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Curriculum Development in Mathematics Education

Edited by
Lili Zhou and Jane-Jane Lo

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Editorial

Evolving Conversations in Mathematics Curriculum Research

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A curriculum is a living, socially mediated set of materials that both reflects and shapes what counts as mathematics and who is positioned to learn it (Ball & Cohen, 1996; Remillard, 2005; Rezat, 2012, 2024; Smith & Barrett, 2017). Previous mathematics curriculum studies have revealed that what often appear as "differences in content organization" are, in fact, differences in the purposes that curricula serve within distinct educational traditions (e.g., Bütüner, 2021; Yang et al., 2025; Zhou et al., 2022; Lo et al., 2024). This realization has shaped our ongoing inquiry into curriculum research scope, methodology, and impacts, moving beyond surface-level comparisons toward broader interpretations of how and what mathematical knowledge is selected, organized, and experienced (Stein et al., 2007).

For this Special Issue on mathematics curriculum research, we invited diverse contributions that interrogate curriculum's role in mediating learning opportunities, fostering equity, and responding to social and institutional contexts. The contributing authors, representing Croatia (Pleština et al., 2025 [Contribution 6]), the United Kingdom (Golding, 2024 [Contribution 3]), South Africa (Khoza & Ngcobo, 2025 [Contribution 4]), and the United States (Edson et al., 2025 [Contribution 1]; Huntley et al., 2024 [Contribution 2]; Lo & Zhou, 2025 [Contribution 5]; Stone et al., 2025 [Contribution 7]), bring rich international perspectives to this conversation. The eight papers featured here provide theoretical, empirical, and practical insights across nations, curriculum types, and levels of schooling. Together, they present a multifaceted portrait of curriculum as a dynamic and critical component of mathematics education. Collectively, the eight publications illuminate the following three themes: (a) curriculum content and design, (b) curriculum policy, implementation, and influence and equity, and (c) culture and the purpose of curriculum.

Mathematics Curriculum Content and Design

Several papers in this Special Issue delve into the substance and structure of mathematics curriculum content, revealing how curricular design shapes students' learning experiences. Lo and Zhou's (2025) [Contribution 5] study focuses on how textbooks structure a small topic (the relationship between area and perimeter), which highlights how different instructional sequences in two U.S. textbooks can shape students' conceptual understanding. Huntley et al. (2024) [Contribution 2] investigate 63,000 algebra problems in U.S. high school textbooks, identifying substantial differences in cognitive demand, content sequencing, and the integration of real-world contexts. Feedback from textbook authors provides rare insight into how design choices reflect underlying educational philosophies and priorities. Pleština et al. (2025) [Contribution 6] trace the historical evolution of polynomial knowledge within Croatian high school mathematics, showing how shifts in curricular frameworks influence the coherence between algebraic and analytical reason-

ing. These works illuminate the complexity of content design and underscore the diverse epistemological and pedagogical assumptions embedded in curriculum materials.

Curriculum Policy, Implementation, and Influence

Another cluster of studies examines the dynamic interactions between curriculum policy, classroom enactment, and educational outcomes. These contributions foreground the interconnectedness of curriculum design, teacher learning, and student achievement, showing that curricular reform is not merely a matter of content revision but also of professional and institutional mediation. Golding's (2024) [Contribution 3] study provides a longitudinal account of mathematics curriculum implementation in England, illustrating how teachers, edu-businesses, and even learners act as policy mediators who reinterpret and reshape curricular intentions. Similarly, Edson et al. (2025) [Contribution 1] examine teacher perceptions of a field-test version of the fourth edition of Connected Mathematics in the United States, highlighting the importance of iterative curriculum development grounded in classroom realities and emphasizing the interconnectedness of curriculum design, teacher learning, and student outcomes. In the South African context, Khoza and Ngcobo (2025) [Contribution 4] highlight how school leaders' decisions around curriculum adoption can influence instructional coherence across contexts. Together, these papers deepen our understanding of curriculum as a negotiated space that is continually reinterpreted through the practices and perspectives of multiple educational actors.

Equity, Culture, and the Purpose of Curriculum

Another group of papers brings a critical and cultural lens to mathematics curriculum studies, addressing the extent to which curriculum can serve as a vehicle for equity and empowerment. Stone et al. (2025) [Contribution 7] develop and apply the Culturally Responsive Mathematics Materials Evidence Tool (CRM-MET) to analyze middle school curricula, revealing substantial gaps in how power, participation, and cultural identity are represented. Zelkowski et al. (2025) [Contribution 8] describe a long-term, structured professional development model that empowers teachers as leaders and change agents in high-need schools, demonstrating that curriculum engagement can promote both instructional growth and social justice-oriented leadership. These studies collectively argue that curriculum is not neutral; rather, it embodies values, voices, and visions of who can participate in mathematics and how.

This Special Issue contributes to the evolving conversation about what mathematics curriculum is and what it can become (Herbst et al., 2023). It invites researchers, teachers, and policymakers to view curriculum not as a finished product but as an evolving practice shaped by educators' interpretation and institutional structures. Echoing the insights of Aguirre et al. (2024) and Fan et al. (2025), the works collected here remind us that curriculum research is most powerful when it bridges theory and practice, challenges taken-for-granted assumptions, and envisions mathematics learning as an inclusive and transformative endeavor. We hope this Special Issue inspires continued dialogue, critical reflection, and collective action toward more coherent, equitable, and responsive mathematics curricula worldwide.

Conflicts of Interest: The authors declare no conflicts of interest.

List of Contributions

The following eight articles are featured in this Special Issue, Curriculum Development in Mathematics Education:

1. Edson, A. J., Wald, S., & Phillips, E. D. (2025). The design, field testing, and evaluation of a contextual, problem-based curriculum: feedback analysis from mathematics

teachers on the field test version of connected Mathematics® 4. *Education Sciences*, *15*(5), 628.
2. Huntley, M. A., Terrell, M. S., & Fonger, N. L. (2024). A content analysis of the algebra strand of six commercially available US high school textbook series. *Education Sciences*, *14*(8), 845.
3. Golding, J. (2024). Teachers, learners and edu-business co-constructing mathematics curriculum implementation: An insider's lens in cross-phase longitudinal research. *Education Sciences*, *14*(12), 1322.
4. Khoza, M. M., & Ngcobo, A. Z. (2025). Forces Influencing technical mathematics curriculum implementation: Departmental heads' understanding of their practices to enact roles and responsibilities. *Education Sciences*, *15*(1), 103.
5. Lo, J. J., & Zhou, L. (2025). Analysis of curricular treatment of the relationship between area and perimeter in two US curricula. *Education Sciences*, *15*(10), 1342.
6. Pleština, J., Milin Šipuš, Ž., & Bašić, M. (2025). Polynomials—Unifying or fragmenting high school mathematics? *Education Sciences*, *15*(7), 854.
7. Stone, R., Smith, E. P., & Ebner, R. J. (2025). Culturally responsive mathematics and curriculum materials: Present realities and imagined futures. *Education Sciences*, *15*(9), 1246.
8. Zelkowski, J., Bergeron, B., Gleason, J., Makowski, M., & Petrulis, R. (2025). Developing secondary mathematics teacher leaders: A multi-year curriculum for inservice teacher excellence. *Education Sciences*, *15*(7), 788.

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Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, *75*(2), 211–246. [CrossRef]

Rezat, S. (2012). Interactions of teachers' and students' use of mathematics textbooks. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *Mathematics curriculum material and teacher development: From text to 'lived' resources* (pp. 231–246). Springer.

Rezat, S. (2024). Research on curriculum resources in mathematics education: A survey of the field. *ZDM–Mathematics Education*, *56*(2), 223–237. [CrossRef]

Smith, J. P., III, & Barrett, J. E. (2017). Learning and teaching measurement: Coordinating quantity and number. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 355–385). National Council of Teachers of Mathematics.

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Yang, X., Chan, M. C. E., Kaur, B., Deng, J., Luo, J., & Wen, Y. (2025). International comparative studies in mathematics education: A scoping review of the literature from 2014 to 2023. *ZDM–Mathematics Education*, *57*, 745–761. [CrossRef]

Zhou, L., Liu, J., & Lo, J. (2022). A comparison of U.S. and Chinese geometry standards through the lens of van Hiele levels. *International Journal of Education in Mathematics, Science, and Technology (IJEMST)*, *10*(1), 38–56. [CrossRef]

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Article

Analysis of Curricular Treatment of the Relationship Between Area and Perimeter in Two U.S. Curricula

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Abstract: This study examines how two widely used elementary mathematics curricula, *Bridges in Mathematics* and *Eureka Math*, support grade 3 students' conceptual understanding of the relationship between area and perimeter. Drawing on the mathematical treatment and emphasis component of the analytic framework, we identified distinct instructional strategies and learning opportunities. Findings indicate distinct instructional strategies and learning opportunities: *Bridges in Mathematics* emphasizes hands-on exploration, pattern recognition, and student-led reasoning using real-world contexts. In contrast, *Eureka Math* employs a more structured and symbolic approach, using multiplication, factor pairs, and line plots to support generalization and data-driven reasoning. Both curricula share strengths such as the use of visual supports, real-world contexts, and attention to student reasoning, yet they differ in how they scaffold conceptual development. Rather than recommending one curriculum over the other, the study highlights how each curriculum sequences ideas and supports mathematical reasoning, offering insights into curricular design and the learning experiences they foster.

Keywords: area and perimeter; elementary mathematics curricula; learning opportunities; mathematical treatment and emphasis

1. Introduction

Mathematics textbooks, curriculum materials, and instructional resources have long been a cornerstone of classrooms globally (Valverde et al., 2002). They mediate curriculum policy (Valverde et al., 2002), support teachers' instructional decisions (Stein et al., 2007) and promote curricular coherence across diverse educational settings (Schmidt et al., 2005). In recent decades, growing public scrutiny of school mathematics has intensified the prominence of curriculum materials in educational policy and practice, fueling a multi-billion-dollar international industry.

Following the release of the Common Core State Standards for Mathematics curriculum in the United States (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), K–12 mathematics curricula in the U.S. have gone through major revisions in terms of content sequence, grade placements, and pedagogical approach to align with the national standards. Understanding the extent of alignment between mathematics textbooks and the Common Core State Standards for Mathematics (CCSSM) is significant for both scholarly inquiry and practical decisions related to curriculum adoption and teacher professional development (Porter et al., 2011; Zhou et al., 2022, 2023).

Prior studies on the alignment of mathematics textbooks with the CCSSM reveal a complex landscape, often challenging publishers' claims of full adherence. A major theme emerging from these studies is that textbooks frequently exhibit significant areas of misalignment, particularly in their emphasis on procedural knowledge over the conceptual understanding and mathematical practices central to the CCSSM. For instance, Polikoff (2015) found that many textbooks systematically overemphasize procedures and memorization, failing to adequately address the higher levels of cognitive demands the standards promote. Hong et al. (2019) similarly found gaps in the coverage and sequencing of area measurement topics, which raise concerns about whether current curricula adequately support the development of key mathematical ideas.

Despite the decades of research, misconceptions about the relationship between area and perimeter persist among learners and educators. Studies consistently show that both pre-service and in-service teachers often hold the mistaken belief that a larger perimeter necessarily corresponds to a larger area (Ma, 2010; Livy et al., 2012; Tan Sisman & Aksu, 2016; Widjaja & Vale, 2021). Such misconceptions hinder flexible problem solving and suggest that instructional interventions may not be fully addressing the conceptual complexity of this topic.

CCSSM standard calls for instruction that supports students in recognizing the relationships between area and perimeter:

Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters (CCSS.MATH.CONTENT.3.MD.D.8).

This makes the concept particularly important for developing mathematical reasoning and spatial understanding in elementary grades.

While previous curriculum analyses focus on broad topics such as geometry and measurement (Polikoff, 2015; Hong et al., 2019), few have investigated the learning opportunities for students and teachers afforded by curricula at the concept level. The relationship between area and perimeter offers a compelling case study because it requires coordination of multiple representations, reasoning about counterexamples, and careful distinction between related but distinct measures.

The present study aims to address this gap by conducting a detailed comparison of two widely used curricula, *Bridges in Mathematics* and *Eureka Math*. These curricula were selected because they have high adoption rates nationally (EdReports, n.d.-a, n.d.-b; Kaufman et al., 2017), and both explicitly target the Grade 3 standards (i.e., CCSS.MATH.CONTENT.3.MD.D.8) for area and perimeter relationships. Understanding how these curricular materials structure learning opportunities for this concept can inform teachers' instructional decisions, curriculum developers' design choices, and researchers' understanding of curriculum affordances. It is important to note that this analysis is limited to the design and content of the curriculum materials themselves and does not evaluate classroom implementation or student learning outcomes.

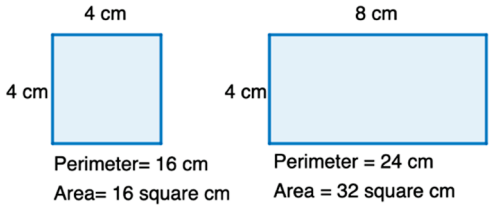
2. Literature and Theoretical Perspectives

2.1. Misconceptions and Challenges in Understanding Area–Perimeter Relationships

Research has shown both pre-service and in-service teachers often misunderstand the relationship. Livy et al. (2012) found that 72% of a group of pre-service primary teachers from Australia thought that an increase in perimeter would always result in an increase in area. Ma (2010) presented the question shown in Figure 1 to a group of 23 U.S. and 72 Chinese elementary school teachers. Notably, 96% of the U.S. teachers and 31% of the

Chinese teachers believed the student's claim was correct—either immediately or after some reflection.

Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you the following picture to prove what she is doing:



How would you respond to this student?

Figure 1. Task prompt exploring misconceptions about the relationship between area and perimeter (Ma, 2010, p. 91).

An assumption about the existence of a "direct proportional relationship" between area and perimeter was reported among students of different ages as well as both pre-service and in-service teachers (e.g., Livy et al., 2012; Lo et al., 2019; Machaba, 2016; Tan Sisman & Aksu, 2016; Widjaja & Vale, 2021). For example, in Tan Sisman and Aksu's (2016) study, only 5% of the 445 Turkish sixth-grade students participating in the study were able to correctly answer the following question, shown in Figure 2, that examines the relationship between the area and perimeter. Most of the students (62%) believed that since both shapes are made up of the same pieces, their perimeters are equal, indicating a lack of understanding that perimeter is a measure of boundary length and can change when parts of a shape's boundary become internal after partitioning or rearrangement. Likewise, Lo et al. (2019) found that approximately 76.5% of U.S. seventh graders inaccurately responded 'it gets shorter' when questioned about the perimeter of a piece of paper after a single piece is removed. The actual change in perimeter (shorter, longer, or unchanged) is contingent upon the location and shape of the cut.

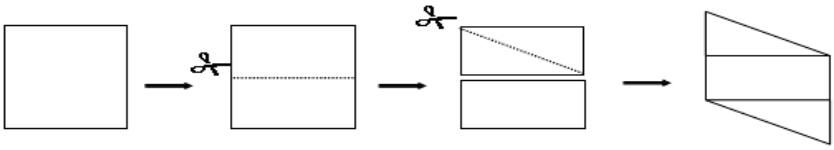


Figure A **Figure B**

A square-shaped paper, as shown in Figure A, is folded in the middle and is cut in half. Then, one of the pieces is cut again as shown above. Figure B is formed by putting the pieces together as shown in the picture. Based on the information given, choose one of three comments about the figures and explain your reasoning.

Comment 1: The perimeter of Figure A is greater than the perimeter of Figure B.

☐ **I agree, because**.....

Comment 2: The perimeters of Figure A and Figure B are equal.

☐ **I agree, because**.....

Comment 3: The perimeter of Figure B is greater than the perimeter of Figure A.

☐ **I agree, because** .....

Figure 2. Task prompt exploring misconceptions about the relationship between area and perimeter (Tan Sisman & Aksu, 2016, Appendix C).

These findings illustrate persistent difficulty in understanding that the perimeter, as a property of the boundary, is independent of its internal area. Additionally, errors could stem from pervasive intuitive rules, such as "more A, then more B," often seen across various significant STEM concepts (Tirosh & Stavy, 1999). The difficulty is compounded by shared variables, i.e., width and length. J. P. Smith and Barrett (2017) concluded in their review that "Overall, research has not yet produced a compelling explanation for this challenge or an effective instructional response" (p. 365). Furthermore, the enduring belief in a linear relationship between area and perimeter might also be attributed to a weak appreciation for the power of counterexamples in mathematical proof. Indeed, Widjaja and Vale (2021) found that a majority of 82 Australian students in grades 4–6 lacked the understanding of how one counterexample can disprove a mathematical statement.

Although instructional materials often ask students to compare rectangles with same areas and different perimeters (or vice versa) as suggested by the CCSSM, this approach appears to have limited long-term effectiveness, as seen by the difficulties faced by pre-service and in-service teachers discussed earlier. The insufficient impact of using rectangles may be due to its strong emphasis on numerical and logical data, whereas a true conceptual understanding of area, perimeter, and their interrelationship is fundamentally tied to intuition, spatial reasoning, and everyday experiences (Machaba, 2016). This study aims to contribute to this conversation by examining how two U.S. mathematics curricula support grade 3 students in developing a robust understanding of the relationship between area and perimeter.

2.2. Mathematics Curricula Analytical Lens

Curriculum materials shape how mathematical ideas are introduced, sequenced, and emphasized in classrooms, thereby influencing the opportunities students have to learn (Valverde et al., 2002; Stein et al., 2007). Effective curriculum goes beyond simply listing topics covered and delves into the *nature* and *quality* of engagement students have with mathematical content (Dietiker & Richman, 2021; Lo et al., 2024; Zhou et al., 2023).

In mathematics curriculum research, "learning opportunities" are recognized as a multifaceted for understanding and evaluating the quality of engagement students have with mathematical content (Remillard & Kim, 2000). Examining curricula at learning opportunities can illuminate the design decisions and pedagogical affordances that either support or constrain deep conceptual understanding.

Remillard and Kim (2000) conceptualize Mathematical Treatment and Emphasis of curricular materials as "aspects of mathematics emphasized and how mathematics is organized and presented for student learning" (p. 11). This construct highlights how mathematical content is presented, sequenced, and framed; the precision of definitions; the choice and variety of examples and non-examples; and the integration of visual, symbolic, and contextual representations.

Building on this conceptualization, Kim and Remillard (2000), in their comparative analysis of five curricula across grades 3–5 with a focus on number and operations, operationalized Mathematical Treatment and Emphasis into four analytic dimensions: scope and sequence, cognitive demand of student tasks, ongoing practice, and representation.

Our theoretical perspective was informed by this prior work, but we adapted it to align with the aims and scope of our study. Specifically, because our analysis focuses on a specific mathematical topic—the relationship between area and perimeter—in grade 3, rather than on broader curricular domains, we modified the analytic framework to attend more closely to topic-specific features. We therefore examine Mathematical Treatment and Emphasis through the following four adapted dimensions: overall lesson structures,

introductory lesson, mathematical emphasis across the lesson sequence, and mathematical representation.

Overall lesson structures indicate captures how lessons are organized and displayed in relation to learning pathways (Remillard & Kim, 2000). Examining lesson structures provides insight into the extent of instructional coherence and how the overall sequencing supports student learning. Introductory lessons play a pivotal role in shaping the tone and trajectory of student learning (Lo et al., 2024). We attend to how the first encounters establish key connections, situate the mathematics in context, and expect approaches students may use. Mathematical emphasis across the lesson sequence considers how emphases are sustained, shifted, or deepened as lessons progress (Kim & Remillard, 2000). Mathematical representation provides a critical analytic lens for analyzing the ways in which the mathematical content is conveyed within the curricula (Kim & Remillard, 2000).

We chose not to include cognitive demand of student tasks as a separate analytic dimension. Although we recognize the importance of task demand in shaping student learning, our analysis is limited to classwork that directly mediates instruction, discourse, and in-class learning opportunities. In reviewing the tasks using M. S. Smith and Stein's (1998) framework, we found that even tasks classified at lower levels (e.g., memorization) often served as essential foundations for subsequent reasoning and instruction. Categorizing tasks strictly by cognitive demand would therefore obscure their pedagogical role within the lesson sequence and add limited analytic value for this study.

The four dimensions of Mathematical Treatment and Emphasis informs our analysis of how the *Bridges in Mathematics* and *Eureka Math* curricula structure content, guide instruction, and support teachers in developing third-grade students' understanding of the relationship between area and perimeter. By comparing these two textbooks, researchers and educators can gain a comprehensive understanding of the affordances and limitations of different elementary mathematics curricula. The insights can guide teachers in making informed instructional decisions, adapting tasks, and using textbooks more effectively to support student learning and mathematical understanding. Specifically, we address the following overarching question: To what extent do the *Bridges in Mathematics* and *Eureka Math* curricula provide distinct or overlapping learning opportunities for third-grade students to develop a conceptual understanding of the relationship between area and perimeter?

3. Methods

Comparative analyses of curricula are well-established approach for examining the treatment of particular mathematical topics or domains, either cross-national or within a single context. Cai and Cirillo (2014) raised questions for curricula studies "what text in textbook(s) should we analyze and How much of that text should we analyze?" (p. 136). To investigate how the selected curricula address standards related to the relationship between area and perimeter, the two authors engaged in a careful selection process followed by iterative discussions to confirm the final data sources. The following section details the data sources and outlines the analytical schemes used in this study.

3.1. Data Source

This study examined two grade 3 U.S. mathematics curricula: *Bridges in Mathematics*, published by The Math Learning Center (2024), and *Eureka Math*, published by Great Minds (2015). The primary data source was teacher guides accompanying each curriculum.

Bridges in Mathematics is a widely used K–5 curriculum designed to foster conceptual understanding and problem solving while supporting procedural fluency. It emphasizes

active, hands-on learning through the use of visual models and manipulatives to help students build mathematical understanding.

Eureka Math (formerly known as *EngageNY*) is a fully CCSSM-aligned K–12 curriculum originally created and maintained by the New York State Education Department. As of 2017, it had been downloaded more than 66 million times by educators across the country since the website launched in 2011 (Kaufman et al., 2017). The curriculum is available in multiple languages, including English, Spanish, and Chinese.

The selection of these two textbooks was not arbitrary. It was based on three key consideration: widespread use, accessibility for research, and conceptual similarity. First, these textbooks are widely used, and both strongly align with the CCSSM (EdReports, n.d.-a, n.d.-b). This popularity and alignment make them excellent subjects for a study aiming to provide relevant, broadly applicable findings. Additionally, both curricular materials are accessible for research purposes upon online requests via their publishers' websites. Such accessibility allows for a thorough and replicable research process. Other researchers can easily verify our findings and conduct similar investigations, which is crucial for the transparency and reproducibility of academic research.

Finally, a quick look at their core student materials reveals they both consist of problem sets with dedicated space for student work (as shown in Figure 3); this initial resemblance is precisely what makes them good candidates for a comparative study. Both curricula officially state they prioritize the development of conceptual understanding and procedural fluency through problem-solving.

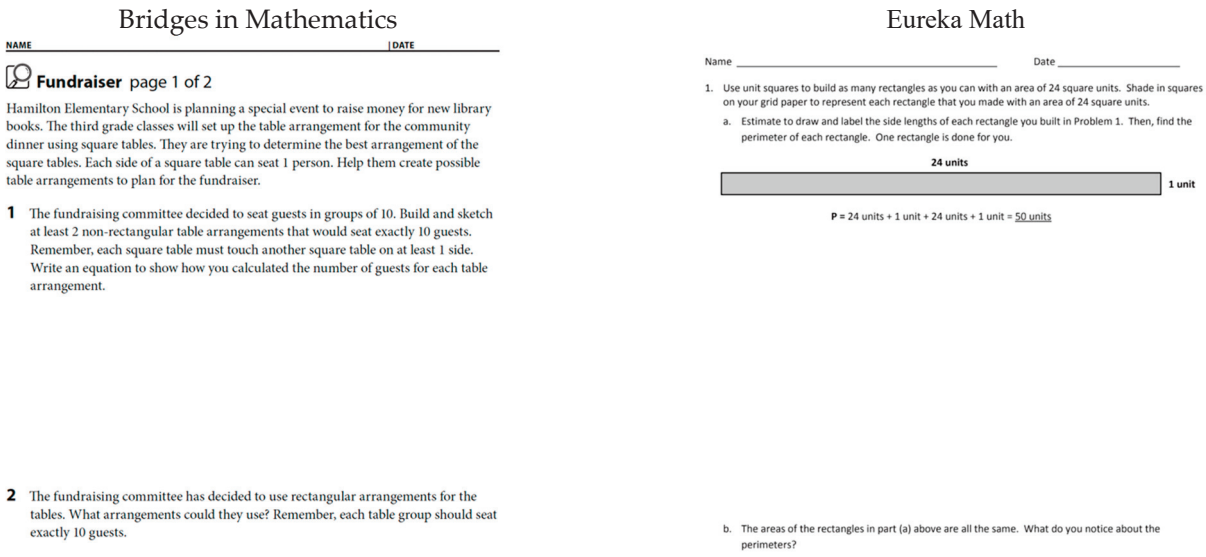


Figure 3. Sample of Student Workbook page from each curriculum.

Our preliminary analysis, however, has unearthed numerous distinctions in their instructional approaches and design elements that are not immediately obvious. By delving into these specific differences, we can move beyond the "what" of their similar goals to investigate the "how" of their distinct methods. This comparative approach will provide crucial insights into how specific curricular design choices are structured. While linking these features directly to student learning outcomes is beyond the scope of this study, our findings will offer a foundational understanding of the curricular differences that may inform future research on that topic. Ultimately, this research will equip educators with the knowledge to make more discerning choices when selecting the most effective resources for their students.

3.2. Analyzed Lessons

Both curricula have three-tier hierarchy structures: Unit ⟶ Module ⟶ Session in *Bridges in Mathematics* and Module ⟶ Topic ⟶ Lesson in *Eureka Math*. For the purposes of this analysis, we refer to the smallest lesson units—sessions in *Bridges* and lessons in *Eureka*—as "lessons." Both curricula recommend approximately 60 min of instruction per day.

Bridges in Mathematics dedicates five constructive lessons within a single module to the relationship between area and perimeter. *Eureka Math* addresses this relationship across two different "Topics." The first Topic also includes five lessons, while the second Topic features additional lessons that involve creating a robot poster project, which reinforces understanding of the distinction between area and perimeter. However, as the teacher's guide explicitly designates these robot poster lessons as optional, they are excluded from this study.

Both curricula introduce the definitions and application of area concepts earlier than perimeter concepts. For instance, *Bridges in Mathematics* introduces area in Unit 5 and perimeter in Unit 6, while *Eureka Math* presents area in Module 4 and perimeter in Module 7. A key difference, however, lies in when the formal definition of perimeter is introduced within the context of the analyzed lessons. *Bridges in Mathematics* presents this definition in the first analyzed lesson, whereas *Eureka Math* introduces it just prior to the analyzed lessons.

3.3. Data Analysis

We conducted a lesson-by-lesson textual analysis of each curriculum, focusing on the lessons on the relationship between area and perimeter using the Constant Comparisons Method (Stake, 2000). Our analytic framework was guided by four adapted dimensions of Mathematical Treatment and Emphasis: (1) overall lesson structures, (2) introductory lesson, (3) mathematical emphasis across the lesson sequence, and (4) mathematical representation.

The analytic process involved three phases. In the first stage, both authors jointly reviewed all relevant lessons from *Bridges in Mathematics* and *Eureka Math* via online meetings. During this phase, we coded lesson materials along the four dimensions, focusing especially on lesson structures (i.e., the placement of area–perimeter content within the larger unit and the overarching instructional flow) and introductory lessons (i.e., how the topic was launched through contextual tasks, manipulatives such as unit squares and tiles, or guided definitions). Because this phase was conducted collaboratively, we encountered only minimal disparities in our initial interpretations. Any differences in interpretation were discussed and resolved during meetings.

In the second stage, each author independently re-examined the selected lessons, focusing more closely on mathematical emphases across the sequence and representations. Detailed analytic notes captured the kinds of reasoning promoted (e.g., distinguishing between area and perimeter, or exploring shapes with equal perimeters but different areas), as well as the use of visual, symbolic, and manipulative models to support exploration, discussion, and formalization. We paid particular attention to the questions posed to students, the objectives stated, the activities designed, and the mathematical conclusions students were guided to draw, such as recognizing the non-direct relationship between area and perimeter.

In the third stage, we met repeatedly to compare interpretations and refine the coding scheme through iterative discussion until consensus was reached. This collaborative reliability check confirmed the consistency of our coding scheme and interpretations. Finally, all coded data were compiled into structured tables for each curriculum and synthesized into cross-curriculum matrices. This process enabled us to identify convergences and

divergences in how *Bridges in Mathematics* and *Eureka Math* address the area–perimeter relationship, in both mathematical and pedagogical terms.

4. Results

In this section, we first describe the overall lesson structures, followed by an examination of the introductory lessons that open each curriculum's treatment of the relationship between area and perimeter. These descriptions illustrate the lesson flow and the forms of student engagement emphasized in each program. We then compare the mathematical emphases, both explicit and implicit, across the lesson sequences. Finally, we analyze the use of mathematical representations in each curriculum.

4.1. Lesson Structures

The two curricula follow a consistent structure across their respective lessons. In *Bridges in Mathematics*, each lesson typically features one to two major "Problems and Investigations" as the core new learning experiences. Additional support is provided through "Daily Practice" and "Homework Connections." A unique component is the "Work Place" station, where students engage in independent or collaborative activities using concrete or visual models.

Eureka Math lessons include five key components. They begin with Fluency Practice, which consists of quick exercises designed to build speed and accuracy with foundational skills. This is followed by an Application Problem, a real-world task that connects new concepts to prior learning. The core of the lesson is the Concept Development section, where new mathematical ideas are introduced and explored through guided discovery. Afterward, students participate in a Student Debrief, a reflective discussion to articulate their understanding and clarify misconceptions. Finally, each lesson concludes with an Exit Ticket, a brief formative assessment used to evaluate students' grasp of the lesson objectives and to inform subsequent instruction. Table 1 summarizes the structural components of each curriculum's typical lesson.

Table 1. Comparison of typical lesson structures in *Bridges in Mathematics* and *Eureka Math.*

Feature/Curriculum	*Bridges in Mathematics*	*Eureka Math*
Core Learning	"Problems and Investigations"—1–2 major tasks serving as the core new learning experiences	"Concept Development"—main instructional portion with guided discovery and discussion
Practice/Review	"Daily Practice" and "Homework Connections" for reinforcement	"Fluency Practice"—quick and targeted exercises designed to build fluency with foundational skills
Reflection/Closure	Not a distinct component, but teachers' guides include reflection prompts	"Student Debrief"—closing discussion for reflection and clarification
Unique Feature	"Work Place"—activity and game-based learning stations with concrete and visual models	"Application Problem"—connecting new concepts to real world context "Exit Ticket"—daily formative assessment

In our primary analysis, we focused on the Core Learning and Reflection/Closure components. Routine Practice/Review activities were excluded, as they primarily reinforce procedural fluency. However, we included Work Place activities (*Bridges*) and Application Problems (*Eureka*) when they directly addressed the relationship between area and perimeter.

4.2. Introductory Lessons in the Two Curricula

4.2.1. Introduction Lesson in Bridges in Mathematics

In *Bridges in Mathematics*, the introduction of the relationship between area and perimeter in Unit 6, Module 3 begins with an open investigation using table arrangements to informally explore "same and different" in mathematical attributes. Prior to this, students had informal exposure to perimeter through tasks involving finding the total distance around shapes.

The lesson then transitions to a "Fundraiser" task, where students use square tables to create arrangements that seat exactly 10 guests. This leads students to think about how combining tables affects perimeter. Students are encouraged to manipulate tiles and line units to test their ideas and provide reasoning.

Later in the lesson, the concept of perimeter is formally introduced as the distance around the shape. A key pedagogical emphasis is to guide students to conclude that rectangles with different areas can have the same perimeter through constructing examples with tiles and line units. The lesson design promotes active student participation through hands-on modeling, problem solving, and collaborative discussion.

4.2.2. Introduction Lesson in Eureka Math

Eureka Math introduces the area–perimeter relationship in Module 7, Topics D and E, starting with teacher-led fluency practices that review multiplication facts and finding the area and perimeter of basic rectangles. This is followed by an application problem involving arrays to engage students to start thinking of different kinds of arrays associated with a specific number of dots.

The main activity involves students working in pairs to find all possible rectangles made from a fixed number of unit square tiles (e.g., 18 tiles), followed by a comparison of the perimeters of these area-equivalent rectangles. Through this process, students observe that rectangles with the same area can have different perimeters depending on their shape, with students noting that "long and skinny" rectangles tend to have greater perimeters.

To reinforce understanding, students complete a problem set in which they generate rectangles with a given area and compare their perimeters. The lesson concludes with a student debrief that encourages reflection, particularly through a prompt asking why squares tend to have the smallest perimeter for a fixed area, followed by an Exit Ticket that assesses students' ability to determine missing side lengths and calculate perimeters for rectangles with a known area.

4.3. Lesson Sequence and Mathematical Emphasis

4.3.1. Bridges in Mathematics

In *Bridges in Mathematics*, the relationship between area and perimeter is developed over five lessons in Unit 6, Module 3, progressing from informal exploration to formalization and deeper investigation. The pedagogical approach emphasizes hands-on activities, systematic recording, and collaborative inquiry. Table 2 provides a lesson-by-lesson overview of how *Bridges in Mathematics* develops the relationship between area and perimeter across five sessions (labeled as U6M3S1 to U6M3S5, referring to Unit 6, Module 3, Sessions 1–5). Each session's main activities are summarized alongside the corresponding mathematical emphases.

Table 2. Mathematical focus in Bridges in Mathematics.

	Main Activities	Main Emphases
U6M3S1	Table arrangement for 10 guests using different configurations	Define perimeter as the distance around a shape
U6M3S2	Build rectangular tables that seat exactly 20 guests	Rectangles of the same perimeter can have different areas
U6M3S3	Open investigation "Can rectangles have the same area but different perimeters?"	Rectangles with the same area can have different perimeters
U6M3S4	Explore table configurations including both rectangular and non-rectangle shapes of an area of 30 square units and a perimeter of 26 units.	Compare shapes with same area and same perimeter
U6M3S5	Estimate and reason about the area and perimeter various of rectangles; Work Place: build rectangles	Distinguish area vs. perimeter units; deepen understanding via comparative strategies

In Session 1, students explore the perimeter informally through table arrangements, where the number of "chairs" represents the perimeter and the number of "tables" represents the area. This foundational task introduces both concepts in a real-world context. Session 2 builds on this by having students construct rectangular tables to seat 20 guests. Through this, students discover that rectangles with the same perimeter can have different areas and begin identifying and recording patterns, such as the sum of the width and length equaling half the perimeter.

Session 3 transitions into an open investigation in which students are challenged to determine whether rectangles can have the same area but different perimeters. Using 24 square tables, they generate multiple configurations and reflect on why and how the perimeter changes. In Session 4, students revisit their findings and use equations to calculate area and perimeter. They also explore shapes with a given area (30 square units) and perimeter (26 units), extending their understanding beyond rectangles to include non-rectangular configurations.

Finally, Session 5 reinforces the distinction between area and perimeter by focusing on units of measurement (e.g., linear vs. square units) and invites students to compare strategies and reasoning through structured practice and a Work Place game. Across all five sessions, the curriculum systematically builds conceptual understanding, guiding students to explore, notice patterns, and articulate relationships between area and perimeter.

4.3.2. Eureka Math

In *Eureka Math*'s Module 7, Lessons 18 through 22 meticulously explore the relationship between area and perimeter, employing a pedagogical approach that blends guided discovery with systematic exploration and explicit reflection. Table 3 outlines the mathematical focus of five lessons (labeled as M7TDL18 to M7TDL22, referring Module 7, Topic D, Lessons 18–22) in *Eureka Math* that explicitly address the relationship between area and perimeter.

Table 3. Mathematical focus in Eureka Math.

	Main Activities	Main Emphases
M7TDL18	Application problem; Build rectangles with given unit squares (18, 24, 16, 15, 12)	Rectangles of the same area can have different perimeters.
M7TDL19	Application problem; build rectangles with areas 13–20; create line plots	The number of possible rectangles related to factor pairs of the area and recorded using line plots.
M7TDL20	Application problem; Build rectangles by using given perimeters (15, 12, 14, 8, 22)	Rectangles of the same perimeter can have different areas.
M7TDL21	Construct rectangles with specific perimeters (10, 14, 16, 20); identify square possibilities	Compare areas of rectangles with the same perimeter; squares tend to have the smallest perimeter.
M7TDL22	Create and compare line plots for area and perimeter; assess generalization	The same numerical values for area and perimeter can yield different numbers of possible rectangles.

Lesson 18 begins with an application problem involving arrays and guides students to construct rectangles using a fixed number of unit squares. This task emphasizes the connection between area and multiplication while prompting students to observe how the perimeter varies depending on how the squares are arranged. In Lesson 19, students build rectangles using given areas (13–20 square units) and represent their findings in line plots. This activity highlights the link between the number of rectangle configurations and the factor pairs of a number, noting that prime numbers (e.g., 13 or 17) yield fewer rectangles than composite numbers (e.g., 12, 16, or 18).

Lessons 20 and 21 shift the focus to the perimeter. Students use application problems and systematic strategies—such as halving the perimeter to find possible length–width combinations—to explore how different rectangles can share the same perimeter while differing in area. These lessons reinforce the idea that rectangles with the same perimeter can vary in shape and that squares tend to have the smallest perimeter for a given area.

Lesson 22 synthesizes these learnings by having students create and compare line plots representing the number of rectangles for given areas versus given perimeters. This comparison explicitly drives home the conclusion that there is no general direct relationship between area and perimeter, as the data do not consistently show a connection.

Eureka Math uses a carefully chosen range of numbers across the lessons to support exploration and generalization. For example, students work with values such as 8, 10, 12, 13, 14, 15, 16, 17, 18, 20, 22, and 24. These numbers are selected to highlight specific relationships—such as factorability, symmetry, and parity—that deepen students' understanding of area and perimeter. The deliberate use of numerical variation allows students to make meaningful comparisons and notice patterns, further enriching the learning experience.

4.4. Mathematical Representation

4.4.1. Bridges in Mathematics

Modeling is a central pedagogical strength of *Bridges in Mathematics*, enabling students to transition between real-world situations and formal mathematical reasoning. The authentic task of arranging tables to seat guests is relatable, visual, and tangible. The curriculum begins to pose authentic mathematical questions: "How many guests can this

arrangement seat? Will this arrangement fit in a given space?" These informal inquiries mark the entry point into mathematics.

Students start by physically modeling table configurations, using tile manipulatives to explore different arrangements to fit a given number of guests. As students explore these arrangements, they gradually shift from intuitive reasoning to explicit use of mathematical tools and concepts—counting unit squares, measuring side lengths, and calculating area and perimeter. Modeling thus becomes a method of mathematizing everyday experience: real-life configurations are translated into numerical representations, visual arrangement, and formal definitions. Students are supported in this transition through structured lesson steps.

Building on its use of modeling, *Bridges in Mathematics* further distinguishes itself by encouraging students to move beyond conventional shapes (i.e., rectangles in this context) and explore the broader world of rectilinear figures and irregular configurations. This openness is not an incidental feature, but a direct result of the curriculum's commitment to contextualized modeling. As students work within the scenario of arranging tables to seat guests, they naturally encounter configurations that extend beyond simple rectangles, such as L-shapes or other composite figures.

4.4.2. Eureka Math

Eureka Math offers a sequenced progression that supports deep conceptual development. Each lesson builds on prior explorations, helping students gradually develop an understanding of the relationship between area and perimeter. Rather than introducing topics in isolation, the curriculum ties each new idea to previous learning, creating a coherent learning trajectory. This structure helps students make sense of mathematical ideas incrementally while reinforcing core understandings over time.

Throughout these lessons, *Eureka Math* uses application problems, systematic construction tasks, and reflective discussions to build students' conceptual understanding of the independent nature of area and perimeter. Line plots are employed as a powerful visual tool to help students move beyond isolated examples and recognize broader patterns—and non-patterns—in mathematical relationships.

Specifically, in Lesson 22, *Eureka Math* has students create a new line plot showing the number of rectangles possible for a specific *perimeter* and then compare it to the area-based line plot. This direct comparison is critical for leading students to the conclusion that there is no general rule about a connection between perimeter and area. Figure 4 illustrates how students compared two line plots—one representing fixed areas (top) and one representing fixed perimeters (bottom)—to analyze whether a generalizable relationship exists between area and perimeter.

