$

Let τ be the scalar curvature of g. Then, τ is given by

$$\tau=\frac{1}{2}\{n(2n+1)(\mathcal{H}+3)+n(\mathcal{H}-1)\}-\frac{7}{2}\operatorname{trace}H^2. \tag{86}$$

Lemma 1. *In a nearly Sasakian manifold, the eigenvalues of the operator H^2 are constant.*

Proof. The proof follows from a direct calculation using (14). □

For any vector field X, Y, Z, and W, one has

$$
\begin{aligned}
(\nabla_X \mathrm{Ric})(Y,Z) = & \frac{n+1}{2}X(\mathcal{H})g(Y,Z) - \left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \operatorname{trace} H^2\right\}\{\eta(Z)(\nabla_X\eta)Y \\
& + \eta(Y)(\nabla_X\eta)Z\} + \frac{5}{2}\{g((\nabla_X H)Y, HZ) + g(HY, (\nabla_X H)Z)\} \\
= & \frac{n+1}{2}X(\mathcal{H})g(Y,Z) + \left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \operatorname{trace} H^2\right\}\{\eta(Z)g(\phi X, Y) \\
& + \eta(Z)g(HX,Y) + \eta(Y)g(\phi X, Z) + \eta(Y)g(HX,Z)\} \\
& + \frac{5}{2}\Big\{\eta(Y)g(H^2X, HZ) - \eta(Y)g(\phi HX, HZ) + \eta(Z)g(H^2X, HY) \\
& - \eta(Z)g(\phi HX, HY)\Big\}.
\end{aligned} \tag{87}
$$

Let $\{E_i\}_{1\le i\le 2n+1}$ be an arbitrary local orthonormal frame field on M. Then,

$$
\nabla_X\tau = 2\sum_{i=1}^{2n+1}(\nabla_{E_i}Ric)(X,E_i) = (n+1)X(\mathcal{H}). \tag{88}
$$

On the other hand, by using (86) and Lemma 1, one obtains

$$
\nabla_X\tau = \frac{1}{2}\{n(2n+1)X(\mathcal{H}) + nX(\mathcal{H})\} - \frac{1}{2}X(\operatorname{trace} H^2) = n(n+1)X(\mathcal{H}). \tag{89}
$$

From the relations (88) and (89), we have

$$
n(n+1)X(\mathcal{H}) = (n+1)X(\mathcal{H}),\ \forall\, X \in \Gamma(TM).
$$

This leads to $(n-1)X(\mathcal{H}) = 0$. If $n > 1$, and the nearly Sasakian manifold M is connected, then H is constant for M. Therefore, according to Ogiue [16], we obtain the following theorem.

Theorem 7. *Let M be a $(2n+1)$-dimensional nearly Sasakian manifold ($n > 1$). If the ϕ-holomorphic sectional curvature at any point of M is independent of the choice of the ϕ-holomorphic section, then it is constant for M, and the curvature tensor R is given by*

$$
\begin{aligned}
R(X,Y)Z = & \frac{\mathcal{H}+3}{4}\{g(Z,Y)X - g(X,Z)Y\} + \frac{\mathcal{H}-1}{4}\{\eta(X)\eta(Z)Y - \eta(Z)\eta(Y)X \\
& + \eta(Y)g(X,Z)\xi - \eta(X)g(Z,Y)\xi + g(Z,\phi Y)\phi X + g(X,\phi Z)\phi Y + 2g(X,\phi Y)\phi Z\} \\
& - \frac{1}{4}\{g(HZ,Y)HX + g(HX,Z)HY + 2g(HX,Y)HZ - g(HZ,\phi Y)\phi HX \\
& - g(HX,\phi Z)\phi HY - 2g(HX,\phi Y)\phi HZ\} + \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X\} \\
& + \{\eta(Y)g(H^2Z,X) - \eta(X)g(H^2Z,Y)\}\xi
\end{aligned} \tag{90}
$$

for any vector fields X, Y, and Z of M.

Note that a complete and simply connected nearly Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H}$ is said to be a *nearly Sasakian space form*. Thus, we obtain the following result.

Theorem 8. *Let M be a $(2n+1)$-dimensional complete and simply connected nearly Sasakian manifold ($n > 1$). Then, M is a nearly Sasakian space form if and only if the curvature tensor R is given by (90).*

Next, we introduce another class of nearly Sasakian manifolds with a Codazzi-type Ricci tensor in which the condition (26) is naturally derived.

With regard to a *Codazzi-type Ricci tensor*, we mean a Ricci tensor Ric satisfying the Codazzi equation; that is,

$$(\nabla_X \mathrm{Ric})(Y,Z) = (\nabla_Y \mathrm{Ric})(X,Z), \;\; \forall\, X,\; Y,\; Z \in \Gamma(TM). \tag{91}$$

A manifold with such a tensor is called a *Codazzi-type Ricci manifold*.

Now, from (87) and (91), one has

$$\begin{aligned}
&\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \text{trace}\, H^2\right\}\{2\eta(Z)g(\phi X, Y + 2\eta(Z)g(HX,Y) \\
&+\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z) + \eta(Y)g(HX,Z) - \eta(X)g(HY,Z)\} \\
&+\frac{5}{2}\Big\{\eta(Y)g(H^2X,HZ) - \eta(X)g(H^2Y,HZ) + \eta(X)g(\phi HY,HZ) \\
&-\eta(Y)g(\phi HX,HZ) + 2\eta(Z)g(H^2X,HY) - 2\eta(Z)g(\phi HX,HY)\Big\} = 0.
\end{aligned} \tag{92}$$

Letting $X = \xi$ in this equation yields

$$\begin{aligned}
&\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \text{trace}\, H^2\right\}\{-g(\phi Y,Z) - g(HY,Z)\} \\
&+\frac{5}{2}\Big\{-g(H^2Y,HZ) + g(\phi HY,HZ)\Big\} = 0.
\end{aligned} \tag{93}$$

If $Z = \phi Y$, then this equation becomes

$$\frac{2}{5}\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \text{trace}\, H^2\right\}\{g(Y,Y) - \eta(Y)\eta(Y)\} + g(HY,HY) = 0. \tag{94}$$

We set

$$\mu = \frac{2}{5}\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \text{trace}\, H^2\right\}. \tag{95}$$

Then, relation (94) leads to

$$H^2 = \mu\{\mathbb{I} - \eta \otimes \xi\}. \tag{96}$$

In accordance with Lemma 1 and Theorem 8, the function μ defined in (95) on a nearly Sasakian space form is a constant. This is achieved by taking into account the reasoning that led to (26), $\mu = -\lambda^2$. This means that μ is non-positive. According to Theorem 4.1 in [5], we have the following result.

Theorem 9. *A Codazzi-type Ricci nearly Sasakian space form is either a Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} = 1$ or is a 5-dimensional proper nearly Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} > 1$.*

Proof. The last assertion follows from (95), (96), and the sign of μ. □

Note that $H\xi = 0$, i.e., 0 is an eigenvalue of H^2. Also, since the operator H is skew-symmetric, the non-vanishing eigenvalues of H^2 are negative, as proven by (26). Thus, the spectrum of H^2 is of the type

$$\mathrm{Spec}(H^2) = \{0, -\lambda^2, \cdots, -\lambda^2\}, \;\; \lambda \neq 0.$$

Let $\mathbb{R}\xi$, $D(0)$, and $D(-\lambda^2)$ denote the distribution of dimension 1 generated by ξ and the distributions of the eigenvectors corresponding to the eigenvalues 0 and $-\lambda^2$, respectively.

If X is an eigenvector of H^2 with a corresponding eigenvalue of $-\lambda^2$, then, from (4), we have

$$H^2\nabla_X\xi = -(\nabla_X H^2)\xi = -(\phi + H)H^2X = -\lambda^2\nabla_X\xi. \tag{97}$$

This means that $\nabla_X\xi$ is an eigenvector corresponding to the eigenvalue $-\lambda^2$. Given that the relation (14) leads to $\nabla_\xi H^2 = 0$, we have

$$H^2\nabla_\xi X = \nabla_\xi H^2 X = -\xi(\lambda^2)X - \lambda^2\nabla_\xi X = -\lambda^2\nabla_\xi X. \tag{98}$$

Thus, $\nabla_\xi X$ is also an eigenvector corresponding to the eigenvalue $-\lambda^2$. If vector fields X and Y are both eigenvectors with the eigenvalue $-\lambda^2$ and are orthogonal to ξ, then, from (14), one obtains

$$H^2(\nabla_X Y) = \nabla_X H^2 Y - (\nabla_X H^2)Y = -\lambda^2\nabla_X Y + \lambda^2 g(\phi X + HX, Y)\xi. \tag{99}$$

If $\lambda = 0$, the $\nabla_X Y$ belongs to $D(0)$. If $\lambda \neq 0$, one obtains

$$H^2(\phi^2\nabla_X Y) = \phi^2(H^2\nabla_X Y) = -\lambda^2\phi^2(\nabla_X Y), \tag{100}$$

and, thus,

$$\nabla_X Y = -\phi^2\nabla_X Y + \eta(\nabla_X Y)\xi \in \mathbb{R}\xi \oplus D(-\lambda^2).$$

Note that, if X is an eigenvector of H^2 with an eigenvalue $-\lambda^2$, then the vector fields X, ϕX, HX, and $H\phi X$ are mutually orthogonal, and they are also eigenvectors of H^2 with the corresponding eigenvalue $-\lambda^2$. By using Theorem 9, 0 becomes a simple eigenvalue, and the multiplicity of the eigenvalue $-\lambda^2$ is 4. Therefore, we obtain the following result.

Theorem 10. *Let M be a Codazzi-type Ricci nearly Sasakian space form. Then, the spectrum of H^2 has the form*

$$\mathrm{Spec}(H^2) = \{0, -\lambda^2,\ -\lambda^2,\ -\lambda^2,\ -\lambda^2\},\ \ \lambda \neq 0,$$

where 0 is a simple eigenvalue, and $-\lambda^2$ is an eigenvalue of multiplicity 4. Moreover, the distributions $D(0)$ and $\mathbb{R}\xi \oplus D(-\lambda^2)$ are integrable with totally geodesic leaves.

Cappelletti-Montano and Dileo proved in ([Theorem 4.3] in [2]) that there is a one-to-one correspondence between a nearly Sasakian space form and $SU(2)$ structures. The latter induces a Sasaki–Einstein structure (see [2] for more details). Therefore, we have the following result.

Theorem 11. *A Codazzi-type Ricci nearly Sasakian space form carries a Sasaki–Einstein structure.*

A similar conclusion from Theorem 11 can also be induced from some of the results found in Section 3. In [Theorem 6.1] in [4], Olszak proved, under the condition (22), that a proper nearly Sasakian space form is a 5-dimensional manifold of a constant sectional curvature. Next, we prove otherwise using the projectively flat notion. First of all, we note that the class of Codazzi-type Ricci manifolds is a subclass of projectively flat manifolds (see [Proposition 5] in [17] for more details). The concept of projectively flat is defined via a tensor called the projective curvature tensor. This plays a role as an important tensor in differential geometry. A manifold M is said to be locally projectively flat if there is a one-to-one correspondence between each coordinate system of M and a subspace of a Euclidean space $\mathbb{E}$ such that any geodesic of M corresponds to a straight line in $\mathbb{E}$. As known in ([17], p. 411), the Levi–Civita connection of a non-degenerate metric g is locally projectively flat if and only if g has a constant sectional curvature.

For $n \geq 1$, a nearly Sasakian manifold M is locally projectively flat if and only if the projective curvature tensor $\mathcal{P}$ vanishes, where $\mathcal{P}$ is given by (see [17])

$$\mathcal{P}(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{Ric(Y,Z)X - Ric(X,Z)Y\} \tag{101}$$

for any vector fields X, Y, and Z of M.

Theorem 12. *A proper nearly Sasakian space form is not of constant sectional curvature.*

Proof. Let M be a proper nearly Sasakian space form. If we assume that M is of a constant sectional curvature, then it is locally projectively flat; that is, the projective curvature tensor $\mathcal{P}$ in (101) vanishes. A direct calculation of (101) leads to

$$\begin{aligned}
&2nR(X,Y)Z - \{Ric(Y,Z)X - Ric(X,Z)Y\} = -\frac{\mathcal{H}-1}{2}\{g(Z,Y)X - g(X,Z)Y\} \\
&+ \left\{\text{trace}\, H^2 + \frac{\mathcal{H}-1}{2}\right\}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} + \frac{n(\mathcal{H}-1)}{2}\{\eta(Y)g(X,Z)\xi \\
&- \eta(X)g(Z,Y)\xi + g(Z,\phi Y)\phi X + g(X,\phi Z)\phi Y + 2g(X,\phi Y)\phi Z\} \\
&- \frac{n}{2}\{g(HZ,Y)HX + g(HX,Z)HY + 2g(HX,Y)HZ - g(HZ,\phi Y)\phi HX \\
&- g(HX,\phi Z)\phi HY - 2g(HX,\phi Y)\phi HZ\} + \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X \\
&+ 2n\{\eta(Y)g(H^2Z,X) - \eta(X)g(H^2Z,Y)\}\xi - \frac{5}{2}g(HY,HZ)X + \frac{5}{2}g(HX,HZ)Y.
\end{aligned} \tag{102}$$

Now, by putting $Y = Z \in \Gamma(D)$ into (102) and considering $X \in \Gamma(D)$ such that $g(X,Y) = 0$, we have

$$2ng(\mathcal{P}(X,Y)Y,Y) = \frac{5}{2}g(HX,HY)g(Y,Y). \tag{103}$$

Since M is locally projectively flat, then (103) vanishes; that is,

$$0 = g(HX,HY)g(Y,Y), \quad \forall\, X \in \Gamma(D).$$

This implies that $H^2Y = 0$, as $g(Y,Y) \neq 0, \forall\, Y \in \Gamma(D)$. Since $H\xi = 0$, thus $H^2 = 0$ for M, which is a contradiction, as M is non-Sasakian. This completes the proof. □

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Article

Geometry of CR-Slant Warped Products in Nearly Kaehler Manifolds

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Abstract: Recently, we studied CR-slant warped products $B_1 \times_f M_\perp$, where $B_1 = M_T \times M_\theta$ is the Riemannian product of holomorphic and proper slant submanifolds and $M_\perp$ is a totally real submanifold in a nearly Kaehler manifold. In the continuation, in this paper, we study $B_2 \times_f M_\theta$, where $B_2 = M_T \times M_\perp$ is a CR-product of a nearly Kaehler manifold and establish Chen's inequality for the squared norm of the second fundamental form. Some special cases of Chen's inequality are given.

Keywords: CR-product; CR-warped product; CR-slant warped product; Chen's inequality; nearly Kaehler manifolds

MSC: 53B05; 53B20; 53C25; 53C40

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1. Introduction

A submanifold M of an almost Hermitian manifold $\tilde{M}$ is called a *complex submanifold* of $\tilde{M}$ if its tangent space remains the same under the action of almost complex structure J. On contrary, M is called a *totally real submanifold* if J carries each tangent space of M into the corresponding normal space (see [1]). A submanifold M of $\tilde{M}$ is called a *CR-submanifold* (or *Cauchy–Riemann submanifold*) [2] if there exists a complex distribution $\mathfrak{D}$ on M whose orthogonal complementary distribution $\mathfrak{D}^\perp$ is a totally real distribution, i.e., $J\mathfrak{D}_p^\perp \subset T_p^\perp N, \forall p \in M$.

A CR-submanifold is called a *CR-product* [3] if it is a Riemannian product of a complex submanifold M_T and a totally real submanifold $M_\perp$. For basic properties of CR-products in Käher manifolds, see, e.g., [2–5]. In [6,7], the second author introduced and investigated fundamental properties of a much larger class of CR-submanifolds; namely, the class of CR-warped product submanifolds. It was proved in [6] that there are no CR-warped product submanifolds in a Kaehler manifold $\tilde{M}$ which are of the form $M_\perp \times_f M_T$, where $M_\perp$ is totally real and M_T is complex in $\tilde{M}$. On the other hand, a CR-submanifold M is called a *CR-warped product* [6] if it is the warped product $M_T \times_f M_\perp$ of a complex submanifold M_T and a totally real submanifold $M_\perp$, where f is the warping function.

The second author proved in [6] that every CR-warped product $M_T \times_f M_\perp$ in an arbitrary Kaehler manifold satisfies the basic inequality,

$$\|h\|^2 \geq 2p\|\nabla(\ln f)\|^2,$$

where p is the dimension of $M_\perp$, $\|h\|^2$ is the squared norm of the second fundamental form h, and $\nabla(\ln f)$ is the gradient of $\ln f$. The second author also classified all CR-warped products in complex space form satisfying the equality of the inequality in [6,7]. For further results in this respect, see [4,5,8–14].

CR-slant warped product submanifolds of the form $B_1 \times_f M_\perp$ in a nearly Kaehler manifold $\tilde{M}$ were studied in [14], where $B_1 = M_T \times M_\theta$ is the Riemannian product of a complex submanifold and a proper slant submanifold of $\tilde{M}$. In fact, the following Chen type inequality was established in [14].

Theorem 1 ([14])**.** *Let $M = B_1 \times_f M_\perp$ be a CR-slant warped product submanifold of a nearly Kaehler manifold $\tilde{M}$, where $B_1 = M_T \times M_\theta$ is the Riemannian product of complex and proper slant submanifolds of $\tilde{M}$. If M is $\mathfrak{D}^\perp \oplus \mathfrak{D}^\theta$-mixed totally geodesic in $\tilde{M}$, then:*

(i) *The second fundamental form h satisfies*

$$\|h\|^2 \geq 2s\|\vec{\nabla}^T(\ln f)\|^2 + s\cot^2\theta\|\vec{\nabla}^\theta(\ln f)\|^2 \tag{1}$$

where $s = \dim M_\perp$ and $\vec{\nabla}^T(\ln f)$ and $\vec{\nabla}^\theta(\ln f)$ denote the gradient components of $\ln f$ along M_T and M_θ, respectively.

(ii) *If the equality sign in* (1) *holds identically, then M_T and M_θ are totally geodesic, B_1 is mixed totally geodesic in $\tilde{M}$ and $M_\perp$ is totally umbilical in $\tilde{M}$.*

In the sequel, we study in this paper CR-slant warped product submanifolds of the form $M = B_2 \times_f M_\theta^{n_3}$, where $B_2 = M_T^{n_1} \times M_\perp^{n_2}$ is a CR-product and $M_\theta^{n_3}$ is an n_3-dimensional proper θ-slant submanifold in a nearly Kaehler manifold $\tilde{M}^{2m}$. We prove that the second fundamental form h of M satisfies the following inequality

$$\|h\|^2 \geq \frac{1}{9}n_3\cos^2\theta\|\vec{\nabla}^\perp(\ln f)\|^2 + 2n_3\left(1 + \frac{10}{9}\cot^2\theta\right)\|\vec{\nabla}^T(\ln f)\|^2$$

where $\vec{\nabla}^\perp(\ln f)$ and $\vec{\nabla}^T(\ln f)$ are the gradients of $\ln f$ along $M_\perp$ and M_T, respectively. In this paper, we also discuss the equality case of this inequality. Several immediate consequences of this inequality are also given.

2. Basic Definitions and Formulas

Let $\tilde{M}^{2m}$ be an almost Hermitian manifold endowed with an almost complex structure J and a Riemannian metric $\tilde{g}$, such that

$$J^2(X) = -X, \quad \tilde{g}(JX, JY) = \tilde{g}(X, Y) \tag{2}$$

for any $X, Y \in \Gamma(T\tilde{M}^{2m})$, where $\Gamma(T\tilde{M}^{2m})$ denotes the Lie algebra of vector fields on $\tilde{M}^{2m}$. In addition, an almost Hermitian manifold is called *Kaehler manifold* if

$$(\tilde{\nabla}_X J)Y = 0, \ \forall X, Y \in \Gamma(T\tilde{M}^{2m}),$$

where $\tilde{\nabla}$ is the Levi–Civita connection on $\tilde{M}^{2m}$. Furthermore, an almost Hermitian manifold $\tilde{M}^{2m}$ is *nearly Kaehler* if $(\tilde{\nabla}_X J)X = 0$, $\forall X \in \Gamma(T\tilde{M}^{2m})$, equivalently

$$(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0, \ \forall X, Y \in \Gamma(T\tilde{M}^{2m}). \tag{3}$$

Clearly, every Kaehler manifold is nearly Kaehler but the converse is not true in general. The best known example of a nearly Kaehler non-Kaehlerian manifold is 6-dimensional sphere $\mathbb{S}^6$. For further results on nearly Kaehler manifolds, see, e.g., [15–19].

Let M be a Riemannian manifold isometrically immersed in $\tilde{M}^{2m}$. We denote the metric $\tilde{g}$ and the induced metric g on M by the same symbol g. The Gauss and Weingarten formulas are, respectively, given by (see, e.g., [4,5])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{4}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{5}$$

for vector fields $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^{\perp}M)$, where $\Gamma(T^{\perp}M)$ denotes the set of all vector fields normal to M and ∇ and $\nabla^{\perp}$ denote the induced connections on the tangent and normal bundles of M, respectively, and h is the second fundamental form A is the shape operator of M; and they are related by

$$g(A_{\xi}X, Y) = g(h(X, Y), \xi) \tag{6}$$

for any vector fields $X, Y \in \Gamma(TM)$ and any normal vector $\xi \in \Gamma(T^{\perp}M)$. A submanifold M in $\tilde{M}^{2m}$ is called *totally geodesic* if the second fundamental form h vanishes identically on M. Furthermore, M is called *totally umbilical* if h satisfies

$$h(X, Y) = g(X, Y)H, \tag{7}$$

where H is the mean curvature vector M defined by $H = \frac{1}{n}\mathrm{trace}\, h$, $n = \dim M$.

For each vector field X tangent to M, we write

$$JX = PX + FX, \tag{8}$$

where PX and FX are the tangential and normal components of JX.

Definition 1 ([20,21])**.** *A submanifold M of an almost Hermitian manifold $\tilde{M}$ is called* slant *if for each $p \in M$, the Wirtinger angle $\theta(X)$ between JX and T_pM is constant on M, i.e., it does not depend on the choice of $X \in T_pM$ and $p \in M$. In this case, θ is called the slant angle of M.*

Complex and totally real submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold is called *proper* if it is neither complex nor totally real.

More generally, a distribution $\mathfrak{D}$ on M is called a *slant distribution* if the angle $\theta(X)$ between JX and $\mathfrak{D}_p$ is independent of the choice of $p \in M$ for any $0 \neq X \in \mathfrak{D}_p$. The second author shown that a submanifold M of $\tilde{M}$ is slant if, and only if, we have [20]

$$P^2X = -(\cos^2\theta)X, \quad X \in \Gamma(TM). \tag{9}$$

Clearly, it follows from (8) and (9) that

$$g(PX, PY) = (\cos^2\theta)g(X, Y), \quad g(FX, FY) = (\sin^2\theta)g(X, Y), \tag{10}$$

for any vector fields X, Y tangent to M.

Definition 2. *A submanifold M of an almost Hermitian manifold $\tilde{M}$ is called* CR-slant *if there exist mutually orthogonal distributions $\mathfrak{D}$, $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$, such that the tangent bundle is spanned by*

$$TM = \mathfrak{D} \oplus \mathfrak{D}^{\perp} \oplus \mathfrak{D}^{\theta}, \tag{11}$$

where $\mathfrak{D}$, $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$ are complex, totally real, and proper slant distributions.

The normal bundle of a CR-slant submanifold M is decomposed by

$$T^{\perp}M = J\mathfrak{D}^{\perp} \oplus F\mathfrak{D}^{\theta} \oplus \nu, \tag{12}$$

where ν is an invariant normal sub-bundle of the normal bundle $T^{\perp}M$. A CR-slant product submanifold M is called *semi-slant mixed-totally geodesic* (resp., *hemi-slant mixed-totally geodesic*) if its second fundamental form satisfies

$$h(X_1, X_2) = 0 \quad \forall\, X_1 \in \Gamma(\mathfrak{D}), \ \ \forall\, X_2 \in \Gamma(\mathfrak{D}^{\theta})$$
$$(resp., \ \ h(X_2, X_3) = 0 \quad \forall\, X_2 \in \Gamma(\mathfrak{D}^{\theta}), \ \ \forall\, X_3 \in \Gamma(\mathfrak{D}^{\perp})).$$

3. CR-Slant Warped Products $(M_T \times M_\perp) \times_f M_\theta$

In this section, first we recall the definition of warped product manifolds which are the generalizations of Riemannian products. In 1969, Bishop and O'Neill [22] introduced the notion of warped product manifolds as follows:

Definition 3. *A warped product $B \times_f F$ of two Riemannian manifolds (B, g_B) and (F, g_F) is the product manifold $M = B \times F$ equipped with the product structure*

$$g_M(X, Y) = g_B(\pi_{1*}X, \pi_{1*}Y) + (f \circ \pi_1)^2 g_F(\pi_{2*}X, \pi_{2*}Y),$$

where $f : B \to (0, \infty)$ and $\pi_1 : M \to B,\ \pi_2 : M \to F$ are projection maps given by $\pi_1(p, q) = p$ and $\pi_2(p, q) = q$ for any $(p, q) \in B \times F$ and $*$ denotes the symbol for tangent map.

The function f is called warping function, if f is constant, then M is simply a Riemannian product. It is known that, for any vector field X on B and a vector field Z on F, we have [22,23]

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{13}$$

where ∇ is the Levi–Civita connection on M. Further, it is well known that the base manifold B is totally geodesic and the fiber F is totally umbilical in M.

Next, we define CR-slant warped products $(M_T \times M_\perp) \times_f M_\theta$ as follows.

Definition 4. *A submanifold M of an almost Hermitian manifold $\tilde{M}$ is said to be* CR-slant warped product submanifold *if it is a warped product of CR-product $M_T \times M_\perp$ and a proper θ-slant submanifold M_θ of $\tilde{M}$.*

In [14], we studied CR-slant warped product submanifolds of the form $B_1 \times_f M_\perp$, where $B_1 = M_T \times M_\theta$. In this section, we study CR-slant warped products of the form $B_2 \times_f M_\theta$, where $B_2 = M_T \times M_\perp$. For this, we use the following conventions, $X_1, Y_1, \cdots$ are vector fields on $\mathfrak{D}$ and $X_2, Y_2, \cdots$ are vector fields on $\mathfrak{D}^\theta$, while $X_3, Y_3, \cdots$ are vector fields on $\mathfrak{D}^\perp$.

First, we have the following preparatory lemmas.

Lemma 1. *On a CR-slant warped product submanifold $M = B_2 \times_f M_\theta$ of a nearly Kaehler manifold $\tilde{M}$, we have*

(i) $g(h(X_1, Y_1), FX_2) = 0$,
(ii) $2g(h(X_3, Y_3), FX_2) = g(h(X_2, X_3), JY_3) + g(h(X_2, Y_3), JX_3)$.

for any $X_1, Y_1 \in \Gamma(TM_T),\ X_2 \in \Gamma(TM_\theta)$ and $X_3, Y_3 \in \Gamma(TM_\perp)$, where $B_2 = M_T \times M_\perp$ is the CR-product submanifold in $\tilde{M}$.

Proof. The first part is easy to prove by using (3), (4) and (13). For the second part, we have

$$\begin{aligned} g(h(X_3, Y_3), FX_2) =& g(\tilde{\nabla}_{X_3} Y_3, JX_2) + g(\tilde{\nabla}_{X_3} PX_2, Y_3) \\ & - g(J\nabla_{X_3} Y_3, X_2) + g(\nabla_{X_3} Y_3, PX_2) \end{aligned}$$

for any $X_2 \in \Gamma(TM_\theta)$ and $X_3, Y_3 \in \Gamma(TM_\perp)$. Since $\nabla_{X_3} Y_3 \in \Gamma(TM_\perp)$, then using orthogonality of vector fields and covariant derivative property of J with (13), we find

$$\begin{aligned} g(h(X_3, Y_3), FX_2) &= g((\tilde{\nabla}_{X_3} J)Y_3, X_2) - g(\tilde{\nabla}_{X_3} JY_3, X_2) + X_3(\ln f) g(PX_2, Y_3) \\ &= g((\tilde{\nabla}_{X_3} J)Y_3, X_2) + g(h(X_2, X_3), JY_3). \end{aligned} \tag{14}$$

Similarly, by interchanging X_3 with Y_3 in (14), we brain

$$g(h(X_3, Y_3), FX_2) = g((\tilde{\nabla}_{Y_3} J)X_3, X_2) + g(h(X_2, Y_3), JX_3). \tag{15}$$

Hence, the second part immediately follows from (14) and (15). □

Lemma 2. *Let $M = B_2 \times_f M_\theta$ be a CR-slant warped product submanifold of a nearly Kaehler manifold $\tilde{M}$, such that $B_2 = M_T \times M_\perp$ is the CR-product submanifold in $\tilde{M}$. Then, we have*

$$g(h(X_1, X_3), FX_2) = \frac{1}{2} g(h(X_1, X_2), JX_3), \tag{16}$$

for any $X_1 \in \Gamma(TM_T)$, $X_2 \in \Gamma(TM_\theta)$ and $X_3 \in \Gamma(TM_\perp)$.

Proof. For any $X_1 \in \Gamma(TM_T)$, $X_2 \in \Gamma(TM_\theta)$ and $X_3 \in \Gamma(TM_\perp)$, we have

$$g(h(X_1, X_3), FX_2) = g((\tilde{\nabla}_{X_3} J)X_1, X_2) - g(\tilde{\nabla}_{X_3} JX_1, X_2) = g((\tilde{\nabla}_{X_3} J)X_1, X_2). \tag{17}$$

On the other hand, we know that

$$g(h(X_1, X_3), FX_2) = g((\tilde{\nabla}_{X_1} J)X_3, X_2) - g(\tilde{\nabla}_{X_1} JX_3, X_2) + g(X_3, \tilde{\nabla}_{X_1} PX_2). \tag{18}$$

Then, the lemma follows from (17) and (18) with the help of (3) and (13). □

Lemma 3. *For a proper CR-slant warped product $M = B_2 \times_f M_\theta$, such that $B_2 = M_T \times M_\perp$ in a nearly Kaehler manifold $\tilde{M}$, we have*

$$g(h(JX_1, X_2), FY_2) = X_1(\ln f) g(X_2, Y_2) + \frac{1}{3} JX_1(\ln f) g(X_2, PY_2), \tag{19}$$

for any $X_1 \in \Gamma(TM_T)$, $X_2, Y_2 \in \Gamma(TM_\theta)$.

Proof. From (4) and (13), we have

$$g(h(X_1, X_2), FY_2) = g((\tilde{\nabla}_{X_2} J)X_1, Y_2) - JX_1(\ln f) g(X_2, Y_2), \tag{20}$$

for any orthogonal vector fields $X_1 \in \Gamma(TM_T)$, $X_2, Y_2 \in \Gamma(TM_\theta)$. On the other hand, we derive

$$g(h(X_1, X_2), FY_2) = g((\tilde{\nabla}_{X_1} J)X_2, Y_2) - X_1(\ln f) g(PX_2, Y_2) + g(h(X_1, Y_2), FX_2). \tag{21}$$

Then, from (20) and (21), we find

$$2g(h(X_1, X_2), FY_2) = X_1(\ln f) g(X_2, PY_2) - JX_1(\ln f) g(X_2, Y_2) + g(h(X_1, Y_2), FX_2). \tag{22}$$

Interchanging X_2 by Y_2, we obtain

$$2g(h(X_1, Y_2), FX_2) = X_1(\ln f) g(PX_2, Y_2) - JX_1(\ln f) g(X_2, Y_2) + g(h(X_1, X_2), FY_2). \tag{23}$$

Then, from (22) and (23), we derive

$$g(h(X_1, X_2), FY_2) = -JX_1(\ln f) g(X_2, Y_2) + \frac{1}{3} X_1(\ln f) g(X_2, PY_2). \tag{24}$$

Hence, (19) follows immediately by interchanging X_1 with JX_1 in (24), which proves the lemma completely. □

The following relations are immediate consequences of (19).

$$g(h(JX_1, PX_2), FY_2) = X_1(\ln f)g(PX_2, Y_2) + \frac{1}{3}\cos^2\theta JX_1(\ln f)g(X_2, Y_2), \quad (25)$$

$$g(h(JX_1, PX_2), FPY_2) = \cos^2\theta X_1(\ln f)g(X_2, Y_2) + \frac{1}{3}\cos^2\theta JX_1(\ln f)g(X_2, PY_2), \quad (26)$$

$$g(h(JX_1, X_2), FPY_2) = X_1(\ln f)g(X_2, PY_2) - \frac{1}{3}\cos^2\theta JX_1(\ln f)g(X_2, Y_2). \quad (27)$$

Lemma 4. *Let $M = B_2 \times_f M_\theta$ be a CR-slant warped product submanifold of a nearly Kaehler manifold $\tilde{M}$ such that $B_2 = M_T \times M_\perp$ is the CR-product submanifold in $\tilde{M}$. Then, we have*

$$g(h(X_2, Y_2), JX_3) = g(h(X_2, X_3), FY_2) + \frac{1}{3}X_3(\ln f)g(X_2, PY_2) \quad (28)$$

for any $X_2, Y_2 \in \Gamma(TM_\theta)$ and $X_3 \in \Gamma(TM_\perp)$.

Proof. From the definition of covariant derivative with (4) and (8), we have

$$g(h(X_2, X_3), FY_2) = g((\tilde{\nabla}_{X_3}J)X_2, Y_2) - g(\tilde{\nabla}_{X_3}PX_2, Y_2) - g(\tilde{\nabla}_{X_3}FX_2, Y_2) - g(\tilde{\nabla}_{X_3}X_2, PY_2).$$

Again, using (4), (5), and (13), we find

$$g(h(X_2, X_3), FY_2) = g((\tilde{\nabla}_{X_3}J)X_2, Y_2) + g(h(Y_2, X_3), FX_2). \quad (29)$$

On the other hand, we derive

$$\begin{aligned} g(h(X_2, X_3), FY_2) &= g((\tilde{\nabla}_{X_2}J)X_3, Y_2) - g(\tilde{\nabla}_{X_2}JX_3, Y_2) - g(\tilde{\nabla}_{X_2}X_3, PY_2) \\ &= g((\tilde{\nabla}_{X_2}J)X_3, Y_2) + g(h(X_2, Y_2), JX_3) - X_3(\ln f)g(X_2, PY_2). \end{aligned} \quad (30)$$

Then, from (29) and (30), we obtain

$$2g(h(X_2, X_3), FY_2) = g(h(X_2, Y_2), JX_3) + g(h(Y_2, X_3), FX_2) - X_3(\ln f)g(X_2, PY_2). \quad (31)$$

Interchanging X_2 by Y_2, we obtain

$$2g(h(Y_2, X_3), FX_2) = g(h(X_2, Y_2), JX_3) + g(h(X_2, X_3), FY_2) + X_3(\ln f)g(X_2, PY_2). \quad (32)$$

Then, from (31) and (32), we obtain (28); which proves the Lemma completely. □

4. Chen's Inequality and Its Consequences

In this section, first we prove the following main result by using Lemma 3.

Theorem 2. *Let $M = B_2 \times_f M_\theta$ be a proper CR-slant warped product submanifold of a nearly Kaehler manifold $\tilde{M}$. Then, M is a Riemannian product if, and only if, either M is semi-slant mixed-totally geodesic, (i.e., $h(X_1, X_2) = 0, \forall X_1 \in \Gamma(\mathfrak{D}), X_2 \in \Gamma(\mathfrak{D}^\theta)$) or $h(\mathfrak{D}, \mathfrak{D}^\theta)$ is orthogonal to $F\mathfrak{D}^\theta$.*

Proof. From Lemma 3, we find

$$g(h(JX_1, X_2), FY_2) = \frac{1}{3}JX_1(\ln f)g(X_2, PY_2) + X_1(\ln f)g(X_2, Y_2), \quad (33)$$

for any $X_1 \in \Gamma(\mathfrak{D}),\ X_2, Y_2 \in \Gamma(\mathfrak{D}^\theta)$. Then, from (27) and (33), we derive

$$g(h(JX_1, X_2), FY_2) + \frac{1}{3}g(h(X_1, X_2), FPY_2) = \left(1 - \frac{1}{9}\cos^2\theta\right)X_1(\ln f)g(X_2, Y_2). \quad (34)$$

If M is semi-slant mixed totally geodesic or $h(\mathfrak{D},\mathfrak{D}^\theta)$ is orthogonal to $F\mathfrak{D}^\theta$ then from (34), we find

$$\left(1-\frac{1}{9}\cos^2\theta\right)X_1(\ln f)g(X_2,Y_2)=0.$$

Since g is a Riemannian metric and $-1\leq\cos\theta\leq 1$, then from above equation we obtain $X_1(\ln f)=0$, i.e., f is constant along M_T.

Conversely, if f is constant then again from (34), we obtain

$$g(h(JX_1,X_2),FY_2)+\frac{1}{3}g(h(X_1,X_2),FPY_2)=0. \tag{35}$$

Interchanging X_1 by JX_1 and Y_2 by PY_2 in (35), we derive

$$g(h(X_1,X_2),FPY_2)+\frac{1}{3}\cos^2\theta g(h(JX_1,X_2),FY_2)=0. \tag{36}$$

Then, from (35) and (36), we obtain

$$\left(1-\frac{1}{9}\cos^2\theta\right)g(h(JX_1,X_2),FY_2)=0. \tag{37}$$

Since $-1\leq\cos\theta\leq 1$ for any value of $\theta\in\mathbb{R}$, thus we find either $h(\mathfrak{D},\mathfrak{D}^\theta)=\{0\}$ or $h(\mathfrak{D},\mathfrak{D}^\theta)$ is orthogonal to $F\mathfrak{D}^\theta$, which completes the proof. □

Next, we derive the Chen's inequality for CR-slant wanted products $M=B_2\times_f M_\theta$, where $B_2=M_T\times M_\perp$ is a CR-product in a nearly Kaehler manifold.

Theorem 3. *Let $M=\left(M_T^{n_1}\times M_\perp^{n_2}\right)\times_f M_\theta^{n_3}$ be a CR-slant warped product submanifold of a nearly Kaehler manifold $\tilde{M}$, such that M is hemi-slant mixed-totally geodesic. Then, the squared norm of the second fundamental form satisfies*

$$\|h\|^2\geq\frac{1}{9}n_3\cos^2\theta\|\vec{\nabla}^\perp(\ln f)\|^2+2n_3\left(1+\frac{10}{9}\cot^2\theta\right)\|\vec{\nabla}^T(\ln f)\|^2 \tag{38}$$

where $\vec{\nabla}^T(\ln f)$ and $\vec{\nabla}^\perp(\ln f)$ denote the gradient components of $\ln f$ along M_T and $M_\perp$, respectively.

Furthermore, if the equality holds in (38), then $M_T\times M_\perp$ is totally geodesic and M_θ is totally umbilical in $\tilde{M}$. Moreover, M is not a semi-slant mixed totally geodesic submanifold of $\tilde{M}$.

Proof. If we denote the tangent bundles of M_T, $M_\perp$ and M_θ by $\mathfrak{D}$, $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$, respectively; then we use the following frame fields for the CR-slant warped product

$$\begin{aligned}
&\mathfrak{D}=\mathrm{Span}\{e_1,\cdots,e_p,\,e_{p+1}=Je_1,\cdots,e_{n_1}=e_{2p}=Je_p\},\\
&\mathfrak{D}^\perp=\mathrm{Span}\{e_{n_1+1}=\hat{e}_1,\cdots,e_{n_1+n_2}=\hat{e}_{n_2}\},\\
&\mathfrak{D}^\theta=\mathrm{Span}\{e_{n_1+n_2+1}=e_1^*,\cdots,e_{n_1+n_2+q}=e_q^*,e_{n_1+n_2+q+1}=\sec\theta Pe_1^*,\cdots,\\
&\qquad e_n=e_{2q}^*=\sec\theta Pe_q^*\}.
\end{aligned}$$

Additionally, the normal bundle frame will be

$$\begin{aligned}
&J\mathfrak{D}^\perp=\mathrm{Span}\{e_{n+1}=\tilde{e}_1=J\hat{e}_1,\cdots,e_{n+n_2}=\tilde{e}_{n_2}=J\hat{e}_{n_2}\},\\
&F\mathfrak{D}^\theta=\mathrm{Span}\{e_{n+n_2+1}=\tilde{e}_{n_2+1}=E_1^*=\csc\theta Fe_1^*,\cdots,e_{n+n_2+q}=\tilde{e}_{n_2+q}=E_q^*=\csc\theta Fe_q^*,\\
&\qquad e_{n+n_2+q+1}=\tilde{e}_{n_2+q+1}=E_{q+1}^*=\csc\theta\sec\theta FPe_1^*,\cdots,\\
&e_{n+n_2+n_3}=\tilde{e}_{n_2+n_3}=E_{n_3}^*=\csc\theta\sec\theta FPe_q^*\},\\
&\nu=\mathrm{Span}\{e_{n+n_2+n_3+1}=\tilde{e}_{n_2+n_3+1},\cdots,e_{2m}=\tilde{e}_{2m-n-n_2-n_3}\}.
\end{aligned}$$

From the definition of h, we find

$$\|h\|^2 = \|h(\mathfrak{D},\mathfrak{D})\|^2 + \|h(\mathfrak{D}^\perp,\mathfrak{D}^\perp)\|^2 + \|h(\mathfrak{D}^\theta,\mathfrak{D}^\theta)\|^2 + 2\Big(\|h(\mathfrak{D},\mathfrak{D}^\perp)\|^2 + \|h(\mathfrak{D},\mathfrak{D}^\theta)\|^2 + \|h(\mathfrak{D}^\perp,\mathfrak{D}^\theta)\|^2\Big). \tag{39}$$

Using the frame fields and preparatory lemmas, we expand each term of (39) as follows:

$$\|h(\mathfrak{D},\mathfrak{D})\|^2 = \sum_{k=1}^{n_2}\sum_{i,j=1}^{n_1}\big(g(h(e_i,e_j),J\hat{e}_k)\big)^2 + \sum_{k=1}^{n_3}\sum_{i,j=1}^{n_1}\big(g(h(e_i,e_j),E_k^*)\big)^2 + \sum_{k=1}^{2m-n-n_2-n_3}\sum_{i,j=1}^{n_1}\big(g(h(e_i,e_j),\tilde{e}_k)\big)^2.$$

Leaving the ν-components terms and the is no warped product relation for the first term, then from Lemma 1 (i), we obtain

$$\|h(\mathfrak{D},\mathfrak{D})\|^2 \geq 0. \tag{40}$$

Similarly, for the second term of (39), we derive

$$\|h(\mathfrak{D}^\perp,\mathfrak{D}^\perp)\|^2 = \sum_{k=1}^{n_2}\sum_{i,j=1}^{n_2}\big(g(h(\hat{e}_i,\hat{e}_j),J\hat{e}_k)\big)^2 + \sum_{k=1}^{n_3}\sum_{i,j=1}^{n_2}\big(g(h(\hat{e}_i,\hat{e}_j),E_k^*)\big)^2 + \sum_{k=1}^{2m-n-n_2-n_3}\sum_{i,j=1}^{n_2}\big(g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_k)\big)^2.$$

Using Lemma 1 (ii) with the given hemi-slant totally geodesic condition and leaving the first and last positive terms, we find

$$\|h(\mathfrak{D}^\perp,\mathfrak{D}^\perp)\|^2 \geq 0. \tag{41}$$

For the third term of (39), we find

$$\|h(\mathfrak{D}^\theta,\mathfrak{D}^\theta)\|^2 = \sum_{k=1}^{n_2}\sum_{i,j=1}^{n_3}\Big(g(h(e_i^*,e_j^*),J\hat{e}_k)\Big)^2 + \sum_{k=1}^{n_3}\sum_{i,j=1}^{n_3}\Big(g(h(e_i^*,e_j^*),E_k^*)\Big)^2 + \sum_{k=1}^{2n-n_2-n_3}\sum_{i,j=1}^{n_3}\Big(g(h(e_i^*,e_j^*),\tilde{e}_k)\Big)^2.$$

Leaving the last two positive terms and using Lemma 4 with mixed totally geodesic condition, we obtain

$$\|h(\mathfrak{D}^\theta,\mathfrak{D}^\theta)\|^2 \geq \frac{2q}{9}\cos^2\theta\sum_{k=1}^{n_2}(e_k(\ln f))^2 = \frac{1}{9}n_3\cos^2\theta\|\vec{\nabla}^\perp(\ln f)\|^2. \tag{42}$$

Similarly, we derive the other terms of (39) as follows

$$\|h(\mathfrak{D},\mathfrak{D}^\perp)\|^2 = \sum_{k,j=1}^{n_2}\sum_{i=1}^{n_1}\big(g(h(e_i,\hat{e}_j),J\hat{e}_k)\big)^2 + \sum_{k=1}^{n_3}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\big(g(h(e_i,\hat{e}_j),E_k^*)\big)^2 + \sum_{k=1}^{2m-n-n_2-n_3}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\big(g(h(e_i,\hat{e}_j),\tilde{e}_k)\big)^2.$$

There is no relation for the first positive term in terms of warped products and leaving the last ν-components term. Then, using Lemma 2, we derive

$$\|h(\mathfrak{D},\mathfrak{D}^{\perp})\|^2 \geq \frac{1}{4}\sum_{j=1}^{n_2}\sum_{i=1}^{n_1}\sum_{k=1}^{n_3}\big(g(h(e_i,e_k^*),J\hat{e}_j)\big)^2 \geq 0. \tag{43}$$

On the other hand, we also have

$$\|h(\mathfrak{D},\mathfrak{D}^{\theta})\|^2 = \sum_{k=1}^{n_2}\sum_{j=1}^{n_3}\sum_{i=1}^{n_1}\Big(g(h(e_i,e_j^*),J\hat{e}_k)\Big)^2 + \sum_{k,j=1}^{n_3}\sum_{i=1}^{n_1}\Big(g(h(e_i,e_j^*),E_k^*)\Big)^2$$
$$+ \sum_{k=1}^{2m-n-n_2-n_3}\sum_{i=1}^{n_1}\sum_{j=1}^{n_3}\Big(g(h(e_i,e_j^*),\tilde{e}_k)\Big)^2.$$

For the first term we use (43) and omit the ν-components terms and using frame fields of $\mathfrak{D}^{\theta}$ and $F\mathfrak{D}^{\theta}$, we derive

$$\begin{aligned}\|h(\mathfrak{D},\mathfrak{D}^{\theta})\|^2 \geq \csc^2\theta\sum_{k,j=1}^{q}\sum_{i=1}^{n_1}\Big(g(h(e_i,e_j^*),Fe_k^*)\Big)^2 \\ + \csc^2\theta\sec^2\theta\sum_{k,j=1}^{q}\sum_{i=1}^{n_1}\Big(g(h(e_i,Te_j^*),Fe_k^*)\Big)^2 \\ + \csc^2\theta\sec^2\theta\sum_{k,j=1}^{q}\sum_{i=1}^{n_1}\Big(g(h(e_i,e_j^*),FTe_k^*)\Big)^2 \\ + \csc^2\theta\sec^4\theta\sum_{k,j=1}^{q}\sum_{i=1}^{n_1}\Big(g(h(e_i,Te_j^*),FTe_k^*)\Big)^2.\end{aligned}$$

Using Lemma 3 with (24)–(27), we obtain

$$\begin{aligned}\|h(\mathfrak{D},\mathfrak{D}^{\theta})\|^2 &\geq 2q\csc^2\theta\sum_{i=1}^{p}\left[(Je_i(\ln f))^2 + (e_i(\ln f))^2\right] \\ &+ \frac{2q}{9}\cot^2\theta\sum_{i=1}^{p}\left[(Je_i(\ln f))^2 + (e_i(\ln f))^2\right] \\ &= 2q\csc^2\theta\sum_{i=1}^{n_1}(e_i(\ln f))^2 + \frac{2q}{9}\cot^2\theta\sum_{i=1}^{n_1}(e_i(\ln f))^2 \\ &= n_3\left(\csc^2\theta + \frac{1}{9}n_3\cot^2\theta\right)\|\vec{\nabla}^T(\ln f)\|^2.\end{aligned} \tag{44}$$

Last term of (39) is identically zero by the hemi-slant mixed totally geodesic condition. Then, for all values of h from (40)–(44), finally we obtain the required inequality (38).

For the equality case, since M is $\mathfrak{D}^{\perp}\oplus\mathfrak{D}^{\theta}$-mixed totally geodesic, i.e.,

$$h(\mathfrak{D}^{\perp},\mathfrak{D}^{\theta}) = \{0\}. \tag{45}$$

Form the leaving and vanishing terms, we also find

$$\begin{aligned}&h(\mathfrak{D},\mathfrak{D}) = \{0\},\ \ h(\mathfrak{D}^{\perp},\mathfrak{D}^{\perp}) = \{0\},\ \ h(\mathfrak{D},\mathfrak{D}^{\perp}) = \{0\},\\ &h(\mathfrak{D}^{\theta},\mathfrak{D}^{\theta}) \subseteq J\mathfrak{D}^{\perp},\ h(\mathfrak{D},\mathfrak{D}^{\theta}) \subseteq F\mathfrak{D}^{\theta}.\end{aligned} \tag{46}$$

Then, $M_T \times M_{\perp}$ is totally geodesic and M_{θ} is totally umbilical in $\tilde{M}$ due to the fact that $M_T \times M_{\perp}$ is totally geodesic and M_{θ} is totally umbilical in M [6,22] with equality holding case of (46). Furthermore, due to Theorem 2 and Lemma 2, we observe that M is not a $\mathfrak{D} \oplus \mathfrak{D}^\theta$-mixed totally geodesic submanifold of $\tilde{M}$. Hence, the proof is complete. □

Now, we give the following consequences of Theorem 3.

A warped submanifold of the form $M = M_\theta \times_f M_\perp$ in a nearly Kaehler manifold $\tilde{M}$ is called *hemi-slant* if $M_\perp$ is a totally real submanifold and M_θ is a proper slant submanifold.

If $\dim M_T = 0$ in Theorem 3, then we have

Theorem 4. *Let $M = M_\perp^{n_1} \times_f M_\theta^{n_2}$ be a mixed totally geodesic hemi-slant warped product submanifold in a nearly Kaehler manifold $\tilde{M}$. Then*

(i) The second fundamental form h of M satisfies

$$||h||^2 \geq \frac{1}{9} n_2 \cos^2\theta\, ||\vec{\nabla}^\perp(\ln f)||^2, \tag{47}$$

where $\vec{\nabla}^\perp(\ln f)$ is the gradient of $\ln f$ along $M_\perp$.

(ii) If the equality sign of (47) holds identically, then $M_\perp$ and M_θ are totally geodesic and totally umbilical submanifolds of $\tilde{M}$, respectively.

On the other hand, if $M_\perp = \{0\}$, we have the following special case of Theorem 3.