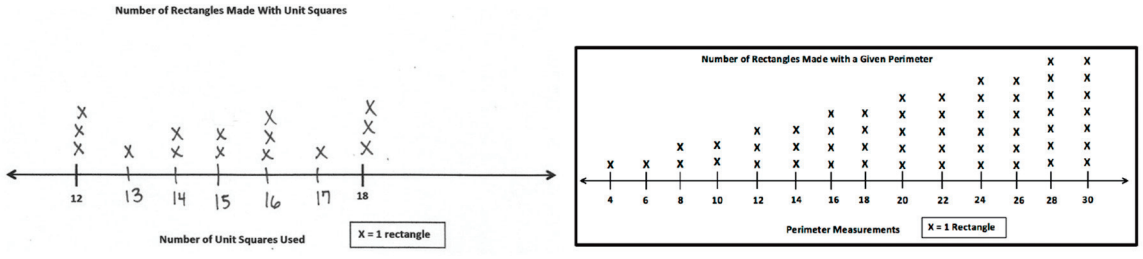


Figure 4. Comparison of the number of rectangles made with unit squares by area (**left**) and perimeter (**right**).

By visually contrasting the distributions and patterns on these two types of line plots, students can concretely grasp that these two measurements are independent, even if they might coincidentally share numerical values in some instances. This use of line plots

transforms individual calculations into meaningful data sets, enabling students to engage in higher-level analysis and generalize about the complex relationship (or lack thereof) between area and perimeter.

5. Discussion

The compared lessons in *Eureka Math* and *Bridges in Mathematics* aim to address the following standards:

Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters (CCSS.MATH.CONTENT.3.MD.D.8).

Both curricula support students in understanding the inconsistency between area and perimeter and offer rich opportunities to observe these relationships across these lessons. This reflects long-standing arguments that curriculum materials are not neutral but embody particular ways of highlighting, sequencing, and representing mathematical ideas (Remillard, 2005).

The two textbooks describe the topic from different angles (Remillard & Kim, 2000; J. P. Smith & Barrett, 2017). *Eureka Math* begins with "same area, different perimeter," then moves to "same perimeter, different area" and ultimately guides students to compare whether having the same numerical value for area and perimeter results in the same number of possible rectangles. While *Bridges in Mathematics* does not limit shapes to rectangles and also includes rectilinear figures, it begins with "same perimeter, different areas," then moves to "same area, different perimeter," followed by "same area and same perimeter can still result in different shapes." Finally, it highlights that even for the same rectangle, the numerical values of area and perimeter can be different. These differences illustrate how curricular representations shape the pathways through which students encounter core mathematical relationships (Schmidt et al., 2005; Stein et al., 2007).

5.1. Shared Learning Opportunities Across Two Textbooks

Both curricula emphasize hands-on, visual representations to support sense-making (Ma, 2010; Valverde et al., 2002). Whether students are building configurations with tiles or drawing on grid paper, the materials and tasks are designed to make mathematical ideas tangible. Prior research underscores that visual and tactile models are critical mediators between intuitive reasoning and formal abstraction. This visual and tactile engagement helps students identify patterns, make comparisons, and explore the consequences of changing dimensions (e.g., Livy et al., 2012).

Another key similarity lies in both curricula's use of real-world contexts to situate learning. In *Bridges in Mathematics*, students engage with table arrangements for seating guests across all five lessons, while *Eureka Math*, although less anchored in a single narrative context, also presents authentic problem settings. Embedding tasks in meaningful contexts has been shown to promote student engagement and transfer of mathematical ideas. These real-life scenarios help students connect abstract measurements to meaningful situations, supporting intuitive thinking as a foundation for formal mathematical reasoning.

Both programs incorporate scaffolded learning sequences—beginning with simpler tasks and gradually increasing in complexity. Students might start with counting units or constructing simple rectangles, then move toward more advanced reasoning, such as identifying patterns in factor pairs or comparing different configurations with the same perimeter. This progression supports students in developing fluency and flexibility in their thinking.

Finally, both *Bridges in Mathematics* and *Eureka Math* are attentive to student reasoning and discourse. They provide teachers with prompts, side notes, and suggested questions to encourage explanation, justification, and peer discussion. These practices align with the Standards for Mathematical Practice, particularly "construct viable arguments and critique the reasoning of others," and help foster a classroom culture where mathematical thinking is made visible and valued. Prior studies confirm that curriculum guidance around discourse strongly influences whether teachers position students as sense-makers or passive learners (Remillard, 2005; M. S. Smith & Stein, 1998).

5.2. Distinct Learning Opportunities in Each Textbook

As discussed earlier, both curricula did not deliver abstract insights of the relationship between area and perimeter as facts but constructed through hands-on engagement, guided reflection, and repeated exposure across multiple lessons. *Bridges in Mathematics* emphasizes concrete and visual modeling, using manipulatives like tiles, diagrams, and hands-on arrangements to help students physically and visually grasp the meaning of space and boundary. In contrast, *Eureka Math* emphasizes symbolic and numerical reasoning, guiding students to connect area with multiplication and perimeter with addition. For example, students use factor pairs to determine rectangle dimensions for a given area and apply addition to calculate the perimeter. These distinct approaches reflect different entry points into the same core ideas, providing multiple pathways for students to build meaningful and transferable understanding. These different emphases reflect what Schmidt et al. (2005) describe as curricular "coherence"—the degree to which mathematical ideas are linked and developed systematically. Bridges highlights horizontal coherence across representations and contexts, while Eureka emphasizes vertical coherence through progressive formalization. Furthermore, they each has unique features that will be discussed further.

5.2.1. Modeling as a Bridge Between Real Life and Mathematics in Bridges in Mathematics

The *Bridges in Mathematics* curriculum leverages modeling as a central process for developing mathematical understanding, guiding students from real-life contexts to abstract mathematical reasoning. Instead of presenting abstract problems, the curriculum initiates learning through authentic tasks, such as arranging tables for guests, which are relatable and tangible. Students begin by physically manipulating tiles to represent different table configurations, naturally leading them to explore and solve problems related to seating, space, area, and perimeter. This process supports students in translating real-life situations into visual arrangements and numerical representations. Through structured lessons, students gradually transition from intuitive reasoning to explicit use of mathematical tools, such as counting unit squares and measuring side lengths. Ultimately, this approach helps students abstract mathematical relationships from concrete experiences.

A key strength of the *Bridges in Mathematics* is its openness to non-rectangular shapes, which is a direct outcome of its focus on contextualized modeling. As students model real-life scenarios, they naturally encounter irregular, composite shapes, such as L-shapes. This approach is significant because it reflects the complexity of the real world and challenges students to think beyond idealized, textbook-perfect figures. By working with these non-standard shapes, students deepen their conceptual understanding of area and perimeter as attributes of all two-dimensional shapes, not just rectangles. This requires them to develop more nuanced strategies, such as calculating area in parts and carefully considering all boundaries when determining perimeter. Unlike curricula that might guide students toward finding all factors for a given area, Bridges in Mathematics poses open-ended tasks, inviting students to discover multiple solutions. This approach validates diverse

problem-solving strategies and ensures that students see irregular shapes as meaningful mathematical objects rooted in everyday logic.

5.2.2. Developing Strategic Thinking Through Cross-Domain Connections in Eureka Math

Eureka Math fosters strategic reasoning by creating a coherent learning trajectory that connects geometry with other mathematical domains. The curriculum goes beyond introducing concepts in isolation, instead building upon students' prior knowledge of topics like multiplication, factors, and odd and even numbers to deepen their understanding of area and perimeter. This integration allows students to apply familiar ideas in new contexts, strengthening their ability to reason mathematically. For example, students are guided to recognize how the commutative property of multiplication relates to rectangles, understanding that a 1×18 rectangle has the same area as an 18×1 rectangle. This approach is further exemplified by the curriculum's use of line plots, which serve as a bridge between measurement and statistics. Students organize and display data about the rectangles they have created, enabling them to visualize patterns, identify extremal values, and engage in data interpretation. This not only enhances their understanding of geometry but also subtly introduces them to early statistical concepts, fostering a deeper, cross-domain understanding of mathematics.

Finally, *Eureka Math* emphasizes the development of flexible and transferable problem-solving skills. The curriculum prompts students to reflect on the strategies they've used in previous lessons and adapt them for new situations. For instance, when transitioning from problems with a given area to those with a given perimeter, the curriculum poses reflective questions that guide students to connect known strategies (e.g., using factor pairs) to new contexts (e.g., addition facts). This strategic continuity reinforces prior learning and prepares students to develop and apply flexible strategies across different mathematical contexts. The curriculum also encourages reasoning about optimization by asking students to explore which rectangle with a fixed area has the smallest perimeter or vice versa, laying the groundwork for more advanced concepts related to efficiency and extremal values.

6. Conclusions

This comparative analysis of *Bridges in Mathematics* and *Eureka Math* reveals that both curricula offer valuable, yet distinct, opportunities for students to develop a conceptual understanding of area and perimeter. Each curriculum embodies a particular pedagogical orientation: *Bridges in Mathematics* encourages open-ended exploration, student-generated strategies, and learning through intuitive, real-world contexts; *Eureka Math* emphasizes mathematical precision, logical progression, and structured reasoning. This echoes prior analyses showing that different curricula encode distinct visions of mathematical learning and teacher enactment (Remillard & Kim, 2000; Valverde et al., 2002).

Bridges in Mathematics is structured around discrete activities, with step-by-step instructions provided for each session. The numbered steps offer a clear sequence of activities and questions, which supports teachers in implementing the lesson in an organized manner. This structure can be particularly helpful for novice teachers, as it provides a clear roadmap for pacing, instructional moves, and transitions between activities. For teachers who are still developing their confidence, this level of guidance offers structure and pedagogical routines. However, consistent with Davis and Krajcik (2005), such guided routines may also constrain teacher adaptation and limit opportunities for deep conceptual coherence.

However, this linear structure also presents some limitations. It may obscure the conceptual coherence or the interconnected development of mathematical ideas within and across lessons. While the activity sequence often reflects a progression in format or engagement, it does not always make the progression in mathematical depth or abstraction

explicit. As a result, while the structure supports smooth lesson delivery, it may underemphasize the kind of intentional scaffolding necessary for fostering deep and connected mathematical understanding.

Eureka Math integrates teacher guidance directly into the lesson flow. Each lesson unfolds through a consistent and visibly tiered structure—Fluency Practice, Concept Development, Application Problem, and Student Debrief—creating a tight alignment between instructional goals and task design. This structure not only provides clarity around the intended learning trajectory but also makes the progression of mathematical ideas more transparent for both teachers and students (Schmidt et al., 2005). The design supports the gradual development of understanding, moving from foundational skills to new concept exploration, and then applying those concepts in varied contexts.

This coherence also places greater demands on the teacher. To implement the curriculum effectively, educators must have a clear vision of the broader mathematical landscape, including how each lesson fits into the larger learning arc (Lo et al., 2024). Teachers are expected to be familiar with the curriculum's long-term progression and recognize the conceptual thread development through lessons, anticipate student thinking, and facilitate discourse that highlights underlying structures. This resonates with research showing that teacher expertise mediates how curricular materials are enacted in classrooms (Remillard, 2005; Stein et al., 2007). While classroom implementation was beyond the scope of the present study, future research could investigate how teachers interpret and enact these curricular structures in practice, and what forms of implementation best support student understanding.

Understanding these curricular design rationales is essential for teachers. When educators recognize how tasks are sequenced, how mathematical ideas are developed, and what mathematical thinking is highlighted, they are better positioned to make informed instructional decisions during teaching (Zhou et al., 2023). Rather than prescribing which curriculum teachers should adopt, we recommend that educators develop a deep familiarity with the structure, mathematical approaches, and pedagogical emphases of the curriculum they are using.

This comparative analysis contributes to broader curriculum research by illustrating how different design choices create distinctive learning opportunities. By examining these diverse pedagogical models, this study offers rich opportunities to investigate the tradeoffs and benefits of each approach, which could then inform the development of more intentional and effective educational programs. Building on these findings, future research could investigate how students from each curriculum program respond to specific, non-standard problems that test their conceptual understanding beyond rote procedures. For example, a study could present students with the task in Figure 2 used by Tan Sisman and Aksu's study, discussed earlier. Analyzing student responses to this question would reveal how each curriculum's approach to modeling and strategic thinking influences a student's ability to reason about the relationship between area and perimeter.

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Culturally Responsive Mathematics and Curriculum Materials: Present Realities and Imagined Futures

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Abstract: This study applies a culturally responsive lens to the analysis of middle school (i.e., grades for students aged 11–13) mathematics curriculum materials. Based on previous conceptual frameworks that describe Culturally Responsive Mathematics (CRM) as a multidimensional construct, we developed a tool, the CRM Materials Evidence Tool (CRM-MET), to indicate the extent of explicit guidance within written curriculum materials toward different dimensions of CRM. Six sets of middle school curriculum materials were analyzed using the CRM-MET, with results demonstrating distinct differences in how these materials attended to different dimensions of CRM. This analysis also indicated that there are notable gaps across all analyzed curricula, especially regarding more critical dimensions of CRM focused on power and participation. These results suggest that systems aimed at supporting teachers' orientation toward and implementation of CRM can benefit from curriculum materials, but that the standardized nature of curriculum may also constrain the role of such materials in supporting CRM. We provide suggestions around how curriculum designers and school leaders might use curriculum strategically to support CRM given these findings, while recognizing policy constraints that may challenge such efforts.

Keywords: curriculum; culturally responsive mathematics teaching; equity

1. Introduction

If curriculum materials aim to guide teachers and support student learning (Remillard, 2018), it is critical to identify features of the materials that will indicate for educators and other stakeholders the quality of that guidance. This paper investigates curriculum materials through the lens of Culturally Responsive Mathematics (CRM) as a means for promoting Culturally Responsive Mathematics Teaching (CRMT), which is a multidimensional construct that encompasses "a set of specific pedagogical knowledges, dispositions, and practices that privilege mathematics, mathematical thinking, cultural and linguistic funds of knowledge, and issues of power and social justice in mathematics education" (Zavala & Aguirre, 2024, p. 6). Although this definition describes a construct that encompasses more than just teaching practices, in this paper we differentiate CRM when describing the pedagogical knowledges, dispositions, and practices from CRMT to describe specifically the teaching practices that flow from CRM. We argue that identifying the ways that written curricula adhere to the broader aims of CRM can strengthen conversations on curriculum quality and the role of curriculum in supporting CRMT and culturally responsive learning opportunities for all students.

In the United States, the push toward high-quality mathematics standards in the 1980s-1990s, along with a rise in policy and curriculum design toward alignment with such

standards, shifted research from framing curriculum as a barrier to student learning to investigating ways in which such materials might support standards-based instruction and student learning (Stein et al., 2007). Such investigations showed promising results around how standards-based curricula have the potential to support student achievement (e.g., Chappell, 2003; Choppin et al., 2022; Putnam, 2003; Stein & Kaufman, 2010). However, achievement on standards-based assessments is only one, narrow representation of student learning, and an overemphasis on only standardized achievement outcomes can perpetuate dominant narratives about who can be a mathematical thinker and what mathematical activity even entails (Gutiérrez, 2012; Gutiérrez et al., 2023; Raymond, 2018). It is therefore insufficient for investigations of curricular quality to be confined to curriculum's ability to influence standardized test scores (Buxton, 2006).

Studies have explored how conceptions of equity and cultural responsiveness are embedded within district mathematics vision-setting and reform efforts, suggesting inconsistent attention to such concepts across educational infrastructures (Comstock et al., 2024; Marshall & Khalifa, 2018). Given curriculum materials' unique position to communicate directly to a wide range of teachers (Remillard & Kim, 2020), it is important to consider these materials' role in supporting CRM as a critical element of educational infrastructures. In these ways, curriculum designers, school leaders, and other stakeholders could explicitly consider cultural and contextual considerations for curriculum to support the practices and dispositions embedded within CRM.

Researchers have designed tools to identify and describe the extent to which mathematics teachers' lessons adhere to multiple dimensions of CRM (Aguirre & Zavala, 2013; Zavala & Aguirre, 2024). This study adapts these tools to the context of curriculum materials to answer the following questions.

(1) To what extent are cultural and contextual considerations included in mathematics curriculum materials?
(2) In what ways do mathematics curricula that do consider cultural and contextual considerations address such issues?

2. Literature Review

Curriculum, or the "substance or content of teaching and learning" (Stein et al., 2007, p. 321), exists in many forms throughout the education system, including national or (in the United States) state standards all the way down to the mathematical experiences of students in the classroom (Remillard & Heck, 2014). Curriculum resources play a key role in communicating to teachers specifics about how standards can be addressed (Remillard, 2018). The different manifestations of curriculum paint a picture of what teaching mathematics should look like in that school system.

In the United States, individual states or even school districts have a certain degree of autonomy in selecting curricula, often a set of materials created by a private publisher, for use in their schools. These materials (in the form of instructional resources, teachers' guides, workbooks, etc.) can therefore serve as a tool for teachers to use and adapt as they plan their instruction (Brown, 2011). Teachers' adaptations are necessary to turn standardized resources into learning opportunities, but the quality of instructions and guidance they receive from the mathematics curriculum is also an important factor in their teaching choices (Choppin et al., 2022; Gay, 2002; Stein & Kaufman, 2010).

As such, investigations of curriculum have recognized that there are distinctions between the intended curriculum—the instructional materials as designed—and the enacted curriculum, or how these materials are actually experienced in the classroom (Gehrke et al., 1992; Remillard, 2005). To this end, Remillard and Heck (2014) define the mathematics curriculum both "as a *plan for the experiences* that learners will encounter, as well as the *actual*

experiences they do encounter, that are designed to help them reach specified mathematics objectives" (p. 707, emphasis in original). Teachers must lean on their own content and pedagogical expertise and interpret the intentions behind curriculum materials as they adapt these materials to their students' specific needs and contexts (Ben-Peretz, 1990). It is therefore "critical that curriculum developers pay careful attention to the multiple ways that their materials communicate with the teacher" (Remillard, 2005, p. 240), as the forces that drive the design of instructional materials are not guaranteed to align with teachers' contexts.

This participatory perspective toward curriculum, which sees educators as a "Collaborator with curriculum materials" (Remillard, 2005, p. 217), requires greater consideration of teachers' role in enacting curriculum. Remillard (2000) describes the importance of such materials not simply speaking *through* teachers but speaking *to* them through guiding resources that recognize teachers as astute interpreters of the texts. This consideration underscores the importance of analyzing not only the supposed quality of student-facing materials, but also the included structures and resources designed to support teachers' enactment of the materials in their own context and given their own experiences.

Although research on mathematics curricula has often focused on standards alignment and the inclusion of rigorous tasks (Stein et al., 1996, 2007), some investigations have also considered the relationship between curriculum and CRM. These studies have often emphasized this participatory perspective of curriculum (Remillard, 2005) and attributed the success of curriculum materials' effect on culturally responsive teaching to the adaptation and practices of the teacher alongside materials themselves (Boaler, 2002; Kisker et al., 2012). Many of these studies have looked at curricula that were developed in collaboration with teachers specifically for their community of students. However, because the adoption of mass-produced curriculum materials of varying quality is prevalent across public school systems in the United States (Center for Education Market Dynamics, 2023), it is also important to consider how these types of materials may be primed for such teacher adaptation.

The very notion of "high quality" within mathematics curriculum materials also deserves critical examination. Despite aligning with notions of "high-quality instruction," such materials often reinforce larger narratives about mathematics that play into systems of inequity already present in the school system and society (Gutiérrez, 2017b), and even reform-oriented curriculum materials require training and support to enact equitably and with fidelity (Boaler, 2002). What's more, school district messaging in support of equitable mathematics may not be matched with district allocation of actual resources to support equitable mathematics teaching (Comstock et al., 2024). As such, it is incumbent for researchers and school leaders to strategically find opportunities to support teachers in recognizing these harmful narratives and imagining alternatives (Gutiérrez et al., 2023). Explicitly analyzing curriculum materials through the lens of CRM can therefore play an important role in developing a constellation of support for fostering CRMT in classrooms.

Fortunately, frameworks around CRM have been developed that can support such investigations. There are many aspects of culturally responsive teaching, ranging from how mathematical thinking is facilitated to how language, culture, and social justice is addressed (Aguirre & Zavala, 2013). Zavala and Aguirre (2024) have organized such knowledges, dispositions, and practices into the Culturally Responsive Mathematics Teaching Tool (CRMT2), which is in part designed to support teachers in analyzing curriculum materials to adapt for their own contexts. This tool categorizes CRMT into three conceptually related (but not wholly distinct) strands that form the basis of our present investigation: Knowledges & Identities, Rigor & Support, and Power & Participation.

Promoting access and equity in mathematics requires instruction that leverages students' *knowledges and identities* (National Council of Teachers of Mathematics, 2014; Zavala & Aguirre, 2024). This can start with bringing in topics that are connected to students' lives, but must go beyond simply adapting standard word problems. Mathematics rooted in real-life is often messier than the mathematics seen in classrooms. Simplifying contextualized math into a word problem with one process or solution can limit a student's ability to utilize their personal funds of knowledge as they have to prioritize the sterilized classroom mathematics over contexts with which they are familiar (Boaler, 1993). This game of "make believe" deprioritizes students' knowledges and identities by sterilizing cultural references into a neat and simplified word problem. To truly understand students' funds of knowledge, students and their communities need to be utilized as a resource. Teachers and other actors within the education systems need to position themselves as learners, seeking out information, connections, and relationships with the adults and communities that surround students in their whole identity and experiences (González et al., 2006; Moll et al., 1992; Wachira & Mburu, 2019). Teachers and school administrators need to recognize and respect the cultures and identities of students if they want to truly capitalize on the students' knowledge.

CRM also highlights the need for mathematical experiences that provide *rigor and support* for all students (Zavala & Aguirre, 2024). Open-ended tasks with many ways to solve, or with many correct answers, are often considered more mathematically rigorous in that they help to sustain high cognitive demand (Stein et al., 1996). Rigorous mathematics moves away from procedural tasks and centers the students' thinking and knowledge. However, students who are considered behind are still often asked to perform more procedural tasks, limiting their access to rigorous, high-level mathematical thinking (Gutiérrez, 2008). Therefore, it is critical to provide all students with scaffolds that support students' mathematical reasoning and do not inadvertently lower the rigor of the task (Boaler, 2002; Stein et al., 1996). In these ways, it is important for teachers to attend to aspects of rigor as well as support for all students to engage in such rigorous environments.

Power and participation are also important aspects of CRM because they require consideration of how cultural dynamics influence how students interact and are perceived (Zavala & Aguirre, 2024). These are often embodied in how schools support and respond to students' participation, leadership, and collaboration. This is especially important in heterogeneous classrooms as peer-to-peer dynamics can reinforce inequitable participation due to the perceived status of students (Curtis et al., 2021). To develop collaboration and participation, teachers can highlight the power, intelligence, and input of students who are perceived as having lower status (Horn, 2012). This will help students in seeing themselves, and for their classmates to see them, as mathematical thinkers (Cohen & Lotan, 1995; Ladson-Billings, 1995). Acknowledging the ways that societal systems affect student experiences and participation in mathematics classrooms is critical to supporting equitable mathematics teaching.

Together, funds of knowledge, students' whole identities, power dynamics, participation in heterogeneous classrooms, and the facilitation of rigorous mathematical experiences are all connected (Zavala & Aguirre, 2024). Teachers need to respect students' full identities before they can be expected to push themselves academically (Ladson-Billings, 1995). Before facilitating equitable groupwork, there needs to be a move away from procedural mathematics (Cohen & Lotan, 2014). This web of ideas provides no linear path for teachers to follow to become culturally responsive in their mathematics teaching but instead informs both a professional and personal journey. Curriculum materials can provide structures, routines, and examples of what CRM could look, sound, and feel like in the classroom, inspiring and supporting teachers to try new ways to approach their teaching. We argue

that bringing these ideas into all levers of teacher support, including curriculum, is key to developing a more equitable and culturally responsive education system.

Theoretical Framework

This study uses a socio-political lens when approaching mathematics education (Gutiérrez, 2013) and adapts the CRMT2 framework (Zavala & Aguirre, 2024) for the purpose of curriculum materials analysis. These two frameworks inform each other as they both consider how the positionality of education reinforces the cultural systems of power present in our society. Without acknowledging this power dynamic within mathematics education, teachers are destined to repeat it. Gutiérrez (2013) speaks more broadly, describing how students' identities are undermined by more traditional mathematics practices, especially students of color and students with language diversity. Zavala and Aguirre (2024) propose a more specific framework intended to support teachers in "attend[ing] to teaching mathematics with strong, deep, and meaningful connections to students and their communities" (p. 18). We adapt this framework for the present study, using it as a steppingstone from CRM broadly to CRMT, and to create a similar tool as CRMT2 for mathematics curriculum resources. As such, we see this work as centered around broader knowledges, dispositions, and practices aligned with CRM, with the aim of fostering classrooms engaged in CRMT.

Though the standardized nature of curriculum may limit its ability to adapt to the cultural context of each classroom, it still holds great power as a tool that teachers use to design and adapt their instruction. This study considers the teacher-tools framework that assumes that teachers should not be expected to follow the curriculum as a script but instead use it as a tool to help make decisions (Brown, 2011). Given the need and inevitability of teacher adaptation, even reform-oriented curricula benefit from being situated within a larger system of supports for teachers to make the best use of such materials (Boaler, 2002). We therefore see this work as one means for empowering mathematics teachers to act upon their beliefs and ideals of culturally responsive teaching.

3. Materials and Methods

3.1. Context and Setting

This paper extends from the Analysis of Middle School Math Systems (AMS), a study that was conducted in partnership with four large urban school districts (serving more than 2500 schools and 1,500,000 students) during the 2021–2022 and 2022–2023 school years, involving grades six through eight (grades for students aged approximately 11–13 in the United States). While the study aimed to assess multiple facets of school systems, one primary goal was to understand how mathematics curriculum materials contributed to or hindered teachers' enactment of culturally responsive teaching practices.

The four school districts (a local, geographic administrative unit for schools in the United States) with whom AMS partnered each brought their own policy, demographic, and instructional contexts and were in a range of stages in formulating and implementing a vision for middle school mathematics. Because the United States does not maintain a national curriculum, individual schools often partner with private companies to purchase sets of curriculum materials in line with state learning standards (Center for Education Market Dynamics, 2023). Within the districts taking part in the AMS study, there were six focal curricula of interest: Illustrative Mathematics (IM), Into Math, Eureka Math, California (CA) Math, Big Ideas, and Key Elements of Mathematics Success (KEMS). These are described in the bullets below.

- IM: A grade K–8 core curriculum published by LearnZillion/Imagine Learning that is available in digital and print versions. Open-Up Resources also offers a free digital program for grade 6–8 that is authored by Illustrative Mathematics.
- Into Math: A grade K–8 core curriculum published by HMH and includes print-based curriculum components as well as digital and interactive versions. Due to copyright restrictions, Into Math materials cannot be shared publicly.
- Eureka Math: A grade pre-K (PK)–12 core curriculum that includes comprehensive print and digital curriculum materials and professional development. The entire PK–12 Eureka Math curriculum, along with a variety of instructional materials and support resources, can be downloaded at no charge.
- CA Math: A core curriculum for grade 6–8 published by McGraw Hill Education that is based on Glencoe Math but revised to align with California's state mathematics standards (available in digital and print). Due to copyright restrictions, CA Math materials cannot be shared publicly.
- Big Ideas: A grade K–12 core curriculum published by Big Ideas Learning that includes print-based curriculum components and digital versions. Due to copyright, Big Ideas materials cannot be shared publicly.
- KEMS: A grade 3–8 core curriculum developed by National Training Network, a mathematics professional development company that provides professional development and coaching services to mathematics teachers. Due to copyright, Into KEMs materials cannot be shared publicly.

3.2. Instruments

In line with the AMS study's emphasis on supporting diverse populations of students, we sought to expand the conception of "high quality" curricula by investigating the ways in which middle school math curricula not only aligned to college- and career-readiness standards, but also provided guidance to support CRM. To understand whether and how the study curricula facilitate culturally responsive instruction and create opportunities for students to demonstrate their learning in holistic ways, we adapted the CRMT2 Lesson Analysis Tool (Zavala & Aguirre, 2024) to create an instrument that we call the CRM materials evidence tool (CRM-MET). While the CRMT2 framework was originally constructed by Zavala and Aguirre (2024) as a tool for analyzing classroom instruction, they also describe how the CRMT2 framework can be applied "to analyze or adapt existing lessons within the curriculum prior to instruction" (p. 25). In line with this use case for the CRMT2 framework, we adapted this tool as a means of quantitatively scoring the presence of guidance around CRM within curriculum materials.

To adapt this tool, the lead developer first reviewed the literature that informed Aguirre and Zavala's (2013) CRMT lesson analysis tool and defined ways these dimensions could arise in written curricula. For example, curricula can include guidance to build on students' cultural and community funds of knowledge (CfoK) (González et al., 2006; Moll et al., 1992) by reminding teachers to ask about and reference students' community and home knowledge, culture, or experiences. Then the curriculum could encourage teachers to adapt problems or situations to align with their classroom. In line with Aguirre and Zavala's (2013) original lesson analysis tool, this original draft of the CRM-MET included seven dimensions focused on mathematical thinking, language, culture, and social justice in mathematics.

The CRM-MET was then piloted on a subset of 27 lessons, which allowed coders to identify additional examples of how these dimensions were arising in the focus curricula and to resolve disagreements. For example, evidence for *CFoK* was revised to include more explicit guidance to coders, including the following:

- A potentially relatable problem context is not sufficient–to represent CFoK, there must be an indication that students are asked to think about how they themselves, or other students, could relate to the mathematized issue.
- Examples include the mathematics being situated in something specific in a community, and as an actual mathematical issue that would emerge from the situation. Sometimes that context is specific to the unique community represented in the school. Other times, a context might be more general or familiar to students, even though it is not unique to their specific community.

The dimension of *rehumanizing mathematics* (Gutiérrez, 2018) was similarly revised to include clarifications to coders about visual imagery in curriculum materials, noting that "Images on the curriculum page of non-white individuals, or use of names that are more typical of non-white cultures, does not count as evidence, unless these identities are a central aspect of the problem." Further descriptions of the final CRM-MET dimensions and examples of evidence are described in Sections 3.2.1–3.2.9.

To provide further evidence of test content validity (American Educational Research Association et al., 2014), Maria Zavala, as one of the authors of the original lesson analysis tool (Aguirre & Zavala, 2013) and the (at the time unpublished) CRMT2 framework (Zavala & Aguirre, 2024), provided an expert review of the tool. Through these discussions, two additional dimensions from the CRMT2 framework—*honoring student ideas and thinking* and *cognitive demand*—were also added to the CRM-MET. These revisions allowed for greater alignment of the tool with such emerging external work involving CRM. Further psychometric analyses have not yet been conducted, but rubrics and associated training materials are available upon request.

We used the CRM-MET to assess curricula across Zavala and Aguirre's (2024) nine dimensions that measure the three broad strands of knowledges and identities, rigor and support, and power and participation. Each dimension was measured on a rubric that defined the type of curriculum evidence necessary, within a lesson, to meet thresholds for opportunities to enact CRMT. Table 1 describes the general rubric for these thresholds across all dimensions, while specific forms of guidance pertaining to each dimension are described below in Sections 3.2.1–3.2.9. Each rubric also included an overarching question that captured essential elements of the CRMT dimensions (Table 2).

Table 1. Coding Rubric for CRMT Evidence Across Dimensions.

Rubric Score	Extent of Evidence
1. No evidence	No guidance across lessons within a dimension.
2. Fragile/Marginal evidence	Brief instance of guidance (e.g., during the warm-up or closing) that may not include all students.
3. Explicit evidence	One instance of explicit guidance (e.g., during one of the lesson's main math tasks) for all students.
4. Sustained, explicit evidence	At least two instances of explicit guidance, with at least one including all students.
5. Strong, centered evidence	The entire lesson includes guidance pertaining to the relevant dimension.

In these ways, we defined evidence for how a curriculum can provide guidance, support, or opportunities for teachers to implement culturally responsive strategies and pedagogies. We recognize that such evidence does not guarantee enactment practices aligned with the CRMT2 dimensions in the classroom, and address this in our discussion. However, this tool does offer opportunities to analyze mathematics curriculum materials in a novel way, which is a contribution of this work. As such, we describe the dimensions of

this instrument in detail below, include examples for how these dimensions are represented in curricula, and provide examples for ways a lesson might include explicit evidence.

Table 2. Culturally Responsive Measurement Strands, Dimensions, and Essential Curricular Questions.

Strands	Dimensions	Essential Question
Knowledges and Identities	Community and Cultural Funds of Knowledge (CFoK)	How does the lesson as written help students connect mathematics with meaningful issues or situations in their lives?
	(Re) Humanizing	How does the lesson as written support creativity, broaden what counts as mathematical knowledge, and affirm positive mathematics identities for all students?
	Student Ideas and Thinking	How does the lesson as written create opportunities to elicit, express, and build on student mathematical thinking in a variety of ways?
Rigor and Support	Cognitive Demand	How does the lesson as written enable all students to closely explore and analyze mathematics concept(s), procedure(s), and problem-solving or reasoning strategies?
	Scaffolding Up	How does the lesson as written maintain high rigor with strong support for all students?
	Affirming Multilingualism	How does the lesson as written position multilingual learners as competent learners in mathematics activities?
Power and Participation	Distributing Intellectual Authority	How does the lesson as written distribute mathematics authority and make space for a variety of forms of knowledge and communication?
	Disrupting Power	How does the lesson as written disrupt status differences, entrenched stereotypes, and inequitable power relationships present in all mathematics classrooms?
	Taking Action	How does the lesson as written support students' use of mathematics to analyze, critique, and address power relationships and injustice in their lives (economic, social, environmental, legal, political, patriarchal)?

Note. The strands and dimensions for this framework are adopted from Zavala and Aguirre's (2024) CRMT2 framework.

3.2.1. Build on Students' Cultural and Community Funds of Knowledge

Cultural and community funds of knowledge are the everyday knowledge and activities in students' homes and communities that they understand and relate to (González et al., 2006; Moll et al., 1992). Culturally responsive teaching sees these as resources students bring to the classroom. Culturally responsive teachers understand that students bring ways of thinking about, reasoning with, and understanding math based on their backgrounds and the extent to which they engage in different mathematical activities (Zavala & Aguirre, 2024). For example, students may know how to calculate fares when riding the bus or have experience measuring ingredients when cooking. Math curricula can tap into students' cultural and community funds of knowledge by encouraging teachers to practice the following:

- Inquire about students' backgrounds and experiences and draw on those during math lessons.
- Ask students to reflect on instances where they might have seen a math concept at work in their own life and adapt it into a problem for the class.
- Reference students' community and home knowledge, culture, or experiences to make math instruction more relatable and meaningful to them.

For a lesson to include explicit guidance relating to students' cultural and community funds of knowledge, the lesson could encourage teachers to adapt one of the main math activities in the lesson to better connect with their students' community and home knowledge or culture. If the curricula included such guidance for every activity in a lesson, then it would include strong, centered guidance on cultural and community funds of knowledge.

3.2.2. Promote Rehumanization

Rehumanizing mathematics involves broadening conceptions of what counts as mathematical knowledge (Zavala & Aguirre, 2024). Gutiérrez (2018) emphasized the verbiage of *re*humanizing in recognition of the humane ways that diverse cultures have traditionally engaged with mathematics over time, stating that "we do not need to invent something new; we simply need to return to full presence that which tends to get erased through the process of schooling" (p. 3). Rehumanizing thus acknowledges that mathematics is a human activity that involves both thinking and feeling: math is not just an abstract set of rules and procedures. Rehumanizing makes clear that all students can succeed in mathematics by showing them that mathematics and strong mathematicians are found in cultures around the world (Battey & Leyva, 2016). This literature suggests that curricula can rehumanize math by prompting teachers within the curriculum materials to implement the following practices:

- Affirm positive math identities for all races, genders, and ethnicities by using math problems that honor students' cultures and identity markers.
- Expand students' views of mathematics by highlighting that math is more than abstract memorization. Math involves problem solving and reasoning that draw upon students' empathy, senses, and feelings.
- Represent the diversity of mathematicians by introducing Black, Indigenous, Latinx, and other mathematicians to students by referencing their literature as examples in textbooks and by referencing websites such as www.lathisms.org (accessed 10 September 2025) or www.mathematicallygiftedandblack.com (accessed 10 September 2025).

For a lesson to include explicit guidance relating to promoting rehumanization, the lesson could introduce how a main math activity connects to Black, Indigenous, Latinx, and other mathematicians or situate these cultures within the activity.

3.2.3. Promote and Value Diverse Student Thinking and Ideas

Research suggests that students attain deeper levels of conceptual understanding in math when they discuss their reasoning and compare different approaches to solving problems (Bennett, 2014; Curtis et al., 2021). In CRMT, attending to student thinking is even more important, given their diverse backgrounds and variety of ways they might approach problems (Zavala & Aguirre, 2024). This literature suggests that curricula can promote student thinking by directing teachers to do the following:

- Prompt students to share their reasoning, ask questions of one another, discuss each other's ideas, and build shared understanding.
- Explicitly encourage students to use multiple forms of communication, including hand gestures, pictures or drawings, and diverse verbal responses.

For a lesson to include explicit guidance relating to student thinking and ideas, one of the main math activities would ask teachers to elicit student thinking in an effort to make student thinking public that develops shared understanding about mathematical ideas.

3.2.4. Include Math Tasks That Require High Levels of Cognitive Demand

Cognitive demand refers to the type of thinking a lesson requires of students to complete mathematical tasks (Stein et al., 1996). Engaging in challenging content is important for every student—and particularly for historically marginalized students who have not had equal access to demanding content in the past (Zavala & Aguirre, 2024). Research shows that some teachers have lower expectations for historically marginalized students and that these lower expectations may lead to worse student outcomes (Jussim & Harber, 2005). This literature suggests that curricula can support teachers to engage all students in cognitively demanding problems:

- Include tasks designed to allow all students to use complex, non-algorithmic thinking.
- Provide tips for teachers to give all students opportunities to engage in intellectually and conceptually challenging math tasks that emphasize underlying concepts, patterns, and properties.

For a lesson to include explicit guidance relating to math tasks that require high levels of cognitive demand, half of the activities within the lesson would involve complex, non-algorithmic thinking or engage students in procedures that are connected back to conceptual understanding.

3.2.5. Scaffolding Up to Maintain Rigor and High Standards for All Students

Scaffolding is a common instructional practice to support students' access to the content when they may be struggling to understand it (Zavala & Aguirre, 2024). Although scaffolding is a valuable instructional practice, it can risk lowering the rigor of mathematical tasks for some students (Athanases & de Oliveira, 2014). This literature suggests that curricula can help all students access rigorous content through scaffolding with the following:

- Include a variety of scaffolding techniques designed to help students achieve success with the math tasks in each lesson that have higher levels of cognitive demand.
- Provide suggestions for teachers to scaffold students into formal mathematics by accessing their everyday knowledge and personal experiences. For example, children's understanding of how to share snacks with friends can be a scaffold for solving equal sharing problems.

For a lesson to include explicit guidance relating to maintaining rigor, the lesson would include at least two pieces of guidance for how teachers could consider the diverse learning needs of their students. Should the guidance include differentiation strategies, it should reconnect students who are assigned less cognitively demanding initial content to make connections to more rigorous content.

3.2.6. Affirm Multilingualism

Classroom discourse provides students with opportunities to develop problem solving abilities and develop mathematical competence (White, 2003). Multilingual learners across the English-learning spectrum in particular need meaningful opportunities to engage in mathematical reasoning such as describing patterns, using representations, and making generalizations (Zavala & Aguirre, 2024; Moschkovich, 2013). This literature suggests that curricula can address the needs of multilingual learners by prompting teachers to do the following:

- Support students to develop their mathematics skills while working in their home language.
- Immerse multilingual children in language-rich mathematics while using strategies to develop academic language, such as reframing everyday language explanations with math terms.
- Encourage students to use multiple forms of communication to show their thinking, including graphic organizers, manipulatives, equations, drawings, labels, and other writing.

For a lesson to include explicit guidance relating to affirming multilingualism, the lesson would include at least two pieces of guidance for how teachers could address varying multilingual learners' needs with at least one suggestion relying on students' home languages.

3.2.7. Distribute Intellectual Authority

Teaching approaches that center on students pave the way for them to actively engage in meaningful mathematics conversations and deeper mathematical understanding (Michael, 2006). Students can draw on their own experiences and knowledge to make sense of and connect with the math concepts they are learning. Using student-centered approaches helps students see that teachers are not the only source of knowledge and honors the various forms of knowledge students bring to the classroom (Zavala & Aguirre, 2024; Wachira & Mburu, 2019). This literature suggests that curricula can support distributing authority in several ways, which include the following:

- Provide teachers with suggestions to promote students as mathematical authorities and use students' ideas to drive mathematical conversations. For example, when students ask questions to confirm their thinking, prompt teachers to direct the question to the rest of the class to consider.
- Provide frequent opportunities for students to engage in group or paired activities that support them discussing and building on each other's ideas.

For a lesson to include explicit guidance relating to distributing intellectual authority, the lesson would be designed for a teacher to facilitate the lesson, but share authority of math knowledge with students at least once.

3.2.8. Disrupt Status and Power

Curricula can help overcome typical power imbalances in the classroom to ensure all students are empowered participants, regardless of their backgrounds. This is particularly important when creating opportunities for students to work together and share intellectual authority (see Horn, 2012). Popular or outgoing students may speak most often, or the top performers in class may get more attention from the teacher (Cohen & Lotan, 1995). Therefore, curricula should support teachers to intentionally attend to students' unique mathematical, cultural, and linguistic strengths (Zavala & Aguirre, 2024). This literature suggests that curricula can ensure all students participate in meaningful ways:

- Include strategies teachers can use to address status imbalances, such as confronting stereotypes or using inclusive talk that builds up students and encourages multiple approaches to math.
- Remind teachers to build up students as mathematical authorities to each other and encourage teachers to lift up important mathematical thinking from students whose peers do not yet see them as math resources.

For a lesson to include explicit guidance relating to disrupting status and power, the lesson would include at least one strategy to minimize status differences among students.

3.2.9. Analyze and Act

Students learn mathematics in the context of meaningful problems from their lives and communities (Boaler, 1993; Zavala & Aguirre, 2024). As such, curricula can support students' understanding by including math tasks that are related to an issue with which students will strongly connect (e.g., Gutstein, 2003, 2016). This literature suggests that curricula can provide opportunities for students to address and act on a problem:

- Include current or historical issues of injustice or social justice that may resonate within students' communities.
- Prompt teachers to tailor problems around issues in the local community. For example, a textbook problem could ask students to discuss different ways to distribute donations among families at a local food bank.

For a lesson to include explicit guidance relating to analyzing and acting, the lesson would contain one activity that uses mathematics to analyze a socio-political context or economic, social, legal issues.

3.3. Analysis

Within each curriculum's grade 6 teacher materials, we selected four units that covered the following topics: number sense, operations, measurement, and data displays. We chose these topics because each focus curriculum included comparable units of instruction that covered each of these topics. Within each unit, we coded nine 50-min lessons—three from the first third of each unit, three from the second third, and the last three lessons of the unit (excluding lessons that encompassed strictly review or summative assessments). The exception to this was the KEMS curriculum: because these materials were designed for 80- to 90-min lessons, we coded six lessons to code an amount of content equivalent to that of the other curricula, two from the beginning, middle, and end. This resulted in a total of 203 lessons, for which two to three coders independently reviewed the teacher's guide and noted evidence or opportunities to enact CRM in line with the CRM-MET tool.

Prior to engaging in analysis, each coder participated in professional development focused on the theoretical background of the CRM-MET tool, in particular the CRMT2 dimensions described by Zavala and Aguirre (2024). The team then engaged in training with the CRM-MET tool itself, analyzing a set of pilot materials drawn from different units of the focus curricula and meeting to discuss and resolve differences. Every member of the coding team had experience in K-12 education (e.g., mathematics teaching experience) or education policy (e.g., experience with curriculum analysis).

Each coder would independently annotate the teacher's guide materials for the selected lessons, taking note of the extent of evidence in line with each of the nine dimensions of CRM within the CRM-MET tool. As shown in Table 3, these annotations attended to both the extent of evidence within particular resources and instructions within the materials, as well as indications of repeated evidence of particular dimensions (e.g., multiple scaffolds provided throughout the teacher's guide).

The pairs or triads of coders then met to discuss and resolve any disagreements and synthesized the ratings and rationale for each CRMT2 dimension, with initial ratings showing approximately 80% agreement across all rubrics and lessons. For example, in the Illustrative Mathematics (2019b) example shown in Table 3, with the full lesson coding annotations shown in the Appendix A, coders resolved the extent of evidence scores and provided rationale for each CRMT dimension as follows:

- For Student Ideas and Thinking, the lesson scored 5 because it included opportunities to make student thinking public and develop shared understanding throughout each part of the lesson.

- For Cognitive Demand, the lesson scored 5 because the entire task involved doing mathematics; that is, it required complex, non-algorithmic thinking to explore the concept of volume, fractions, and how they relate to cost.
- For Scaffolding Up, the lesson scored 5 because it included more than four instructions to the teacher to plan activities or supports in ways that maintain the rigor of the task for all students.
- For Affirming Multilingualism, the lesson scored 3 because it included at least one instruction in how to address varying multilanguage learners' needs but did not encourage teachers to value students' home languages.
- For Distributing Intellectual Authority, the lesson scored 5 because students were generating mathematical knowledge, reasoning, and solution strategies on their own while the teacher served as a facilitator for the class.
- For each of the remaining domains (that is, Community and Cultural Funds of Knowledge, (Re)Humanizing, Disrupting Power, and Taking Action), the tool scored 1 because the lesson included no instructions or evidence to support the domains.

Table 3. Excerpt of Coding Annotations for Illustrative Mathematics (2019b) Lesson.

Lesson (Teacher's Guide)	Coding Annotations
. . . Scaffold, particularly for students with disabilities, "To help get students started, display sentence frames such as "_____ jewelry boxes can fit into one shipping box because . . ."	Fourth scaffold for students
"After the small groups have reached a consensus, have students share and discuss the variety of orientations and calculations. Or, the students could complete a gallery walk, depending on the amount of time left. Select 1–2 groups to share their recommended box size(s) needed to ship all 270 boxes." "Consider the following questions as discussion starters: How did the choice of jewelry box orientation affect how many would fit into each shipping box?" . . .	Meaningful instruction to make student thinking public and develop shared understanding. The entire math task the students worked on could be classified as "Doing Mathematics" that required complex, non-algorithmic thinking.

Note. See Appendix A for the full coding annotations of this lesson. The quoted teacher's guide materials are drawn from Illustrative Mathematics (2019b) Grade 6, Unit 4, Lesson 17. IM 6–8 Math was originally developed by Open Up Resources and authored by Illustrative Mathematics®, and is copyright 2017–2019 by Open Up Resources. It is licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0). OUR's 6–8 Math Curriculum is available at https://openupresources.org/math-curriculum/ (accessed 10 September 2025). Adaptations and updates to IM 6–8 Math are copyright 2019 by Illustrative Mathematics (https://www.illustrativemathematics.org/; accessed 10 September 2025).

After coding of the 203 selected lessons was completed, the extent of CRM evidence scores were aggregated at the unit and curriculum level to allow for descriptive analysis and reporting of the findings. This allowed for comparative analysis of each CRMT2 dimension across each of the sets of curriculum materials, as well as an overall "score" of the observed CRM evidence within each curriculum calculated by averaging across all nine dimensions. These quantitative data, along with the qualitative coding annotations and rationales, form the basis of our results.

4. Results

We designed and used the CRM-MET to understand the ways and extent to which mathematics curriculum materials provided written guidance aligned with dimensions of the CRMT2 tool (Zavala & Aguirre, 2024). We found that there were meaningful differences in how different curriculum materials provided such guidance, but overall limited evidence

that such materials consistently provide support aligned with dimensions of CRMT. We did find specific dimensions of CRMT at which particular curricula included more consistent written guidance, but also some dimensions, in particular aligned with the strand of Power and Participation, with limited or no guidance within these materials. We discuss our findings in line with our two research questions below.

4.1. To What Extent Are Cultural and Contextual Considerations Included in Mathematics Curriculum?

CRM-MET ratings reflected the prevalence of guidance within the teacher-facing curriculum materials that aligned with particular dimensions of CRMT. All six curricula offered fragile or marginal explicit guidance regarding CRMT (see Figure 1). On average, the curricula provide at least one brief instruction (for example, to a subset of students or during a short task such as a warm-up) to address CRM practices in each lesson. Illustrative Mathematics (IM) held the highest average score across CRMT2 dimensions, at 2.69, suggesting that the CRM guidance in IM lessons usually included all students in the class and took place during the main activities in a lesson. IM lessons were more narrative than other curriculum materials that were more transactional, which may have provided more opportunities to include guidance that addressed the CRMT2 dimensions. No curriculum achieved an average rating of 3 or higher, meaning that the analyzed materials did not tend to provide explicit guidance that addressed CRM across their lessons.

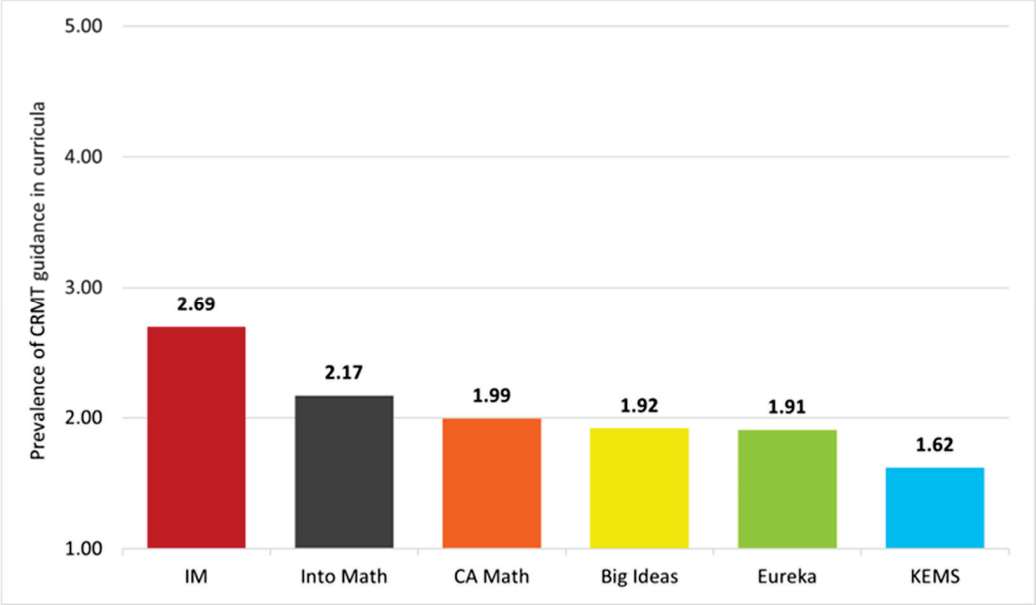

Figure 1. Prevalence of CRMT guidance in curricula.

When looking across the nine dimensions of CRMT, we found that the curricula provided more instances of explicit guidance to teachers in more traditional areas of instructional reform, such as Cognitive Demand, Student Ideas and Thinking, and Distributing Intellectual Authority (Figure 2). These CRMT dimensions emphasize student understanding of underlying concepts, patterns, and properties, communication, and ownership. In CRMT dimensions related to students' identities, power, and participation, the curricula provided little or no guidance. For example, all six curricula scored at or near 1 (no evidence), on average, for Community and Cultural Funds of Knowledge, (Re)Humanizing, Disrupting Power, and Taking Action. This meant that the curricula typically provided no guidance for how teachers could enact or adapt the materials to recognize and affirm students' identities, broaden students' conceptions of themselves as doers of mathematics, or use mathematics to address community or social change.

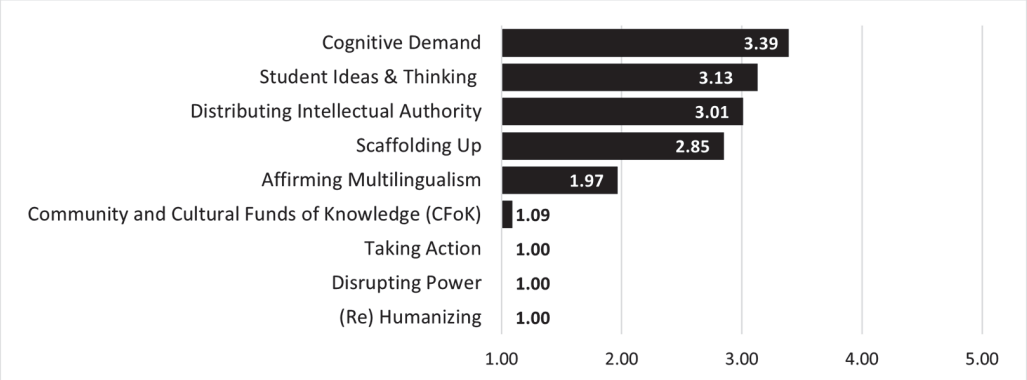

Figure 2. Average prevalence of CRM guidance in curricula by dimension.

When focusing on the three dimensions embedded within the Knowledges & Identities strand, these materials mostly attended to Honoring Student Thinking and Ideas (Figure 3). IM included the most guidance to elicit students' ideas and thinking (an average rating of 4.25) while CA Math included the least (an average of 2.47). All curricula demonstrated only marginal evidence in connecting the mathematics concepts with student's cultural and community funds of knowledge. None of the materials were found to help rehumanize mathematics by highlighting that mathematics is more than abstract memorization and should draw upon students' empathy, senses, and feelings.

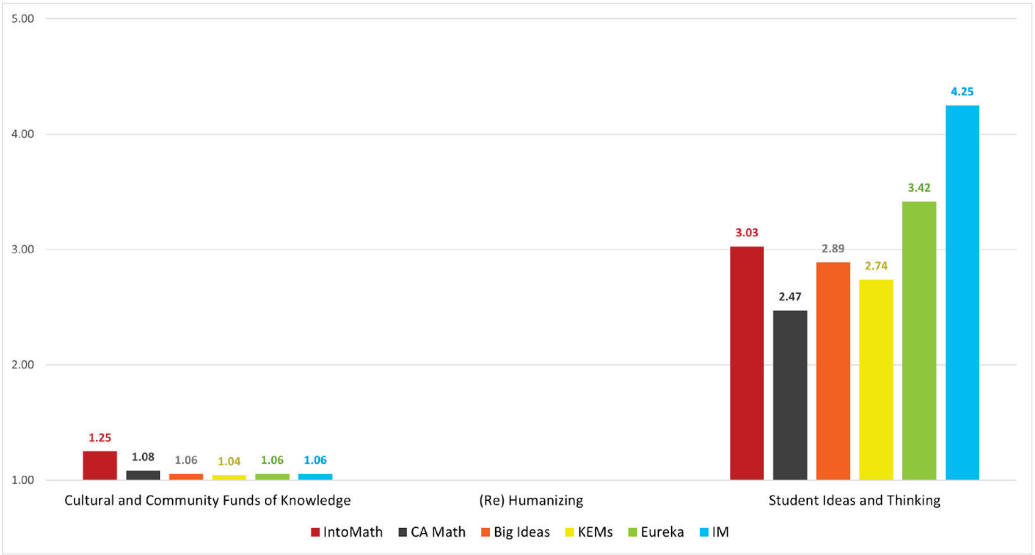

Figure 3. Knowledges & Identities strand ratings, by curriculum.

Within the Rigor & Support strand, on average, curricula included the most guidance to sustain a higher level of cognitive demand on the math tasks included in the curricula (Figure 4). Most curricula also included, on average, at least one instance of explicit guidance within each analyzed lesson to scaffold mathematical tasks for all students and, in particular, to affirm multilingualism. However, the analyzed materials within the KEMs curricula often did not include guidance for these two dimensions.

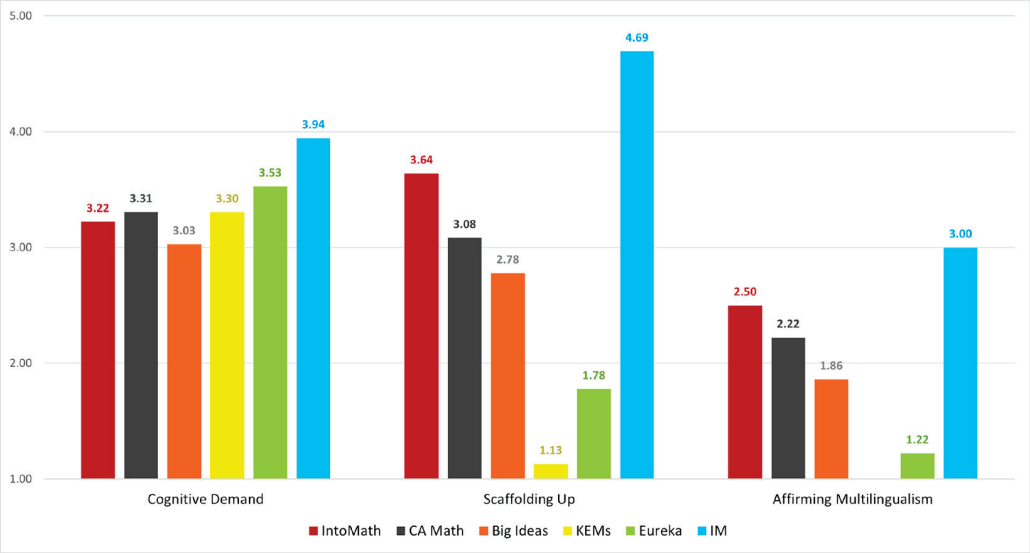

Figure 4. Rigor & Support strand ratings, by curriculum.

On average, curricula included at least one instance of explicit guidance for teachers to encourage students to take ownership over their learning by having them connect their initial thinking to mathematical concepts before teaching procedures (Figure 5). However, most curricula did not include any guidance to address status imbalances nor issues of injustice or social justice. In fact, only one curriculum (CA Math) included an instruction to address power imbalances in one of the lessons reviewed, and only one other curriculum (Into Math) included a mathematical task that addressed social justice in one of the lessons reviewed. These specific examples are described further in the results to our second research question below.

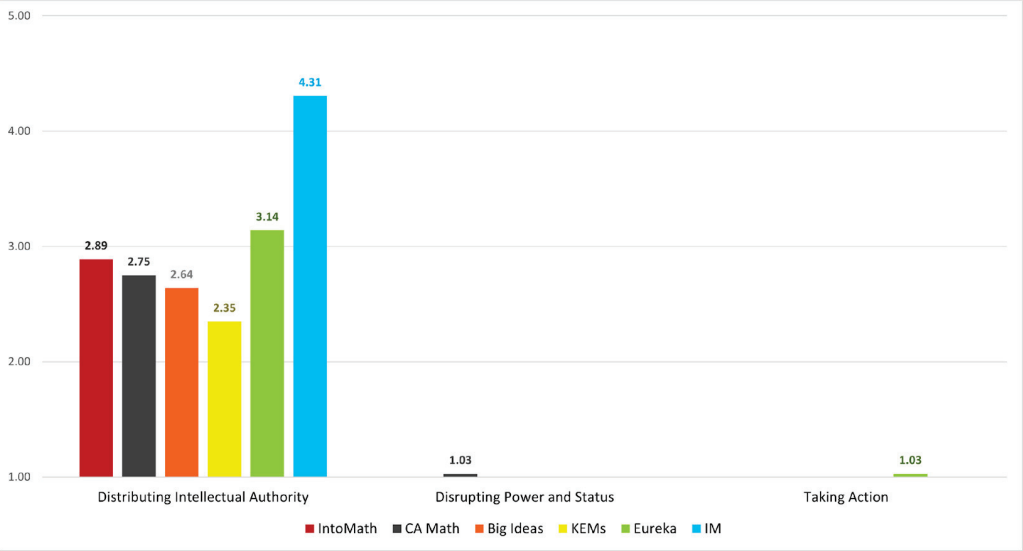

Figure 5. Power & Participation strand ratings, by curriculum.

Across these results, we found that the analyzed curriculum materials were attending to dimensions of CRMT, albeit not always consistently across the analyzed materials. On average, most curricula were found to have included only fragile or marginal evidence of attention to the CRMT2 dimensions. There are also notable differences in which dimensions particular curricula addressed, and notable absences of attention to particular dimensions, namely CFoK, Taking Action, Disrupting Power, and (Re)Humanizing Mathematics.

4.2. In What Ways Do Mathematics Curricula That Do Consider Cultural and Contextual Considerations Address Such Issues?

For our second research question, we examined how curriculum materials did address different dimensions of CRMT. These results consider not only the presence of guidance around the dimensions, but also the extent of evidence when guidance was observed. We describe trends in such guidance for each of the nine dimensions of CRMT and note trends in limitations of instructional guidance that may have held curricula back from attaining higher ratings for particular dimensions.

Of the 203 lessons reviewed, at most only marginal guidance to reference students' cultural and community funds of knowledge was observed (19 lessons; nine in Into Math, three in CA Math, two in Big Ideas, one in KEMs, two in Eureka, and two in IM). For example, in the opener of a lesson from one of the target curricula, there was guidance to have student pairs discuss and share ideas of real-life situations where they may need to determine the area, allowing for students to share examples of when they had considered area in their own life. However, the remainder of the lesson drew upon traditional area examples that provided rectangles and squares and asked students to find their area without grounding these shapes in the student's lives. To include guidance that permeates the lesson, the curricula could have, for instance, included guidance that prompted the teacher to customize the remaining math tasks in the lesson by including shapes inspired by the examples students shared for calculating area in their lives.

There were ten lessons where eliciting student ideas and thinking permeated the entire lesson (one in Big Ideas and nine in IM). These lessons encouraged students to share their reasoning, ask questions of one another, or discuss each other's ideas during each task. For example, one lesson from IM included three explicit tasks about representing ratios with tables. During the first task, designed to represent growth in a tile pattern, the curriculum instructed the teacher to do the following: "Invite students to share their responses and reasoning. Record and display the different ways of thinking for all to see. If possible, record the relevant reasoning on or near the images themselves. After each explanation, ask the class if they agree or disagree and to explain alternative ways of thinking, referring back to what is happening in the images each time." (Illustrative Mathematics, 2019a). The second and third task included similar instructions, emphasizing that the teacher should have students who solved the problem in specific different ways share and to have the class discuss advantages and challenges of using those methods.

There were two lessons that included tasks that predominantly involved doing mathematics, a high level of cognitive demand (Stein et al., 1996), one in CA Math and one from IM. Doing mathematics requires complex, non-algorithmic thinking, such as exploring the nature of mathematical concepts, processes, or relationships. It often requires students to access relevant knowledge and experiences and make appropriate uses of them in working through the task. For example, one curriculum encouraged students to research physical fitness by surveying students about the sports or physical activities they do in a week; researching physical activities and the calories burned per hour; creating a jogging schedule; looking up calories in fast food items; and planning for one day's meals. Within all of these activities students would have explored means and various data displays (e.g., box plots, line graph, etc.). To share with the class, students were also then encouraged to get creative by writing an article for a food or health section of an online website or to act as a doctor and create a presentation prompting physical fitness.

There were 37 lessons that included more than three instances of explicit guidance to differentiate instruction in ways that maintain rigor for all students (eight in Into Math, two in CA Math, one in Big Ideas, one in Eureka, and 25 in IM). These instructions address supporting access to rigorous content for all students and encourage teachers to design

scaffolds that do not lower the level of rigor or ensure ways to reconnect student to rigorous content. For example, one lesson began with a problem of the day and the curriculum included instructions, based on students' responses, to jump into an interactive reteach or to complete the prerequisite skills activity. The lesson also included small-group options and math center options for students on track, almost there, and ready for more. Moreover, the lesson described a specific differentiation activity for English language learners that also could benefit all students, included various depth of knowledge questions, and brought students back together for a cumulative activity and practice problems at the conclusion of the lesson.

There was only one lesson that included explicit guidance for positioning multilingual learners as competent learners in mathematics activities. The lesson encouraged teachers to use word walls and sentence stems for vocabulary as well as modeling how to underline key words, write down equivalent words, and to draw lines between elements to make connections. The lesson encouraged teachers to have students write down equivalent words in both English and their native language to ensure connections between the two. In order to achieve a rating of strong, centered evidence for the affirming multilingualism dimension, a lesson would have had to include at least two instructions on how to attend to multilingual learners' needs and at least two encouragements to the teacher to value multilingual learners' home language.

There were 18 lessons where the entire lesson distributed intellectual authority between the teacher and the students (one in CA Math, one in Big Ideas, one in Eureka, and 15 in IM). In these lessons, the teacher acts as a facilitator such that the authority of math knowledge primarily resides with students. For example, one Eureka Math lesson included instructions such as, "Students should be leading the discussion in order for them to be prepared to complete the exercises", "Students complete the volume and surface area problems in small groups", and having students lead the closing discussion.

There was only one lesson that included marginal guidance to disrupt power and status by minimizing status differences among students. The lesson included guidance for the teacher to provide each student three counters. After each student contributed to the discussion, they would place one counter in the center of the table so that all students have an opportunity to contribute three times each.

There was only one lesson that included one activity or problem that uses mathematics to analyze a socio-political context or economic, social, legal issues that may concern students from non-dominant subgroups. This Eureka lesson had an activity to analyze data regarding the Supreme Court and its chief justices' year appointed and length of term. The activity included an instruction to "...make connections to social studies. Ask students if they know what cases are before the current Supreme Court, whether they think any data would be involved in those cases, and whether any of the analysis techniques might involve what they have been learning about statistics."

5. Discussion

Culturally responsive teaching necessitates being responsive to the students in the classroom. Curricula is often written with the intention that it can be used across classrooms, districts, counties, and states. This creates a tension where the materials cannot anticipate the culture or community of the classroom in which the lesson will be delivered, and where it is difficult know how the teacher will blend the instruction on the page with the culture of the students in the classroom. However, curriculum materials can provide teachers with guidance and structure to support and encourage the use of culturally responsive pedagogies in any context. The absence of explicit guidance on culturally responsive pedagogies in a textbook does not mean teachers are unable or do not layer on culturally

responsive teaching; it simply means the textbook is not providing explicit guidance or support on how or when to be culturally responsive during the math lesson.

Our findings shed light on the extent to which and the ways in which middle school mathematics curriculum materials address dimensions of CRMT. These results demonstrate that standards-based mathematics curriculum materials do offer some guidance to support teacher dispositions and practices in line with certain aspects of CRMT, but that such guidance is largely sporadic and not consistent across different curricula. This suggests that curriculum materials can serve as an important part of a constellation of support for teachers but certainly not as the only means of fostering knowledges, dispositions, and practices at the heart of CRMT. Such findings have implications for educational leaders and curriculum designers in considering the affordances and limitations of curriculum materials in supporting dimensions of CRMT, but also suggest pathways for school leaders, educators, and advocates in considering which aspects of CRMT might require additional support beyond the written curriculum.

5.1. Limitations

The CRM-MET rubrics help in describing the extent to which current curriculum materials provide guidance around different dimensions of CRMT. However, there are inherently issues of power in who is represented in curriculum materials, and it is critical for students to have the opportunity "to see themselves in the curriculum or analyze the world around them" (Gutiérrez, 2012, p. 30). Therefore, this rubric alone is not intended to foster CRM in the classroom. Rather, pairing the CRM-MET with other resources such as Zavala and Aguirre's (2024) CRMT2 rubric (from which it was adapted) can help add nuance and depth to CRM-focused work.

Additionally, our analyses solely focused on selecting lessons from select units across each curriculum's grade 6 teacher materials. We did not review other supplementary material (such as a guide for students with disabilities or English language learners). While we assumed these trends may be similar across units and grade-levels, we did not test that hypothesis, limiting the generalizability of these findings. Future research may investigate how other grade-levels or units attend to culturally responsive math teaching.

Moreover, access to certain curriculum materials is not uniform across the nation. For example, materials developed by private developers may not be accessible to under-resourced districts. This also limits the generalizability of these findings.

5.2. Implications for Research and Practice

It is important to note that we not only included analyses that averaged guidance across dimensions, but also provided guidance on when and how these different curricula attended to each dimension of CRMT. We recognize that it is unlikely and unrealistic to expect a single lesson to score high across all dimensions. It may be more realistic to expect a range of scores across an entire unit, perhaps encouraging publishers and developers to include guidance that emphasizes different dimensions, and that ultimately attends to all CRMT dimensions across a single unit. Moreover, there isn't going to be a single curriculum that includes strong, centered guidance across the board, at least not based on our findings. While IM included the most guidance, on average, to distribute intellectual authority, attend to multilingual learners needs, differentiate instruction, illicit students' ideas and thinking, and included tasks that required more complex thinking, Into Math and CA Math included more guidance to attend to students' cultural and community funds of knowledge. The implication is that no single curriculum is likely to meet all of a district's needs, and the best curriculum for each district will be a function of district priorities.

Districts planning or considering adopting new curricula do, however, need information to understand how each set of curriculum materials aligns with district priorities. Ideally, districts would be easily able to use a single comprehensive information source to make curriculum decisions. However, there isn't a source for this information yet, so there is no way of knowing about the quality of curricula beyond broader analyses such as EdReports. Understanding the extent to which various curricula attend to dominant and critical dimensions of CRMT could be instrumental for administrators in selecting and implementing professional learning opportunities or providing teachers with context for how they might adapt their lessons. The CRM-MET in this way can provide valuable insight to school leaders in the adoption of new curricula, or how they might best target teacher support around CRMT given the affordances and limitations of their current curriculum.

5.3. Future Directions for Curriculum Design and Adaptation

It is important to consider culturally responsive pedagogies because the content covered and pedagogical approach that mathematics teachers use is heavily influenced by the curriculum (Stein et al., 2007). Curriculum developers ultimately hold great power in guiding the narratives about what does and does not count as mathematics in our educational systems (Gutiérrez et al., 2023). As such, the decisions that curriculum developers make regarding the inclusion or avoidance of guidance aligned with CRM can have a meaningful impact on the constellation of support that teachers have in fostering and sustaining CRMT. In our review, it is clear that while curricula often do provide guidance on more traditionally dominant aspects of instruction (e.g., maintaining rigor, scaffolding up), guidance on the critical elements is limited.

In first addressing more dominant aspects of instruction, several considerations do still stand out regarding curriculum design. First, almost twenty percent of all analyzed lessons in our focus curricula included more than three instructions for differentiation. While this dimension had the most lessons that scored the highest category on the rubric, it is important to note that even more lessons could have scored higher in this category. Many such lessons did not also include guidance on how to help students reconnect with rigorous content, a requirement for higher ratings of evidence for this dimension. By adjusting the content of a lesson to support students below grade-level, teachers may meet students where they are at, but without reconnecting students with the rigorous content a teacher is essentially creating a microcosm of a tracked environment.

Second, 96 lessons (out of 203) did include at least two instructions to attend to multilingual learners' needs (a rating of three—explicit evidence—on our rubric). However, none of these lessons encouraged the use of students' native language, a prerequisite for attaining a sustained (4) or strong (5) rating. Other than using vocabulary in English and students' native language to make connections, curricula could provide additional suggestions or could use digital technologies that connect mathematical concepts in students' native language. In the meantime, teachers could consider instructional resources and strategies (e.g., Chval et al., 2021) that not only describe how to adapt materials to engage multilingual learners through reading, writing, discourse, and gesture, but also on how to position multilingual learners as capable mathematical thinkers and doers in the classroom. As noted by Zavala and Aguirre (2024), supporting multilingual learners ultimately "is about using all the resources, even if you as a teacher don't have every resource, to reach out to families and communities" (p. 85). This dimension is rooted in asset-orientation and relationship-building, and thus beginning from a place of affirmation–regardless of support built into curriculum materials–is essential.

Curricula also only provided brief instructions to attend to students' cultural and community funds of knowledge, most likely because the students in classrooms across

a range of geographical, cultural, and political settings vary. The curricula appeared to usually attend to these funds of knowledge in introductory questions framing a process in the real-world, such as asking students to consider how the topic at hand has related to their own personal experiences. However, curricula could do more by including guidance for how teachers can adapt problems, based on their students' interests and aspirations, and perhaps share examples of such adaptations. This would frame the curriculum as more pliable, underscoring the importance of critical examination and responsiveness necessary for teachers to more equitably use such standardized materials (e.g., Gutiérrez, 2012). Regardless of curriculum guidance, teachers should also be critical observers, considering how they might adapt task contexts to their particular students. Neihaus et al. (2023), for example, describe a cycle through which teachers can identify and revise "problematic faves," or tasks that have valuable mathematical ideas but for which the contexts–sans revision–might be inauthentic or damaging to students' identities and experiences. Consistently building in such critical review practices could help teachers address these observed limitations in curriculum materials.

Considering more critical elements of CRM, there were no observed lessons that included any guidance for rehumanizing mathematics. Given the relatively stronger findings concerning the strand of Rigor & Support, lessons that attended to this strand—such as by providing higher cognitive demand tasks or scaffolds for students—could serve as an entryway toward seeing such rigorous mathematics in new, broader, and more diverse terms (i.e., rehumanizing). Gutiérrez (2018) describes eight specific concepts that can contribute to rehumanizing mathematics: "(1) participation/positioning, (2) cultures/histories, (3) windows/mirrors, (4) living practice, (5) creation, (6) broadening mathematics, (7) body/emotions, and (8) ownership" (p. 4). Curriculum designers and teachers alike could consider how such concepts could strengthen the emphasis on more academically rigorous mathematics by providing students with opportunities to see their full selves as capable mathematical thinkers within the context of such content.

Curriculum materials offering windows and mirrors (Style, 1996) through which students can broaden their conception of what mathematics is and who can be a mathematician is especially a means by which developers and teachers alike could support rehumanizing mathematics. Numerous resources exist that demonstrate that mathematical contributions are confined to no particular culture, race, or gender [e.g., https://mathematicallygiftedandblack.com/ (accessed 10 September 2025) and https://www.lathisms.org/ (accessed 10 September 2025)], which could be integrated into materials to emphasize such contributions and explore related concepts (e.g., Lumpkin & Strong, 1995). Curriculum materials could also integrate contexts from a greater variety of cultures, such as referencing Chekutnak, a stick dice game played by the Cree from the Piapot Reserve (see https://www.aboriginalperspectives.uregina.ca/workshops/workshop2011/background.shtml; (accessed 10 September 2025)), as a means for exploring percentages and probability. Ultimately, such approaches, along with the practices described by Gutiérrez (2018), orient toward an emerging vision of mathematics that Gutiérrez (2017a) describes as *living mathematx*, which "accedes that all knowledge is based on particular worldviews and ways of knowing that close down other possible choices; that is, knowledge is a political process, not a neutral product" (p. 18). Such critical reorientations toward the nature of mathematics teaching and learning (and mathematics itself) could spur more meaningful opportunities at rehumanizing the role of curriculum in mathematics education.

Such approaches could also serve to disrupt issues of status and power, another dimension of CRMT that was only observed as brief guidance in one lesson from the focus curricula. Because status ultimately reflects students' *perception* of their own academic capability and social desirability (Horn, 2012), teachers could address this dimension by con-

sidering how students collaborate and demonstrate their understanding around the lessons that compose these curriculum materials. For instance, Zavala and Aguirre (2024) describe three strategies—humanizing assessment (focusing on asset or strength based formative assessments), assigning competence (publicly stating students' mathematical contributions), and co-constructing group norms (establishing a groupwork focus on collaboration and respect)—that can support this dimension. In short, the absence of guidance within curriculum materials need not preclude teachers from taking charge in disrupting status and power in their classrooms by supplementing their instruction with such strategies.

There was also only one lesson that included a brief, optional instruction to explore a socio-political context, the supreme court. While there may not always be an opportunity in a lesson to analyze socio-political contexts or economic, social, legal issues that may concern students from non-dominant subgroups, when there is, these activities may permeate an entire lesson. Curricula could build on existing resources that address how to explore, understand, and respond to issues of social injustice with mathematics (e.g., Berry et al., 2020) to better attend to issues of power relevant to students. For example, students in Michigan–or in other locales impacted by failing public water systems–could engage with the context of the Flint water crisis as an avenue for understanding the role of mathematics in issues of justice (e.g., modeling the water needs of citizens during such crises). Indeed, encouraging teachers to first engage in the exploration of such topics could help in fostering educators committed to mathematics for social justice (Aguirre et al., 2019). Another example would be to explore functions in tables and arrays using electoral votes and popular votes to understand if one is a function of the other (see https://skewthescript.org/; accessed 10 September 2025). As we note below, we recognize the fraught nature of such a focus in the present political climate, which may indicate a limitation of the present model of private, profit-driven curriculum development.

5.4. Navigating Culturally Responsive Mathematics Teaching in the Present Climate

It is ultimately impossible to consider issues of CRM and curriculum without considering the broader policy and political climate around education, particularly in the context of the United States where many of these curricula have been adopted (Center for Education Market Dynamics, 2023). Curriculum developers hold significant power in shaping dominant narratives about what counts as mathematics (Gutiérrez et al., 2023), and visions of mathematics instruction heralded by district leaders often emphasize dominant aspects of equitable mathematics while downplaying critical dimensions of equity (Comstock et al., 2024). Additionally, the design of a standards-based curriculum could never alone foster the individualized knowledges, dispositions, and practices necessary for CRMT (Zavala & Aguirre, 2024). As we have laid out in this investigation, curriculum can offer opportunities and guidance around CRM, but we recognize that it is ultimately through the alignment of curriculum with other educational resources, along with teacher and school leader desire and ability to foster culturally responsive practices, that CRMT can be achieved. For such reasons, it is necessary to contextualize the role of curriculum materials in light of the broader climate of educational policy.

Of particular concern are recent legislative threats and federal executive orders that seek to dismantle diversity, equity, and inclusion policies and programs (see Ostrager et al., 2025). Given CRM's emphasis on valuing (1) a diversity of student backgrounds and mathematical perspectives, (2) equitable practices that promote success for traditionally marginalized student populations, and (3) an inclusive classroom environment for all learners, it must be assumed that efforts to promote CRM will be impacted by such actions. Indeed, this political environment is understood as not only shuttering public policies and programs that might support diversity, equity, and inclusion, but also creating a

chilling effect on the private sector as well (Ng et al., 2025). We therefore recognize that the challenges in calling for greater inclusion of guidance aligned with CRM from private curriculum developers will be further exacerbated in the present climate.

As such, we see investigations such as this, which take a critical look at curriculum design, as necessary in illuminating the affordances and limitations of standards-based curriculum materials. To engage in critical conversations that both foster greater attention toward CRMT in classrooms and counter potential rollbacks of the limited progress made within the domain of curriculum design, it is vital for education stakeholders and advocates to have tangible tools to address such issues. In this way, tools like the CRM-MET, in coordination with other analytical tools and frameworks, can help provide measurement and language to advocate for policies and programs that support CRMT, and hold to account actors who might attempt to counter such progress.

6. Conclusions

The CRM-MET provides not only guidance to teachers and school leaders regarding curricular decisions, but also is a charge to curriculum developers to consider ways in which their products might foster (or hinder) culturally responsive instruction. This extends considerations of curricular quality beyond a focus on alignment to standards to also address the ways in which these materials might better encourage opportunities for CRM. Including support structures aligned with different dimensions of culturally responsive teaching does not guarantee enactment of such structures in the classroom but may, over time, afford teachers opportunities to focus and strengthen their own culturally responsive practices, in line with CRM instructional tools (e.g., Zavala & Aguirre, 2024).

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Abbreviations

The following abbreviations are used in this manuscript:

CRM	Culturally Responsive Mathematics
CRMT	Culturally Responsive Mathematics Instruction
IM	Illustrative Mathematics
KEMS	Key Elements of Mathematics Success
CFoK	Community and Cultural Funds of Knowledge

Appendix A

The following coding annotations are drawn from IM Grade 6, Unit 4, Lesson 17. The latest version of this lesson (updated after the time of coding) can be found on the IM website (https://im.kendallhunt.com/MS/teachers/1/4/17/index.html; accessed 10 September 2025).

Lesson (Teacher's Guide)	Coding Annotations
This lesson is broken up into three main parts. The objective of the lesson is for students to use volume and fractions to calculate shipping costs for a variety of boxes to determine which is the most economical option for delivering 270 jewelry boxes.	
In the first part of the lesson, students should make sense of the task and determine what they need to know and do to find the most economical shipping box combination. Encourage students to work together to systematically minimize omissions and errors and to create drawings or models. Reflective questions you can ask students include, "Which orientations are possible? How much empty space would result?"	
Give students 1–2 min to read the task individually and ask any clarifying questions. Model the task by putting a small box inside a larger box in different orientations. After individual review, place students in groups of 4 and give them 5 min to individually brainstorm what information is needed to solve the task. Then, give another 5 min to plan in groups, followed by time to measure boxes or research box options and dimensions for themselves.	
Scaffold for English Language Learners : Three Reads. Use this routine to orient students to the context of the problem. In the first read, students read the problem with the goal of comprehending the situation (an artist is packing jewelry boxes to ship to a store.). Clarify any unknown language, such as a "flat rate" box or shipping rates, as needed. For the second read, ask students to identify the quantities and mathematical relationships (number of necklaces ordered, the dimensions of the jewelry box). After the final read, ask students to brainstorm possible strategies they may use to solve the problem.	First scaffold for students (first for English Language Learners)
After this first step, reconvene as a class and ask the groups a few steps they took to answer the questions or to plan for completing the task. Highlight any ideas students might have about making the problem-solving process more efficient and systematic. If not already mentioned by students, suggest that each group divide up the calculations to be done so each person is responsible for one shipping box.	Meaningful instruction to make student thinking public and develop shared understanding.
In part 2, students will now calculate the cost of shipping the boxes in each of the larger USPS boxes. Notice groups using different strategies for division with fractions, asking students to think about the different ways they have used fractions in calculations. If students are stuck, remind them that drawing the boxes or modeling them out of paper might help to visualize how to calculate a solution.	
Students will stay in their groups of four, selecting one of each of the different sized shipping boxes. Give students quiet time to work through a few box orientations and calculations. Two additional scaffolds (particularly for English Language Learners and Students with Disabilities) you can draw from, as necessary, include.	Second and third scaffold for students (second for English Language Learners)

Lesson (Teacher's Guide)	Coding Annotations
1. Provide students with multiple opportunities to clarify their explanations through conversation. Give students time to meet with 2–3 partners to share and get feedback on their work. Display prompts for feedback that students can use to help their partner strengthen and clarify their ideas. For example, "Your explanation tells me.", "Can you say more about why you.?", and "A detail (or word) you could add is _____, because." Give students 3–4 min to revise their initial draft based on feedback from their peers. 2. Provide a project checklist that chunks the various steps of this activity into a set of manageable tasks.	Meaningful instruction to make student thinking public and develop shared understanding.
In the last phase of the lesson, students will present, reflect on, and revise their work within their small group. They will discuss their decisions, accuracy, and revise any steps taken, if needed. The goal will be for the group to decide on the shipping box size or combination of sizes that will be most economical for shipping 270 boxes.	
Still working in the same group of 4, give students 10–12 min to discuss each group member's work and make revisions as needed. Display the following questions and ask students to use them to guide their discussion: How many different ways can the jewelry boxes fit into each shipping box? How does the orientation of the jewelry boxes affect how they fit within the shipping boxes? Do some shipping boxes have more wasted space than others? Why? Can you use diagrams to show and compare the unused spaces in different configurations? Are there ways to reduce the amount of wasted space when shipping exactly 270 jewelry boxes? How does the orientation of the jewelry boxes affect the cost of shipping with each shipping box? Is there a way to increase the number of jewelry boxes that will fit into a shipping box? How?	
Once each group member has shared, give students 4–5 min to decide on the least expensive option to ship 270 boxes and to write down ideas explaining why they selected that option.	
Scaffold, particularly for students with disabilities, "To help get students started, display sentence frames such as "_____ jewelry boxes can fit into one shipping box because."	Fourth scaffold for students
After the small groups have reached a consensus, have students share and discuss the variety of orientations and calculations. Or, the students could complete a gallery walk, depending on the amount of time left. Select 1–2 groups to share their recommended box size(s) needed to ship all 270 boxes.	Meaningful instruction to make student thinking public and develop shared understanding.