Theorem 5 ([24])**.** *Let $M = M_T^{n_1} \times_f M_\theta^{n_2}$ be a semi-slant warped product submanifold in a nearly Kaehler manifold $\tilde{M}$. Then, we have*

(i) The second fundamental form h and the warping function f satisfy

$$\|h\|^2 \geq 2n_2\left(1 + \frac{10}{9}\cot^2\theta\right)\|\vec{\nabla}^T(\ln f)\|^2. \tag{48}$$

where $\vec{\nabla}^T \ln f$ is gradient of $\ln f$ along M_T.

(ii) If the equality sign in (48) holds identically, then M_T is totally geodesic and M_θ is totally umbilical in $\tilde{M}$. Moreover, M is a minimal submanifold in $\tilde{M}$.

Furthermore, if $\dim M_\perp = 0$ and $\theta = \frac{\pi}{2}$ in Theorem 3, then $M = M_T^{n_1} \times_f M_\perp^{n_2}$ is a CR-warped product submanifold of a nearly Kaehler manifold $\tilde{M}$ and they were studied in [25] and, hence, the main Theorem 4.2 of [25] is a special case of Theorem 3.

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mathematics

Article

Quarter-Symmetric Metric Connection on a Cosymplectic Manifold

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Abstract: We study the quarter-symmetric metric A-connection on a cosymplectic manifold. Observing linearly independent curvature tensors with respect to the quarter-symmetric metric A-connection, we construct the Weyl projective curvature tensor on a cosymplectic manifold. In this way, we obtain new conditions for the manifold to be projectively flat. At the end of the paper, we define η-Einstein cosymplectic manifolds of the θ-th kind and prove that they coincide with the η-Einstein cosymplectic manifold.

Keywords: almost-contact manifold; cosymplectic manifold; co-Kähler manifold; quarter-symmetric connection; η-Einstein manifold

MSC: 53B05; 53B35; 53C05; 53C15

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1. Introduction

This paper deals with almost-contact metric manifolds and cosymplectic manifolds. A cosymplectic manifold is an almost-contact metric manifold with a normality condition and with the 2-form F and 1-form η both closed, according to Blair's definition [1]. Lately, the name co-Kähler manifolds has also been used for such manifolds, since they are odd-dimensional analogs of Kähler manifolds [2]. A trivial example of cosymplectic manifolds is given by a product of a Kähler manifold with a circle or line (for instance, see [1,3]). Moreover, there is an example of a compact cosymplectic manifold that is not a global product of a compact Kähler manifold with a circle [4]. In [5], the author studied contact, concircular, recurrent, and torse-forming vector fields on cosymplectic manifolds. Note that a different definition of cosymplectic manifolds was used in some papers (for instance, see [2,6,7]).

Here, we investigate the application of quarter-symmetric metric connections on almost-contact metric manifolds and cosymplectic manifolds. The quarter-symmetric connection in differentiable manifolds was introduced by S. Golab [8]. The systematic study of the quarter-symmetric metric connection was continued by S. C. Rastogi in [9,10]. Many authors studied the quarter-symmetric metric connection on almost-contact metric manifolds and their special manifolds. The properties of the torsion tensor of the quarter-symmetric metric connection on almost-contact metric manifolds were studied in [11]. In [12], the authors studied the existence of almost-pseudo-symmetric and Ricci-symmetric Sasakian manifolds admitting a quarter-symmetric metric connection. The existence of weakly symmetric and weakly Ricci-symmetric Sasakian manifolds admitting a quarter-symmetric metric connection was studied in [13]. The papers [14–16] are devoted to studying some special types of K-contact manifolds with respect to the quarter-symmetric metric connection.

Among other things, the $*$-conformal η-Ricci–Yamabe soliton admitting a quarter-symmetric metric connection on α-cosymplectic manifolds was studied in [17]. If $\alpha = 0$, then the α-cosymplectic manifold reduces to the cosymplectic manifold. Additionally, if the characteristic vector field ξ is projective on an α-cosymplectic manifold, then it is a cosymplectic manifold [18]. On the other hand, if $\alpha \in \mathbb{R} \setminus \{0\}$, then the α-cosymplectic manifold is α-Kenmotsu (see [19]). On the α-Kenmotsu manifold, the characteristic vector field ξ is never projective [18].

If $\mathcal{M}$ is an n-dimensional locally symmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection, then the scalar curvature of the Levi–Civita connection of $\mathcal{M}$ is a negative constant (see Theorem 2 in [20]). In the 3-dimensional Kenmotsu manifold, the η-parallel and cyclic-parallel Ricci tensor with respect to the quarter-symmetric metric connection and the Levi–Civita connection are equivalent [21].

The present paper can be considered a continuation of [22,23], where some curvature properties of quarter-symmetric metric connections on a generalized Riemannian manifold and Kähler manifold were studied. Our goal is to continue to determine new results and geometric structures on cosymplectic manifolds, as well as to apply these results to obtain special examples of the mentioned manifold.

2. Almost-Contact Metric Manifolds

Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold, where $\mathcal{M}$ is an n-dimensional differentiable manifold, G is a non-symmetric (0,2) tensor (the so-called generalized Riemannian metric), g is the symmetric part of G, and F is the skew-symmetric part of G. The tensor A is defined as a tensor associated with the tensor F, i.e.,

$$F(X, Y) = g(AX, Y). \tag{1}$$

Depending on the properties of (1,1) tensor A, we can observe various examples of the generalized Riemannian manifold, such as the almost-Hermitian, almost-para-Hermitian, almost-contact, and almost-para-contact manifolds (see [24]).

An almost-contact metric manifold$(\mathcal{M}, g, A, \eta, \xi)$ is an n-dimensional differentiable manifold $\mathcal{M}$ (where $n = 2k + 1$) equipped with an almost-contact structure A and a characteristic (or Reeb) vector field ξ dual to η with respect to g, $\eta(\xi) = 1$, $\eta(X) = g(X, \xi)$, which satisfies

$$A^2 = -I + \eta \otimes \xi, \quad A\xi = 0, \quad \eta \circ A = 0 \tag{2}$$

and

$$g(AX, AY) = g(X, Y) - \eta(X)\eta(Y). \tag{3}$$

The symmetric metric g that satisfies the previous relationship is called compatibly metric with the almost-contact structure. The fundamental 2-form F, defined by (1), is a degenerate of $F(X, \xi) = 0$ and has a rank of $2k$. It can be easily shown that the generalized metric $G = g + F$ and the fundamental 2-form F satisfy the following relationships:

$$\begin{aligned}
&G(X, \xi) = \eta(X), \quad G(\xi, \xi) = 1, \\
&F(AX, Y) = -F(X, AY), \quad F(AX, AY) = F(X, Y), \\
&G(AX, Y) = -G(X, AY), \quad G(AX, AY) = G(X, Y) - \eta(X)\eta(Y).
\end{aligned}$$

An almost-contact manifold is said to be normal if the corresponding complex structure on $\mathcal{M} \times \mathbb{R}$ is integrable, which is equivalent to the condition $N^{ac} = N + \mathrm{d}\eta \otimes \xi = 0$, where N denotes the Nijenhuis tensor of structure tensor A and d denotes the exterior derivative. An almost-contact metric manifold is said to be an almost-cosymplectic manifold if the 2-form F and 1-form η are both closed, i.e., $\mathrm{d}F = 0$ and $\mathrm{d}\eta = 0$ [3]. If an almost-cosymplectic manifold is normal, then it is called a cosymplectic (or a co-Kähler) manifold [2,3]. An almost-contact metric manifold is cosymplectic if and only if $\overset{g}{\nabla} A = 0$ (for instance, see p. 95 in [25]).

The present paper deals with applying quarter-symmetric connections on almost-contact metric manifolds. A linear connection $\overset{1}{\nabla}$ is said to be quarter-symmetric if its torsion tensor is of the form

$$\overset{1}{T}(X,Y) = \eta(Y)AX - \eta(X)AY.$$

The quarter-symmetric connection $\overset{1}{\nabla}$ preserving the generalized Riemannian metric G, $\overset{1}{\nabla}G = 0$, is called the *quarter-symmetric G-metric connection*, and it is determined by the following equations (see [22]):

$$\overset{1}{\nabla}_X Y = \overset{g}{\nabla}_X Y - \eta(X)AY \tag{4}$$

and

$$\overset{1}{\nabla}g = 0, \quad \overset{1}{\nabla}A = \overset{g}{\nabla}A = 0, \tag{5}$$

where $\overset{g}{\nabla}$ is a Levi–Civita connection. The symmetric connection $\overset{0}{\nabla}$ and the dual connection $\overset{2}{\nabla}$ of the quarter-symmetric connection (4) are given by

$$\overset{0}{\nabla}_X Y = \overset{g}{\nabla}_X Y - \frac{1}{2}\eta(X)AY - \frac{1}{2}\eta(Y)AX, \tag{6}$$

$$\overset{2}{\nabla}_X Y = \overset{g}{\nabla}_X Y - \eta(Y)AX. \tag{7}$$

In [24], it is proved that for the G-metric connection $\overset{1}{\nabla}$ on the almost-contact metric manifold, $\overset{1}{\nabla}\eta = \overset{1}{\nabla}\xi = 0$ holds. Taking into account Equation (5), it follows that on the almost-contact metric manifold, the torsion tensor is parallel to the connection $\overset{1}{\nabla}$, i.e., $\overset{1}{\nabla}\overset{1}{T} = 0$. In the paper [11], quarter-symmetric metric connections (4) are studied on almost-contact metric manifolds, and the properties of the torsion tensor $\overset{1}{T}$ are presented.

From Equation (5), we see that structure tensor A is parallel with respect to the Levi–Civita connection, and it implies the following statement.

Theorem 1. *The almost-contact metric manifold $(\mathcal{M}, g, A, \eta, \xi)$ with a quarter-symmetric connection (4) preserving the generalized Riemannian metric G is a cosymplectic (co-Kähler) manifold.*

Following the previous theorem, further consideration can be given to the cosymplectic (i.e., co-Kähler) manifold. The term "generalized metric (i.e., G-metric) connection" is equivalent to the term "metric A-connection".

In the cosymplectic manifold, it also holds that $\overset{g}{\nabla}\eta = \overset{g}{\nabla}\xi = 0$ (see [3]). Moreover, the Reeb vector ξ is Killing, and its dual 1-form η is harmonic (see Lemma 1.2 in [26]). The Riemannian curvature tensor $\overset{g}{R}$ of the Levi–Civita connection on the cosymplectic manifold $(\mathcal{M}, g, A, \eta, \xi)$ satisfies the following relationships (for instance see [3,7,27,28]):

$$\overset{g}{R}(X,Y)AZ = A\overset{g}{R}(X,Y)Z, \quad \overset{g}{R}(AX,AY)Z = \overset{g}{R}(X,Y)Z, \tag{8}$$

$$\eta(\overset{g}{R}(X,Y)Z) = 0, \quad \overset{g}{R}(X,Y)\xi = \overset{g}{R}(X,\xi)Z = 0, \tag{9}$$

$$\overset{g}{Ric}(AX,AY) = \overset{g}{Ric}(X,Y), \quad \overset{g}{Ric}(X,\xi) = 0, \quad \overset{g}{Q}\xi = 0, \tag{10}$$

where $\overset{g}{Ric}(Y,Z) = Trace\{X \to \overset{g}{R}(X,Y)Z\}$ is the Ricci tensor and $\overset{g}{Q}$ is the Ricci operator defined by $\overset{g}{Ric}(X,Y) = g(\overset{g}{Q}X,Y)$. Additionally, the Ricci operator $\overset{g}{Q}$ commutes with the structure tensor A, i.e., $A\overset{g}{Q} = \overset{g}{Q}A$ (see [7] or [27]).

3. Curvature Properties of Quarter-Symmetric Metric A-Connection on Cosymplectic Manifold

The six linearly independent curvature tensors can be observed with respect to a non-symmetric connection [29]. The curvature tensors $\overset{0}{R}, \overset{1}{R}, \dots, \overset{5}{R}$ with respect to the quarter-symmetric connection (4) on the generalized Riemannian manifold are presented in [22] by Equations (2.6), (2.9)–(2.13). Considering cosymplectic manifold properties (more precisely, Equation (2) and $\overset{g}{\nabla}\eta = 0$), the curvature tensors with respect to the quarter-symmetric connection (4) take the following form:

$$\overset{\gamma}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z, \ \gamma = 1,2,3, \tag{11}$$

$$\overset{0}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z + \frac{1}{4}\eta(Z)(\eta(Y)X - \eta(X)Y), \tag{12}$$

$$\overset{4}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z + \eta(Z)(\eta(Y)X - \eta(X)Y), \tag{13}$$

$$\overset{5}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z + \frac{1}{2}\eta(Y)(\eta(Z)X - \eta(X)Z). \tag{14}$$

Since the curvature tensors $\overset{1}{R}$ and $\overset{2}{R}$ of the connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, respectively, coincide with the Riemannian curvature tensor $\overset{g}{R}$ of the Levi–Civita connection (see Equation (11)), the following theorem holds.

Theorem 2. *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold, let $\overset{g}{\nabla}$ be a Levi-Civita connection, let $\overset{1}{\nabla}$ be a quarter-symmetric metric A-connection (4), and let $\overset{2}{\nabla}$ be its dual connection given by (7). The Riemannian curvature tensor $\overset{g}{R}$ is invariant under connection transformations $\overset{g}{\nabla} \to \overset{1}{\nabla}$ and $\overset{g}{\nabla} \to \overset{2}{\nabla}$.*

On the other hand, for the transformation of connections $\overset{g}{\nabla} \to \overset{0}{\nabla}$, we will prove the following theorem.

Theorem 3. *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold, let $\overset{g}{\nabla}$ be a Levi–Civita connection, and let $\overset{0}{\nabla}$ be a symmetric connection given by (6). The Riemannian curvature tensor $\overset{g}{R}$ cannot be invariant under the connection transformation $\overset{g}{\nabla} \to \overset{0}{\nabla}$.*

Proof. If we assume that $\overset{g}{R}$ is invariant under the connection transformation $\overset{g}{\nabla} \to \overset{0}{\nabla}$, then $\overset{0}{R} = \overset{g}{R}$ holds. Based on Equation (12), we have $\eta(Y)X - \eta(X)Y = 0$. Furthermore, by contracting, we obtain $(n-1)\eta(Y) = 0$, which is impossible. □

From Equations (11)–(14), we can easily conclude that all curvature tensors are skew-symmetric by X and Y, except tensor $\overset{5}{R}$. On the other hand, all curvature tensors $\overset{0}{R}, \overset{1}{R}, \dots, \overset{5}{R}$ have the cyclic-symmetry property. Since tensors $\overset{1}{R}$, $\overset{2}{R}$ and $\overset{3}{R}$ coincide with $\overset{g}{R}$, it is clear that they have the same properties. In the further discussion, we will study only the properties of the curvature tensors $\overset{0}{R}$, $\overset{4}{R}$, and $\overset{5}{R}$. Using the properties satisfied by the Riemannian curvature tensor $\overset{g}{R}$, the following relationships can be easily proven:

$$\eta(\overset{\theta}{R}(X,Y)Z) = 0, \ \overset{\theta}{R}(AX,AY)Z = \overset{g}{R}(X,Y)Z, \ \ \theta = 0,4,5,$$
$$\overset{0}{R}(X,Y)AZ = \overset{4}{R}(X,Y)AZ = \overset{g}{R}(X,Y)AZ, \ \ \overset{5}{R}(X,AY)Z = \overset{g}{R}(X,AY)Z$$

and

$$A\overset{0}{R}(X,Y)Z = \overset{0}{R}(X,Y)AZ + \frac{1}{4}\eta(Z)\overset{1}{T}(X,Y),$$
$$A\overset{4}{R}(X,Y)Z = \overset{4}{R}(X,Y)AZ + \eta(Z)\overset{1}{T}(X,Y),$$
$$A\overset{5}{R}(X,Y)Z = \overset{5}{R}(X,Y)AZ + \frac{1}{2}\eta(Y)\eta(Z)AX.$$

The curvature tensors $\overset{0}{R}$, $\overset{4}{R}$, and $\overset{5}{R}$ and the Reeb vector field ξ satisfy

$$4\overset{0}{R}(X,Y)\xi = \overset{4}{R}(X,Y)\xi = 2\overset{5}{R}(X,\xi)Y = \eta(Y)X - \eta(X)Y,$$
$$4\overset{0}{R}(X,\xi)Y = \overset{4}{R}(X,\xi)Y = 2\overset{5}{R}(X,Y)\xi = -\eta(Y)A^2X,$$
$$\overset{0}{R}(\xi,\xi)X = \overset{4}{R}(\xi,\xi)X = \overset{5}{R}(\xi,X)\xi = 0.$$

By contracting with respect to X in Equations (12)–(14), we obtain the corresponding Ricci tensors, as follows:

$$\overset{0}{Ric} = \overset{g}{Ric} + \frac{n-1}{4}\eta \otimes \eta, \tag{15}$$
$$\overset{4}{Ric} = \overset{g}{Ric} + (n-1)\eta \otimes \eta, \tag{16}$$
$$\overset{5}{Ric} = \overset{g}{Ric} + \frac{n-1}{2}\eta \otimes \eta. \tag{17}$$

We see that all Ricci tensors are symmetric and satisfy the following properties:

$$\overset{\theta}{Ric}(AX,AY) = \overset{g}{Ric}(X,Y), \ \ \theta = 0,4,5, \tag{18}$$
$$4\overset{0}{Ric}(X,\xi) = \overset{4}{Ric}(X,\xi) = 2\overset{5}{Ric}(X,\xi) = (n-1)\eta(X), \tag{19}$$
$$4\overset{0}{Ric}(\xi,\xi) = \overset{4}{Ric}(\xi,\xi) = 2\overset{5}{Ric}(\xi,\xi) = n-1. \tag{20}$$

Using the above results, we prove the following theorem.

Theorem 4. *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric A-connection (4). Then, the Ricci operators $\overset{\theta}{Q}$, $\overset{\theta}{Ric}(X,Y) = g(\overset{\theta}{Q}X,Y)$ commute with the structure tensor A.*

Proof. From Equations (15)–(17), we obtain the corresponding Ricci operators,

$$\overset{0}{Q} = \overset{g}{Q} + \frac{n-1}{4}\eta \otimes \xi, \ \ \overset{4}{Q} = \overset{g}{Q} + (n-1)\eta \otimes \xi, \ \ \overset{5}{Q} = \overset{g}{Q} + \frac{n-1}{2}\eta \otimes \xi. \tag{21}$$

Taking into account Equation (2), we have

$$A\overset{\theta}{Q} = A\overset{g}{Q} \quad \text{and} \quad \overset{\theta}{Q}A = \overset{g}{Q}A.$$

Considering that the Ricci operator $\overset{g}{Q}$ commutes with A, we have thus proved the statement. □

From the relationship (21), we obtain corresponding curvature scalars, which satisfy

$$4(\overset{0}{r} - \overset{g}{r}) = \overset{4}{r} - \overset{g}{r} = 2(\overset{5}{r} - \overset{g}{r}) = n - 1 \tag{22}$$

and with this, we have proved the following theorem.

Theorem 5. *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric A-connection (4). Then, the differences $\overset{\theta}{r} - \overset{g}{r}$ are constant, where $\overset{\theta}{r}$ and $\overset{g}{r}$ denote curvature scalars, $\theta = 0, 4, 5$.*

Using the equations of the curvature tensors $\overset{0}{R}$, $\overset{4}{R}$, and $\overset{5}{R}$ and the properties of the Riemannian curvature tensor $\overset{g}{R}$, we can easily prove that these tensors cannot be zero.

Theorem 6. *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric A-connection (4). The curvature tensors $\overset{0}{R}$, $\overset{4}{R}$, and $\overset{5}{R}$ given by (12)–(14) are non-zero.*

Proof. If we assume that $\overset{0}{R} = 0$, then from Equation (12), we have

$$\overset{g}{R}(X, Y)Z = \frac{1}{4}\eta(Z)(\eta(X)Y - \eta(Y)X).$$

If we use the equation $\overset{g}{R}(X, Y)\xi = 0$, we obtain that $\eta(X)Y - \eta(Y)X = 0$, which is impossible. The same is proved for the tensors $\overset{4}{R}$, $\overset{5}{R}$. □

4. Projectively Flat Cosymplectic Manifold

In this section, we will study the Weyl projective curvature tensor of the Levi–Civita connection on a cosymplectic manifold. Namely, using curvature tensors of a quarter-symmetric connection (4), we will construct tensors that coincide with the Weyl projective curvature tensor.

Theorem 7. *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric A-connection (4). The following holds:*

$$\overset{\theta}{W}(X, Y)Z = \overset{g}{W}(X, Y)Z, \quad \theta = 0, 4, \tag{23}$$

$$\overset{5}{W}(X, Y)Z = \overset{g}{W}(X, Z)Y + \overset{g}{R}(Z, Y)X, \tag{24}$$

where $\overset{g}{W}$ is the Weyl projective curvature tensor of the Levi–Civita connection given by

$$\overset{g}{W}(X, Y)Z = \overset{g}{R}(X, Y)Z + \frac{1}{n-1}(\overset{g}{Ric}(X, Z)Y - \overset{g}{Ric}(Y, Z)X), \tag{25}$$

and $\overset{0}{W}$, $\overset{4}{W}$, *and* $\overset{5}{W}$ *are given by*

$$\overset{\theta}{W}(X,Y)Z = \overset{\theta}{R}(X,Y)Z + \frac{1}{n-1}(\overset{\theta}{Ric}(X,Z)Y - \overset{\theta}{Ric}(Y,Z)X), \ \ \theta = 0,4, \tag{26}$$

$$\overset{5}{W}(X,Y)Z = \overset{5}{R}(X,Y)Z + \frac{1}{n-1}(\overset{5}{Ric}(X,Y)Z - \overset{5}{Ric}(Z,Y)X). \tag{27}$$

Proof. We prove the equality for tensor $\overset{5}{W}$. From Equation (17), we have

$$\frac{1}{2}\eta \otimes \eta = \frac{1}{n-1}(\overset{5}{Ric} - \overset{g}{Ric}).$$

By substituting the previous equation in (14), the curvature tensor of the fifth kind takes the form

$$\overset{5}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z + \frac{1}{n-1}(\overset{5}{Ric}(Y,Z)X - \overset{g}{Ric}(Y,Z)X - \overset{5}{Ric}(X,Y)Z + \overset{g}{Ric}(X,Y)Z),$$

and after rearranging, we obtain

$$\begin{aligned}\overset{5}{W}(X,Y)Z &= \overset{g}{R}(X,Y)Z + \frac{1}{n-1}(\overset{g}{Ric}(X,Y)Z - \overset{g}{Ric}(Z,Y)X) \\ &= \overset{g}{R}(X,Z)Y + \frac{1}{n-1}(\overset{g}{Ric}(X,Y)Z - \overset{g}{Ric}(Z,Y)X) - \overset{g}{R}(X,Z)Y + \overset{g}{R}(X,Y)Z \\ &= \overset{g}{W}(X,Z)Y + \overset{g}{R}(Z,X)Y + \overset{g}{R}(X,Y)Z = \overset{g}{W}(X,Z)Y + \overset{g}{R}(Z,Y)X,\end{aligned}$$

where $\overset{5}{W}$ is given by (27) and where we used the skew-symmetry and cyclic-symmetry properties of $\overset{g}{R}$. □

Tensor $\overset{0}{W}$ is the projective curvature tensor with respect to the connection $\overset{0}{\nabla}$, and since it coincides with the Weyl projective curvature tensor of the Levi–Civita connection, we can formulate the following statement.

Theorem 8. *Let* $(\mathcal{M}, g, A, \eta, \xi)$ *be a cosymplectic manifold, let* $\overset{g}{\nabla}$ *be a Levi–Civita connection, and let* $\overset{0}{\nabla}$ *be a symmetric connection given by (6). The Weyl projective curvature tensor* $\overset{g}{W}$ *is invariant under the connection transformation* $\overset{g}{\nabla} \to \overset{0}{\nabla}$.

Additionally, since the Riemannian curvature tensor $\overset{g}{R}$ coincides with $\overset{1}{R}$ and $\overset{2}{R}$, the Weyl projective curvature tensor is invariant under connection transformations $\overset{g}{\nabla} \to \overset{1}{\nabla}$ and $\overset{g}{\nabla} \to \overset{2}{\nabla}$.

Given that we have constructed the Weyl projective curvature tensor $\overset{g}{W}$ on a cosymplectic manifold, we will examine what happens when this manifold is projectively flat. If we assume that $\overset{g}{W} = 0$, then it holds that

$$\overset{g}{R}(X,Y)Z = \frac{1}{n-1}(\overset{g}{Ric}(Y,Z)X - \overset{g}{Ric}(X,Z)Y). \tag{28}$$

Based on the properties of the Riemannian curvature tensor $\overset{g}{R}$ and the Ricci tensor $\overset{g}{Ric}$ on the cosymplectic manifold, i.e., using Equations (9) and (10), we have

$$0 = \overset{g}{R}(X,\xi)Z = \frac{1}{n-1}(\overset{g}{Ric}(\xi,Z)X - \overset{g}{Ric}(X,Z)\xi) = -\frac{1}{n-1}\overset{g}{Ric}(X,Z)\xi$$

from where we obtain $\overset{g}{Ric} = 0$. By substituting the last equality in (28), we obtain $\overset{g}{R} = 0$. In this way, we have proved the following assertion.

Theorem 9. *A cosymplectic manifold is projectively flat if and only if it is flat.*

Considering the previous results for the quarter-symmetric metric A-connection (4), we have the following corollary.

Corollary 1. *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric A-connection (4). The manifold is projectively flat if and only if the tensors $\overset{\theta}{W}$, $\theta = 0,4,5$, given by (26) and (27), vanish.*

Proof. Considering that $\overset{0}{W}$ and $\overset{4}{W}$ coincide with $\overset{g}{W}$, the statement is clear for those two tensors. Let us now prove the statement for tensor $\overset{5}{W}$. If the manifold is projectively flat, then it is also flat, so it follows that $\overset{g}{W} = \overset{g}{R} = 0$. Furthermore, from Equation (24), we obtain $\overset{5}{W} = 0$.

On the other hand, if $\overset{5}{W} = 0$, then from Equation (24), we have $\overset{g}{W}(X,Z)Y + \overset{g}{R}(Z,Y)X = 0$, from where it follows that

$$\overset{g}{R}(X,Y)Z = \frac{1}{n-1}(\overset{g}{Ric}(Z,Y)X - \overset{g}{Ric}(X,Y)Z). \tag{29}$$

Taking into account Equations (9) and (10), we obtain

$$0 = \overset{g}{R}(X,Y)\xi = \frac{1}{n-1}(\overset{g}{Ric}(\xi,Y)X - \overset{g}{Ric}(X,Y)\xi) = -\frac{1}{n-1}\overset{g}{Ric}(X,Y)\xi,$$

from which it follows that $\overset{g}{Ric} = 0$. By substituting this equality in (29), we obtain that the manifold is flat, which implies that it is also projectively flat. This completes the proof of the theorem. □

Remark 1. *Theorem 9 can be considered as a consequence of the statements from [1,30]. Namely, the manifold is projectively flat if and only if it is of constant curvature (see pp. 84–85 in [30]). A cosymplectic manifold of constant curvature is flat (see [1,3]). Therefore, we can conclude that a projectively flat cosymplectic manifold is flat. Here, we have given explicit proof.*

5. η-Einstein Cosymplectic Manifold

A cosymplectic manifold is η-Einstein if

$$\overset{g}{Ric} = ag + b\eta \otimes \eta, \tag{30}$$

where a, b are smooth functions. If we use the Ricci tensor property $\overset{g}{Ric}(X,\xi) = 0$, then from the previous equation, we have $a + b = 0$, and the Ricci tensor takes the form

$\overset{g}{Ric} = a(g - \eta \otimes \eta)$. By contracting the last equality, we obtain that $\overset{g}{r} = a(n-1)$. Thus, the η-Einstein cosymplectic manifold satisfies the following equation (see [31,32]):

$$\overset{g}{Ric} = \frac{\overset{g}{r}}{n-1}(g - \eta \otimes \eta). \tag{31}$$

We see that the η-Einstein cosymplectic manifold is Ricci flat if and only if $\overset{g}{r} = 0$ [31]. Moreover, in [32], it has been proved that the curvature scalar $\overset{g}{r}$ is constant in the case of the η-Einstein cosymplectic manifold of a dimension of 5 or higher. An example of the η-Einstein cosymplectic manifold is the cosymplectic manifold of constant ϕ-sectional curvature c, whose Ricci tensor is given by $\overset{g}{Ric} = \frac{c(k+1)}{2}(g - \eta \otimes \eta)$ (see Equation (2.4) in [33]). Additionally, any 3-dimensional cosymplectic manifold is η-Einstein [34,35] and of quasi-constant curvature [31]. For $b = 0$ in (30), we have an Einstein manifold. Any Einstein cosymplectic manifold is Ricci flat [36].

Considering that the Ricci tensors $\overset{0}{Ric}, \overset{4}{Ric}, \overset{5}{Ric}$ given by (15)–(17) are symmetric, we will now define special classes of the cosymplectic manifold with a quarter-symmetric metric A-connection (4).

Definition 1. *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric A-connection (4). The manifold is η-Einstein of the θ-th kind, $\theta = 0, 4, 5$ if*

$$\overset{\theta}{Ric} = \overset{\theta}{a}g + \overset{\theta}{b}\eta \otimes \eta, \quad \theta = 0, 4, 5,$$

where $\overset{\theta}{a}$ and $\overset{\theta}{b}$ are smooth functions.

By contracting the previous equation, we obtain

$$\overset{\theta}{r} = \overset{\theta}{a}n + \overset{\theta}{b}, \quad \theta = 0, 4, 5.$$

Based on the properties of the Ricci tensor $\overset{0}{Ric}$ (see Equation (20)), we have $4(\overset{0}{a} + \overset{0}{b}) = n - 1$. By solving the system of equations

$$\overset{0}{r} = \overset{0}{a}n + \overset{0}{b}, \quad 4(\overset{0}{a} + \overset{0}{b}) = n - 1,$$

we obtain

$$\overset{0}{a} = \frac{4\overset{0}{r} - n + 1}{4(n-1)}, \quad \overset{0}{b} = \frac{n(n-1) - 4\overset{0}{r}}{4(n-1)}.$$

Similarly,

$$\overset{4}{a} = \frac{\overset{4}{r} - n + 1}{n-1}, \quad \overset{4}{b} = \frac{n(n-1) - \overset{4}{r}}{n-1},$$
$$\overset{5}{a} = \frac{2\overset{5}{r} - n + 1}{2(n-1)}, \quad \overset{5}{b} = \frac{n(n-1) - 2\overset{5}{r}}{2(n-1)}.$$

Consequently, the η-Einstein cosymplectic manifold of the θ-th kind $\theta = 0, 4, 5$ takes the form

$$\overset{0}{Ric} = \frac{4\overset{0}{r} - n + 1}{4(n-1)} g + \frac{n(n-1) - 4\overset{0}{r}}{4(n-1)} \eta \otimes \eta,$$
$$\overset{4}{Ric} = \frac{\overset{4}{r} - n + 1}{n-1} g + \frac{n(n-1) - \overset{4}{r}}{n-1} \eta \otimes \eta,$$
$$\overset{5}{Ric} = \frac{2\overset{5}{r} - n + 1}{2(n-1)} g + \frac{n(n-1) - 2\overset{5}{r}}{2(n-1)} \eta \otimes \eta.$$

Using Equation (22), the η-Einstein cosymplectic manifold of the θ-th kind $\theta = 0, 4, 5$ can be written in terms of the scalar curvature $\overset{g}{r}$

$$\overset{0}{Ric} = \frac{\overset{g}{r}}{n-1} g + \frac{(n-1)^2 - 4\overset{g}{r}}{4(n-1)} \eta \otimes \eta, \tag{32}$$

$$\overset{4}{Ric} = \frac{\overset{g}{r}}{n-1} g + \frac{(n-1)^2 - \overset{g}{r}}{n-1} \eta \otimes \eta, \tag{33}$$

$$\overset{5}{Ric} = \frac{\overset{g}{r}}{n-1} g + \frac{(n-1)^2 - 2\overset{g}{r}}{2(n-1)} \eta \otimes \eta. \tag{34}$$

Theorem 10. *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric A-connection (4). The manifold is η-Einstein if and only if it is η-Einstein of the θ-th kind, $\theta = 0, 4, 5$.*

Proof. We prove the theorem for $\theta = 0$. If the manifold is η-Einstein, then by substituting Equation (31) in (15), we obtain Equation (32), and therefore the manifold is η-Einstein of the zeroth kind. On the other hand, if the manifold is η-Einstein of the zeroth kind, then from Equations (15) and (32), we have

$$\frac{\overset{g}{r}}{n-1} g + \frac{(n-1)^2 - 4\overset{g}{r}}{4(n-1)} \eta \otimes \eta = \overset{g}{Ric} + \frac{n-1}{4} \eta \otimes \eta,$$

from which we obtain

$$\overset{g}{Ric} = \frac{\overset{g}{r}}{n-1} (g - \eta \otimes \eta),$$

which means that the cosymplectic manifold is η-Einstein with respect to the Riemannian metric g. □

6. Results and Discussion

The paper discussed the application of a quarter-symmetric connection on almost-contact metric manifolds. We proved that an almost-contact metric manifold with a quarter-symmetric G-metric connection is actually a cosymplectic manifold. Based on the properties of the Riemannian curvature tensor, we also observed the properties of the curvature tensor with respect to the quarter-symmetric metric A-connection. Invariants for certain connection transformations were also found. For example, the Riemannian curvature tensor is invariant under the transformation of the Levi–Civita connection to a quarter-symmetric metric A-connection (4) or to its dual connection (7).

The Weyl projective curvature tensor $\overset{g}{W}$ is well known as a geodesic mapping invariant of Riemannian manifolds (for instance, see [37]). We found a way to construct it on a cosymplectic manifold. More precisely, using the curvature tensors with respect to the quarter-symmetric metric A-connection on the cosymplectic manifold, we constructed tensors that coincide with the Weyl projective curvature tensor $\overset{g}{W}$. We proved that a cosymplectic manifold is projectively flat if and only if it is flat.

At the end of the paper, we constructed examples of η-Einstein cosymplectic manifolds. Namely, with respect to the Ricci tensors on the cosymplectic manifold with a quarter-symmetric metric A-connection, we defined the η-Einstein manifold of the θ-th kind, $\theta = 0, 4, 5$, and we demonstrated that these manifolds coincide with the η-Einstein (with respect to the Riemannian metric).

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mathematics

Article

The Heat Equation on Submanifolds in Lie Groups and Random Motions on Spheres

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Abstract: We studied the random variable $V_t = \mathrm{vol}_{S^2}(g_t B \cap B)$, where B is a disc on the sphere S^2 centered at the north pole and $(g_t)_{t\geq 0}$ is the Brownian motion on the special orthogonal group $SO(3)$ starting at the identity. We applied the results of the theory of compact Lie groups to evaluate the expectation of V_t for $0 \leq t \leq \tau$, where τ is the first time when V_t vanishes. We obtained an integral formula using the heat equation on some Riemannian submanifold Γ_B seen as the support of the function $f(g) = \mathrm{vol}_{S^2}(gB \cap B)$ immersed in $SO(3)$. The integral formula depends on the mean curvature of Γ_B and the diameter of B.

Keywords: Brownian motion; Lie group; heat kernel; Riemannian manifold

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1. Introduction

We studied the behavior of the shape of a body under random transformations.The random motion of a particle on the unit sphere S^2 in $\mathbb{R}^3$ can be used to model the tracking of animals equipped with a transmitter, which has a range given by a disc B of a certain radius depending on the power of the signal issued by a radar. The most-common way to model an erratic motion at least for a sufficiently small body is the Brownian motion on the sphere S^2. The reason for that is the property that such processes have no memory, which means that the motion in the future only depends on the present, not on the past. There are at least two ways to simulate a Brownian motion on the sphere [1]. The most-natural one is to use the Brownian motion of the sphere S^2; its exact density is well-known and has been computed explicitly by Yosida [2]. Another way to simulate a Brownian motion on the sphere is by using the group action point of view. Indeed, we fix a point, say the north pole N, then choose a Brownian motion valued in the group of direct isometries of the sphere S^2, namely the group $SO(3)$. The required Markov process $X_t = r_t(N)$ will give rise to a random motion on the sphere, which differs from the Brownian motion on the sphere, which starts from N. The second point of view requires the exact density of the Brownian motion of $SO(3)$. Fortunately, this theory is well-developed now and can be recast in the Fourier theory of compact Lie groups using unitary representations and Peter–Weyl decomposition. This point of view has been used by M. Liao in order to deduce the stochastic property of the random motion of a rigid body subject to white noise perturbation [3]. It is possible to use Levy processes instead of the Brownian motion, but those have points of discontinuity, while we are considering continuous motions. This was recently performed by S. Albeverio and M. Gordina for matrix Lie groups such asthe special linear group and the Heisenberg group [4]. In our case, we deal with the compact Lie groups for which the complete picture is completely understood using unitary representations and their characters.

Set Up and Main Result

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\omega \in \Omega \mapsto (r_t(\omega))_{t\geq 0}$ be a continuous Brownian motion starting at identity and valued in the group $SO(3)$ of rotations in $\mathbb{R}^3$. Here, $SO(3)$ is seen as a compact Lie group acting on S^2, the unit sphere in $\mathbb{R}^3$, and the action is transitive. We equip S^2 with a volume area measure denoted as $volS^2$, which has density $f(\theta, \varphi) = \sin \phi d\varphi d\theta$ with respect to the Lebesgue measure on $[0, 2\pi) \times [-\pi/2, \pi/2]$. This measure is invariant under the action of $SO(3)$. We studied the following real-valued continuous stochastic process:

$$V_t(\omega) := vol_{S^2}(r_t(\omega)B \cap B)$$

where B is a Borel subset of S^2 with $vol_{S^2}(B) > 0$. In particular, $V_0(\omega) = vol_{S^2}(B)$ since $r_0(\omega) = I_3$.

For each $g \in SO(3)$, let us consider the function on $SO(3)$ corresponding to $(V_t)_{t\geq 0}$ given by

$$f(g) = vol_{S^2}(gB \cap B).$$

Thus, the random process $(V_t)_{t\geq 0}$ is just the image of the Brownian process $(g_t)_{t\geq 0}$ under the map $f : SO(3) \to \mathbb{R}_{\geq 0}$. We are more particularly interested in the Brownian motion (g_t) valued in $SO(3)$, but stopped at the boundary of the support of f. Namely, if $\tau = \inf\{t > 0 : vol_{S^2}(g_t B \cap B) = 0\}$ is the corresponding stopping time, then $(g_t)_{t\geq 0}$ will be the Brownian motion valued in $SO(3)$, which starts at identity and stops at time τ. The unit sphere S^2 can be equipped with the spherical distance given by

$$d_{S^2}(x, y) = \arccos(\langle x, y\rangle)$$

where $\langle x, y\rangle$ is just the Euclidean inner product in $\mathbb{R}^3$. Until the end, we assume that B is the spherical disc with the north pole $N = (0, 0, 1)$ as its center and with diameter $\mathrm{diam}(B)$. Using the property of $(g_{t\wedge\tau})_{t\geq 0}$, we are able to prove a closed formula for the expectation of $(V_t)_{t\geq 0}$.

Theorem 1. *Let B be the spherical disc with the north pole $N = (0, 0, 1)$ as its center in S^2, and let $(g_t)_{t\geq 0}$ be a Brownian motion on $SO(3)$, which starts at the identity and stops at $\tau = \inf\{t > 0 : \mathrm{vol}_{S^2}(g_t B \cap B) = 0\}$. Then, the expectation of $V_{t\wedge\tau} = \mathrm{vol}_{S^2}(g_{t\wedge\tau} B \cap B)$ is given by*

$$\mathbb{E}[V_{t\wedge\tau}] = \frac{4}{\pi^2\sqrt{\pi t}} \int_0^{\pi} \mathcal{J}(t, \theta) e^{L_t(\theta)} \sin^2(\theta/2) d\theta$$

where, for each $0 \leq t \leq \tau$, $\mathcal{J}(t, \theta) = J_0 + \sum_{n\geq 1} (2n+1) e^{-n(n+1)t/2} \chi_n(\theta) J_n$ with

$$J_n = \int_0^{\mathrm{diam}(B)} f(\beta) \chi_n(\beta) \sin^2(\beta/2)\, d\beta$$

and $\chi_n(u) = \dfrac{\sin((2n+1)u/2)}{\sin(u/2)}$ for all $n \geq 0$ and where L_t is a function that depends on the mean curvature of the support of f.

2. Motivation and Literature Review

The Brownian motion is the most-natural way to encode a random motion. It has all the properties that make it the most-unpredictable behavior possible, and it is the most-suitable candidate to model molecular rotations in fluids (FPL model). The probability density function of a Brownian motion satisfies the heat equation. We are interested in the rotational Brownian motion, that is the Brownian motion on the sphere. This kind of random process has been well-studied in the past, and it is still an active area of research. For instance, let us mention the work of Furry [5], Favro [6], Ivanov [7], and Hubbard [8].

For nice surveys on rotational Brownian motions, we invite the reader to read the survey of Valiev and Ivanov [9] and McClung [10] for the rotational Fokker–Planck equation. The problem we are interested in is geometrical. Given a subset B on the two-dimensional sphere, say a cloud, one can use Brownian rotation to move the cloud B. The question is to give the expectation of the volume of the intersection of the cloud with its translation. To treat this question, we need to introduce the Brownian motion on the Lie group $SO(3)$ corresponding to the group of positively oriented rotations on the sphere; this is the aim of Section 3. The point of view taken is to treat a Lie group as a Riemannian manifold; for such a class of spaces, the Brownian motion was studied for instance by Graham [11], van Kampen [12], and Risken [13]. For the general theory of the Brownian motion on manifolds, we refer to the classical book of Elworthy ([14]). The study of such stochastic models has many applications in physics. Let us mention the work of Castro-Villarreal et al. [15,16], Novikov et al. [17], Gómez et al. [18] and Yang-Li [19].

3. The Heat Kernel in $SO(3)$

In this section, we review the spectral theory of the Laplace operator of $SO(3)$ within the theory of compact Lie groups (see, e.g., [20–23]).

3.1. The Lie Group $SO(3)$

The group of isometries of the sphere S^2 is the group of all the space transformations g such that $\langle gx, gy\rangle_{S^2} = \langle x, y\rangle_{S^2}$ for any $x, y \in S^2$. Using duality, such isometries have to satisfy the relation $g^t g = 1$. The group of all such transformations is denoted $O(3)$ and is called the orthogonal group in three dimensions. The orthogonality relation $g^t g = 1$ implies that $\det g = \pm 1$. The elements of $O(3)$ such that $\det g = 1$ preserve the orientation (i.e., act with the positive Jacobian) and form what we call the special orthogonal group given by

$$SO(3) = \{g \in \mathrm{SL}_3(\mathbb{R}) \mid g^t g = I_3\}.$$

The group $SO(3)$ is a maximal Lie compact subgroup of $\mathrm{SL}_3(\mathbb{R})$; in particular, it has a Lie group structure. The Lie algebra of $SO(3)$, namely the tangent space at $g = I_3$, is given by

$$\mathfrak{so}(3) = \{X \in \mathcal{M}_3(\mathbb{R}) \mid X^t = -X\}$$

which consists of skew-symmetric matrices. A basis of $\mathfrak{so}(3)$ is given by the following three matrices:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be seen that $\mathfrak{so}(3)$ is closed under the Lie bracket by noting the following commutation relations:

$$[X_1, X_2] = X_3 \quad [X_2, X_3] = X_1 \quad [X_3, X_1] = X_2.$$

A Lie group closely related to $SO(3)$ is the group $SU(2)$ of unitary matrices of size two:

$$SU(2) = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} : z_1, z_2 \in \mathbb{C}, \ |z_1|^2 + |z_2|^2 = 1 \right\}.$$

The Lie algebra of $SU(2)$ is denoted $\mathfrak{su}(2)$, and it is generated by the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

which satisfy the commutation relations $[\sigma_1, \sigma_2] = 2\sigma_3$, $[\sigma_2, \sigma_3] = 2\sigma_3$, and $[\sigma_3, \sigma_1] = 2\sigma_2$. The $SU(2)$ group is homeomorphic to the unit sphere in $\mathbb{C}^2$. As a consequence, $SU(2)$ is simply connectedand compact. The group $SO(3)$ is not simply connected; its universal covering is given by $SU(2)$. More precisely, $SO(2)$ has a two-sheet universal covering realized by the adjoint representation of $SU(2)$:

$$\mathrm{Ad} : SU(2) \to SO(3)$$

given by $Ad(X)g = X^{-1}gX$. The kernel of this map is given by the center of $SU(2)$, which is $\{\pm I_2\}$. It is less trivial to find the image of this map, and it can be proven that it is surjective. The latter fact can be checked by working on the Lie algebra level. Indeed, the differential of the adjoint map at the identity is given by $ad : \mathfrak{su}(2) \to \mathfrak{so}(3)$, $X \mapsto adX = [X, .]$. In particular,

$$SO(3) \simeq SU(2)/\{\pm I_2\}.$$

3.2. Euler Parametrization and Haar Measure on $SO(3)$

For our purposes, we need a precise description of the group $SO(3)$ in terms of the Euler angles. This will give a well-suited parametrization of the elements of the group in order to perform the analysis. The group $SO(3)$ is a compact Lie group, which is given by

$$SO(3) = \{g \in \mathrm{SL}n(3) \mid g^t g = I_3\}.$$

The tangent space of G at some $g \in G$ is just the set of matrices of the form gX, where X is some element in $\mathfrak{so}(3)$. The exponential map $\exp : \mathfrak{so}(3) \to SO(3)$ is surjective. We use the polar coordinate for an element X of the Lie algebra $\mathfrak{so}(3)$; indeed, such an X can be written as $X = \theta T(u, v, w)$ with $(u, v, w) \in S^2$ and where

$$T(u, v, w) = \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix}.$$

Any element of $g \in SO(3)$ can be written in the form:

$$g = e^{\psi X_3} e^{\theta X_1} e^{\phi X_3} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1)$$

The normalized Haar measure is, therefore, given by (see [20])

$$\mu_{SO(3)}(d\theta, d\psi, d\phi) = \frac{2}{\pi^2} \sin^2(\frac{\theta}{2}) \sin\psi d\psi d\phi. \quad (2)$$

3.3. Brownian Motion on a Riemannian Manifold

For an introduction to Brownian motion on manifolds, we refer to the book of Elworthy [14]. There are several ways to construct the Brownian motion on a Lie group G. An elegant one consists of defining the density function of the Brownian motion as the solution of the heat equation on G. In fact, only the underlying structure of the Riemannian manifold on G is needed. Let us assume more generally that we are given an n-dimensional Riemannian (M, h), where the metric h is a symmetric bilinear form on the tangent bundle h. Given local coordinates $(x_1, \ldots, x_n)$ of a point $x \in M$ with a local frame $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$, which forms a basis of $T_x(M)$, the metric is then locally determined by its coefficients $h_{ij} = h(\partial/\partial x_i, \partial\, \partial x_j)$ giving the length element:

$$ds^2 = \sum_{i,j} h_{ij} dx_i \otimes dx_j.$$

Let $\mathcal{C}(TM)$ denote the space of smooth sections of the tangent bundle, then one can define a covariant derivative using a connection $\nabla : \mathcal{C}(TM) \times \mathcal{C}(TM) \to \mathcal{C}(TM)$ depending on the metric h. This connection assigns to a pair of vectors fields $X, Y \in \mathcal{C}(TM)$ the vector field

$\nabla_X Y$, which may be seen as the derivative of Y. The torsion associated with the connection is the quantity defined by

$$T'(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{3}$$

In local coordinates, the connection is essentially characterized by its values on the basis $(\partial_{x_1}, \ldots, \partial_{x_n})$ of TM:

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \sum_{1 \leqslant k \leqslant n} \Gamma_{ij}^k \partial_{x_k}$$

where the coefficients Γ_{ij}^k of the connection, known as the Christoffel symbols, can be computed explicitly by using the first derivatives of the metric components h_{ij}. The gradient of a smooth function f associated with the metric h is defined by the relation $h(X, \mathrm{grad} f) = df(X)$ for any $X \in \mathcal{C}(TM)$. The central role in the theory of heat diffusion is played by the Laplace–Beltrami operator on (M,h), which is defined in local coordinates by

$$\Delta_{M,h} = \frac{1}{\sqrt{h}} \sum_{i,j} \frac{\partial}{\partial x_i} \left[\sqrt{h} h^{ij} \frac{\partial}{\partial x_j} \right]$$

where h and h^{ij} are, respectively, the determinant and the inverse of the coordinates of the metric tensor (h_{ij}) in the local chart. With the Laplace–Beltrami operator one can associate the heat equation on M with an initial condition f:

$$\begin{cases} \frac{1}{2}\Delta_M u(t,x) + \partial_t u(t,x) = 0 \\ \\ u(0,x) = f(x) \qquad \text{on} \quad M. \end{cases} \tag{4}$$

In the compact case, which is our main concern, this equation always has a smooth solution denoted by $p_t(x)$. The Brownian motion $(X_t)_{t \geq 0}$ on M is just a Feller process, which has a transition operator of the form:

$$P_t f(x) = \mathbb{E}[f(X_t) | X_0 = x]$$

for any f continuous with compact support on M. The kernel associated with this operator is given by $p_t(x,y)$; this quantity is the probability that the Brownian is at y at time t conditioned on the fact that it started at x. It satisfies the relation:

$$P_t f(x) = \mathbb{E}[f(X_t) | X_0 = x] = \int_M p_t(x,y) f(y) dy.$$

3.4. The Density Probability of the Brownian Motion in $SO(3)$

The notion of Brownian motion on a compact Lie group will be directly derived from the setting of the previous section, in that a Lie group has a structure of the Riemannian manifold. The aim is to find an explicit formulation of the solution of the heat equation in $SO(3)$. Before, we need to find the expression of the Laplace operator. The fact that the Brownian motion in a compact Lie group can be constructed from a solution of the heat kernel was developed by K. Ito [24]. In this case, one can do much better than proving the existence; indeed, using Fourier analysis on $SO(3)$, it is possible to give an explicit formula for the density $(p_t)_{t>0}$. Let us first recall that the Lie algebra of $SO(3)$ is generated by three matrices X_1, X_2, and X_3, which give rise to the three corresponding differential operators $\widetilde{X_i}$ $(i = 1,2,3)$, which act on the set of functions on $SO(3)$ via the rule:

$$(\widetilde{X_i}.f)(x) = \frac{d}{ds} f(e^{sX_i} x)_{|s=0} \quad i = 1,2,3.$$

The kth iteration of an operator X is just written $\widetilde{X}^k$. The Levi-Civita connection on $SO(3)$ is given by

$$\nabla_X Y = \frac{1}{4}[X, Y]$$

for any two vector fields X, Y [25]. This defines a Riemannian metric on $SO(3)$, which is just the identity. This gives the simple expression for the Laplace operator:

$$\Delta_{SO(3)} = \widetilde{X_1}^2 + \widetilde{X_2}^2 + \widetilde{X_3}^2.$$

The density of the Brownian motion on G starting at identity is given by the solution in $L^2(G) \cap \mathcal{C}_c^2(G)$ of the heat equation with initial data in L^2:

$$\begin{cases} \frac{1}{2}\Delta_{SO(3)} u(t, x) = -\partial_t u(t, x) \\ \\ u(id_G, x) = f(x) \qquad \text{on} \quad G. \end{cases} \tag{5}$$

3.5. Root Decomposition of the Lie Algebra of a Compact Lie Group

The explicit description of a solution to the problem (5) is well understood in the setting of compact Lie groups, which consists of a vast generalization of the L^2-theory of n-dimensional tori $\mathbb{R}^n/2\pi\mathbb{Z}^n$. Let us be given a compact Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G over the complex numbers and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. In particular, since $\mathfrak{h}$ is abelian, the operators $ad(H) = [H, .]$ commute with each other for all $h \in \mathfrak{h}$. A general fact from linear algebra implies that all the operators $ad(H)$ $(H \in \mathfrak{h})$ are diagonalizable over the same basis. Thus, for any $X \in \mathfrak{g}$ and $H \in \mathfrak{h}$, there exists a not necessarily real eigenvalue $\alpha(H)$ such that

$$ad(H)X = [H, X] = \alpha(H)X.$$

This gives rises to a well-defined map $\alpha : \mathfrak{h} \to \mathbb{C}$, which is linear and, thus, can be seen as an element of the dual of $\mathfrak{h}$. The set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is the set of all $\alpha \in \mathfrak{h}^*$ coming this way. We denote by $\mathcal{R}$ the set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. One can define a definite negative bilinear on $\mathfrak{g} \times \mathfrak{g}$, by the following rule $B(X, Y) = \text{tr}(ad_X \circ ad_X)$. For each root $\alpha \in \mathcal{R}$ and any $H \in \mathfrak{h}$, there exists a unique $H_\alpha \in \mathfrak{h}$ such that $\alpha(H) = B(H, H_\alpha)$. Let us set $\mathfrak{h}_0 = \oplus_{\alpha \in \mathcal{R}} \mathbb{Q} H_\alpha$ as the $\mathbb{Q}$-span of H_α $(\alpha \in \mathfrak{h})$. One can define a positive definite inner product on $\mathfrak{h}_0^*$, by the rule:

$$(\alpha, \beta) = B(H_\alpha, H_\beta) \text{ for every } \alpha, \beta \in \mathcal{R}.$$

If we fix a set $H_1, \ldots, H_s$ that spans $\mathfrak{h}_0$, we say that an element α of $\mathfrak{h}_0$ is positive if there exists an integer $1 \leqslant j \leqslant s$ such that $\alpha(H_1) = \ldots, \alpha(H_{j-1}) = 0$ and $\alpha(H_j) > 0$. We denote $\alpha > 0$, and we denote by $\mathcal{R}^+$ the set of positive roots. If $\alpha \in \mathcal{R}$, then $-\alpha \in \mathcal{R}$. We have the following decomposition into eigenspaces:

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-.$$

where $\mathfrak{n}^+$ (respectively $\mathfrak{n}^-$) is the direct sum $\oplus_{\alpha \in \mathcal{R}^+} \mathfrak{g}\alpha$ (respectively, $\oplus \alpha \in \mathcal{R}^+ \mathfrak{g}\alpha$). For any given irreducible representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, there exists a nonzero vector $v \in V$ and $\Lambda \in \mathfrak{h}_0^*$ such that $\rho(H)v = \Lambda(H)v$ and $\dfrac{2(\Lambda, \lambda)}{(\lambda, \lambda)}$ is a nonnegative integer for each $\lambda \in \mathcal{R}^+$. The vector v is called the highest weight vector, and Λ is the highest weight of the representation ρ. Actually, the highest weights of an irreducible representation characterize completely the equivalence class of an irreducible representation of $\mathfrak{g}$. For connected compact Lie groups, Abelian subgroups are just tori in the usual sense, and Cartan subgroups are replaced by the notion of maximal tori. In particular, this opens the way to the the generalization of Fourier analysis to compact Lie groups. In tori, the key role is played by irreducible characters, which are traces of the irreducible representations rather than the highest weights. The reason behind this is that maximal tori are conjugate in G, i.e., they form a unique orbit under the conjugation action. Thus, the trace of a representation restricted to any maximal torus is constant on the conjugacy class.