Lesson (Teacher's Guide)	Coding Annotations
Consider the following questions as discussion starters : How did the choice of jewelry box orientation affect how many would fit into each shipping box? How did the quantity of jewelry boxes (270) affect the choice of shipping box size? How did you calculate how many jewelry boxes would fit in a box? Did you multiply the lengths of the jewelry boxes or divide the lengths of the shipping boxes? Did the size of fractions affect how you performed division? What methods did you use to divide? How did you confirm or check your calculations? If you had a chance to solve a similar problem, what might you do differently to improve the efficiency or accuracy of your work?"	The entire math task the students worked on could be classified as "Doing Mathematics" that required complex, non-algorithmic thinking.
Scaffold, particularly for English Language Learners , "Invite students to restate what they heard the group present using mathematical language. Consider providing students time to restate what they heard to a partner, before selecting one or two students to share with the class. This will provide additional opportunities for all students to speak."	Fifth scaffold for students (third for English Language Learners)
To wrap up this culminating lesson, consider highlighting instances of math modeling by asking questions such as: When did you have to make assumptions to make the problem solving possible or more manageable? What assumptions did you make? Was there any missing information you had to find out before you could proceed? Were there times when you had to change course or strategy because the approach you had chosen was not productive?	Throughout the lesson, the authority of math knowledge resides with the students. They work through understanding what they need to know, plan for, execute and share about the most economical way to ship 270 boxes.

Note. The quoted teacher's guide materials are drawn from Illustrative Mathematics (2019b) Grade 6, Unit 4, Lesson 17. IM 6–8 Math was originally developed by Open Up Resources and authored by Illustrative Mathematics®, and is copyright 2017–2019 by Open Up Resources. It is licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0). OUR's 6–8 Math Curriculum is available at https://openupresources.org/math-curriculum/ (accessed 10 September 2025). Adaptations and updates to IM 6–8 Math are copyright 2019 by Illustrative Mathematics (https://www.illustrativemathematics.org/; (accessed 10 September 2025). Text highlighted in green represents coding annotations associated with the Student Ideas and Thinking domain. Text highlighted in blue represents coding annotations as-sociated with the High Cognitive Demand dimension. Text highlighted in yellow rep-resents coding annotations associated with the Distributing Intellectual Authority do-main. Text highlighted in pink represents coding annotations associated with the Maintaining Rigor and Affirming Multilingualism domains.

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Article

Polynomials—Unifying or Fragmenting High School Mathematics?

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Abstract: This paper presents research on the origin, scope, evolution, and rationale of knowledge about polynomials in high school mathematics. Within the framework of the Anthropological Theory of the Didactic, Croatian high school curricula and textbooks were analyzed, and four models of knowledge to be taught were identified in the period following the formal abandonment of New Math principles. None of the identified models provides a unified discourse that integrates knowledge about polynomials transposed from scholarly domains of algebra and mathematical analysis. In relation to other curricular content the knowledge about polynomials has two-fold importance: (1) contributing to the development of various techniques related to high school algebra and calculus; (2) serving as a fundamental example in the formation of the notion of a function. Thus, the observed reduction in polynomial content over the analyzed period affects both practical and theoretical knowledge. The findings suggest that curricular changes have primarily focused on the selection of knowledge, with scarce adaptations of knowledge to be taught compared to the knowledge before each curricular change. This has led to a persistent gap between algebraic and analytical approaches to polynomials, potentially influencing the learned knowledge even among the highest-achieving students. Despite polynomials' epistemological and didactical potential to bridge high school algebra and calculus, their restriction to specific forms of algebraic expressions and linear and quadratic functions contributes more to the fragmentation of high school mathematics.

Keywords: polynomials; high school mathematics; Anthropological Theory of the Didactic (ATD); curriculum; textbooks analysis

1. Introduction

Research on the transition from high school to undergraduate mathematics studies concerning the notion of a polynomial (Pleština & Milin Šipuš, 2022), highlights the relevance of epistemological inquiry into polynomial-related knowledge in high school mathematics. The fundamental role of polynomials in the discipline of mathematics (Barbeau, 2003; Stillwell, 2010) may induce significance of polynomial-related knowledge in the high school curriculum, as well as their interconnections with other curricular content. However, there is a gap in research directly addressing educational treatment of polynomials (Pleština, 2023). In existing studies, the approach to the polynomials is often not directly addressed, and there is a tendency to focus either on the algebraic or analytical aspects of polynomials, leaving a significant gap in understanding their comprehensive treatment within high school education.

To address this issue, this study undertakes a case analysis of the Croatian high school curriculum and textbooks, focusing on the period following the formal abandonment of New Math principles in Croatia. It examines the hypothesis that the variability in definitions of polynomials in high school education, which Bolondi et al. (2020) observed in primarily algebraic high school textbooks, reflects different approaches to polynomials across different mathematical domains of scholarly knowledge, particularly algebra and mathematical analysis.

The chosen theoretical framework, the Anthropological Theory of the Didactic (ATD), allows for an epistemological analysis of polynomial-related knowledge within the institution of Croatian high school education. The ATD enables addressing the relationship between scholarly knowledge and knowledge to be taught, as well as an examination of what should be taught about polynomials and for what purpose. The knowledge to be taught within the institution of Croatian high school education is delineated by the curriculum, which outlines the content and its organization, and by textbooks, which realize the curriculum. Therefore, both serve as resources for examining how scholarly knowledge is selected and adapted for high school teaching.

Although this study focuses on knowledge to be taught, insights from a questionnaire conducted among students who achieved excellent grades in mathematics and participated in mathematics competitions, who are therefore expected to demonstrate the highest level of learned knowledge, illustrate how epistemological gaps in the knowledge to be taught can constrain learned knowledge. While the sample is small and specific, the fact that constraints are evident even among the most successful students suggests that similar or even greater limitations may affect the learned knowledge by other students in the system.

1.1. Reference Studies

While research specifically focused on polynomials in high school education is relatively scarce, studies indicate that knowledge about polynomials is integrated into high school curricula in various countries.

In the 1980s, Dreyfus and Eisenberg (1990) observed that polynomial-related topics were marginalized in school curricula, a trend likely linked to the abandonment of the New Math in the United States, which subsequently influenced the educational systems of other countries. They argued for the importance of including polynomials and their associated theory in school education, beyond mere equation solving or root finding.

According to Weiss (2020), "polynomials occupy a central role in the secondary curriculum" (p. 76) in the United States as part of algebra, precalculus, and calculus. This curricular position could suggest their potential to serve as a unifying notion, connecting different curricular components.

However, Sultan and Artzt (2018), in a book for high school mathematics teachers, noted that the word "polynomial" is "probably the most misunderstood word in secondary school mathematics" (p. 81). Their observation could suggest that students encounter polynomials through various approaches, yet the relationships between these approaches remain hidden to students.

Adu-Gyamfi et al. (2017) and Llanos et al. (2015) examine polynomials within the precalculus part of the curriculum, while Bolondi et al. (2020) focus on polynomials in high school algebra

Adu-Gyamfi et al. (2017) investigated the understanding of graphical and algebraic representations of polynomials among high school students aged 15–17 in the United States. In their study, the notion of a polynomial encompasses terms like "polynomial relation", "polynomial function", "linear function", "quadratic function", and "polynomial", without explicitly addressing their relationships. Llanos et al. (2015) investigated the same topic

among students aged 14–17 in Argentina, employing the instructional format of study and research paths, a tool from the ATD. In their study polynomials are approached as functions, and they used terms "polynomial" and "polynomial function". Both studies primarily focus on polynomials of the first and second degrees (i.e., linear and quadratic functions), yet the reason for this emphasis is not explicitly stated.

Bolondi et al. (2020), in their study on definition schemes, analyzed high school algebra textbooks from Italy, Spain, and North America for students in grades 9–11. They identified various definitions of polynomials based on notions such as "variable, algebraic expression, literal equation, algebraic sum, and monomials" (p. 5).

That the notion of an algebraic expression lacks a functional role in the modelling process is observed by Strømskag and Chevallard (2022) in their analysis of the notion of a formula in Norwegian- and English-language textbooks. They also observe that "we commonly speak of algebraic expressions without specifying what these expressions do express" (p. 392).

These studies highlight multiple aspects of polynomials in high school education, collectively calling for further epistemological investigation into polynomial-related knowledge. In particular, they raise questions about the interconnections and organization of this knowledge, its relationship with other notions in high school mathematics, as well as the notion of a polynomial in scholarly mathematics.

The process of didactic transposition, as a methodological tool within the ATD, enables the identification and analysis of what elements of disciplinary mathematical knowledge are selected and how they are adapted for teaching within a given institution. Strømskag and Chevallard (2022) examined the didactic transposition of algebraic knowledge into curricula and textbooks. Barbé et al. (2005), in their analysis of the transposition of the notion of limits into Spanish high school education, showed that the knowledge to be taught imposes constraints on teachers' practice and, ultimately, on the learned knowledge.

As the literature review has shown, knowledge about polynomials is embedded within high school algebra, precalculus, and calculus. Therefore, analyzing the didactic transposition of polynomial-related knowledge enables an understanding of high school mathematics both as a whole and in terms of the relationships between its constituent parts. Since the didactic transposition of polynomials in high school education has not yet been addressed, this study aims to fill that gap by examining the case of the didactic transposition of knowledge about polynomials in Croatian high school education.

1.2. What Is a Polynomial in the Discipline of Mathematics?

Polynomials occupy a fundamental position within the discipline of mathematics (Barbeau, 2003; Stillwell, 2010). In algebra, when endowed with standard operations, polynomials have the structure of a ring and a vector space. The ring of polynomials over a field shares key properties with the ring of integers: division with remainder, the Euclidean algorithm, and the existence of greatest common divisors expressible as linear combinations. The notion of an irreducible polynomial generalizes the notion of a prime number, and the Fundamental Theorem of Algebra may be seen as an analogue of the Fundamental Theorem of Arithmetic. Consequently, certain algebraic techniques for solving algebraic equations arise from these properties. Polynomials also exhibit strong connections to number theory through modular arithmetic, Diophantine equations, Fermat's little theorem, and more advanced topics such as Galois theory. In real and complex analysis, polynomials are continuous, differentiable, smooth, integrable, and analytic functions. They are foundational in approximation theory, where key results include Taylor's theorem and the Stone–Weierstrass theorem. Polynomials are also of central importance in numerical mathematics, particularly through Horner's algorithm, polynomial interpolation, and

root-finding methods. Due to their geometric interpretability, computational simplicity, and numerical stability of those of degree at most three, polynomials are important for mathematical modeling.

The notion of a polynomial in the discipline of mathematics encompasses two approaches to polynomials, originating from two fundamental domains of mathematics: algebra and mathematical analysis (Markushevich, 1991). In this paper, one perspective on a polynomial is referred to as the *formal-algebraic approach* and is denoted as D_A, while the second perspective is referred to as the *functional approach* and denoted by D_F.

In scholarly algebra, a polynomial (over a commutative ring R with unit 1) is a formal expression, i.e., a finite formal sum of powers

$$a_0 + a_1 x + \cdots + a_n x^n, \tag{1}$$

where every a_i is an element of the commutative ring R, and x is a formal variable (indeterminate). The meaning of the formal expression (1) is explained by the following definition (Lang, 2005). Let $R[x]$ be the set of sequences $(a_0, a_1, \ldots)$, where $a_i \in R$ such that all but finitely many are zero $0 \in R$, i.e., sequences of the form $(a_0, a_1, \ldots, a_n, 0, 0, \ldots)$ are considered. The elements of the set $R[x]$ are referred to as polynomials over R, and $a_0, a_1, \ldots$ are called the coefficients of the polynomial $(a_0, a_1, \ldots)$. Polynomials are added componentwise, and multiplication is performed according to the convolution rule. If the polynomial $(0, 1, 0, \ldots)$ is denoted with x, the polynomials x^r, $r \in \mathbb{N}, r \geq 2$, can be inductively defined. It can be shown that $x^r = (0, 0, \ldots, 0, 1, 0, \ldots)$, that is, the polynomial x^r has the coefficient $a_r = 1$, and all other coefficients are 0. Furthermore, the mapping that associates each $a \in R$ with the polynomial $(a, 0, 0, \ldots)$ is an injective ring homomorphism. Since a can be identified with $(a, 0, 0, \ldots)$, the definition of polynomial multiplication implies that $a(a_0, a_1, \ldots, a_n, 0, \ldots) = (aa_0, aa_1, \ldots, aa_n, 0, \ldots)$. Finally, the polynomial $(a_0, a_1, \ldots, a_n, 0, \ldots)$ can be expressed in the standard form (1). Hence, this perspective on a polynomial exemplifies the formal-algebraic approach D_A.

In mathematical analysis, polynomials are real (or complex) functions of one or more real (or complex) variables. To illustrate, for $a_0, a_1, \cdots, a_n \in \mathbb{R}$ the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n,$$

for every $x \in \mathbb{R}$, is a real polynomial of one real variable.

The functional approach D_F is also present in the domain of algebra but always in relation to D_A. Assuming that R is a subring of a commutative ring R', for each polynomial $a_0 + a_1 x + \cdots + a_n x^n \in R[x]$, an induced function $f : R' \longrightarrow R'$ is defined by $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, for every $x \in R'$ (Lang, 2005).

The identification of a polynomial and an induced function is generally not possible. For the finite field $F_2 = \{0, 1\}$ $(\mathbb{Z}/2\mathbb{Z})$, let us consider polynomials 0, $x^2 + x \in F_2[x]$ in the variable x over F_2, and their induced functions: $f : F_2 \longrightarrow F_2$ defined by $f(x) = 0$, for all $x \in F_2$, and $g : F_2 \longrightarrow F_2$ defined by $g(x) = x^2 + x$, for all $x \in F_2$. In the D_A approach, the notation 0 represents $(0, 0, 0, \ldots)$, and $x^2 + x$ represents $(0, 1, 1, 0, \ldots)$; thus, these polynomials are not equal. However, the functions f and g are equal since $f(0) = g(0)$ and $f(1) = g(1)$. Although an isomorphism can be established between the ring of polynomials with coefficients from an infinite field and the ring of corresponding induced functions, these are distinct mathematical objects with different meanings for their corresponding notions (variable, equality, operations).

In the cited works on mathematical education, the notion of a polynomial is restricted to polynomials with real (or possibly complex) coefficients, which enables the implicit use of the identification between the D_A and D_F approaches. However, the implicit identification

of these approaches, resulting in different mathematical objects, can be particularly delicate for the development of the general notion of the function.

1.3. Theoretical Framework

The ATD is a theoretical framework for research in mathematics education, which highlights the institutional dependency of mathematical and didactic activities (Bosch et al., 2020; Chevallard & Bosch, 2020a). An *institution* in the ATD is a key term; it refers to any social structure wherein the didactic process unfolds or is facilitated (Chevallard & Bosch, 2020a). For example, research findings by Bolondi et al. (2020) and Sultan and Artzt (2018) indicate the dependence of knowledge about polynomials on the institution of high school textbooks.

The institutional dependency of knowledge is elucidated through the process of *didactic transposition* (Bosch & Gascón, 2006; Chevallard & Bosch, 2020b), which delineates institutional construction of knowledge into four stages: 1. *scholarly knowledge* generated within the academic community or by experts; 2. *knowledge to be taught* determined by the curriculum and textbooks; 3. *taught knowledge* by teachers within an observed institution; 4. *learned knowledge* by students. *External didactic transposition* entails the process of selecting and adapting scholarly knowledge into knowledge to be taught (Bosch & Winsløw, 2020; Bosch et al., 2021), while *internal didactic transposition* encompasses the remaining steps of didactic transposition that lead to learned knowledge (Chevallard, 1991).

For the analysis and description of knowledge in the process of didactic transposition, the notion of a *praxeology* is used as the basic unit for analyzing mathematical activities (Barbé et al., 2005). *Praxeology* p is represented by an ordered quadruple $[T/\tau/\theta/\Theta]$, where: T is the *type of tasks*, τ is the *technique* used to solve a task of type T, θ is the *technology* that is the discourse of p and explains τ, and Θ is the *theory* that is a formal argument and thus justifies θ (Chevallard & Bosch, 2020a). The ordered pair $[T/\tau]$ is called a *praxis block* of praxeology p and answers the question of knowing *how* to solve a task of type T. The ordered pair $[\theta/\Theta]$ is called the logos block of p and answers the question of knowing *why* the technique τ can solve a task of type T.

For example, the task $\frac{1}{9}x^2 + \frac{7}{12}x - \frac{1}{2} = 0$ belongs to the type of task "to solve quadratic equations" and can be solved using the technique of applying the formula for the solutions of a quadratic equation in radicals. The general form of this formula constitutes the technology, while its proof constitutes the theory of the praxeology.

Although all praxeologies include a technological and a theoretical component, scholarly praxeologies are typically characterized by a developed logos. In contrast, didactic transposition often involves the reduction or omission of certain elements of the scholarly logos, particularly the theory. For instance, while proof serves as a central form of justification in scholarly mathematics, it is frequently absent in the knowledge to be taught at the school level. Therefore, analyzing the structure of logos in praxeologies contributes to understanding the epistemic transformations that occur in the adaptation of disciplinary knowledge into school mathematics.

Praxeologies can be unified into new praxeologies (mathematical organizations) with a common technology or theory (Barbé et al., 2005). Praxeology is considered a *punctual praxeology* if it contains only one type of tasks. Punctual praxeologies that share a common logos block constitute a *local praxeology*. Local praxeologies that share a common theory constitute a *regional praxeology*.

The praxeology in which the type of tasks is "to solve quadratic equations" is an example of a punctual praxeology. This praxeology can be unified with another punctual praxeology whose type of task is "to discuss the solutions of a quadratic equation depending on a parameter", which can be solved using a technique based on the relationship between

the discriminant and the roots of the quadratic equation. These punctual praxeologies, which share a common logos block, constitute a local praxeology. If a theory is established regarding the relationship between the roots of a quadratic equation and the zeros of a quadratic function, these praxeologies can be further unified into a regional praxeology on quadratic functions, which may also include types of tasks such as "to draw the graph of a quadratic function", "to determine the extremum and intervals of monotonicity of a quadratic function", "to solve quadratic inequalities", among others.

In the examination of the limit of a function in high school, Barbé et al. (2005) illustrated how the classification of praxeology can reveal the incompleteness and lack of coherence in the knowledge to be taught, imposing significant demands on teachers' practice. Therefore, external didactic transposition provides insights into the *conditions and constraints* that shape the learning and teaching processes within the observed institution (Barbé et al., 2005).

During the analysis of the didactic transposition process for a particular object of knowledge, which includes praxeological analysis, the researcher creates, tests, and supplements the *epistemological model* of that knowledge with respect to reference institutions (Barbé et al., 2005; Chevallard & Bosch, 2020b). This enables an understanding and questioning of the dominant epistemology within the institution (Lucas et al., 2019). While "the epistemological model describes what is to be studied", "educational ends point out why" (Gascón & Nicolás, 2022, p. 1155). The scholarly knowledge, briefly outlined in Section 1.2, along with the results of research in mathematics education, formed the basis for the development of a reference epistemological model through which the curriculum and textbooks were analyzed in this study.

2. Materials and Methods

2.1. Research Objectives

This paper examines the external didactic transposition of knowledge about polynomials in high school education. It addresses the selection, organization, and evolution of polynomial-related knowledge within high school mathematics, including its coherence and its relation to scholarly knowledge. To investigate these questions in a concrete context, the study takes the case of Croatian high school education between 1994 and 2024 as the object of analysis. The knowledge to be taught about polynomials, as defined by the Croatian curricula and textbooks, has been examined to address the following questions:

1. What scholarly knowledge about polynomials has been transposed into high school education and why?
2. How do polynomial-related praxeologies develop through high school education? How do they interact with other praxeologies?
3. Have there been changes in the knowledge to be taught about polynomials over time, and if so, what are the possible implications of these changes?

2.2. Educational Context, Curricula and Textbook Analysis

After completing eight years of primary education (ages 7–14), Croatian students advance to high school, which includes vocational schools and grammar schools. Grammar school programs, lasting four years and targeting students aged 15–18, prepare students for university education. Due to the importance of polynomials in university education (Pleština & Milin Šipuš, 2022; Pleština, 2023), this study focuses on grammar schools as the reference institution. Grammar school programs can be divided into two groups: those with 3–5 h of mathematics lessons per week and those with 5–7 h per week, with the number of hours potentially varying from one grade to another within the same program. Textbooks are written according to this division. Additionally, Croatian regulations require that textbooks align with curriculum documents.

The analysis of curricula and textbooks has also provided insights into the process of external didactic transposition in Croatia. The curriculum specifies which notions are taught and in what order, but it usually does not detail how each notion is introduced. While the curriculum defines knowledge selection, textbooks provide insight into how the selected knowledge is adapted to be taught (Figure 1).

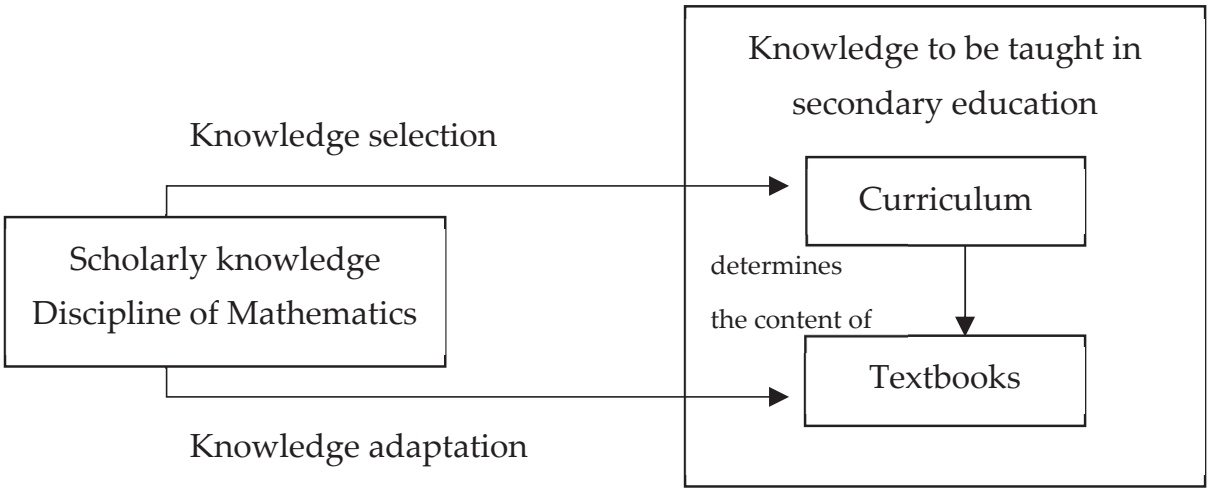

Figure 1. The process of external didactic transposition in Croatia.

The period under study began in 1994 with the implementation of the curriculum (Ministarstvo Kulture i Prosvjete, 1994) that nominally renounces all the characteristics of the New Math movement (OECD, 1961). This curriculum, with some non-binding recommendations in 2003 (Ministarstvo Znanosti, Obrazovanja i Športa, 2003) and 2011 (Ministarstvo Znanosti, Obrazovanja i Športa, 2011), defined the knowledge to be taught until the reform in 2019 (Ministarstvo Znanosti i Obrazovanja, 2019). In 1994, there was a significant re-conceptualization of textbooks, where the knowledge to be taught was distanced from the more formal approach that had dominated in the previous decades. The non-binding curricular document from 2003 did not significantly influence textbooks regarding knowledge about polynomials, although it may have affected the taught knowledge, as it, along with the 2011 document, indicated a shift towards reducing polynomial content that was fully implemented in the 2019 curriculum.

The analysis of curriculum documents aimed to identify polynomial-related mathematical organizations. However, curriculum documents varied in how informative they were in determining the praxis and logos blocks of the praxeologies that constitute these organizations. The 1994 document included content descriptions, often using scholarly terminology, which determine how praxeologies would be structured in textbooks aligned with this document. In terms of defining praxis blocks, the 1994 document was less informative, as it rarely specified types of tasks and techniques. The subsequent curriculum documents provided more explicit guidance on the types of tasks and techniques. Since the identified mathematical organizations were not fully defined by the curriculum documents, a praxeological analysis of textbooks was necessary.

For each program and for each period (1994–2003, 2003–2011, 2011–2019, 2019–present) defined by the curricular documents, a set of the most frequently used high school textbooks, as identified by the Ministry, was subjected to praxeological analysis. Moreover, the analyzed textbooks constituted the reference literature for the state graduation exam, indicating the aim of ensuring that both taught knowledge and learned knowledge align with the knowledge presented in the textbooks.

Each lesson in the textbook was examined sequentially, following the prescribed curriculum order. Textbooks begin each lesson with theoretical notions, followed by solved examples that clarify these notions. Subsequently, the lesson includes exercises, where only final answers are provided at the end of the textbook. The analysis of each lesson comprised two phases. In the first phase, solved examples and theoretical notions underwent praxeological analysis. Types of tasks and associated techniques (praxis blocks) were identified from the analyzed examples, whereas explanations in the examples and theoretical notions informed technology and theory (the corresponding logos block). In the second phase, the authors of this paper solved tasks using techniques identified in the first phase while seeking to justify these techniques with theoretical knowledge from the textbooks. This process complemented the types of tasks by assessing whether the techniques could be elucidated by theoretical knowledge, essentially examining whether praxis and logos are consistent. By comparing praxeologies, their evolution was examined in relation to the scholarly knowledge presented in the scholarly mathematical literature (Markushevich, 1991; Barbeau, 2003; Stillwell, 2010; Lang, 2005). Figure 2 presents a definition, a solved example, and several tasks related to the division of polynomials, illustrating how the source of data[1] was utilized in formulating the praxeologies.

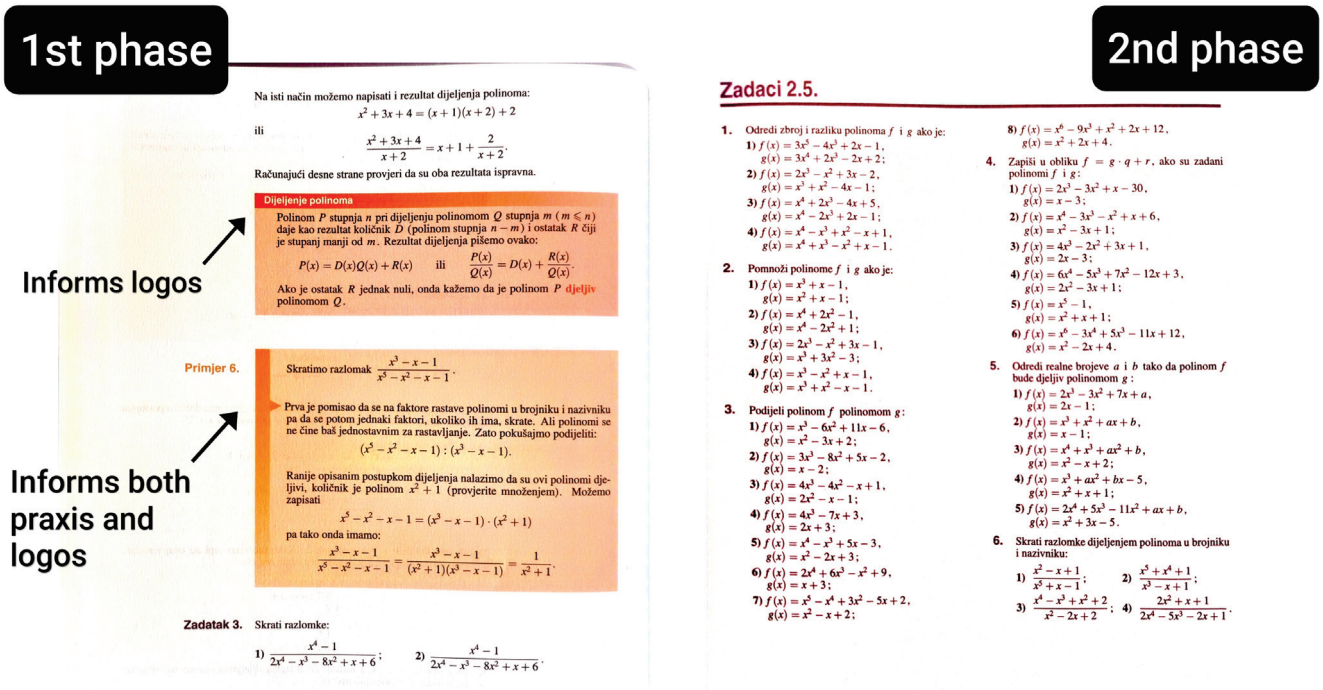

Figure 2. The scheme of textbook analysis.

The analysis of curricula and textbooks has led to the identification of models of knowledge to be taught about polynomials, differing in the scope of the notion, based on the following criteria: (1) the transposition of the definition of a polynomial, (2) praxeologies about polynomials that constitute these models, and (3) the interaction of praxeologies about polynomials with those concerning other content. The models are presented in Section 3.

2.3. Questionnaire Design and Implementation

Examples of how the knowledge to be taught can impose conditions on the learned knowledge were gathered through a questionnaire conducted with 14 final-year high school students (aged 18) during their preparation for the county-level mathematics competition. Afterwards, they completed their high school education under the curriculum introduced in

2019. Six students were in the 5–7 h per week mathematics program, and eight were in the 3–5 h per week program. Although the sample size was small, the fact that these students achieved excellent grades in mathematics and participated in mathematics competitions suggests that their knowledge should be closest to the targeted institutional knowledge, and perhaps even exceeds it.

Based on a thorough analysis of the curriculum and textbooks, the following questionnaire was designed:

1. Simplify the expression
$$\frac{x^5+x^4+1}{x^3-x+1}.$$
Explain the steps you took.
2. Solve the equation
$$\frac{x^5+x^4+1}{x^3-x+1}=0.$$
Explain the steps you took.
3. Sketch the graph of the real function f defined by
$$f(x)=\frac{x^5+x^4+1}{x^3-x+1}.$$
Explain the steps you took.

The aim of the tasks was to examine the activation of knowledge about polynomials, as well as the justifications, within three mathematical organizations: (1) on algebraic expressions, (2) on equations, (3) on functions.

In all three tasks, the polynomials $g(x)=x^5+x^4+1$ and $h(x)=x^3-x+1$ appear. The roots of these polynomials are not easy to determine, making it difficult to factor each of them into linear and quadratic polynomials without using a Computer Algebra System (CAS). Although both polynomials have odd degrees and therefore each has at least one real root, it is not immediately clear whether they share any common root. The polynomial $h(x)$ divides the $g(x)$, meaning that $\frac{g(x)}{h(x)}=x^2+x+1$. This implies that all roots of $h(x)$ are also roots of $g(x)$, with $g(x)$ having two additional complex roots, which are the roots of the polynomial x^2+x+1.

In the first task, students are expected to determine x^2+x+1, and we may observe whether they use long division of polynomials or some other technique.

In the second task, we may observe whether students reason that $\frac{g(x)}{h(x)}=0$ if and only if $g(x)=0$, which is incorrect in this case since $h(x)$ and $g(x)$ have some roots in common, and these numbers are not the solutions of the equation $\frac{g(x)}{h(x)}=0$.

In the third task, students should consider the polynomials as real functions of a real variable, i.e., $g,h:\mathbb{R}\longrightarrow\mathbb{R}$ defined as $g(x)=x^5+x^4+1$ for all $x\in\mathbb{R}$, and $h(x)=x^3-x+1$ for all $x\in\mathbb{R}$. They should conclude that the function f is not defined at the zero of the function h, i.e., the real root of the polynomial $h(x)$.

We consider that the prompt to graph the function f will direct students' focus toward the denominator in the function's definition. As a result, it is likely that students will consider the roots of $h(x)$ in the third task, even if they overlook them in the second task. Moreover, by including or excluding the real root of the $h(x)$ in the domain of the function f and the corresponding point in the graph of f, students will demonstrate how they interpret the difference between the algebraic expression $\frac{x^5+x^4+1}{x^3-x+1}$ and the quotient $\frac{g}{h}$ of the functions g and h.

In the questionnaire, students were allowed to use any mathematical software they preferred, such as CAS and graphing calculators. However, they were still required to write out and explain their techniques. Their responses were subjected to praxeological analysis, with a focus on the techniques and justifications that they provided. Students' responses were compared and interpreted in relation to the models of knowledge to be taught, as determined by the analysis of the curriculum and textbooks. This provides an illustration of how taught knowledge translates into learned knowledge, and how students navigate different mathematical organizations based on varying approaches to polynomials.

3. Results

By analyzing the external didactic transposition in Croatian high school education, four models of knowledge to be taught about polynomials have been identified. Models $\mathcal{M}_1$ and $\mathcal{M}_2$, which have existed since 1994, correspond to programs with 3–5 h of mathematics lessons per week. Under the 2019 curriculum, $\mathcal{M}_2$ was completely abolished, leaving $\mathcal{M}_1$ as the sole model for this program. For the program with 5–7 h, model $\mathcal{M}_3$ has been present since 1994 and has been completely replaced by model $\mathcal{M}_4$ in 2019. The period between 2011 and 2019 was a transitional phase in which models $\mathcal{M}_2$ and $\mathcal{M}_3$ were gradually phased out. According to data from the Ministry regarding the distribution of students across high school programs, it can be inferred that before 2011, $\mathcal{M}_2$ was the prevailing model, whereas after 2019, $\mathcal{M}_1$ definitively became the prevailing model of knowledge to be taught.

The models are composed of mathematical organizations, that is, praxeologies, which are classified according to the extent to which they are grounded in knowledge about polynomials. Praxeologies whose praxis and logos blocks are explicitly based on knowledge about polynomials, and whose logos blocks include some form of definition of a polynomial, are referred to as *praxeologies where a polynomial is the object of knowledge*. Conversely, praxeologies whose logos block does not explicitly contain a definition of a polynomial, but where the polynomial appears merely as an instance of a more general notion (e.g., a function), are referred to as *praxeologies where a polynomial is a component of knowledge*. The latter are typically developed on the basis of the former.

Based on this classification, the results are presented as follows. Section 3.1 addresses the didactic transposition of definitions of polynomials, which form the foundation of the logos blocks of praxeologies where the polynomial is the object of knowledge, discussed in Section 3.2. Section 3.3 presents praxeologies where the polynomial is a component of knowledge, which activate some of the praxeologies introduced in Section 3.2.

3.1. Transposition of the Definitions of a Polynomial

In Croatian high school education, we encounter the definitions of linear and quadratic functions, as well as the following three types of definitions of a polynomial.

Definition 1. *A polynomial is an expression composed of variables and constants, obtained only by addition (subtraction) and multiplication. A monomial is the simplest expression obtained by the multiplication of constants and variables. The sum of two monomials is a binomial. The sum of three monomials is a trinomial.*

Definition 2. *A polynomial is an expression of the form*

$$a_0 + a_1 x + \cdots + a_n x^n,$$

where $a_0,\ a_1, \ldots, a_n \in \mathbb{R}$.

Definition 3. *Let $a_0, a_1, \ldots, a_n \in \mathbb{R}$. A polynomial is a function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

for every $x \in \mathbb{R}$.

Models $\mathcal{M}_1 - \mathcal{M}_4$ of knowledge to be taught differ with respect to the definitions of a polynomial. The representations of the definitions in the models are given in Table 1.

Table 1. Representations of the definitions in the models.

Definitions Models	Definition 1	Definition 2	Linear and Quadratic Function	Definition 3
$\mathcal{M}_1$	Yes	No	Yes	No
$\mathcal{M}_2$	Yes	Yes	Yes	No
$\mathcal{M}_3$	Yes	Yes	Yes	Yes
$\mathcal{M}_4$	Yes	No	Yes	Yes

Definition 3, included in models $\mathcal{M}_3$ and $\mathcal{M}_4$, as well as the definitions of linear and quadratic functions that are part of all models, are special cases of the functional approach D_F to polynomials. Definition 1, which is present in all the models, and Definition 2, which is present in $\mathcal{M}_2$ and $\mathcal{M}_3$, indicate that D_F is not the only approach to polynomials in Croatian high school education. Definition 2 can be justified by the formal-algebraic approach D_A to polynomials. Although not explicitly stated in the analyzed textbooks, Definition 1 encompasses polynomials in multiple variables, whereas Definition 2 addresses polynomials in one variable. Definitions 1 and 2 do not define a variable. However, explanations in the textbooks imply that a variable is a *general real number that is not predetermined*, whereas a *constant is a specific real number*. Hence, it cannot be said that these definitions *fully* belong to the scholarly algebra. They may be regarded as a form of *generalized arithmetic* (Usiskin, 1988), designed to support students in building knowledge based on the properties of real numbers.

In all the models, the notion of an algebraic expression, defined as any expression consisting of variables and constants, obtained via the four basic algebraic operations and the use of parentheses, appears. The notion of an algebraic fraction, defined as the quotient of polynomials, is also present. From the definition of an algebraic expression and Definition 1, it follows that a polynomial is a special type of algebraic expression. In scholarly mathematics, the terms algebraic expression and algebraic fraction are used informally. From the perspective of mathematical logic (Weiss, 2020), the definition of an algebraic expression specifies only the alphabet. It does not specify what constitutes a well-defined sequence of symbols from the given alphabet, i.e., a well-defined string referred to as an algebraic expression. Not every finite sequence of symbols from the observed alphabet should be categorized as an algebraic expression (e.g., $x + *y+$), a point that Bolondi et al. (2020) have also noted. The distinction between an arbitrary string and an algebraic expression is only implicit in the models and it is built upon the generalization of operations with real numbers. Thus, the interpretation of strings is implicitly based on the school definition of a variable as a general real number.

3.2. Praxeologies Where a Polynomial Is the Object of Knowledge

Models $\mathcal{M}_1 - \mathcal{M}_4$ of knowledge to be taught about polynomials are composed of some of the following praxeologies, presented in the order prescribed by the curricula: $p_{1.1}$

on polynomials, algebraic expressions, and algebraic fractions; $p_{1.2}$ on formal operations on polynomials in one variable; $p_{1.3}$ on linear functions; $p_{2.1}$ on quadratic equations; $p_{2.2}$ on quadratic functions; and $p_{2.3}$ on polynomials and algebraic equations. The representations of these praxeologies in the models are given in Table 2.

Table 2. Representations of the praxeologies in the models.

Praxeologies Models	$p_{1.1}$	$p_{1.2}$	$p_{1.3}$	$p_{2.1}$	$p_{2.2}$	$p_{2.3}$
$\mathcal{M}_1$	Yes	No	Yes	Yes	Yes	No
$\mathcal{M}_2$	Yes	Yes	Yes	Yes	Yes	No
$\mathcal{M}_3$	Yes	Yes	Yes	Yes	Yes	Yes
$\mathcal{M}_4$	Yes	No	Yes	Yes	Yes	Partially

3.2.1. Computing with Polynomials

Local praxeology $p_{1.1}$ unites praxeologies $p_{1.1.A}$ on polynomials in one or more variables and $p_{1.1.B}$ on algebraic fractions, through discourse on algebraic expressions, implicitly grounded in real numbers.

Praxeology $p_{1.1.A}$ (Table 3) consists of two punctual praxeologies (one generated with the type of tasks $T_{1.1.A.1}$, and the other with $T_{1.1.A.2}$), united by the discourse on polynomials defined with Definition 1. The definition of a variable as *a general real number*, and the introduction of operations with polynomials, which are based on the generalization of some properties of addition and multiplication with real numbers, can be justified with the aim of *generalizing the knowledge of arithmetic* (Usiskin, 1988) acquired through the previous level of education. Except for the application of institutionalized formulas ($\tau_{1.1.A.2}$), the logos of $p_{1.1.A}$ does not address when and why factorisation is possible or not. The fact that $p_{1.1.A}$ is not limited to polynomials in one variable raises the question of how to factor polynomials in multiple variables. In scholarly knowledge, to factor polynomials in multiple variables, they are often treated as polynomials in one variable with polynomials as coefficients. This approach is challenging to transpose into high school, considering that Definition 1 encompasses polynomials in both single and multiple variables. Consequently, the techniques of $p_{1.1.A}$ for factorising polynomials rely on techniques such as grouping polynomials (justified by the associativity of addition of real numbers), extracting common factors (justified by the distributivity of multiplication over addition of real numbers), and recognising institutionalised formulas about binomials and trinomials.

Table 3. Praxeology $p_{1.1.A}$ on polynomials in one or more variables.

Types of tasks	$T_{1.1.A.1}$: to calculate with polynomials.	$T_{1.1.A.2}$: to factor polynomials.
Techniques	$\tau_{1.1.A.1}$: rules for calculation with polynomials.	$\tau_{1.1.A.2}$: square and cube of binomials, differences of squares, difference and sum of cubes, some criteria for factorization of quadratic trinomial, collecting like terms.
Technology $\theta_{1.1.A}$	Properties of operations with real numbers. Exponent rules. Algebraic expression. Definition 1. GCF of polynomials.	
Theory $\Theta_{1.1.A}$	Real numbers.	

Praxeology $p_{1.1.B}$ (Table 4) consists of three punctual praxeologies (each generated by one type of tasks $T_{1.1.B.1}$, $T_{1.1.B.2}$, $T_{1.1.B.3}$) unified by the notion of an algebraic fraction. Since addition, multiplication, and simplification of algebraic fractions are introduced as

generalized techniques for operations with rational numbers, the discourse of $p_{1.1.B}$ also exhibits features of generalized arithmetic.

Table 4. Praxeology $p_{1.1.B}$ on algebraic fractions.

Types of tasks	$T_{1.1.B.1}$: to calculate with algebraic fractions.	$T_{1.1.B.2}$: to simplify algebraic fractions.	$T_{1.1.B.3}$: to find value of an algebraic fraction for given values of the variables.
Techniques	$\tau_{1.1.B.1}$: rules for calculation (includes $\tau_{1.1.A.1}$) with algebraic fractions.	$\tau_{1.1.B.2}$: factorization of polynomials ($p_{1.1.A}$) in numerator and denominator and then reducing their greatest common factor.	$\tau_{1.1.B.3}$: plugging values instead of variables.
Technology $\theta_{1.1.B}$	Calculation with fractions. LCM of polynomials. Algebraic fraction.		
Theory $\Theta_{1.1.B}$	Real numbers.		

The knowledge of $p_{1.1.A}$ on factorisation forms the basis for the technique $\tau_{1.1.B.2}$ of simplifying algebraic fractions, which is essentially a method for dividing polynomials. Factoring polynomials yields a form from which the greatest common factor (GCF) of the polynomials in the numerator and denominator of the algebraic fraction, as well as the least common multiple (LCM) of polynomials in the denominator of algebraic fractions that need to be added or subtracted, can be identified. The terms GCF and LCM of polynomials inherently imply an order on the set of polynomials. However, since the degree of a polynomial is not discussed in $p_{1.1}$, this order remains implicit. Therefore, recognition of the GCF and LCM of polynomials is based on an intuitive understanding of the complexity of expressions (syntax) and some analogies with the GCF and LCM of integers, which again points to the aspect of generalized arithmetic.

Although there are examples indicating for which values the variable of an algebraic fraction is undefined, when simplifying algebraic fractions ($T_{1.1.B.2}$), the roots of the polynomial in the denominator of the algebraic fraction are generally disregarded. Instead, the variable is treated solely as a symbol, that is, indeterminate. For example, the claim $\frac{20x+16}{25x^2-16} = \frac{4}{5x-4}$, without specifying the set to which the variable x belongs, exemplifies the approach D_A.

The technique $\tau_{1.1.B.3}$ for determining the value of an algebraic expression ($T_{1.1.B.3}$) for a given real number is partly justified by the discourse on the variable as a general real number. However, this discourse does not apply to tasks where an algebraic expression is substituted for a variable, for example, in a task where the value of the expression $\frac{(x-a)^2+(x-b)^2}{(x-a)^2-(x-b)^2}$ for $x = \frac{a^2+b^2}{2a}$ needs to be calculated. Therefore, according to the functionality, the variable in the praxis block $[T_{1.1.B.3}/\tau_{1.1.B.3}]$ is treated as a placeholder and indeterminate.

Praxeology $p_{1.2}$ (Table 5) introduces division with a remainder for polynomials in one variable based on the analogy with integer division, which is a consequence of the transposition of the analogy between the ring of polynomials and the ring of integers.

Long division serves as a technique for type of tasks $T_{1.2.2}$ related to the simplification of algebraic fractions in one variable; thus, $p_{1.2}$ can complete the praxis block of $p_{1.1}$. For instance, the fraction $\frac{x^5+x^4+1}{x^3-x+1}$ can be simplified by dividing polynomials. However, in terms of the logos blocks of $p_{1.1}$ and $p_{1.2}$, there is no discourse connecting the technique of division based on factorisation with the technique of long division. A common discourse would facilitate the comparison of the reliability and efficiency of these techniques depending on the specific task.

Table 5. Praxeology $p_{1.2}$ on formal operations on polynomials in one variable.

Types of tasks	$T_{1.2.1}$: to calculate with polynomials in one variable	$T_{1.2.2}$: to simplify algebraic fractions in one variable.
Techniques	$\tau_{1.2.1}$: addition, subtraction, multiplication, and (long) division of polynomials in one variable.	$\tau_{1.2.2}$: dividing the polynomials in numerator and denominator or dividing them by their greatest common divisor.
Technology $\theta_{1.2}$	Degree of a polynomial. Zero polynomial. Division with a reminder.	
Theory $\Theta_{1.2}$	Definition 2.	

Although we indicated that Definition 2 closely aligns with the formal-algebraic approach D_A, the discourse sometimes acknowledges that the notation

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

implies that the polynomial's values depend on the choice of variable x, which originates from the functional approach D_F. Given that there are no types of tasks in the praxis block of $p_{1.2}$ that would necessitate this discourse, this can be a way to connect $p_{1.2}$ with types of tasks about calculating the value of an algebraic fraction in $p_{1.1}$. On the other hand, the logos of $p_{1.2}$ is not grounded in the generalization of the properties of real numbers, as is the case for the logos of $p_{1.1}$, nor is the variable treated as a general real number. Instead, both the praxis and logos of $p_{1.2}$, are characterised by treating the variable solely as a symbol, which is a characteristic of the formal-algebraic approach D_A.

3.2.2. Linear and Quadratic Equations, Inequalities, and Functions

Praxeology $p_{1.3}$ (Table 6) comprises knowledge about solving linear equations and inequalities, accompanied by a discourse on linear functions and their graphs.

Table 6. Praxeology $p_{1.3}$ on linear functions.

Types of tasks	$T_{1.3.1}$: to solve a linear equation or a system of linear equations in two or three unknowns.	$T_{1.3.2}$: to solve a linear inequality or a system of linear inequalities in two variables.	$T_{1.3.3}$: to sketch and analyse a graph of (piecewise) linear functions.
Techniques	$\tau_{1.3.1.1}$: substitution. $\tau_{1.3.1.2}$: opposite coefficients. $\tau_{1.3.1.3}$: graphical method.	$\tau_{1.3.2}$: graphical method.	$\tau_{1.3.3}$: drawing (part of) a line (two points/point and slope coefficient), translation and mirroring of a graph; roots, intervals of increase/decrease.
Technology $\theta_{1.3}$	Line in the Cartesian plane. Relative positions of two lines. Open and closed half-planes.		
Theory $\Theta_{1.3}$	Linear function and its graph.		

The discourse results from the transposition of some knowledge about the functional approach D_F to real first-degree polynomials and marks the beginning of developing analytical notions such as the monotonicity of functions, the zero of a function, and the graph of a function.

While this discourse, through the notion of a graph of function, provides techniques and justifications for solving equations and inequalities via the graphical method, solving linear equations via the substitution method and the method of opposite coefficients is not supported by the discourse, indicating the incompleteness of $p_{1.3}$. Additionally, although

$p_{1.1}$ and $p_{1.2}$ are activated in the praxis of $p_{1.3}$ when simplifying equations and inequalities, i.e., bringing equations and inequalities into the appropriate form to which the techniques of $p_{1.3}$ can be applied, there is no elucidation of the interrelation between the notions of a polynomial introduced in $p_{1.1}$ and $p_{1.2}$ and the notion of a linear function introduced in $p_{1.3}$. Thus, given this absence of discourse unifying knowledge across these praxeologies, they do not converge into a broader mathematical organization, such as a new local or regional praxeology.

Knowledge about quadratic equations, inequalities, and functions is organized into two praxeologies: $p_{2.1}$ on quadratic equations and $p_{2.2}$ on quadratic functions. Praxeology $p_{2.1}$ (Table 7) is characterized by the D_A approach to the second-degree polynomial—the quadratic trinomial. It complements the discourse of $p_{1.1}$ on factorisation by transposing the Fundamental Theorem of Algebra (FTA) for quadratic trinomials and provides an efficient technique for factorizing such polynomials via the quadratic formula.

Table 7. Praxeology $p_{2.1}$ on quadratic equations.

Types of tasks	$T_{2.1.1}$: to solve quadratic equations.	$T_{2.1.2}$: to discuss the solutions of a quadratic equation depending on the parameter.	$T_{2.1.3}$: to solve algebraic and rational equations.	$T_{2.1.4}$: to solve a system of linear and quadratic equations in two unknowns.
Techniques	$\tau_{2.1.1.1}$: factorization. $\tau_{2.1.1.2}$: applying the formula in radicals. $\tau_{2.1.1.3}$: Viète's formulas.	$\tau_{2.1.2}$: use of the discriminant.	$\tau_{2.1.3}$: reduction to solving linear or quadratic equations.	$\tau_{2.1.4}$: substitution from the linear equation.
Technology $\theta_{2.1}$	Discriminant and solutions of a quadratic equation.			
Theory $\Theta_{2.1}$	Proof of the formula for solutions in radicals of a quadratic equation. FTA for quadratic trinomial. Proof of Viète's formulas and Rational Root Theorem for quadratic trinomial.			

Unlike $p_{2.1}$, praxeology $p_{2.2}$ (Table 8) is characterized by the D_F approaching the second-degree polynomial. Consequently, $p_{2.1}$ and $p_{2.2}$ diverge in their approaches to polynomials. Only discourse that identifies the zero point of the polynomial introduced as a function and solutions of the quadratic equation serves as a potential point of unification. However, since the quadratic function is defined exclusively as a real function, this identification introduces inconsistencies when the zero point of the polynomial is a complex number, as it is not an element of the domain of the function. Like in the case of linear functions and inequalities, the graph of a quadratic function also provides techniques and discourse for solving types of tasks involving quadratic inequalities. Through $p_{2.2}$, the analytical knowledge of $p_{1.3}$ continues to evolve, introducing new notions such as the extremum of a quadratic function.

Table 8. Praxeology $p_{2.2}$ on quadratic functions.

Types of tasks	$T_{2.2.1}$: to draw the graph of a quadratic function.	$T_{2.2.2}$: to determine the extremum and intervals of monotonicity of a quadratic function.	$T_{2.2.3}$: to solve (system of) quadratic inequalities.	$T_{2.2.4}$: to find the intersection of line and parabola.
Techniques	$\tau_{2.2.1}$: the sign of the leading coefficient, vertex, zero points	$\tau_{2.2.2}$: from the graph of the function.	$\tau_{2.2.3.1}$: algebraically. $\tau_{2.2.3.2}$: graphically.	$\tau_{2.2.4.1}$: algebraically. $\tau_{2.2.4.2}$: graphically.
Technology $\theta_{2.2}$	Zero points of function. Parabola and graph of a quadratic function.			
Theory $\Theta_{2.2}$	Quadratic function and its graph.			

3.2.3. Polynomials and Algebraic Equations of Arbitrary Degree

Praxeology $p_{2.3}$ (Table 9) focuses on polynomials of arbitrary degree with the functional D_F approach.

Table 9. Praxeology $p_{2.3}$ on polynomials and algebraic equations of arbitrary degree.

Types of tasks	$T_{2.3.1}$: to calculate with polynomials in one variable	$T_{2.3.2}$: to divide expression into partial fractions.	$T_{2.3.3}$: to solve an algebraic equation or determine the zero points of a polynomial.	$T_{2.3.4}$ *: to sketch the graph of the polynomial.
Techniques	$\tau_{2.3.1}$: addition, subtraction, multiplication, and (long) division with (remainder) of polynomials in one variable.	$\tau_{2.3.2}$: factorization of polynomials in the denominator and multiplicity of its zero points, equality of polynomials.	$\tau_{2.3.3.1}$: factorization. $\tau_{2.3.3.2}$: leading coefficient and free coefficient divisors, division with a linear polynomial. $\tau_{2.3.3.3}$: splitting method. $\tau_{2.3.3.4}$ *: iteration method.	$\tau_{2.3.4}$ *: factorization, zero points (multiplicity), extrema, translations, and properties (monotony, parity) of polynomials.
Technology $\theta_{2.3}$	Algebra of polynomials—functional approach. Zero points of polynomials and solutions of algebraic equations. Properties of zero points (multiplicity, complex conjugate pair, divisibility by a linear polynomial) of polynomials. Characterization of zero polynomial. Viète's formulas. Criterion of opposite signs			
Theory $\Theta_{2.3}$	Polynomial as a real function of one real variable. Definition and characterization of equality of polynomials. Definition of a rational function. Little Bézout Theorem. FTA and its consequences. Rational Root Theorem. Horner's algorithm and its applications *.			

* The components of praxeology $p_{2.3}$ are excluded from model $\mathcal{M}_4$.

Compared with other praxeologies, the logos of $p_{2.3}$ is notably more developed, as evidenced by the numerous slightly adapted results, including proofs, from scholarly knowledge, which not only explain the technique but also construct it. The discourse of $p_{1.1}$ and $p_{1.2}$ is incomplete regarding basic notions like polynomial equality, whereas the logos of $p_{2.3}$ defines it as the equality of functions and characterizes it by the degree and coefficients in the canonical representation of the polynomial. Polynomial algebra is also introduced based on the general notion of a function, that is, pointwise. However, the functional approach to polynomial algebra is not utilized in the praxis block $[T_{2.3.1}/\tau_{2.3.1}]$, due to the selection of tasks of type $T_{2.3.1}$, which can be solved using algebraic techniques from praxeology $p_{1.2}$, without the need for a functional approach.

Praxeology $p_{2.3}$ establishes a connection between polynomials and algebraic equations, but it faces the same issue with complex zeros as $p_{2.2}$. The developed logos of $p_{2.3}$ enables

the determination of real solutions for any algebraic equation, either exactly ($\tau_{2.3.3.1}$ and $\tau_{2.3.3.2}$) or approximately ($\tau_{2.3.3.3}$ and $\tau_{2.3.3.4}$), and allows the sketching of graphs of polynomials ($\tau_{2.3.4}$) without the need for limits and differential calculus. Without techniques $\tau_{2.3.3.3}$ and $\tau_{2.3.3.4}$ equations like $x^3 - x + 1 = 0$ cannot be solved without CAS tools. The approximate solving of algebraic equations and sketching graphs of polynomials without differential calculus is just one of the differences between model $\mathcal{M}_3$, which fully incorporates $p_{2.3}$, and model $\mathcal{M}_4$, which includes it without the techniques $\tau_{2.3.3.3}$, $\tau_{2.3.3.4}$ and $\tau_{2.3.4}$ (the components of praxeology $p_{2.3}$ that $\mathcal{M}_4$ does not include are marked with * in Table 9).

3.3. Praxeologies Where a Polynomial Is a Component of Knowledge

In this section, the possibilities and limitations of the models $\mathcal{M}_1 - \mathcal{M}_4$ are presented in relation to the content that activates knowledge about polynomials presented in Section 3.2. Knowledge about polynomials discussed in this section is typically mobilized as a part of technique associated with a broader type of tasks that is not specific to polynomials. More precisely, a praxeology $[T_A/\tau_A/\theta_A/\Theta_A]$ from Section 3.2 *contributes to the construction of techniques* of a praxeology $[T_B/\tau_B/\theta_B/\Theta_B]$ if solving the type of tasks T_B with technique τ_B requires solving the type of tasks T_A with technique τ_A.

Praxeologies where a polynomial is component of knowledge are grouped into organizations concerning: (1) equations and inequalities; (2) differential and integral calculus; and (3) other mathematical domains, such as analytical geometry, combinatorics, and statistics.

3.3.1. Equations and Inequalities

The absence of $p_{1.2}$ in model $\mathcal{M}_1$ hinders the development of the technique of praxeology p_{ae} about algebraic equations based on the technique of polynomial division. Even in $\mathcal{M}_2$, where $p_{1.2}$ exists, insufficient discourse on the interrelation of polynomials and algebraic equations obstructs the activation of division in the praxis block of p_{ae}. For example, solving an equation like $3x^3 - 4x^2 + 5x + 6 = 0$ is challenging in $\mathcal{M}_1$ using only factorization techniques of $p_{1.1}$. Addressing this equation in $\mathcal{M}_2$ remains difficult because $p_{1.2}$ lacks knowledge about the Rational Root Theorem (RRT). In $\mathcal{M}_3$ and $\mathcal{M}_4$, the equation can be solved since the RRT is part of the logos of $p_{2.3}$. It provides a technique for determining the root $-\frac{2}{3}$ and then employs division to derive the polynomial $x^2 - 2x + 3$, for which solutions can be determined via the techniques of $p_{2.1}$. Thus, the lack of $p_{2.3}$ in $\mathcal{M}_1$ and $\mathcal{M}_2$ strongly impacts the development of techniques of p_{ae} and the selection of tasks (Table 10).

Table 10. Examples of tasks in the praxis block of p_{ae} and techniques of $p_{1.1}$ utilized during their solving.

Types of tasks	To solve an algebraic equation.	To solve a rational equation.
Task	$\left(\frac{x}{2} + \frac{3}{4}\right)^2 - \left(\frac{x}{3} - \frac{1}{4}\right)^2 = \left(1 + \frac{x}{6}\right)^2$	$\frac{3x-1}{6x-3} + \frac{1}{1-4x^2} = \frac{x}{2x+1}$
Techniques	$\tau_{1.1.A.2}$	$\tau_{1.1.A.2}$, $\tau_{1.1.B.1}$, $\tau_{1.1.B.3}$
Results of implementing techniques	$\frac{1}{9}x^2 + \frac{7}{12}x - \frac{1}{2} = 0 \Longleftrightarrow$ $\frac{1}{36}(x+6)(4\,x-3) = 0$	$4x(2x+1) = 0$, for $x \neq \frac{1}{2}$ and $x \neq -\frac{1}{2}$

The discourse explaining techniques of p_{ae} hardly activates the discourse of both $p_{1.1}$, where the D_A approach is dominant, and $p_{1.3}$, where the D_F approach is dominant. Instead, the discourse of p_{ae} is based on the property of real numbers that *if* $a, b \in \mathbb{R}$ *such that* $a \cdot b = 0$, *then it follows that* $a = 0$ *or* $b = 0$, which is a transposition of the scholarly knowledge that a field has no zero divisors. This justification again points to the ambiguity of the variable in the discourses of $p_{1.1}$ and $p_{1.3}$. Although in the scholarly knowledge,

techniques of p_{ae} are justified by the knowledge that the polynomial ring has no zero divisors, this is difficult to transpose given the level and incoherence of the discourses of $p_{1.1}$ and $p_{1.3}$.

Solving exponential, logarithmic, and trigonometric equations reduces to solving algebraic equations through appropriate substitution (Table 11); thus, the limitations of techniques and task selection of p_{ae} in $\mathcal{M}_1$ and $\mathcal{M}_2$ also lead to a reduction in the choice of tasks of these types.

Table 11. Examples of equations that can be reduced to algebraic equations.

Task	Technique of Reduction	Algebraic Equation
$2^x + 2^{1-x} = 3$	$t = 2^x$	$t^2 - 3t + 2 = 0$
$\log(x-1) + \log(x-2) = 2log(x-3)$	$\log(x) + \log(y) = \log(xy)$ $ylog(x) = log(x)^y$	$(x-1)(x-2) = (x-3)^2$
$sin^2x + 3sinxcosx + 2cos^2x = 0$	Division by cos^2x and substitution $t = tanx$	$t^2 + 3t + 2 = 0$
$2sinx - 3cosx = 1$	$t = tan\frac{x}{2}$ $sinx = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$	$t^2 + 2t - 2 = 0$

Unlike p_{ae}, which does not utilise the discourse of $p_{1.3}$ on linear functions, praxeology p_{ai} on algebraic inequalities employs it as justification for solving inequalities via graphical methods. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ only include knowledge about sketching graphs of linear ($p_{1.3}$) and quadratic ($p_{2.2}$) functions without using differential calculus (which is taught later), it is necessary to first represent polynomials in algebraic inequalities as products of first- and second-degree polynomials. Justifying this technique requires transposing knowledge about the FTA.

Praxeology p_{ae} is activated through the praxis of praxeology p_{ag} on analytical geometry by solving systems of algebraic equations. However, polynomials in two variables are part of the logos of p_{ag}. They are addressed through the classification of second-order curves, which emphasizes the polynomial form over the functional approach.

3.3.2. Differential and Integral Calculus

A polynomial as a real function of one real variable is part of the praxis blocks of praxeologies p_f on the general notion of a function, p_{der} on differential calculus, and p_{int} on integral calculus.

In $\mathcal{M}_1$, the notion of a polynomial is narrowed down to specific forms of algebraic expressions, linear and quadratic functions. Consequently, the definition of a rational function is not explicitly covered in $\mathcal{M}_1$, and examples are limited to rational functions where both the numerator and denominator are polynomials of degree less than or equal to two, along with specific forms of degree three polynomials that can be factorised via techniques from $p_{1.1}$. For example, the limit $\lim\limits_{x\to 1} \frac{x^3+x-2}{x^3-x^2-x+1}$ can be solved in model $\mathcal{M}_1$ because polynomials can be factorized via the sum and difference of cubes formulas. On the other hand, $\lim\limits_{x\to \frac{-2}{3}} \frac{3x^5-x^4-5x^3+19x^2-4x-12}{3x^3-4x^2+5x+6}$ is challenging to solve in models $\mathcal{M}_1$ and $\mathcal{M}_2$, which do not include a discourse on the rational roots of polynomials. In $\mathcal{M}_3$ and $\mathcal{M}_4$, this discourse with division allows the previous limit to be simplified to $\lim\limits_{x\to -\frac{2}{3}} x^2 + x - 2 = -\frac{20}{9}$. The absence of the division technique in $\mathcal{M}_1$ and the lack of discourse in $\mathcal{M}_2$ affect the reduction in techniques for determining asymptotes (Katalenić et al., 2023) and sketching the graphs of rational functions.

The absence of discourse on polynomial equality in $\mathcal{M}_1$ and $\mathcal{M}_2$ prevents the existence of the technique of decomposing a rational function into partial fractions via the method of undetermined coefficients. This technique, along with division, is essential for solving integration problems involving rational functions. For example, even though the polynomial $x^3 - x^2 - x + 1$ can be factorized via techniques present in models $\mathcal{M}_1$ and $\mathcal{M}_2$, the integral $\int \frac{dx}{x^3-x^2-x+1}$ cannot be solved via the techniques from these models due to their lack of the partial fraction decomposition technique.

3.3.3. Other Mathematical Domains

Praxeologies based on certain, due to a lack of discourse, isolated knowledge about polynomials are diverse and relate to number theory, combinatorics (e.g., the Binomial Theorem), statistics (e.g., linear regression), etc. For example, the technique for converting from one numerical system to another, derived from the application of Horner's algorithm, exists in all models. However, in models $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_4$, it is not connected to polynomials due to the absence of the transposition of Horner's algorithm and thus remains isolated praxis.

3.4. Examples of How Knowledge to Be Taught Can Condition Learned Knowledge

This section provides examples of how the knowledge to be taught can shape students' learned knowledge. Table 12 presents the task completion rates from the questionnaire conducted among 14 students, as described in Section 2.3.

Table 12. Completion rates for the tasks from the questionnaire.

i	The Task Q_i in the Questionnaire	The Proportion of Students Who Solved Task Q_i Correctly
1	Simplify the expression $\frac{x^5+x^4+1}{x^3-x+1}$.	11/14
2	Solve the equation $\frac{x^5+x^4+1}{x^3-x+1} = 0$.	11/14
3	Sketch the graph of the function f defined by $f(x) = \frac{x^5+x^4+1}{x^3-x+1}$.	2/14

Task Q_1 to simplify the expression $\frac{x^5+x^4+1}{x^3-x+1}$ is of type $T_{1.1.B.2}$ of praxeology $p_{1.1}$ and $T_{1.2.2}$ of praxeology $p_{1.2}$. Using techniques from $p_{1.1}$ that do not include polynomial root analysis but only manipulation of algebraic expressions, nine students factored the polynomial in the numerator, obtaining

$$x^5 + x^4 + 1 = \left(x^3 - x + 1\right)\left(x^2 + x + 1\right)$$

from which they further concluded that

$$\frac{x^5 + x^4 + 1}{x^3 - x + 1} = x^2 + x + 1$$

Only two out of these nine students provided an explanation of their techniques, and in their explanations, they did not use terms such as polynomial, roots of polynomials, polynomial division, GCF, etc. (Table 13, third row and second column). The remaining seven students who successfully solved the task by factoring only wrote out the calculation

steps. Such responses can be explained by a lack of logos in $p_{1.1}$, as the textbooks do not sufficiently emphasize the *why* behind the procedures (Tables 3 and 4).

Table 13. Analysis of the responses to the first question in the questionnaire.

Techniques	Factorization	Long division
The proportion of students who solved task Q_1 correctly	9/14	2/14
Students' explanations	"I assumed that $x^3 - x + 1$ would be a factor in the numerator, so I just fit it." "My goal was to obtain the same expression in the numerator and denominator."	"Due to polynomial division."

Two students used the long division technique, which is part of the praxeologies $p_{1.2}$ and $p_{2.3}$. For this task, long division is a more straightforward and efficient technique than factorization, as it eliminates the need for trial and error or reliance on CAS tools.

Three students attempted to solve Q_1 through factorisation using techniques of praxeology $p_{1.1}$ but failed, despite the option to use CAS tools, which could have provided them with an idea of how to approach the problem. They explained that at school, they were not permitted to use any tools like CAS for such tasks. These three students also failed to solve the remaining two tasks in the questionnaire, which confirms that solving equations and sketching the graph of a rational function rely on the techniques from praxeologies $p_{1.1}$ and $p_{1.2}$.

All students who solved Q_1 also solved Q_2 by using the result obtained in Q_1. They reduced solving the equation $\frac{x^5+x^4+1}{x^3-x+1} = 0$ to solving the equation $x^2 + x + 1 = 0$ which has no solutions in the set of real numbers. All students solved the resulting quadratic equation via the technique of praxeology $p_{2.1}$. However, only three of those eleven students explained their procedure. One student wrote that "$x^3 - x + 1$ must not be zero" but did not determine the roots of the polynomial. The two students who used the long division technique in Q_1 demonstrated knowledge that the polynomials $x^5 + x^4 + 1$ and $x^3 - x + 1$ share the same real root, which they determined using CAS tools rather than using approximate techniques ($\tau_{2.3.3.3}$, $\tau_{2.3.3.4}$) of praxeology $p_{2.3}$. Therefore, it can be concluded that these students demonstrated knowledge aligned with model $\mathcal{M}_4$.

Only those two students who used the long division technique showed in their response to Q_3 that the function f defined by $f(x) = \frac{x^5+x^4+1}{x^3-x+1}$ is not defined for the real root of the polynomial $x^3 - x + 1$ (Figure 3A). The remaining nine students sketched the graph of the quadratic function defined by $x^2 + x + 1$ via techniques of praxeology $p_{2.2}$ and wrote that $f(x) = x^2 + x + 1$, not distinguishing this function from the given function (Figure 3B). This issue also appears in the graph generated by mathematical software (Figure 4), which ignores the fact that the function f is not defined at the real root of the polynomial $x^3 - x + 1$.

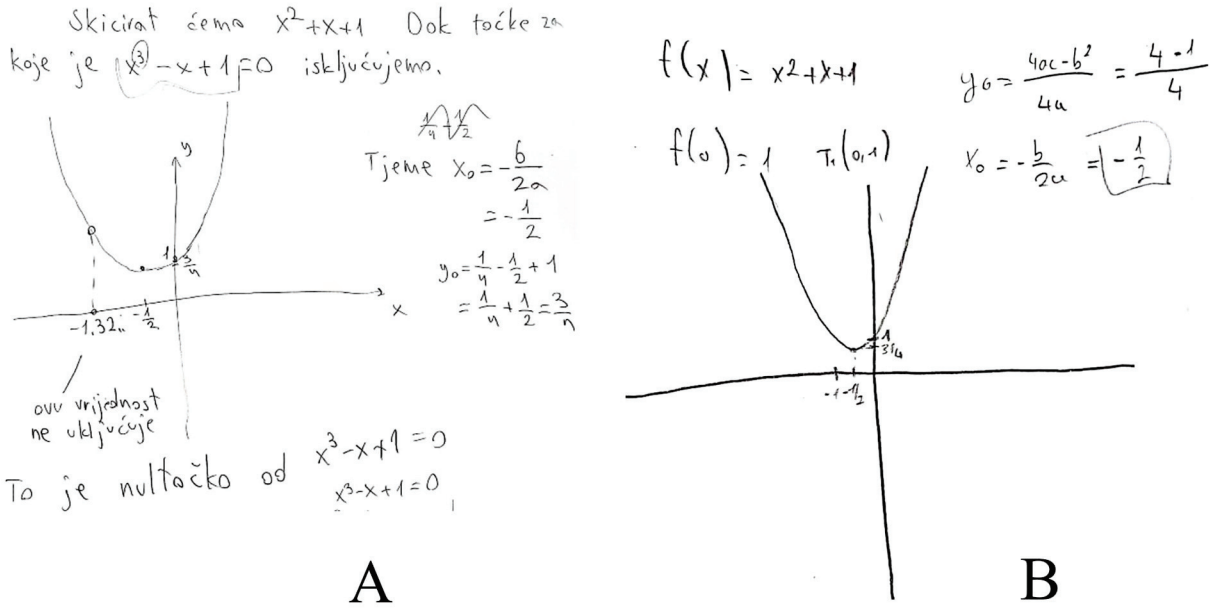

Figure 3. Two types of responses to question Q_3 about sketching the graph of a function that is not defined at the real root of the polynomial $x^3 - x + 1$: response type (**A**) was provided by two students, while response type (**B**) was given by nine students.

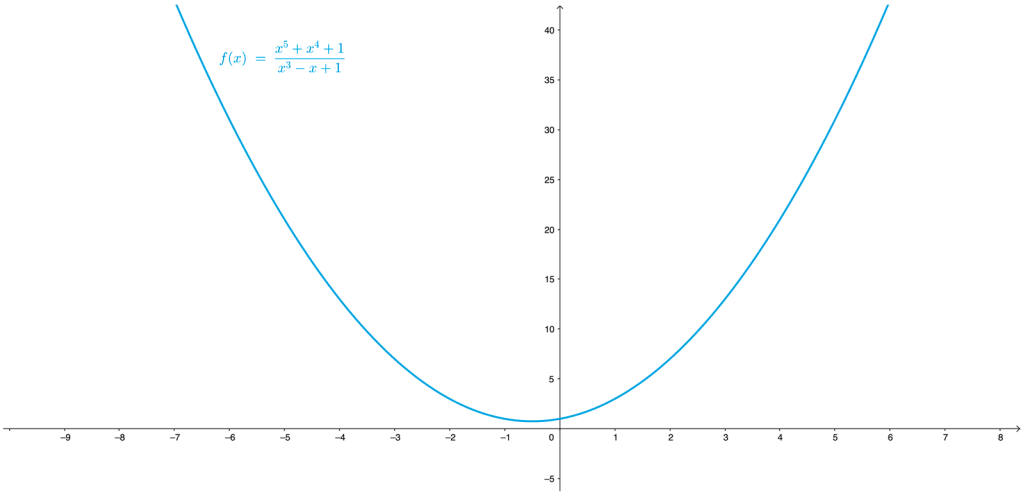

Figure 4. Graph of the function f defined by $f(x) = \frac{x^5+x^4+1}{x^3-x+1}$, generated by mathematical software.

4. Discussion and Conclusions

4.1. The Development of Knowledge to Be Taught and Its Connection to Scholarly Knowledge

4.1.1. Summary of Models of Knowledge to Be Taught

The four models of knowledge to be taught, namely $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_4$, reflect varying levels of development in the treatment of polynomials within Croatian high school education, which, due to the interconnectedness of polynomial-related knowledge with other mathematical content, has implications for the overall high school mathematics.

Model $\mathcal{M}_1$ represents the least developed model of knowledge to be taught. In this model, the polynomial is treated as a special algebraic expression, without an explicit general form (Definition 1), alongside knowledge about linear and quadratic functions. Model $\mathcal{M}_2$ includes all the elements of $\mathcal{M}_1$ but additionally introduces a definition of the polynomial that provides its general form as a polynomial of degree n (Definition 2). In $\mathcal{M}_1$, computational techniques involving polynomials are presented solely as generalized arithmetic, and polynomial division with remainder is not defined. In contrast, $\mathcal{M}_2$

supplements this with a formal approach to polynomial operations, including polynomial division. Neither $\mathcal{M}_1$ nor $\mathcal{M}_2$ has a sufficiently developed logos to support solving algebraic equations using methods other than polynomial factorization and applying the formula for solving quadratic equations in radicals. This limitation has consequences for solving equations and inequalities in general, as well as for analytic geometry. The fact that both $\mathcal{M}_1$ and $\mathcal{M}_2$ address only linear and quadratic functions within a functional approach constrains the development of the notion of rational functions, which poses a significant limitation for the development of calculus.

Model $\mathcal{M}_3$ contains all the knowledge present in $\mathcal{M}_2$, and $\mathcal{M}_4$ contains all the knowledge present in $\mathcal{M}_1$. In addition, both $\mathcal{M}_3$ and $\mathcal{M}_4$ include knowledge about polynomials of arbitrary degree as real functions of a real variable (Definition 3), which enables the development of calculus. $\mathcal{M}_3$ has a sufficiently developed logos to support the exact or approximate determination of the roots of any algebraic equation, whereas $\mathcal{M}_4$ does not support approximate root determination. $\mathcal{M}_3$ also enables techniques for sketching the graph of a polynomial of arbitrary degree without using differential or integral calculus, while $\mathcal{M}_4$ does not support this.

4.1.2. Transposition of Functional and Algebraic Approaches to Polynomials

The identified models of knowledge to be taught, $\mathcal{M}_1 - \mathcal{M}_4$, comprise two types of praxeologies: $p_{1.1} - p_{2.3}$, where the polynomial is the *object of knowledge* (Section 3.2) and those where the polynomial is a *component of knowledge* (Section 3.3).

The discourse of praxeologies $p_{1.3}$ (on linear functions), $p_{2.2}$ (on quadratic functions), and $p_{2.3}$ (on polynomials and algebraic equations of arbitrary degree) follows from the transposition of the *functional approach* D_F to polynomials. In contrast, the discourse of praxeology $p_{2.1}$ (on quadratic equations) is a consequence of the *formal-algebraic approach* D_A to polynomials. Praxeologies $p_{1.1}$ and $p_{1.2}$, where the first encounters with polynomials occur, result from a more complex process of transposition, combining elements of both the D_F and D_A approaches.

The praxeology $p_{1.1}$ results from the transposition of scholarly knowledge about real numbers, identification of formal polynomials and induced functions, and the analogy between the ring of polynomials and the ring of integers. This transposition leads to the school notion of an algebraic expression and school definition of a polynomial focusing on syntax, while semantics is determined by defining the variable as a *general real number*. Although the properties of operations with algebraic expressions and polynomials are attempted to be justified through the generalization of operations with numbers (except for tasks where it is necessary to determine the value of an algebraic expression for a given real number), the environment of real numbers is not evident in the praxis block of $p_{1.1}$, and the variable is treated solely as a *symbol without meaning*, which leans towards the D_A approach. The inconsistency between the praxis and the logos blocks of $p_{1.1}$ has direct consequences for the types of tasks and the validity of techniques, indicating the *fragility of knowledge in the transposition process* (Kang & Kilpatrick, 1992). If it is defined that a fraction can be simplified when the numerator and the denominator are not coprime, and if this knowledge is extended to algebraic fractions through analogies between integers and polynomials, then the validity of techniques depends on how the variable is treated, that is, on the adopted approach to the polynomial. This can be illustrated by the following example. Consider the algebraic fraction $\frac{4y+xy}{xy-2y}$. It may be concluded that the expression can be simplified if polynomials are treated formally, with x and y treated as indeterminate. In that case, $\frac{4y+xy}{xy-2y} = \frac{y(4+x)}{y(x-2)} = \frac{4+x}{x-2}$, since y is a common factor in both the numerator and the denominator. However, if the question is whether this fraction can be simplified for all real numbers for which it is defined, it must be observed that the expression is defined for

$x = 1$ and $y = 1$, but in this case, the value of the algebraic fraction is $\frac{5}{-1}$, and the numbers 5 and -1 are coprime.

In high school mathematics, in both the praxis and logos blocks, the objects of knowledge are exclusively polynomials with integer, rational, real, or complex coefficients, allowing implicit identification of polynomials as algebraic expressions (originating from the D_A approach) and polynomials introduced as functions (originating from the D_F approach). However, none of the models contain a discourse that would explain this identification and thereby unify praxeologies in the mathematical organization with a common and coherent discourse, which can represent a significant constraint on teachers' didactic practice, as shown by Barbé et al. (2005) in the case of limits of functions. The lack of discourse creates tension not only between the definitions but also in the most basic techniques:

1. Polynomial equalities and operations with polynomials as algebraic expressions (equalizing or adding coefficients with the same powers), versus operations with polynomials as functions (defined pointwise);
2. Roots of a polynomial as an algebraic expression with real coefficients (which appear also as complex numbers), versus zeros of a polynomial as a real function (only real numbers).

Considering praxeologies where the polynomial is a component of knowledge, the formal-algebraic approach D_A appears to be unnecessary in any of the models. Therefore, the definition of an algebraic expression, as well as the definition of a polynomial as an expression of a certain form, can be interpreted as a transitional phase towards the functional approach, culminating in differential and integral calculus, with the notion of a function at its core. Until then, content on some individual elementary functions and analytical notions has been adopted, starting with linear functions and continuing with quadratic functions. The dominant focus on linear and quadratic functions in $\mathcal{M}_1$ and $\mathcal{M}_2$ could be justified by the FTA and its implications, but this connection is not evident given the selected knowledge in the models.

Generally, the knowledge about equations, which is not accompanied by the discourse on functions, precedes the knowledge about inequalities, which is accompanied by the discourse on functions and their graphs. The organization of content in which topics transposed from the domain of algebra precede topics transposed from the domain of analysis point out one of the *raisons d'être* of polynomials in high school—the development of algebraic techniques on which the techniques of praxeologies based on functions will further develop. Specifically, $p_{1.1}$ and $p_{1.2}$, for which the phenomenon of *emptiness of interpretation* is characteristic due to the status of the variable in the praxis blocks, and the inconsistency of praxis and logos blocks, demonstrate their reason for existence in the construction of techniques of praxeologies that accompany them, ranging from equation solving to differential and integral calculus. Generally, the dominant role of praxeologies where the polynomial is the *object of knowledge* is in the construction of techniques rather than the discourse for praxeologies where a polynomial is the *component of knowledge*.

4.1.3. Illustration of the Connection Between Knowledge to Be Taught and Learned Knowledge

Although the sample of students surveyed in the questionnaire is small, it includes top mathematics students whose knowledge represents the maximum that can be expected in the institution. Their responses are closest to institutional knowledge, and in some cases may even go beyond it. This makes the findings useful for illustrating possible limitations of the knowledge to be taught, even though they cannot be generalized, as other students are likely to demonstrate less knowledge and possibly different kinds of errors.