Since any element of G is contained in a maximal torus, central functions on G are completely characterized by their restriction to a maximal torus. In particular, this applies to the characters of irreducible representations. A character is characterized by the sets of highest weight of a maximal torus, that is for each highest weight λ, one has a corresponding irreducible character χ_λ of G.

3.6. Computation of the Characters

Since all maximal tori are conjugate under G, the action of G on the set of all maximal tori is transitive. If we fix a representative torus T for this action, the Weyl group is by definition $W(T) = N(T)/T$, where $N(T) = \{g \in G : g^{-1}Tg = T\}$ is the normalizer of T. Concretely, the elements of the Weyl group are generated by a finite set of reflections with respect to the hyperplanes $F_\alpha = \{\beta \in \mathcal{R} : (\alpha, \beta) = 0\}$. The set $C_\alpha = \{\beta \in \mathcal{R} : (\alpha, \beta) > 0\}$ is called the Weyl chamber associated with the root α. An important fact is that the Weyl group permutes Weyl chambers. For each $w \in W$, let us denote by N_w the number of reflections in the decomposition of w. The irreducible character corresponding to the highest weight λ evaluated for $H \in \mathrm{Lie}(T)$ is as follows:

$$\chi_\lambda(e^H) = \frac{\sum_{w\in W}(-1)^{N_w} e^{iw(\lambda+\rho)H}}{\sum_{w\in W}(-1)^{N_w} e^{iw\rho H}}$$

where ρ is the half sum of the positive roots. The dimension of the corresponding irreducible representation is given by

$$d_\lambda = \frac{\prod_{\alpha\in\mathcal{R}^+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\mathcal{R}^+}(\rho,\alpha)}.$$

For each highest root λ and $g \in G$, one has

$$\Delta_G \chi_\lambda(g) = c(\lambda)\chi_\lambda(g) \tag{6}$$

with the corresponding eigenvalues being

$$c(\lambda) = (\lambda+\rho, \lambda+\rho) - (\lambda, \lambda). \tag{7}$$

3.7. Solution of the Heat Equation for Compact Lie Groups

We solve Equation (5) for an initial data f, which is a trace class function in $L^2(G)$, that is $f(hgh^{-1})$ for any $g, h \in G$. Under this assumption, the Peter–Weyl theorem gives us the Fourier expansion of f, which takes the following nice form

$$f(g) = \sum_{\lambda\in\Lambda^+} \sqrt{d_\lambda}\chi_\lambda(g)$$

where the equality is to be considered in the L^2 sense. Now, we set the following map $p_t : G \times G \to \mathbb{R}$ for each $t > 0$.

$$p_t^G(k, g) = \sum_{\lambda\in\Lambda_+} \frac{1}{\sqrt{d_\lambda}}\chi_\lambda(k)\chi_\lambda(g)e^{-tc(\lambda)} \quad (k, g \in G). \tag{8}$$

We claim that this function is a solution of (5). Indeed,

$$\Delta_G p_t^G(k, g) = \sum_{\lambda\in\Lambda_+} \frac{1}{\sqrt{d_\lambda}}\chi_\lambda(k)\Delta_G(\chi_\lambda(g))e^{-tc(\lambda)}.$$

Using (6), one can deduce that

$$\Delta_G p_t^G(k,g) = \sum_{\lambda \in \Lambda_+} \frac{1}{\sqrt{d_\lambda}} \chi_\lambda(k)\chi_\lambda(g)c(\lambda)e^{-tc(\lambda)} = -\frac{\partial}{\partial t}[p_t^G(k,g)].$$

The solution is called the *heat kernel* of the compact Lie group G. We also call the heat kernel the one variable function $p_t(g) = p_t(I_3, g)$, the only ambiguity being that kernels in operators theory are defined on the product of the space with itself; indeed, the kernel defines an operator of $L^2(G)$ called the heat operator:

$$P_t f(k) = \int_G p_t(k,g) f(g) dg.$$

A notation of common use for $p_t(I_3, g)$ is $p_t(g)$, which is also called the heat kernel, and we employ both terms with no risk of confusion.

3.8. Solution of the Heat Equation for SO(3)

Now, we are able to give the explicit form of the heat kernel for $G = SO(3)$. In most of the presentations in the literature, it is always derived from the case $G = SU(2)$, for which the situation is much clearer due to the fact that it is simply connected. Here, we follow the presentation of M. Liao (Liao gives the formula for Levy processes, and it is easy to deduce the Brownian case, which corresponds to continuous Levy trajectories, which have a null Levy measure and the infinitesimal generator L, being the half of the Laplacian of G.) (Example 4.20 [23]). We recall that this construction is only valid if f is a conjugate invariant, which is the case for us. The equivalence classes of irreducible unitary representations of $SO(3)$ are indexed by the set of nonnegative integers $\{n = 0, 1, 2, 3, \ldots\}$, and the corresponding characters are trace class functions depending only the conjugacy class of a rotation depending only on an angle θ and given by

$$\chi_n(g) = \chi_n(\theta) = \frac{\sin((2n+1)\theta/2)}{\sin(\theta/2)}.$$

The expanded form of the heat kernel of $SO(3)$ is given by

$$p_t^{\mathrm{SO}(3)}(g) = p_t^{\mathrm{SO}(3)}(\theta) = 1 + \sum_{n \geq 1} (2n+1) e^{-atn(n+1)} \frac{\sin((2n+1)\theta/2)}{\sin(\theta/2)}. \tag{9}$$

with the corresponding kernel given by

$$p_t^{\mathrm{SO}(3)}(h,g) = p_t^{\mathrm{SO}(3)}(\beta,\theta) = 1 + \sum_{n \geq 1} (2n+1) e^{-n(n+1)t/2} \chi_n(\theta)\chi_n(\beta). \tag{10}$$

For a such that the infinitesimal generator is $L = a\Delta_G$, for the Brownian motion, we took $a = 1/2$. Thus, the density distribution of the Brownian motion on G is

$$p_t^{\mathrm{SO}(3)}(\theta) = 1 + \sum_{n \geq 1} (2n+1) e^{-n(n+1)t/2} \frac{\sin((2n+1)\theta/2)}{\sin(\theta/2)}. \tag{11}$$

The action of the heat operator relative to $L = \frac{1}{2}\Delta$ on the space of the L^2-integrable function of G are conjugate invariant. Thus, using (2), it takes the following form:

$$P_t^G f(I_3) = \int_{\mathrm{SO}(3)} p_t^{\mathrm{SO}(3)}(g) f(g) \mu_{\mathrm{SO}(3)}(dg) = \frac{2}{\pi} \int_0^\pi p_t^{\mathrm{SO}(3)}(\theta) f(\theta) \sin^2(\theta/2)\, d\theta. \tag{12}$$

In other terms, this means that if $(g_t)_{t\geq 0}$ is a Brownian motion on SO(3) starting at the identity, which is conjugate invariant, we have

$$\mathbb{E}[f(g_t)] = \frac{2}{\pi}\int_0^{\pi} p_t^{\mathrm{SO}(3)}(\theta) f(\theta)\sin^2(\theta/2)\, d\theta. \tag{13}$$

4. The Brownian Motion on the Support of f

4.1. The Support of f

We introduce the support of f, which for us will be the following set:

$$\Gamma_B = \mathrm{supp} f = \{g \in SO(3) | f(g) = vol_{S^2}(gB \cap B) \geq 0\}.$$

The function f vanishes as soon as $d_{S^2}(gN, N) \geqslant \mathrm{diam}(B)$, which amounts to saying that f is supported by those g such that $\arccos\langle gN, N\rangle < \mathrm{diam}(B)$. Noting that cos is decreasing in the interval $[0, \pi]$, the latter condition is equivalent to $\langle gN, N\rangle > \cos\mathrm{diam}(B)$. The support of f is

$$\Gamma_B = \{g \in SO(3) | \langle gN, N\rangle \geqslant \cos\mathrm{diam}(B)\}.$$

The support of f, namely Γ_B, is then a closed subset of $SO(3)$, but not a Lie subgroup. The boundary of Γ_B is denoted Σ_B and is simply given by

$$\Sigma_B = \{g \in SO(3) | \langle gN, N\rangle = \cos\mathrm{diam}(B)\}.$$

The subset Σ_B can be seen as a smooth hypersurface of $SO(3)$ of equation $\theta(g) = \langle gN, N\rangle = \cos\mathrm{diam}(B)$. Reminding that N is the north pole, we readily obtain that $\theta(g) = g_{33}$. Thus,

$$\Sigma_B = \{g \in SO(3) | g_{33} = \cos\mathrm{diam}(B)\}.$$

Using the Euler parametrization of the rotations $(\theta(g), \varphi(g), \psi(g))$, we know that

$$g_{33} = \cos\theta(g).$$

This shows that the boundary of the support of f is then given by

$$\Sigma_B = \{g \in SO(3) | \theta(g) = \mathrm{diam}(B)\}.$$

4.2. The Support Γ Seen as Submanifold Embedded in $SO(3)$

There are several ways to construct a Brownian process on a Lie group viewed as a Riemannian manifold. The more suitable way in our case is to introduce the density of such a process, which is given by the solution of the heat equation on the support of f viewed as a Riemannian manifold. Indeed, the support Γ_B can be endowed with a structure of the Riemannian submanifold embedded in G with the induced metric of $SO(3)$. In particular, from this induced metric, one is able to extract the Laplace–Beltrami operator Δ_{Γ_B}. The reason we are interested in this operator is that it encodes the property of the Brownian motion killed outside Γ_B, in that the density of a Brownian process on such a submanifold is the solution of the heat operator associated with Γ_B. In other words, one can say that the infinitesimal generator of $(g_t)_{t\geq 0}$ stopped outside the support of f is just $\frac{1}{2}\Delta\Gamma_B$.

Tangent Space of the Submanifold Γ_B.

The set Γ_B is the set of all $g \in G$ such that

$$\theta(g) = \langle gN, N\rangle \leqslant \cos\mathrm{diam}(B).$$

Since we are going to work locally, it is more suitable to look at Γ_B as a union of level sets of ϕ, namely

$$\Gamma_B = \bigcup_{0 \leqslant \gamma \leqslant \cos \operatorname{diam}(B)} \{g \in G | \theta(g) = \gamma\}.$$

For each $\gamma \in [0, \cos \operatorname{diam}(B)]$, we first need to show that the level set:

$$\Gamma_B[\gamma] := \theta^{-1}(\gamma) = \{g \in G | \theta(g) = \gamma\}$$

is a smooth submanifold in G. A sufficient condition is that θ is a submersion, i.e., the differential is surjective at any point (see [26]). We check this fact in the following lemma.

Lemma 1. *The map $\theta : G \to \mathbb{R}$ is submersion, in particular the level sets of θ are smooth immersed submanifolds of G.*

Proof. Let us compute the differential of θ at a point $g \in \Gamma_f$ in the direction given by a vector field $X \in T_g(G)$. This is given by

$$(d\theta)_g X = (\widetilde{X}.\theta)(g) = \frac{d}{dt}\theta(e^{tX}g)|t = 0.$$

Thus,

$$(d\theta)_g X = \lim t \to 0 \frac{\theta(e^{tX}g) - \theta(g)}{t} = \lim_{t \to 0} \langle \frac{(e^{tX} - I_3)}{t} gN, N \rangle.$$

One has,

$$\lim_{t \to 0} \frac{e^{tX} - I_3}{t} = \lim_{t \to 0} \frac{1}{t}\left(tX + \frac{(tX)^2}{2} + \frac{(tX)^3}{3!} + \dots\right) = X.$$

Therefore, we obtain

$$(d\theta)_g X = \langle XgN, N \rangle = \theta(Xg).g_{33}.$$

The kernel is given by

$$\operatorname{Ker}(d\theta)_g = \{X \in Tg(G)) \mid \theta(Xg) = 0\}$$

$$= \{X \in T_g(G)) \mid \langle XgN, N \rangle = 0\}$$

$$= \{X \in T_g(G)) \mid (Xg)_{33} = 0\}.$$

Thus,

$$\operatorname{Ker}(d\phi)_g = \{X \in Tg(G) \mid X_{31}g_{13} + X_{32}g_{23} + X_{33}g_{33} = 0\}.$$

This $\operatorname{Ker}(d\theta)_g$ is a hyperplane of $Tg(G)$ being of codimension one as the kernel of a linear form on $T_g(G)$. In particular,

$$\operatorname{rank}(d\theta)_g = \dim T_g(G) - \dim \operatorname{Ker}(d\theta)_g = 1.$$

Hence, for every $g \in G$, $(d\theta)_g$ is surjective since it is real-valued. □

Proposition 1. *There exists a vector field Z such that, for every $g \in G$,*

$$T_g(G) = T_g(\Gamma) \oplus \mathbb{R}Z_3.$$

In particular, Γ_B is a smooth hypersurface in G with the normal direction given by Z_3.

Proof. Since

$$\theta(g) = \langle gN, N \rangle = \langle gN, N \rangle$$

the tangent space of Γ_B is given by

$$T_g(\Gamma_B) = \{X \in T_g(G) \mid (Xg)_{33} = 0\}.$$

A vector field $X \in T_g(G$ is then in $T_g(\Gamma_B)$ if $X = Yg$ for some $Y \in \mathfrak{g}$ and, thus, if $(Xg)_{33} = (g^2Y)_{33} = 0$. The later condition gives

$$(g^2)_{31}Y_{13} + (g^2)_{32}Y_{23} + (g^2)_{33}Y_{33} = 0.$$

Since $Y \in \mathfrak{so}(3)$, it is skew-symmetric, hence of the form:

$$Y = \begin{pmatrix} 0 & -Y_{21} & -Y_{31} \\ Y_{21} & 0 & -Y_{32} \\ Y_{31} & Y_{32} & 0 \end{pmatrix}$$

In particular, $Y_{33} = 0$, and therefore, one has the equation

$$(g^2)_{31}Y_{31} + (g^2)_{32}Y_{32} = 0.$$

This gives the relation $Y_{31} = -\rho(g)Y_{32}$, where

$$\rho(g) = \frac{(g^2)_{32}}{(g^2)_{31}} = \frac{g_{31}g_{12} + g_{32}g_{22} + g_{33}g_{32}}{g_{31}g_{11} + g_{32}g_{21} + g_{33}g_{31}}.$$

By recasting in Y, one obtains

$$Y = \begin{pmatrix} 0 & -Y_{21} & \rho(g)Y_{32} \\ Y_{21} & 0 & -Y_{32} \\ -\rho(g)Y_{32} & Y_{32} & 0 \end{pmatrix} = Y_{21}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + Y_{32}\begin{pmatrix} 0 & 0 & \rho(g) \\ 0 & 0 & -1 \\ -\rho(g) & 1 & 0 \end{pmatrix}.$$

Thus, we obtain that the elements X of the tangent space $T_g(\Gamma_B)$ are of the form:

$$X = \mathbb{R}Z_1 + \mathbb{R}Z_2$$

where Z_1 and Z_2 are two vector fields given by $Z_1(g) = gX_3$ and $Z_2(g) = gX_1 + \rho(g)gX_2$; here, $(X_i)_{1\leqslant i\leqslant 3}$ is the basis of $\mathfrak{g} = \mathfrak{so}(3)$ given in §2.1. Hence, the tangent space at the point g of the submanifold ΓB is given by

$$T_g(\Gamma_B) = \mathrm{span}\{Z_1(g), Z_2(g)\}.$$

The normal bundle $\mathcal{N}(\Gamma_B)$ is the orthogonal complement of the tangent bundle in $T(G)$ with respect to the Killing inner product given by $\beta(X,Y) = \mathrm{Tr}(XY)$

$$T(G) = T(\Gamma_B) \oplus \mathcal{N}(\Gamma_B).$$

We already know that, for every $g \in \Gamma_B$, $\dim \mathcal{N}g(\Gamma B) = 1$. Its generator Z_3 satisfies the two conditions:

$$\mathrm{Tr}(Z_3Z_1) = 0 \text{ and } \mathrm{Tr}(Z_3Z_2) = 0.$$

□

4.3. Laplace–Beltrami Operator on Γ_B.

As remarked earlier, the support of f, Γ_B, is not a Lie group, but only a submanifold of G. For this reason, we cannot write the Laplace operator of Γ_B as squares of differential operators afforded to the basis of $\mathfrak{so}(3)$. The structure of the Riemannian manifold on Γ_B allows overcoming this issue. Indeed, there is a canonical way to obtain an expression of the Laplace–Beltrami operator of a submanifold as a function of the Laplace operator of the underlying manifold and the coefficients of the second fundamental form of Γ_B embedded in G.

There is a useful formula that allows expressing the Laplace operator of a submanifold as function of the following.

Lemma 2. *Let us assume that we have an n-dimensional Riemannian manifold M and a k-dimensional Riemannian submanifold N immersed in M. Let us denote by ∇_M, Δ_M (respectively, ∇_N, Δ_N) the connections and the Laplace operator on M (respectively, N). Suppose $(X_{k+1}, \ldots, X_n)$ is an orthonormal basis of the normal bundle of N, and H denotes the mean curvature vector of N in M.*
Then, for any $f \in \mathcal{C}^\infty(M)$, one has

$$\Delta_N f_{|N} = (\Delta_M f)_{|N} + Hf - \sum_{i=k+1}^{n} \nabla_M^2 f(X_i, X_i). \tag{14}$$

Proof. See, e.g., Lemma 2 in [27]. □

We applied the lemma to the case when $M = SO(3)$ and $N = \Gamma_B$ and with Z_3 as the generator of the normal bundle of $N = \Gamma_B$ (here $n = 3$ and $k = 2$), then for any $f \in \mathcal{C}^\infty(SO(3))$, we have

$$\Delta_{\Gamma_B} f_{|\Gamma_B} = (\Delta_{SO(3)} f)_{|\Gamma_B} + H_B f - \nabla^{(2)}_{SO(3)} f(Z_3, Z_3) \tag{15}$$

where H_B denotes the mean curvature of Γ_B in $SO(3)$. The last term can be simplified; indeed, the second covariant derivative is by definition equal to

$$\nabla^{(2)}_{SO(3)} f(Z_3, Z_3) = \nabla_{Z_3} \nabla_{Z_3} f - \nabla_{\nabla_{Z_3} Z_3} f.$$

The Levi-Civita connection on $SO(3)$ is just $\nabla_X Y = \frac{1}{2}[X, Y]$ for any vector fields X, Y. Thus,

$$\nabla^{(2)}_{SO(3)} f(Z_3, Z_3) = Z_3^2 f - \nabla_{\frac{1}{2}[Z_3, Z_3]} f = Z_3^2 f.$$

Let us denote by C_G the Casimir operator of G, the unique generator of the center of the enveloping algebra $U(\mathfrak{g})$, which is nothing but the Laplace operator on G. To sum up, we obtained the following.

Proposition 2. *The Laplace operator of the submanifold Γ_B takes the following form:*

$$\Delta_\Gamma = C_G + H_\Gamma - Z_3^2.$$

4.4. The Heat Kernel on Γ_B.

The density probability distribution of the Brownian motion in Γ_B is determined by the heat kernel of the Markov semi-group operator $P^\Gamma t = e^{-t\Delta\Gamma}$ acting on $L^2(\Gamma_B)$. The Casimir operator C_G lies in the center of the enveloping algebra $U(\mathfrak{g})$; in particular, it commutes with Z_3^2, i.e., $[C_G, Z_3^2] = 0$. Thus, the commutation relation and Proposition 2 give the identity:

$$P_t^\Gamma = e^{-tC_G} e^{tZ_3^2} e^{-th} \tag{16}$$

where h is the mean curvature scalar of Γ_B seen as the embedded Riemannian submanifold in G. The diffusion operator $P^\Gamma t$ has a heat kernel function $p_t^\Gamma : G \times G \to \mathbb{R}$ characterized by the following relation:

$$P_t^\Gamma f(g) = \int_\Gamma p_t^\Gamma(k, g) f(k) dk$$

for every $g \in \Gamma$ and $f \in L^2(\Gamma)$. The value $p_t^\Gamma(k, g)$ gives exactly the probability of the Brownian motion in Γ to be at k at time t provided it started at g. We need to make this probability transition explicit as a function ofthe one in G and in the normal direction given by the operator Z_3^2. Recall that the normal direction is given by the vector field Z_3, and thus, the normal direction at the point $g \in G$ is just given by $Z_3(g)$ and $\pi(g)$ is the normal component of $g \in G$ onto the geodesic $Z = \{e^{tZ_3} : t \in \mathbb{R}\}$. The element $\alpha(g)$ is a unique element of Lie (Z) such that $e^{\alpha(g)} = \pi(g)$. Since Z is a one-dimensional one, the element $\alpha(g) = \log(\pi(g))$ can be seen as an element of $\mathbb{R}$. To compute $\alpha(g)$, it suffices to consider the vector fields Z_1, Z_2, and Z_3, which form the basis of $\mathfrak{g}$. Indeed, let $g \in G$ be given; since the exponential is surjective onto G, there exists a $X \in \mathfrak{g}$ such that $g = e^X$. Now, writing the decomposition of X with respect to the basis $\{Z_i, i = 1, 2, 3\}$, we get that $X = z_1 Z_1 + z_2 Z_2 + z_3 Z_3$ for some real numbers z_i $(i = 1, 2, 3)$. Thus, we have

$$\alpha(g) = \alpha(e^X) = z_3.$$

The relation (16) gives

$$P^\Gamma t f(\gamma) = \int_\Gamma p_t^\Gamma(k, \gamma) f(k) dk = \int_G p_t^G(\gamma, g) e^{-th(g)} \left(\int_\mathbb{R} p_t^Z(s, \alpha(g)) f(e^{sZ_3}) ds \right) dg.$$

Finally, taking $f = \delta_{I_3}$, we obtain the solution of the heat equation in Δ with the initial condition $u(0^+, x) = \delta_{I_3}(x)$. Thus, the heat kernel of Δ_Γ is given by

$$p_t^\Gamma(\gamma) = p_t^\Gamma(I_3, \gamma) = \int_G p_t^G(\gamma, g) e^{-th(g)} p_t^Z(0, \alpha(g)) dg. \tag{17}$$

Now, Z is a the trajectory of a both-sided geodesic with initial velocity Z_3 in $SO(3)$. In particular, it is a totally geodesic submanifold and, therefore, minimal in $SO(3)$. Hence, the heat kernel on Z is just the one-dimensional heat kernel:

$$p^Z(z) = \frac{1}{\sqrt{\pi t}} e^{z^2/2t}.$$

The heat kernel of Δ_Γ takes the following form:

$$p_t^\Gamma(\gamma) = \frac{1}{\sqrt{\pi t}} \int_G p_t^G(\gamma, g) e^{-th(g)} e^{\alpha(g)^2/2t} dg. \tag{18}$$

Proof of Theorem 1. Now, we come to our initial problem, namely the study of the random process $V_t = vol_{S^2}(g_t B \cap B)$ for $t \geq 0$, where $(g_t)_{t \geqslant 0}$ is the Brownian motion, which is stopped when it hits the boundary of the support of f. More precisely, we define the stopping time:

$$\tau = \inf\{t > 0 \; : \; g_t \in \partial\Gamma_B\}.$$

Thus, the Brownian motion starting at identity and killed outside $\Gamma = \text{supp} f$ has its density given by

$$p_t^\Gamma(k) = \frac{1}{\sqrt{\pi t}} \int_G p_t^G(k, g) e^{-th(g)} e^{\alpha(g)^2/2t} dg. \tag{19}$$

The expectation of $(V_{t \wedge \tau})_{t \geq 0} = (f(g_{t \wedge \tau}))$ is

$$\mathbb{E}[V_{t \wedge \tau}] = \mathbb{E}[V_t \mid t < \tau] = \int_\Gamma p_t^\Gamma(k) f(k) dk.$$

Using (19), we obtain

$$\mathbb{E}[V_{t \wedge \tau}] = \frac{1}{\sqrt{\pi t}} \int_\Gamma \int_G p_t^G(k, g) f(k) e^{L_t(g)} dg \, dk$$

where $L_t(g) = -th(g) + \alpha(g)^2/2t$ for every $g \in G$. The function L_t defines a map that is $SO(3)$ invariant. Furthermore, the function f is conjugate invariant; indeed, for any $g, h \in G$, we have

$$f(hgh^{-1}) = \mathrm{vol}_{S^2}(hgh^{-1}B \cap B) = \mathrm{vol}_{S^2}(gh^{-1}B \cap h^{-1}B) = \mathrm{vol}_{S^2}(gB \cap B) = f(g).$$

The last equality is justified by vol_{S^2} being $SO(3)$ invariant. Thus, f is entirely determined by its values at a rotation of a given axis, thus depending only on the angle β,

$$f(\beta) = f(R_\beta).$$

The support of f is, thus, given by the interval $0 \leq \beta \leq \mathrm{diam}(B)$.

$$\mathbb{E}[V_{t\wedge\tau}] = \frac{4}{\pi^2\sqrt{\pi t}} \int_{\beta=0}^{\mathrm{diam}(B)} \int_{\alpha=0}^{\pi} p_t^G(\beta,\theta) f(\beta) e^{L_t(\theta)} \sin^2(\theta/2) \sin^2(\beta/2) d\theta\, d\beta.$$

Let us set

$$\mathcal{J}(t,\theta) = \int_{\beta=0}^{\mathrm{diam}(B)} p_t(\beta,\theta) f(\beta) \sin^2(\beta) d\beta.$$

Then, using the heat kernel expansion (9):

$$\mathcal{J}(t,\theta) = \int_{\beta=0}^{\mathrm{diam}(B)} \left(1 + \sum_{n\geq 1} (2n+1) e^{-n(n+1)t/2} \chi_n(\theta)\chi_n(\beta)\right) f(\beta) \sin^2(\beta/2) d\beta.$$

Let us denote $J_n = \displaystyle\int_0^{\mathrm{diam}(B)} f(\beta)\chi_n(\beta) \sin^2(\beta/2)\, d\beta$ for $n \geq 0$; therefore,

$$\mathcal{J}(t,\theta) = J_0 + \sum_{n\geq 1} (2n+1) e^{-n(n+1)t/2} \chi_n(\theta) J_n.$$

Finally, using Fubini's theorem, one has

$$\mathbb{E}[V_{t\wedge\tau}] = \frac{4}{\pi^2\sqrt{\pi t}} \int_0^{\pi} \mathcal{J}(t,\theta) e^{L_t(\theta)} \sin^2(\theta/2) d\theta.$$

This proves Theorem 1. □

5. Conclusions

Using all the variety of mathematical tools coming from the theory of the Brownian motions on manifolds, we were able to derive an integral expression for the expectation of the volume intersection of a subset of the sphere S^2 with its translation. Such results could be applied to concrete problems in physics and dynamical 3D image processing. A natural generalization of our result would be to try to find an analog of Theorem 1 by replacing Brownian motions on Lie groups by Levy processes, which are stochastic processes, which can have jump discontinuities using the recent results of Albeverio and Gordina [4].

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mathematics

Article

A Note on the Geometry of RW Space-Times

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Abstract: A conformally flat GRW space-time is a perfect fluid RW space-time. In this note, we investigated the influence of many differential curvature conditions, such as the existence of recurrent and semi-symmetric curvature tensors. In each case, the form of the Ricci curvature tensor, the energy–momentum tensor, the energy density, the pressure of the fluid, and the equation of state are determined and interpreted. For example, it is demonstrated that a Ricci semi-symmetric RW space-time reduces to Einstein space-time or a Ricci recurrent RW space-time, and the perfect fluid space-time is referred to as Yang pure space-time or dark matter era.

Keywords: RW space-times; Ricci semi-symmetric; Ricci recurrent; projective curvature tensor

MSC: 53C25; 83F05

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1. Introduction

One of the most significant areas of research in both mathematics and physics is the geometry of generalized Robertson–Walker (or GRW) space-times. A warped product manifold with a one-dimensional base manifold serves as the representation of a GRW space-time. The term Friedmann–Lemaitre–Robertson–Walker metrics, which accurately captures the contributions of different scientists to this issue, is currently used in physics for Robertson–Walker-type metrics. There are many exciting decomposition theorems on Lorentzian manifolds. The author of [1] described a particularly remarkable decomposition of a Lorentzian manifold to a GRW space-time. The existence of a time-like concircular vector field is sufficient for a Lorentzian manifold to be a GRW space-time. This condition becomes weaker as follows in the presence of another condition [2]. If a unit time-like torse-forming vector field ω^i that is an eigenvector of the Ricci tensor S_{ij} exists on a Lorentzian manifold M, then M is a GRW space-time. By a unit time-like torse-forming, we mean that there is a scalar function φ on M such that

$$\nabla_k\omega_j = \varphi\Big(\omega_k\omega_j + g_{kj}\Big), \tag{1}$$

$$\omega^i\omega_i = -1. \tag{2}$$

The factor φ coincides with the Hubble's parameter H on a GRW space-time M. How rapidly the universe is expanding is determined by Hubble's parameter H (for a description of H and further information, see [3]). This torse-forming vector field is also an eigenvector of the Ricci tensor S_{ij}, that is, $\omega^i S_{ij} = \psi\omega_j$ where ψ is the corresponding eigenvalue of ω_j [1,2,4]. In [5], a GRW space-time Ricci tensor has been established to be

$$S_{ij} = \frac{S-\psi}{n-1} g_{ij} + \frac{S-n\psi}{n-1} \omega_i \omega_j + (n-2)\omega^k \omega^l C_{kijl} \tag{3}$$

where C_{kijl} is the Weyl conformal curvature tensor and S is the scalar curvature. The classical Robertson–Walker (or RW) space-time is a conformally flat GRW space-time, which allows the Ricci curvature form to change as

$$S_{ij} = \frac{S-\psi}{n-1} g_{ij} + \frac{S-n\psi}{n-1} \omega_i \omega_j. \tag{4}$$

On the other hand, the Ricci tensor of a perfect fluid space-time has the form

$$S_{ij} = \alpha g_{ij} + \beta \tau_i \tau_j. \tag{5}$$

Accordingto this equation, an RW space-time is a perfect fluid space-time where

$$\alpha = \frac{S-\psi}{n-1}, \beta = \frac{S-n\psi}{n-1}, \tau_i = \omega_i. \tag{6}$$

For further information about perfect fluid space-times and characterization of GRW space-times and RW space-times, the reader is recommended to read [2,4–7]. An algebraic curvature condition is that a space-time is a perfect fluid space-time [8]. Manifolds having this algebraic curvature criterion are known as quasi-Einstein manifolds in differential geometry [9]. However, there are additional types of differential curvature conditions that can be used, such as the existence of recurrent and semi-symmetric curvature tensors. Many alternative differential curvature conditions are examined in this article by using Riemann and Ricci curvature tensors. In each case, the form of the Ricci tensor, energy–momentum tensor, pressure, energy density and equation of state of the perfect fluid is given.

2. Notes on RW Space-Times

It is easy to obtain the scalar curvature of RW space-time, the eigenvalue of the Ricci tensor corresponding to ω and the divergence of the one form ω as

$$S = n\alpha - \beta, \ \psi = \alpha - \beta \tag{7}$$

$$\nabla^j \omega_j = (n-1)\varphi \tag{8}$$

It should be observed that the form (5) on an RW space-time has a perfect fluid structure that is unique up to a sign. For this, we assume that there exists a vector field v that is time-like and

$$S_{ij} = \bar{\alpha} g_{ij} + \bar{\beta} v_i v_j.$$

Then,

$$\omega^i S_{ij} = \bar{\alpha}\omega_j + \bar{\beta}\left(\omega^i v_i\right) v_j$$

$$(\psi - \bar{\alpha})\omega_j = \bar{\beta}\left(\omega^i v_i\right) v_j.$$

Since any two time-like vectors can not be orthogonal to each other, $\psi - \bar{\alpha} = \bar{\beta} = 0$; that is, M is Einstein, or $\omega_j = \pm v_j$.

Einstein's field equations without cosmological constant are

$$S_{ij} - \frac{S}{2} g_{ij} = kT_{ij}$$

where T_{ij} is the energy–momentum tensor, and k is the gravitational constant. Thus,

$$\alpha g_{ij} + \beta\omega_i\omega_j - \frac{S}{2} g_{ij} = kT_{ij}$$

$$\left(\alpha - \frac{S}{2}\right) g_{ij} + \beta\omega_i\omega_j = kT_{ij}. \tag{9}$$

However, the energy–momentum tensor for a perfect fluid space-time with velocity vector field ω is given by

$$T_{ij} = pg_{ij} + (p+\mu)\omega_i\omega_j \tag{10}$$

where p and μ are pressure and energy density, respectively [10]. Equations (9) and (10) show that

$$\begin{aligned} kp &= \alpha - \frac{S}{2} = \frac{S-\psi}{n-1} - \frac{S}{2} = \frac{(3-n)S-2\psi}{2(n-1)} && (11)\\ k(p+\mu) &= \beta = \frac{S-n\psi}{n-1} && (12)\\ k\mu &= \frac{S-n\psi}{n-1} - \frac{(3-n)S-2\psi}{2(n-1)} = \frac{S-2\psi}{2}. && (13) \end{aligned}$$

3. Ricci Curvature Conditions on RW Space-Times

3.1. Semi-Symmetric Ricci Curvature

A space-time is called Ricci semi-symmetric if [11]

$$\nabla_l\nabla_k S_{ij} - \nabla_k\nabla_l S_{ij} = 0.$$

Taking the covariant derivative of Equation (5) twice, one obtains

$$\begin{aligned} \nabla_l\nabla_k S_{ij} - \nabla_k\nabla_l S_{ij} &= \nabla_l\nabla_k\big(\alpha g_{ij} + \beta\omega_i\omega_j\big) - \nabla_k\nabla_l\big(\alpha g_{ij} + \beta\omega_i\omega_j\big)\\ &= \beta\nabla_l\nabla_k\big(\omega_i\omega_j\big) - \beta\nabla_k\nabla_l\big(\omega_i\omega_j\big)\\ &= \beta\big[\omega_i(\nabla_l\nabla_k - \nabla_k\nabla_l)\omega_j + \omega_j(\nabla_l\nabla_k - \nabla_k\nabla_l)\omega_i\big]. && (14) \end{aligned}$$

It is clear that this equation implies that an RW space-time is Ricci semi-symmetric if and only if either $\beta = 0$ or $\nabla_l\nabla_k\omega_j = \nabla_k\nabla_l\omega_j$. Let us consider the first condition. It is noted that $\beta = 0$ implies that an RW space-time is Einstein. The converse is also true. Assume that the space-time is Einstein, then

$$\begin{aligned} \frac{S}{n}g_{ij} &= \alpha g_{ij} + (n\alpha - S)\omega_i\omega_j,\\ \frac{S}{n}\omega_j &= (\alpha - n\alpha + S)\omega_j,\\ \frac{S}{n} &= \alpha - n\alpha + S, \end{aligned}$$

that is, $\alpha = \frac{S}{n}$. Equation (5) yields $\beta = n\alpha - S$ and consequently $\beta = 0$. The second condition is equivalent to $\omega_h S^h_{ilk} = 0$

Theorem 1. *An RW space-time M is Ricci semi-symmetric if and only if M is Einstein or* $\omega_h S^h_{ilk} = 0$.

Now, assume that $\beta = 0$. Then an RW space-time is Einstein and the eigenvalue is $\psi = \frac{S}{n}$. Let us rewrite the Ricci tensor and the energy–momentum tensor for a perfect fluid space-time in the case $\beta = 0$ as

$$\begin{aligned} S_{ij} &= \frac{S}{n}g_{ij} && (15)\\ T_{ij} &= pg_{ij} + (p+\mu)\omega_i\omega_j && (16) \end{aligned}$$

where

$$\alpha = \psi = \frac{S}{n}. \tag{17}$$

The equation of state in this case

$$
\begin{aligned}
k(p+\mu) &= 0, &(18)\\
kp &= -\left(\frac{n-2}{2n}\right)S, &(19)\\
k\mu &= \left(\frac{n-2}{2n}\right)S. &(20)\\
kT_{ij} &= -\left(\frac{n-2}{2n}\right)Sg_{ij}. &(21)
\end{aligned}
$$

That is, the perfect fluid is referred to as the dark energy. On the other hand, if $\nabla_l\nabla_k\omega_j = \nabla_k\nabla_l\omega_j$, then

$$\omega_h S^h_{ilk} = 0.$$

This equation yields

$$\omega_h S^h_k = 0.$$

Using Equation (5), it is

$$0 = (\alpha - \beta)\omega_j$$

and consequently $\psi = 0, \alpha = \beta$, and in this case, it is

$$S_{ij} = \frac{S}{n-1}(g_{ij} + \omega_i\omega_j). \tag{22}$$

Equations (9) and (10) show that

$$
\begin{aligned}
kT_{ij} &= -\frac{(n-3)}{2(n-1)}Sg_{ij} + \frac{S}{n-1}\omega_i\omega_j, &(23)\\
kp &= -\frac{(n-3)}{2(n-1)}S, &(24)\\
k\mu &= \frac{S}{2}, &(25)\\
k(p+\mu) &= \frac{S}{n-1}. &(26)
\end{aligned}
$$

Theorem 2. *Let M be a Ricci semi-symmetric RW space-time. Then, M satisfies one of the following:*

1. *($\beta = 0$) M is Einstein. The Ricci tensor and the equation of state take the form of Equation (15) and Equations (18)–(21). The perfect fluid is referred to as dark matter era.*
2. *($\alpha = \beta$) The Ricci tensor, the energy–momentum tensor, and the equation of state take the form of Equations (22)–(26).*

Remark 1. *Notably, an RW space-time is a perfect fluid space-time. Dark matter era refers to perfect fluid space with the equation of state $p + \mu = 0$ [12]. However, so far, according to [13], a four-dimensional perfect fluid space-time with $p + \mu \neq 0$ is RW space-time if and only if it is a Yang pure space-time. These space-times are identified by a Ricci tensor, which is a Codazzi tensor [13].*

Corollary 1. *A four-dimensional Ricci semi-symmetric RW space-time is a Yang pure space-time given that $\beta \neq 0$.*

3.2. Generalized Recurrent Ricci Curvature

A space-time M is called generalized Ricci recurrent if there are two $1-$form a and b such that

$$(\nabla_X S)(Y,Z) = a(X)S(Y,Z) + b(X)g(Y,Z) \tag{27}$$

where $X, Y, Z \in \mathfrak{X}(M)$ and a, b are called the corresponding recurrence $1-$forms. In local coordinates, one may write

$$\nabla_l S_{ij} = a_l S_{ij} + b_l g_{ij}. \tag{28}$$

Two contractions of this equation by g^{ij} and g^{li} yield

$$\nabla_l S = S a_l + n b_l \tag{29}$$
$$\frac{1}{2}\nabla_j S = a_l S^l_j + b_j. \tag{30}$$

A third contraction with ω^i infers

$$\omega^i \nabla_l S_{ij} = a_l \left(\omega^i S_{ij}\right) + b_l \omega_j. \tag{31}$$

Since ω^i is an eigenvector of the Ricci tensor, it is

$$\nabla_l \left(\omega^i S_{ij}\right) - S_{ij}\nabla_l \omega^i = (\psi a_l + b_l)\omega_j \tag{32}$$
$$\nabla_l (\psi\omega_j) - S_{ij}\nabla_l \omega^i = (\psi a_l + b_l)\omega_j. \tag{33}$$

Now, we may insert the definition of the Ricci tensor as

$$\nabla_l (\psi\omega_j) - (\alpha g_{ij} + \beta\omega_i\omega_j)\nabla_l \omega^i = (\psi a_l + b_l)\omega_j \tag{34}$$
$$\nabla_l (\psi\omega_j) - \alpha\nabla_l \omega_j = (\psi a_l + b_l)\omega_j \tag{35}$$
$$(\nabla_l \psi)\omega_j + (\psi - \alpha)\nabla_l \omega_j = (\psi a_l + b_l)\omega_j. \tag{36}$$

By multiplying both sides by ω^j, one obtains

$$(\nabla_l \psi) = \psi a_l + b_l. \tag{37}$$

Back substitution in Equation (36) results in

$$(\psi - \alpha)\nabla_l \omega_j = 0. \tag{38}$$

Thus, we have two cases, namely, $\psi - \alpha = 0$ and $\nabla_l \omega_j = 0$. The first case $\psi - \alpha = 0$ implies that $\beta = 0$ and so M is Einstein, and the Ricci tensor and the equation of state take the form of Equation (15) and Equations (18)–(21). To consider the second case $\nabla_l \omega_j = 0$, it is clear that $\varphi = 0$. In this case, the perfect fluid is called static. One may use the fact that

$$\psi = (n-1)\left(\varphi^2 + \dot{\varphi}\right)$$

where $\dot{\varphi} = \omega^k \nabla_k \varphi$ to obtain $\psi = 0$, that is $\alpha = \beta$. In this case, the Ricci tensor, the energy–momentum tensor, and the equation of state take the form

$$\begin{aligned}
S_{ij} &= \frac{S}{n-1}\left(g_{ij} + \omega_i\omega_j\right), \\
kT_{ij} &= \frac{S}{2n-2}\left(-(n-3)g_{ij} + 2\omega_i\omega_j\right), \\
k(p+\mu) &= \frac{S}{n-1}, \\
kp &= -\frac{(n-3)S}{2(n-1)}, \\
k\mu &= \frac{S}{2}.
\end{aligned}$$

The covariant derivative of the Ricci tensor is now given by

$$\nabla_l S_{ij} = \frac{\nabla_l S}{n-1}\big(g_{ij} + \omega_i \omega_j\big).$$

Using the defining property of the generalized Ricci recurrent tensor, it is

$$a_l S_{ij} + b_l g_{ij} = \frac{\nabla_l S}{n-1}\big(g_{ij} + \omega_i \omega_j\big).$$

Now, the definition of the Ricci tensor yields

$$a_l \frac{S}{n-1}\big(g_{ij} + \omega_i \omega_j\big) + b_l g_{ij} = \frac{\nabla_l S}{n-1}\big(g_{ij} + \omega_i \omega_j\big).$$

One may simplify this equation as

$$\begin{aligned} 0 &= \left(\frac{\nabla_l S}{n-1} - a_l \frac{S}{n-1} - b_l\right) g_{ij} + \left(\frac{\nabla_l S}{n-1} - a_l \frac{S}{n-1}\right)\omega_i \omega_j \\ 0 &= (\nabla_l S - S a_l - (n-1) b_l) g_{ij} + (\nabla_l S - S a_l)\omega_i \omega_j. \end{aligned}$$

Different contractions of this equation infer

$$\begin{aligned} b_l &= 0, \\ 0 &= \nabla_l S - S a_l - n b_l. \\ S a_l &= \nabla_l S \end{aligned}$$

The defining equation of the generalized Ricci recurrent RW space-time reduces to

$$\nabla_l S_{ij} = a_l S_{ij}$$

For a non-zero scalar curvature S, it is

$$\nabla_l S_{ij} = \nabla_l (\ln S) S_{ij}.$$

Theorem 3. *Let M be a generalized Ricci recurrent RW space-time. Then M reduces to be Einstein or to a Ricci recurrent RW space-time of the form*

$$\nabla_l S_{ij} = \nabla_l (\ln S) S_{ij}.$$

Moreover, M satisfies one of the following:

1. *M is Einstein. The Ricci tensor and the equation of state take the form of Equations (15) and (18)–(21).*
2. *M is a static perfect fluid, and the Ricci tensor, the energy–momentum tensor, and the equation of state take the form of Equations (22)–(26).*

A space-time M is called Ricci recurrent if there is a $1-$form a such that

$$(\nabla_X S)(Y, Z) = a(X) S(Y, Z)$$

where X, Y, Z are vector fields on M and a is called the recurrence $1-$form. In local coordinates, one may write

$$\nabla_l S_{ij} = a_l S_{ij}.$$

It should be noted that a Ricci recurrent space-time is a generalized Ricci recurrent space-time. Let M be a Ricci recurrent RW space-time. Then M reduces to be Einstein or to a Ricci recurrent RW space-time of the form

$$\nabla_l S_{ij} = \nabla_l (\ln S) S_{ij}.$$

Moreover, M is either Einstein and Equations (15)–(21) hold or M is a static perfect fluid and the Ricci tensor, the energy–momentum tensor, and the equation of state take the form of Equations (22)–(26).

A space-time M is called Ricci symmetric if [14]

$$(\nabla_X S)(Y, Z) = 0$$

where X, Y, Z are vector fields on M. In local coordinates, one may write

$$\nabla_l S_{ij} = 0.$$

It should be noted that a Ricci symmetric space-time is a Ricci recurrent space-time. In the Ricci flat case, it is easy to show that only one case of the above result will hold, namely, M is Einstein.

Corollary 2. *Let M be a Ricci symmetric RW space-times. Then M reduces to be Einstein, and the Ricci tensor and the equation of state take the form of Equations (15) and (18)–(21).*

3.3. Codazzi Type of Ricci Tensor

The RW space-time is of Codazzi type of Ricci tensor if

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) \tag{39}$$

where X, Y, Z are vector fields on M. In local coordinates, it is

$$\nabla_k S_{ij} = \nabla_i S_{kj}. \tag{40}$$

To obtain contraction of this equation by $\omega^i \omega^k$, let us first evaluate both sides as

$$\begin{aligned} \omega^i \omega^k \nabla_k S_{ij} &= \omega^i \left(\dot{\alpha} g_{ij} + \dot{\beta} \omega_i \omega_j + \beta \dot{\omega}_i \omega_j + \beta \omega_i \dot{\omega}_j \right) \\ &= \dot{\alpha} \omega_j - \dot{\beta} \omega_j - \beta \dot{\omega}_j \\ &= \dot{\psi} \omega_j - \beta \dot{\omega}_j \end{aligned} \tag{41}$$

and

$$\begin{aligned} \omega^i \omega^k \nabla_i S_{kj} &= \omega^i \nabla_i \left(\omega^k S_{kj} \right) - \omega^i S_{kj} \nabla_i \omega^k \\ &= \omega^i \nabla_i (\psi \omega_j) - \omega^i S_{kj} \nabla_i \omega^k \\ &= \dot{\psi} \omega_j + \psi \dot{\omega}_j - \omega^i \varphi S_{kj} \left(\delta_i^k + \omega_i \omega^k \right) \\ &= \dot{\psi} \omega_j + \psi \dot{\omega}_j - \omega^i \varphi (S_{ij} + \psi \omega_i \omega_j) \\ &= \dot{\psi} \omega_j + \psi \dot{\omega}_j - \omega^i \varphi (\psi \omega_j - \psi \omega_j) \\ &= \dot{\psi} \omega_j + \psi \dot{\omega}_j \end{aligned} \tag{42}$$

The above equations imply

$$\begin{aligned} \dot{\psi} \omega_j - \beta \dot{\omega}_j &= \dot{\psi} \omega_j + \psi \dot{\omega}_j \\ (\psi + \beta) \dot{\omega}_j &= 0 \end{aligned}$$

Therefore, $\psi + \beta = 0$ or $\dot{\omega}_j = 0$. The first case infers $\alpha = 0$ $\psi = -\beta = S$ and consequently

$$S_{ij} = -S\omega_i\omega_j. \tag{43}$$

Space-times with this form of Ricci curvature are called Ricci simple space-times [15]. The energy–momentum tensor, the pressure and energy density are consequently given by

$$\begin{aligned} kT_{ij} &= -\frac{S}{2}g_{ij} - \frac{3S}{2}\omega_i\omega_j, & (44)\\ kp &= -\frac{S}{2}, & (45)\\ k\mu &= -S, & (46)\\ k(p+\mu) &= -\frac{3S}{2}. & (47) \end{aligned}$$

The second condition implies that the fluid acceleration is zero and the velocity vector field is geodesic.