The knowledge demonstrated by nearly all surveyed students aligns with model $\mathcal{M}_1$, with only two students exhibiting knowledge consistent with model $\mathcal{M}_4$. Since students were free to choose the techniques for solving the tasks in the questionnaire, it cannot be concluded that those whose knowledge aligns with $\mathcal{M}_1$ do not also possess some knowledge exclusive to model $\mathcal{M}_4$. However, while $\mathcal{M}_1$ encompasses a subset of knowledge compared to $\mathcal{M}_4$, students can compensate for certain techniques, such as long division, by using other techniques related to algebraic expressions and CAS tools. What is missing in students' work are the interpretations of techniques in different contexts and their justifications, leading to incorrect techniques that are built on algebra and thus ultimately hindering the development of the notion of a function. In particular, the technique of simplifying algebraic expressions, which lacks a clear interpretation in its isolated form, takes on interpretations when applied to equations and functions. However, these interpretations elude students. Students perform operations with algebraic expressions and polynomials, but without addressing notions consistently. They tend to approach functions algebraically rather than pointwise, and accordingly the initial understanding of polynomials as algebraic expressions conditions the knowledge about polynomials as functions. If this separation of mathematical organizations with different approaches to polynomials induces gaps in knowledge among the high achieving students, it is reasonable to expect that such difficulties would be visible, and maybe even more pronounced, among other students.

This insufficiently developed discourse on polynomials as both algebraic expressions and functions, along with the absence of a unifying discourse integrating these approaches, which are characteristics of all models, hinders the interpretation of techniques in different contexts, with the functional context being both the most delicate and essential for further learning in mathematics.

4.2. Changes in Knowledge to Be Taught During the Analyzed Period and Their Implications

Following previous discussion based on the analysis of curricula and textbooks, the following rationales for knowledge about polynomials in high school education over the observed period have been identified:

i. Interconnectedness of knowledge. As with scholarly knowledge, knowledge about polynomials is closely intertwined with knowledge about equations, inequalities, analytical geometry, statistics, and differential and integral calculus. High school education begins with generalizing operations with real numbers through operations with polynomials, and based on these algebraic techniques, techniques in the aforementioned domains are developed;

ii. Evolution of a function and analytical approach. Polynomials participate in the gradual evolution of the notion of a function, which occurs almost inductively, starting with linear functions and then progressing to quadratic functions. The inclusion of polynomials of higher degrees allows for the development of analytical notions through their graphs before introducing new techniques based on limits and differential calculus;

iii. Analogies with integers. Analogies between integers and polynomials (properties of operations, division with remainder, Euclidean algorithm, Fundamental Theorem of Arithmetic and Fundamental Theorem of Algebra, prime numbers and irreducible polynomials, GCF and LCM) demonstrate how different objects can share similar properties.

Although the rationales *i*–*iii* are present throughout the entire thirty-year period, the reduction in polynomial content over the years diminishes the significance of reason *iii* and affects reason *i* due to the reduction in techniques. This reflects subtle shifts in high

school education priorities. The unifying potential of algebra, highlighted by the New Math movement, has diminished. The lack of techniques increases the need for CAS tools, while the availability of such tools may in turn encourage the omission of certain techniques and contribute to the reshaping of the knowledge to be taught.

The dominant characteristic of polynomial teaching knowledge in high school remains consistent throughout the observed period: the lack of discourse that would unify, or at least reduce, the tension between the algebraic approach to polynomials as formal expressions and the analytical approach to polynomials as real or complex functions. The fact that teaching knowledge throughout the whole analyzed period exhibits phenomena of incompleteness and incoherence, along with the rationale for polynomials, with only variations in the emphasis of these reasons, suggests that despite the modifications, curriculum changes have not significantly altered the dominant epistemology (Lucas et al., 2019) or the *educational ends* and *paradigms* of high school education (Gascón & Nicolás, 2022). Since a complete analysis of the paradigm would require examining the taught knowledge, teaching practices, and the ecology of knowledge, the collected data suggest that, despite changes, the curricula continue to uphold the *paradigm of visiting the works* (Chevallard, 2015).

Instead, *external didactic transposition*—that is, the very selection and adaptation of knowledge for teaching (Bosch et al., 2021)—has unfolded as an *iterative process*, with knowledge being selected and adapted from already transposed knowledge. The phenomenon of *iteration* in the process of didactic transposition of high school knowledge about polynomials could also be related to other notions and is in line with the *refined model of external didactic transposition* presented by Bosch and Winsløw (2020) for undergraduate education—new university textbooks "draw more on previous textbooks than on scholarly sources" (Bosch & Winsløw, 2020, p. 379).

The phenomenon of iteration in the process of external didactic transposition calls for further investigation, particularly in understanding how to prevent greater curricular fragmentation, as well as how to establish connections when existing links are weakened or lost due to the marginalization or exclusion of certain knowledge, as seen in the case of polynomials.

The reduction in polynomial content over the years has led to even greater fragmentation: the notion of the polynomial has diminished, splitting into linear and quadratic functions on the one hand, and being subsumed into the school notion of an algebraic expression on the other, which is a syntactic notion lacking semantic interpretation. Interpretations of algebraic expressions occur only with the acquisition of new content, such as equations, inequalities, functions, limits, and differential and integral calculus. However, the absence of a discourse on the interference of learned operations on algebraic expressions without interpretations, and the new content whose techniques are built on these operations, may lead not only to a misunderstanding of techniques in specific contexts but also to the incorrect development of new techniques. CAS tools may both contribute to content reduction and assist in bridging the gaps caused by such reduction. However, when students' knowledge is based primarily on techniques without underlying notions, they may struggle to critically evaluate the results produced by CAS and to recognize the limitations of these results.

Although this study focuses on the Croatian case, it raises a more general question about the role and status of polynomials in current high school mathematics. The lack of cohesion between content transposed from algebra and analysis, observed here, may not be specific to the Croatian context. The topics of other research on polynomials worldwide, suggests that similar forms of fragmentation may be present in other educational systems as well. The Croatian case thus serves as a concrete example through which a broader

epistemological phenomenon may be hypothesized, which could be further explored through comparative research across different countries.

Addressing the gaps left by the curriculum and textbooks and fostering a more cohesive and interconnected approach to polynomials in high school mathematics could present a considerable challenge for teachers. Given that knowledge about polynomials is deeply interwoven with various mathematical domains, these disruptions extend beyond polynomials themselves, potentially affecting students' knowledge of related notions. Future research could explore how teachers perceive the fragmentation of curricula content, the extent to which they can bridge these discontinuities, and the didactic strategies they employ or could employ to mitigate these challenges.

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Abbreviations

The following abbreviations are used in this manuscript:

ATD	Anthropological Theory of the Didactic
D_A	formal-algebraic approach to polynomials
D_F	functional approach to polynomials
CAS	Computer Algebra System

Note

[1] Dakić, B. & Elezović, N. (2009). *Matematika 1, 1. dio*, 3rd. ed. Element, pp. 80–81.

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Article

Developing Secondary Mathematics Teacher Leaders: A Multi-Year Curriculum for Inservice Teacher Excellence

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Abstract: In response to systemic inequities in mathematics education, we developed and evaluated a five-year, multi-phase curriculum model to cultivate effective secondary mathematics teacher leaders. Supported by NSF Noyce Master Teacher Fellowships, the APLUS in MATH (APLUS in Math: Alabama Practitioner Leaders for Underserved Schools in Mathematics) program engaged 22 inservice teachers through graduate coursework, National Board Certification preparation, and leadership project development. Using a mixed-methods design, we analyzed data from classroom observations (MCOP^2), National Board Certification assessments, course performance ratings, and teacher leadership project proposals. Results indicate significant improvements in instructional practices, content knowledge, and leadership readiness. Findings underscore the importance for sustained, structured professional development to prepare teachers as instructional experts and change agents in high-need educational contexts.

Keywords: mathematics teacher leadership; national boards; classroom observations; graduate coursework

1. Introduction

Effective mathematics education is essential for preparing students to thrive in an increasingly STEM-driven economy (NCTM, 2014; Taguma et al., 2023). However, persistent challenges, such as inequitable access to high-quality instruction and insufficient professional development opportunities, underscore the urgent need for teacher leaders who can advocate for and implement systemic improvements in mathematics teaching and learning, especially in high-need schools and districts (NCTM, 2018). These challenges have been further intensified by the ongoing teacher shortages and increasing complexity of instructional demands in post-pandemic learning environments.

Recognizing this need, the National Science Foundation (NSF) in the United States has prioritized initiatives like the Noyce Master Teacher Fellowship (MTF) programs to transform certified STEM educators into highly skilled teacher leaders. The APLUS in Math program, funded by the NSF, was designed to align with this approach by preparing inservice secondary mathematics teachers for teacher leadership roles through a structured three-phase curriculum. Launched in 2019, the program engaged participants in graduate coursework, National Board Certification preparation, and teacher leadership

project development, fostering both individual growth and collective impact in high-need educational contexts.

Grounded in Wenger's (1999) Communities of Practice framework and the National Board for Professional Teaching Standard's Five Core Propositions (NBPTS, 2016), the program's curriculum was intentionally structured to cultivate professional identity, instructional expertise, and leadership capacity. The Communities of Practice model informed the creation of collaborative structures such as professional learning communities (PLCs), peer mentoring, and shared inquiry through leadership project design. These structures supported the development of teacher identity through active participation, reflection, and increasing responsibility within the community, an approach consistent with identity-as-practice theories (Ntow & Adler, 2019).

Similarly, the NBPTS Core Propositions served as both an instructional and an evaluative foundation. The program embedded the propositions across coursework and leadership development components, aligning teacher learning activities with principles of equity, reflective practice, and high standards for student learning. As Spagnolo et al. (2022) have shown, formative engagement with standards-aligned practices in structured training programs supports teachers in internalizing leadership roles and driving instructional improvement. By drawing on these complementary frameworks, the program aimed to support participants in not only developing skills, but also assuming identities as mathematics teacher leaders committed to excellence in high-need settings.

This study examines the effectiveness of the first two phases of the curriculum through quantitative and qualitative analyses, while also exploring the readiness of teachers to enter the third phase as teacher leaders equipped to implement high-impact projects. By offering evidence-based recommendations for curriculum development and implementation, this research contributes to the growing body of work on empowering mathematics teacher leaders and enhancing educational outcomes in underserved communities.

1.1. Mathematics Teacher Leadership

The concept of mathematics teacher leadership has gained increasing attention as educators and policymakers recognize its potential to drive systemic improvement in mathematics education. Teacher leaders extend their influence beyond their classroom responsibilities by mentoring colleagues, advocating for effective practices, and contributing to professional communities (Clemans et al., 2012; Hunzicker, 2012; Sarrell et al., 2024; Sinha & Hanuscin, 2017; York-Barr & Duke, 2004). Within mathematics education specifically, teacher leaders are uniquely positioned to address content-specific challenges, such as implementing rigorous standards, fostering mathematical reasoning, and promoting equitable teaching practices (NCTM, 2014, 2018; Vale et al., 2023).

Effective mathematics teacher leaders often hold formal leadership positions (e.g., department chairs, instructional coaches, curriculum specialists) and exercise informal leadership roles (e.g., mentoring peers, leading collaborative lesson planning). Regardless of their titles, successful and effective mathematics teacher leaders possess deep content knowledge, pedagogical expertise, and interpersonal skills that enable them to influence their colleagues constructively (Lumpkin et al., 2014; Stronge, 2018; Sublette, 2013; Surrette, 2020; Wenner & Campbell, 2017).

The development of teacher leaders requires sustained, targeted professional development. Programs such as the NSF Noyce Master Teacher Fellowship emphasize multi-year opportunities that integrate advanced coursework, certification pathways, and leadership training. The literature consistently suggests that sustained, structured experiences are critical for preparing teachers to navigate the complexities of leadership roles while maintaining a focus on improving student outcomes (Darling-Hammond et al., 2017; Gading,

2024; Groothuijsen et al., 2018; Hitt & Tucker, 2016). Despite these supports, challenges to fostering mathematics teacher leadership remain. Common barriers include limited time and resources, inadequate administrative support, and varying perceptions of leadership roles within school communities. Additionally, the need for culturally responsive leadership that addresses the diverse backgrounds of students and schools, particularly in high-need districts, is increasingly emphasized in the literature (Brown et al., 2022; Dickson, 2023; Klar & Brewer, 2013).

Although still a developing area of research, existing studies underscore the transformative potential of mathematics teacher leadership. When supported through systemic initiatives and ongoing professional development, teacher leaders can lead instructional change and foster collaborative cultures that improve mathematics teaching and learning at scale.

1.2. Fostering Mathematics Teacher Leadership and Its Challenges

Fostering mathematics teacher leadership is a critical strategy for enhancing the quality of mathematics education and addressing systemic inequities in schools. Teacher leaders play pivotal roles in driving instructional change, mentoring colleagues, and shaping school-wide or district-wide practices (Borko et al., 2021; Campbell & Lee, 2017; Huggins et al., 2017). They translate complex mathematical concepts into effective teaching practices while promoting critical thinking, problem-solving, and equitable learning outcomes. However, cultivating such leadership requires intentional professional development and systemic support (Hopkins et al., 2013; Parfitt, 2022).

Effective teacher leadership programs emphasize a blend of content mastery, pedagogical expertise, and leadership skills. Multi-year initiatives, such as the NSF Noyce Master Teacher Fellowships, offer structured opportunities that combine graduate coursework, National Board Certification preparation, and leadership project development. These experiences equip teachers to lead professional development, mentor peers, and implement impactful instructional reforms.

Collaboration is also a key component. Professional learning communities (PLCs) and collaborative lesson planning provide opportunities for teachers to develop and apply leadership skills in authentic contexts (Ansari & Asad, 2024). Moreover, leadership training must include strategies for effective communication, conflict resolution, and cultural competency for teacher leaders to navigate diverse school environments successfully (Ghamrawi et al., 2024; Sorge et al., 2023). Despite these promising models, challenges persist. Time constraints and heavy teaching loads limit teachers' capacity to engage in leadership roles (Bellibaş et al., 2024; Jotkoff, 2022). Additionally, school systems often lack formal structures to support teacher leadership, resulting in unclear roles for teacher leaders (Bryant & Walker, 2022; Hoy & Miskel, 2013).

Teacher leadership programs must be responsive to the unique needs of high-need schools, where systemic barriers to success are prevalent. Teacher leaders in these contexts must address not only academic challenges but also the socioeconomic factors, as well as community cultural factors shaping student learning (Theoharis, 2024).

Lastly, in addition to balancing the required components and remaining responsive to individual needs, professional development for teacher leaders requires securing buy-in from administrators and peers. This is a common obstacle, as teacher leaders can face resistance when introducing new practices (Printy & Marks, 2006). Building trust and establishing credibility takes time but is critical for effective leadership.

In short, fostering mathematics teacher leadership holds transformative potential, but it requires a systemic approach (Campbell & Lee, 2017; Smith et al., 2025). Schools, districts, and policymakers must provide sustained support, clear pathways for leadership roles

and culturally responsive professional development to overcome these challenges and maximize the impact of teacher leaders.

1.3. Approaches and Structures for Developing Secondary Mathematics Teacher Leaders

Developing effective secondary mathematics teacher leaders requires intentional approaches and well-designed structures that address the unique demands of mathematics education. Beyond improving individual teaching practices, these initiatives aim to empower teachers to influence colleagues, drive instructional change, and foster collaborative professional cultures within schools and districts (Lai & Cheung, 2015). A foundational approach involves strengthening deep content knowledge and pedagogical expertise. Secondary mathematics teachers must be equipped to handle complex subject matter and facilitate rich mathematical reasoning and problem-solving. Graduate-level coursework and professional development programs tailored to the nuances of secondary mathematics instruction play a critical role. The NSF Noyce Master Teacher Fellowships exemplify this approach by combining advanced coursework with leadership development.

Structured certification pathways, such as National Board Certification, also support teacher leadership by validating instructional excellence and demonstrating a commitment to professional growth. Preparing for these certifications also fosters reflective practices and the use of evidence-based teaching strategies, which are critical for leading peers effectively.

Leadership development must also emphasize interpersonal and organizational skills. Workshops and training sessions on mentoring, conflict resolution, and facilitating PLCs help teacher leaders develop the competencies needed to guide their colleagues effectively (Downing Murley et al., 2008). Leadership projects embedded within these programs provide authentic opportunities for teachers to apply these skills in real-world educational contexts (Duşe, 2020). Sustaining teacher leadership requires collaborative structures and systemic support. Schools and districts should establish PLCs, interdisciplinary teams, and mentorship networks that promote shared learning and continuous improvement (Hunzicker, 2012). Teacher leaders thrive in environments that provide protected time for leadership activities, clear role definitions, and recognition of their contributions (Cherkowski, 2018).

By integrating robust content-focused professional development, leadership skill-building, and supportive organizational structures, schools and districts can cultivate a cadre of secondary mathematics teacher leaders capable of driving lasting improvements in teaching and learning.

2. Mathematics Teacher Leader Program Curriculum

The curriculum for this project was designed to develop mathematics teacher leaders through a multi-phase approach aligned with the research on effective teacher leadership preparation, outlined in the previous section. The planned curriculum was guided by five overarching goals:

- **Goal 1:** Teachers will become instructional experts in their schools/districts by working to improve and master their own instructional practices over time.
- **Goal 2:** Teachers will increase their mathematical content knowledge to levels requisite for leading future professional development in their schools/districts.
- **Goal 3:** Through the collaborative support of administrators, teachers will become leaders to serve as mathematics department chairs, instructional coaches, school/district action-researchers, and/or mentors of early-career teachers through an induction program.
- **Goal 4:** Teachers will assist in building a high-capacity network for exceptional quality clinical experiences for preservice mathematics teachers.

- **Goal 5:** Teachers will learn and emerge to enter leadership roles with the Alabama Council of Teachers of Mathematics' annual conference and national conferences.

This paper focuses primarily on teachers' accomplishments related to Goals 1 and 2, as well as their readiness to begin the work of Goal 3. Findings related to Goals 4 and 5 are not addressed in this study. These goals and the resultant curriculum reflect a deliberate alignment between advanced content knowledge development, leadership capacity building, and the practical application of skills in real-world contexts. Grounded in the Communities of Practice framework (Wenger, 1999) and the National Board for Professional Teaching Standards' Five Core Propositions, the program provided a coherent and structured pathway for teachers to grow as instructional leaders and change agents. Our research question of focus is as follows: *How effective is a multi-year professional development program in preparing inservice secondary mathematics teachers to improve their instructional practices, strengthen their content knowledge, and develop readiness for teacher leadership roles?*

2.1. Description of the Three-Phase Curriculum Design

The curriculum was organized into a three-phase design spanning five years. Each phase aligned with the program's overarching goals to develop teachers as instructional experts, content leaders, and emerging teacher leaders.

Phase 1 (Graduate Coursework—14 months) focused on building instructional expertise (Goal 1) and deepening mathematical content knowledge (Goal 2). Teachers completed six graduate-level courses designed to strengthen both pedagogical practices and content mastery:

Access & Equity in Mathematics Education: Explored strategies for creating equitable learning environments by examining personal instructional beliefs and addressing systemic barriers to student success.

Advanced Algebra and Number Theory: Deepened teachers' understanding of mathematical structures and connections across the K–12 curriculum.

Instruction and Supervision in Mentoring and Coaching: Introduced adult learning theory and practical strategies for mentoring and instructional coaching, laying the groundwork for teacher leadership.

Teaching, Learning, and Curriculum in Mathematics Education: Focused on research-based instructional strategies and curriculum design aligned with national standards.

Technology and Assessment in Mathematics Education: Developed teachers' capacity to integrate technology effectively and apply formative assessment practices to support student learning.

Advanced Geometry and Data Analysis: Strengthened teachers' knowledge of geometry, statistics, and data analysis, critical for leading professional development in content areas.

Phase 1 also established a Professional Learning Community (PLC) among participating teachers, fostering collaborative relationships and preparing them for the leadership components in later phases.

In Phase 2 (National Board Certification and Collaborative Leadership Preparation—2 years), teachers worked collaboratively to pursue National Board Certification (NBCT), a rigorous, nationally recognized credential that emphasizes accomplished teaching and reflective practice. Teachers who achieved NBCT status in the first year of Phase 2 served as mentors for peers still pursuing certification, directly supporting Goal 3.

This phase reinforced teachers' leadership capacity through structured PLCs, mentorship experiences, and preparation for leading school- or district-level projects in Phase 3. This last phase centers on the implementation of teacher leadership projects developed in

collaboration with school and district administrators. At the time of this study, Phase 3 was ongoing, and is not the focus of this report.

2.1.1. Brief Descriptions of the Six Graduate Courses, Phase 1

Course 1a—Access & Equity in Mathematics Education: This course focused on developing teachers' understanding of how cultural, socioeconomic, and behavioral factors impact student learning. Teachers critically examined their instructional beliefs and developed strategies to create equitable and accessible mathematics learning environments. Key assignments included an instructional beliefs reflection, an equity action plan, and a professional presentation on cultural responsiveness (supports Goal 1 and Goal 3).

Course 1b—Advanced Algebra and Number Theory: Teachers explored the historical development of number systems and advanced algebraic concepts, with an emphasis on problem-solving and mathematical reasoning. The course connected abstract mathematical structures to the K–12 curriculum and prepared teachers to lead content-focused professional development. Assignments included content mastery through Khan Academy modules and reflective analysis of teaching practices (supports Goal 2).

Course 2—Instruction and Supervision in Mentoring and Coaching: This course introduced adult learning theory and effective strategies for mentoring and coaching peers. Teachers developed skills in providing constructive feedback, facilitating professional growth, and leading PLCs. Assignments included structured coaching simulations, data-informed feedback exercises, and reflection on adult learning principles (supports Goal 3).

Course 3—Teaching, Learning, and Curriculum in Mathematics Education: Teachers examined historical and contemporary research on mathematics instruction and curriculum design, focusing on strategies that promote active student engagement and standards-based instruction. Major projects included curriculum design aligned to student learning needs and reflective critiques of instructional practices (supports Goal 1).

Course 4a—Technology and Assessment in Mathematics Education: This course developed teachers' capacity to integrate technology tools effectively in mathematics instruction and to design meaningful formative assessments. Teachers also explored strategies for leading technology-focused professional development for colleagues. Key assignments included technology integration plans and professional development workshops (supports Goals 1 and 3).

Course 4b—Advanced Geometry and Data Analysis: Teachers deepened their understanding of geometry, statistics, and data analysis to strengthen their instructional content knowledge. The course emphasized applying statistical reasoning and data-driven decision-making in classroom contexts. Assignments included content mastery modules and data analysis projects using real-world datasets (supports Goal 2).

2.1.2. Brief Description of the Two Years of Phase 2

Phase 2 spanned two academic years and focused on advancing teachers' leadership development through the pursuit of National Board Certification (NBCT) and participation in collaborative professional learning communities (PLCs). This phase directly supported the program's goals of developing instructional expertise (Goal 1), deepening content knowledge (Goal 2), and fostering emerging leadership skills (Goal 3).

Teachers worked collaboratively in PLCs to prepare for the four components of the National Board Certification process, engaging in regular peer feedback and shared reflection on their instructional practices. Those who achieved NBCT status in the first year of Phase 2 served as mentors for their peers in the second year, providing direct leadership experience and reinforcing the program's emphasis on teacher-led professional growth.

Phase 2 also prepared teachers for Phase 3 leadership projects. During the summer transition between Phases 2 and 3, teachers worked in collaboration with school administrators to develop leadership project proposals aimed at addressing instructional challenges and improving educational outcomes in their local contexts. These projects represented the culminating application of the leadership skills developed throughout the first two phases of the program.

3. Research Methods

This study employed a convergent mixed-methods design to evaluate the effectiveness of the program's first two phases in meeting its goals of improving instructional expertise, deepening content knowledge, and preparing teachers for leadership roles (Katz-Buonincontro, 2024). Data were collected from multiple sources to capture both quantitative outcomes and qualitative insights.

Specifically, the data sources include:

1. Classroom Observations: Three observations per year per teacher using the Mathematical Classroom Observation Protocol for Practices (MCOP^2; Gleason et al., 2017).
2. National Board Certification Scores: Teachers' performance on Components 1–4.
3. Course Performance Ratings: Quantifiable assessments from Phase 1 coursework.
4. Teacher Leadership Project Proposals: Evaluations of proposals for Phase 3 leadership projects as assessed by the project leadership team.

3.1. Participants

The study involved 22 inservice secondary mathematics teachers across five different school districts who participated in the APLUS in Math program over a five-year period. All participants engaged in the full Phase 1 graduate coursework, Phase 2 to pursue National Board Certification, and entered Phase 3 teacher leadership project implementation.

3.2. Data Collection Procedures and Sources

Data were collected systematically across the project phases to capture changes in teacher instructional practices and dimensions of pedagogical content knowledge. The four main data sources used for this study are each briefly described in the sections that follow.

3.2.1. Classroom Observations with the MCOP^2

The MCOP^2 was designed to capture the two sides of the classroom, capturing student engagement in mathematical practices (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) during instruction and teachers' facilitation of strong effective teaching practices during lesson enactment (NCTM, 2014). The MCOP^2 development was framed by examining the classroom as a community of learners in which teachers and students comprise the community (Wenger, 1999). The MCOP^2 has a long history of published works providing strength to validity arguments for its use to capture and measure teacher facilitation and student engagement (Gleason et al., 2015, 2017; Zelkowski & Gleason, 2016; Zelkowski et al., 2024). Classroom observations allow for understanding patterns of improvement (or not) across the two constructs of measurement by the MCOP^2.

Data was collected with the MCOP^2 with three observations per teacher per academic school year. Each observation was conducted by a trained doctoral graduate student. Observations began in the first two months of the school year, and each of the second and third observations occurred 2–3 months apart. Three data points provided a yearly rating for teachers and their classrooms.

3.2.2. National Boards Portfolio Scores

Teachers' performance on Components 1–4 of the National Board Certification process provided standardized measures of content knowledge, instructional expertise, and reflective practice. A supplementary file is provided that describes a greater depth of each component and the scoring scales. Component 1 is a content knowledge computerized assessment. Component 2 is centered on assessing a teacher's differentiation of instruction. Component 3 is centered on assessing a teacher's teaching practice and learning environment. Component 4 is centered on assessing a teacher's ability to be effective and a reflective practitioner.

3.2.3. Course Grades with Quantifiable Differences Across Teachers

Project faculty rated teacher performance across the six Phase 1 graduate courses using a four-level scale. The four-level scale measured the demonstrated teacher performance as:

1. Growth less than the level expected of a teacher with experience and assuming teacher leadership in no capacity.
2. Growth to the level expected of a teacher with experience and assuming teacher leadership in limited capacities.
3. Growth beyond the level expected of a teacher with experience and assuming teacher leadership in multiple capacities.
4. Growth at an exceptional level expected of a teacher with experience and assuming teaching leadership in any capacity.

To ensure consistency in performance ratings and reduce subjectivity, project faculty employed the use of descriptive rubrics for overall course performance relative to course assessments administered across the beginning, middle, and end of the courses. Thresholds for course growth ratings (e.g., "acceptable," "beyond expected," "exceptional") were based on program-defined benchmarks that reflected anticipated competencies of teacher leaders. These thresholds were informed by prior literature on teacher development and internal alignment to NBCT scoring expectations.

3.2.4. Teacher Leadership Project Proposal Ideas for Entering Phase 3

Teachers submitted leadership project proposals at the end of Phase 2, which were evaluated by the project leadership team for potential impact and alignment with leadership goals. Each teacher leadership project was categorized as follows:

1. Initial proposed ideas of the project demonstrate potential impact.
2. Initial proposed ideas of the project demonstrate a significant potential impact.
3. Initial proposed ideas of the project demonstrate a very significant potential impact.

These ratings provided snapshots of teachers' initial ideas of teacher leadership.

3.3. Data Analyses

Our analyses focused on assessing how well the curriculum met Goals 1 (instructional expertise) and 2 (content knowledge) and prepared teachers for leadership roles (Goal 3). We employed both quantitative methods—such as comparing MCOP^2 scores and NBCT components—and qualitative assessments from course and leadership project proposal evaluations. While Goals 4 and 5 will be addressed in subsequent papers upon project completion, our current analysis provides a comprehensive picture of the curriculum's impact in the first two phases.

3.3.1. Analyses for Goal 1, Teachers Become Instructional Experts to Lead

Overall, the accomplishment of a teacher in this project becoming an NBCT is effectively a single-measure outcome. Nationally, a total of only about 1–3% of all teachers

earn this status. Beyond the accomplishment, we examine the relationship between actual classroom practices via the MCOP^2 during Phases 1 and 2, and the outcomes of Components 2, 3, and 4 on National Board components. Additionally, three courses in Phase 1 evaluated teachers early on as to their level of accomplishment in individual courses. While these ratings certainly would change as time progresses, these data sources were integrated into the analysis. Triangulation was achieved by comparing patterns across these instruments to identify consistency in teacher development outcomes. For instance, growth observed in MCOP^2 scores was examined in relation to NBCT scores for instructional practice (Components 2–4) and corroborated by course-based performance ratings in pedagogical domains. Similarly, teacher leadership readiness was assessed by aligning qualitative ratings of leadership project proposals with overall Phase 1 course performance and NBCT certification outcomes.

3.3.2. Analyses for Goal 2, Teachers Develop Strong Content Knowledge to Lead

Component 1 on National Boards includes three individualized constructed response items on Algebra, Geometry, and Data Analysis, as well as a score on the 45-question multiple choice content knowledge test spanning Contexts for Mathematics, Problem Solving/Number Sense, and Modeling and Analysis. Our analysis includes overall preparedness as measured by the National Boards rubrics in each of the three content areas, as well as the overall component score on content knowledge. Additionally, analyses of the MCOP^2 teacher facilitation construct played a significant role in understanding, while teaching practices were present in higher-/lower–content–knowledge teachers to lead specific professional practice development.

3.3.3. Analyses for Goal 3, Teachers Become Leaders

At the conclusion of Phase 2, the instructional faculty assigned categorical readiness for their proposed leadership projects entering Phase 3. The analysis consists of using measures across Goals 1 and 2 in consideration of the categorical readiness of the impact potential of leadership entering Phase 3. A correlational relationship and a comparative assessment of performance and project potential are presented in the results.

4. Findings

In this section we present our findings from the data analyses.

4.1. Instructional Expertise Outcomes (Goal 1)

We considered two high-validity evidence measures to evaluate the effectiveness of our curriculum in preparing teachers to become instructional experts, as well as our instructor-assigned growth ratings based on Phase 1 course performances. The first was their efforts to become National Board-Certified Teachers (NBCTs) with national standardized scoring rubrics. The second was with the observation protocol the MCOP^2. Additionally, our team's third measure was based on course performances in the pedagogical graduate coursework. To evaluate teacher growth in instructional practice, we triangulated data from the MCOP^2 classroom observation protocol, National Board Components 2–4, and pedagogical coursework ratings.

Table 1 presents the average scores of the participants on NBCT Components 2, 3, and 4, which, as previously described, are focused on instructional practices. Teachers' average scores on NBCT instructional components ranged from 2.601 to 2.685, indicating clear evidence of accomplished teaching. Median and mode scores were at or above 3.000 across all components (component 4 median 2.813), suggesting that the majority of participants met or exceeded national benchmarks. The means represent scores' bottom-end range of "clear evidence of accomplished teaching practice". Likewise, the median and mode scores

are slightly more convincing of "clear evidence of accomplished teaching practice" in the upper half and majority of teachers. When factoring in the standard deviation, one could conclude that variation of a half-point reflects the range of teacher performance as slightly "limited" to "clear, consistent, and convincing" evidence of accomplished teaching.

Table 1. National Board Pedagogical Component 2, 3, 4 findings.

	Component 2	Component 3	Component 4
Mean	2.601	2.685	2.602
Median	3.000	3.000	2.813
Mode	3.000	3.000	3.000
St. Dev.	0.572	0.520	0.535

Note. Data reflects the highest score for each teacher if a component was submitted more than once.

Next, we consider classroom practice, as evidenced by live, multiple classroom observations, of the teachers with the MCOP^2 (Table 2). The MCOP^2 is scored on a 0-1-2-3 rubric on 16 items (See Gleason et al., 2015, 2017). Results from MCOP^2 observations showed statistically significant improvement in both teacher facilitation and student engagement from baseline to post-Phase 1. Notably, growth in teacher facilitation continued into Phase 2, while student engagement plateaued, suggesting that early gains were sustained through the certification process. Generally, the data shows tremendous growth (Phase 1 to Phase 2) and consistency some improvement after Phase 1.

Table 2. Mathematics Classroom Observation for Practices Protocol [MCOP^2] means.

	Baseline	Post-Phase 1	Post-Phase 2
Student Engagement	1.830	2.409 **	2.475 ^{ns}
Teacher Facilitation	1.536	2.193 **	2.413 *
Total MCOP^2	1.683	2.301 **	2.444 ^{ns}

Note. The MCOP^2 scores represent the mean across all teachers, where baseline scores were collected in a teachers' first school year in the project; post-Phase 1 scores were collected in the immediate school year after graduate coursework, and post-Phase 2 scores were collected after two years of work towards National Boards. * $p < 0.05$, ** $p < 0.01$, ^{ns} non-significant.

Across the three pedagogy-focused courses (Table 3), over two-thirds of teachers demonstrated growth "beyond" or "exceptional" relative to expectations for teacher leaders. These results affirm the effectiveness of the graduate coursework in advancing teachers' instructional expertise. While five teachers demonstrated mostly acceptable growth towards teacher leadership, only one teacher with a life tragedy event did not demonstrate expected growth in the last course.

Table 3. Teacher pedagogical course growth in Phase 1 courses.

	Course 1a	Course 3	Course 4a	Mean
3-Rating Exceptional	3	7	7	6.667
2-Rating Beyond Expected	13	12	12	12.333
1-Rating Expected/Acceptable	6	3	4	4.667
0-Rating Less than Expected	0	0	1	0.333

Note. Course 1a is access and equity; Course 2 is curriculum, teaching, and learning; Course 3 is tools, technology, and assessment. See Section 3.2.3 for greater descriptions. Zero rating reflects less-than-expected growth. One: rating is acceptable, two: beyond expected, and three: exceptional. N = 22.

4.2. Content Knowledge Outcomes (Goal 2)

Content knowledge development was assessed using NBCT Component 1 results and Phase 1 mathematics course ratings. There are multiple measures within the National

Boards Component 1 content knowledge assessment. The constructed response section requires teachers to demonstrate content and pedagogical knowledge of mathematics by addressing a specific scenario involving a student's work or a mathematical problem in the areas of Algebra and Functions, Geometry, and Data Analysis and Statistics. Teachers showed strong performance in all domains, with the highest average scores in data analysis/statistics (M = 3.398) and solid results in algebra and geometry (Table 4). The combined score distribution indicated that 21 of 22 teachers demonstrated "clear" or better evidence of content mastery.

Table 4. National Board content knowledge findings.

	Algebra and Function #	Geometry	Data Analysis and Statistics	Selected Response ^
Mean	3.091	3.170	3.398	3.193
Median	3.000	3.000	3.438	3.205
Mode	3.000	4.000	4.000	^
St. Dev.	0.532	0.705	0.560	0.575

Note. Data reflects the highest score for each teacher if a component was submitted more than once. # There are slight variations in the Early Adolescence and Adolescence and Young Adulthood math component tests. Families of Functions extend for the high school teachers beyond middle school teachers. There is also a slight shift in Calculus and Trigonometry between both tests. ^ Selected response is a continuous score, so the mode is not calculable since all are different.

Lastly, we considered growth as measured within the two content knowledge courses in Phase 1 of the project. As shown in Table 5, fifteen teachers were rated as having achieved "beyond expected" or "exceptional" growth in content knowledge across two content courses. These ratings suggest the curriculum's alignment with NBCT standards effectively deepened teachers' mathematical understanding (see Section 3.2.3).

Table 5. Teacher content knowledge growth in Phase 1 courses.

	Course 1b	Course 4b	Mean
3-Rating Exceptional	9	9	9.000
2-Rating Beyond Expected	7	5	6.000
1-Rating Expected/Acceptable	4	3	3.500
0-Rating Less than Expected	1	4	2.500

Note. Course 1b is algebraic and number; Course 2b geometry and data analysis, stats, and probability. See Section 3.2.3 for depth descriptions. Zero rating reflects less-than-expected growth. One rating is acceptable; two: beyond expected; and three: exceptional. N = 22.

4.3. Teacher Leadership Readiness Outcomes (Goal 3)

Teacher leadership development was assessed through three sources: performance in the leadership-focused Phase 1 course, Phase 2 leadership project proposals, and overall program performance across Phases 1 and 2 entering Phase 3.

In the Phase 1 teacher leadership course focused on adult learning and professional collaboration, 15 of 22 teachers were rated as demonstrating "beyond expected" or "exceptional" growth in leadership development (Table 6). These ratings reflected teachers' ability to mentor peers, develop and lead professional learning communities, and design early leadership activities.

At the conclusion of Phase 2, teachers submitted leadership project proposals collaboratively developed with their school administrators. These were evaluated for potential impact (Table 7). Nearly 70% of teachers (15 of 22) proposed projects rated as having "significant" or "very significant" potential to impact instruction, collaboration, or student outcomes.

Table 6. Teacher leadership growth in Phase 1 course, overall growth, and project potential.

	Course 2
3-Rating Exceptional	5
2-Rating Beyond Expected	10
1-Rating Expected/Acceptable	7
0-Rating Less than Expected	0

Note. Course 2 is the course focused on adult learners, mentoring, coaching, and teacher leadership in PLCs. See Section 3.2.3 for descriptions. Zero rating reflects less-than-expected growth. One rating is acceptable; two: beyond expected; and three: exceptional. N = 22.

Table 7. Teacher overall teacher leadership growth in Phases 1 and 2, overall performance.

	Mean Rating	**Project Potential**	**Comparison**	**NBCT**	**Overall**
Teacher1	1.500	1	Under	No	Not
Teacher2	1.833	2	Expected or Better	Yes	Successful
Teacher3	1.000	1	Expected or Better	Yes	Borderline
Teacher4	0.833	2	Expected or Better	No	Not
Teacher5	2.500	3	Expected or Better	Yes	High Success
Teacher6	2.167	2	Under	Yes	Successful
Teacher7	3.000	3	Expected or Better	Yes	High Success
Teacher8	2.333	2	Under	Yes	Successful
Teacher9	2.833	3	Expected or Better	Yes	High Success
Teacher10	2.000	3	Expected or Better	Yes	High Success
Teacher11	1.333	2	Expected or Better	Yes	Successful
Teacher12	2.833	3	Expected or Better	Yes	High Success
Teacher13	1.667	1	Under	Yes	Borderline
Teacher14	1.833	2	Expected or Better	Yes	Successful
Teacher15	2.500	3	Expected or Better	Yes	High Success
Teacher16	1.667	1	Under	No	Borderline
Teacher17	1.500	2	Expected or Better	Yes	Successful
Teacher18	1.833	1	Under	No	Borderline
Teacher19	2.667	2	Under	Yes	Successful
Teacher20	1.833	1	Under	Yes	Borderline
Teacher21	2.500	1	Under	Yes	Successful
Teacher22	1.833	2	Expected or Better	Yes	Successful
MEAN	2.000	1.950	13 of 22	18 of 22	20 of 22

Note. Mean ratings represent the mean across all six graduate courses related to Goals 1, 2, and 3 at the individual teacher level in terms of growth towards teacher leadership. See Section 3.2.3 for descriptions. Project potential is a 1-2-3 level proposed project on the potential to impact students, their school, and/or teachers; see Section 3.2.4. Comparison is whether the teachers' proposed project to start Phase 3 was rated "at or higher" than their performance in Phase 1 coursework. NBCT is whether achieved or not in Phase 2. The overall column presents our qualitative categorical assignment for all 22 teachers for teacher leadership potential because of Phases 1 and 2. Correlation coefficient between mean rating and project potential: 0.579.

Cross-analysis of teachers' NBCT status, course growth averages, and leadership project quality revealed that 18 of 22 teachers were rated as successful or highly successful in leadership preparation. A positive correlation (r = 0.579) was found between mean growth in coursework and leadership project impact potential, underscoring the value of sustained instructional and content development as foundations for effective leadership.

5. Discussion

Findings from the evaluation of the first two phases of the APLUS in Math program curriculum suggest that the multi-year curriculum was largely successful in supporting teachers' development across three domains, including instructional expertise, content knowledge, and teacher leadership.

In relation to instructional practice, significant improvements were evident across multiple indicators. The MCOP^2 observation tool showed increased teacher facilitation and student engagement scores from pre- to post-Phase 1. These findings were corroborated by NBCT Component 2–4 scores and coursework evaluations. This triangulated evidence highlights the value of structured pedagogical coursework, paired with reflective certification processes, in strengthening instructional capacity among experienced teachers. The observed plateau in student engagement during Phase 2, while not necessarily negative, may suggest the need for sustained support beyond initial professional development phases. These outcomes affirm the importance of sustained, well-aligned content-focused professional development in transforming instructional practice, as supported by prior research (Darling-Hammond et al., 2017; Lai & Cheung, 2015).

Regarding content knowledge, the data indicates strong alignment between the Phase 1 mathematics courses and the mathematical knowledge domains assessed in NBCT Component 1. Teachers demonstrated improved understanding across algebra, geometry, and data analysis, which were emphasized in both the curriculum and the certification framework. This alignment underscores the importance of intentional curriculum design that bridges advanced content learning with national teaching standards. These findings align with prior studies emphasizing that deep content knowledge is foundational for teachers to effectively lead instructional improvements (Hitt & Tucker, 2016; Hunzicker, 2012; Campbell & Lee, 2017).

In terms of leadership development, teachers' performance in coursework and leadership project proposals suggests that the program effectively prepared most participants for initiating school-based leadership efforts. However, the range of project proposal quality indicates that some participants may benefit from additional scaffolding or mentoring before implementing high-impact initiatives. These findings point to the value of Phase 3 in providing continued guidance as teachers enact their projects and transition into formal leadership roles. These findings reinforce the role of structured leadership preparation and mentoring in building teacher leadership capacity (Borko et al., 2021; Sinha & Hanuscin, 2017).

Despite these positive outcomes, some teachers demonstrated only partial readiness for leadership roles. In these cases, readiness was often limited by external factors such as administrative constraints or personal circumstances rather than programmatic shortcomings. These findings highlight the importance of ongoing administrative support and flexible leadership pathways to accommodate varying personal and professional contexts (Jotkoff, 2022; Klar & Brewer, 2013). Such variability underscores the need for additional systemic supports including protected time for leadership activities and more explicit role definitions within school structures.

5.1. Limitations

This study is limited by its small sample size and regional focus, which may limit the generalizability of the findings. Additionally, leadership readiness was assessed primarily through faculty ratings and project proposal evaluations, which, while informative, introduce some subjectivity. Finally, as Phase 3 implementation is ongoing, the long-term impacts of teacher leadership projects have not yet been fully evaluated.

5.2. Implications for Practice and Policy

This study offers several implications for the design and implementation of professional learning programs that aim to cultivate mathematics teacher leaders. First, the success of the APLUS in Math program in fostering instructional and content growth suggests that multi-phase models—combining structured coursework, national certification,

and applied leadership experiences—can be highly effective in preparing experienced teachers for leadership roles in high-need schools. Such models allow for the gradual development of professional capacity, integrating theory, reflection, and practice in a coherent sequence.

Second, the alignment between NBCT components and the program's curriculum demonstrates the value of anchoring professional learning in widely recognized instructional standards. Embedding the Five Core Propositions into the coursework and leadership development activities helped ensure consistency and rigor, while also fostering teachers' commitment to continuous improvement and equity-focused instruction. This alignment can serve as a replicable design principle for other districts or institutions seeking to support instructional leadership.

Third, the emphasis on Communities of Practice fostered collaborative learning, distributed expertise, and shared inquiry—critical components for sustaining teacher leadership beyond the duration of the program. Professional learning communities and structured mentoring relationships provided a platform for identity development, agency, and mutual accountability, reinforcing research on identity-as-practice and leadership emergence.

Finally, the variability in leadership project proposal quality highlights the importance of sustained support in the transition from leadership preparation to leadership enactment. Institutions and school districts implementing similar models should consider providing ongoing coaching, administrator engagement, and structured peer feedback mechanisms to increase the likelihood of successful implementation and long-term impact. As districts and states continue to grapple with teacher shortages and persistent equity gaps, investing in leadership development for experienced teachers represents a strategic and scalable approach to improving instruction and retaining highly qualified educators in underserved communities.

Overall, the integration of instructional, content, and leadership preparation in a coherent, multi-year model appears to be a promising approach to cultivating mathematics teacher leaders. By embedding professional identity development, reflective practice, and collaborative learning structures into the program, the APLUS in Math project demonstrates how theoretical frameworks can be operationalized to support meaningful, sustained growth among inservice secondary mathematics teachers.

6. Conclusions

The APLUS in Math program, supported through the NSF Noyce Track 3 Master Teacher Fellowship, represents a sustained and theoretically grounded effort to prepare experienced mathematics teachers for leadership roles in high-need educational contexts. Through its three-phase design anchored in graduate coursework, National Board Certification preparation, and leadership project development, the program provided a structured, multi-year pathway for professional growth across instructional, content, and leadership domains.

Findings from the first two phases indicate that most participants experienced measurable gains in instructional effectiveness and content knowledge, with many demonstrating readiness to lead instructional improvement efforts in their school communities. The integration of Communities of Practice (Wenger, 1999) and the Five Core Propositions of the National Board for Professional Teaching Standards contributed to the coherence of the program by aligning teacher identity development with rigorous standards for teaching and leadership. These frameworks also supported collaboration, reflection, and distributed expertise as central components of leadership cultivation.

The results contribute to the growing body of evidence that sustained and structured professional development is critical for developing teacher leaders capable of promoting eq-

uity and improving instruction in high-need contexts (Borko et al., 2021; Darling-Hammond et al., 2017; Sinha & Hanuscin, 2017). Our findings are consistent with prior research emphasizing the importance of intentional, multi-phase programs in supporting teacher leadership (Darling-Hammond et al., 2017; Hopkins et al., 2013). While most participants demonstrated strong preparation, some variation in leadership project quality suggests that additional scaffolding may be needed to ensure that all teachers are equipped to design and implement high-impact initiatives.

Importantly, the success of such programs depends not only on thoughtful curriculum design but also on systemic support such as administrative encouragement, protected time for leadership responsibilities, and clearly defined leadership pathways (Jotkoff, 2022; Klar & Brewer, 2013). Addressing these structural factors is essential for sustaining teacher leadership at scale and ensuring its integration into school improvement systems. By investing in the development of mathematics teacher leaders, educational systems can create more equitable and effective learning environments that support both teacher and student success.

Ultimately, this study contributes to the emerging literature on mathematics teacher leadership by providing empirical evidence of a program's effectiveness and offering actionable recommendations for scaling and sustaining high-quality leadership development in high-need educational settings.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/educsci15070788/s1, National Boards Portfolio Scores.

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Data Availability Statement: The datasets presented in this article are not readily available publicly because of the confidentiality of the small number of participants, dates of records, and locations which could compromise the participants' identities in addition to our IRB requirements for waived consent. Requests to access the datasets should be directed to the corresponding lead author.

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Article

The Design, Field Testing, and Evaluation of a Contextual, Problem-Based Curriculum: Feedback Analysis from Mathematics Teachers on the Field Test Version of *Connected Mathematics®4*

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Abstract: For over forty years, each edition of the *Connected Mathematics®* curriculum reflects the understanding that teaching and learning are not distinct—"what to teach" and "how to teach it" are inextricably linked. Each edition goes through iterative cycles of design, development, field testing, data feedback, and revision. Grounded in the theoretical and empirical curriculum design and enactment tensions that emerge in problem-based mathematics classrooms, this study reports on teachers' perceived effectiveness of implementing and enacting the field test version of *Connected Mathematics®*4. Analysis of the survey revealed that the problems in the field test version of the fourth edition promote student engagement and learning in mathematics, problem-solving, mathematical connections to real-life applications, and multiple solution strategies. We also discuss implications for how the curriculum design, development, and professional learning experiences involving teachers and their students help provide cohesive and effectively sequenced materials to support students and teachers in rich mathematical problem-solving experiences.

Keywords: teacher perceptions; student engagement; curriculum development; middle grades mathematics

1. Introduction

Mathematics curriculum—the content, organization, sequencing, and methods of instruction—forms the foundation of mathematics education. As Ball and Cohen (1996) argued, the relationship between curriculum materials and teachers has often been underexamined in mathematics education. Curriculum materials are not effective in isolation; developers frequently assume that textbooks are static resources that can operate nearly independently of teacher input (Dow, 1991; Remillard, 2005). However, teachers play a central role as mediators between curriculum materials and student learning. Providing critical feedback that informs the ongoing development of these resources and regular communication between curriculum designers and teachers are essential to ensure materials are responsive to classroom realities (Ben-Peretz, 1990; Pepin, 2019; Stein & Kim, 2009). Thus, the quality and effectiveness of curriculum materials matter for improving mathematics teaching.

As research has consistently shown, the effectiveness of curriculum materials is deeply influenced by how teachers interact with, adapt, and modify them, which in turn affects student outcomes (Borko & Putnam, 1996; Cohen & Hill, 2000; Silver & Smith, 1996). Recent

research has highlighted the importance of teacher feedback in adapting and improving curriculum materials during development. Teachers are not merely implementers of curriculum but active participants in the design process who offer essential insights on how materials can be adjusted to better meet classroom needs (Jones & Pepin, 2016; Remillard et al., 2009). Teachers rely heavily on textbooks and curriculum materials for the "what" and "how" of teaching mathematics (Robitaille & Travers, 1992), but they can use these materials in varied ways. These different approaches result in unique kinds of interactions between teachers and students, focusing on sequences of mathematics problems that may not necessarily resemble the interpretations and decisions made by the teacher in planning (Lloyd, 1999; Remillard, 2005). Stein and Kim (2009) emphasized that teacher engagement with curriculum materials directly shapes how these materials are enacted in the classroom. Teachers must "perceive and interpret existing resources, evaluate the constraints of the classroom setting, balance tradeoffs, and devise strategies—all in the pursuit of their instructional goals" (Remillard & Heck, 2014, p. 18). Thus, research and practice underscore the dynamic relationship between teachers and curriculum materials, suggesting that teachers are crucial in the development, use, and refinement of effective teacher resources.

Building on the literature that emphasizes the importance of the teacher–curriculum relationship, we explore the role of teacher input in the development of a new edition of a set of problem-based curriculum materials. We argue that involving teachers in the design process is crucial for generating materials that are both educative for teacher learning and effective in the classroom. Classroom materials must support teacher learning. As observed by Ball and Cohen, "[u]nlike frameworks, objectives, assessments, and other mechanisms that seek to guide curriculum, instructional materials are concrete and daily. They are the stuff of lessons and units, of what teachers and students do. . . Not only are curriculum materials well positioned to influence individual teachers' work but, unlike many other innovations, textbooks are already 'scaled up' and part of the routine of schools. They have 'reach' in the system" (1996, p. 6). Curriculum materials can help teachers enhance their content knowledge and refine their teaching strategies (Brown, 2009; Davis & Krajcik, 2005). Teachers also need materials that are flexible enough to allow for adaptation, ensuring that diverse classroom contexts and student needs are addressed (Jones & Pepin, 2016; Remillard et al., 2009). These interactions between teachers and materials create a dynamic feedback loop that can lead to improved curriculum designs and more effective teaching practices (Remillard & Heck, 2014).

This paper examines how teacher feedback was integrated into the development of the field test version of the Connected Mathematics Project (CMP) middle grades *Connected Mathematics®4* (CMP4; Phillips et al., 2025a, 2025b, 2025c) curriculum. Through a survey of teachers who field-tested the materials, we explore how their feedback influenced the iterative process of curriculum design and enactment. This study addresses the often-overlooked need for curriculum materials to be extensively field-tested in diverse classroom settings using design research methodologies, coupled with sustained professional learning. By focusing on the teachers' perceptions of both the design and classroom enactment of the materials, this work foregrounds the teacher–curriculum relationship and contributes to the growing body of research emphasizing the importance of teacher input in the development of effective educational resources and the improvement of student learning outcomes. These contributions are particularly significant for CMP, which evolves across editions by refining and building upon established foundations; incorporating field-based teacher insights is therefore essential to ensuring the curriculum remains relevant, effective, and responsive to classroom needs.

1.1. Conceptual Framework

As shown in Figure 1, Remillard and Heck (2014) outlined a conceptual model that situates the important role of instructional materials for mathematics classrooms within a broader curriculum system connecting policy, design, and enactment in educational research. The model recognizes the various ways in which curricula are used across different educational contexts and defines the mathematics curriculum as the "plan for the experiences that learners will encounter, as well as the actual experiences they do encounter, that are designed to help them reach specified mathematics objectives" (p. 707). In this framework, instructional materials are understood as "resources designed to support or supplement instruction, including textbooks, curriculum guides, descriptions of mathematical tasks, and instructional software" (p. 707). These materials are crucial in shaping how the curriculum is enacted in the classroom and, by extension, influencing both teaching practices and student learning outcomes.

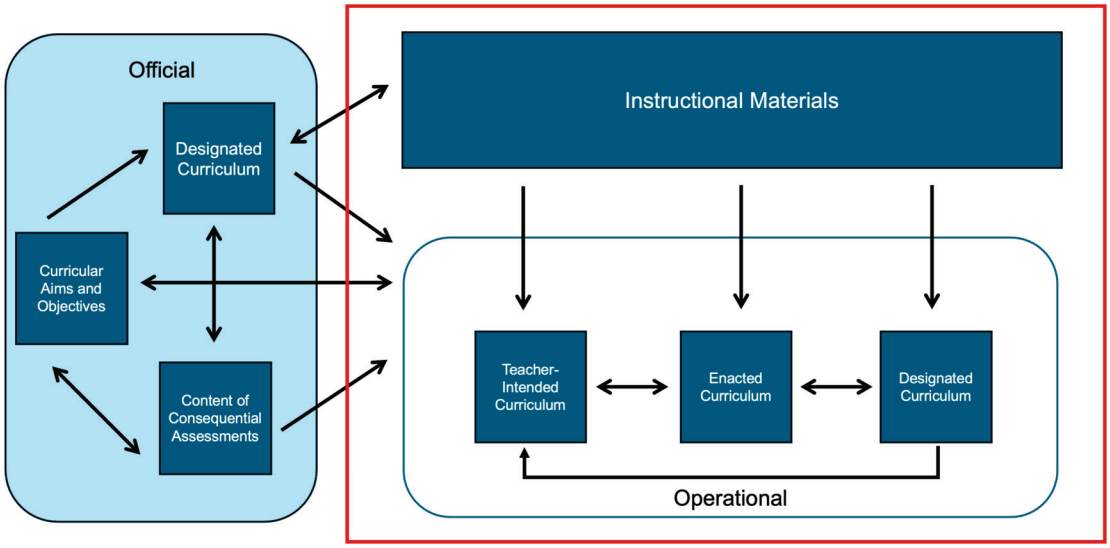


Figure 1. Visual representation of the curriculum policy, design, and enactment model. Adapted from Remillard and Heck (2014, p. 709).

In our work, we position our research as focusing on the interaction between instructional materials and the operational curriculum (see red rectangular box in Figure 1). According to Remillard and Heck (2014), the operational curriculum refers to the "actual enacted curriculum in the classroom," which includes how teachers implement the curriculum, interpret materials, and adapt them based on their unique teaching contexts and the needs of their students. While we acknowledge the insights of Remillard and Heck (2014), we argue in this study that the relationship between instructional materials and the operational curriculum is a dynamic, reciprocal relationship—especially during the processes of designing, field testing, evaluating, and revising curriculum materials. This iterative and continuous feedback loop between the instructional materials and the operational curriculum allows for ongoing adaptation, ensuring that the curriculum remains responsive to both instructional goals and evolving student learning outcomes.

1.2. Theoretical Perspective

To investigate the dynamic interplay between instructional materials and the operational curriculum, we draw on a theoretical perspective grounded in research and practice. Building on the theoretical and empirical underpinnings outlined by Edson et al. (2019) that examine the complex interactions between curriculum design, teacher enactment,

and the development of problem-based curriculum materials, our framework highlights how curriculum materials and teaching practices inform and shape one another and the ways in which teachers engage with and adapt these materials over time in mathematics classrooms. As Edson et al. (2019) emphasize, the design of problem-based curricula requires careful attention to identifying important mathematical ideas and embedding them within contextual problems that promote reasoning, problem-solving, and conceptual understanding (Lappan & Phillips, 2009).

To bridge the theoretical and practical dimensions, Figure 2 visualizes the key curriculum design and enactment tensions that arise when developing and implementing a problem-based curriculum. These tensions illustrate the reciprocal relationship between theory and practice in mathematics classrooms. In Figure 2, the curriculum design tensions focus on decisions related to the selection and organization of mathematical content and tasks. These include balancing open and closed tasks, ensuring coherence in the curriculum, and determining the appropriate timing for closure on mathematical concepts. Meanwhile, the teacher enactment tensions center on the challenges teachers face as they bring the curriculum into real-time classroom settings. Many teachers, especially those with limited experience in problem-based environments, often adhere to a mastery-oriented view of mathematics, where they are shown how to solve a problem and then practice the procedure in isolated lessons (Edson et al., 2019). When teachers lack experience with problem-based learning, they may struggle to envision how to create a learning experience that encourages deeper engagement with mathematics. This adds an additional layer of complexity to their efforts to implement the curriculum effectively. The center column of the theoretical framework highlights the features of curriculum materials and strategies that curriculum developers can incorporate to address these interrelated tensions. The goal is to foster effective curriculum enactment that supports meaningful student learning. This framework offers a comprehensive lens through which to examine the evolving relationship between curriculum development and classroom practice, illustrating how curriculum materials evolve in response to teacher feedback and classroom dynamics (Edson et al., 2019).

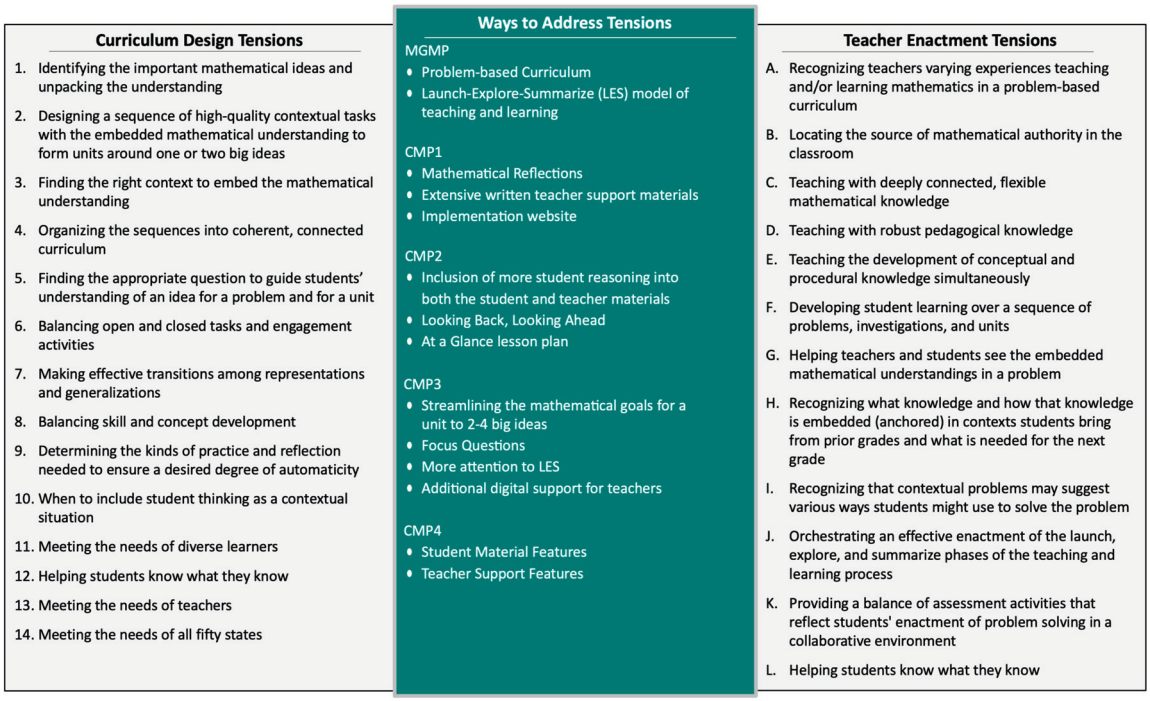


Figure 2. Curriculum design and teacher enactment tensions. Adapted from Edson et al. (2019).

The theoretical framework presented by Edson et al. (2019) has evolved since its original version, reflecting several key revisions that align with ongoing improvements to the curriculum and an increased emphasis on teacher feedback. As the authors continue to refine their problem-based curriculum, they underscore the importance of balancing skill development with conceptual understanding and addressing the needs of diverse learners. These changes are especially evident in the curriculum design tensions, where more attention is now given to structuring tasks that not only promote procedural knowledge but also foster a deeper understanding of underlying mathematical concepts. Moreover, the teacher enactment tensions have expanded to include a stronger focus on recognizing the embedded mathematical understandings within contextual tasks and supporting teachers in leveraging their mathematical knowledge to guide students through the problem-solving process (Choppin, 2011). One notable addition to the framework is the emphasis on teachers' ability to recognize and respond to students' mathematical strategies and reasoning, a key component for fostering an inquiry-driven learning environment. In response to research indicating that teachers often struggle to visualize and implement problem-based learning, Edson et al. (2019) now place greater emphasis on providing teachers with ongoing support in recognizing the embedded mathematics within tasks and designing inquiry-driven lessons. These revisions reflect a more integrated approach to supporting teachers in the effective enactment of problem-based curriculum materials, ensuring that the curriculum continues to evolve to meet the needs of both teachers and students.

1.3. Research Question

While there is substantial research on the teacher–curriculum relationship (Ball & Cohen, 1996), the specific role of teacher feedback in shaping problem-based curricula has not been thoroughly explored. Given that problem-based curricula are designed to foster deeper conceptual understanding and student engagement, understanding how teachers interact with and modify these materials is essential for improving their effectiveness. The objective of this study is to examine the teacher–curriculum relationship in mathematics classrooms by focusing on how teacher feedback from their classroom experiences influences the development of problem-based curriculum materials, particularly in middle grades mathematics. Drawing on the conceptual framework that emphasizes the interaction between curriculum materials and teacher practices, as well as the theoretical framework that highlights the challenges inherent in curriculum design and enactment as described by Edson et al. (2019), another objective of this study is to uncover how teachers perceive the effectiveness and challenges of implementing these materials in real classroom settings. Specifically, the research is guided by the following question:

> How do field test teachers perceive the effectiveness of implementing and enacting a contextualized, problem-based middle grades mathematics curriculum and its design features?

Through teacher perceptions, this research aims to provide insights into the relationship between curriculum design, teacher practices, and student outcomes, with the goal of contributing to the ongoing refinement of curriculum materials in response to teacher feedback and classroom dynamics.

2. Materials and Methods

2.1. Methodological Approach

The overarching approach for this development of the curriculum materials follows an iterative, research-informed process, drawing on principles from design-oriented methodologies such as design studies (Edelson, 2002), design experiments (The Design-Based Research Collective, 2003), and developmental research (Richey et al., 2004). These ap-

proaches share a focus on refining educational interventions through cycles of design, implementation, analysis, and revision in classroom contexts (The Design-Based Research Collective, 2003; Edelson, 2002; Richey et al., 2004). These approaches aim to produce new theories, artifacts, and practices that impact teaching and learning in real-world settings (Barab & Squire, 2004). The research reported in this study was conducted within the larger design research project, focusing on field test teachers' perspectives on the effectiveness of enacting a contextualized, problem-based middle grades mathematics curriculum and its design features.

Connected Mathematics®4 (CMP4; Phillips et al., 2025a, 2025b, 2025c) builds on the foundation set by the Middle Grades Mathematics Project (MGMP). As described in Edson et al. (2019), MGMP (Lappan et al., 1985) helped teachers move from traditional instruction to problem-based learning, with a focus on key mathematical ideas and sequences of problems designed to build understanding. The Launch–Explore–Summarize model developed by MGMP supported this shift and addressed curriculum design tensions, such as identifying core mathematical concepts and sequencing tasks to support deep understanding. It also helped address teacher enactment tensions, such as recognizing varying levels of teacher experience and ensuring effective use of the instructional model.

MGMP teachers reported struggling to apply what they learned from the units across their full curriculum. In response, three of the MGMP authors developed CMP1 (Lappan et al., 1998), which introduced new features to help teachers implement problem-based instruction in a complete mathematics curriculum for grades 6, 7, and 8. Each grade consisted of eight units, each with 2–6 investigations. Each investigation consisted of 1–6 problems. As described in Edson et al. (2019), the addition of Mathematical Reflections at the end of each investigation allowed both students and teachers to pause and assess understanding, addressing curriculum design tensions related to knowing when to close a lesson. CMP1, using the Launch–Explore–Summarize model, also provided extensive teacher support, helping teachers facilitate rich discourse and better navigate the complexities of problem-based teaching, responding to enactment tensions around developing strong pedagogical practices.

CMP2 (Lappan et al., 2006) was developed based on research and classroom experience with CMP1. This version refined problems and teacher support materials to help students develop a deeper understanding. As described in Edson et al. (2019), features like Looking Back, Looking Ahead addressed curriculum design tensions around organizing a coherent sequence of lessons, while the At a Glance feature streamlined lesson planning for teachers, addressing enactment tensions related to managing increasing complexity in materials.

In CMP3 (Lappan et al., 2014), student problems and teacher support materials were enriched with examples of student thinking, helping teachers anticipate how students might approach problems. As described in Edson et al. (2019), this addressed enactment tensions around recognizing and responding to diverse student reasoning. CMP3 also introduced Focus Questions to help teachers maintain focus on key mathematical ideas and guide lesson planning, addressing curriculum design tensions around choosing the right questions to guide student understanding. A one-page resource provided a clearer view of the mathematical progression across units, supporting teachers in managing the flow of concepts and reinforcing the curriculum design tension of developing a connected curriculum.

2.2. The Iterative Development and Field Testing of CMP4

The development and testing of CMP4 field test materials followed a thorough, iterative process involving over 500 teachers over 25 U.S. states and six additional countries (see Table 1). Throughout this process, field test teachers were primarily recruited from those already using the CMP3 materials, with ongoing professional learning and feedback

integrated at each stage. The development process began with an initial phase that involved both the CMP team and the publisher. In this initial phase, the CMP development team began conversations to discuss initial changes to the structure and scope of the curriculum. These discussions focused on key topics such as the table of contents, scope and sequence, and unit organization. A major area of focus was the integration of the CMP STEM Problem format (described later in Section 2.3), which was being explored as part of a digital collaborative platform with the *Connected Mathematics®4* seventh-grade curriculum. The early work aimed to align the vision for CMP4 with necessary educational goals, ensuring the curriculum would be both innovative and feasible for classroom implementation. These foundational conversations helped to address some of the curriculum design tensions identified in earlier phases of CMP development.

Table 1. Overview of the field test teachers for the field test version of CMP4.

Year	Total Teachers	Approximate Number of Unique Schools	States Represented (United States)	Countries Represented
2019–2020	11	10	AZ, IL, ME, MI, NY, OH, TX	United States
2020–2021	13	6	MI, NY, OH	United States
2021–2022	343	155	CA, DE, IL, IN, KS, MA, ME, MI, MN, MO, NC, NJ, NM, NY, OH, PA, TX, VA, VT, WI	Colombia, England, and the United States
2022–2023	515	409	AZ, CA, IL, IN, KS, MA, ME, MI, MN, MO, NC, NJ, NM, NY, OH, PA, TX, VA, VT, WI	Brazil, Colombia, England, the Netherlands, the United States, and Vietnam
2023–2024	319	138	AL, AZ, CA, IL, KY, MD, ME, MI, NC, NJ, NY, OH, PA, TN, UT, VA, VT, WI	Brazil, England, South Korea, the Netherlands, the United States, and Vietnam

In Year 1 (2018–2019), formal field testing began, focusing on a select group of sixth-grade units. Early field testing helped to evaluate the new features, including the CMP STEM Problem format (described later in Section 2.3) and the instructional support materials. Feedback from this first round of testing was used to identify areas for improvement and revise the curriculum accordingly. During this year, the CMP team worked closely with a few teachers. A project-developed digital platform funded by the National Science Foundation was also used in parallel with seventh-grade teachers, providing a space for teachers to test the curriculum in a collaborative, digital format. This allowed for real-time adjustments and feedback as teachers engaged with the project-developed digital platform and worked on the same units digitally. The project-developed platform differs from the platform provided by the publisher.

By Year 2 (2019–2020), the field testing expanded significantly, including units from grades 6, 7, and 8. The group of field test teachers included 11 teachers from 10 schools across 7 states. The teachers continued to provide valuable feedback on new features, such as the inclusion of more student reasoning examples and the Focus Questions, which helped guide lesson planning and enactment. Feedback revealed areas for improvement in supporting diverse learners and streamlining the teacher support materials to make lesson planning more manageable. These insights led to revisions and updates, with ongoing collaboration to ensure the materials met the needs of both teachers and students.

In Year 3 (2020–2021), all units for grades 6, 7, and 8 were included in the field testing. This year saw a broadening of feedback, as more teachers participated in testing and provided in-depth insights on how well the curriculum supported problem-based learning

and engaged students in rich mathematical problem-solving experiences. The CMP team also continued to make refinements based on feedback related to the teacher's experience, particularly around managing the increased amount of material and supporting teachers in effective lesson planning and enactment. The project-developed digital platform continued to play a key role in allowing teachers to collaborate and implement the curriculum in a flexible, digital format. The field test in 2020–2021 included 13 teachers from 6 schools across 3 states.

Year 4 (2021–2022) saw the field test units from all grades continuing to evolve. The field test in 2021–2022 included 343 teachers from 155 schools across 18 states, as well as international participants from Colombia and England. Feedback from these participants led to refinements in the curriculum, emphasizing a more cohesive structure, smoother transitions between units, and clearer guidance on orchestrating the problem-solving process. Teachers noted that having a clearer vision of the progression from one unit to the next helped improve both teaching and student learning. The continued involvement of teachers in the field test ensured that the materials remained grounded in classroom realities and continued to evolve in ways that best supported both students and teachers. The project-developed digital platform continued to provide real-time feedback and offered insights into how the curriculum was being implemented across different classrooms.

By Year 5 (2022–2023), the CMP4 units for grades 6–8 were fully field-tested, and significant revisions had been made to the materials. The curriculum was near final draft status, with ongoing minor adjustments. The field test in 2022–2023 included 515 teachers from 409 schools across 22 states, as well as international teachers from Brazil, Colombia, the Netherlands, England, and Vietnam. Surveys and in-depth feedback (through written unit reports and virtual monthly meetings) confirmed that new features—such as the Focus Questions and Mathematical Reflections—were positively impacting both teaching practices and student learning outcomes. The CMP team continued working closely to finalize the materials for wider dissemination, ensuring they met the needs of teachers and aligned with curriculum standards.

In the 2023–2024 academic year, the CMP4 units for all grades were revised and improved based on the feedback received in the previous year. The feedback from the final phase of field testing helped refine the curriculum materials, with a strong emphasis on ensuring that students could build on prior knowledge and deepen their understanding across the full set of units. The units were now ready for broader implementation, having undergone years of feedback, testing, and refinement to become a cohesive, problem-based mathematics curriculum. The field test in 2023–2024 included 318 teachers from 137 schools across 15 U.S. states, as well as international educators from Brazil, England, the Netherlands, and Vietnam. Extensive feedback was provided by teachers in their unit reports and in monthly virtual meetings.

Professional learning for CMP4 field test teachers over the past four years began with a face-to-face meeting that included a small number of CMP3 teachers, teacher leaders, and field test teachers using the project-developed digital platform. From there, professional learning experiences were provided through a combination of webinars, monthly virtual meetings, and informal virtual conversations. These sessions started with introductory meetings and unit overviews, and over time, they progressively focused on specific units and teaching strategies. The professional learning over the years centered on several key aspects, such as an overview of CMP4, which included its philosophy, new student features, and teacher support, along with grade-level overviews that highlighted the development of key mathematical concepts. Teachers were also guided through the instructional model (Launch–Explore–Summarize) and given detailed information on available teacher resources. Unit overviews were provided both synchronously and asynchronously to

accommodate varying schedules. Monthly virtual meetings allowed teachers to ask questions, provide feedback, and receive ongoing support. Additionally, extensive written teacher support was made available, offering more in-depth guidance on each instructional phase, including planning, teaching (Launch, Explore, and Summarize), and reflection, with a particular focus on helping teachers deepen their understanding of embedded mathematical concepts. All virtual sessions and webinars were recorded, providing teachers with the flexibility to revisit the material at their own pace. As the years progressed, the professional learning offerings expanded to meet the growing needs of teachers as they implemented CMP4, ensuring they received the necessary support for the successful enactment of the curriculum.

At the end of each field test year, feedback from participating teachers played a critical role in refining the CMP4 materials. Initially, in Year 1, feedback was collected from a small group of teachers who tested the CMP STEM Problem format on the project-developed digital platform. This feedback focused on individual problems and how they were implemented in the digital context. Over the course of the subsequent years, the feedback process evolved to include more extensive forms, which accompanied each unit being tested. The feedback was organized into several key categories, including "big picture" unit feedback, which allowed teachers to reflect on the overall design and flow of the units, as well as problem-by-problem feedback, which delved into specific challenges or successes teachers encountered with tasks. Other categories included feedback on the context of the problems, their applications, connections, and extensions, as well as assessments and teacher support materials. Year-end grade-level feedback helped to gauge the progression and coherence across units for each grade, and a final "big question" invited teachers to reflect on the overarching impact of the materials on student learning. Additionally, the feedback included insights into the effectiveness of the Mathematical Reflections feature, which teachers used to support student understanding throughout the units. This comprehensive feedback loop, collected and analyzed across multiple years, ensured that the materials continuously improved to better support teachers and engage students in rich mathematical problem-solving experiences.

2.3. CMP4 and Its Design Features

The Connected Mathematics Project (CMP) materials are created to support teachers and students in developing deeper mathematical understanding and reasoning. *Connected Mathematics®4* (Phillips et al., 2025a, 2025b, 2025c) extends the successes of its predecessors, CMP1 (Lappan et al., 1998), CMP2 (Lappan et al., 2006), and CMP3 (Lappan et al., 2014). Each revision of CMP was extensively field-tested in its development phases. After the release of each edition, the CMP development team continues to interact with schools. The development team seeks iterative and in-depth input and reviews from teachers, families, administrators, mathematics educators, mathematicians, educational researchers, and experts in reading, special education, equity, and multilingual learners. Most importantly, the enthusiasm of the teachers and the creative, productive mathematical thinking of their students provide the fuel for the authors to think even more deeply about "Can we do more?" The result is "yes!" This continuous interaction with the field and design research is a critical process in the development of CMP4 and is the foundation of the success of CMP, which has withstood the pressures of various political changes since CMP1 was released in 1996.

Connected Mathematics®4 provides seven student units for sixth grade and eight student units each in seventh and eighth grades. The eighth-grade units offer the possibility of teaching an eighth-grade course or a first-year high school course focused on algebra. Each unit is organized around a big mathematical idea or cluster of related ideas, such as

variables and patterns, area and perimeter, ratio and proportion, linear relationships, or nonlinear relationships. The format of the student material promotes student engagement with an exploration of important mathematical concepts and related skills and procedures. Students develop strategies and conceptual understanding by solving problems and discussing their solutions in class.

The CMP4 student material consists of the following components:

Mathematical Goals and Looking Ahead. The mathematical goals guide the development of the big ideas of mathematics for the unit. Each unit opens with three interesting problem situations to draw students into the unit, pique their curiosity and joy in mathematics, and point to the kinds of ideas they will investigate. This is followed by a set of focusing questions that reflect the mathematical goals of the unit. Students can use these questions to help track their progress through the mathematical goals. Students can revisit these questions as part of their reflections on their learning.

Investigations. Mathematics learning is focused on one or two big ideas of mathematics developed through carefully sequenced investigations. Each unit builds toward the mathematical goal of the unit and is comprised of two to four investigations. Each investigation includes the following key elements:

- 2–4 CMP STEM Problems.
- Did You Know? This component connects the context and/or mathematics to real-life anecdotes.
- Mathematical Reflections. These contain one overarching question that guides the development of students' understanding of the big mathematical idea of the unit.
- Applications–Connections–Extensions (ACEs). This component allows students to reinforce, connect, or extend their understandings.

Student Notebooks. This notebook provides space for students to record their strategies and understandings as they work through a unit.

In this section, we address some of the ways CMP4 has attended to the curriculum design issues (see Figure 3). The design features that address some of the curriculum and enactment tensions are indicated in the middle column labeled CMP4. The numbers next to each design feature refer to a "Curriculum Design Tension" and the letter refers to a "Teacher Enactment Tension". The bulk of the tensions were addressed by the redesign of the problem format (Figure 4). It includes a more open challenge, which provides multiple ways of accessing it and provides insights into the embedded mathematical understandings.

CMP4's new CMP STEM Problem format (see Figure 4) promotes learning and problem-solving that resembles the work of STEM professionals. It attends to some of the curriculum design tensions as the context of the problem in the Initial Challenge is more engaging and open, and thus allows for more students to access the challenge. The What If…? feature provides a balance between open and more closed questions in the Initial Challenge, thus guiding the students' understanding of the embedded mathematical idea in the Initial Challenge feature. The Now What Do You Know? feature of the problem provides an opportunity for students to know what they know. Collectively, the format promotes students' problem-solving strategies and hence their ability as doers, creators, and communicators of mathematics. In addition, the CMP STEM Problem format provides teachers with flexibility to carry out equitable practices that help address the individual needs of all students. CMP4, using the CMP STEM Problem format and other features, has streamlined the mathematical focus of each unit, resulting in a 17–20% reduction in the number of problems across the grades.

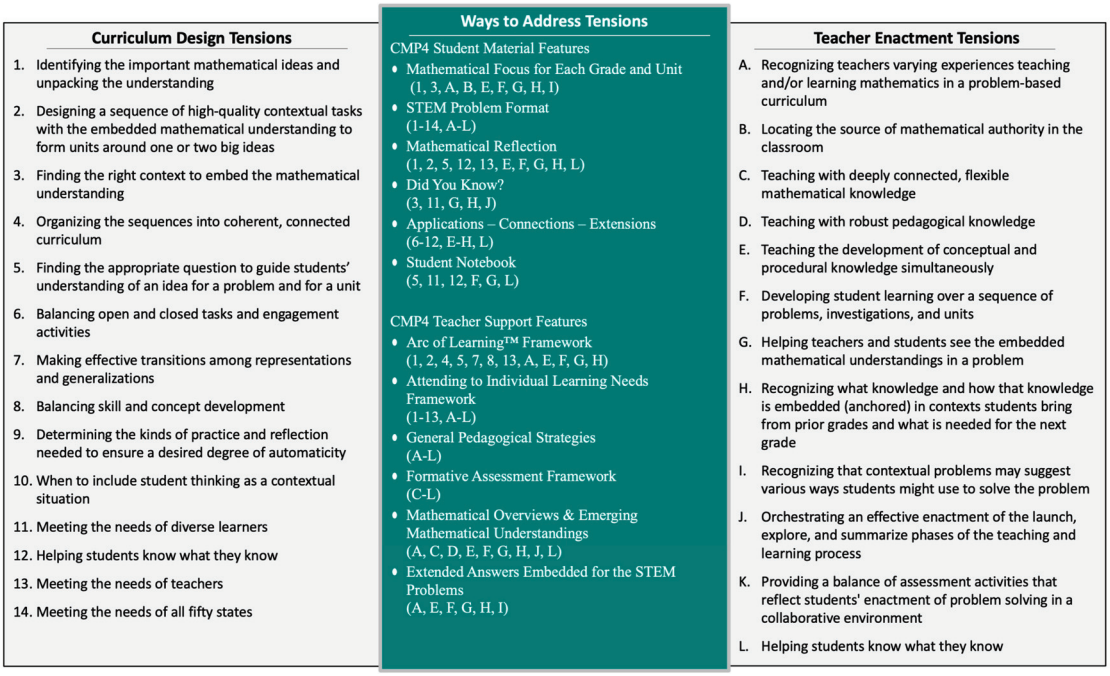


Figure 3. Examples of new features that address curriculum design and teacher enactment tensions. Adapted from Edson et al. (2019).