Theorem 4. *Let M be an RW space-time admitting a Codazzi type of Ricci tensor. Then, the velocity vector field is geodesic or M is Ricci simple and*

$$\begin{aligned} S_{ij} &= -S\omega_i\omega_j, & (48)\\ kT_{ij} &= -\frac{S}{2}g_{ij} - \frac{3S}{2}\omega_i\omega_j, & (49)\\ k(p+\mu) &= -\frac{3S}{2}, & (50)\\ kp &= -\frac{S}{2}, & (51)\\ k\mu &= -S. & (52) \end{aligned}$$

4. Riemann Curvature Tensor on RW Space-Times

The Riemann curvature tensor of an RW space-time is completely determined by the vector ω as follows. It is clear that the conformal curvature tensor is null and so

$$\begin{aligned} 0 &= \mathcal{C}_{jklm}\\ &= S_{jklm} + \frac{1}{n-2}\Big[g_{jm}S_{kl} - g_{km}S_{jl} + g_{kl}S_{jm} - g_{jl}S_{km}\Big]\\ &\quad - \frac{S}{(n-1)(n-2)}\Big[g_{jm}g_{kl} - g_{km}g_{jl}\Big]. \end{aligned} \tag{53}$$

Now, the Riemann curvature tensor has the form

$$\begin{aligned} S_{jklm} &= \frac{S}{(n-1)(n-2)}\Big[g_{jm}g_{kl} - g_{km}g_{jl}\Big]\\ &\quad - \frac{1}{n-2}\Big[g_{jm}S_{kl} - g_{km}S_{jl} + g_{kl}S_{jm} - g_{jl}S_{km}\Big] \end{aligned} \tag{54}$$

Using the form of the Ricci curvature tensor, one obtains

$$\begin{aligned} S_{jklm} &= \frac{S - 2(n-1)\alpha}{(n-1)(n-2)}\Big[g_{jm}g_{kl} - g_{km}g_{jl}\Big]\\ &\quad - \frac{\beta}{n-2}\Big[g_{jm}\omega_k\omega_l - g_{km}\omega_j\omega_l + g_{kl}\omega_j\omega_m - g_{jl}\omega_k\omega_m\Big]. \end{aligned} \tag{55}$$

It is clear that

$$\begin{aligned}\nabla_r(\omega_k\omega_l) &= \nabla_r(\omega_k)\omega_l + \omega_k\nabla_r(\omega_l)\\ &= \varphi(g_{rk}+\omega_r\omega_k)\omega_l + \varphi(g_{rl}+\omega_r\omega_l)\omega_k\\ &= \varphi(g_{rk}\omega_l + g_{rl}\omega_k + 2\omega_r\omega_k\omega_l)\end{aligned}$$

After lengthy computations using this equation, the covariant derivative of the Riemann curvature tensor may be finally rewritten as

$$\begin{aligned}\nabla_r S_{jklm} &= \frac{\nabla_r S - 2(n-1)\nabla_r\alpha}{(n-1)(n-2)}\Big[g_{jm}g_{kl} - g_{km}g_{jl}\Big]\\ &\quad -\frac{\nabla_r\beta}{n-2}\Big[g_{jm}\omega_k\omega_l - g_{km}\omega_j\omega_l + g_{kl}\omega_j\omega_m - g_{jl}\omega_k\omega_m\Big]\\ &\quad -\frac{\varphi\beta}{n-2}\big[g_{jm}g_{rk}\omega_l + g_{jm}g_{rl}\omega_k + 2g_{jm}\omega_r\omega_k\omega_l - g_{km}g_{rj}\omega_l\big]\\ &\quad -\frac{\varphi\beta}{n-2}\big[-g_{km}g_{rl}\omega_j - 2g_{km}\omega_r\omega_j\omega_l + g_{kl}g_{rj}\omega_m + g_{kl}g_{rm}\omega_j\big]\\ &\quad -\frac{\varphi\beta}{n-2}\Big[2g_{kl}\omega_r\omega_j\omega_m - g_{jl}g_{rk}\omega_m - g_{jl}g_{rm}\omega_k - 2g_{jl}\omega_r\omega_k\omega_m\Big] \qquad (56)\end{aligned}$$

4.1. Locally Symmetric RW Space-Time

Assume that an RW space-time is symmetric, that is, $\nabla_r S_{jklm} = 0$ [16], and consequently, the scalar curvature is constant and $n\nabla\alpha = \nabla\beta$. Thus,

$$\begin{aligned}0 &= \frac{-2\nabla_r\alpha}{(n-2)}\Big[g_{jm}g_{kl} - g_{km}g_{jl}\Big]\\ &\quad -\frac{n\nabla_r\alpha}{n-2}\Big[g_{jm}\omega_k\omega_l - g_{km}\omega_j\omega_l + g_{kl}\omega_j\omega_m - g_{jl}\omega_k\omega_m\Big]\\ &\quad -\frac{\varphi\beta}{n-2}\big[g_{jm}g_{rk}\omega_l + g_{jm}g_{rl}\omega_k + 2g_{jm}\omega_r\omega_k\omega_l - g_{km}g_{rj}\omega_l\big]\\ &\quad -\frac{\varphi\beta}{n-2}\big[-g_{km}g_{rl}\omega_j - 2g_{km}\omega_r\omega_j\omega_l + g_{kl}g_{rj}\omega_m + g_{kl}g_{rm}\omega_j\big]\\ &\quad -\frac{\varphi\beta}{n-2}\Big[2g_{kl}\omega_r\omega_j\omega_m - g_{jl}g_{rk}\omega_m - g_{jl}g_{rm}\omega_k - 2g_{jl}\omega_r\omega_k\omega_m\Big].\end{aligned}$$

By multiplying this equation by g^{jl}, it is

$$0 = \nabla_r\alpha[n\omega_k\omega_m + g_{km}] + \varphi\beta[g_{rk}\omega_m + g_{rm}\omega_k + 2\omega_r\omega_k\omega_m]. \qquad (57)$$

A last contraction with $\omega^k\omega^m$ gives us $\nabla_r\alpha = 0$. Back substitution in the above equation yields

$$\begin{aligned}0 &= \varphi\beta[g_{rk}\omega_m + g_{rm}\omega_k + 2\omega_r\omega_k\omega_m] \qquad (58)\\ &= \varphi\beta[\omega_m\nabla_r u_k + \omega_k\nabla_r u_m] \qquad (59)\\ &= \varphi\beta[\omega_m\nabla_r\omega_k + \omega_k\nabla_r\omega_m] \qquad (60)\end{aligned}$$

From this equation, it is easy to show that either $\beta = 0$ or $\nabla_r\omega_m = 0$. The first case implies that the space-time is Einstein, and the second case infers the space-time is static. In the first case, the Riemann curvature tensor becomes

$$\begin{aligned}S_{jklm} &= \frac{S-2(n-1)\alpha}{(n-1)(n-2)}\Big[g_{jm}g_{kl} - g_{km}g_{jl}\Big] \qquad (61)\\ &= \frac{S}{n(n-1)}\Big[g_{km}g_{jl} - g_{jm}g_{kl}\Big]. \qquad (62)\end{aligned}$$

Therefore, RW space-time has constant curvature. A simple contraction of this equation implies $\alpha = 0$. Now, the manifold is Ricci flat and consequently is flat.

Theorem 5. *Let M be a locally symmetric RW space-time. Then,*

1. *M has a constant curvature. The Riemann tensor, the Ricci tensor, and the equation of state take the form*

$$\begin{aligned} S_{jklm} &= \frac{S}{n(n-1)}\Big[g_{km}g_{jl} - g_{jm}g_{kl}\Big]. \\ S_{ij} &= \frac{S}{n}g_{ij}, \\ kT_{ij} &= -\left(\frac{n-2}{2n}\right)Sg_{ij}, \\ kp &= -k\mu = -\left(\frac{n-2}{2n}\right)S, \\ k(p+\mu) &= 0. \end{aligned}$$

2. *M is a static space-time.*

4.2. Recurrent RW Space-Times

A space-time is called recurrent if there is one form a such that

$$(\nabla_U S)(X,Y,Z,W) = a(U)S(X,Y,Z,W).$$

In local coordinates, it is

$$\nabla_r S_{jklm} = a_r S_{jklm}.$$

Thus, an RW space-time is recurrent if

$$\begin{aligned} a_r S_{jklm} &= \nabla_r S_{jklm} \\ &= \frac{\nabla_r S - (n-1)\nabla_r \alpha}{(n-1)(n-2)}\Big[g_{jm}g_{kl} - g_{km}g_{jl}\Big] \\ &\quad -\frac{\nabla_r \beta}{n-2}\Big[g_{jm}\omega_k\omega_l - g_{km}\omega_j\omega_l + g_{kl}\omega_j\omega_m - g_{jl}\omega_k\omega_m\Big] \\ &\quad -\frac{\varphi\beta}{n-2}[g_{jm}g_{rk}\omega_l + g_{jm}g_{rl}\omega_k + 2g_{jm}\omega_r\omega_k\omega_l - g_{km}g_{rj}\omega_l] \\ &\quad -\frac{\varphi\beta}{n-2}[-g_{km}g_{rl}\omega_j - 2g_{km}\omega_r\omega_j\omega_l + g_{kl}g_{rj}\omega_m + g_{kl}g_{rm}\omega_j] \\ &\quad -\frac{\varphi\beta}{n-2}\Big[2g_{kl}\omega_r\omega_j\omega_m - g_{jl}g_{rk}\omega_m - g_{jl}g_{rm}\omega_k - 2g_{jl}\omega_r\omega_k\omega_m\Big] \end{aligned} \tag{63}$$

Using the calculations in the above subsection, one obtains

$$a_r S_{km} = -\frac{\nabla_r \alpha}{n-2}g_{km} + \nabla_r \beta\omega_k\omega_m + \varphi\beta[g_{rk}\omega_m + g_{rm}\omega_k + 2\omega_r\omega_k\omega_m].$$

Using two contractions with g^{km} and $\omega^k\omega^m$, this equation infers

$$\begin{aligned} a_r S &= -\frac{n\nabla_r \alpha}{n-2} - \nabla_r \beta \\ a_r \psi\omega_k &= -\frac{\nabla_r \alpha}{n-2}\omega_k - \nabla_r \beta\omega_k + \varphi\beta[-g_{rk} - \omega_r\omega_k] \\ a_r \psi &= -\frac{\nabla_r \alpha}{n-2} - \nabla_r \beta \end{aligned}$$

The subtraction of these two equations implies

$$\begin{aligned} a_r(S-\psi) &= -\frac{n-1}{n-2}\nabla_r\alpha \\ a_r(n-1)\alpha &= -\frac{n-1}{n-2}\nabla_r\alpha \\ a_r\alpha &= -\frac{1}{n-2}\nabla_r\alpha. \end{aligned}$$

Theorem 6. *Let M be a recurrent RW space-time. Then, M is Ricci simple or the recurrence form is given by*

$$a_r = -\frac{1}{n-2}\frac{1}{\alpha}\nabla_r\alpha.$$

4.3. Harmonic RW Space-Time

A contraction of $\nabla_r S_{jklm}$ with g^{rj} infers

$$\begin{aligned} \nabla^j S_{jklm} &= \frac{\nabla_m S-(n-1)\nabla_m\alpha}{(n-1)(n-2)}g_{kl} - \frac{\nabla_l S-(n-1)\nabla_l\alpha}{(n-1)(n-2)}g_{km} \\ &\quad -\frac{\nabla_m\beta}{n-2}\omega_k\omega_l + \frac{\dot{\beta}}{n-2}g_{km}\omega_l - \frac{\dot{\beta}}{n-2}g_{kl}\omega_m + \frac{\nabla_l\beta}{n-2}\omega_k\omega_m \\ &\quad -\frac{\varphi\beta}{n-2}[g_{km}\omega_l + g_{lm}\omega_k + 2\omega_m\omega_k\omega_l - ng_{km}\omega_l] \\ &\quad -\frac{\varphi\beta}{n-2}[-g_{km}\omega_l + 2g_{km}\omega_l + ng_{kl}\omega_m + g_{kl}\omega_m] \\ &\quad -\frac{\varphi\beta}{n-2}[-2g_{kl}\omega_m - g_{lk}\omega_m - g_{lm}\omega_k - 2\omega_l\omega_k\omega_m] \end{aligned} \tag{64}$$

Thus, the divergence of the Riemann tensor is give by

$$\begin{aligned} \nabla^j S_{jklm} &= \frac{1}{(n-1)(n-2)}((\nabla_m\psi)g_{kl} - (\nabla_l\psi)g_{km}) \\ &\quad +\left(\frac{\dot{\beta}}{n-2}+\varphi\beta\right)(g_{km}\omega_l - g_{kl}\omega_m) \\ &\quad -\frac{1}{n-2}((\nabla_m\beta)\omega_k\omega_l - (\nabla_l\beta)\omega_k\omega_m). \end{aligned} \tag{65}$$

Assume that M is harmonic, that is,

$$\begin{aligned} 0 &= \nabla^j S_{jklm} \\ &= \frac{1}{(n-1)(n-2)}((\nabla_m\psi)g_{kl} - (\nabla_l\psi)g_{km}) \\ &\quad +\left(\frac{\dot{\beta}}{n-2}+\varphi\beta\right)(g_{km}\omega_l - g_{kl}\omega_m) \\ &\quad -\frac{1}{n-2}((\nabla_m\beta)\omega_k\omega_l - (\nabla_l\beta)\omega_k\omega_m). \end{aligned} \tag{66}$$

Therefore, one obtains

$$\begin{aligned} 0 &= \frac{1}{n-2}(\nabla_m\psi + \nabla_m\beta) + \big(-\dot{\beta} + \varphi\beta(1-n)\big)\omega_m \\ 0 &= \frac{n\nabla_m\alpha}{n-2} - \big(\dot{\beta} + (n-1)\varphi\beta\big)\omega_m \\ 0 &= \frac{n\dot{\alpha}}{n-2} + \big(\dot{\beta} + (n-1)\varphi\beta\big) \\ 0 &= n\dot{\alpha} + (n-2)\big(\dot{\beta} + (n-1)\varphi\beta\big) \end{aligned} \tag{67}$$

However, a harmonic RW space-time has a divergence free Ricci tensor, that is,

$$\begin{aligned} 0 &= \nabla^j S_{jk} \\ 0 &= \omega^k \nabla^j S_{jk} = \nabla^j\Big(\omega^k S_{jk}\Big) - S_{jk}\nabla^j\omega^k \\ &= \nabla^j(\psi\omega_j) - \varphi S_{jk}\Big(g^{jk} + \omega^j\omega^k\Big) \\ &= \psi\nabla^j(\omega_j) + \omega_j\nabla^j(\psi) - \varphi(S-\psi) \\ &= \psi\varphi(n-1) + \dot{\psi} - \varphi(n\alpha - \beta - \alpha + \beta) \\ &= \dot{\psi} - (n-1)\varphi\alpha + \psi\varphi(n-1) \\ &= \dot{\psi} - (n-1)\varphi\beta. \end{aligned} \tag{68}$$

Thus, $\dot{\psi} = (n-1)\varphi\beta$. Equation (68) now becomes

$$\begin{aligned} 0 &= n\dot{\alpha} + (n-2)\big(\dot{\beta} + \dot{\psi}\big) \\ &= (2n-2)\dot{\alpha}. \end{aligned}$$

Hence, $\dot{\alpha} = 0$, $\dot{\beta} = -\dot{\psi} = -(n-1)\varphi\beta$ and Equation (67) reduce to

$$\begin{aligned} 0 = &\frac{1}{(n-1)}((\nabla_m\psi)g_{kl} - (\nabla_l\psi)g_{km}) \\ &-\frac{1}{n-2}\varphi\beta(g_{km}\omega_l - g_{kl}\omega_m) \\ &-((\nabla_m\beta)\omega_k\omega_l - (\nabla_l\beta)\omega_k\omega_m). \end{aligned} \tag{69}$$

A contraction by g^{kl} implies

$$\begin{aligned} 0 &= \nabla_m\psi + \frac{n-1}{n-2}\varphi\beta\omega_m + \nabla_m\beta + \dot{\beta}\omega_m \\ &= \nabla_m\alpha + \left(1 - \frac{1}{n-2}\right)\dot{\beta}\omega_m \\ &= \nabla_m\alpha + \frac{n-3}{n-2}\dot{\beta}\omega_m \end{aligned} \tag{70}$$

Again, transfecting this equation by ω^m yields

$$0 = \dot{\alpha} - \frac{n-3}{n-2}\dot{\beta} = -\frac{n-3}{n-2}\dot{\beta} = \frac{n-3}{n-2}(n-1)\varphi\beta.$$

Therefore, $\beta = 0$ or $\varphi = 0$.

Theorem 7. *Let M be a harmonic RW space-time. Then, M is Einstein or M is a static space-time.*

5. Conclusions

A conformally flat GRW space-time satisfies an algebraic curvature condition; namely, it is a perfect fluid RW space-time. The existence of one of the differential curvature conditions (i.e., semi-symmetric Ricci curvature, generalized recurrent Ricci curvature tensor, recurrent Ricci curvature tensor, parallel Ricci curvature tensor, Codazzi Ricci tensor, locally symmetric, and harmonic Riemann curvature tensor) implies the RW space-time has a constant curvature or is a static space-time.

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mathematics

Article

Li–Yau-Type Gradient Estimate along Geometric Flow

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Abstract: In this article we derive a Li–Yau-type gradient estimate for a generalized weighted parabolic heat equation with potential on a weighted Riemannian manifold evolving by a geometric flow. As an application, a Harnack-type inequality is also derived in the end.

Keywords: gradient estimate; weighted Laplacian; parabolic equation; geometric flow

MSC: 53C21; 53E20; 35B45

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1. Introduction

The gradient estimation for both elliptic and parabolic equations plays a significant role in geometric analysis. Harnack estimation is also one of the powerful tools in heat kernel analysis. The local and global behavior of positive solutions of nonlinear elliptic equations on $\mathbb{R}^n$ $(n > 2)$ near an isolated singularity were studied by Gidas and Spruck [1]. In [2], Hamilton proved a Harnack estimate on the Riemannian manifold for Ricci flow with a weakly positive curvature operator, which was later used in solving the Poincaré conjecture. Li and Yau [3] established parabolic gradient estimates on solutions to the linear heat equation

$$(\Delta - \partial_t)u \quad = \quad q(x,t)u \tag{1}$$

on Riemannian manifold having Ricci curvature bounded from below, where $q(x,t)$ is C^2 in first variable x and C^1 in second variable t, where C^2 and C^1 denote the space of all twice differentiable and one-time differentiable functions, respectively. After a remarkable work by Perelman [4–6] in Ricci flow, this topic gained massive importance. Thus, this topic becomes one of the important tools in geometric analysis and modern PDE theory. In [7], Jiyau Li considered the heat-type equation

$$(\Delta - \partial_t)u(x,t) + h(x,t)u^{\alpha}(x,t) \quad = \quad 0 \tag{2}$$

on $M \times [0,\infty)$, where $h(x,t)$, is a function on $M \times [0,\infty)$, which is C^2 in the first variable and C^1 in the second variable, $\alpha \in \mathbb{R}$ and derived the gradient estimates and Harnack inequalities for a positive solution to the above nonlinear parabolic equation. This equation represents a simple ecological model for population dynamics, where $u(x,t)$ is the population density at time t.

Wu [8] studied gradient estimates for the nonlinear parabolic equation

$$(\Delta_{\phi} - \partial_t)u + \mu(x,t)u + p(x,t)u^{\alpha} + q(x,t)u^{\beta} \quad = \quad 0, \tag{3}$$

where Δ_ϕ is the weighted Laplacian, $p(x,t)$, $q(x,t)$ are C^2 in x and C^1 in t. Abolarinwa et al. [9–12] studied gradient and Harnack estimates for various nonlinear parabolic equations. In [13], Dung et al. studied various gradient estimations for solutions of the following f-heat type equations

$$u_t = \Delta_f u + au\log u + bu + Cu^p + Du^{-q} \tag{4}$$

$$\text{and } u_t = \Delta_f u + Ce^{pu} + De^{-pu} + E, \tag{5}$$

where $a, b \in \mathbb{R}$ and C, D, E are smooth functions, on a complete smooth metric measure space $(M, g, e^{-f}dv)$ with Bakry–Émery Ricci curvature bounded from below. In [14], Azami studied gradient estimates for a weighted parabolic equation

$$(\Delta_\phi - \partial_t)u(x,t) = q(x,t)u^{a+1}(x,t) + p(x,t)A(u(x,t)) \tag{6}$$

evolving under the geometric flow, where $p(x,t), q(x,t), A(u(x,t))$ are C^2 in x and C^1 in t. Thereafter many authors studied the geometric aspect of analysis on the Riemannian manifold, see [15–23] and the references therein. Recently, Hui et al. studied Hamilton-Souplet-Zhang type gradient estimation for nonlinear weighted parabolic equation in [24], the same estimation for a system of equations in [25] and Saha et al. [26] studied first eigenvalue of weighted p-Laplacian along the Cotton flow.

Motivated by the above works in this paper we consider a generalized non-linear parabolic equation with potential by

$$\Delta_\phi u = \frac{\partial u}{\partial t} + A(u)p(x,t) + B(u)q(x,t) + \xi(x,t)u(x,t), \tag{7}$$

where $p(x,t), q(x,t)$ and $\xi(x,t)$ are C^2 functions of x, t. We derive a Li–Yau-type gradient estimate for a positive solution of (7) on a weighted Riemannian manifold which evolves under an abstract geometric flow.

In particular, if we consider $A(u) = u^\alpha,\ B(u) = u^\beta,\ \xi = \mu(x,t)$ then (7) reduces to (3), which was studied by Wu [8]. If we take $A(u) = u\log u,\ B(u) = u,\ \xi = Cu^p + Du^q$ then (7) reduces to (4) and if $A(u) = Ce^{pu},\ B(u) = De^{-pu},\ \xi = \frac{E}{u}$ then (7) reduces to (5), both of which were studied by Dung et al.[13]. The generalized Lichnerowicz type equation studied by Dung [13] comes from our Equation (7) by considering $A(u) = u^\alpha \log u,\ B(u) = u^\beta$ and p, q, ξ are suitable constants. Finally for $B(u) = u^{a+1}$ and $\xi = 0$ we have (6), which was studied by Azami [14]. Thus, our Equation (7) generalizes all the cases.

2. Preliminaries

Let us consider an n-dimensional closed weighted Riemannian manifold $(M^n, g, e^{-\phi}d\mu)$, where $e^{-\phi}d\mu$ is the weighted volume measure, g is Riemannian metric and $\phi \in C^2(M)$. Choose $\{e_1, e_2, \cdots, e_n\}$ as an orthonormal frame on M. Let $g(t)$ be a one-parameter family of Riemannian metrics evolving along the following abstract geometric flow

$$\frac{\partial}{\partial t} g_{ij}(t) = 2S_{ij}(t), \tag{8}$$

where $S_{ij}(t) := \mathcal{S}(e_i, e_j)(t)$ is smooth symmetric (0, 2)-type tensor on $(M, g(t))$. Let us define one parameter family of functions $S(t) = trace(\mathcal{S})(t) = g^{ij}(t)S_{ij}(t)$ on M. The weighted Laplacian operator is defined by

$$\Delta_\phi = \Delta - \nabla\phi\nabla,$$

where Δ is the Laplace operator and ∇ is the gradient operator. Let $u = e^f$ be a positive solution of (7), then Equation (7) transforms to

$$\Delta_\phi f = \partial_t f - |\nabla f|^2 + \hat{A}(f)p + \hat{B}(f)q + \xi, \tag{9}$$

where $\hat{A}(f) = \frac{A(u)}{u}, \hat{B}(f) = \frac{B(u)}{u}$. We define

$$\hat{A}_f = A'(u) - \frac{A(u)}{u}, \;\; \hat{A}_{ff} = uA''(u) - A'(u) + \frac{A(u)}{u}. \tag{10}$$

Example 1. *Let $u = e^f$ and $A(u) = |u|^{\alpha-1}u$. Therefore $\hat{A}(f) = \frac{A(u)}{u} = e^{(\alpha-1)f}$, which gives*

1. $\hat{A}_f = (\alpha-1)e^{(\alpha-1)f}$
2. $\hat{A}_{ff} = (\alpha-1)^2 e^{(\alpha-1)f}$
3. $\nabla\hat{A} = (\alpha-1)e^{(\alpha-1)f}\nabla f = \hat{A}_f \nabla f$
4. $\Delta\hat{A} = (\alpha-1)^2 e^{(\alpha-1)f}|\nabla f|^2 + (\alpha-1)e^{(\alpha-1)f}\Delta f = \hat{A}_{ff}|\nabla f|^2 + \hat{A}_f \Delta f.$

Let $\bar{f} = \hat{A}p + \hat{B}q + \xi$ so that Equation (9) reduces to

$$\Delta_\phi f = -|\nabla f|^2 + f_t + \bar{f}. \tag{11}$$

Definition 1 ([27] Bakry–Émery Ricci tensor)**.** *For any integer $m > n$, an $(m-n)-$ Bakry–Émery tensor is defined by*

$$Ric_\phi^{m-n} := Ric + Hess\ \phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n},$$

where Hess is the Hessian operator. The case when $m = n$ occurs if and only if ϕ is a constant function. Furthermore, when $m \to \infty$ the $\infty-$Bakry–Émery Ricci tensor is defined by

$$Ric_\phi := Ric + Hess\ \phi.$$

Lemma 1 ([14] Weighted Bochner Formula)**.** *For any smooth function u on a weighted Riemannian manifold $(M, g, e^{-\phi}d\mu)$, we have the weighted version of Bochner formula*

$$\frac{1}{2}\Delta_\phi|\nabla u|^2 = |Hess\ u|^2 + \langle\nabla\Delta_\phi u, \nabla u\rangle + Ric_\phi(\nabla u, \nabla u),$$

where $\langle\cdot,\cdot\rangle$ is the induced inner product by the Riemannian metric g.

Lemma 2 ([14])**.** *Under the geometric flow Equation (8) and for any smooth function u on a weighted Riemannian manifold $(M, g, e^{-\phi}d\mu)$ we have the following evolution formulas*

1. $\frac{\partial}{\partial t}|\nabla u|^2 = -2\mathcal{S}(\nabla u, \nabla u) + 2\langle\nabla u, \nabla u_t\rangle$,
2. $\frac{\partial}{\partial t}(\Delta_\phi u) = \Delta_\phi u_t - 2S^{ij}\nabla_i\nabla_j u - \langle 2div\ \mathcal{S} - \nabla S, \nabla u\rangle + 2\mathcal{S}(\nabla\phi, \nabla u) - \langle\nabla u, \nabla\phi_t\rangle$, *where $div\ \mathcal{S}$ denotes the divergence of $\mathcal{S}$ and $S^{ij} = g^{ik}g^{jl}S_{kl}$.*

Let $T > 0$ be any real number. For any two points $x, y \in M$ and for any $t \in [0, T]$, the quantity $d(x, y, t)$ denotes the geodesic distance between x and y under the metric $g(t)$. For any fixed $x_0 \in M$ and $R > 0$ we define a compact set

$$Q_{2R,T} = \{(x,t) : d(x, x_0, t) \le 2R, 0 \le t \le T\} \subset M^n \times (-\infty, +\infty). \tag{12}$$

Now for $u > 0$ we define some non-negative real numbers

$$
\begin{array}{lll}
\lambda_1 := \sup_{Q_{2R,T}} |\hat{A}| & \lambda_2 := \sup_{Q_{2R,T}} |\hat{A}_f| & \lambda_3 := \sup_{Q_{2R,T}} |\hat{A}_{ff}| \\
\Lambda_1 := \sup_{M\times[0,T]} |\hat{A}| & \Lambda_2 := \sup_{M\times[0,T]} |\hat{A}_f| & \Lambda_3 := \sup_{M\times[0,T]} |\hat{A}_{ff}| \\
b_1 := \sup_{Q_{2R,T}} |\hat{B}| & b_2 := \sup_{Q_{2R,T}} |\hat{B}_f| & b_3 := \sup_{Q_{2R,T}} |\hat{B}_{ff}| \\
B_1 := \sup_{M\times[0,T]} |\hat{B}| & B_2 := \sup_{M\times[0,T]} |\hat{B}_f| & B_3 := \sup_{M\times[0,T]} |\hat{B}_{ff}| \\
\sigma_1 := \sup_{Q_{2R,T}} |q| & \sigma_2 := \sup_{Q_{2R,T}} |\nabla q| & \sigma_3 := \sup_{Q_{2R,T}} |\Delta_\phi q| \\
\Sigma_1 := \sup_{M\times[0,T]} |q| & \Sigma_2 := \sup_{M\times[0,T]} |\nabla q| & \Sigma_3 := \sup_{M\times[0,T]} |\Delta_\phi q| \\
\gamma_1 := \sup_{Q_{2R,T}} |p| & \gamma_2 := \sup_{Q_{2R,T}} |\nabla p| & \gamma_3 := \sup_{Q_{2R,T}} |\Delta_\phi p| \\
\Gamma_1 := \sup_{M\times[0,T]} |p| & \Gamma_2 := \sup_{M\times[0,T]} |\nabla p| & \Gamma_3 := \sup_{M\times[0,T]} |\Delta_\phi p| \\
\theta_1 := \sup_{Q_{2R,T}} |\nabla \phi| & \theta_2 := \sup_{Q_{2R,T}} |\nabla \phi_t| & \Theta_1 := \sup_{M\times[0,T]} |\nabla \phi| \\
\Theta_2 := \sup_{M\times[0,T]} |\nabla \phi_t| & m_1 := \sup_{Q_{2R,T}} |\nabla \xi| & m_2 := \sup_{Q_{2R,T}} |\Delta_\phi \xi| \\
m_3 := \sup_{Q_{2R,T}} |\xi| & M_1 := \sup_{M\times[0,T]} |\nabla \xi| & M_2 := \sup_{M\times[0,T]} |\Delta_\phi \xi| \\
M_3 := \sup_{M\times[0,T]} |\xi| & &
\end{array}
$$

Lemma 3 ([14])**.** *For any smooth function f on an n-dimensional Riemannian manifold $(M^n, g, e^{-\phi} d\mu)$ and $m > n$ we have the following relation connecting Hessian and weighted Laplacian*

$$
|Hess\ f|^2 \geq \frac{(\Delta_\phi f)^2}{m} - \frac{1}{m-n}\langle \nabla f, \nabla \phi \rangle^2. \tag{13}
$$

Proof. Let $m > n$. Then we see that

$$
\begin{aligned}
0 &\leq \left(\sqrt{\frac{m-n}{mn}}\Delta f + \sqrt{\frac{n}{m(m-n)}}\langle \nabla f, \nabla \phi \rangle \right)^2 \\
&= (\frac{1}{n} - \frac{1}{m})(\Delta f)^2 + \frac{2}{m}\Delta f \langle \nabla f, \nabla \phi \rangle + (\frac{1}{m-n} - \frac{1}{m})\langle \nabla f, \nabla \phi \rangle^2 \\
&\leq |\text{Hess } f|^2 - \frac{1}{m}\left((\Delta f)^2 - 2\Delta f \langle \nabla f, \nabla \phi \rangle + \langle \nabla f, \nabla \phi \rangle^2 \right) + \frac{1}{m-n}\langle \nabla f, \nabla \phi \rangle^2 \\
&= |\text{Hess } f|^2 - \frac{(\Delta_\phi f)^2}{m} + \frac{1}{m-n}\langle \nabla f, \nabla \phi \rangle^2.
\end{aligned}
$$

Thus $|\text{Hess } f|^2 \geq \frac{(\Delta_\phi f)^2}{m} - \frac{1}{m-n}\langle \nabla f, \nabla \phi \rangle^2$. □

Lemma 4 ([28] Young's inequality)**.** *If a, b are nonnegative real numbers and $p > 1$, $q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ then*

$$
ab \leq \frac{a^p}{p} + \frac{b^q}{q}.
$$

Let $\alpha > 0$ be any real number. Put $a = \alpha a$ and $b = \frac{b}{\alpha}$ in the above expression we get Peter-Paul type inequality

$$
ab \leq \alpha^p \frac{a^p}{p} + \frac{b^q}{\alpha^q q}. \tag{14}
$$

If we put $a = a\sqrt{2\alpha}$, $b = \frac{b}{\sqrt{2\alpha}}$, $p = q = 2$ in Young's inequality then we have the well known Peter-Paul inequality

$$ab \leq \alpha a^2 + \frac{b^2}{4\alpha}. \tag{15}$$

In this paper we use these inequalities with a suitable choice of α.

3. Li-iYau-Type Gradient Estimation

In this section, we are going to derive a bound for the quantity $\frac{|\nabla u|^2}{u^2}$ on a compact domain $Q_{2R,T}$ of M, where u satisfies (7). This estimation is known as local Li–Yau-type estimation. After that, we derive global Li–Yau-type estimation on the whole of M. This method enables us to find the heat ratio between two points on a manifold by deriving a Harnack-type inequality. For this, we fix a point $x_0 \in M$ and let $R > 0$ be a real number. Let u be a positive solution to (7) in $Q_{2R,T}$.

Theorem 1. *If k_1, k_2, k_3, k_4 are positive constants such that*

$$Ric_{\phi}^{m-n} \geq -(m-1)k_1 g,\ -k_2 g \leq \mathcal{S} \leq k_3 g,\ |\nabla S| \leq k_4$$

on $Q_{2R,T}$, then for any solution u of (7), any $\lambda > 1$ and $\delta \in (0,1)$ we have

$$\frac{|\nabla u|^2}{u^2} - \lambda\left(\frac{u_t}{u} + \frac{A(u)}{u}p + \frac{B(u)}{u}q + \xi\right) \leq \frac{m\lambda^2}{2t(1-\lambda\epsilon)} + \frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_1} + \tilde{E_1}, \tag{16}$$

where

$$\begin{aligned}
\tilde{D_1} &= \frac{c_0}{R}(m-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2k_2 + \frac{m\lambda^2 c_1}{4(1-\lambda\epsilon)(\lambda-1)R^2} + \frac{1}{\lambda}, \\
\tilde{E_1} &= \left(\frac{m\lambda^2}{2(1-\lambda\epsilon)}E_1\right)^{\frac{1}{2}}, \\
E_1 &= \frac{m\lambda^2}{4(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\bar{C_1}^2 + 2\lambda k_2\epsilon\theta_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 \\
&+ \frac{9}{8}n\lambda^2 k_4 + (\lambda_1\gamma_3 + b_1\sigma_3) + \frac{3}{4}\left(\frac{2m\lambda^2}{(1-\lambda\epsilon)(\lambda-1)\delta}\right)^{\frac{1}{3}}(2\lambda_2\gamma_2 \\
&+ 2b_2\sigma_2)^{\frac{4}{3}} + m_2 + \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}\lambda^{\frac{4}{3}}\theta_2^{\frac{4}{3}} \\
&+ \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}(\lambda_1\gamma_2 + b_1\sigma_2 + m_1)^{\frac{4}{3}}, \\
\bar{C_1} &= \frac{\lambda k_2}{2\epsilon} + 2k_4 + \lambda_3\gamma_1 + b_3\sigma_1 + 2(\lambda-1)k_3 + \frac{\lambda-1}{\lambda} + \gamma_1\lambda_2 + \sigma_1 b_2 \\
&\quad + 2(1-\lambda\epsilon)(m-1)k_1.
\end{aligned}$$

To prove the theorem we need the following lemma.

Lemma 5. *If $u = e^f$ is a positive solution to (7) and $F := t(|\nabla f|^2 - \lambda(f_t + \bar{f}))$, where $\bar{f} = \hat{A}p + \hat{B}q + \xi$ then for any $\epsilon \in (0, \frac{1}{\lambda})$ and assuming conditions of Theorem 1 we have*

$$
\begin{aligned}
(\Delta_\phi - \partial_t)F \quad \geq \quad & 2t(1-\lambda\epsilon)\frac{(\Delta_\phi f)^2}{m} - \frac{\lambda t k_2}{2\epsilon}|\nabla f|^2 - 2\lambda t k_2 \epsilon |\nabla\phi|^2 \\
- \quad & 2t(1-\lambda\epsilon)(m-1)k_1|\nabla f|^2 - 2\nabla F \nabla f - \frac{F}{t} - 2t(\lambda-1)k_3|\nabla f|^2 \\
- \quad & \frac{n\lambda t}{2\epsilon}(k_2+k_3)^2 - 3\lambda t\sqrt{n}k_4|\nabla f|^2 + \mathcal{H},
\end{aligned} \tag{17}
$$

where $\mathcal{H} = -2t(\lambda-1)\nabla\bar{f}\nabla f - \lambda t \nabla f \nabla\phi_t - \lambda t \Delta_\phi \bar{f}$.

Proof. Let u be a solution of (7) and consider $F := t(|\nabla f|^2 - \lambda(f_t + \bar{f}))$, where $\bar{f} = \hat{A}p + \hat{B}q + \xi$. Hence

$$
\frac{F}{t} = |\nabla f|^2 - \lambda(f_t + \bar{f}) \tag{18}
$$

and applying Lemma 1 (Weighted Bochner formula) we have

$$
\begin{aligned}
\Delta_\phi F \quad = \quad & 2t|\text{Hess } f|^2 + 2t\langle \nabla\Delta_\phi f, \nabla f\rangle + 2tRic_\phi(\nabla f, \nabla f) - \lambda t \Delta_\phi f_t \\
- \quad & \lambda t \Delta_\phi \bar{f}.
\end{aligned} \tag{19}
$$

Now $\Delta_\phi f = -\frac{F}{t} - (\lambda-1)(f_t + \bar{f})$, so $\nabla\Delta_\phi f = -\frac{\nabla F}{t} - (\lambda-1)(\nabla f_t + \nabla\bar{f})$.
Hence

$$
\begin{aligned}
\Delta_\phi F \quad = \quad & 2t|\text{Hess } f|^2 - 2\nabla F\nabla f - 2t(\lambda-1)(\nabla f_t + \nabla\bar{f})\nabla f + 2tRic_\phi(\nabla f, \nabla f) \\
- \quad & \lambda t \Delta_\phi f_t - \lambda t \Delta_\phi \bar{f}.
\end{aligned} \tag{20}
$$

Furthermore,

$$
\partial_t(\Delta_\phi f) = \frac{F}{t^2} - \frac{F_t}{t} - (\lambda-1)(f_{tt} + \bar{f}_t). \tag{21}
$$

Using (21) on (20) we get

$$
\begin{aligned}
\Delta_\phi F \quad = \quad & 2t|\text{Hess } f|^2 - 2\nabla F\nabla f - 2t(\lambda-1)(\nabla f_t + \nabla\bar{f})\nabla f + 2tRic_\phi(\nabla f, \nabla f) \\
- \quad & \frac{\lambda F}{t} + \lambda F_t + \lambda(\lambda-1)t(f_{tt} + \bar{f}_t) - 2\lambda t\langle \mathcal{S}, \text{Hess } f\rangle - 2\lambda t\langle \text{div}\mathcal{S} - \frac{1}{2}\nabla S, \nabla f\rangle \\
+ \quad & 2\lambda t\mathcal{S}(\nabla\phi, \nabla f) - \lambda t\langle \nabla f, \nabla\phi_t\rangle - \lambda t\Delta_\phi \bar{f},
\end{aligned} \tag{22}
$$

and

$$
\partial_t F = \frac{F}{t} + t(\partial_t|\nabla f|^2 - \lambda(f_{tt} + \bar{f}_t)). \tag{23}
$$

From (22) and (23) we get

$$
\begin{aligned}
(\Delta_\phi - \partial_t)F \quad = \quad & 2t|\text{Hess } f|^2 - 2\nabla F\nabla f - 2t(\lambda-1)(\nabla f_t + \nabla\bar{f})\nabla f \\
+ \quad & 2tRic_\phi(\nabla f, \nabla f) - 2t(\lambda-1)\mathcal{S}(\nabla f, \nabla f) \\
+ \quad & 2t(\lambda-1)\nabla f\nabla f_t - 2\lambda t\langle \mathcal{S}, \text{Hess } f\rangle \\
- \quad & 2\lambda t\langle \text{div } \mathcal{S} - \frac{1}{2}\nabla S, \nabla f\rangle + 2\lambda t\mathcal{S}(\nabla\phi, \nabla f) - \lambda t\nabla f\nabla\phi_t \\
- \quad & \lambda t\Delta_\phi \bar{f} - \frac{F}{t}.
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\text{or, } (\Delta_\phi - \partial_t)F \quad = \quad & 2t|\text{Hess } f|^2 + 2tRic_\phi(\nabla f, \nabla f) - 2\nabla F\nabla f - \frac{F}{t} \\
- \quad & 2t(\lambda-1)\mathcal{S}(\nabla f, \nabla f) + 2\lambda t\mathcal{S}(\nabla\phi, \nabla f) - 2\lambda t\langle \mathcal{S}, \text{Hess } f\rangle \\
- \quad & 2\lambda t\langle \text{div } \mathcal{S} - \frac{1}{2}\nabla S, \nabla f\rangle + \mathcal{H},
\end{aligned} \tag{25}
$$

where $\mathcal{H} = -2t(\lambda-1)\nabla\bar{f}\nabla f - \lambda t\nabla f\nabla\phi_t - \lambda t\Delta_\phi\bar{f}$.
Given that

$$-(k_2+k_3)g_{ij} \leq S_{ij} \leq (k_2+k_3)g_{ij}, \tag{26}$$

which implies

$$|S|^2 \leq n(k_2+k_3)^2, \tag{27}$$

as S_{ij} is a symmetric tensor.

Following [14], for any $\epsilon \in (0, \frac{1}{\lambda})$ using Young's inequality, we have

$$\langle \mathcal{S}, \text{Hess } f\rangle \leq \epsilon|\text{Hess } f|^2 + \frac{n}{4\epsilon}(k_2+k_3)^2, \tag{28}$$

$$2\lambda t\mathcal{S}(\nabla\phi, \nabla f) \geq -\frac{\lambda t k_2}{2\epsilon}|\nabla f|^2 - 2\lambda t k_2\epsilon|\nabla\phi|^2. \tag{29}$$

Also

$$|\text{div } S_{ij} - \frac{1}{2}\nabla S| \leq \frac{3}{2}\sqrt{n}k_4. \tag{30}$$

Using Lemma 3, Equations (26)–(30) and bounds of Ric_ϕ^{m-n}, $\mathcal{S}$ in (25) we have (17). □

Proof of Theorem 1. Consider a C^2-function ψ on $[0,\infty)$,

$$\psi(s) = \begin{cases} 1, \ s \in [0,1], \\ 0, \ s \in [2,\infty), \end{cases}$$

and it satisfies $\psi(s) \in [0,1]$, $-c_0 \leq \psi'(s) \leq 0$, $\psi''(s) \geq -c_1$ and $\frac{|\psi''(s)|^2}{\psi(s)} \leq c_1$, where c_1 is a constant and for $R \geq 1$ we defined a function

$$\eta(x,t) = \psi\left(\frac{r(x,t)}{R}\right),$$

where $r(x,t) = d(x, x_0, t)$. Applying the same argument as in [3] we can apply a maximum principle and use Calabi's trick [29] to assume everywhere smoothness of $\eta(x,t)$, as $\psi(s)$ is Lipschitz.

By generalized Laplacian comparison theorem [14], we have

1. $\Delta_\phi r(x) \leq (m-1)\sqrt{k_1}\coth(\sqrt{k_1}r(x))$,
2. $\Delta_\phi\eta \geq -\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) - \frac{c_1}{R^2}$,
3. $\frac{|\nabla\eta|^2}{\eta} \leq \frac{c_1}{R^2}$.

Let $G = \eta F$. Fix any $T_1 \in (0,T]$ and assume G achieves maximum at $(x_0,t_0) \in Q_{2R,T_1}$. If $G(x_0,t_0) \leq 0$ then the result is trivial and hence nothing to be proved, so assume that $G(x_0,t_0) \geq 0$.

Thus, at (x_0,t_0) we have

$$\nabla G = 0, \quad \Delta G \leq 0, \quad \partial_t G \geq 0.$$

Therefore

$$\nabla F = -\frac{F}{\eta}\nabla\eta \tag{31}$$

and

$$0 \geq (\Delta_\phi - \partial_t)G = F(\Delta_\phi - \partial_t)\eta + \eta(\Delta_\phi - \partial_t)F + 2\langle\nabla\eta, \nabla F\rangle. \tag{32}$$

By [16], there is a constant c_2 such that

$$-F\eta_t \geq -c_2k_2F. \tag{33}$$

Using (31) and (33) in (32) we get

$$0 \geq -\left(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{3c_1}{R^2}+c_2k_2\right)F+\eta(\Delta_\phi-\partial_t)F. \tag{34}$$

Following [14,20,23], we set

$$\xi = \frac{|\nabla f|^2}{F}\bigg|_{(x_0,t_0)} \geq 0,$$

then at (x_0, t_0) we have

$$\begin{aligned} |\nabla f| &= \sqrt{\xi F}, &(35)\\ (\xi - \frac{t_0\xi-1}{\lambda t_0})F &= |\nabla f|^2-(f_t+\bar{f}), &(36)\\ \eta\langle\nabla f,\nabla F\rangle &\leq \frac{\sqrt{c_1}}{R}\eta^{\frac{1}{2}}F|\nabla f|, &(37)\\ 3\lambda\sqrt{n}k_4|\nabla f| &\leq 2k_4|\nabla f|^2+\frac{9}{8}n\lambda^2k_4. &(38) \end{aligned}$$

Using Lemma 5 in (34) we have

$$\begin{aligned} 0 \geq & -\left(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{3c_1}{R^2}+c_2k_2\right)F+\frac{2\eta t_0(1-\lambda\epsilon)}{m}(\Delta_\phi f)^2-\frac{\lambda\eta t_0k_2}{2\epsilon}|\nabla f^2| \\ & - 2\lambda t_0\eta k_2\epsilon|\nabla\phi|^2-2\eta t_0(1-\lambda\epsilon)(m-1)k_1|\nabla f|^2-2\eta\nabla F\nabla f-\frac{\eta F}{t_0} \\ & - 2t_0(\lambda-1)\eta k_3|\nabla f|^2-\frac{n\eta\lambda t_0}{2\epsilon}(k_2+k_3)^2-3\eta\lambda t_0\sqrt{n}k_4|\nabla f|+\eta\mathcal{H}. \end{aligned} \tag{39}$$

Multiplying (39) with ηt_0 and using results from (35)–(38) we get

$$\begin{aligned} 0 \geq & -\left(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{3c_1}{R^2}+c_2k_2\right)Gt_0+\frac{2\eta^2t_0^2(1-\lambda\epsilon)}{m}(\Delta_\phi f)^2 \\ & - \frac{\lambda\eta t_0^2k_2\xi}{2\epsilon}G-2\lambda\eta^2t_0^2k_2\epsilon|\nabla\phi|^2-2\eta\xi t_0^2(1-\lambda\epsilon)(m-1)k_1G-2t_0\frac{\sqrt{c_1}}{R}G^{\frac{3}{2}}\xi^{\frac{1}{2}} \\ & - \eta G-2\eta^2t_0^2(\lambda-1)k_3|\nabla f|^2-\frac{n\eta^2t_0^2\lambda}{2\epsilon}(k_2+k_3)^2-2k_4t_0^2\xi G\eta \\ & - \frac{9}{8}n\lambda^2\eta^2k_4t_0^2+\eta^2t_0\mathcal{H}. \end{aligned} \tag{40}$$

Now we use Young's inequality by choosing suitable values for a, b, α, p, q as in Lemma 4. Set $a = \frac{2\sqrt{c_1}}{R}G^{\frac{1}{2}}$, $b = G\xi^{\frac{1}{2}}$, $p = 2$, $q = 2$, $\alpha = \frac{m\lambda^2}{4(1-\lambda\epsilon)(\lambda-1)}$ and apply Lemma 4 (Young's inequality) we get

$$2t_0\frac{\sqrt{c_1}}{R}G^{\frac{3}{2}}\xi^{\frac{1}{2}} \leq \frac{4(1-\lambda\epsilon)}{m\lambda^2}(\lambda-1)\xi G^2t_0+\frac{m\lambda^2c_1t_0G}{4(1-\lambda\epsilon)(\lambda-1)R^2}. \tag{41}$$

Cauchy–Schwarz inequality gives

$$\begin{aligned} \eta^2\lambda\langle\nabla f,\nabla\phi_t\rangle &\leq \eta^2\lambda|\nabla f||\nabla\phi_t| \\ &\leq \lambda\theta_2G^{\frac{1}{2}}\xi^{\frac{1}{2}}. \end{aligned}$$

Set $a = \lambda\theta_2$, $b = \xi^{\frac{1}{2}} G^{\frac{1}{2}}$, $p = \frac{4}{3}$, $q = 4$, $\alpha = \left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{4}}$ and apply Lemma 4 we get

$$\eta^2\lambda\langle\nabla f, \nabla\phi_t\rangle \leq \frac{(1-\lambda\epsilon)(1-\delta)}{2m\lambda^2}(\lambda-1)^2\xi^2G^2 + \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\epsilon\lambda)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}\lambda^{\frac{4}{3}}\theta_2^{\frac{4}{3}}, \text{ for all } \delta \in (0,1). \tag{42}$$

We have $\bar{f} = \hat{A}p + \hat{B}q + \xi$. Hence

$$\nabla\bar{f} = p\hat{A}_f\nabla f + q\hat{B}_f\nabla f + \hat{A}\nabla p + \hat{B}\nabla q + \nabla\xi,$$

$$\begin{aligned}\Delta\bar{f} &= \nabla\nabla\bar{f} \\ &= (\hat{A}\Delta p + \hat{B}\Delta q) + p(\hat{A}_{ff}|\nabla f|^2 + \hat{A}_f\Delta f) + q(\hat{B}_{ff}|\nabla f|^2 + \hat{B}_f\Delta f) \\ &+ 2(\hat{A}_f\langle\nabla f, \nabla p\rangle + \hat{B}_f\langle\nabla f, \nabla q\rangle) + \Delta\xi.\end{aligned}$$

Hence

$$\begin{aligned}\Delta_\phi\bar{f} &= \eta^2(\Delta\bar{f} - \nabla\phi\nabla\bar{f}) \\ &= (\hat{A}\Delta_\phi p + \hat{B}\Delta_\phi q) + (p\hat{A}_f + q\hat{B}_f)\Delta_\phi f + (p\hat{A}_{ff} + q\hat{B}_{ff})|\nabla f|^2 \\ &+ 2(\hat{A}_f\langle\nabla p, \nabla f\rangle + \hat{B}_f\langle\nabla q, \nabla f\rangle) + \Delta_\phi\xi.\end{aligned} \tag{43}$$

Again

$$\begin{aligned}2\eta^2\hat{A}_f\langle\nabla p, \nabla f\rangle &\leq 2\lambda_2\eta^2|\nabla p||\nabla f|, \text{ using Cauchy-Schwarz inequality} \\ &\leq 2\lambda_2\gamma_2\eta^2|\nabla f| \\ &\leq 2\lambda_2\gamma_2\xi^{\frac{1}{2}}G^{\frac{1}{2}}.\end{aligned} \tag{44}$$

Similarly

$$2\eta^2\hat{B}_f\langle\nabla q, \nabla f\rangle \leq 2b_2\sigma_2\xi^{\frac{1}{2}}G^{\frac{1}{2}}. \tag{45}$$

Adding (45) and (45) gives

$$2\eta^2(\hat{A}_f\langle\nabla p, \nabla f\rangle + \hat{B}_f\langle\nabla q, \nabla f\rangle) \leq 2(\gamma_2\lambda_2 + b_2\sigma_2)\xi^{\frac{1}{2}}G^{\frac{1}{2}}. \tag{46}$$

Using (46) in (43) and applying Young's inequality with $a = 2(\gamma_2\lambda_2 + b_2\sigma_2)$, $b = \xi^{\frac{1}{2}}G^{\frac{1}{2}}$, $p = \frac{4}{3}$, $q = 4$ and $\alpha = \left(\frac{2m\lambda^2}{(1-\lambda\epsilon)\delta(\lambda-1)}\right)^{\frac{1}{4}}$ we obtain

$$\begin{aligned}\eta^2\Delta_\phi\bar{f} &\leq (\lambda_1\gamma_3 + b_1\sigma_3) + (\gamma_1\lambda_2 + \sigma_1 b_2)\eta^2\Delta_\phi f \\ &+ (\lambda_3\gamma_1 + b_3\sigma_1)\xi G + \frac{2(1-\lambda\epsilon)\delta}{m\lambda^2}(\lambda-1)^2\xi^2G^2 \\ &+ \frac{3}{4}\left(\frac{2m\lambda^2}{(1-\lambda\epsilon)\delta(\lambda-1)}\right)^{\frac{1}{3}}(2\lambda_2\gamma_2 + 2b_2\sigma_2)^{\frac{4}{3}} + m_2.\end{aligned} \tag{47}$$

Similarly we get

$$\eta^2\langle\nabla\bar{f},\nabla f\rangle \leq \frac{(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}{2m\lambda^2}\xi^2G^2 + \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}(\lambda_1\gamma_2+b_1\sigma_2+m_1)^{\frac{4}{3}} + (\gamma_1\lambda_2+\sigma_1 b_2)\xi G. \tag{48}$$

Equations (47) and (48) are the quantities that estimates $\mathcal{H}$.
From (36) we have

$$\begin{aligned}\Delta_\phi f &= |\nabla f|^2-(f_t+\bar{f})\\ &= \left(\xi-\frac{t_0\xi-1}{\lambda t_0}\right)F.\end{aligned}$$

Thus

$$\frac{2\eta^2t_0^2(1-\lambda\epsilon)}{m}(\Delta_\phi f)^2 = \frac{2(1-\lambda\epsilon)}{m\lambda^2}G^2-\frac{4\xi t_0(1-\lambda\epsilon)(\lambda-1)}{m\lambda^2}G^2 + \frac{2(1-\lambda\epsilon)}{m\lambda^2}\xi^2t_0^2(\lambda-1)^2G^2 \tag{49}$$

and

$$\eta^2\Delta_\phi f = -\frac{1}{\lambda t_0}G-\frac{t_0(\lambda-1)}{\lambda t_0}\xi G. \tag{50}$$

Set

$$\bar{C_1} := \left\{\frac{\lambda k_2}{2\epsilon}+2k_4+\lambda_3\gamma_1+b_3\sigma_1+2(\lambda-1)k_3+\frac{\lambda-1}{\lambda}+\gamma_1\lambda_2+\sigma_1 b_2+2(1-\lambda\epsilon)(m-1)k_1\right\}$$

and apply Peter-Paul inequality with $a=\xi G$, $b=\bar{C_1}$, $\alpha=\frac{m\lambda^2}{(1-\epsilon\lambda)(1-\delta)(\lambda-1)^2}$ we get

$$\bar{C_1}\xi G \leq \frac{(1-\epsilon\lambda)(1-\delta)(\lambda-1)^2}{m\lambda^2}\xi^2G^2+\frac{m\lambda^2}{4(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\bar{C_1}^2. \tag{51}$$

Set

$$D_1 := 1+t_0\left(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{3c_1}{R^2}+c_2k_2+\frac{m\lambda^2c_1}{4(1-\lambda\epsilon)(\lambda-1)R^2}+\frac{1}{\lambda}\right), \tag{52}$$

$$\begin{aligned}E_1 := {}& \frac{m\lambda^2}{4(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\bar{C_1}^2+2\lambda k_2\epsilon\theta_1^2\\ &+ \frac{n\lambda}{2\epsilon}(k_2+k_3)^2+\frac{9}{8}n\lambda^2k_4+(\lambda_1\gamma_3+b_1\sigma_3)+\frac{3}{4}\left(\frac{2m\lambda^2}{(1-\lambda\epsilon)(\lambda-1)\delta}\right)^{\frac{1}{3}}(2\lambda_2\gamma_2\\ &+ 2b_2\sigma_2)^{\frac{4}{3}}+m_2+\frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}\lambda^{\frac{4}{3}}\theta_2^{\frac{4}{3}}\\ &+ \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}(\lambda_1\gamma_2+b_1\sigma_2+m_1)^{\frac{4}{3}}.\end{aligned} \tag{53}$$

Using (41) to (52) in (40) we obtain

$$0 \geq \frac{2(1-\lambda\epsilon)}{m\lambda^2}G^2 - D_1 G - t_0^2 E_1. \tag{54}$$

For a positive number p and two non-negative numbers q, r, the quadratic inequality of the form $px^2 - qx - r \leq 0$ implies that $x \leq \frac{q}{p} + \sqrt{\frac{r}{p}}$.

So at (x_0, t_0) we have

$$G \leq D_1 \frac{m\lambda^2}{2(1-\lambda\epsilon)} + t_0 \sqrt{\frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)}}. \tag{55}$$

Since $\eta(x,t) = 1$ whenever $d(x, x_0, T_1) \leq R$, hence

$$\frac{F(x,T_1)}{T_1} = (|\nabla f|^2 - \lambda(f_t + \bar{f}))\Big|_{(x,T_1)} \leq \frac{G(x_0,t_0)}{T_1} \leq \frac{1}{T_1}\left(D_1 \frac{m\lambda^2}{2(1-\lambda\epsilon)} + t_0 \sqrt{\frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)}}\right).$$

Since $t_0 \leq T_1$, so

$$\frac{1}{T_1}\left(D_1 \frac{m\lambda^2}{2(1-\lambda\epsilon)} + t_0 \sqrt{\frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)}}\right) \leq \frac{m\lambda^2}{2T_1(1-\lambda\epsilon)} + \frac{\tilde{D}}{T_1}\frac{m\lambda^2}{2(1-\lambda\epsilon)} + \sqrt{\frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)}},$$

where $\tilde{D} = t_0 \tilde{D_1}$ and $\tilde{D_1} = \left(\frac{c_0}{R}(m-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2 k_2\right) + \frac{m\lambda^2 c_1}{4(1-\lambda\epsilon)(\lambda-1)R^2} + \frac{1}{\lambda}$ satisfying $\frac{\tilde{D}}{T_1} \leq \tilde{D_1}$. Since T_1 is arbitrary so

$$|\nabla f|^2 - \lambda(f_t + \hat{A}p + \hat{B}q + \xi) \leq \frac{m\lambda^2}{2t(1-\lambda\epsilon)} + \frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_1} + \tilde{E_1}, \tag{56}$$

where $\tilde{E_1} = \left(\frac{m\lambda^2}{2(1-\lambda\epsilon)}E_1\right)^{\frac{1}{2}}$.

Substituting $f = \log u$ on (56) and using the definition of $\hat{A}, \hat{B}$, we get (16). This completes the proof. □

Corollary 1. *If k_1, k_2, k_3, k_4 are positive constants such that*

$$Ric_\phi^{m-n} \geq -(m-1)k_1 g, \; -k_2 g \leq \mathcal{S} \leq k_3 g, \; |\nabla S| \leq k_4$$

on M, then for any $\lambda > 1$ and $\delta \in (0,1)$ we have

$$\frac{|\nabla u|^2}{u^2} - \lambda\left(\frac{u_t}{u} + \frac{A(u)}{u}p + \frac{B(u)}{u}q + \xi\right) \leq \frac{m\lambda^2}{2t(1-\lambda\epsilon)} + \frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_2} + \tilde{E_2}, \tag{57}$$

where

$$
\begin{aligned}
\tilde{D_2} &= c_2k_2 + \frac{1}{\lambda}, \\
\tilde{E_2} &= \left(\frac{m\lambda^2}{2(1-\lambda\epsilon)}E_2\right)^{\frac{1}{2}}, \\
E_2 &= \frac{m\lambda^2}{4(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\bar{C_2}^2 + 2\lambda k_2\epsilon\Theta_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 \\
&+ \frac{9}{8}n\lambda^2k_4 + (\Lambda_1\Gamma_3 + B_1\Sigma_3) + \frac{3}{4}\left(\frac{2m\lambda^2}{(1-\lambda\epsilon)(\lambda-1)\delta}\right)^{\frac{1}{3}}(2\Lambda_2\Gamma_2 + 2B_2\Sigma_2)^{\frac{4}{3}} \\
&+ M_2 + \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}\lambda^{\frac{4}{3}}\Theta_2^{\frac{4}{3}} \\
&+ \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}(\Lambda_1\Gamma_2 + B_1\Sigma_2 + M_1)^{\frac{4}{3}}, \\
\bar{C_2} &= \left(\frac{\lambda k_2}{2\epsilon} + 2k_4 + \Lambda_3\Gamma_1 + B_3\Sigma_1 + 2(\lambda-1)k_3 + \frac{\lambda-1}{\lambda} + 2(1-\lambda\epsilon)(m-1)k_1\right).
\end{aligned}
$$

Proof. We know $g(t)$ is uniformly equivalent to the initial metric $g(0)$. For a fixed $\delta \in (0,1)$ if we let R tend to $+\infty$ then we obtain our result. □

Theorem 2. *If k_1, k_2, k_3, k_4 are positive constants such that*

$$Ric_\phi^{m-n} \geq -(m-1)k_1 g,\ -k_2 g \leq \mathcal{S} \leq k_3 g,\ |\nabla S| \leq k_4$$

on M and let u be a positive solution to (7) under the flow (8) then we have the Harnack inequality

$$u(y_1,s_1) \leq u(y_2,s_2)\left(\frac{s_2}{s_1}\right)^{\frac{m\lambda}{2(1-\lambda\epsilon)}} \exp\left\{\frac{\lambda}{4}\mathcal{I}(s_1,s_2) + (s_2-s_1)(\Lambda_1\Gamma_1 + B_1\Sigma_1 + M_3 + \frac{1}{\lambda}\tilde{F_2})\right\}, \tag{58}$$

where $\mathcal{I}(s_1,s_2) = \inf_{\zeta}\int_{s_1}^{s_2}|\zeta'(t)|^2dt$ and $\zeta : [s_1,s_2] \to M$ is a path joining the points (y_1,s_1), (y_2,s_2) in $M \times [0,T]$ and $\tilde{F_2} = \frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_2} + \tilde{E_2}$.

Proof. Set $\tilde{F_2} = \frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_2} + \tilde{E_2}$ then (57) becomes

$$\frac{|\nabla u|^2}{u^2} - \lambda\left(\frac{u_t}{u} + \frac{A(u)}{u}p + \frac{B(u)}{u}q + \xi\right) \leq \frac{m\lambda^2}{2t(1-\lambda\epsilon)} + \tilde{F_2}. \tag{59}$$

For $u = e^f$ we have

$$|\nabla f|^2 - \lambda\left(f_t + \hat{A}p + \hat{B}q + \xi\right) \leq \frac{m\lambda^2}{2t(1-\lambda\epsilon)} + \tilde{F_2}. \tag{60}$$

Let $(y_1,s_1),(y_2,s_2) \in M \times [0,T]$ be such that $s_1 < s_2$. Take a geodesic path $\zeta : [s_1,s_2] \to M$ satisfying $\zeta(s_1) = y_1,\ \zeta(s_2) = y_2$. Using (60) we obtain

$$\begin{aligned} f(y_1,s_1)-f(y_2,s_2) &= -\int_{s_1}^{s_2}\frac{d}{dt}f(\zeta(t),t)dt \\ &= -\int_{s_1}^{s_2}\partial_t f dt-\int_{s_1}^{s_2}\langle\nabla f,\zeta'(t)\rangle dt \\ &\leq \frac{m\lambda}{2(1-\lambda\epsilon)}\ln(\frac{s_2}{s_1})+(s_2-s_1)(\Lambda_1\Gamma_1+B_1\Sigma_1+M_3+\frac{1}{\lambda}\tilde{F_2}) \\ &- \int_{s_1}^{s_2}\frac{1}{\lambda}|\nabla f|^2 dt-\int_{s_1}^{s_2}\langle\nabla f,\zeta'(t)\rangle dt. \end{aligned} \tag{61}$$

Now using the relation $-ax^2-bx\leq\frac{b^2}{4a}$, we set $x=\nabla f$, $a=\frac{1}{\lambda}$ and $b=\zeta'(t)$ we get

$$\begin{aligned} f(y_1,s_1)-f(y_2,s_2) &\leq \frac{m\lambda}{2(1-\lambda\epsilon)}\ln(\frac{s_2}{s_1})-\int_{s_1}^{s_2}\frac{\lambda|\zeta'(t)|^2}{4}dt \\ &+ (s_2-s_1)(\Lambda_1\Gamma_1+B_1\Sigma_1+M_3+\frac{1}{\lambda}\tilde{F_2}). \end{aligned} \tag{62}$$

Take infimum of (62) over all possible curves ζ on M and put $f=\ln u$ to obtain (58). □

4. Conclusions

In this paper, we have established Li–Yau-type estimate for a positive solution of the equation

$$\Delta_\phi u=\frac{\partial u}{\partial t}+A(u)p(x,t)+B(u)q(x,t)+\xi(x,t)u(x,t),$$

along the flow $\partial_t g_{ij}=2S_{ij}$ and related Harnack type inequality. In particular if $\xi(x,t)=0$, $B(u)=u^{a+1}$ then the results are same as in Section 2 of [14]. Thus, our paper generalizes some results of [14].

Further $A(u)=B(u)=\xi(u)=0$ gives the classical Li–Yau-type estimate for positive solution of the weighted heat equation

$$\Delta_\phi u=\partial_t u \tag{63}$$

under the geometric flow $\partial_t g_{ij}=2S_{ij}$. To obtain this estimate we put

1. $A(u)=B(u)=\xi(x,t)=0$
2. $\lambda_1=\lambda_2=\lambda_3=0$
3. $b_1=b_2=b_3=0$
4. $p(x,t)=q(x,t)=0$

in (16) and get

$$\frac{|\nabla u|^2}{u^2}-\lambda\frac{u_t}{u} \leq \frac{m\lambda^2}{2t(1-\lambda\epsilon)}+\frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_3}+\tilde{E_3}, \tag{64}$$

where

$$\begin{aligned} \tilde{D_3} &= \frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{3c_1}{R^2}+c_2k_2+\frac{m\lambda^2 c_1}{4(a-\lambda\epsilon)(\lambda-1)R^2}+\frac{1}{\lambda}, \\ \tilde{E_3} &= \left(\frac{m\lambda^2}{2(1-\lambda\epsilon)}E_3\right)^{\frac{1}{2}}, \end{aligned}$$

$$E_3 = \frac{m\lambda^2}{4(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\bar{C_3}^2 + 2\lambda k_2\epsilon\theta_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 + \frac{9}{8}n\lambda^2 k_4 + \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}\lambda^{\frac{4}{3}}\theta_2^{\frac{4}{3}},$$

$$\bar{C_3} = \frac{\lambda k_2}{2\epsilon} + 2k_4 + 2(\lambda-1)k_3 + \frac{\lambda-1}{\lambda} + 2(1-\lambda\epsilon)(m-1)k_1.$$

Here if we let $R \to +\infty$ then we get the classical Li–Yau-type global gradient estimate for (63) along the flow $\partial_t g_{ij} = 2S_{ij}$. The key ingredient in this estimation is the assumption of bounds for the weight function ϕ and its derivative $|\nabla\phi|$ (see Preliminaries section), it would be interesting if one can derive Li–Yau-type estimation for a positive solution u of (7) without assuming bounds for ϕ, $|\nabla\phi|$. One can consider this problem as a future work for this article.

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mathematics

Article

The λ-Point Map between Two Legendre Plane Curves

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Abstract: The λ-point map between two Legendre plane curves, which is a map from the plane into the plane, is introduced. The singularity of this map is studied through this paper and many known plane map singularities are realized as special cases of this construction. Precisely, the corank one and corank two singularities of the λ-point map between two Legendre plane curves are investigated and the geometric conditions for this map to have corank one singularities, such as fold, cusp, swallowtail, lips, and beaks are obtained. Additionally, the geometric conditions for the λ-point map to have a sharksfin singularity, which is a corank two singularity, are obtained.

Keywords: Legendre curve; singularity; cusp; fold; swallowtail; lips; beaks; sharksfin

MSC: 53A04; 57R45; 58K05

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1. Introduction

The singularity theory is useful for studying the differential geometry of curves and surfaces and lots of geometric features can be studied from the singularity theory viewpoint (cf. [1–3]). One of the main subjects in the singularity theory of smooth maps is the classifications of the singularities of maps germs from the plane into the plane. This is because of its applications in several areas. For the applications of plane maps we refer the reader to [1–4]. In 1955, Whitney proved that, in general, maps from the plane into the plane have fold and cusp singularities. The classification of maps germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with a corank one singularities was studied by J.H. Rieger in [5]. Some of these singularities are shown in Table 1. In 2010, K. Saji obtained the criteria for lips, beaks, and swallowtail singularities of smooth maps germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with a corank one singularities.

Table 1. Classification of $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

Name	Normal Form
fold	(x, y^2)
cusp	$(x, xy + y^3)$
lips	$(x, x^2y + y^3)$
beaks	$(x, x^2y - y^3)$
swallowtail	$(x, xy + y^4)$

The criteria for sharksfin and deltoid singularities, which are corank two singularities, of maps germ from the plane into the plane was investigated by Kabata and Saji [6]. In this paper, we introduce the λ-point map between two Legendre plane curves (Definition 6). Additionally, we study the classification of corank one (respect corank two) singularities of this map. In the beginning, we review some basic definitions and results through the second section which will be used in this paper. In the third section, we give the geometric conditions for the λ-point map between two Legendre plane curves to have fold, cusp, lips, beaks and swallowtail singularities when γ_1 (respect γ_2) is regular at $s_1 = 0$ (respect $s_2 = 0$) (Theorem 2). Additionally, we give the geometric conditions for the λ-point map between two Legendre plane curves to have fold and beaks singularities when one of the two curves is singular (Theorem 3). In the forth section, we give the geometric conditions for the λ-point map between two Legendre plane curves with a corank two singularity to have sharksfin singularity (Theorem 6). In the final section, we give three examples to illustrate some obtained results in this research. Precisely, for corank one singularity we give two examples for the λ-point map between two Legendre plane curves to have fold and beaks singularities and the third example deals with the sharksfin, which is a corank two, singularity of this map.

Throughout this paper, the definitions and results are provided for smooth maps.

2. Preliminaries

In this section, we review some definitions and results for Legendre plane curves and the singularity of maps from the plane into the plane. Additionally, we introduce the λ-point map between two Legendre plane curves.

Definition 1. *Let I be an interval of $\mathbb{R}$. The map $(\gamma, \omega) : I \to \mathbb{R}^2 \times S^1$ is called a Legendre curve if $\gamma'(s) \cdot \omega(s) = 0$ for all $s \in I$, where S^1 is the unit circle and $\omega : I \to S^1$ is a smooth unit vector field.*

The Frenet formula of a Legendre plane curve is given by

$$\omega'(s) = \ell(s)\mu(s),$$

$$\mu'(s) = -\ell(s)\omega(s),$$

where prime is the derivative with respect to the parameter s, $\ell(s) = \omega'(s) \cdot \mu(s)$ and $\mu(s) = J(\omega(s))$, such that J is the counterclockwise rotation by $\dfrac{\pi}{2}$. We call the pair $\{\omega(s), \mu(s)\}$ a moving frame of a Legendre plane curve γ. Furthermore, there exists a smooth function $\beta(s)$, such that $\beta(s) = \gamma'(s) \cdot \mu(s)$. We call the pair $(\ell(s), \beta(s))$ the curvature of this curve. For more information about the Legendre plane curves, we refer the reader to [7–12].

Definition 2. *A singular point of a map germ $h : (U \subseteq \mathbb{R}^l, 0) \to (\mathbb{R}^m, 0)$ is a point $p \in U$ which satisfies that $rank(dh)(p) < min(l, m)$, where dh is the Jacobin matrix of h.*

The set of singular points of h is denoted by $S(h) \subset \mathbb{R}^l$. We say that $q \in S(h)$ is of corank α if the rank of the Jacobin matrix of h at q is equal to $min(l, m) - \alpha$.

Definition 3. *Two map germs $h_1, h_2 : (\mathbb{R}^l, 0) \to (\mathbb{R}^m, 0)$ are said to be $\mathcal{A}$-equivalent if there exist smooth diffeomorphisms $\varphi_1 : (\mathbb{R}^l, 0) \to (\mathbb{R}^l, 0)$ and $\varphi_2 : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$, such that the following diagram commutes.*

$$\begin{array}{ccc} (\mathbb{R}^l, 0) & \xrightarrow{h_1} & (\mathbb{R}^m, 0) \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ (\mathbb{R}^l, 0) & \xrightarrow{h_2} & (\mathbb{R}^m, 0) \end{array}$$

In other words, $h_2 \circ \varphi_1 = \varphi_2 \circ h_1$ holds.

Definition 4 ([13])**.** *For a positive integer n, the n-jet of a differentiable map $\mathcal{F}$ at point p is the Taylor expansion at p truncated to the degree n which is denoted by $j^n\mathcal{F}$.*

Definition 5 ([14])**.** *A map germ $h : (U \subseteq \mathbb{R}^l, q) \to (\mathbb{R}^m, 0)$ is said to be n-determined whenever $j^n h(q) = j^n k(q)$ for any $k : (U \subseteq \mathbb{R}^l, q) \to (\mathbb{R}^m, 0)$, then k is $\mathcal{A}$-equivalent to h.*

For example, the lips and beaks are three-determined, whereas swallowtail is four-determined (see [5]). Let $h : (U \subseteq \mathbb{R}^2, q) \to (\mathbb{R}^2, 0)$ be a map germ with a corank one singularity at a point $q \in U$. Then there exist a neighborhood C of q and a non-zero vector field (null vector field) ρ, such that $dh(\rho)(q) = 0$ holds for any $q \in S(h) \cap C$. Let (s_1, s_2) be coordinates of U. We define the discriminant function Ω of h by $\Omega(s_1, s_2) = det\left(\frac{\partial h}{\partial s_1}, \frac{\partial h}{\partial s_2}\right)(s_1, s_2)$. A singular point $q \in S(h)$ is a non-degenerate if $d\Omega(q) \neq 0$ and it is a degenerate if $d\Omega(q) = 0$. Note that a non-degenerate singular point is of corank one. The normal forms of some simple generic singularities of corank one of maps from the plane into the plane are shown in Table 1.

We end this section by introducing the λ-point map between two Legendre plane curves.

Definition 6. *Let $\gamma_i : I_i \subseteq \mathbb{R} \to \mathbb{R}^2 \times \mathbb{S}^1$ $(i = 1, 2)$ be two Legendre plane curves. The λ-point map between γ_1 and γ_2 is a map $M : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$M(s_1, s_2) = (1 - \lambda)\gamma_1(s_1) + \lambda\gamma_2(s_2),$$

where $U = I_1 \times I_2$ and $\lambda \in (0, 1)$.

Note that M in the above definition is more general than the midpoint map of a smooth plane curve γ which is defined by $M(s_1, s_2) = \frac{1}{2}(\gamma_1(s_1) + \gamma_2(s_2))$, where γ_1 and γ_2 are two smooth parts of γ parametrized by s_1 and s_2, respectively. For more details on the midpoint map, we refer the reader to [15].

3. Classification of Corank One Singularities of λ-Point Map between Two Legendre Plane Curves

The classification of corank one singularities of the λ-point map between two Legendre curves in plane breaks naturally into two cases depending on the regularity of γ_1 and γ_2.

Lemma 1. *Let M be the λ-point map between two Legendre plane curves γ_1 and γ_2. Then M is parametrized by a corank one singularity at (s_{1_0}, s_{2_0}) if, and only if, one of the following cases holds:*

1. $\beta_1(s_{1_0}) \neq 0$, $\beta_2(s_{2_0}) \neq 0$ *and* $\omega_1(s_{1_0}) = \pm\omega_2(s_{2_0})$.
2. $\beta_1(s_{1_0}) \neq 0$ *or* $\beta_2(s_{2_0}) \neq 0$.

First, we will review the criteria for the fold, the cusp, the beaks, the lips and the swallowtail singularities which are the generic singularities of corank one of maps from the plane into the plane.

Theorem 1 ([14])**.** *Let $h : (U \subseteq \mathbb{R}^2, q) \to (\mathbb{R}^2, 0)$ be a map germ and $q \in S(h)$. Then at q*

1. *h is $\mathcal{A}$-equivalent to fold if, and only if, $\rho\Omega(q) \neq 0$.*
2. *h is $\mathcal{A}$-equivalent to cusp if, and only if, q is non-degenerate, $\rho\Omega(q) = 0$ and $\rho^2\Omega(q) \neq 0$.*
3. *h is $\mathcal{A}$-equivalent to lips if, and only if, q is of corank one, $d\Omega(q) = 0$ and Ω has a Morse type critical point of index 0 or 2 at q, namely $det(\mathcal{H}_\Omega(q)) > 0$.*
4. *h is $\mathcal{A}$-equivalent to beaks if, and only if, q is of corank one, $d\Omega(q) = 0$ and Ω has a Morse type critical point of index 1 at q, namely $det(\mathcal{H}_\Omega(q)) < 0$ and $\rho^2\Omega(q) \neq 0$.*
5. *h is $\mathcal{A}$-equivalent to swallowtail if, and only if, $d\Omega(q) \neq 0$, $\rho\Omega(q) = \rho^2\Omega(q) = 0$ and $\rho^3\Omega(q) \neq 0$.*

The expression $\rho\Omega$ means the directional derivative of Ω in the direction of the vector field ρ and $\mathcal{H}_\Omega$ is the Hessian matrix of Ω.

3.1. The Case When $\beta_1(0) \neq 0$, $\beta_2(0) \neq 0$ and $\omega_1(0) = \pm\omega_2(0)$

In this section, we study the corank one singularity of the λ-point map between two Legendre plane curves when γ_1 (respect γ_2) is regular at $s_1 = 0$ (respect $s_2 = 0$), that means $\beta_1(0) \neq 0$, $\beta_2(0) \neq 0$, and $\omega_1(0) = \pm\omega_2(0)$.

Lemma 2. *Let M be the λ-point map between two Legendre plane curves γ_1 and γ_2, such that $\beta_1(0) \neq 0$, $\beta_2(0) \neq 0$ and $\omega_1(0) = \pm\omega_2(0)$. The singular point $(0,0)$ is a non-degenerate if, and only if, $\ell_i(0) \neq 0$, $i = 1,2$.*

Proof. The proof of this lemma is obvious. □

We now give the main result of this section.

Theorem 2. *Let M be the λ-point map between two Legendre plane curves γ_1 and γ_2. Suppose that $\beta_1(0) \neq 0$, $\beta_2(0) \neq 0$ and $\omega_1(0) = \pm\omega_2(0)$. Then at $(0,0)$*

1. *M is $\mathcal{A}$-equivalent to fold if, and only if, $\left(\dfrac{\lambda}{1-\lambda}\right)\dfrac{\beta_2(0)}{\beta_1(0)}\ell_1(0) \neq \mp\ell_2(0)$.*
2. *M is $\mathcal{A}$-equivalent to cusp if, and only if, $\left(\dfrac{\lambda}{1-\lambda}\right)\dfrac{\beta_2(0)}{\beta_1(0)}\ell_1(0) = \mp\ell_2(0)$ and $\beta_2\left(\dfrac{\beta_1}{\ell_1}\right)' \neq \beta_1\left(\dfrac{\beta_2}{\ell_2}\right)'$ at $(0,0)$.*
3. *M is $\mathcal{A}$-equivalent to lips if, and only if, $\ell_i(0) = 0$, $i = 1,2$, and $\ell_1'(0)\ell_2'(0) < 0$.*
4. *M is $\mathcal{A}$-equivalent to beaks if, and only if, $\ell_i(0) = 0$, $i = 1,2$, $\ell_1'(0)\ell_2'(0) > 0$ and $\left(\dfrac{\lambda}{1-\lambda}\right)^2\dfrac{\beta_2^2(0)}{\beta_1^2(0)}\ell_1'(0) \neq \ell_2'(0)$.*
5. *M is $\mathcal{A}$-equivalent to swallowtail if, and only if, $\left(\dfrac{\lambda}{1-\lambda}\right)\dfrac{\beta_2(0)}{\beta_1(0)}\ell_1(0) = \mp\ell_2(0)$, $\ell_i(0) \neq 0$ $(i = 1,2)$, $\beta_2\left(\dfrac{\beta_1}{\ell_1}\right)' = \beta_1\left(\dfrac{\beta_2}{\ell_2}\right)'$ at $(0,0)$ and*
$$-\frac{\beta_1''\beta_2\ell_2^3}{\ell_1^2} - 3\frac{\beta_1'\beta_2'\ell_2^2}{\ell_1} + \beta_1\beta_2''\ell_2 - 3\frac{\beta_1'\beta_2\ell_2\ell_2'}{\ell_1} + \frac{\beta_1\beta_2\ell_1''\ell_2^3}{\ell_1^3} + 6\frac{\beta_1\beta_2'\ell_1'\ell_2^2}{\ell_1^2}$$
$$-3\beta_1\beta_2'\ell_2' - \beta_1\beta_2\ell_2'' + 3\frac{\beta_1(\beta_2')^2\ell_2}{\beta_2} \neq 0 \text{ at } (0,0).$$

Proof. Let $M(s_1,s_2) = (1-\lambda)\gamma(s_1) + \gamma(s_2)$ be the λ-point map between two Legendre plane curves. Suppose that $\beta_1(0) \neq 0$, $\beta_2(0) \neq 0$ and $\omega_1(0) = -\omega_2(0)$.

We choose vector field ρ, such that $dM|_{(0,0)}(\rho) = 0$, thus we take $\rho = \dfrac{\lambda\beta_2(0)}{(1-\lambda)\beta_1(0)}\dfrac{\partial}{\partial s_1} + \dfrac{\partial}{\partial s_2}$. We can prove that $\Omega(s_1,s_2) = -\beta_1(s_1)\beta_2(s_2)\omega_1(s_1)\cdot\mu_2(s_2)$.

For simplicity we omit s_1 and s_2, hence $\Omega = -\beta_1\beta_2\omega_1\cdot\mu_2$. By a straightforward calculations at $(0,0)$, we have

$$\frac{\partial\Omega}{\partial s_1}|_{(0,0)} = \beta_1(0)\beta_2(0)\ell_1(0), \frac{\partial\Omega}{\partial s_2}|_{(0.0)} = -\beta_1(0)\beta_2(0)\ell_2(0),$$

$$\rho\Omega|_{(0,0)} = \beta_2(0)\left(\frac{\lambda}{1-\lambda}\beta_2(0)\ell_1(0) - \beta_1(0)\ell_2(0)\right),$$

$$\rho^2\Omega|_{(0,0)} = \frac{-2\lambda}{1-\lambda}\frac{\beta_1'(0)\beta_2^2(0)\ell_2(0)}{\beta_1(0)} + \frac{\lambda^2}{(1-\lambda)^2}\frac{\beta_2^3(0)\ell_1'(0)}{\beta_1(0)} + \frac{3\lambda}{1-\lambda}\beta_2(0)\beta_2'(0)\ell_1(0)$$
$$+ \frac{\lambda^2}{(1-\lambda)^2}\frac{\beta_1'(0)\beta_2^3(0)\ell_1(0)}{\beta_1^2(0)} - 2\beta_1(0)\beta_2'(0)\ell_2(0) - \beta_1(0)\beta_2(0)\ell_2'(0),$$

$$
\begin{aligned}
\rho^3\Omega\big|_{(0,0)} =& \frac{2\lambda^3}{(1-\lambda)^3}\frac{\beta_1''(0)\beta_2^4(0)\ell_1(0)}{\beta_1^3(0)} + \frac{-3\lambda^3}{(1-\lambda)^3}\frac{(\beta_1')^2(0)\beta_2^4(0)\ell_1(0)}{\beta_1^4(0)} + \frac{6\lambda^2}{(1-\lambda)^2}\frac{\beta_1'(0)\beta_2'(0)\beta_2^2(0)\ell_1(0)}{\beta_1^2(0)} \\
&+ \frac{3\lambda^2}{(1-\lambda)^2}\frac{\beta_2^3(0)\ell_1^2(0)\ell_2(0)}{\beta_1(0)} - \frac{\lambda^3}{(1-\lambda)^3}\frac{\beta_2^4(0)\ell_1^3(0)}{\beta_1^2(0)} - \frac{3\lambda}{1-\lambda}\beta_2^2(0)\ell_1(0)\ell_2^2(0) \\
&+ \frac{4\lambda}{1-\lambda}\beta_2(0)\beta_2''(0)\ell_1(0) - \frac{3\lambda^2}{(1-\lambda)^2}\frac{\beta_1''(0)\beta_2^3(0)\ell_2(0)}{\beta_1^2(0)} + \frac{3\lambda^2}{(1-\lambda)^2}\frac{(\beta_1')^2(0)\beta_2^3(0)\ell_2(0)}{\beta_1^3(0)} \\
&- \frac{3\lambda}{1-\lambda}\frac{\beta_1'(0)\beta_2^2(0)\ell_2'(0)}{\beta_1(0)} - \frac{9\lambda}{1-\lambda}\frac{\beta_2(0)\beta_1'(0)\beta_2'(0)\ell_2(0)}{\beta_1(0)} + \frac{\lambda^3}{(1-\lambda)^3}\frac{\beta_2^4(0)\ell_1''(0)}{\beta_1^2(0)} \\
&+ \frac{6\lambda^2}{(1-\lambda)^2}\frac{\beta_2^2(0)\beta_2'(0)\ell_1'(0)}{\beta_1(0)} + \frac{3\lambda}{1-\lambda}(\beta_2')^2(0)\ell_1(0) + \beta_1(0)\beta_2(0)\ell_2^3(0) \\
&- 3\beta_1(0)\beta_2''(0)\ell_2(0) - 3\beta_1(0)\beta_2'(0)\ell_2'(0) - \beta_1(0)\beta_2(0)\ell_2''(0),
\end{aligned}
$$

and

$$
\mathcal{H}_\Omega(0,0) = \begin{pmatrix} 2\beta_1'(0)\beta_2(0)\ell_1(0) + \beta_1(0)\beta_2(0)\ell_1'(0) & \beta_1(0)\beta_2'(0)\ell_1(0) - \beta_1'(0)\beta_2(0)\ell_2(0) \\ \\ \beta_1(0)\beta_2'(0)\ell_1(0) - \beta_1'(0)\beta_2(0)\ell_2(0) & -(2\beta_1(0)\beta_2'(0)\ell_2(0) + \beta_1(0)\beta_2(0)\ell_2'(0)) \end{pmatrix}.
$$

Hence,

$$
\begin{aligned}
det(\mathcal{H}_\Omega(0,0)) =& -(2\beta_1'(0)\beta_2(0)\ell_1(0) + \beta_1(0)\beta_2(0)\ell_1'(0))(2\beta_1(0)\beta_2'(0)\ell_2(0) + \beta_1(0)\beta_2(0)\ell_2'(0)) \\
&- (\beta_1(0)\beta_2'(0)\ell_1(0) - \beta_1'(0)\beta_2(0)\ell_2(0))^2.
\end{aligned}
$$

Therefore, applying Theorem 1 the results of this theorem hold. By a similar argument, we prove the case when $\omega_1(0) = \omega_2(0)$ by choosing

$$
\rho = \frac{\lambda\beta_2(0)}{(1-\lambda)\beta_1(0)}\frac{\partial}{\partial s_1} - \frac{\partial}{\partial s_2}.
$$

□

Note that The results in [15] related to the midpoint map are special cases of Theorem 2.

3.2. The Case When $\beta_1(0) = 0$ and $\beta_2(0) \neq 0$

In this section, we study the case when one of the two curves is singular. Precisely, $\beta_1(0) = 0$ and $\beta_2(0) \neq 0$. We give the conditions for the λ-point map between two Legendre plane curves to have fold and beaks singularities in the following theorem.

Theorem 3. *Let M be the λ-point map between two Legendre plane curves γ_1 and γ_2, such that $\beta_1(0) = 0$ and $\beta_2(0) \neq 0$.*

1. *If $\omega_1(0) \neq \pm\omega_2(0)$, then at $(0,0)$ M is $\mathcal{A}$-equivalent to fold if, and only if, $\beta_1'(0) \neq 0$.*
2. *If $\omega_1(0) = \pm\omega_2(0)$, then at $(0,0)$ M is $\mathcal{A}$-equivalent to beaks if, and only if, $\beta_1'(0) \neq 0$ and $\ell_i(0) \neq 0$, $i = 1,2$.*

Proof. Let $M(s_1, s_2) = (1-\lambda)\gamma_1(s_1) + \gamma_2(s_2)$ be the λ-point map between two Legendre plane curves. Suppose that $\beta_1(0) = 0$ and $\beta_2(0) \neq 0$. We prove this theorem by using Theorem 1. Now we choose vector field ρ, such that $dM\big|_{(0,0)}(\rho) = 0$, so we take $\rho = \dfrac{\partial}{\partial s_1}$.

Then, by a straightforward calculations, we have

$$
\frac{\partial\Omega}{\partial s_1} = -\beta_1'\beta_2\omega_1\cdot\mu_2 - \beta_1\beta_2\ell_1\omega_1\cdot\omega_2, \frac{\partial\Omega}{\partial s_2} = -\beta_1\beta_2'\omega_1\cdot\mu_2 + \beta_1\beta_2\ell_2\omega_1\cdot\omega_2,
$$

$$\rho\Omega = -\beta_1'\beta_2\omega_1 \cdot \mu_2 - \beta_1\beta_2\ell_1\omega_1 \cdot \omega_2, \rho^2\Omega = (\beta_1\beta_2\ell_1^2 - \beta_1''\beta_2)\omega_1 \cdot \mu_2 - (2\beta_1'\beta_2\ell_1 + \beta_1\beta_2\ell_1')\omega_1.\omega_2,$$
$$\rho^3\Omega = (3\beta_1'\beta_2\ell_1^2 + 3\beta_1\beta_2\ell_1\ell_1' - \beta_1'''\beta_2)\omega_1 \cdot \mu_2 + (\beta_1\beta_2\ell_1^3 - \beta_1\beta_2\ell_1'' - 3\beta_1''\beta_2\ell_1 - 3\beta_1'\beta_2\ell_1')\omega_1 \cdot \omega_2$$

and

$$\begin{aligned} det(\mathcal{H}_\Omega) =& \big((\beta_1\beta_2\ell_1^2 - \beta_1''\beta_2)\omega_1 \cdot \mu_2 - (2\beta_1'\beta_2\ell_1 + \beta_1\beta_2\ell_1')\omega_1 \cdot \omega_2\big)\big((\beta_1\beta_2\ell_2^2 - \beta_1\beta_2'')\omega_1 \cdot \mu_2 \\ &+ (2\beta_1\beta_2'\ell_2 + \beta_1\beta_2\ell_2')\omega_1 \cdot \omega_2\big) - \big(- (\beta_1'\beta_2' + \beta_1\beta_2\ell_1\ell_2)\omega_1 \cdot \mu_2 + (\beta_1'\beta_2\ell_2 - \beta_1\beta_2'\ell_1)\omega_1 \cdot \omega_2\big)^2. \end{aligned}$$

Thus, we have

$$\frac{\partial\Omega}{\partial s_1}\Big|_{(0,0)} = -\beta_1'(0)\beta_2(0)\omega_1(0) \cdot \mu_2(0), \frac{\partial\Omega}{\partial s_2}\Big|_{(0,0)} = 0,$$
$$\rho\Omega\big|_{(0,0)} = -\beta_1'(0)\beta_2(0)\omega_1(0) \cdot \mu_2(0), \rho^2\Omega\big|_{(0,0)} = -\beta_1''(0)\beta_2(0)\omega_1(0) \cdot \mu_2(0) - 2\beta_1'(0)\beta_2(0)\ell_1(0)\omega_1(0).\omega_2(0),$$
$$\rho^3\Omega\big|_{(0.0)} = \big(3\beta_1'(0)\beta_2(0)\ell_1^2(0) - \beta_1'''(0)\beta_2(0)\big)\omega_1(0) \cdot \mu_2(0) - 3\big(\beta_1''(0)\beta_2(0)\ell_1(0) + \beta_1'(0)\beta_2(0)\ell_1'(0)\big)\omega_1(0) \cdot \omega_2(0)$$

and

$$det(\mathcal{H}_\Omega(0,0)) = -(-\beta_1'(0)\beta_2'(0)\omega_1(0) \cdot \mu_2(0) + \beta_1'(0)\beta_2(0)\ell_2(0)\omega_1(0) \cdot \omega_2(0))^2.$$

Therefore, applying Theorem 1 we obtain the result. □

Given proof of the above theorem, we have the following theorem.

Theorem 4. *Let M be the λ-point map between two Legendre plane curves γ_1 and γ_2 satisfying $\beta_1(0) = 0$ and $\beta_2(0) \neq 0$. Then at $(0,0)$, M cannot be $\mathcal{A}$-equivalent to cusp, or lips or swallowtails singularity.*

4. Classification of Corank Two Singularities of λ-Point Map between Two Legendre Plane Curves

The criteria for sharksfin and deltoid singularities, which are generic singularities of corank two of maps from the plane into the plane (cf. [16]), have been obtain by kabata and Saji in [6].

Let $h : (U \subseteq \mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a map germ with a corank two singularity at $(0,0)$. We call the function $\Omega : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ which is defined by $\Omega(s_1, s_2) = det\left(\frac{\partial h}{\partial s_1}, \frac{\partial h}{\partial s_2}\right)(s_1, s_2)$ a discriminant of singularities. The zeros of Ω are all the singular points of h. We define non-zero vector fields ρ_1, ρ_2 at a non-degenerate critical point of Ω which is a solution of the Hesse quadric of Ω at $(0,0)$. Recall that a vector field (ρ_{11}, ρ_{12}) is a solution of the Hesse quadric of Ω

$$\begin{pmatrix}\rho_{11} & \rho_{12}\end{pmatrix}\mathcal{H}_\Omega(0,0)\begin{pmatrix}\rho_{11} \\ \rho_{12}\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}.$$

Our goal in this section is to give the geometric conditions for the λ-point map between two Legendre plan curves to have sharksfin singularity. The normal forms of sharksfin and deltoid singularities are $(xy, x^2 + y^2 + x^3)$ and $(xy, -x^2 + y^2 + x^3)$, respectively. We state the criteria for sharksfin and deltoid singularities.

Theorem 5 ([6])**.** *Let $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a map germ with a corank two singularity at $(0,0)$ and suppose that Ω have a non-degenerate critical point at $(0,0)$.*

Then h is a sharksfin (respectively, deltoid) at $(0,0)$ if, and only if, $det(\mathcal{H}_\Omega(0,0)) < 0$ (respectively, $det(\mathcal{H}_\Omega(0,0)) > 0$), $det(\rho_1^2 h, \rho_1^3 h)(0,0) \neq 0$ and $det(\rho_2^2 h, \rho_2^3 h)(0,0) \neq 0$. Here, $\rho\Omega$ means the directional derivative of Ω in the direction of the vector field ρ, and $\rho^i h = \rho(\rho^{i-1} h)$.

Lemma 3. *Let M be the λ-point map between two Legendre plane curves γ_1 and γ_2. Then M is parametrized by a corank two singularity at $(0,0)$ if, and only if, $\beta_i(0) = 0$, $i = 1, 2$.*

Proof. The proof of this lemma is obvious. □

Lemma 4. *Let M be the λ-point map between two Legendre plane curves γ_1 and γ_2, such that $\beta_i(0) = 0$, $i = 1, 2$. Then*

1. *$(0, 0)$ is a critical point of Ω.*
2. *$(0, 0)$ is a non-degenerate critical point of Ω if, and only if, $\beta_1'(0) \neq 0$, $\beta_2'(0) \neq 0$ and $\omega_1(0) \neq \pm\omega_2(0)$.*

Proof. We define the discriminant of singularities $\Omega : \mathbb{R}^2 \to \mathbb{R}$ of M by

$$\Omega(s_1, s_2) = det\left(\frac{\partial M}{\partial s_1}, \frac{\partial M}{\partial s_2}\right) = -\beta_1(s_1)\beta_2(s_2)\omega_1(s_1) \cdot \mu_2(s_2).$$

It is easy to check that $(0, 0)$ is a critical point of Ω. A point $(0, 0)$ is a non-degenerate if, and only if, $det(\mathcal{H}_\Omega(0, 0)) \neq 0$. Now

$$\mathcal{H}_\Omega(0,0) = \begin{pmatrix} \frac{\partial^2 \Omega}{\partial^2 s_1}\Big|_{(0,0)} & \frac{\partial^2 \Omega}{\partial s_1 \partial s_2}\Big|_{(0,0)} \\ \frac{\partial^2 \Omega}{\partial s_1 \partial s_2}\Big|_{(0,0)} & \frac{\partial^2 \Omega}{\partial^2 s_2}\Big|_{(0,0)} \end{pmatrix} = \begin{pmatrix} 0 & -\beta_1'(0)\beta_2'(0)\omega_1(0)\cdot\mu_2(0) \\ -\beta_1'(0)\beta_2'(0)\omega_1(0)\cdot\mu_2(0) & 0 \end{pmatrix}$$

Thus,

$$det(\mathcal{H}_\Omega(0,0)) = -(\beta_1'(0)\beta_2'(0)\omega_1(0)\cdot\mu_2(0))^2 \neq 0$$

if and only if $\beta_1'(0)\beta_2'(0)\omega_1(0)\cdot\mu_2(0) \neq 0$. □

Now we introduce the main theorem of this section.

Theorem 6. *Let M be the λ-point map between two Legendre plane curves γ_1 and γ_2 with a corank two singularity at $(0, 0)$ and let $(0, 0)$ be non-degenerate critical point of Ω. Then M is $\mathcal{A}$-equivalent to a sharksfin if, and only if, $\ell_i(0) \neq 0$, $i = 1, 2$.*

Proof. Let $M(s_1, s_2) = (1 - \lambda)\gamma_1(s_1) + \lambda\gamma_2(s_2)$ be the λ-point map between two Legendre plane curves γ_1 and γ_2, such that $rank(dM)\big|_{(0,0)} = 0$.

We will use Theorem 5 to prove this theorem. From Lemma 3 we have $\beta_1(0) = \beta_2(0) = 0$. Now we have

$$\mathcal{H}_\Omega(0,0) = \begin{pmatrix} 0 & -\beta_1'(0)\beta_2'(0)\omega_1(0)\cdot\mu_2(0) \\ -\beta_1'(0)\beta_2'(0)\omega_1(0)\cdot\mu_2(0) & 0 \end{pmatrix}.$$

Thus, $det(\mathcal{H}_\Omega(0, 0)) = -(\beta_1'(0)\beta_2'(0)\omega_1(0) \cdot \mu_2(0))^2 < 0$.

Now we choose vector fields $\rho_1 = \dfrac{\partial}{\partial s_1}$ and $\rho_2 = \dfrac{\partial}{\partial s_2}$ which satisfy the Hesse quadric of Ω at $(0, 0)$. Calculations show that

$$\rho_1^2 M\big|_{(0,0)} = (1-\lambda)\beta_1'(0)\mu_1(0),$$

$$\rho_1^3 M\big|_{(0,0)} = (1-\lambda)(\beta_1''(0)\mu_1(0) - 2\beta_1'(0)\ell_1(0)\omega_1(0)),$$

$$\rho_2^2 M\big|_{(0,0)} = \lambda\beta_2'(0)\mu_2(0)$$

and

$$\rho_2^3 M\big|_{(0,0)} = \lambda(\beta_2''(0)\mu_2(0) - 2\beta_2'(0)\ell_2(0)\omega_2(0)).$$

Now, $det(\rho_1^2 M, \rho_1^3 M) \neq 0$ if, and only if, $\beta'^2_1(0)\ell_1(0) \neq 0$. Additionally, $det(\rho_2^2 M, \rho_2^3 M) \neq 0$ if, and only if, $\beta'^2_2(0)\ell_2(0) \neq 0$. □

Given the proof of the above theorem, we have the following theorem.

Theorem 7. *The λ-point map between two Legendre plane curves γ_1 and γ_2 with a corank two singularity at $(0,0)$ cannot be $\mathcal{A}$-equivalent to a deltoid at $(0,0)$.*

5. Examples

In this section, we present three examples for the λ-point map between two Legendre plane curves to have fold, beaks, and sharksfin singularities.

Example 1. *We give an example for part 1 of Theorem 2. Take $\gamma_1(s_1) = (2s_1, 4s_1^2 + s_1^4)$, $\gamma_2(s_2) = (-s_2, s_2^2 - s_2^5)$, and $\lambda = \frac{1}{4}$. Then, the λ- point map between γ_1 and γ_2 is given by $M(s_1, s_2) = \left(\frac{3}{2}s_1 - \frac{1}{4}s_2, \frac{3}{4}(4s_1^2 + s_1^4) + \frac{1}{4}(s_2^2 - s_2^5)\right)$. Clearly, M is singular at $(0,0)$, and direct calculation shows that $\beta_1(s_1) = 2\sqrt{1 + (4s_1 + 2s_1^3)^2}$, $\ell_1(s_1) = \dfrac{4 + s_1^2}{1 + (4s_1 + 2s_1^3)^2}$, $\omega_1(s_1) = \left(\frac{4s_1 + 2s_1^3}{\sqrt{1+(4s_1+2s_1^3)^2}}, \frac{-1}{\sqrt{1+(4s_1+2s_1^3)^2}}\right)$, $\beta_2(s_2) = \sqrt{1 + (2s_2 - 5s_2^4)^2}$, $\ell_2(s_2) = \dfrac{20s_2^2 - 2}{1 + (2s_2 - 5s_2^4)^2}$, and $\omega_2(s_2) = \left(\frac{2s_2 - 5s_2^4}{\sqrt{1+(2s_2-5s_2^4)^2}}, \frac{1}{\sqrt{1+(2s_2-5s_2^4)^2}}\right)$. Now at $s_1 = 0$ and $s_2 = 0$, we have $\beta_1(0) = 2$, $\beta_2(0) = 1$, $\ell_1(0) = 4$, $\ell_2(0) = -2$, $\omega_1(0) = (0, -1) = -\omega_2(0)$, and $\left(\frac{\lambda}{1-\lambda}\right)\frac{\beta_2(0)\ell_1(0)}{\beta_1(0)} = \frac{2}{3} \neq \ell_2(0)$. Thus, M is $\mathcal{A}$-equivalent to fold at $(0,0)$.*

Example 2. *This example is dedicated to part 2 of Theorem 3. Let $\gamma_1(s_1) = (s_1^2, s_1^3)$, $\gamma_2(s_2) = (s_2, s_2^2)$, and $\lambda = \frac{1}{5}$. Then the λ-point map between γ_1 and γ_2 is given by $M(s_1, s_2) = (\frac{4}{5}s_1^2 + \frac{1}{5}s_2, \frac{4}{5}s_1^3 + \frac{1}{5}s_2^2)$. Direct calculation shows that $\beta_1(s_1) = s_1\sqrt{4 + 9s_1^2}$, $\beta_2(s_2) = \sqrt{1 + 4s_2^2}$, $\ell_1(s_1) = \frac{6}{4+9s_1^2}$, $\ell_2(s_2) = \frac{2}{1+4s_2^2}$, $\omega_1(s_1) = \left(\frac{3s_1}{\sqrt{4+9s_1^2}}, \frac{-2}{\sqrt{4+9s_1^2}}\right)$, and $\omega_2(s_2) = \left(\frac{2s_2}{1+4s_2^2}, \frac{-1}{1+4s_2^2}\right)$. At $(s_1, s_2) = (0,0)$, M as a corank one singularity and $\beta_1(0) = 0$, $\beta_1'(0) = 2$, $\beta_2(0) = 1$, $\ell_1(0) = \frac{3}{2}$, $\ell_2(0) = 2$, and $\omega_1(0) = (0,1) = \omega_2(0)$. Therefore, M is $\mathcal{A}$-equivalent to beaks at $(0,0)$.*

Example 3. *This example illustrates the result in Theorem 6. Let $\gamma_1(s_1) = (\frac{1}{2}s_1^2, \frac{1}{3}s_1^3)$, $\gamma_2(s_2) = (2s_2^3, -s_2^2)$, and $\lambda = \frac{1}{3}$. The λ- point map between γ_1 and γ_2 is given by $M(s_1, s_2) = (\frac{1}{3}s_1^2 + \frac{2}{3}s_2^3, \frac{2}{9}s_1^3 - \frac{1}{3}s_2^2)$ and $\Omega(s_1, s_2) = -2s_1s_2(1 + 3s_1s_2)$. It is clear that $(0,0)$ is a non-degenerate critical point of Ω and M has a corank two singularity at $(0,0)$. Calculation shows that $\beta_1(s_1) = s_1\sqrt{1 + s_1^2}$, $\ell_1(s_1) = \frac{1}{1+s_1^2}$, $\omega_1(s_1) = \left(\frac{s_1}{\sqrt{1+s_1^2}}, \frac{-1}{\sqrt{1+s_1^2}}\right)$, $\beta_2(s_2) = -2s_2\sqrt{1 + 9s_2^2}$, $\ell_2(s_2) = \frac{3}{1+9s_2^2}$, and $\omega_2(s_2) = \left(\frac{1}{\sqrt{1+9s_2^2}}, \frac{3s_2}{\sqrt{1+9s_2^2}}\right)$. At $(s_1, s_2) = (0,0)$, we have $\ell_1(0) = 1$ and $\ell_2(0) = 3$. Therefore, M is $\mathcal{A}$-equivalent to sharksfin at $(0,0)$.*

6. Conclusions

Throughout this paper we introduce the λ-point map between two Legendre plane curves. The classifications of this map have been investigated for corank one and two singularities. All results obtained in this research are more general and many known plane map's singularities are realized as special cases of these results. Moreover, three non-trivial examples are given throughout this research to illustrate some of the obtained results.

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Article

ζ-Conformally Flat LP-Kenmotsu Manifolds and Ricci–Yamabe Solitons

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Abstract: In the present paper, we characterize m-dimensional ζ-conformally flat LP-Kenmotsu manifolds (briefly, $(LPK)_m$) equipped with the Ricci–Yamabe solitons (RYS) and gradient Ricci–Yamabe solitons (GRYS). It is proven that the scalar curvature r of an $(LPK)_m$ admitting an RYS satisfies the Poisson equation $\Delta r = \frac{4(m-1)}{\delta}\{\beta(m-1)+\rho\} + 2(m-3)r - 4m(m-1)(m-2)$, where $\rho, \delta(\neq 0) \in \mathbb{R}$. In this sequel, the condition for which the scalar curvature of an $(LPK)_m$ admitting an RYS holds the Laplace equation is established. We also give an affirmative answer for the existence of a GRYS on an $(LPK)_m$. Finally, a non-trivial example of an LP-Kenmotsu manifold (LPK) of dimension four is constructed to verify some of our results.

Keywords: Lorentzian manifolds; Ricci–Yamabe solitons; gradient Ricci–Yamabe solitons; perfect fluid spacetime; Einstein manifolds

MSC: 53C25; 53C21; 53C50; 53E20

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1. Introduction

The Ricci solitons (RS) and Yamabe solitons (YS) correspond to self-similar solutions of the Ricci flow, $2S + \frac{\partial}{\partial t}g = 0$, and the Yamabe flow, $\frac{\partial}{\partial t}g = -rg, g(0) = g_0$ (where S denotes the Ricci tensor and r is the scalar curvature of the metric g); they are given by [1,2]

$$\pounds_{\mathcal{E}}g + 2\rho g + 2S = 0, \tag{1}$$

and

$$\pounds_{\mathcal{E}}g = -2(\rho - r)g, \tag{2}$$

respectively, where $\rho \in \mathbb{R}$ (set of real numbers) and $\pounds_{\mathcal{E}}$ stands for the Lie derivative operator along the smooth vector field $\mathcal{E}$ on a semi-Riemannian manifold M of dimension m.

Recently, a scalar combination of Ricci and Yamabe flows was established by Güler and Crasmareanu [3]. This class of geometric flow was named a Ricci–Yamabe (RY) flow of type (β, δ) and was defined by

$$\frac{\partial}{\partial t}g(t) + 2\beta S(g(t)) + \delta r(t)g(t) = 0, \quad g(0) = g_0 \tag{3}$$

for some scalars β and δ.

A solution to the RY flow is called a Ricci–Yamabe soliton (RYS) if it depends only on one parameter group of diffeomorphism and scaling. An M is said to admit an RYS if

$$\pounds_{\mathcal{E}}g + 2\beta S + (2\rho - \delta r)g = 0, \tag{4}$$

where $\beta, \delta, \rho \in \mathbb{R}$. If $\mathcal{E}$ is the gradient of a smooth function u on M, then Equation (4) is called a gradient Ricci–Yamabe soliton (GRYS) and then Equation (4) transforms to

$$\nabla^2 u + \beta S + (\rho - \frac{\delta r}{2})g = 0, \tag{5}$$

where $\nabla^2 u$ is the Hessian of u and is denoted by $Hess(u) = \nabla\nabla u$. Moreover, we note that a RYS of type $(\beta, 0)$ and of type $(0, \delta)$ are known as β-Ricci soliton and δ-Yamabe soliton, respectively. An RYS is said to be shrinking, steady or expanding if $\rho < 0, = 0$ or > 0, respectively. An RYS is said to be a

- Ricci soliton (RS) [4] if $\beta = 1, \delta = 0$;
- Yamabe soliton (YS) [5] if $\beta = 0, \delta = 1$;
- Einstein soliton [6] if $\beta = -\delta = 1$;
- ϱ-Einstein soliton [7] if $\beta = 1, \delta = -2\varrho$.

On the other hand, the Lorentzian manifold which is one of the most important subclass of pseudo-Riemannian manifolds plays an important role in the development of the theory of relativity and cosmology [8]. In 1989, Matsumoto [9] introduced the notion of LP-Sasakian manifolds, while in 1992, the same notion was independently studied by Mihai and Rosca [10], and they obtained several results on this manifold. Later, such manifolds were studied by many authors. Recently, Haseeb and Prasad defined and studied the Lorentzian para-Kenmotsu manifold [11] as a subclass of Lorentzian paracontact manifold. For more details about the related studies, we recommend the papers [12–26] and the references therein.

As a continuation of this study, we propose a study of the RYS and GRYS in the framework of a ζ-conformally flat $(LPK)_m$. In Section 2, we include some basic results and definitions which are required to study an $(LPK)_m$. Sections 3 and 4 are concerned with the study of a RYS and a GRYS on a ζ-conformally flat $(LPK)_m$, respectively. In Section 5, we construct a non-trivial example of an $(LPK)_4$ and proved that an $(LPK)_4$ is ζ-conformally flat and that a GRYS on an $(LPK)_4$ is trivial.

2. Preliminaries

A differentiable manifold M (where the dimension of M is m) with the structure (f, ζ, ω) is named a Lorentzian almost paracontact manifold, where f, ζ and ω represent a $(1,1)$-type tensor field, a contravariant vector field and a one-form, respectively, on M satisfying [27]

$$\omega(\zeta) = -1 \text{ and } f^2 = \omega \otimes \zeta + I, \tag{6}$$

which yields

$$f\zeta = 0, \quad \omega \circ f = 0, \ \text{rank}(f) = m - 1. \tag{7}$$

Let the Lorentzian metric g of M fulfill

$$g(\cdot, \zeta) = \omega(\cdot) \text{ and } g(f\,\cdot, f\,\cdot) = g(\cdot\,,\cdot) + \omega(\cdot)\omega(\cdot). \tag{8}$$

Then, the structure (f, ζ, ω, g) is said to be an almost paracontact structure and M is called an almost paracontact metric manifold.
Define the second fundamental form Φ as

$$\Phi(\mathcal{E}_1, \mathcal{E}_2) = \Phi(\mathcal{E}_2, \mathcal{E}_1) = g(\mathcal{E}_1, f\mathcal{E}_2) \tag{9}$$

for any vector fields $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of vector fields on M. If

$$d\omega(\mathcal{E}_1, \mathcal{E}_2) = \Phi(\mathcal{E}_1, \mathcal{E}_2), \tag{10}$$

where d is an exterior derivative, then (M, f, ζ, ω, g) is termed as a paracontact metric manifold.

If the vector field ζ is a Killing vector field, then the (para)contact structure is called a K-(para)contact. In such a situation, we have

$$\nabla_{\mathcal{E}_1}\zeta = f\mathcal{E}_1. \quad (11)$$

Definition 1. *A Lorentzian almost paracontact manifold M is called an $(LPK)_m$ if [11]*

$$(\nabla_{\mathcal{E}_1} f)\mathcal{E}_2 = -g(f\mathcal{E}_1, \mathcal{E}_2)\zeta - \omega(\mathcal{E}_2)f\mathcal{E}_1 \quad (12)$$

for any $\mathcal{E}_1, \mathcal{E}_2$ on M.

In an $(LPK)_m$, we have

$$\nabla_{\mathcal{E}_1}\zeta + \mathcal{E}_1 + \omega(\mathcal{E}_1)\zeta = 0, \quad (13)$$

$$(\nabla_{\mathcal{E}_1}\omega)\mathcal{E}_2 + g(\mathcal{E}_1, \mathcal{E}_2) + \omega(\mathcal{E}_1)\omega(\mathcal{E}_2) = 0, \quad (14)$$

where ∇ stands for the Levi–Civita connection with respect to g.

Furthermore, in an $(LPK)_m$, the following relations hold [11]:

$$g(\mathcal{R}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3, \zeta) = \omega(\mathcal{R}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3) = g(\mathcal{E}_2, \mathcal{E}_3)\omega(\mathcal{E}_1) - g(\mathcal{E}_1, \mathcal{E}_3)\omega(\mathcal{E}_2), \quad (15)$$

$$\mathcal{R}(\zeta, \mathcal{E}_1)\mathcal{E}_2 = -\mathcal{R}(\mathcal{E}_1, \zeta)\mathcal{E}_2 = g(\mathcal{E}_1, \mathcal{E}_2)\zeta - \omega(\mathcal{E}_2)\mathcal{E}_1, \quad (16)$$

$$\mathcal{R}(\mathcal{E}_1, \mathcal{E}_2)\zeta = \omega(\mathcal{E}_2)\mathcal{E}_1 - \omega(\mathcal{E}_1)\mathcal{E}_2, \quad (17)$$

$$\mathcal{R}(\zeta, \mathcal{E}_1)\zeta = \mathcal{E}_1 + \omega(\mathcal{E}_1)\zeta, \quad (18)$$

$$S(\mathcal{E}_1, \zeta) = (m-1)\omega(\mathcal{E}_1), \;\; S(\zeta, \zeta) = -(m-1), \quad (19)$$

$$\mathcal{Q}\zeta = (m-1)\zeta, \quad (20)$$

for any $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ on an $(LPK)_m$, where $\mathcal{R}$ is the curvature tensor and $\mathcal{Q}$ is the Ricci operator of $(LPK)_m$.

Definition 2. *An $(LPK)_m$ is said to be a perfect fluid spacetime if its $(0,2)$-type Ricci tensor $S(\neq 0)$ satisfies the following condition*

$$S(\mathcal{E}_1, \mathcal{E}_2) = \sigma_1 g(\mathcal{E}_1, \mathcal{E}_2) + \sigma_2\omega(\mathcal{E}_1)\omega(\mathcal{E}_2), \quad (21)$$

for smooth functions σ_1 and σ_2, where ω is a one-form such that $g(\mathcal{E}_1, \zeta) = \omega(\mathcal{E}_1)$, for all vector field $\mathcal{E}_1$, associated to the unit timelike vector field ζ. The one-form ω is called the associated one-form and ζ is called the velocity vector field. For more details, we refer the reader to [28–34] and the references therein.

An $(LPK)_m$ is said to be ζ-conformally flat if the conformal curvature tensor $\mathcal{C}$ [35] defined by

$$\begin{aligned} \mathcal{C}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 &= \mathcal{R}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 - \tfrac{1}{m-2}\{S(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_1 - S(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_2 + g(\mathcal{E}_2, \mathcal{E}_3)\mathcal{Q}\mathcal{E}_1 \\ &\quad - g(\mathcal{E}_1, \mathcal{E}_3)\mathcal{Q}\mathcal{E}_2\} + \tfrac{r}{(m-1)(m-2)}\{g(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_1 - g(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_2\}, \end{aligned} \quad (22)$$

$\forall\, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ on the $(LPK)_m$ satisfies the relation $\mathcal{C}(\mathcal{E}_1, \mathcal{E}_2)\zeta = 0$.

Setting $\mathcal{E}_2 = \mathcal{E}_3 = \zeta$ in Equation (22) and then following Equations (6), (8), (17), (19) and (20), we infer that

$$\mathcal{Q} = \left(\frac{r}{m-1} - 1\right)I + \left(\frac{r}{m-1} - m\right)\omega \otimes \zeta, \quad (23)$$

which yields that an $(LPK)_m$ is a perfect fluid spacetime. Thus, we write

Proposition 1. *Every ζ-conformally flat $(LPK)_m$ is a perfect fluid spacetime.*

Lemma 1. *In a ζ-conformally flat $(LPK)_m$, we have*

$$\zeta(r) = 2(r - m(m-1)), \tag{24}$$

$$\mathcal{E}_1(r) = -2(r - m(m-1))\omega(\mathcal{E}_1), \tag{25}$$

$$\omega(\nabla_\zeta Dr) = 4(r - m(m-1)) \tag{26}$$

for any $\mathcal{E}_1$ on the $(LPK)_m$.

Proof. The covariant differentiation of Equation (23) with respect to $\mathcal{E}_2$ and the use of Equations (13) and (14) lead to

$$\begin{aligned}(\nabla_{\mathcal{E}_2}\mathcal{Q})\mathcal{E}_1 &= \frac{\mathcal{E}_2(r)}{m-1}(\mathcal{E}_1 + \omega(\mathcal{E}_1)\zeta) \\ &- (\frac{r}{m-1} - m)(g(\mathcal{E}_1,\mathcal{E}_2)\zeta + \omega(\mathcal{E}_1)\mathcal{E}_2 + 2\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\zeta).\end{aligned} \tag{27}$$

Taking the inner product of Equation (27) with $\mathcal{E}_3$, we have

$$\begin{aligned}g\Big((\nabla_{\mathcal{E}_2}\mathcal{Q})\mathcal{E}_1,\mathcal{E}_3\Big) &= \tfrac{\mathcal{E}_2(r)}{m-1}(g(\mathcal{E}_1,\mathcal{E}_3) + \omega(\mathcal{E}_1)\omega(\mathcal{E}_3)) \\ &- (\tfrac{r}{m-1} - m)(g(\mathcal{E}_1,\mathcal{E}_2)\omega(\mathcal{E}_3) + \omega(\mathcal{E}_1)g(\mathcal{E}_2,\mathcal{E}_3) + 2\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\omega(\mathcal{E}_3)).\end{aligned} \tag{28}$$

Let $\{\ell_1, \ell_2, \ell_3 \ldots \ell_{m-1}, \ell_m = \zeta\}$ be the orthonormal basis of the tangent space at each point of an $(LPK)_m$. By putting $\mathcal{E}_2 = \mathcal{E}_3 = \ell_i$ and taking the summation over $i(1 \le i \le m)$, we find

$$\mathcal{E}_1(r) = \frac{2(m-1)}{m-3}\{\frac{\zeta(r)}{m-1} - (r - m(m-1))\}\omega(\mathcal{E}_1), \tag{29}$$

where the trace $\{\mathcal{E}_2 \to (\nabla_{\mathcal{E}_2}\mathcal{Q})\mathcal{E}_1\} = \frac{1}{2}\mathcal{E}_1(r)$ is used.

Replacing $\mathcal{E}_1$ by ζ in Equation (29) and using Equation (6) gives Equation (24). Next, by using Equation (24) in Equation (29), we easily obtain Equation (25). By the covariant differentiation of Equation (24) with respect to ζ and using Equation (13), Equation (26) follows. □

Remark 1. *From the relation (24), it is noticed that if a ζ-conformally flat $(LPK)_m$ has a constant scalar curvature, then $r = m(m-1)$.*

3. RYS on a ζ-Conformally Flat $(LPK)_m$

Let the metric of a ζ-conformally flat $(LPK)_m$ be an RYS, then, in view of Equation (23), Equation (4) takes the form

$$\begin{aligned}(\pounds_{\mathcal{E}} g)(\mathcal{E}_1,\mathcal{E}_2) &= -2\{\beta(\tfrac{r}{m-1} - 1) + (\rho - \tfrac{\delta r}{2})\}g(\mathcal{E}_1,\mathcal{E}_2) \\ &\quad -2\beta(\tfrac{r}{m-1} - m)\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\end{aligned} \tag{30}$$

for any $\mathcal{E}_1$, $\mathcal{E}_2$ on $(LPK)_m$.

Taking the covariant derivative of Equation (30) with respect to $\mathcal{E}_3$, we find

$$\begin{aligned}(\nabla_{\mathcal{E}_3}\pounds_{\mathcal{E}} g)(\mathcal{E}_1,\mathcal{E}_2) &= -2(\frac{\beta}{m-1} - \frac{\delta}{2})\mathcal{E}_3(r)g(\mathcal{E}_1,\mathcal{E}_2) - \frac{2\beta}{m-1}\mathcal{E}_3(r)\omega(\mathcal{E}_1)\omega(\mathcal{E}_2) \\ &\quad +2\beta(\frac{r}{m-1} - m)(g(\mathcal{E}_1,\mathcal{E}_3)\omega(\mathcal{E}_2) + g(\mathcal{E}_2,\mathcal{E}_3)\omega(\mathcal{E}_1) \\ &\quad +2\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\omega(\mathcal{E}_3)).\end{aligned} \tag{31}$$

Since $\nabla g = 0$, then the formula [36]

$$(\pounds_{\mathcal{E}}\nabla_{\mathcal{E}_1}g - \nabla_{\mathcal{E}_1}\pounds_{\mathcal{E}}g - \nabla_{[\mathcal{E},\mathcal{E}_1]}g)(\mathcal{E}_2,\mathcal{E}_3) = -g((\pounds_{\mathcal{E}}\nabla)(\mathcal{E}_1,\mathcal{E}_2),\mathcal{E}_3) - g((\pounds_{\mathcal{E}}\nabla)(\mathcal{E}_1,\mathcal{E}_3),\mathcal{E}_2) \tag{32}$$

becomes

$$(\nabla_{\mathcal{E}_1}\pounds_{\mathcal{E}}g)(\mathcal{E}_2,\mathcal{E}_3) = g((\pounds_{\mathcal{E}}\nabla)(\mathcal{E}_1,\mathcal{E}_2),\mathcal{E}_3) + g((\pounds_{\mathcal{E}}\nabla)(\mathcal{E}_1,\mathcal{E}_3),\mathcal{E}_2). \tag{33}$$

Moreover, since $\pounds_{\mathcal{E}}\nabla$ is symmetric, then we have

$$2g((\pounds_{\mathcal{E}}\nabla)(\mathcal{E}_1,\mathcal{E}_2),\mathcal{E}_3) = (\nabla_{\mathcal{E}_1}\pounds_{\mathcal{E}}g)(\mathcal{E}_2,\mathcal{E}_3) + (\nabla_{\mathcal{E}_2}\pounds_{\mathcal{E}}g)(\mathcal{E}_1,\mathcal{E}_3) - (\nabla_{\mathcal{E}_3}\pounds_{\mathcal{E}}g)(\mathcal{E}_1,\mathcal{E}_2). \tag{34}$$

By using Equation (31) in the the last equation, we arrive at

$$\begin{aligned} 2g((\pounds_{\mathcal{E}}\nabla)(\mathcal{E}_1,\mathcal{E}_2),\mathcal{E}_3) &= -\mathcal{E}_1(r)\{2(\tfrac{\beta}{m-1} - \tfrac{\delta}{2})g(\mathcal{E}_2,\mathcal{E}_3) + \tfrac{2\beta}{m-1}\omega(\mathcal{E}_2)\omega(\mathcal{E}_3)\} \\ &\quad -\mathcal{E}_2(r)\{2(\tfrac{\beta}{m-1} - \tfrac{\delta}{2})g(\mathcal{E}_1,\mathcal{E}_3) + \tfrac{2\beta}{m-1}\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\} \\ &\quad +\mathcal{E}_3(r)\{2(\tfrac{\beta}{m-1} - \tfrac{\delta}{2})g(\mathcal{E}_1,\mathcal{E}_2) + \tfrac{2\beta}{m-1}\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\} \\ &\quad +4\beta(\tfrac{r}{m-1} - m)\{\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\omega(\mathcal{E}_3) + g(\mathcal{E}_1,\mathcal{E}_2)\omega(\mathcal{E}_3)\}, \end{aligned} \tag{35}$$

and from Equation (35), it follows that

$$\begin{aligned} 2(\pounds_{\mathcal{E}}\nabla)(\mathcal{E}_1,\mathcal{E}_2) &= -\mathcal{E}_1(r)\{2(\tfrac{\beta}{m-1} - \tfrac{\delta}{2})\mathcal{E}_2 + \tfrac{2\beta}{m-1}\omega(\mathcal{E}_2)\zeta\} \\ &\quad -\mathcal{E}_2(r)\{2(\tfrac{\beta}{m-1} - \tfrac{\delta}{2})\mathcal{E}_1 + \tfrac{2\beta}{m-1}\omega(\mathcal{E}_1)\zeta\} \\ &\quad +D(r)\{2(\tfrac{\beta}{m-1} - \tfrac{\delta}{2})g(\mathcal{E}_1,\mathcal{E}_2) + \tfrac{2\beta}{m-1}\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\} \\ &\quad +4\beta(\tfrac{r}{m-1} - m)\{g(\mathcal{E}_1,\mathcal{E}_2)\zeta + \omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\zeta\}. \end{aligned} \tag{36}$$

Putting $\mathcal{E}_1 = \zeta$ in Equation (36), then using Equations (6), (8) and (24), we find

$$\begin{aligned} 2(\pounds_{\mathcal{E}}\nabla)(\mathcal{E}_2,\zeta) &= \delta g(Dr,\mathcal{E}_2)\zeta - \delta D(r)\omega(\mathcal{E}_2) \\ &\quad -2(r - m(m-1))\{2(\tfrac{\beta}{m-1} - \tfrac{\delta}{2})\mathcal{E}_2 + \tfrac{2\beta}{m-1}\omega(\mathcal{E}_2)\zeta\}. \end{aligned} \tag{37}$$

The covariant differentiation of Equation (37) along $\mathcal{E}_1$ and the use of Equations (6), (8) and (37) give

$$\begin{aligned} 2(\nabla_{\mathcal{E}_1}\pounds_{\mathcal{E}}\nabla)(\mathcal{E}_2,\zeta) &= -3g(Dr,\mathcal{E}_1)\{2(\frac{\beta}{m-1} - \frac{\delta}{2})\mathcal{E}_2 + \frac{2\beta}{m-1}\omega(\mathcal{E}_2)\zeta\} \\ &\quad -\frac{2\beta}{m-1}g(Dr,\mathcal{E}_2)(\mathcal{E}_1 + \omega(\mathcal{E}_1)\zeta) + \frac{2\beta}{m-1}D(r)\{g(\mathcal{E}_1,\mathcal{E}_2) \\ &\quad +\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\} + \frac{4(r - m(m-1))}{m-1}\{\beta\omega(\mathcal{E}_2)\mathcal{E}_1 \\ &\quad -(\beta - \frac{\delta(m-1)}{2})\omega(\mathcal{E}_1)\mathcal{E}_2 + 2\beta g(\mathcal{E}_1,\mathcal{E}_2)\zeta + 2\beta\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\zeta\} \\ &\quad +\delta g(\nabla_{\mathcal{E}_1}Dr,\mathcal{E}_2)\zeta - \delta(\nabla_{\mathcal{E}_1}Dr)\omega(\mathcal{E}_2). \end{aligned} \tag{38}$$

Again, from [36], we have

$$(\pounds_{\mathcal{E}}\mathcal{R})(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 = (\nabla_{\mathcal{E}_1}\pounds_{\mathcal{E}}\nabla)(\mathcal{E}_2,\mathcal{E}_3) - (\nabla_{\mathcal{E}_2}\pounds_{\mathcal{E}}\nabla)(\mathcal{E}_1,\mathcal{E}_3). \tag{39}$$

By putting $\mathcal{E}_3 = \zeta$ and using Equation (38), Equation (39) takes the form

$$\begin{aligned} 2(\pounds_{\mathcal{E}}\mathcal{R})(\mathcal{E}_1,\mathcal{E}_2)\zeta &= g(Dr,\mathcal{E}_1)\{(3\delta - \frac{4\beta}{m-1})\mathcal{E}_2 - \frac{4\beta}{m-1}\omega(\mathcal{E}_2)\zeta\} \\ &\quad - g(Dr,\mathcal{E}_2)\{(3\delta - \frac{4\beta}{m-1})\mathcal{E}_1 - \frac{4\beta}{m-1}\omega(\mathcal{E}_1)\zeta\} \\ &\quad - \frac{4(r-m(m-1))}{m-1}\{(2\beta - \frac{\delta(m-1)}{2})\omega(\mathcal{E}_1)\mathcal{E}_2 \\ &\quad - (2\beta - \frac{\delta(m-1)}{2})\omega(\mathcal{E}_2)\mathcal{E}_1\} + \delta g(\nabla_{\mathcal{E}_1}Dr,\mathcal{E}_2)\zeta \\ &\quad - \delta g(\nabla_{\mathcal{E}_2}Dr,\mathcal{E}_1)\zeta + \delta(\nabla_{\mathcal{E}_2}Dr)\omega(\mathcal{E}_1) - \delta(\nabla_{\mathcal{E}_1}Dr)\omega(\mathcal{E}_2). \end{aligned} \tag{40}$$

Taking the inner product of Equation (40) with $\mathcal{E}_4$, we have

$$\begin{aligned} 2g\Big((\pounds_{\mathcal{E}}\mathcal{R})(\mathcal{E}_1,\mathcal{E}_2)\zeta,\mathcal{E}_4\Big) &= g(Dr,\mathcal{E}_1)\{(3\delta - \tfrac{4\beta}{m-1})g(\mathcal{E}_2,\mathcal{E}_4) - \tfrac{4\beta}{m-1}\omega(\mathcal{E}_2)\omega(\mathcal{E}_4)\} \\ &\quad - g(Dr,\mathcal{E}_2)\{(3\delta - \tfrac{4\beta}{m-1})g(\mathcal{E}_1,\mathcal{E}_4) - \tfrac{4\beta}{m-1}\omega(\mathcal{E}_1)\omega(\mathcal{E}_4)\} \\ &\quad - \tfrac{4(r-m(m-1))}{m-1}\{(2\beta - \tfrac{\delta(m-1)}{2})\omega(\mathcal{E}_1)g(\mathcal{E}_2,\mathcal{E}_4) \\ &\quad - (2\beta - \tfrac{\delta(m-1)}{2})\omega(\mathcal{E}_2)g(\mathcal{E}_1,\mathcal{E}_4)\} + \delta g(\nabla_{\mathcal{E}_1}Dr,\mathcal{E}_2)\omega(\mathcal{E}_4) \\ &\quad - \delta g(\nabla_{\mathcal{E}_2}Dr,\mathcal{E}_1)\omega(\mathcal{E}_4) + \delta g((\nabla_{\mathcal{E}_2}Dr),\mathcal{E}_4)\omega(\mathcal{E}_1) \\ &\quad - \delta g((\nabla_{\mathcal{E}_1}Dr),\mathcal{E}_4)\omega(\mathcal{E}_2). \end{aligned} \tag{41}$$

Let $\{\ell_1, \ell_2, \ell_3 \ldots \ell_{m-1}, \ell_m = \zeta\}$ be the orthonormal basis of the tangent space at each point of $(LPK)_m$. By putting $\mathcal{E}_1 = \mathcal{E}_4 = \ell_i$ and taking the summation over $i(1 \leq i \leq m)$, we find

$$\begin{aligned} 2(\pounds_{\mathcal{E}}S)(\mathcal{E}_2,\zeta) &= \{\frac{4\beta(m-2)}{m-1} - 3(m-1)\delta\}\mathcal{E}_2(r) \\ &\quad + 2(r-m(m-1))\{\frac{4\beta(m-2)}{m-1} - (m-1)\delta\}\omega(\mathcal{E}_2) \\ &\quad + \delta g(\nabla_\zeta Dr,\mathcal{E}_2) - \delta(\Delta r)\omega(\mathcal{E}_2), \end{aligned} \tag{42}$$

where Equation (24) is used and Δ appears for the Laplacian of g. By putting $\mathcal{E}_2 = \zeta$ in Equation (42), then using Equations (6), (24) and (26), we find

$$2(\pounds_{\mathcal{E}}S)(\zeta,\zeta) = -4(m-2)(r-m(m-1))\delta + \delta(\Delta r). \tag{43}$$

Taking the Lie derivative of Equation (19) along $\mathcal{E}$, we have

$$(\pounds_{\mathcal{E}}S)(\zeta,\zeta) = -2(m-1)\omega(\pounds_{\mathcal{E}}\zeta). \tag{44}$$

By putting $\mathcal{E}_2 = \zeta$ in Equation (30), we have

$$(\pounds_{\mathcal{E}}g)(\mathcal{E}_1,\zeta) = -\{2\beta(m-1) + 2\rho - \delta r\}\omega(\mathcal{E}_1). \tag{45}$$

The Lie derivative of $g(\zeta,\zeta) + 1 = 0$ leads to

$$(\pounds_{\mathcal{E}}g)(\zeta,\zeta) = -2\omega(\pounds_{\mathcal{E}}\zeta) \tag{46}$$

Now combining Equations (43)–(46), we deduce

$$\Delta r = \Psi, \tag{47}$$

where $\Psi = \frac{4(m-1)}{\delta}\{\beta(m-1) + \rho\} + 2(m-3)r - 4m(m-1)(m-2)$, $\delta \neq 0$.

An M of dimension m satisfies Poisson's equation if $\Delta\vartheta = \Psi$ holds for smooth functions ϑ and Ψ on M. Poisson's equation reduces to the Laplace equation if $\Psi = 0$.

This definition, together with Equation (47), states the following:

Theorem 1. *Let the metric of a ζ-conformally flat $(LPK)_m$ be an RYS $(g, \mathcal{E}, \rho, \beta, \delta)$. Then, the scalar curvature of $(LPK)_m$ satisfies the Poisson Equation (47).*

Corollary 1. *The scalar curvature of a ζ-conformally flat $(LPK)_m, m(>3)$ admitting an RYS $(g, \mathcal{E}, \rho, \beta, \delta)$ satisfies the Laplace equation if and only if $r = \frac{2(m-1)}{m-3}[m(m-2) - \frac{(\beta(m-1)+\rho)}{\delta}]$.*

Let a ζ-conformally flat $(LPK)_m, m(>3)$ admit an RYS $(g, \mathcal{E}, \rho, \beta, \delta)$. If r satisfies the Laplace equation, then $r = \frac{2(m-1)}{m-3}[m(m-2) - \frac{(\beta(m-1)+\rho)}{\delta}] =$ constant. This equation together with Remark 1 gives $\rho = (m-1)(\frac{\delta m}{2} - \beta)$. Thus, we state:

Corollary 2. *Let the metric of a ζ-conformally flat $(LPK)_m, m(>3)$ be an RYS $(g, \mathcal{E}, \rho, \beta, \delta)$ and suppose that its scalar curvature satisfies the Laplace equation. Then, we have:*

Values of β, δ	*Soliton type*	*Conditions for* $(g, \mathcal{E}, \rho, \beta, \delta)$ *to be expanding, shrinking or steady*
$\beta = 0, \delta = 1$	*Yamabe soliton*	$(g, \mathcal{E}, \rho, \beta, \delta)$ *is expanding.*
$\beta = 1, \delta = -1$	*Einstein soliton*	$(g, \mathcal{E}, \rho, \beta, \delta)$ *is shrinking.*

4. GRYS on a ζ-Conformally Flat $(LPK)_m$

Let the metric of a ζ-conformally flat $(LPK)_m$ be a GRYS. Then Equation (5) can be written as

$$\nabla_{\mathcal{E}_1} Du + \beta \mathcal{Q}\mathcal{E}_1 + (\rho - \frac{\delta r}{2})\mathcal{E}_1 = 0 \tag{48}$$

for all $\mathcal{E}_1$ on $(LPK)_m$, where D appears for the gradient operator of g. The covariant derivative of Equation (48) along $\mathcal{E}_2$ leads to

$$\nabla_{\mathcal{E}_2}\nabla_{\mathcal{E}_1} Du = -\beta\{(\nabla_{\mathcal{E}_2}\mathcal{Q})\mathcal{E}_1 + \mathcal{Q}(\nabla_{\mathcal{E}_2}\mathcal{E}_1)\} + \delta\frac{\mathcal{E}_2(r)}{2}\mathcal{E}_1 - (\rho - \frac{\delta r}{2})\nabla_{\mathcal{E}_2}\mathcal{E}_1. \tag{49}$$

Interchanging $\mathcal{E}_1$ and $\mathcal{E}_2$ in Equation (49), we have

$$\nabla_{\mathcal{E}_1}\nabla_{\mathcal{E}_2} Du = -\beta\{(\nabla_{\mathcal{E}_1}\mathcal{Q})\mathcal{E}_2 + \mathcal{Q}(\nabla_{\mathcal{E}_1}\mathcal{E}_2)\} + \delta\frac{\mathcal{E}_1(r)}{2}\mathcal{E}_2 - (\rho - \frac{\delta r}{2})\nabla_{\mathcal{E}_1}\mathcal{E}_2. \tag{50}$$

On account of Equations (49) and (50), we easily find

$$R(\mathcal{E}_1, \mathcal{E}_2)Du = \beta\{(\nabla_{\mathcal{E}_2}\mathcal{Q})\mathcal{E}_1 - (\nabla_{\mathcal{E}_1}\mathcal{Q})\mathcal{E}_2\} + \frac{\delta}{2}\{\mathcal{E}_1(r)\mathcal{E}_2 - \mathcal{E}_2(r)\mathcal{E}_1\}. \tag{51}$$

Taking the inner product of Equation (51) with $\mathcal{E}_3$, we have

$$\begin{aligned} g(R(\mathcal{E}_1, \mathcal{E}_2)Du, \mathcal{E}_3) &= \beta\{g((\nabla_{\mathcal{E}_2}\mathcal{Q})\mathcal{E}_1, \mathcal{E}_3) - g((\nabla_{\mathcal{E}_1}\mathcal{Q})\mathcal{E}_2, \mathcal{E}_3)\} \\ &\quad + \tfrac{\delta}{2}\{\mathcal{E}_1(r)g(\mathcal{E}_2, \mathcal{E}_3) - \mathcal{E}_2(r)g(\mathcal{E}_1, \mathcal{E}_3)\}. \end{aligned} \tag{52}$$

Let $\{\ell_1, \ell_2, \ell_3 \ldots \ell_{m-1}, \ell_m = \zeta\}$ be the orthonormal basis of the tangent space at each point of an $(LPK)_m$. By putting $\mathcal{E}_1 = \mathcal{E}_3 = \ell_i$ and taking the summation over $i(1 \le i \le m)$, we find

$$S(\mathcal{E}_2, Du) = \{\frac{\beta - (m-1)\delta}{2}\}\mathcal{E}_2(r). \tag{53}$$

From Equation (23), we can write

$$S(\mathcal{E}_2, Du) = (\frac{r}{m-1} - 1)\mathcal{E}_2(u) + (\frac{r}{m-1} - m)\omega(u)\zeta(u). \tag{54}$$

From Equations (53) and (54), we find

$$(\beta-(m-1)\delta)\mathcal{E}_2(r)=2(\frac{r}{m-1}-1)\mathcal{E}_2(u)+2(\frac{r}{m-1}-m)\omega(\mathcal{E}_2)\zeta(u). \quad (55)$$

By putting $\mathcal{E}_2=\zeta$ in Equation (55), then using Equations (6) and (24), we find

$$\zeta(u)=\{\frac{\beta-(m-1)\delta}{m-1}\}(r-m(m-1)). \quad (56)$$

By the use of Equation (56) in Equation (55), we have

$$\begin{aligned}(\beta-(m-1)\delta)\mathcal{E}_2(r) &= 2(\tfrac{r}{m-1}-1)\mathcal{E}_2(u)\\ &\quad +2(\beta-(m-1)\delta)(\tfrac{r}{m-1}-m)^2\omega(\mathcal{E}_2).\end{aligned} \quad (57)$$

Taking the covariant derivative of Equation (57) along $\mathcal{E}_1$, we find

$$\begin{aligned}&(\beta-(m-1)\delta)g(\nabla_{\mathcal{E}_1}Dr,\mathcal{E}_2)\\ &=\tfrac{2\mathcal{E}_1(r)}{m-1}\mathcal{E}_2(u)+2(\tfrac{r}{m-1}-1)g(\nabla_{\mathcal{E}_1}Du,\mathcal{E}_2)\\ &+\tfrac{2(r-m(m-1))(\beta-(m-1)\delta)}{(m-1)^2}\mathcal{E}_1(r)\omega(\mathcal{E}_2)\\ &-\tfrac{2(r-m(m-1))^2(\beta-(m-1)\delta)}{(m-1)^2}\{g(\mathcal{E}_1,\mathcal{E}_2)+\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\}\end{aligned} \quad (58)$$

Interchanging $\mathcal{E}_1$ and $\mathcal{E}_2$ in Equation (58), we have

$$\begin{aligned}&(\beta-(m-1)\delta)g(\nabla_{\mathcal{E}_2}Dr,\mathcal{E}_1)\\ &=\tfrac{2\mathcal{E}_2(r)}{m-1}\mathcal{E}_1(u)+2(\tfrac{r}{m-1}-1)g(\nabla_{\mathcal{E}_2}Du,\mathcal{E}_1)\\ &+\tfrac{2(r-m(m-1))(\beta-(m-1)\delta)}{(m-1)^2}\mathcal{E}_2(r)\omega(\mathcal{E}_1)\\ &-\tfrac{2(r-m(m-1))^2(\beta-(m-1)\delta)}{(m-1)^2}\{g(\mathcal{E}_1,\mathcal{E}_2)+\omega(\mathcal{E}_1)\omega(\mathcal{E}_2)\}\end{aligned} \quad (59)$$

The equality of Equations (58) and (59) yields

$$\begin{aligned}&(m-1)\mathcal{E}_1(r)\mathcal{E}_2(u)+2(r-m(m-1))(\beta-(m-1)\delta)\mathcal{E}_1(r)\omega(\mathcal{E}_2)\\ &-(m-1)\mathcal{E}_2(r)\mathcal{E}_1(u)-2(r-m(m-1))(\beta-(m-1)\delta)\mathcal{E}_2(r)\omega(\mathcal{E}_1)=0,\end{aligned} \quad (60)$$

from which, by substituting $\mathcal{E}_2=\zeta$ and following Equations (6), (24) and (56), from Equation (60), we infer

$$\begin{aligned}&(r-m(m-1))\{(\beta-(m-1)\delta)\mathcal{E}_1(r)+2(m-1)\mathcal{E}_1(u)\\ &\quad +4(r-m(m-1))(\beta-(m-1)\delta)\omega(\mathcal{E}_1)\}=0.\end{aligned} \quad (61)$$

Thus, we have either $(\beta-(m-1)\delta)\mathcal{E}_1(r)+2(m-1)\mathcal{E}_1(u)+4(r-m(m-1))(\beta-(m-1)\delta)\omega(\mathcal{E}_1)=0$, or $r=m(m-1)$. For the second case $r=m(m-1)$, Equations (56) and (57) yield that u is constant and hence the GRYS on a ζ-conformally flat $(LPK)_m$ is trivial. Moreover, a ζ-conformally flat $(LPK)_m$ is an Einstein manifold and its scalar curvature is constant. On the other hand, if r is non-constant, that is, $r\neq m(m-1)$ and $(\beta-(m-1)\delta)\mathcal{E}_1(r)=-2(m-1)\mathcal{E}_1(u)-4(r-m(m-1))(\beta-(m-1)\delta)\omega(\mathcal{E}_1)$, in view of Equation (57), it becomes

$$(r+(m-1)(m-2))\{(m-1)\mathcal{E}_1(u)+(r-m(m-1))(\beta-(m-1)\delta)\omega(\mathcal{E}_1)\}=0. \quad (62)$$

From Equation (62), it follows that either $(m-1)\mathcal{E}_1(u)+(r-m(m-1))(\beta-(m-1)\delta)\omega(\mathcal{E}_1)=0$ or $r=-(m-1)(m-2)=$ constant, which is inadmissible (by hypothesis).

Thus, we have,

$$\mathcal{E}_1(u) = -\frac{1}{m-1}(r - m(m-1))(\beta - (m-1)\delta)\omega(\mathcal{E}_1) \Longleftrightarrow$$

$$Du = -\frac{1}{m-1}(r - m(m-1))(\beta - (m-1)\delta)\zeta = -\zeta(u)\zeta. \quad (63)$$

This shows that the gradient of u is pointwise collinear with the velocity vector field ζ.

Now, taking the covariant derivative of Equation (63) with respect to $\mathcal{E}_1$, then using Equations (13) and (48), we find

$$\beta \mathcal{Q}\mathcal{E}_1 + (\rho - \frac{\delta r}{2})\mathcal{E}_1 = \mathcal{E}_1(\zeta(u))\zeta - \zeta(u)(\mathcal{E}_1 + \omega(\mathcal{E}_1)\zeta), \quad (64)$$

which forms a perfect fluid spacetime.

Now, by replacing $\mathcal{E}_1$ by ζ in Equation (64), then using Equations (6), (20) and (56), we find

$$\rho = \frac{(4\beta - 3(m-1)\delta)r}{2(m-1)} - \{(3m-1)\beta - 2m(m-1)\delta\}, \quad (65)$$

which yields that the scalar curvature of the $(LPK)_m$ is constant. This contradicts our hypothesis that r is non-constant. Thus, the only possibility is $r = m(m-1)$. By considering the above facts, we have the following results:

Theorem 2. *An $(LPK)_m$ admitting a GRYS is an Einstein spacetime and the GRYS is trivial.*

Corollary 3. *If the metric of an $(LPK)_m$ is a gradient Ricci soliton, then the $(LPK)_m$ has a constant scalar curvature.*

Equations (23) and (64) together with Theorem 2 reduce to

$$\mathcal{Q} = (m-1)I, \quad (66)$$

and

$$\beta\mathcal{Q} + \rho - \frac{\delta m(m-1)}{2} = \omega \otimes \zeta. \quad (67)$$

The above Equation (66) and Equation (67) lead to $\rho = \frac{m-1}{2}\{\delta m - 2\beta - \frac{2}{m-1}\}$. Thus, the GRYS on the manifold is expanding, shrinking or steady if $m\delta > 2\beta + \frac{2}{m-1}, m\delta < 2\beta + \frac{2}{m-1}$ or $m\delta = 2\beta + \frac{2}{m-1}$. Now, we state:

Corollary 4. *A GRYS on an $(LPK)_m$ is either expanding or shrinking or steady if either $m\delta > 2\beta + \frac{2}{m-1}$, $m\delta < 2\beta + \frac{2}{m-1}$ or $m\delta = 2\beta + \frac{2}{m-1}$.*

Corollary 5. *Let the metric of an $(LPK)_m$ be a GRYS $(g, Du, \rho, \beta, \delta)$. Then, we have*

Values of β, δ	*Soliton type*	*Soliton constant*	*$(g, Du, \rho, \beta, \delta)$ to be expanding, shrinking or steady*
$\beta = 1, \delta = 0$	*Ricci soliton*	$\rho = -m$	*$(g, Du, \rho, \beta, \delta)$ is shrinking*
$\beta = 0, \delta = 1$	*Yamabe soliton*	$\rho = \frac{(m-2)(m+1)}{2}$, *provided* $m > 2$	*$(g, Du, \rho, \beta, \delta)$ is expanding*
$\beta = 1, \delta = -1$	*Einstein soliton*	$\rho = -\frac{1}{2}m(m-3)$, *provided* $m > 3$	*$(g, Du, \rho, \beta, \delta)$ is shrinking*
$\beta = 1, \delta = -2\rho$	*ρ-Einstein soliton*	$\rho = -m(m-1)[\varrho + \frac{1}{m-1}]$	*(i) $(g, Du, \rho, \beta, \delta)$ is shrinking if $\varrho(m-1)+1 > 0$* *(ii) $(g, Du, \rho, \beta, \delta)$ is expanding if $\varrho(m-1)+1 < 0$* *(iii) $(g, Du, \rho, \beta, \delta)$ is shrinking if $\varrho(m-1)+1 = 0$*

5. Example of Lorentzian Para-Kenmotsu Manifold

Let $M^4 = \{(x,y,z,\kappa) \in R^4 : \kappa > 0\}$ be a manifold of dimension four, where (x,y,z,κ) are the standard coordinates in R^4. Let ℓ_1, ℓ_2, ℓ_3 and ℓ_4 be the vector fields on M^4 given by

$$\ell_1 = \kappa\frac{\partial}{\partial x}, \ \ \ell_2 = \kappa\frac{\partial}{\partial y}, \ \ \ell_3 = \kappa\frac{\partial}{\partial z}, \ \ \ell_4 = \kappa\frac{\partial}{\partial \kappa} = \zeta, \tag{68}$$

which are linearly independent at each point of M^4. Let g be the Lorentzian metric defined by

$$g(\ell_i,\ell_j) = \begin{cases} 1, \ \ 1 \le i = j \le 3, \\ -1, \ \ i = j = 4, \\ 0, \ \ 1 \le i \ne j \le 4. \end{cases} \tag{69}$$

Let the one-form ω be defined by $\omega(\mathcal{E}_1) = g(\mathcal{E}_1,\ell_4) = g(\mathcal{E}_1,\zeta)$ for all $\mathcal{E}_1 \in \mathfrak{X}(M^4)$, and let f be the $(1,1)$-tensor field defined by

$$f\ell_1 = -\ell_1, \ \ f\ell_2 = -\ell_2, \ \ f\ell_3 = -\ell_3, \ \ f\ell_4 = 0. \tag{70}$$

By using the linearity of f and g, we have

$$\omega(\zeta) = g(\zeta,\zeta) = -1, \ \ f^2\mathcal{E}_1 = \mathcal{E}_1 + \omega(\mathcal{E}_1)\zeta \ \text{ and } \ g(f\mathcal{E}_1, f\mathcal{E}_2) = g(\mathcal{E}_1,\mathcal{E}_2) + \omega(\mathcal{E}_1)\omega(\mathcal{E}_2) \tag{71}$$

for all $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{X}(M^4)$. Thus, for $\ell_4 = \zeta$, the structure (f,ζ,ω,g) defines a Lorentzian almost paracontact metric structure on M^4.

Then, we have

$$[\ell_i,\ell_4] = \begin{cases} -\ell_i, \quad \text{for} \quad 1 \le i \le 3, \\ 0 \quad \text{otherwise}. \end{cases} \tag{72}$$

By using Koszul's formula, we can easily find

$$\begin{aligned} &\nabla_{\ell_1}\ell_1 = -\ell_4, \ \ \nabla_{\ell_1}\ell_2 = 0, \ \ \nabla_{\ell_1}\ell_3 = 0, \ \ \nabla_{\ell_1}\ell_4 = -\ell_1, \\ &\nabla_{\ell_2}\ell_1 = 0, \ \ \nabla_{\ell_2}\ell_2 = -\ell_4, \ \ \nabla_{\ell_2}\ell_3 = 0, \ \ \nabla_{\ell_2}\ell_4 = -\ell_2, \\ &\nabla_{\ell_3}\ell_1 = 0, \ \ \nabla_{\ell_3}\ell_2 = 0, \ \ \nabla_{\ell_3}\ell_3 = -\ell_4, \ \ \nabla_{\ell_3}\ell_4 = -\ell_3, \\ &\nabla_{\ell_4}\ell_1 = 0, \ \ \nabla_{\ell_4}\ell_2 = 0, \ \ \nabla_{\ell_4}\ell_3 = 0, \ \ \nabla_{\ell_4}\ell_4 = 0. \end{aligned} \tag{73}$$

Moreover, one can easily verify that

$$\nabla_{\mathcal{E}_1}\zeta + \mathcal{E}_1 + \omega(\mathcal{E}_1)\zeta = 0 \quad \text{and} \quad (\nabla_{\mathcal{E}_1} f)\mathcal{E}_2 + g(f\mathcal{E}_1,\mathcal{E}_2)\zeta + \omega(\mathcal{E}_2) f\mathcal{E}_1 = 0. \tag{74}$$

Therefore, M^4 is an LP-Kenmotsu manifold.

The non-vanishing components of $\mathcal{R}$ are obtained as follows:

$$\begin{aligned} &\mathcal{R}(\ell_1,\ell_2)\ell_1 = -\ell_2, \ \ \mathcal{R}(\ell_1,\ell_2)\ell_2 = \ell_1, \ \ \mathcal{R}(\ell_1,\ell_3)\ell_1 = -\ell_3, \ \ \mathcal{R}(\ell_1,\ell_3)\ell_3 = \ell_1, \\ &\mathcal{R}(\ell_1,\ell_4)\ell_1 = -\ell_4, \ \ \mathcal{R}(\ell_1,\ell_4)\ell_4 = -\ell_1, \ \ \mathcal{R}(\ell_2,\ell_3)\ell_2 = -\ell_3, \ \ \mathcal{R}(\ell_2,\ell_3)\ell_3 = \ell_2, \\ &\mathcal{R}(\ell_2,\ell_4)\ell_2 = -\ell_4, \ \ \mathcal{R}(\ell_2,\ell_4)\ell_4 = -\ell_2, \ \ \mathcal{R}(\ell_3,\ell_4)\ell_3 = -\ell_4, \ \ \mathcal{R}(\ell_3,\ell_4)\ell_4 = -\ell_3. \end{aligned} \tag{75}$$

Moreover, we calculate $\mathcal{S}$ as follows:

$$S(\ell_1,\ell_1) = 3 = S(\ell_2,\ell_2) = S(\ell_3,\ell_3), \quad S(\ell_4,\ell_4) = -3. \tag{76}$$

Therefore, we have

$$r = S(\ell_1,\ell_1) + S(\ell_2,\ell_2) + S(\ell_3,\ell_3) - S(\ell_4,\ell_4) = 12. \tag{77}$$

Let ℓ_1, ℓ_2, and ℓ_3 be the vector fields given by

$$\begin{cases} \mathcal{E}_1 = a_1\ell_1 + a_2\ell_2 + a_3\ell_3 + a_4\ell_4, \\ \mathcal{E}_2 = b_1\ell_1 + b_2\ell_2 + b_3\ell_3 + b_4\ell_4, \\ \mathcal{E}_3 = c_1\ell_1 + c_2\ell_2 + c_3\ell_3 + c_4\ell_4, \end{cases} \tag{78}$$

where $a_i, b_i, c_i \in \mathbb{R}$, for all $i = 1,2,3,4$.

Putting $\mathcal{E}_3 = \zeta$ and $n = 4$ in Equation (22), we have

$$\begin{aligned} \mathcal{C}(\mathcal{E}_1,\mathcal{E}_2)\zeta = \mathcal{R}(\mathcal{E}_1,\mathcal{E}_2)\zeta - \tfrac{1}{2}\{S(\mathcal{E}_2,\zeta)\mathcal{E}_1 - S(\mathcal{E}_1,\zeta)\mathcal{E}_2 + g(\mathcal{E}_2,\zeta)\mathcal{Q}\mathcal{E}_1 \\ -g(\mathcal{E}_1,\zeta)\mathcal{Q}\mathcal{E}_2\} + \tfrac{r}{6}\{g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1 - g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2\}. \end{aligned} \tag{79}$$

By using the above listed values of $\mathcal{R}$, $\mathcal{S}$ and r, we have

$$\mathcal{R}(\mathcal{E}_1,\mathcal{E}_2)\zeta = (a_4b_1 - a_1b_4)\ell_1 + (a_4b_2 - a_2b_4)\ell_2 + (a_4b_3 - a_3b_4)\ell_3, \tag{80}$$

$$\mathcal{S}(\mathcal{E}_2,\zeta)\mathcal{E}_1 = -3(a_1b_4\ell_1 + a_2b_4\ell_2 + a_3b_4\ell_3 + a_4b_4\ell_4), \tag{81}$$

$$\mathcal{S}(\mathcal{E}_1,\zeta)\mathcal{E}_2 = -3(b_1a_4\ell_1 + b_2a_4\ell_2 + b_3a_4\ell_3 + b_4a_4\ell_4), \tag{82}$$

$$g(\mathcal{E}_1,\zeta) = -a_4, \;\; g(\mathcal{E}_2,\zeta) = -b_4, \tag{83}$$

$$g(\mathcal{E}_2,\zeta)\mathcal{Q}\mathcal{E}_1 = -3(a_1b_4\ell_1 + a_2b_4\ell_2 + a_3b_4\ell_3 - a_4b_4\ell_4), \tag{84}$$

$$g(\mathcal{E}_1,\zeta)\mathcal{Q}\mathcal{E}_2 = -3(b_1a_4\ell_1 + b_2a_4\ell_2 + b_3a_4\ell_3 - b_4a_4\ell_4). \tag{85}$$

It can be easily seen that $C(\mathcal{E}_1,\mathcal{E}_2)\zeta = 0$. Thus, an $(LPK)_4$ is ζ-conformally flat.

Now, by taking $Du = (\ell_1u)\ell_1 + (\ell_2u)\ell_2 + (\ell_3u)\ell_3 + (\ell_4u)\ell_4$, we have

$$\nabla_{\ell_1}Du = (\ell_1(\ell_1u) - (\ell_4u))\ell_1 + (\ell_1(\ell_2u))\ell_2 + (\ell_1(\ell_3u))\ell_3 + (\ell_1(\ell_4u) - (\ell_1u))\ell_4, \tag{86}$$

$$\nabla_{\ell_2}Du = (\ell_2(\ell_1u))\ell_1 + (\ell_2(\ell_2u) - (\ell_4u))\ell_2 + (\ell_2(\ell_3u))\ell_3 + (\ell_2(\ell_4u) - (\ell_2u))\ell_4, \tag{87}$$

$$\nabla_{\ell_3}Du = (\ell_3(\ell_1u))\ell_1 + (\ell_3(\ell_2u))\ell_2 + (\ell_3(\ell_3u) - (\ell_4u))\ell_3 + (\ell_3(\ell_4u) - (\ell_3u))\ell_4, \tag{88}$$

$$\nabla_{\ell_4}Du = (\ell_4(\ell_1u))\ell_1 + (\ell_4(\ell_2u))\ell_2 + (\ell_4(\ell_3u))\ell_3 + (\ell_4(\ell_4u))\ell_4. \tag{89}$$

Thus, by virtue of Equation (48), we obtain

$$\begin{cases} \ell_1(\ell_1 u) - \ell_4 u = -(\rho + 3\beta - 6\delta), \\ \ell_2(\ell_2 u) - \ell_4 u = -(\rho + 3\beta - 6\delta), \\ \ell_3(\ell_3 u) - \ell_4 u = -(\rho + 3\beta - 6\delta), \\ \ell_4(\ell_4 u) = -(\rho + 3\beta - 6\delta), \\ \ell_1(\ell_2 u) = \ell_1(\ell_3 u) = 0, \\ \ell_2(\ell_1 u) = \ell_2(\ell_3 u) = 0, \\ \ell_3(\ell_1 u) = \ell_3(\ell_2 u) = 0, \\ \ell_4(\ell_1 u) = \ell_4(\ell_2 u) = \ell_4(\ell_3 u) = 0, \\ \ell_1(\ell_4 u) - (\ell_1 u) = \ell_2(\ell_4 u) - (\ell_2 u) = 0, \\ \ell_3(\ell_4 u) - (\ell_3 u) = 0. \end{cases} \tag{90}$$

Thus, the equations in Equation (90) are, respectively, equal to

$$k^2 \frac{\partial^2 u}{\partial x^2} - k \frac{\partial u}{\partial k} = -(\rho + 3\beta - 6\delta), \tag{91}$$

$$k^2 \frac{\partial^2 u}{\partial y^2} - k \frac{\partial u}{\partial k} = -(\rho + 3\beta - 6\delta), \tag{92}$$

$$k^2 \frac{\partial^2 u}{\partial z^2} - k \frac{\partial u}{\partial k} = -(\rho + 3\beta - 6\delta), \tag{93}$$

$$k^2 \frac{\partial^2 u}{\partial k^2} + k \frac{\partial u}{\partial k} = -(\rho + 3\beta - 6\delta), \tag{94}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial x \partial z} = 0, \tag{95}$$

$$k^2 \frac{\partial^2 u}{\partial k \partial x} + k \frac{\partial u}{\partial x} = k^2 \frac{\partial^2 u}{\partial k \partial y} + k \frac{\partial u}{\partial y} = k^2 \frac{\partial^2 u}{\partial k \partial z} + k \frac{\partial u}{\partial z} = 0. \tag{96}$$

$$k^2 \frac{\partial^2 u}{\partial x \partial k} - k \frac{\partial u}{\partial x} = k^2 \frac{\partial^2 u}{\partial y \partial k} - k \frac{\partial f}{\partial y} = k^2 \frac{\partial^2 u}{\partial z \partial k} - k \frac{\partial u}{\partial z} = 0. \tag{97}$$

From the above equations, it is observed that u is constant for $\rho = -3\beta + 6\delta$. Hence, Equation (48) is satisfied. Thus, g is a GRYS with the soliton vector field $\mathcal{E} = Du$, where u is constant and $\rho = -3\beta + 6\delta$. This verifies Theorem 2 and Corollary 3.

6. Conclusions

The Ricci flow has been applied as a tool to prove the Poincaré conjecture, geometrization conjecture, differentiable sphere conjecture, uniformalization theorem, etc. It can also be applied to study cancer invasion, avascular tumor growth and decay control, brain surface conformal parameterization, medical imaging (such as the parameterization of a surface, the matching of a surface, splines of a manifold and the formation of a geometric structure on general surfaces), computer graphics, geometric modeling, computer vision, wireless sensor networking, mathematics and physics, etc. It is well known that the Laplace operator is used to study celestial mechanics and measure the flux density of the gradient flow of a function [37]. Several differential equations are expressed in terms of the Laplacian, used to explain various physical problems. The Laplacian appears in problems of computer vision and image processing, electrical and gravitational potentials, the diffusion equation for fluid and heat flow, the de Rham cohomology, the Hodge theory, etc. This manuscript dealt with the study of the Laplacian, and the equations of Poisson and Laplace. We also addressed the existence of a proper gradient Ricci–Yamabe soliton on an $(LPK)_m$.

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Optimal Inequalities for Hemi-Slant Riemannian Submersions

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Abstract: In the present paper, we establish some basic inequalities involving the Ricci and scalar curvature of the vertical and the horizontal distributions for hemi-slant submersions having the total space a complex space form. We also discuss the equality case of the obtained inequalities and provide illustrative examples.

Keywords: Chen–Ricci inequality; Riemannian submersion; hemi-slant submersion; complex space form; Kähler manifold

MSC: 53C15; 53B20

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1. Introduction

As a dual notion to the isometric immersions, O'Neill and Gray introduced independently the concept of Riemannian submersions in [1,2], respectively. Riemannian submersions play an important role in mathematical and theoretical physics, especially due to their usage in the superstring, Yang–Mills, Kaluza–Klein and supergravity theories [3–8]. For more information on Riemannian submersions, we refer to the monographs [9,10].

In [11], Taştan, Şahin and Yanan introduced and investigated hemi-slant submersions from almost Hermitian manifolds onto Riemannian manifolds. This class of submersions appears as a natural generalization of invariant, anti-invariant, semi-invariant and slant submersions, four families of Riemannian submersions with remarkable geometric properties thoroughly investigated by Şahin [12–15]. Later, these submersions were studied for different ambient spaces by various authors who obtained several results regarding their geometry (see, e.g., [16–23]).

One of the most important curvature invariants for a Riemannian manifold (M, g) was introduced by Chen [24] as follows:

$$\delta_M = \tau(p) - \inf(K)(p), \tag{1}$$

where $\tau(p)$ is scalar curvature of M and

$$\inf(K)(p) = \inf\{K(\Pi) : \Pi \text{ is a plane section of } TpM\}. \tag{2}$$

In ref. [25], B.-Y. Chen established a general optimal inequality involving the intrinsic invariant δ_M and the squared mean curvature of a submanifold M isometrically immersed in a real space form $R(c)$ of constant sectional curvature c. This result gave rise to a whole theory, known as the theory of Chen's invariants, which gained an exponential development in the following years (see the monograph [26] and the recent articles [27–32], as well as the references cited therein). The main purpose of this new theory is to prove answers to a fundamental problem in the geometry of submanifolds, namely "*establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold*" [26]. Recently, Chen-like inequalities have been investigated in the setting of Riemannian submersions (see, e.g., [19,33–36]).

Motivated by the studies indicated above, we obtain in this work various Chen-like inequalities for hemi-slant Riemannian submersions from complex space forms onto Riemannian manifolds and discuss the equality case of the obtained inequalities. The paper is organized as follows. In Section 2, we recall the definition and some fundamental properties of hemi-slant submersions. In Section 3, we derive the main inequalities: we first establish a Chen-like inequality involving the Ricci curvature and then state a Chen–Ricci inequality for the vertical and the horizontal distributions of hemi-slant Riemannian submersions with total space a complex space form, and with base an arbitrary Riemannian manifold. We also discuss the equality case of the obtained inequalities. In Section 4 , we provide examples of hemi-slant Riemannian submersions to show that the equality cases of the main inequalities can be attained.

2. Hemi-Slant Riemannian Submersions

In this study, manifolds, mappings, vector fields, sections, and so on, will always be supposed of class C^∞. We first recall the following definition.

Definition 1 ([11])**.** *Let (M, J, g) be an almost Hermitian manifold and (N, g_N) be a Riemannian manifold. A Riemannian submersion $\sigma : (M, J, g) \to (N, g_N)$ is said to be a hemi-slant submersion if there is a distribution $\mathcal{D}^\perp \subset \ker \sigma_*$ such that*

$$\ker \sigma_* = \mathcal{D}^\perp \oplus \mathcal{D}^\theta, \;\; J\left(\mathcal{D}^\perp\right) \subseteq (ker\sigma_*)^\perp,$$

and the angle $\theta = \theta(X)$ between JX and the space $\left(\mathcal{D}^\theta\right)_q$ is constant for nonzero $X \in \left(\mathcal{D}^\theta\right)_q$ and $q \in M$, where $\mathcal{D}^\theta$ is the orthogonal complement of $\mathcal{D}^\perp$ in $ker\sigma_*$. In this case, θ is called the hemi-slant angle of σ. Moreover, the hemi-slant submersion σ is called proper if $\mathcal{D}^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

Throughout this paper, we will assume all horizontal vector fields as basic vector fields. Let $\sigma : (M, g, J) \to (N, g_N)$ be a hemi-slant submersion. For $U \in ker\sigma_*$, we obtain

$$JU = \phi U + \omega U, \tag{3}$$

where $\phi U \in \ker \sigma_*$ and $\omega U \in (\ker \sigma_*)^\perp$. For $Z \in (\ker \sigma_*)^\perp$, we obtain

$$JZ = \mathcal{B}Z + \mathcal{C}Z \tag{4}$$

where $\mathcal{B}Z \in \ker \sigma_*$ and $\mathcal{C}Z \in (\ker \sigma_*)^\perp$. We have

$$(\ker \sigma_*)^\perp = J\mathcal{D}^\perp \oplus \omega\mathcal{D}^\theta \oplus \mu,$$

where μ is the orthogonal complement of $J\mathcal{D}^\perp \oplus \omega\mathcal{D}^\theta$ in $(ker\sigma_*)^\perp$ and is invariant under J. Let us consider the O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ given by [1]

$$\mathcal{T}_\xi \eta = \mathfrak{v}\nabla_{\mathfrak{v}\xi}\mathfrak{h}\eta + \mathfrak{h}\nabla_{\mathfrak{v}\xi}\mathfrak{v}\eta, \;\; \mathcal{A}_\xi \eta = \mathfrak{v}\nabla_{\mathfrak{h}\xi}\mathfrak{h}\eta + \mathfrak{h}\nabla_{\mathfrak{h}\xi}\mathfrak{v}\eta \tag{5}$$

for any vector fields ξ and η on M, where $\mathfrak{v}$ and $\mathfrak{h}$ denote the vertical and horizontal projections of the submersion, and ∇ is the Levi-Civita connection of g. On the other hand, for any $X, Y \in \Gamma((ker\sigma_*)^\perp)$ and $V, W \in \Gamma(ker\sigma_*)$, from (5), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{6}$$

$$\nabla_V X = \mathcal{T}_V X + \mathfrak{h}\nabla_V X, \tag{7}$$

$$\nabla_X V = \mathcal{A}_X V + \mathfrak{v}\nabla_X V, \tag{8}$$

$$\nabla_X Y = \mathfrak{h}\nabla_X Y + \mathcal{A}_X Y, \tag{9}$$

where $\hat{\nabla}_V W = \mathfrak{v}\nabla_V W$.

We denote by R, R', $\hat{R}$ and R^* the Riemannian curvature tensors of Riemannian manifolds M, N, the vertical distribution $ker\sigma_*$ and the horizontal distribution $(ker\sigma_*)^\perp$, respectively. Then, the Gauss–Codazzi type equations are given by [1]

$$R(U,V,F,W) = \hat{R}(U,V,F,W) + g(\mathcal{T}_U W, \mathcal{T}_V F) - g(\mathcal{T}_V W, \mathcal{T}_U F) \tag{10}$$

$$\begin{aligned} R(X,Y,Z,H) = R^*(X,Y,Z,H) - 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) \\ + g(\mathcal{A}_Y Z, \mathcal{A}_X H) - g(\mathcal{A}_X Z, \mathcal{A}_Y H) \end{aligned} \tag{11}$$

$$\begin{aligned} R(X,V,Y,W) = g((\nabla_X \mathcal{T})(V,W),Y) + g((\nabla_V \mathcal{A})(X,Y),W) \\ - g(\mathcal{T}_V X, \mathcal{T}_W Y) + g(\mathcal{A}_Y W, \mathcal{A}_X V) \end{aligned} \tag{12}$$

where

$$\sigma_*(R^*(X,Y)Z)) = R'(\sigma_* X, \sigma_* Y)\sigma_* Z \tag{13}$$

for all $U, V, F, W \in \Gamma(ker\sigma_*)$ and $X, Y, Z, H \in \Gamma((ker\sigma_*)^\perp)$.

Moreover, the mean curvature vector field $\mathcal{H}$ of any fiber of Riemannian submersion σ is given by

$$\mathcal{H} = \frac{1}{r}\sum_{j=1}^{r} \mathcal{T}_{U_j} U_j, \tag{14}$$

where $\{U_1, \ldots, U_r\}$ is an orthonormal basis of the vertical distribution $ker\sigma_*$. Furthermore, σ has totally geodesic fibers if $\mathcal{T}$ vanishes identically.

Let M be an almost Hermitian manifold with an almost complex structure J and a Hermitian metric g. If J is parallel with respect to the Levi–Civita connection ∇ on M, that is

$$(\nabla_X J)Y = 0.$$

for all vector fields X and Y on M, then (M, J, g, ∇) is called a Kähler manifold. A complete and simply connected Kähler manifold M is said to be a complex space form if it has constant holomorphic sectional curvature c. In this case, the complex space form is denoted by $M(c)$. The curvature tensor of the complex space form $M(c)$ is given by

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY + 2g(X,JY)JZ\} \end{aligned} \tag{15}$$

for any $X, Y, Z \in \Gamma(TM)$.

The following theorem gives us a characterization of hemi-slant submersions (see [11]).

Theorem 1. *Let σ be a Riemannian submersion from an almost Hermitian manifold (M, g, J) onto a Riemannian manifold (N, g_N). Then, σ is a hemi-slant Riemannian submersion with hemi-slant angle θ if and only if there exist a distribution $\mathcal{D} \subset \ker\sigma_*$ and a constant $\lambda \in [0,1]$ such that*

(i) $\mathcal{D} = \{U \in \ker\sigma_* | \phi^2 U = -\lambda U\}$;

(ii) $\phi V = 0$, *for all* $V \in \mathcal{D}^{\perp}$, *where* $\mathcal{D}^{\perp}$ *is the orthogonal complement of* $\mathcal{D}$ *in* $\ker \sigma_*$.
Furthermore, we have $\cos^2\theta = \lambda$.

By virtue of (3) and (4), we have the following result.

Lemma 1. *Let* $(M(c), g)$, (N, g_N) *be a complex space form and a Riemannian manifold, respectively. If* $\sigma : M(c) \to N$ *is a hemi-slant Riemannian submersion, then the following relations are valid*

$$g(\phi U, \phi V) = \cos^2\theta g(U, V),$$
$$g(\omega U, \omega V) = \sin^2\theta g(U, V),$$

for any $U, V \in \Gamma(\mathcal{D}^{\theta})$.

3. Chen–Ricci Inequality

In the present section, we aim to obtain some inequalities involving the Ricci curvature and the scalar curvature on the vertical and horizontal distributions for hemi-slant Riemannian submersions from a complex space form to a Riemannian manifold. We will also discuss the equality cases of these inequalities.

Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N) and $\dim(\ker \sigma_*) = r = k_1 + 2k_2$. For every $q \in M$, we consider

$$\{U_1, \dots, U_{k_1}, U_{k_1+1}, U_{k_1+2}, \dots, U_{k_1+2k_2-1}, U_{k_1+2k_2}\}$$

an orthonormal basis of $(\ker \sigma_*)$ and $\{X_1 \dots, X_n\}$ an orthonormal basis of $(\ker \sigma_*)^{\perp}$, respectively, such that $\{U_1, U_2, \dots, U_{k_1}\}$ is an orthonormal basis of $D^{\perp}$, while

$$\{U_{k_1+1}, U_{k_1+2} = \sec\theta\phi U_{k_1+1}, \dots, U_{k_1+2k_2-1}, U_{k_1+2k_2} = \sec\theta\phi U_{k_1+2k_2-1}\}$$

is an orthonormal basis of D^{θ}. We will call this basis an adapted hemi-slant basis of $(\ker \sigma_*)$. Obviously, we have

$$g^2(JU_i, U_{i+1}) = \begin{cases} 0, & \text{for } i \in \{1, \dots, k_1 - 1\}, \\ cos^2\theta, & \text{for } i \in \{k_1 + 1, \dots, k_1 + 2k_2 - 1\} \end{cases}$$

and

$$\sum_{i,j=1}^{r} g^2(JU_i, U_j) = 2k_2 cos^2\theta. \tag{16}$$

Besides from (10), (11) and (15), we have

$$\begin{aligned} \hat{R}(U, V, F, W) = \frac{c}{4}\{&g(V, F)g(U, W) - g(U, F)g(V, W) + g(U, JF)g(JV, W) \\ &- g(V, JF)g(JU, W) + 2g(U, JV)g(JF, W)\} \\ &- g(\mathcal{T}_U W, \mathcal{T}_V F) + g(\mathcal{T}_V W, \mathcal{T}_U F), \end{aligned} \tag{17}$$

for all vector fields $U, V, F, W \in \Gamma(\ker \sigma_*)$ and

$$\begin{aligned} R^*(X, Y, Z, H) = \frac{c}{4}\{&g(Y, Z)g(X, H) - g(X, Z)g(Y, H) + g(JY, Z)g(JX, H) \\ &- g(JX, Z)g(JY, H) + 2g(X, JY)g(JZ, H)\} + 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) \\ &- g(\mathcal{A}_Y Z, \mathcal{A}_X H) + g(\mathcal{A}_X Z, \mathcal{A}_Y H) \end{aligned} \tag{18}$$

for all vector fields $X, Y, Z, H \in \Gamma((\ker \sigma_*)^{\perp})$.

Theorem 2. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\widehat{Ric}(U) \geq \frac{c}{4}(r - 1 + 3\cos^2\theta) - rg(\mathcal{T}_U U, \mathcal{H}) \tag{19}$$

for a unit vector field $U \in \Gamma(\mathcal{D}^\theta)$, where r is the dimension of the vertical distribution. The equality case of (19) *holds identically for any unit vector field $U \in \Gamma(\mathcal{D}^\theta)$ if and only if each fiber is totally geodesic.*

Proof. From (17), we obtain

$$\widehat{Ric}(U) = \frac{c}{4}\left[(r-1)g(U,U) + 3\sum_{i=1}^{r} g^2(U, JU_i)\right] - rg(\mathcal{T}_U U, \mathcal{H}) + \sum_{i=1}^{r} \|\mathcal{T}_U U_i\|^2 \tag{20}$$

where

$$\widehat{Ric}(U) = \sum_{i=1}^{r} \hat{R}(U, U_i, U_i, U). \tag{21}$$

If $U \in \Gamma(\mathcal{D}^\theta)$, then choosing an adapted hemi-slant basis

$$\{U_1, \ldots, U_{k_1}, U_{k_1+1}, U_{k_1+2}, \ldots, U_{k_1+2k_2-1}, U_{k_1+2k_2}\}$$

of $\ker\sigma_*$, one derives

$$\sum_{i=1}^{r} g^2(U, JU_i) = cos^2\theta. \tag{22}$$

Using last equation in (20), we derive (19). On the other hand, it is clear that the equality case of (19) holds identically for any unit vector field $U \in \Gamma(\mathcal{D}^\theta)$ if and only if

$$\mathcal{T}_U U_i = 0,\ i = 1, \ldots, r$$

which means that the fibers are totally geodesic (see [9]). □

In a similar way, using an adapted hemi-slant basis of $\ker\sigma_*$, we obtain the following results.

Theorem 3. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\widehat{Ric}(U) \geq \frac{c}{4}(r - 1) - rg(\mathcal{T}_U U, \mathcal{H}) \tag{23}$$

for a unit vector field $U \in \Gamma(\mathcal{D}^\perp)$. The equality case of (23) *holds identically for any unit vector field $U \in \Gamma(\mathcal{D}^\perp)$ if and only if each fiber is totally geodesic.*

Theorem 4. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\widehat{Ric}(U, V) = \frac{c}{4}(r - 1 + 3\cos^2\theta)g(U, V) - rg(\mathcal{T}_U V, \mathcal{H}) + \sum_{i=1}^{r} g(\mathcal{T}_{U_i} V, \mathcal{T}_U U_i) \tag{24}$$

for $U, V \in \Gamma(\mathcal{D}^\theta)$.

Theorem 5. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then we have*

$$2\widehat{r} = \frac{c}{4}\{r^2 - r + 6k_2 \cos^2\theta\} - r^2\|\mathcal{H}\|^2 + \sum_{i=1}^{r}\|\mathcal{T}_{U_i}U_i\|^2. \tag{25}$$

As a consequence of the last theorem, we derive the following.

Corollary 1. *Let $\pi : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$2\widehat{r} \geq \frac{c}{4}\{r^2 - r + 6k_2 \cos^2\theta\} - r^2\|\mathcal{H}\|^2 \tag{26}$$

The equality case of (26) *holds if and only if each fiber is totally geodesic.*

Proof. The inequality (26) is clear from (25). On the other hand, the equality case of (26) holds if and only if

$$\mathcal{T}_{U_i}U_i = 0,\ i = 1, \ldots, r$$

which implies

$$\mathcal{T}_U U = 0, \tag{27}$$

for all $U \in \Gamma(\ker\sigma_*)$. Replacing U by $U + V$ in (27), where $U, V \in \Gamma(\ker\sigma_*)$, and using the symmetry of the O'Neill tensor $\mathcal{T}$ for vertical vector fields, we obtain $\mathcal{T}_U V = 0$. Hence, the fibers of the submersion are totally geodesic. □

Now, if $\{U_1, \ldots, U_r\}$ is an orthonormal basis of $\ker\sigma_*$ and $\{X_1 \ldots, X_n\}$ is an orthonormal basis of $(\ker\sigma_*)^\perp$, we denote

$$\mathcal{T}_{ij}^s = g(\mathcal{T}_{U_i}U_j, X_s), \tag{28}$$

where $1 \leq i, j \leq r$ and $1 \leq s \leq n$, and

$$\mathcal{A}_{ij}^\alpha = g(\mathcal{A}_{X_i}X_j, U_\alpha), \tag{29}$$

where $1 \leq i, j \leq n$ and $1 \leq \alpha \leq r$. From [35], we use

$$\delta(N) = \sum_{i=1}^{n}\sum_{k=1}^{r} g((\nabla_{X_i}\mathcal{T})_{U_k}U_k, X_i). \tag{30}$$

We also define

$$\|\mathcal{C}\|^2 = \sum_{i,j=1}^{n} g^2(\mathcal{C}X_i, X_j), \tag{31}$$

$$\|\mathcal{B}\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{r} g^2(\mathcal{B}X_i, U_k) \tag{32}$$

and

$$\tau^* = \sum_{1\leq i<j\leq n} R^*(X_i, X_j, X_j, X_i). \tag{33}$$

Moreover, if $X \in \Gamma((\ker\sigma_*)^\perp)$, then

$$\|\mathcal{C}X\|^2 = \sum_{i=1}^{n} g^2(\mathcal{C}X, X_i) \tag{34}$$

and

$$Ric^*(X) = \sum_{i=1}^{n} R^*(X, X_i, X_i, X). \tag{35}$$

From the Binomial theorem, we have the following relation between the components of the O'Neill tensor field $\mathcal{T}$ and the squared mean curvature $\mathcal{H}$:

$$\sum_{s=1}^{n}\sum_{i,j=1}^{r}(\mathcal{T}_{ij}^{s})^2 = \frac{1}{2}r^2\|\mathcal{H}\|^2 + \frac{1}{2}\sum_{s=1}^{n}(\mathcal{T}_{11}^{s} - \mathcal{T}_{22}^{s} - \cdots - \mathcal{T}_{rr}^{s})^2 + 2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^{s})^2 - 2\sum_{s=1}^{n}\sum_{2\le i<j\le r}\Big[\mathcal{T}_{ii}^{s}\mathcal{T}_{jj}^{s} - (\mathcal{T}_{ij}^{s})^2\Big]. \tag{36}$$

Theorem 6. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Suppose U is a unit vertical vector field. Then:*

(i) If $U \in \Gamma(\mathcal{D}^{\perp})$, we have

$$\widehat{Ric}(U) \ge \frac{c}{4}(r-1) - \frac{1}{4}r^2\|\mathcal{H}\|^2. \tag{37}$$

(ii) If $U \in \Gamma(\mathcal{D}^{\theta})$, we have

$$\widehat{Ric}(U) \ge \frac{c}{4}(r-1+3\cos^2\theta) - \frac{1}{4}r^2\|\mathcal{H}\|^2. \tag{38}$$

Moreover, the equality cases of (37) *and* (38) *hold if and only if there exist two orthonormal bases $\{U_1 = U, U_2, \ldots, U_r\}$ and $\{X_1 \ldots, X_n\}$ of $\ker\sigma_*$ and $(\ker\sigma_*)^{\perp}$, respectively, such that $\mathcal{T}_{12}^{\alpha} = \cdots = \mathcal{T}_{1r}^{\alpha} = 0$ and $\mathcal{T}_{11}^{\alpha} = \mathcal{T}_{22}^{\alpha} + \cdots + \mathcal{T}_{rr}^{\alpha}$, for $\alpha = 1, \ldots, n$.*

Proof. Let $\{U_1, \ldots, U_{k_1}, U_{k_1+1}, U_{k_1+2}, \ldots, U_{k_1+2k_2-1}, U_{k_1+2k_2}\}$ be an adapted hemi-slant basis of $(\ker\sigma_*)$.

(i) Due to the fact that one can choose the above adapted hemi-slant basis such that $U_1 = U$, it suffices to prove (38) for $U = U_1$.

Using (28) in (25), we can write

$$2\hat{r} = \frac{c}{4}\{r^2 - r + 6k_2\cos^2\theta\} - r^2\|\mathcal{H}\|^2 + \sum_{\alpha=1}^{n}\sum_{k,s=1}^{r}(\mathcal{T}_{ks}^{\alpha})^2. \tag{39}$$

If (36) is used in (39), then (39) can be rewritten as

$$2\hat{r} = \frac{c}{4}\Big(r^2 - r + 6k_2\cos^2\theta\Big) - \frac{1}{2}r^2\|\mathcal{H}\|^2 + \frac{1}{2}\sum_{\alpha=1}^{n}(\mathcal{T}_{11}^{\alpha} - \mathcal{T}_{22}^{\alpha} - \cdots - \mathcal{T}_{rr}^{\alpha})^2 + 2\sum_{\alpha=1}^{n}\sum_{s=2}^{r}(\mathcal{T}_{1s}^{\alpha})^2 - 2\sum_{\alpha=p+1}^{b_1}\sum_{2\le k<s\le r}^{r}\Big[\mathcal{T}_{kk}^{\alpha}\mathcal{T}_{ss}^{\alpha} - (\mathcal{T}_{ks}^{\alpha})^2\Big]. \tag{40}$$

Thus, from (40) we derive

$$2\hat{r} \ge \frac{c}{4}(r^2 - r + 6k_2\cos^2\theta) - \frac{1}{2}r^2\|\mathcal{H}\|^2 - 2\sum_{\alpha=1}^{n}\sum_{2\le i<j\le r}[\mathcal{T}_{ii}^{\alpha}\mathcal{T}_{jj}^{\alpha} - (\mathcal{T}_{ij}^{\alpha})^2]. \tag{41}$$

Furthermore, taking $U = W = U_i$, $V = F = U_j$ in (10), we obtain

$$2 \sum_{2 \le i < j \le r} R(U_i, U_j, U_j, U_i) = 2 \sum_{2 \le i < j \le r} \hat{R}(U_i, U_j, U_j, U_i) + 2 \sum_{\alpha=1}^{n} \sum_{2 \le i < j \le r} \Big[\mathcal{T}_{ii}^{\alpha} \mathcal{T}_{jj}^{\alpha} - (\mathcal{T}_{ij}^{\alpha})^2 \Big]. \tag{42}$$

Using (42) in (41), we derive

$$2\hat{r} \ge \frac{c}{4}\Big(r^2 - r + 6k_2 \cos^2\theta\Big) - \frac{1}{2} r^2 \|H\|^2 + 2 \sum_{2 \le k < s \le r} \hat{R}(U_k, U_s, U_s, U_k) - 2 \sum_{2 \le k < s \le r} R(U_k, U_s, U_s, U_k). \tag{43}$$

Furthermore, we have

$$2\hat{r} = 2 \sum_{2 \le i < j \le r} \hat{R}(U_i, U_j, U_j, U_i) + 2 \sum_{j=1}^{r} \hat{R}(U_1, U_j, U_j, U_1). \tag{44}$$

Considering (44) in (43), we derive

$$2\widehat{Ric}(U_1) \ge \frac{c}{4}\Big(r^2 - r + 6k_2 \cos^2\theta\Big) - \frac{1}{2} r^2 \|\mathcal{H}\|^2 - 2 \sum_{2 \le k < s \le r} R(U_k, U_s, U_s, U_k). \tag{45}$$

On the other hand, since $M(c)$ is a complex space form, its curvature tensor R satisfies (15) and we get

$$\sum_{2 \le k < s \le r} R(U_k, U_s, U_s, U_k) = \frac{c}{4}\left[\frac{(r-2)(r-1)}{2} + 3 \sum_{2 \le k < s \le r} g^2(JU_k, U_s) \right]. \tag{46}$$

As $U_1 \in \Gamma(\mathcal{D}^{\perp})$, we obtain immediately

$$\sum_{2 \le k < s \le r} g^2(JU_k, U_s) = k_2 \cos^2\theta$$

and therefore (46) can be written as

$$\sum_{2 \le k < s \le r} R(U_k, U_s, U_s, U_k) = \frac{c}{4}\left[\frac{(r-2)(r-1)}{2} + 3k_2 \cos^2\theta \right] \tag{47}$$

Considering now the last equation in (45), we get

$$\widehat{Ric}(U_1) \ge \frac{c}{4}(r-1) - \frac{1}{4} r^2 \|\mathcal{H}\|^2$$

and the conclusion is now clear.

(ii) Due to the fact that in this case one can choose the adapted hemi-slant basis

$$\{U_1, \dots, U_{k_1}, U_{k_1+1}, U_{k_1+2}, \dots, U_{k_1+2k_2-1}, U_{k_1+2k_2}\}$$

such that $U_{k_1+1} = U$, it suffices to prove (38) for $U = U_{k_1+1}$.

With similar arguments as in case (i), we obtain

$$2\widehat{Ric}(U_{k_1+1}) \ge \frac{c}{4}\{r^2 - r + 6k_2 \cos^2\theta\} - \frac{1}{2} r^2 \|\mathcal{H}\|^2 - 2 \sum_{1 \le k < s \le r;\ k,s \ne k_1+1} R(U_k, U_s, U_s, U_k). \tag{48}$$

and

$$\sum_{1\le k<s\le r;\ k,s\neq k_1+1} R(U_k,U_s,U_s,U_k) = \frac{c(r-2)(r-1)}{8} + \frac{3c}{4}\sum_{1\le k<s\le r;\ k,s\neq k_1+1} g^2(JU_k,U_s). \tag{49}$$

As $U_{k_1+1}\in\Gamma(\mathcal{D}^\perp)$, we obtain immediately

$$\sum_{1\le k<s\le r;\ k,s\neq k_1+1} g^2(JU_k,U_s) = (k_2-1)\cos^2\theta$$

and therefore (49) can be written as

$$\sum_{1\le k<s\le r;\ k,s\neq k_1+1} R(U_k,U_s,U_s,U_k) = \frac{c}{4}\left[\frac{(r-2)(r-1)}{2} + 3(k_2-1)\cos^2\theta\right]. \tag{50}$$

Considering now the last equation in (48), we get

$$\widehat{Ric}(U_{k_1+1}) \ge \frac{c}{4}(r-1+3\cos^2\theta) - \frac{1}{4}r^2\|\mathcal{H}\|^2$$

and inequality (38) is clear.

Now, we remark that the equality case of (37) holds if and only if the equality is attained in (41). However, this happens if and only if $\mathcal{T}^\alpha_{11} = \mathcal{T}^\alpha_{22}+\cdots+\mathcal{T}^\alpha_{rr}$ and $\mathcal{T}^\alpha_{1s}=0$, for $s=2,\ldots,r$ and $\alpha=1,\ldots,n$. On the other hand, the equality case of (38) holds if and only, with respect to the hemi-slant adapted basis considered in the proof, we have $\mathcal{T}^\alpha_{k_1+1,k_1+1} = \sum_{i\neq k_1+1}\mathcal{T}^\alpha_{ii}$ and $\mathcal{T}^\alpha_{k_1+1,s}=0$, for $\alpha=1,\ldots,n$ and $s\in\{1\ldots,r\}-\{k_1+1\}$. Using a reordering of the vectors in the basis of $\ker\sigma_*$, we derive the conclusion. □

Theorem 7. *Let $\sigma: M(c)\to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c),g)$ onto a Riemannian manifold (N,g_N). Then, we have*

$$\tau^* \le \frac{c}{8}\Big[n(n-1)+3\|\mathcal{C}\|^2\Big]. \tag{51}$$

Moreover, the equality holds in (51) *if and only if the horizontal distribution* $(\ker\sigma_*)^\perp$ *is integrable.*

Proof. Using the anti-symmetry of $\mathcal{A}$ and (18), we obtain

$$2\tau^* = \frac{c}{4}\left[n(n-1)+3\sum_{i,j=1}^{n} g(\mathcal{C}X_i,X_j)g(\mathcal{C}X_i,X_j)\right] - 3\sum_{i,j=1}^{n} g(\mathcal{A}_{X_i}X_j,\mathcal{A}_{X_i}X_j), \tag{52}$$

where $\{X_1,\ldots,X_n\}$ is an orthonormal basis of $(\ker\sigma_*)^\perp$. Now, using (31) in (52) we obtain

$$2\tau^* = \frac{c}{4}\Big[n(n-1)+3\|\mathcal{C}\|^2\Big] - 3\sum_{i,j=1}^{n}\|\mathcal{A}_{X_i}X_j\|^2. \tag{53}$$

and inequality (51) follows immediately. Moreover, it is clear that the equality case of (51) holds if and only if $\mathcal{A}_{X_i}X_j=0$, for $i,j=1,\ldots,n$ and the proof is now complete due to the fact that the vanishing of the O'Neill tensor $\mathcal{A}$ is equivalent to the integrability of the horizontal distribution (see, e.g., [9]). □

Theorem 8. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). If $\{X_1, \ldots, X_n\}$ is an orthonormal basis of $(\ker \sigma_*)^{\perp}$, then we have*

$$Ric^*(X_1) = \frac{c}{4}\Big[(n-1) + 3\|\mathcal{C}X_1\|^2\Big] - 3\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}^{\alpha}_{1j})^2. \tag{54}$$

Proof. By using (29) in (53), we have

$$2\tau^* = \frac{c}{4}\Big[n(n-1) + 3\|\mathcal{C}\|^2\Big] - 3\sum_{\alpha=1}^{r}\sum_{i,j=1}^{n}(\mathcal{A}^{\alpha}_{ij})^2. \tag{55}$$

Thus, (55) can be written as

$$2\tau^* = \frac{c}{4}\Big[n(n-1) + 3\|\mathcal{C}\|^2\Big] - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}^{\alpha}_{1j})^2 - 6\sum_{\alpha=1}^{r}\sum_{2\le i<j\le n}(\mathcal{A}^{\alpha}_{ij})^2. \tag{56}$$

Moreover, taking $X = H = X_i$, $Y = Z = X_j$ in (11), we obtain

$$\sum_{2\le i<j\le n} R(X_i, X_j, X_j, X_i) = \sum_{2\le i<j\le n} R^*(X_i, X_j, X_j, X_i) + 3\sum_{\alpha=1}^{r}\sum_{2\le i<j\le n}(\mathcal{A}^{\alpha}_{ij})^2. \tag{57}$$

Using (57) in (56), we derive

$$\begin{aligned} 2\tau^* = {} & \frac{c}{4}\Big[n(n-1) + 3\|\mathcal{C}\|^2\Big] - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}^{\alpha}_{1j})^2 \\ & + 2\sum_{2\le i<j\le n} R^*(X_i, X_j, X_j, X_i) - 2\sum_{2\le i<j\le n} R(X_i, X_j, X_j, X_i). \end{aligned} \tag{58}$$

Since $M(c)$ is a complex space form, its curvature tensor R satisfies the equality (15) and we obtain

$$\sum_{2\le i<j\le n} R(X_i, X_j, X_j, X_i) = \frac{c}{8}\left[(n-2)(n-1) + 6\sum_{2\le i<j\le n} g^2(\mathcal{C}X_i, X_j)\right]. \tag{59}$$

Then, from (58) and (59), taking into account that

$$\|\mathcal{C}\|^2 - 2\sum_{2\le i<j\le n} g^2(\mathcal{C}X_i, X_j) = 2\|\mathcal{C}X_1\|^2 \tag{60}$$

we get

$$2Ric^*(X_1) = \frac{c}{2}\Big[(n-1) + 3\|\mathcal{C}X_1\|^2\Big] - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}^{\alpha}_{1j})^2 \tag{61}$$

and equality (54) follows immediately. □

As an outcome of the above result, we have the following.

Theorem 9. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). If X is a unit horizontal vector field, then we have*

$$Ric^*(X) \le \frac{c}{4}\Big[(n-1) + 3\|\mathcal{C}X\|^2\Big]. \tag{62}$$

Moreover, the equality case of the above inequality holds identically for all unit horizontal vector fields if and only if the horizontal distribution is integrable.

Proof. Inequality (62) is clear from Theorem 8 because in (54) we can select $X_1 = X$ to be any arbitrary unit horizontal vector field. This is due to the fact that one can always choose in Theorem 8 an orthonormal basis $\{X_1, \ldots, X_n\}$ of $(\ker \sigma_*)^\perp$ with $X_1 = X$.

Now, if the horizontal distribution is integrable, then $\mathcal{A}_{X_i} X_j = 0$, for $i, j = 1, \ldots, n$, and it is clear that we have equality in (62). Conversely, if the equality case of (62) holds identically for all unit horizontal vector fields, then it follows that $\mathcal{A}^\alpha_{ij} = 0$, for $\alpha = 1, \ldots, r$ and $i, j = 1, \ldots, n$, $i \neq j$, which means $\mathcal{A}_{X_i} X_j = 0$, for all $i \neq j$. However, due to the skew-symmetry of $\mathcal{A}$ for horizontal vector fields, it is obvious that $\mathcal{A}_{X_i} X_i = 0$. Hence, $\mathcal{A}_{X_i} X_j = 0$ for $i, j = 1, \ldots, n$, and therefore the horizontal distribution is integrable. □

Now, we are going to state the Chen–Ricci inequality between the vertical and horizontal distributions for a hemi-slant Riemannian submersion $\sigma : M(c) \to N$ from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Suppose $\{U_1, U_2, \ldots, U_r\}$ is an orthonormal basis of $\ker \sigma_*$ and $\{X_1 \ldots, X_n\}$ is an orthonormal basis of $(\ker \sigma_*)^\perp$. Then, for the scalar curvature τ of $M(c)$, we have

$$2\tau = \sum_{k=1}^{r} Ric(U_k, U_k) + \sum_{s=1}^{n} Ric(X_s, X_s). \tag{63}$$

Further, we can write

$$\begin{aligned} 2\tau = & \sum_{j,k=1}^{r} R(U_j, U_k, U_k, U_j) + \sum_{i=1}^{n} \sum_{k=1}^{r} R(X_i, U_k, U_k, X_i) \\ & + \sum_{i,s=1}^{n} R(X_i, X_s, X_s, X_i) + \sum_{s=1}^{n} \sum_{j=1}^{r} R(U_j, X_s, X_s, U_j). \end{aligned} \tag{64}$$

Next, let us denote as usual (see [35]):

$$\left\| \mathcal{T}^V \right\|^2 = \sum_{i=1}^{n} \sum_{k=1}^{r} g(\mathcal{T}_{U_k} X_i, \mathcal{T}_{U_k} X_i), \tag{65}$$

$$\left\| \mathcal{T}^H \right\|^2 = \sum_{k,j=1}^{r} g(\mathcal{T}_{U_k} U_j, \mathcal{T}_{U_k} U_j), \tag{66}$$

$$\left\| \mathcal{A}^V \right\|^2 = \sum_{i,j=1}^{n} g(\mathcal{A}_{X_i} X_j, \mathcal{A}_{X_i} X_j), \tag{67}$$

$$\left\| \mathcal{A}^H \right\|^2 = \sum_{i=1}^{n} \sum_{k=1}^{r} g(\mathcal{A}_{X_i} U_k, \mathcal{A}_{X_i} U_k). \tag{68}$$

Theorem 10. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Suppose $\{U_1, U_2, \ldots, U_r\}$ is an orthonormal basis of* $\ker \sigma_*$ *and $\{X_1 \ldots, X_n\}$ is an orthonormal basis of* $(\ker \sigma_*)^\perp$.

(i) If $U_1 \in \Gamma(\mathcal{D}^\perp)$, then

$$\begin{aligned} & \frac{c}{2}\Big[nr + n + r - 2 + 3\Big(\|\mathcal{B}\|^2 + \|\mathcal{C}X_1\|^2\Big)\Big] \\ & \leq \widehat{Ric}(U_1) + Ric^*(X_1) + \frac{1}{4} r^2 \|\mathcal{H}\|^2 + 3 \sum_{\alpha=1}^{r} \sum_{s=2}^{n} (\mathcal{A}^\alpha_{1s})^2 \\ & - \delta(N) + \left\| \mathcal{T}^V \right\|^2 - \left\| \mathcal{A}^H \right\|^2. \end{aligned} \tag{69}$$

(ii) If $U_1 \in \Gamma(\mathcal{D}^\theta)$, then

$$\begin{aligned}&\frac{c}{2}\Big[nr+n+r-2+3\Big(2\cos^2\theta+\|\mathcal{B}\|^2+\|\mathcal{C}X_1\|^2\Big)\Big]\\&\leq \widehat{Ric}(U_1)+Ric^*(X_1)+\frac{1}{4}r^2\|\mathcal{H}\|^2+3\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(A^\alpha_{1s})^2\\&-\delta(N)+\left\|\mathcal{T}^V\right\|^2-\left\|\mathcal{A}^H\right\|^2.\end{aligned} \tag{70}$$

The equality case of (69) *and* (70) *holds if and only if*

$$\begin{aligned}&\mathcal{T}^s_{11}=\mathcal{T}^s_{22}+\cdots+\mathcal{T}^s_{rr},\\&\mathcal{T}^s_{1j}=0,\end{aligned}$$

for $s=1,\ldots,n$, $j=2,\ldots,r$.

Proof. Since $M(c)$ is a complex space form, using (64) and (32) we get

$$2\tau=\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+6\|\mathcal{B}\|^2+3\|\mathcal{C}\|^2]. \tag{71}$$

On the other hand, using the Gauss–Codazzi type Equations (10)–(12), we derive

$$\begin{aligned}2\tau&=2\hat{\tau}+2\tau^*+r^2\|\mathcal{H}\|^2-\sum_{k,j=1}^{r}g(\mathcal{T}_{U_k}U_j,\mathcal{T}_{U_k}U_j)+3\sum_{i,s=1}^{n}g(\mathcal{A}_{X_i}X_s,\mathcal{A}_{X_i}X_s)\\&+\sum_{i=1}^{n}\sum_{k=1}^{r}\{g(\mathcal{T}_{U_k}X_i,\mathcal{T}_{U_k}X_i)-g(\mathcal{A}_{X_i}U_k,\mathcal{A}_{X_i}U_k)-g((\nabla_{X_i}\mathcal{T})_{U_k}U_k,X_i)\}\\&+\sum_{s=1}^{n}\sum_{j=1}^{r}\{g(\mathcal{T}_{U_j}X_s,\mathcal{T}_{U_j}X_s)-g(\mathcal{A}_{X_s}U_j,\mathcal{A}_{X_s}U_j)-g((\nabla_{X_s}\mathcal{T})_{U_j}U_j,X_s)\}.\end{aligned} \tag{72}$$

Therefore, using (30) and (36) in (72), we obtain

$$\begin{aligned}2\tau&=2\hat{\tau}+2\tau^*+\frac{1}{2}r^2\|\mathcal{H}\|^2-\frac{1}{2}\sum_{s=1}^{n}(\mathcal{T}^s_{11}-\mathcal{T}^s_{22}-\cdots-\mathcal{T}^s_{rr})^2-2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}^s_{1j})^2\\&+2\sum_{s=1}^{n}\sum_{2\leq i<j\leq r}(\mathcal{T}^s_{ii}\mathcal{T}^s_{jj}-(\mathcal{T}^s_{ij})^2)+6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}^\alpha_{1s})^2+6\sum_{\alpha=1}^{r}\sum_{2\leq i<s\leq n}(\mathcal{A}^\alpha_{is})^2\\&+\sum_{i=1}^{n}\sum_{k=1}^{r}\{g(\mathcal{T}_{U_k}X_i,\mathcal{T}_{U_k}X_i)-g(\mathcal{A}_{X_i}U_k,\mathcal{A}_{X_i}U_k)\}-2\delta(N)\\&+\sum_{s=1}^{n}\sum_{j=1}^{r}(g(\mathcal{T}_{U_j}X_s,\mathcal{T}_{U_j}X_s)-g(\mathcal{A}_{X_s}U_j,\mathcal{A}_{X_s}U_j))\end{aligned} \tag{73}$$

Using now (42), (57) and (71) in (73), we get

$$\begin{aligned}
&\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+3\|\mathcal{C}\|^2+6\|\mathcal{B}\|^2]\\
&=2\widehat{Ric}(U_1)+2Ric^*(X_1)+\frac{1}{2}r^2\|\mathcal{H}\|^2\\
&-\frac{1}{2}\sum_{s=1}^{n}(\mathcal{T}_{11}^s-\mathcal{T}_{22}^s-\cdots-\mathcal{T}_{rr}^s)^2-2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^s)^2\\
&+6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^2+\sum_{i=1}^{n}\sum_{k=1}^{r}\{g(\mathcal{T}_{U_k}X_i,\mathcal{T}_{U_k}X_i)-g(\mathcal{A}_{X_i}U_k,\mathcal{A}_{X_i}U_k)\}\\
&-2\delta(N)+\sum_{s=1}^{n}\sum_{j=1}^{r}\{g(\mathcal{T}_{U_j}X_s,\mathcal{T}_{U_j}X_s)-g(\mathcal{A}_{X_s}U_j,\mathcal{A}_{X_s}U_j)\}\\
&+2\sum_{2\le i<j\le r}R(U_i,U_j,U_j,U_i)+2\sum_{2\le i<j\le n}R(X_i,X_j,X_j,X_i).
\end{aligned}\tag{74}$$

Hence, in view of (65)–(68), the equality (74) implies

$$\begin{aligned}
&\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+3\|\mathcal{C}\|^2+6\|\mathcal{B}\|^2]\\
&\le 2\widehat{Ric}(U_1)+2Ric^*(X_1)+\frac{1}{2}r^2\|\mathcal{H}\|^2\\
&+6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^2+2(\left\|\mathcal{T}^V\right\|^2-\left\|\mathcal{A}^H\right\|^2)-2\delta(N)\\
&+2\sum_{2\le i<j\le r}R(U_i,U_j,U_j,U_i)+2\sum_{2\le i<j\le n}R(X_i,X_j,X_j,X_i).
\end{aligned}\tag{75}$$

If $U_1\in\Gamma(\mathcal{D}^{\perp})$, then considering (47) and (59) in (75), in view of (60) we obtain (69). Similarly, if $U_1\in\Gamma(\mathcal{D}^{\theta})$, then using (50), (59) and (60) in (75), we obtain (70). Finally, the equality of (69) and (70) holds if and only if we have equality in (75), which happens if and only if $\mathcal{T}_{11}^s-\mathcal{T}_{22}^s-\cdots-\mathcal{T}_{rr}^s=0$ and $\mathcal{T}_{1j}^s=0$, for all $s=1,\ldots,n$ and $j=2,\ldots,r$. This completes the proof. □

Remark 1. *If $\sigma: M(c)\to N$ is a proper hemi-slant Riemannian submersion from a complex space form $(M(c),g)$ onto a Riemannian manifold (N,g_N), then from* (71) *and* (72) *we get*

$$\begin{aligned}
&\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+3(\|\mathcal{C}\|^2+2\|\mathcal{B}\|^2)]\\
&=2\hat{\tau}+2\tau^*+r^2\|\mathcal{H}\|^2-\left\|\mathcal{T}^H\right\|^2+3\left\|\mathcal{A}^V\right\|^2\\
&-2\delta(N)+2\left\|\mathcal{T}^V\right\|^2-2\left\|\mathcal{A}^H\right\|^2.
\end{aligned}\tag{76}$$

From (76) *we derive immediately that*

$$\begin{aligned}
2\hat{\tau}+2\tau^*\le{}&\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+3(\|\mathcal{C}\|^2+2\|\mathcal{B}\|^2)]\\
&-r^2\|\mathcal{H}\|^2+\left\|\mathcal{T}^H\right\|^2+2\delta(N)-2\left\|\mathcal{T}^V\right\|^2+2\left\|\mathcal{A}^H\right\|^2,
\end{aligned}\tag{77}$$

and

$$\begin{aligned}
2\hat{\tau}+2\tau^*\ge{}&\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+3(\|\mathcal{C}\|^2+2\|\mathcal{B}\|^2)]\\
&-r^2\|\mathcal{H}\|^2+2\delta(N)-2\left\|\mathcal{T}^V\right\|^2+2\left\|\mathcal{A}^H\right\|^2-3\left\|\mathcal{A}^V\right\|^2.
\end{aligned}\tag{78}$$

Moreover, it is clear that the equality case of (77) *holds for all $p \in M$ if and only if the horizontal distribution $(ker\sigma_*)^\perp$ is integrable, while the equality cases of* (78) *hold for all $p \in M$ if and only if the fibers of σ are totally geodesic submanifolds of $M(c)$. In particular, we deduce the following result.*

Theorem 11. *Let $\sigma : M(c) \to N$ be proper a hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N) with totally geodesic fibers. Then, we have*

$$2\hat{\tau} + 2\tau^* \leq \frac{c}{4}[(n+r)(n+r-1) + 6k_2\cos^2\theta + 3(\|\mathcal{C}\|^2 + 2\|\mathcal{B}\|^2)] + 2\left\|\mathcal{A}^H\right\|^2. \tag{79}$$

Moreover, the equality case of (79) *holds for all $p \in M$ if and only if the horizontal distribution $(ker\sigma_*)^\perp$ is integrable.*

We now recall the following result, which we will use a little later.

Lemma 2 ([37])**.** *Let $a_1, a_2, \ldots, a_n$ be n real numbers $(n > 1)$. Then*

$$\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i=1}^{n} a_i^2$$

with equality iff $a_1 = a_2 = \ldots. = a_n$.

Theorem 12. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\frac{c}{4}[(n+r)(n+r-1) + 6k_2\cos^2\theta + 3(\|\mathcal{C}\|^2 + 2\|\mathcal{B}\|^2)] \leq 2\hat{\tau} + 2\tau^* + r(r-1)\|\mathcal{H}\|^2 + 3\left\|\mathcal{A}^V\right\|^2 - 2\delta(N) + 2\left\|\mathcal{T}^V\right\|^2 - 2\left\|\mathcal{A}^H\right\|^2. \tag{80}$$

Equality case of (80) *holds for all $p \in M$ if and only σ has totally umbilical fibers.*

Proof. From (76) we have

$$\frac{c}{4}[(n+r)(n+r-1) + 6k_2\cos^2\theta + 3(\|\mathcal{C}\|^2 + 2\|\mathcal{B}\|^2)] = 2\hat{\tau} + 2\tau^* + r^2\|\mathcal{H}\|^2 - \sum_{s=1}^{n}\sum_{j=1}^{r}(\mathcal{T}_{jj}^s)^2 - \sum_{s=1}^{n}\sum_{j\neq k}^{r}(\mathcal{T}_{jk}^s)^2 + 3\left\|\mathcal{A}^V\right\|^2 - 2\delta(N) + 2\left\|\mathcal{T}^V\right\|^2 - 2\left\|\mathcal{A}^H\right\|^2. \tag{81}$$

Applying Lemma 2 in (81), we get

$$\frac{c}{4}[(n+r)(n+r-1) + 6k_2\cos^2\theta + 3(\|\mathcal{C}\|^2 + 2\|\mathcal{B}\|^2)] \leq 2\hat{\tau} + 2\tau^* + r^2\|\mathcal{H}\|^2 - \frac{1}{r}\sum_{s=1}^{n}(\sum_{j=1}^{r}\mathcal{T}_{jj}^s)^2 - \sum_{s=1}^{n}\sum_{j\neq k}^{r}(\mathcal{T}_{jk}^s)^2 + 3\left\|\mathcal{A}^V\right\|^2 - 2\delta(N) + 2\left\|\mathcal{T}^V\right\|^2 - 2\left\|\mathcal{A}^H\right\|^2, \tag{82}$$

which gives (80). Equality case of (80) holds for all $p \in M$ if and only if the components of the O'Neill tensor $\mathcal{T}$ satisfy $\mathcal{T}_{11}^s = \mathcal{T}_{22}^s = \cdots = \mathcal{T}_{rr}^s$ and $\mathcal{T}_{jk}^s = 0$, for $s = 1, \ldots, n$, $j, k = 1, \ldots, r$, $j \neq k$. The conclusion is now clear. □

Using a similar proof as in Theorem 12, we deduce the following result.

Theorem 13. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\frac{c}{4}[(n+r)(n+r-1) + 6k_2 \cos^2\theta + 3(\|\mathcal{C}\|^2 + 2\|\mathcal{B}\|^2)] \geq 2\hat{\tau} + 2\tau^* + r^2\|\mathcal{H}\|^2 - \left\|\mathcal{T}^H\right\|^2 + \frac{3}{n}tr(\mathcal{A}^V)^2 - 2\delta(N) + 2\left\|\mathcal{T}^V\right\|^2 - 2\left\|\mathcal{A}^H\right\|^2 \quad (83)$$

The equality case of the above inequality holds for all $p \in M$ if and only if the components of the O'Neill tensor $\mathcal{A}$ with respect to some suitable orthonormal bases of the horizontal and vertical distributions satisfy $\mathcal{A}^s_{11} = \mathcal{A}^s_{22} = \cdots = \mathcal{A}^s_{nn}$ and $\mathcal{A}^s_{ij} = 0$, for $s = 1, \ldots, r$ and $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$.

4. Examples

In this section, we provide examples of hemi-slant Riemannian submersions, illustrating the main results stated above.

From [11], we know that the concept of hemi-slant submersion generalizes in a natural way the notions of invariant, anti-invariant, semi-invariant and slant submersions. More precisely, if we denote the dimension of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ by p_1 and p_2, respectively, then we have the following:

- If $\theta = 0$, then M is a semi-invariant submersion [12].
- If $\theta = 0$ and $p_1 = 0$, then M is an invariant submersion [15,38].
- If $\theta = 0$ and $p_2 = 0$, then M is an anti-invariant submersion [13].
- If $p_1 = 0$, then M is a slant submersion with slant angle θ [14].

We would like to point out that there is a special type of anti-invariant submersion, called Lagrangian submersion, for which the almost complex structure of the total space of the submersion reverses $\ker\sigma_*$ and $(\ker\sigma_*)^\perp$ (see [39]). Examples of invariant, anti-invariant, Lagrangian, semi-invariant, slant and hemi-slant submersions, as well as various interesting results regarding the geometry of these submersions, can be found in [11–15,39]. At this point, we would just like to note that according to Theorem 4.5 of [39], it follows that the horizontal distribution of a Lagrangian submersion with total space a complex space form is integrable. However, such submersions do not provide us suitable examples to illustrate the equality case of the inequalities stated in Theorems 7, 9 and 11, because a Lagrangian submersion is not a proper hemi-slant submersion.

Next, we will construct the first example of proper hemi-slant submersion satisfying the equality case of all inequalities established in the above section.

Example 1. *Consider the Kähler manifold $(\mathbb{R}^6, J, g_1)$ equipped with the canonical Euclidean metric g_1 and the complex structure J given by:*

$$J(x_1, x_2, x_3, x_4, x_5, x_6) = (-x_4, x_6, x_5, x_1, -x_3, -x_2).$$

Define now a map $\sigma : (\mathbb{R}^6, J, g_1) \to)(\mathbb{R}^3, g_2)$ by

$$\sigma(x_1, x_2, x_3, x_4, x_5, x_6) = (\frac{x_1 + x_4}{\sqrt{2}}, -\frac{-x_2 + x_5}{\sqrt{2}}, x_3\cos\alpha + x_6\sin\alpha),$$

where $\alpha \in (0, \frac{\pi}{2})$ and g_2 is the standard canonical Euclidean metric on $\mathbb{R}^3$. Then, it is easy to check that σ is a hemi-slant submersion such that

$$\mathcal{D}^\perp = Sp\{V_1 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_4}\},$$

$$\mathcal{D}^{\theta} = Sp\{V_2 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_5}, V_3 = -\sin\alpha\frac{\partial}{\partial x_3} + \cos\alpha\frac{\partial}{\partial x_6}\}$$

and

$$(ker\sigma_*)^{\perp} = Sp\{H_1 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_4}, H_2 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_5},$$
$$H_3 = \cos\alpha\frac{\partial}{\partial x_3} + \sin\alpha\frac{\partial}{\partial x_6}\}.$$

Moreover, the hemi-slant angle of σ is $\theta = arccos(\frac{sin\alpha + cos\alpha}{\sqrt{2}})$. A straightforward computation shows that fibers of the submersions are totally geodesic and the horizontal distribution is integrable. Hence, we conclude that the inequalities stated in Theorems 2, 3, 6, 7, 9–11 *and* 13 *are satisfied with equality sign.*

Similarly, the following map illustrates the equality case of the above-mentioned inequalities.

Example 2. *Consider the Euclidean space $\mathbb{R}^{10}$ equipped with the standard metric (denoted by g_1) and the compatible almost complex structure J given by*

$$J(x_1, x_2, \ldots, x_9, x_{10}) = (x_7, x_3, -x_2, x_{10}, x_8, -x_9, -x_1, -x_5, x_6, -x_4).$$

Then, $(\mathbb{R}^{10}, J, g_1)$ is a Kähler manifold and we define a map $\sigma : (\mathbb{R}^{10}, J, g_1) \to (\mathbb{R}^5, g_2)$ by

$$\sigma(x_1, x_2, \ldots, x_{10}) = (t_1, t_2, t_3, t_4, t_5)$$

where

$$t_1 = x_5, \ \ t_2 = \sqrt{\frac{1}{1+7^{\sqrt{2}-1}}}x_1 + \sqrt{\frac{1}{1+7^{1-\sqrt{2}}}}x_7, \ \ t_3 = x_2$$
$$t_4 = x_4\tanh\alpha + x_{10}\mathrm{sech}\alpha, \ \ t_5 = \frac{\sqrt{5}}{3}x_3 + \frac{2}{3}x_6,$$

$\alpha \in (0, \frac{\pi}{2})$ and g_2 is the standard canonical Euclidean metric on $\mathbb{R}^5$. A direct computation shows that σ is a hemi-slant submersion with the hemi-slant angle $\theta = \arccos(\frac{\sqrt{5}}{3})$ such that

$$\mathcal{D}^{\perp} = Sp\{V_1 = -\sqrt{\frac{1}{1+7^{1-\sqrt{2}}}}\frac{\partial}{\partial x_1} + \sqrt{\frac{1}{1+7^{\sqrt{2}-1}}}\frac{\partial}{\partial x_7},$$
$$V_2 = -\tanh\alpha\frac{\partial}{\partial x_{10}} + \mathrm{sech}\alpha\frac{\partial}{\partial x_4}, V_3 = \frac{\partial}{\partial x_8}\},$$
$$\mathcal{D}^{\theta} = Sp\{V_4 = -\frac{2}{3}\frac{\partial}{\partial x_3} + \frac{\sqrt{5}}{3}\frac{\partial}{\partial x_6}, V_5 = \frac{\partial}{\partial x_9}\}.$$

and

$$(ker\sigma_*)^{\perp} = Sp\{H_1 = -\sqrt{\frac{1}{1+7^{1-\sqrt{2}}}}\frac{\partial}{\partial x_1} - \sqrt{\frac{1}{1+7^{\sqrt{2}-1}}}\frac{\partial}{\partial x_7},$$
$$H_2 = \tanh\alpha\frac{\partial}{\partial x_4} + \mathrm{sech}\alpha\frac{\partial}{\partial x_{10}},$$
$$H_3 = -\frac{2}{3}\frac{\partial}{\partial x_3} - \frac{\sqrt{5}}{3}\frac{\partial}{\partial x_6}, H_4 = \frac{\partial}{\partial x_2}, H_5 = \frac{\partial}{\partial x_5}\}.$$

We derive immediately that fibers of the submersions are totally geodesic and the horizontal distribution is integrable. Hence, we have again that σ satisfies the equality case of the inequalities stated in Theorems 2, 3, 6, 7, 9–11 *and* 13*.*

5. Conclusions

(Semi-)Riemannian submersions are mathematical objects of high interest in theoretical physics (see, e.g., [9]). A particular class of such submersions was introduced by Taştan et al. [11], as a natural generalization of some important families of Riemannian submersions: invariant, anti-invariant, semi-invariant and slant submersions. In this paper, we prove various optimal inequalities involving basic curvature invariants for hemi-slant Riemannian submersions having as total space a complex space form and discuss the equality case of the obtained inequalities. Finally, we provide examples of hemi-slant Riemannian submersions to show that the equality cases of the main inequalities can be attained. For further research, it would be interesting to obtain Chen-like inequalities for lightlike submersions (see [40]). In this case, it could be necessary to use not only techniques from submanifold theory, but also from singularity theory (see, e.g., [41–44]).

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mathematics

Article

Application of Mixed Generalized Quasi-Einstein Spacetimes in General Relativity

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Abstract: In the present article, some geometric and physical properties of $MG(QE)_n$ were investigated. Moreover, general relativistic viscous fluid $MG(QE)_4$ spacetimes with some physical applications were studied. Finally, through a non-trivial example of $MG(QE)_4$ spacetime, we proved its existence.

Keywords: Einstein manifold; mixed generalized quasi-Einstein manifold; Einstein's field equation; energy–momentum tensor; general relativistic viscous fluid

MSC: 53C25; 53Z05

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1. Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g) of dimension $n(> 2)$ is termed as an Einstein manifold if its $(0, 2)$-type Ricci tensor $Ric(\neq 0)$ satisfies $Ric = \frac{r}{n}$, where r stands for the scalar curvature [1]. In addition to Riemannian geometry, Einstein manifolds also have a vital contribution to the general theory of relativity (GTR).

Approximately two decades ago, Chaki and Maity introduced and studied quasi-Einstein manifolds [2]. An (M^n, g), $(n > 2)$ is said to be a quasi-Einstein manifold $(QE)_n$ if its Ric $(\neq 0)$ realizes the following condition:

$$Ric(U_1, U_2) = ag(U_1, U_2) + bA(U_1)A(U_2), \tag{1}$$

where $a, b \in \mathbb{R}$ such that $b \neq 0$ and $A(\neq 0)$ is the 1-form such that

$$g(U_1, \rho) = A(U_1), \quad g(\rho, \rho) = A(\rho) = 1, \tag{2}$$

for any vector field U_1, and a unit vector field ρ called the generator of (M^n, g). In addition, A is named the associated 1-form. Einstein manifolds form a natural subclass of the class of $(QE)_n$.

Under the study of exact solutions of the Einstein field equations, as well as under the consideration of quasi-umbilical hypersurfaces of semi-Euclidean spaces, $(QE)_n$ came into existence. For instance, the Robertson–Walker spacetimes are $(QE)_n$. Thus, $(QE)_n$ have great importance in GTR.

An (M^n, g), $(n \geq 2)$ is said to be a generalized quasi-Einstein manifold $G(QE)_n$ [3] if its $Ric(\neq 0)$ realizes the following condition:

$$Ric(U_1, U_2) = ag(U_1, U_2) + bA(U_1)A(U_2) + cB(U_1)B(U_2), \tag{3}$$

where a, b, c are non-zero scalars and A, B are two non-zero 1-forms such that

$$g(U_1, \rho) = A(U_1), \quad g(U_1, \sigma) = B(U_1), \tag{4}$$

where ρ and σ are mutually orthogonal unit vector fields, i. e., $g(\rho, \sigma) = 0$. The vector fields ρ and σ are called the generators of the manifold. If $c = 0$, then the manifold reduces to a quasi-Einstein manifold.

In 2007, Bhattacharya, De and Debnath [4] introduced the notion of a mixed generalized quasi-Einstein manifold. A non-flat Riemannian manifold is said to be a mixed generalized quasi-Einstein manifold and is denoted by $MG(QE)_n$, if its $Ric(\neq 0)$ satisfies the following condition:

$$\begin{aligned} Ric(U_1, U_2) = &\, ag(U_1, U_2) + bA(U_1)A(U_2) + cB(U_1)B(U_2) \\ &+ d[A(U_1)B(U_2) + B(U_1)A(U_2)], \end{aligned} \tag{5}$$

where a, b, c, d are non-zero scalars and A, B are two non-zero 1-forms such that

$$g(U_1, \rho) = A(U_1), \quad g(U_1, \sigma) = B(U_1), \tag{6}$$

where ρ and σ are mutually orthogonal unit vector fields and are called the generators of the manifold. Recently, $MG(QE)_n$ have been studied by various geometers in several ways to a different extent, such as [5–8] and many others.

Putting $U_1 = U_2 = e_i$ in (5), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i(1 \leq i \leq n)$, we obtain

$$r = na + b + c. \tag{7}$$

A Lorentzian four-dimensional manifold is said to be a mixed generalized quasi-Einstein spacetime with the generator ρ as the unit timelike vector field if its $Ric(\neq 0)$ satisfies (5). Here, A and B are non-zero 1-forms such that σ is the heat flux vector field perpendicular to the velocity vector field ρ. Therefore, for any vector field U_1, we have

$$\begin{aligned} &g(U_1, \rho) = A(U_1), \quad g(U_1, \sigma) = B(U_1), \\ &g(\rho, \rho) = A(\rho) = -1, \quad g(\sigma, \sigma) = B(\sigma) = 1. \end{aligned} \tag{8}$$

Further, we know that if the Riemannian curvature tensor $\overline{K}$ of type $(0, 4)$ has the form

$$\overline{K}(U_1, U_2, U_3, U_4) = k[g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4)], \tag{9}$$

then the manifold is said to be of constant curvature k. The generalization of this manifold is the manifold of quasi-constant curvature and, in this case, the curvature tensor has the following form:

$$\begin{aligned} \overline{K}(U_1, U_2, U_3, U_4) = &\, f_1[g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4)] \\ &+ f_2[g(U_2, U_3)A(U_1)A(U_4) - g(U_2, U_4)A(U_1)A(U_3) \\ &+ g(U_1, U_4)A(U_2)A(U_3) - g(U_1, U_3)A(U_2)A(U_4)], \end{aligned} \tag{10}$$

where $g(K(U_1, U_2)U_3, U_4) = \overline{K}(U_1, U_2, U_3, U_4)$, K is the curvature tensor of type $(1, 3)$ and f_1, f_2 are scalars, and ρ is a unit vector field defined by

$$g(U_1, \rho) = A(U_1),$$

It can be easily seen that, if the curvature tensor $\overline{K}$ is of the form (10), then the manifold is conformally flat [3]. Thus, a Riemannian or semi-Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor $\overline{K}$ satisfies the relation (10); we denote such a manifold of dimension n by $(QC)_n$.

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be a manifold of generalized quasi-constant curvature if the curvature tensor $\overline{K}$ of type $(0,4)$ satisfies the condition [3]

$$\begin{aligned}\overline{K}(U_1,U_2,U_3,U_4) = {} & f_1[g(U_2,U_3)g(U_1,U_4) - g(U_1,U_3)g(U_2,U_4)] \\ & + f_2[g(U_1,U_4)A(U_2)A(U_3) - g(U_2,U_4)A(U_1)A(U_3) \\ & + g(U_2,U_3)A(U_1)A(U_4) - g(U_1,U_3)A(U_2)A(U_4)] \\ & + f_3[g(U_1,U_4)B(U_2)B(U_3) - g(U_2,U_4)B(U_1)B(U_3) \\ & + g(U_2,U_3)B(U_1)B(U_4) - g(U_1,U_3)B(U_2)B(U_4)],\end{aligned} \tag{11}$$

where f_1, f_2, f_3 are scalars and A, B are two non-zero 1-forms. ρ and σ are orthonormal unit vectors corresponding to A and B such that $g(U_1,\rho) = A(X)$, $g(U_1,\sigma) = B(X)$ and $g(\rho,\sigma) = 0$. Such a manifold is denoted by $G(QC)_n$.

In [9], Bhattacharya and De introduced the notion of mixed generalized quasi-constant curvature. A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be a manifold of mixed generalized quasi-constant curvature if the curvature tensor $\overline{K}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}\overline{K}(U_1,U_2,U_3,U_4) = {} & f_1[g(U_2,U_3)g(U_1,U_4) - g(U_1,U_3)g(U_2,U_4)] \\ & + f_2[g(U_1,U_4)A(U_2)A(U_3) - g(U_2,U_4)A(U_1)A(U_3) \\ & + g(U_2,U_3)A(U_1)A(U_4) - g(U_1,U_3)A(U_2)A(U_4)] \\ & + f_3[g(U_1,U_4)B(U_2)B(U_3) - g(U_2,U_4)B(U_1)B(U_3) \\ & + g(U_2,U_3)B(U_1)B(U_4) - g(U_1,U_3)B(U_2)B(U_4)] \\ & + f_4[\{A(U_2)B(U_3) + B(U_2)A(U_3)\}g(U_1,U_4) \\ & - \{A(U_1)B(U_3) + B(U_1)A(U_3)\}g(U_2,U_4) \\ & + \{A(U_1)B(U_4) + B(U_1)A(U_4)\}g(U_2,U_3) \\ & - \{A(U_2)B(U_4) + B(U_2)A(U_4)\}g(U_1,U_3)],\end{aligned} \tag{12}$$

where f_1, f_2, f_3, f_4 are scalars. A, B are two non-zero 1-forms. ρ and σ are orthonormal unit vectors corresponding to A and B such that $g(U_1,\rho) = A(X)$, $g(U_1,\sigma) = B(X)$ and $g(\rho,\sigma) = 0$. Such a manifold is denoted by $MG(QC)_n$.

The spacetime of general relativity and cosmology is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-,+,+,+)$. The geometry of the Lorentz manifold begins with the study of a causal character of vectors of the manifold. Due to this causality, the Lorentz manifold becomes a convenient choice for the study of general relativity. Spacetimes have been studied by various authors in several ways, such as [10–14] and many others.

2. $MG(QE)_n$ Admitting the Generators ρ and σ as Recurrent Vector Fields

Let us consider the generators ρ and σ corresponding to the associated recurrent 1-forms A and B. Then, we have

$$(D_{U_1}A)(U_2) = \eta(U_1)A(U_2),$$

$$(D_{U_1}B)(U_2) = \varphi(U_1)B(U_2),$$

where η and φ are non-zero 1-forms.

A non-flat Riemannian or semi-Riemannian manifold (M^n, g), $(n > 2)$ is said to be Ricci-recurrent [15,16] if its $Ric(\neq 0)$ satisfies the following condition:

$$(D_{U_1}Ric)(U_2,U_3) = \alpha(U_1)Ric(U_2,U_3), \tag{13}$$

where α is in non-zero 1-form. Since we know that

$$(D_{U_1}Ric)(U_2,U_3) = U_1 Ric(U_2,U_3) - Ric(D_{U_1}U_2,U_3) - Ric(U_2,D_{U_1}U_3), \tag{14}$$

using (14) in (13), it follows that

$$\alpha(U_1)Ric(U_2,U_3) = U_1 Ric(U_2,U_3) - Ric(D_{U_1}U_2,U_3) - Ric(U_2,D_{U_1}U_3). \tag{15}$$

Using (5) in (15), we obtain

$$\begin{aligned}
&\alpha(U_1)[ag(U_2,U_3) + bA(U_2)A(U_3) + cB(U_2)B(U_3)\\
&+ d\{A(U_2)B(U_3) + A(U_3)B(U_2)\}] = U_1[ag(U_2,U_3) + bA(U_2)A(U_3)\\
&+ cB(U_2)B(U_3) + d\{A(U_3)B(U_2) + A(U_2)B(U_3)\}]\\
&- [ag(D_{U_1}U_2,U_3) + bA(D_{U_1}U_2)A(U_3) + cB(D_{U_1}U_2)B(U_3)\\
&+ d\{A(D_{U_1}U_2)B(U_3) + A(U_3)B(D_{U_1}U_2)\}]\\
&- [ag(U_2,D_{U_1}U_3) + bA(U_2)A(D_{U_1}U_3) + cB(U_2)B(D_{U_1}U_3)\\
&+ d\{A(U_2)B(D_{U_1}U_3) + A(D_{U_1}U_3)B(U_2)\}].
\end{aligned} \tag{16}$$

Putting $U_2 = U_3 = \rho$ in (16), we obtain

$$U_1(a+b) - \alpha(U_1)(a+b) = 2(a+b)A(D_{U_1}\rho) + 2dB(D_{U_1}\rho). \tag{17}$$

By using the fact that $A(D_{U_1}\rho) = 0$ and (6) in (17), we have

$$U_1(a+b) - \alpha(U_1)(a+b) = 2dg(D_{U_1}\rho,\sigma), \tag{18}$$

which can be written as

$$U_1(a+b) - \alpha(U_1)(a+b) = -2dA(D_{U_1}\sigma).$$

Thus, we have $A(D_{U_1}\sigma) = 0$ if and only if $U_1(a+b) - \alpha(U_1)(a+b) = 0$. This implies that either $D_{U_1}\sigma \perp \rho$ or σ is a parallel vector field.

Again, putting $U_2 = U_3 = \sigma$ in (16), we have

$$U_1(a+b) - \alpha(U_1)(a+b) = 2(a+c)B(D_{U_1}\sigma) + 2dA(D_{U_1}\sigma). \tag{19}$$

Again, using the fact that $B(D_{U_1}\sigma) = 0$ and (6) in (19), we have

$$U_1(a+b) - \alpha(U_1)(a+b) = 2dg(D_v\sigma,\rho), \tag{20}$$

$$or, \quad U_1(a+b) - \alpha(U_1)(a+b) = -2dB(D_v\rho).$$

Thus, we have $B(D_{U_1}\rho) = 0$ if and only if $U_1(a+b) - \alpha(U_1)(a+b) = 0$. This implies that either $D_{U_1}\rho \perp \sigma$ or ρ is a parallel vector field. Hence, we can state the following theorem:

Theorem 1. *Let a mixed generalized quasi-Einstein manifold $MG(QE)_n$ be Ricci-recurrent; then, the following statements are equivalent:*
(i) ρ and σ are parallel vector fields;
(ii) $U_1(a+b) - \alpha(U_1)(a+b) = 0$ if and only if $D_{U_1}\sigma \perp \rho$;
(iii) $U_1(a+b) - \alpha(U_1)(a+b) = 0$ if and only if $D_{U_1}\rho \perp \sigma$.

3. $MG(QE)_n$ Admitting the Generators ρ and σ as Concurrent Vector Fields

A vector field π is said to be concurrent if it satisfies the following condition [17,18]:

$$D_{U_1}\pi = \xi U_1, \tag{21}$$

where ξ is constant.

Let us consider the generators ρ and σ corresponding to the associated concurrent 1-forms A and B. Then, we have

$$(D_{U_1}A)(U_2) = \lambda g(U_1, U_2), \tag{22}$$

$$and \quad (D_{U_1}B)(U_2) = \mu g(U_1, U_2), \tag{23}$$

where λ and μ are non-zero constants.

Taking the covariant derivative of (5) with respect to U_3, we obtain

$$\begin{aligned}(D_{U_3}Ric)U_2 = {}& b[(D_{U_3}A)(U_1)A(U_2) + A(U_1)(D_{U_3}A)(U_2)] \\ &+ c[(D_{U_3}B)(U_1)B(U_2) + B(U_1)(D_{U_3}B)(U_2)] \\ &+ d[(D_{U_3}A)(U_1)B(U_2) + A(U_1)(D_{U_3}B)(U_2) \\ &+ (D_{U_3}B)(U_1)A(U_2) + B(U_1)(D_{U_3}A)(U_2)].\end{aligned} \tag{24}$$

Using (22) and (23) in (24), it follows that

$$\begin{aligned}(D_{U_3}Ric)(U_1, U_2) = {}& b[\lambda g(U_1, U_3)A(U_2) + \lambda g(U_2, U_3)A(U_1)] \\ &+ c[\mu g(U_1, U_3)B(U_2) + \mu g(U_2, U_3)B(U_1)] \\ &+ d[\lambda g(U_1, U_3)B(U_2) + \mu g(U_1, U_3)A(U_2) \\ &+ \lambda g(U_2, U_3)B(U_1) + \mu g(U_2, U_3)A(U_1)].\end{aligned} \tag{25}$$

Contracting (25) over U_1 and U_2 leads to

$$\partial r(U_3) = A(U_3)[2b\lambda + 2d\mu] + B(U_3)[2c\mu + 2d\lambda]. \tag{26}$$

From (7), it follows that

$$\partial r(U_1) = 0. \tag{27}$$

In view of (27), (26) turns to

$$A(U_3)[2b\lambda + 2d\mu] + B(U_3)[2c\mu + 2d\lambda] = 0. \tag{28}$$

Thus, by virtue of (28), (5) takes the form

$$Ric(U_1, U_2) = ag(U_1, U_2) + \Big[b + c\Big(\frac{(b\lambda + d\mu)}{(c\mu + d\lambda)}\Big)^2 - 2d\frac{(b\lambda + d\mu)}{(c\mu + d\lambda)}\Big]A(U_1)A(U_2) \tag{29}$$

which is a quasi-Einstein manifold. Thus, we can state the following theorem:

Theorem 2. *Let $MG(QE)_n$ be a mixed generalized quasi-Einstein manifold. If the associated vector fields of $MG(QE)_n$ are concurrent and the associated scalars are constants, then the manifold reduces to a quasi-Einstein manifold.*

4. $MG(QE)_n$ Admitting Einstein's Field Equations

The Einstein's field equations with and without cosmological constants are given by

$$Ric(U_1, U_2) - \frac{r}{2}g(U_1, U_2) + \lambda g(U_1, U_2) = \kappa T(U_1, U_2), \tag{30}$$

and

$$Ric(U_1, U_2) - \frac{r}{2}g(U_1, U_2) = \kappa T(U_1, U_2), \tag{31}$$

respectively; κ is a gravitational constant, λ is a cosmological constant, and T is the energy–momentum tensor.

Using (6) in (31), it follows that

$$\begin{aligned}&\left(a - \frac{r}{2}\right)g(U_1, U_2) + bA(U_1)A(U_2) + cB(U_1)B(U_2)\\&+ d[A(U_1)B(U_2) + A(U_2)B(U_1)] = \kappa T(U_1, U_2).\end{aligned} \tag{32}$$

Now, taking the covariant derivative of (32) with respect to U_3, we arrive at

$$\begin{aligned}&b[(D_{U_3}A)(U_1)A(U_2) + A(U_1)(D_{U_3}A)(U_2)]\\&+ c[(D_{U_3}B)(U_1)B(U_2) + B(U_1)(D_{U_3}B)(U_2)]\\&+ d[(D_{U_3}A)(U_1)B(U_2) + A(U_1)(D_{U_3}B)(U_2)\\&+ (D_{U_3}B)(U_1)A(U_2) + B(U_1)(D_{U_3}A)(U_2)] = \kappa(D_{U_3}T)(U_1, U_2).\end{aligned} \tag{33}$$

Thus, we have a result.

Theorem 3. *Let $MG(QE)_n$ admit Einstein's field equation without a cosmological constant. If the associated 1-forms A and B are covariantly constant, then the energy–momentum tensor is also covariantly constant.*

5. $MG(QE)_4$ Spacetime Admitting Space-Matter Tensor

In 1969, Petrov [19] introduced and studied the space–matter tensor $\overline{P}$ of type $(0, 4)$ and defined by

$$\overline{P} = \overline{K} + \frac{\kappa}{2}g \wedge T - \nu G, \tag{34}$$

where $\overline{K}$ is the curvature tensor of type $(0, 4)$, T is the energy–momentum tensor of type $(0, 2)$, κ is the gravitational constant, and ν is the energy density. Furthermore, G and $g \wedge T$ are, respectively, defined by

$$G(U_1, U_2, U_3, U_4) = g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4), \tag{35}$$

and

$$\begin{aligned}(g \wedge T)(U_1, U_2, U_3, U_4) = &\, g(U_2, U_3)T(U_1, U_4) + g(U_1, U_4)T(U_2, U_3)\\&- g(U_1, U_3)T(U_2, U_4) - g(U_2, U_4)T(U_1, U_3),\end{aligned} \tag{36}$$

for all U_1, U_2, U_3, U_4 on M.

Using (35) and (36) in (34), it follows that

$$\begin{aligned}\overline{P}(U_1, U_2, U_3, U_4) = &\,\overline{K}(U_1, U_2, U_3, U_4) + \frac{\kappa}{2}[g(U_2, U_3)T(U_1, U_4)\\&+ g(U_1, U_4)T(U_2, U_3) - g(U_1, U_3)T(U_2, U_4)\\&- g(U_2, U_4)T(U_1, U_3)] - \nu[g(U_2, U_3)g(U_1, U_4)\\&- g(U_1, U_3)g(U_2, U_4)].\end{aligned} \tag{37}$$

If $\overline{P} = 0$, then (37) gives

$$\begin{aligned}\overline{K}(U_1, U_2, U_3, U_4) = &-\frac{\kappa}{2}[g(U_2, U_3)T(U_1, U_4) + g(U_1, U_4)T(U_2, U_3)\\&- g(U_1, U_3)T(U_2, U_4) - g(U_2, U_4)T(U_1, U_3)]\\&+ \nu[g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4)].\end{aligned} \tag{38}$$

In view of (5), from (31), it follows that

$$\begin{aligned}\kappa T(U_1, U_2) = & \left(a - \frac{r}{2}\right) g(U_1, U_2) + bA(U_1)A(U_2) + cB(U_1)B(U_2) \\ & + d[A(U_1)B(U_2) + A(U_2)B(U_1)].\end{aligned} \tag{39}$$

Thus, from (38) and (39), we obtain

$$\begin{aligned}\overline{K}(U_1, U_2, U_3, U_4) = & f_1[g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4)] \\ & + f_2[g(U_1, U_4)A(U_2)A(U_3) - g(U_2, U_4)A(U_1)A(U_3) \\ & + g(U_2, U_3)A(U_1)A(U_4) - g(U_1, U_3)A(U_2)A(U_4)] \\ & + f_3[g(U_1, U_4)B(U_2)B(U_3) - g(U_2, U_4)B(U_1)B(U_3) \\ & + g(U_2, U_3)B(U_1)B(U_4) - g(U_1, U_3)B(U_2)B(U_4)] \\ & + f_4[g(U_1, U_4)\{A(U_2)B(U_3) + B(U_2)A(U_3)\} \\ & - g(U_2, U_4)\{A(U_1)B(U_3) + B(U_1)A(U_3)\} \\ & + g(U_2, U_3)\{A(U_1)B(U_4) + B(U_1)A(U_4)\} \\ & - g(U_1, U_3)\{A(U_2)B(U_4) + B(U_2)A(U_4)\}],\end{aligned} \tag{40}$$

where $f_1 = (\nu - a + \frac{r}{2})$, $f_2 = -\frac{b}{2}$, $f_3 = -\frac{c}{2}$, $f_4 = -\frac{d}{2}$. Thus, we can state the following theorem:

Theorem 4. *For a vanishing space–matter tensor, $MG(QE)_4$ spacetime satisfying Einstein's field equation without a cosmological constant is a $MG(QC)_4$ spacetime.*

Next, we investigate the existence of a sufficient condition under which $MG(QE)_4$ can be a divergence-free space–matter tensor.

From (31) and (37), we obtain

$$\begin{aligned}(div\overline{P})(U_1, U_2, U_3) = & (divK)(U_1, U_2, U_3) + \frac{1}{2}[(D_{U_1}Ric)(U_2, U_3) \\ & - (D_{U_2}Ric)(U_1, U_3)] - g(U_2, U_3)[\frac{1}{4}\partial r(U_1) + \partial\nu(U_1)] \\ & + g(U_1, U_3)[\frac{1}{4}\partial r(U_2) + \partial\nu(U_2)].\end{aligned} \tag{41}$$

By using $(divK)(U_1, U_2, U_3) = (D_{U_1}Ric)(U_2, U_3) - (D_{U_2}Ric)(U_1, U_3)$ in (41), we obtain

$$\begin{aligned}(div\overline{P})(U_1, U_2, U_3) = & \frac{3}{2}[(D_{U_1}Ric)(U_2, U_3) - (D_{U_2}Ric)(U_1, U_3)] \\ & - g(U_2, U_3)[\frac{1}{4}\partial r(U_1) + \partial\nu(U_1)] \\ & + g(U_1, U_3)[\frac{1}{4}\partial r(U_2) + \partial\nu(U_2)].\end{aligned} \tag{42}$$

Let $(div\overline{P})(U_1, U_2, U_3) = 0$; then, contracting (42) over U_2 and U_3, we obtain $\partial\nu(U_1) = 0$, where (27) is used. Hence, we can state the following theorem:

Theorem 5. *For a divergence-free space–matter tensor, the energy density in $MG(QE)_4$ spacetime satisfying Einstein's field equation without a cosmological constant is constant.*

Now, by using (5) in (42), we obtain

$$\begin{aligned}(div\overline{P})(U_1,U_2,U_3) =& \frac{3}{2}[\partial a(U_1)g(U_2,U_3) - \partial a(U_2)g(U_1,U_3)] \\ &+ \frac{3}{2}[\partial b(U_1)A(U_2)A(U_3) - \partial b(U_2)A(U_1)A(U_3)] \\ &+ \frac{3b}{2}[(D_{U_1}A)(U_2)A(U_3) + A(U_2)(D_{U_1}A)(U_3) \\ &- (D_{U_2}A)(U_1)A(U_3) - (D_{U_2}A)(U_3)A(U_1)] \\ &+ \frac{3}{2}[\partial c(U_1)B(U_2)B(U_3) - \partial c(U_2)B(U_1)B(U_3)] \\ &+ \frac{3c}{2}[(D_{U_1}B)(U_2)B(U_3) + B(U_2)(D_{U_1}B)(U_3) \\ &- (D_{U_2}B)(U_1)B(U_3) - (D_{U_2}B)(U_3)B(U_1)] \\ &+ \frac{3}{2}[\partial d(U_1)\{A(U_2)B(U_3) + B(U_2)A(U_3)\} \\ &- \partial d(U_2)\{A(U_1)B(U_3) + B(U_1)A(U_3)\}] \\ &+ \frac{3d}{2}[(D_{U_1}A)(U_2)B(U_3) + A(U_2)(D_{U_1}B)(U_3) \\ &+ (D_{U_1}A)(U_3)B(U_2) + A(U_3)(D_{U_1}B)(U_2) \\ &- (D_{U_2}A)(U_1)B(U_3) - A(U_1)(D_{U_2}B)(U_3) \\ &- (D_{U_2}A)(U_3)B(U_1) - A(U_3)(D_{U_2}B)(U_1)] \\ &- g(U_2,U_3)[\frac{1}{4}\partial r(U_1) + \partial\nu(U_1)] \\ &+ g(U_1,U_3)[\frac{1}{4}\partial r(U_2) + \partial\nu(U_2)]. \end{aligned} \tag{43}$$

By assuming that ν, a, b, c, and d are constants and the generator ρ is a parallel vector field, i.e., $D_{U_1}\rho = 0$, we obtain

$$\partial r(U_1) = 0, \quad \partial\nu(U_1) = 0, \quad (D_{U_1}A)(U_2) = 0. \tag{44}$$

In view of (44), we derive

$$a + b = 0, \quad c = 0, \quad d = 0. \tag{45}$$

Using (44) and (45), (43) reduces to

$$(div\overline{P})(U_1,U_2,U_3) = 0.$$

Thus, we can state the following theorem:

Theorem 6. *In $MG(QE)_4$ spacetimes admitting parallel vector field ρ satisfying Einstein's field equation without a cosmological constant, if the energy density and associated scalars constant are constants, then the divergence of the space–matter tensor vanishes.*

6. $MG(QE)_4$ Spacetime Admitting General Relativistic Viscous Fluid

Ellis [20] defined the energy–momentum tensor for a perfect fluid distribution with heat conduction as

$$\begin{aligned} T(U_1,U_2) =& \omega g(U_1,U_2) + (\nu+\omega)A(U_1)A(U_2) + B(U_1)B(U_2) \\ &+ A(U_1)B(U_2) + A(U_2)B(U_1), \end{aligned} \tag{46}$$

where $g(U_1,\rho) = A(U_1)$, $g(U_1,\sigma) = B(U_1)$, $A(\rho) = -1$, $B(\sigma) > 0$, $g(\rho,\sigma) = 0$, and ν, ω are called the isotropic pressure and the energy density, respectively. σ is the heat conduction vector field perpendicular to the velocity vector field ρ. Assuming a mixed generalized quasi-Einstein spacetime satisfying Einstein's field equation without a cosmological constant whose matter content is viscous fluid, then, from (31) and (46), the Ricci tensor takes the form

$$Ric(U_1, U_2) = (\kappa\omega + \frac{r}{2})g(U_1, U_2) + \kappa(\nu + \omega)A(U_1)A(U_2) + \kappa B(U_1)B(U_2) + \kappa[A(U_1)B(U_2) + A(U_2)B(U_1)]. \tag{47}$$

By comparing (5) and (47), we obtain

$$a = \kappa\omega + \frac{r}{2}, \quad b = \kappa(\nu + \omega), \quad c = \kappa, \quad d = \kappa. \tag{48}$$

Taking a frame field to contract (48) over U_1 and U_2, we obtai

$$r = \kappa(\nu - 3\omega). \tag{49}$$

In view of (49), (47) turns to

$$Ric(U_1, U_2) = \frac{\kappa(\nu - \omega)}{2}g(U_1, U_2) + \kappa(\nu + \omega)A(U_1)A(U_2) + \kappa B(U_1)B(U_2) + \kappa[A(U_1)B(U_2) + A(U_2)B(U_1)]. \tag{50}$$

Now, let R be the Ricci operator given by $g(R(U_1), U_2) = Ric(U_1, U_2)$ and $Ric(R(U_1), U_2) = Ric^2(U_1, U_2)$. Then, we have $A(R(U_1)) = g(R(U_1), \rho) = Ric(U_1, \rho)$ and $B(R(U_1)) = g(R(U_1), \sigma) = Ric(U_1, \sigma)$. Thus, we obtain

$$Ric(R(U_1), U_2) = \frac{\kappa(\nu - \omega)}{2}Ric(U_1, U_2) + \kappa(\nu + \omega)Ric(U_1, \rho)A(U_2) + \kappa Ric(U_1, \sigma)B(U_2) + \kappa[Ric(U_1, \rho)B(U_2) + A(U_2)Ric(U_1, \sigma)]. \tag{51}$$

Now, contracting (51) over U_1 and U_2, we obtain

$$Ric(U_1, U_1) = ||R||^2 = \frac{\kappa(\nu - \omega)r}{2} + \kappa(\nu + \omega)Ric(\rho, \rho) + \kappa Ric(\sigma, \sigma) + \kappa[Ric(\rho, \sigma) + Ric(\sigma, \rho)]. \tag{52}$$

For a mixed generalized quasi-Einstein spacetime, from (5), it follows that

$$Ric(U_1, \rho) = (a - b)A(U_1) - dB(U_1), \quad Ric(U_1, \sigma) = (a + c)B(U_1) + dA(U_1). \tag{53}$$

In view of (48), (49), and (53), we find that

$$Ric(\rho, \rho) = \frac{\kappa(\nu + 3\omega)}{2}, \ Ric(\sigma, \rho) = Ric(\rho, \sigma) = -\kappa, \ Ric(\sigma, \sigma) = \frac{\kappa(\nu - \omega + 2)}{2}. \tag{54}$$

By making use of (54), from (52), it follows that

$$||R||^2 = \kappa^2(\nu^3\omega^2 + \nu + \omega - 3). \tag{55}$$

Thus, we can state the following theorem:

Theorem 7. *If $MG(QE)_4$ spacetime admitting viscous fluid satisfies Einstein's field equation without a cosmological constant, then the square of the length of Ricci operator is $\kappa^2(\nu^3\omega^2 + \nu + \omega - 3)$.*

7. Example of $MG(QE)_4$ Spacetime

In this section, we constructed a non-trivial concrete example to prove the existence of a $MG(QE)_4$ spacetime.

We assume a Lorentzian manifold (M^4, g) endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}du^i du^j = (1+2p)[(du^1)^2 + (du^2)^2 + (du^3)^2 - (du^4)^2], \tag{56}$$

where u^1, u^2, u^3, u^4 are standard coordinates of M^4, i, j = 1, 2, 3, 4, and $p = e^{u^1}k^{-2}$, and k is a non-zero constant. Here, the signature of g is $(+,+,+,-)$, which is Lorentzian. Then, the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{p}{1+2p}, \ \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{-p}{1+2p}. \tag{57}$$

$$\overline{K}_{1212} = \overline{K}_{1313} = \frac{-p}{1+2p}, \ K_{1414} = \frac{p}{1+2p},$$

$$\overline{K}_{3232} = \frac{-p^2}{1+2p}, \ \overline{K}_{4242} = \overline{K}_{4343} = \frac{p^2}{1+2p}$$

and the components are obtained by the symmetry properties.

The non-vanishing components of the Ricci tensors are

$$R_{11} = \frac{3p}{(1+2p)^2}, \ R_{22} = R_{33} = \frac{p}{(1+2p)^2}, \ R_{44} = \frac{-p}{(1+2p)^2},$$

Thus, the scalar curvature r is $\frac{6q(1+q)}{(1+2q)^3}$.

Let us consider the associated scalars a, b, c, and d defined by

$$a = \frac{p}{(1+2p)^3}, \ b = \frac{1}{(1+2p)}, \ c = \frac{-1}{(1+2p)^3}, \ d = \frac{-p}{(1+2p)^2}$$

and the 1-forms are defined by

$$A_1 = B_1 = \sqrt{1+2p}, \quad A_i = B_i = 0 \quad \forall \quad i = 2, 3, 4,$$

where the generators are unit vector fields; then, from (5), we have

$$R_{11} = ag_{11} + bA_1A_1 + cB_1B_1 + d(A_1B_1 + A_1B_1), \tag{58}$$

$$R_{22} = ag_{22} + bA_2A_2 + cB_2B_2 + d(A_2B_2 + A_2B_2), \tag{59}$$

$$R_{33} = ag_{33} + bA_3A_3 + cB_3B_3 + d(A_3B_3 + A_3B_3), \tag{60}$$

$$R_{44} = ag_{44} + bA_4A_4 + cB_4B_4 + d(A_4B_4 + A_4B_4). \tag{61}$$

$$\begin{aligned} Now, R.H.S. \ of \ (58) \quad &= \quad ag_{11} + bA_1A_1 + cB_1B_1 + d(A_1B_1 + A_1B_1) \\ &= \frac{3p}{(1+2p)^2} \\ &= R_{11} \\ &= L.H.S. \ of \ (58). \end{aligned}$$

Similarly, it can easily be show that (59), (60), and (61) are also true. Hence, $(\mathbb{R}^4, g)$ is a $MG(QE)_4$.

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