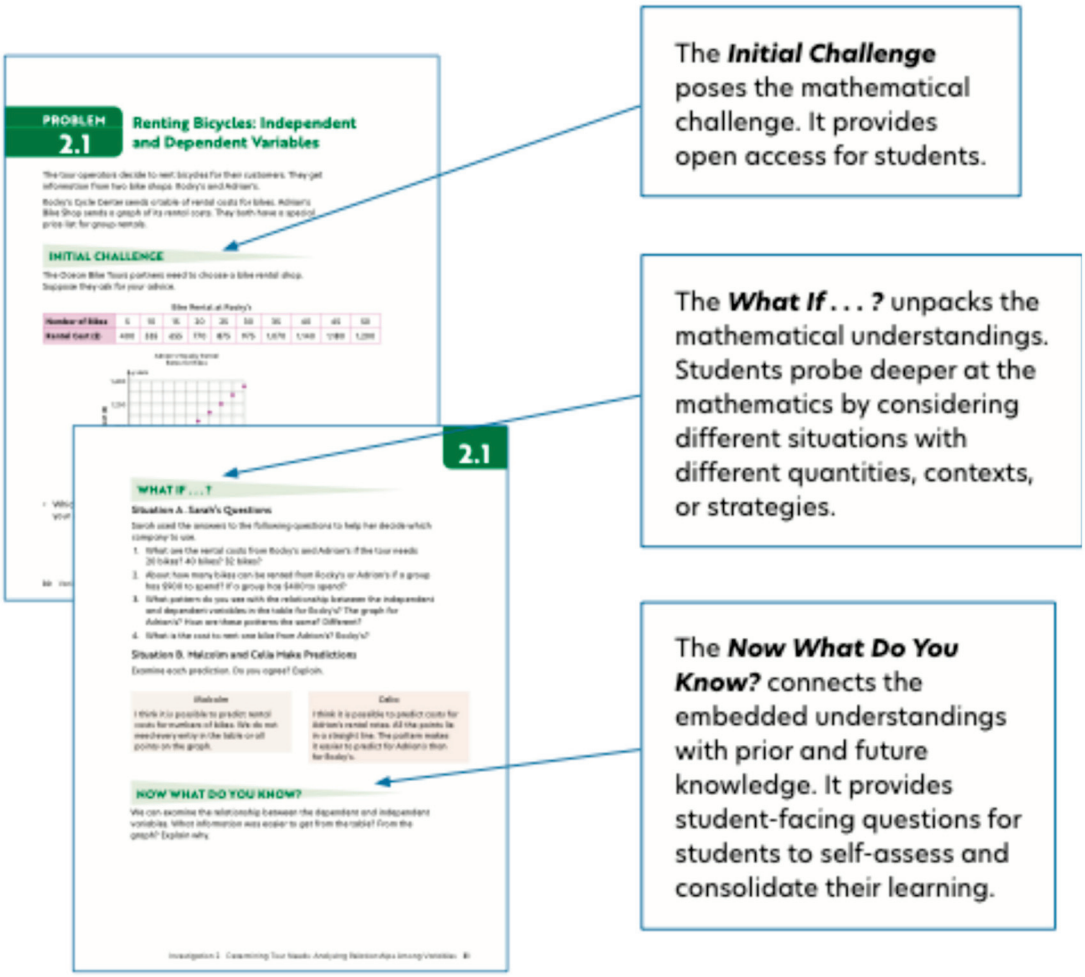


Figure 4. Example of Problem 2.1 using the CMP STEM Problem format from the sixth-grade unit, *Variables and Patterns* (Phillips et al., 2025a, 2025b).

At the end of each investigation, there is now just one overarching Mathematical Reflection question that guides the students' mathematical understanding of the big idea in the unit. Figure 5 illustrates the Mathematical Reflection for the sixth-grade *Variables and Patterns* unit. The Now What Do You Know? feature notes can be used to record students' emerging understanding of the big mathematical idea of the unit that is embedded in the unit's Mathematical Reflection question.

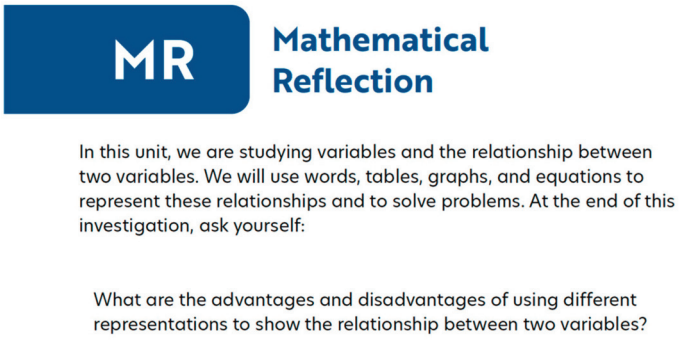


Figure 5. The Mathematical Reflection for the sixth-grade unit, *Variables and Patterns* (Phillips et al., 2025a).

CMP4 has more examples of student thinking as context for promoting student learning. The student work in Problem 2.1 Situation A (see Figure 6) provides student work that helps students see there that is sometimes more than one way to solve a problem (Zane, Bruce, and Gwen's Strategies), and there is not always a direct path to the answer (Yvonne's Strategy). Situation B enhances students' understanding of proportional reasoning by altering the ratio of cupcakes in each box and by changing the size of the box (or number of total cupcakes). It also provides a transition among representations and generalizations among pictures, words, and tables. The Now What Do You Know? question helps students reflect on their understanding of ratios as a tool for solving problems. CMP4 strengthens the use of visual representations to develop and recall understanding of important mathematical ideas.

Figure 7 illustrates the Mathematical Reflection for the sixth-grade *Comparing Quantities* unit. The Now What Do You Know? feature provides student notes for recording their emerging understanding of the big mathematical idea of the unit, which are then consolidated into the Mathematical Reflection question. For example, the Now What Do You Know? feature in Problem 2.1 in Figure 5 can provide students with evidence of students' understanding of ratios at this stage of development in the unit.

There are more problems with embedded card sorts, models, matching, games, and experiments. These formats promote active engagement by requiring students to manipulate, categorize, visualize, and reason about mathematical concepts, rather than passively absorb information. For example, card sorts and matching tasks prompt learners to identify relationships among representations, quantities, or strategies, encouraging them to justify their thinking and articulate reasoning. The use of visual or physical models facilitates students' ability to represent and make sense of abstract mathematical ideas, supporting the development of conceptual understanding. Games and experiments introduce playful yet structured contexts in which students can explore mathematical ideas, offering multiple entry points and promoting inclusive participation. These interactive formats often foster collaboration and discourse, providing opportunities for students to engage in mathematical argumentation and refine their ideas through peer interaction. Additionally, such activities serve as rich sources of formative assessment, allowing teachers to observe students' thinking in action and respond to their needs in real time. For example, in

Figure 8, a game is used to strengthen students' understanding of integers. In Figure 9, a card sort (matching) of graphs, equations, and contexts is used to pull together the key understanding of linear functions.

PROBLEM 2.1

Packaging Cupcakes: Using Ratios

In Problem 1.3, the students at Washington plan to sell two different-size boxes of cupcakes.

Large Box
8 green cupcakes
10 white cupcakes

Small Box
4 green cupcakes
8 white cupcakes

INITIAL CHALLENGE

The students have 40 white cupcakes to package.

If they use large boxes:

- How many green cupcakes will they need?
- How many large boxes will they need?

If they use small boxes:

- How many green cupcakes will they need?
- How many small boxes will they need?

What operations did you use to answer the questions?

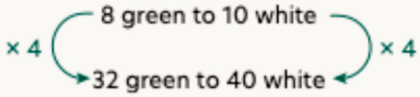
WHAT IF . . . ?

Situation A. Student Strategies

The following are students' strategies for answering the questions in the Initial Challenge. Are the strategies correct? Explain why or why not.

Zane's Strategy

I used ratios and multiplication. I started with the ratio of 8 green to 10 white and then multiplied each quantity by the same number:



18 Comparing Quantities

Figure 6. *Cont.*

2.1

Yvonne's Strategy

I increased the number of each kind of cupcake to get 40 white cupcakes.

8 green to 10 white

+ 30 → 38 green to 40 white ← + 30

Bruce's Strategy

I started with the ratio of 8 green to 10 white. I used several multiplications to get my answer.

8 green to 10 white

× 2 → 16 green to 20 white ← × 2

× 3 → 24 green to 30 white ← × 3

× 4 → 32 green to 40 white ← × 4

Gwen's Strategy

I used tables to find the number of green and white cupcakes I needed for various sizes starting with 1 large box. I stopped when I got to 40 white cupcakes.

Gwen's Table for Number of Cupcakes in Large Boxes

Number of Large Boxes	1	2	3	4	
Number of Green Cupcakes	8	16	24	32	
Number of White Cupcakes	10	20	30	40	

Situation B. Changing the Packaging

Examine each packaging idea, and answer the question.

Changing the Ratio in a Box

Some of the students suggested that the ratio of green to white cupcakes should be changed to 2 to 1 for each box.

If the total cupcakes in the large and small boxes are still 18 and 12, how many green and white cupcakes would there be in each box?

Changing the Size of a Box

The school is considering a grande box that holds 24 cupcakes. The ratio of green to total cupcakes is 2 green for every total of 3.

How many green and white cupcakes are needed?

NOW WHAT DO YOU KNOW?

How are equivalent ratios useful in solving this problem?

Investigation 2 Using Ratios to Solve Problems **19**

Figure 6. Example of student work as a context for student learning from the sixth-grade unit, *Comparing Quantities* (Phillips et al., 2025a).

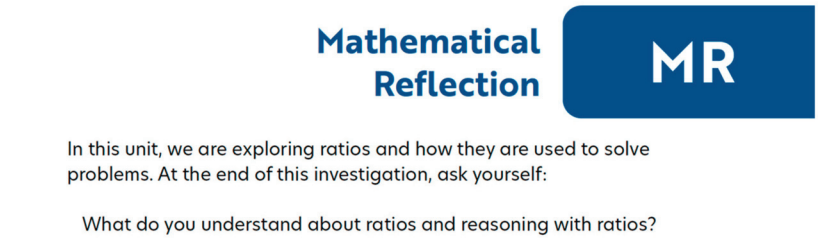


Figure 7. The Mathematical Reflection for the sixth-grade unit, *Comparing Quantities* (Phillips et al., 2025a).

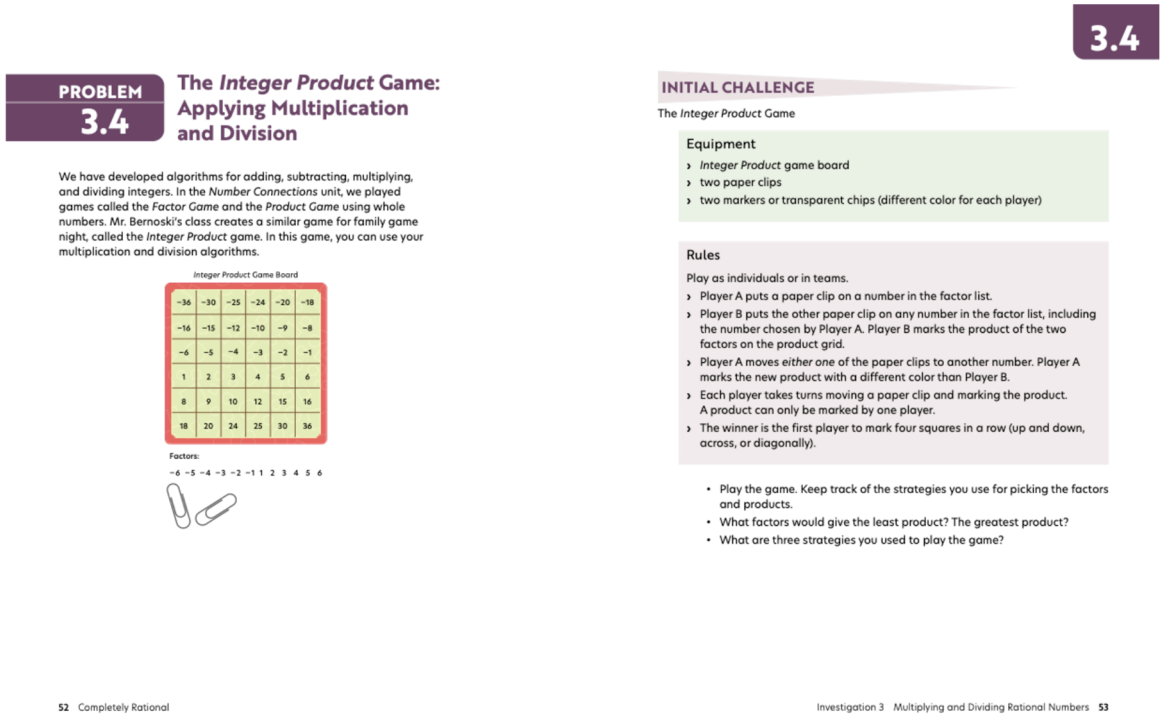


Figure 8. Example of a game to engage students' explorations of integers in the seventh-grade *Completely Rational* unit (Phillips et al., 2025a).

The following activity (see Figure 10) is an experiment that investigates some of the variables that affect the strength of a bridge. In this experiment, the two variables are the thickness of the bridge and weight. The relationship is linear. In Problem 1.2, they repeat the experiment, looking at the length of the bridge and the weight. This relationship is not linear. It is an inverse relationship.

The prior figures in this section illustrate problems from various units. Figure 11 provides an overview of the units for grades six, seven, and eight. In CMP4, there are stronger connections both within and across units and grades. CMP4 strengthens its unique development of algebra and functions, which continues to develop exceptionally strong student algebraic understandings. Starting with *Variables and Patterns* as the first unit in grade 6 allows the use of patterns of change and equivalence, representations, and generalization to naturally arise in all the remaining 22 CMP4 units, ending with the Function Junction unit in eighth grade. The unit titles highlight the main contextual thread and the mathematical focus. The mathematical focus for each grade is highlighted in the first row of Figure 11. The focus of the 8th grade on algebra and functions serves the needs of both grade 8 and High School Course 1.

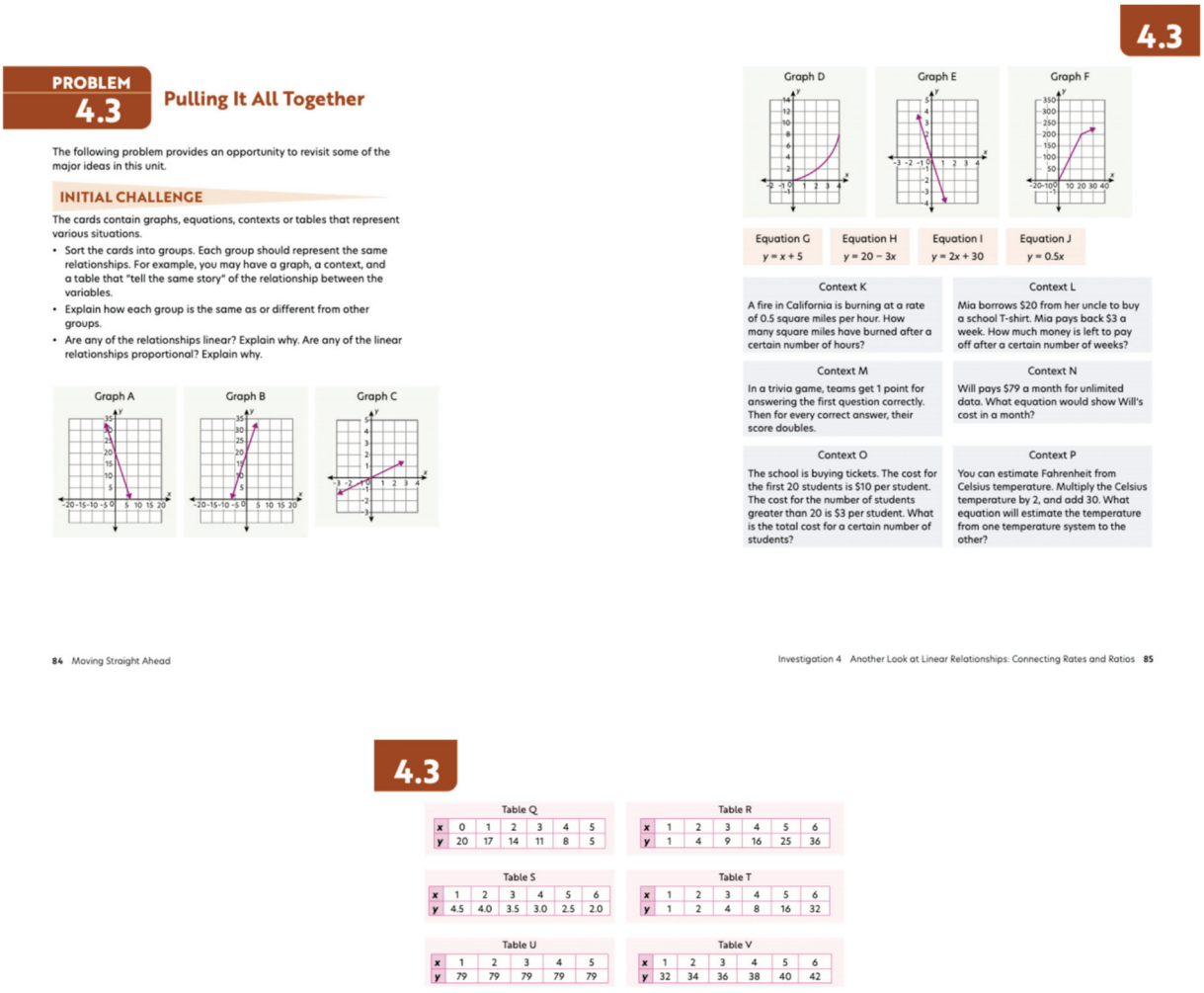


Figure 9. Example of a matching/sorting activity to engage students' understanding of linear functions from the eighth-grade unit *Moving Straight Ahead* (Phillips et al., 2025a).

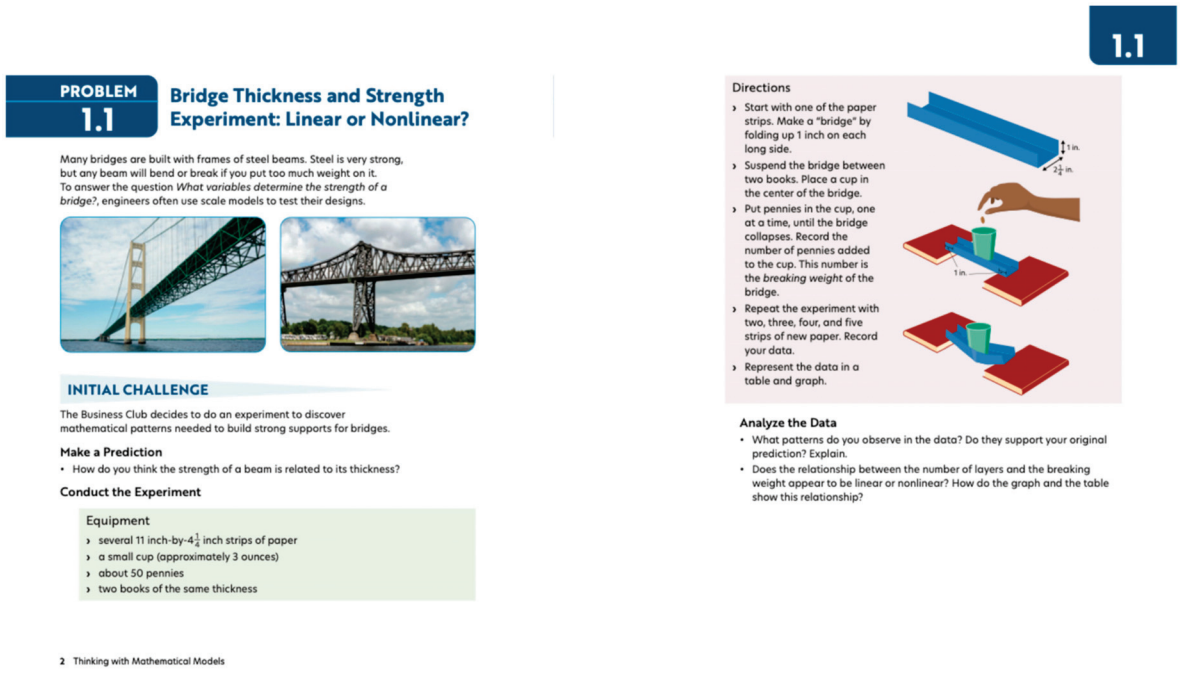


Figure 10. Example of an experiment to engage students' important mathematical understanding in the eighth-grade *Thinking with Mathematical Models* unit (Phillips et al., 2025a).

	GRADE 6	GRADE 7	GRADE 8 or High School Course 1	
Unit	**Mathematical Focus:** Generalizing patterns with a focus on algebra and number	**Mathematical Focus:** Proportional reasoning in the context of number, similarity, algebra, measurement, probability, and statistics	**Mathematical Focus:** Formalizing patterns of change, relationships, and transformations Grade 8: Shortened Pathway HS Course I: All Problems	
1	Variables and Patterns: Introducing Algebraic Reasoning	Shapes and Designs: Generalizing and Using Properties of Geometric Shapes	Thinking with Mathematical Models: Linear Functions and Bivariate Data	
2	Number Connections: Expressing Factors and Multiples Algebraically	Completely Rational: A Focus on Integers	Looking for Pythagoras: The Pythagorean Theorem and Real Numbers	
3	Comparing Quantities: Ratios, Rates, and Equivalence	Stretching and Shrinking: Proportional Reasoning in the Context of Similarity (Scale Drawings)	Growing, Growing, Growing: Linear Versus Exponential Patterns of Change	
4	Bits of Rational: Extending Fraction Operations and Solving Equations	Comparing and Scaling: Proportional Reasoning in the Context of Number	Mars, Gravity, and Painted Cubes: Linear Versus Quadratic Patterns of Change	
5	Covering and Surrounding: Two- and Three-Dimensional Measurement	Moving Straight Ahead: Linear Relationships: Expressions and Equations	Flip, Spin, Slide, and Stretch: Exploring Transformations	
6	Points of Rational: A Focus on Decimals and Algebraic Reasoning	How Likely Is It?: Proportional Relationships in the Context of Probability	Say It with Symbols: Reasoning with Equivalent Expressions and Equations	
7	Data About Us: Statistics and Data Analysis	Filling and Wrapping: Two- and Three-Dimensional Measurement	It's in the System: Systems of Linear Equations and Inequalities	
8		Samples and Populations: Making Comparisons and Predictions	Function Junction: A Deeper Look at Algebra and Functions	
	7 units 26 investigations 84 problems	8 units 27 investigations 88 problems	8th grade: 8 units 29 investigations 88 problems	HS 1: 8 units 32 investigations 112 problems

Figure 11. List of units for grades 6, 7, and 8 (Phillips et al., 2025c).

CMP4 has a stronger, more robust development of proportional reasoning starting with a new grade 6 unit, *Comparing Quantities*, and continuing with several units in grades 7 and 8. CMP4 incorporates aspects of the statistical problem-solving process throughout all units. The first problem in the first unit of each grade starts with an experiment.

To address some of the teacher enactment tensions, CMP4 (Phillips et al., 2025a, 2025b, 2025c) provides more support for teachers in its CMP Arc of Learning™ Framework (Edson et al., 2019) for each unit, CMP Formative Assessment Framework, CMP Attending to Individual Needs Framework, embedded General Pedagogical Strategies, and its new At a Glance teacher support for each problem. These were not robust enough to be fully implemented during field testing. However, the teacher's comments on the student materials provided insights into the teacher support documents.

The Arc of Learning™ Framework provides guidance on how the understanding of the big mathematical idea for the unit evolves. In Figure 12, the Arc of Learning™ Framework is shown for the first unit on proportional reasoning in sixth grade and a third unit of proportional reasoning in seventh grade. The development of proportional reasoning is growing toward abstracting. There are three more seventh-grade units that continue to look at proportional reasoning in the context of linear relationships, probability, and statistics.

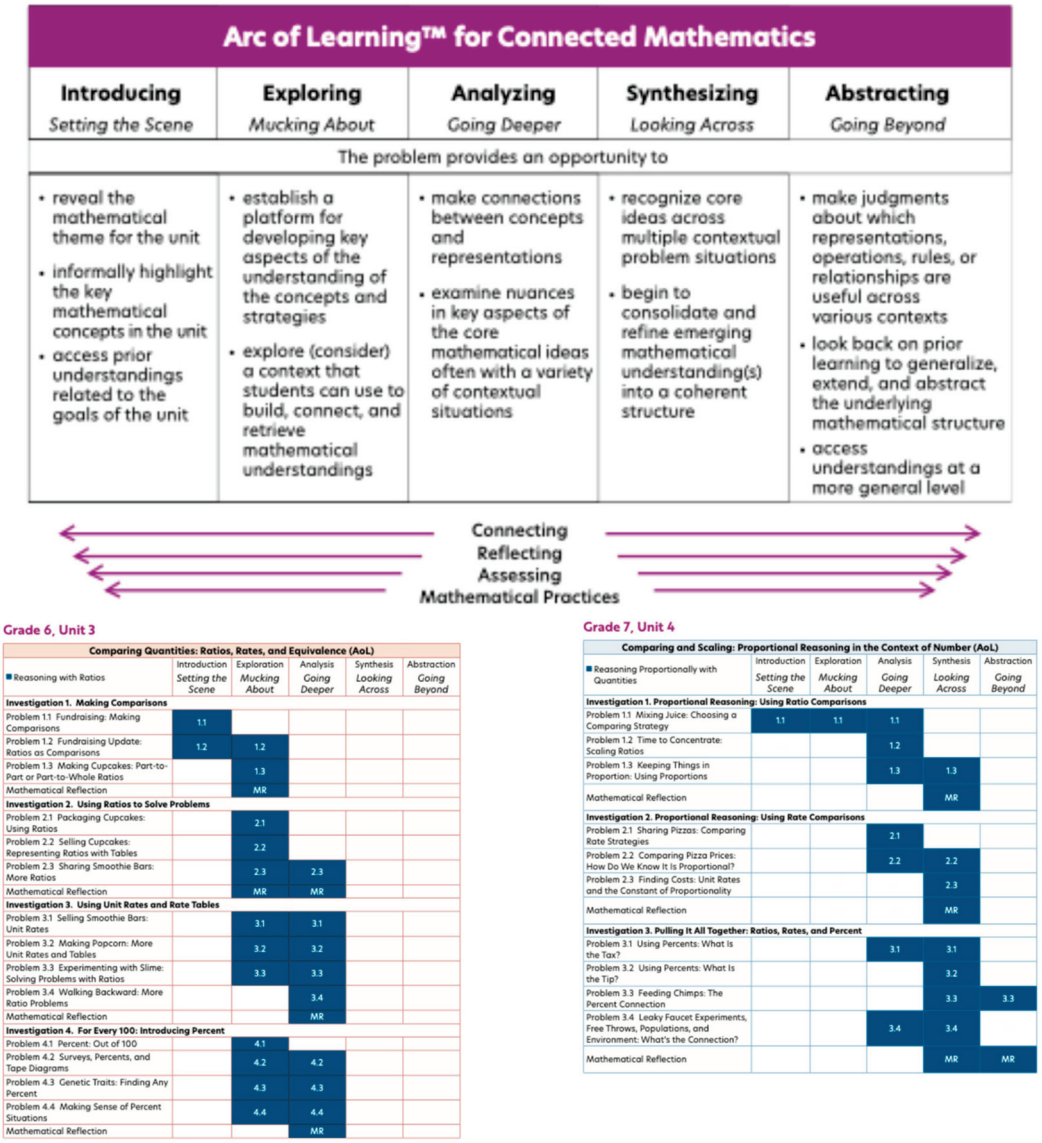


Figure 12. The CMP Arc of Learning™ Framework and the unit-specific Arcs of Learning for the 6th grade *Comparing Quantities* unit and the 7th grade *Comparing and Scaling* unit (Phillips et al., 2025b, 2025c).

The Attending to Individual Learning Needs (AILNs) framework (see Figure 13) communicates to teachers the information learned from research and practice over 40 years of field testing in CMP classrooms. It is consistent with the teaching practices set forth by other organizations. The appropriate component of the AILNs framework is indicated in the Launch, Explore, or Summarize phase of instruction in the At a Glance guide for the teacher.

At a Glance is a two-page format that provides the important information needed to implement a CMP4 problem. More details for the Launch–Explore–Summarize instructional phases of a lesson are provided in an extended Launch–Explore–Summarize section. Figure 14 is an example of an At a Glance for Problem 2.1 in the 6th grade *Comparing Quantities* unit. In addition to supplies, pacing, and grouping information, it also provides a sample of questions that can be used during each phase of the lesson. The appropriate Attending to Individual Learning Needs framework is indicated, as well as a suggested pedagogical strategy. Corresponding applications, connections, and extension task suggestions are also included.

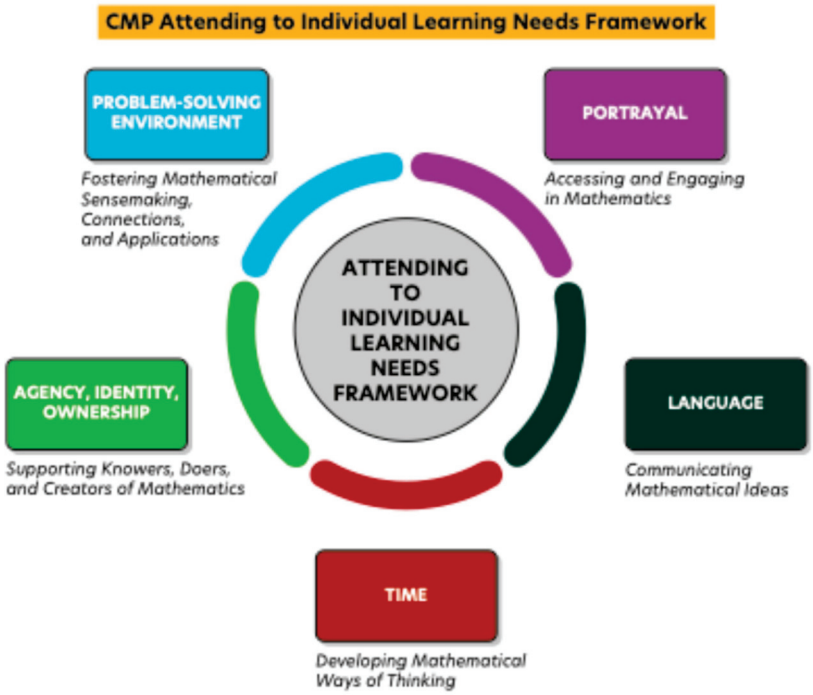


Figure 13. The Attending to Individual Learning Needs framework (Phillips et al., 2025c).

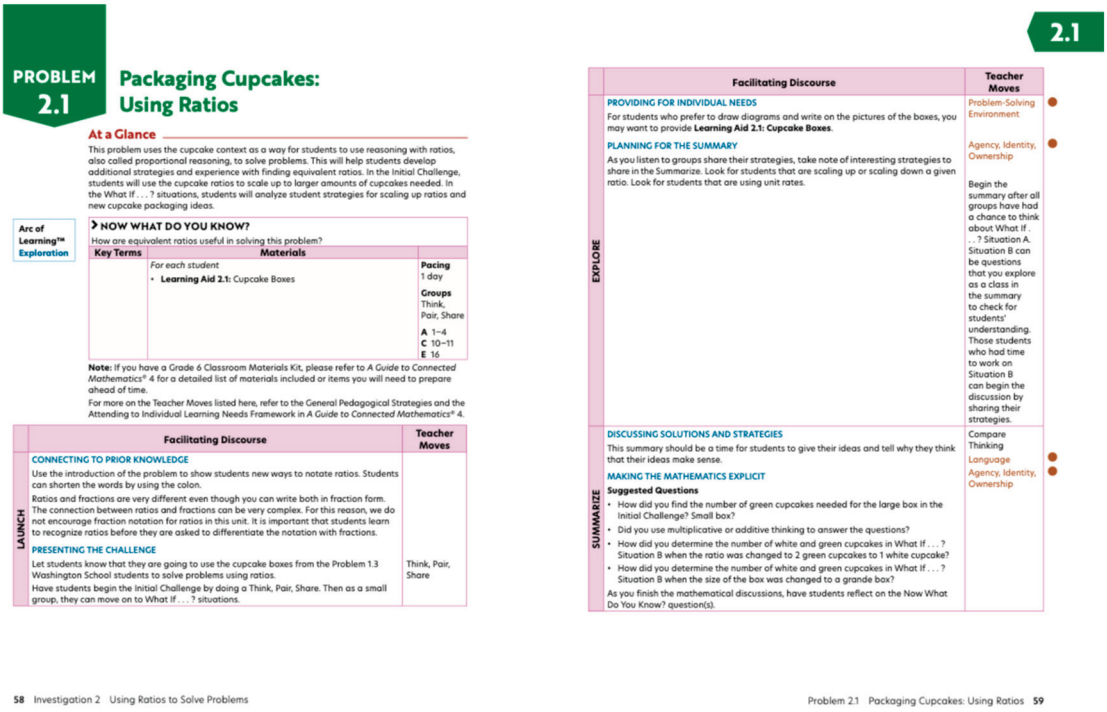


Figure 14. The At a Glance format for Problem 2.1 in the sixth grade *Comparing Quantities* unit (Phillips et al., 2025b).

2.4. Data Collection

At the close of the 2022–2023 school year, a total of 233 teachers, coaches, and administrators who field tested CMP4 completed a survey about their experience with the curriculum. This was the final year of field testing that provided substantial feedback to inform the published version of CMP4. Recruitment of field test teachers throughout the development process primarily focused on CMP3 teachers, coaches, and administrators, reflecting an intentional effort to gather feedback from educators familiar with prior

editions of the curriculum. This provided feedback that was crucial for evaluating how the fourth edition field test materials built upon, refined, or changed previous versions. However, the inclusion of newcomers—such as teachers newly hired into CMP districts and a small number of non-CMP3 schools—was also essential for capturing a broader range of perspectives. Some non-CMP schools participated in field testing, primarily because they requested involvement due to professional learning efforts that made use of the materials with neighboring districts. These educators brought fresh views on the curriculum's usability and effectiveness for schools that had not previously implemented CMP, offering a unique opportunity to assess how CMP4 field test materials performed in diverse instructional settings.

The survey was conducted electronically at the end of the school year, with responses from stakeholders (teachers, coaches, and administrators) representing approximately 70 cities spanning 20 states in the United States, as well as three additional countries. The participating schools represent a wide range of diversity in terms of gender, race, ethnicity, economic status, and/or disability. The socio-cultural diversity of the countries represented brings a variety of perspectives, each shaped by distinct educational contexts, which may influence how teachers and students engage with the field test materials. This diversity adds complexity to the interpretation of the results, as socio-cultural factors likely affect the implementation and effectiveness of the curriculum in different ways. Given the diversity in the socio-cultural and educational contexts, the design of the survey was intentionally developed to capture a broad range of feedback on the curriculum's features, effectiveness, and student engagement across varied instructional settings. The survey was built on prior CMP curriculum development and research efforts in middle school mathematics classrooms, and included items specifically focused on the design features of the fourth edition of the materials. The survey included approximately 10–15 quantitative items (e.g., multiple choice, Likert scale, and frequency questions) addressing the courses taught, frequency of collaboration with teachers and/or coaches, students' levels of math engagement and learning, and the effectiveness of the curriculum and its features. Participants were also asked two open-ended questions: one inviting them to share encouraging stories from teachers, and another asking how they would describe CMP4 to someone unfamiliar with the curriculum. Survey items were developed based on prior CMP research addressing curriculum effectiveness, student engagement, and implementation experiences and reviewed for clarity and alignment with study goals. The results were analyzed using descriptive statistics and inductive coding to identify patterns across diverse instructional settings.

2.5. Data Analysis

The survey was analyzed based on data from both the quantitative and qualitative items answered in the survey. To begin the survey analysis, identifying information was removed from the individual responses, such as first name, last name, and email address. Each quantitative question was then examined for the number of responses and the percentages of each answer. The quantitative questions most closely explored the effectiveness of the curriculum and its embedded features, with a focus on those that promoted mathematics engagement and learning. Effectiveness was measured using a Likert scale in two ways: one scale ranged from 1 ("less effective than last year") to 3 ("about the same as last year") to 5 ("more effective than last year"), while another scale ranged from 1 ("not effective") to 3 ("somewhat effective") to 5 ("very effective"). Simple summary statistics were used to analyze the data, including frequencies and percentages, to examine trends in responses. This analysis allowed the CMP4 team, including curriculum developers, to identify which features were found to be the least and most effective, as well as those that were less well known. The analysis of the qualitative responses focused

on identifying patterns or themes across the responses, with an emphasis on features and stories that supported mathematics engagement, learning, problem-solving, sharing multiple solutions, and working through real-life contexts. Most participants were CMP3 users and therefore had prior familiarity with the CMP materials. While the Likert scales were not formally validated, they were designed to capture key aspects of curriculum effectiveness and engagement based on prior research and expert input. These findings, both quantitative and qualitative, will inform future curriculum revisions and professional learning for teachers.

3. Results

3.1. Result 1: CMP4 Problems Promote Student Engagement and Learning in Mathematics

Our survey analysis showed that over 95% of teachers felt that their students were the same or more engaged in mathematics and critical thinking when field testing CMP4. The survey data showed that 66.52% of teachers (n = 155) indicated CMP4 was more effective in promoting student engagement and critical thinking when compared to the prior year. The prior year's curriculum ranged from earlier versions of CMP to teacher-generated materials. Further, our analysis found that approximately 30% of teachers (n = 65) believed that student engagement and critical thinking were around the same level as last year. Only four teachers (<2%) found that engagement and critical thinking in CMP4 classrooms were less than in the prior year.

CMP4 promotes students' engagement and learning in mathematics by allowing students to be knowers and doers of mathematics. Niss et al. (2016) discuss the intricate relationship between knowing mathematics and being a "doer" of mathematics. According to Anderson (2007), "all students can become mathematics learners, identifying themselves and being recognized by others as capable of doing mathematics" (p. 7). The notion of being a doer of mathematics is furthered by CMP4 Problems, which are "a more hands-on math curriculum that requires more critical thinking and analytical reasoning from students". The focus on CMP4 Problems allows students to engage in small and whole-group collaboration with their classmates. A teacher stated that CMP4 "encourages collaborative learning and problem-solving strategies that leads to students' academic success". Another teacher said CMP4 "is a different approach that leads students to critically think and take ownership of their thinking processes".

Figure 15 addresses the effectiveness of the highlighted design features as perceived by the field test teachers. Our analysis of the survey found that the CMP STEM Problem format was an important curriculum design feature that teachers used to promote student engagement and learning. Out of 233 total respondents, approximately 90% of teachers (n = 199) rated the CMP STEM Problem format to be somewhat effective, effective, or highly effective in promoting engagement and learning (see Figure 16). While many teachers found the CMP STEM Problem format effective, approximately 7% of the teachers did not know this format existed. Teachers found that the CMP STEM Problem format allowed students to extend their thinking and problem-solving skills. These skills are further discussed by a teacher stating, "CMP4 is a math curriculum that engages students in critical thinking, problem solving, and mathematical discourse".

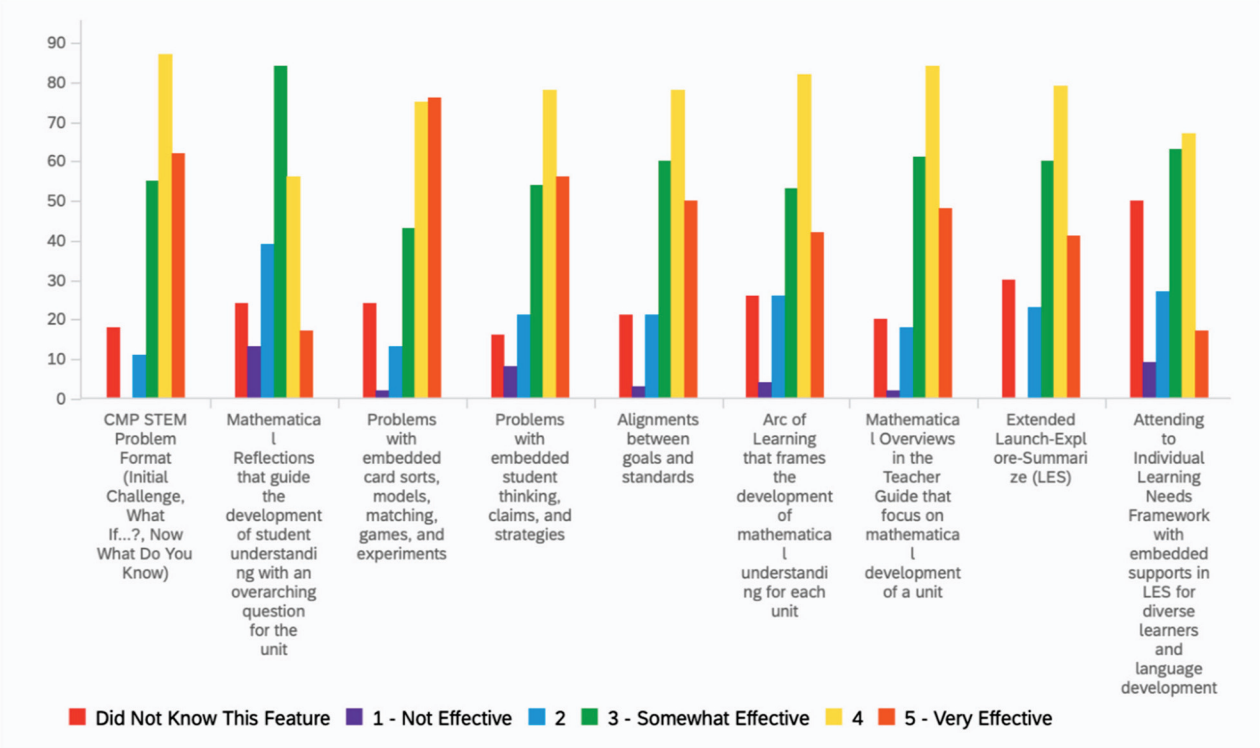


Figure 15. The effectiveness of CMP4 design features as perceived by teachers.

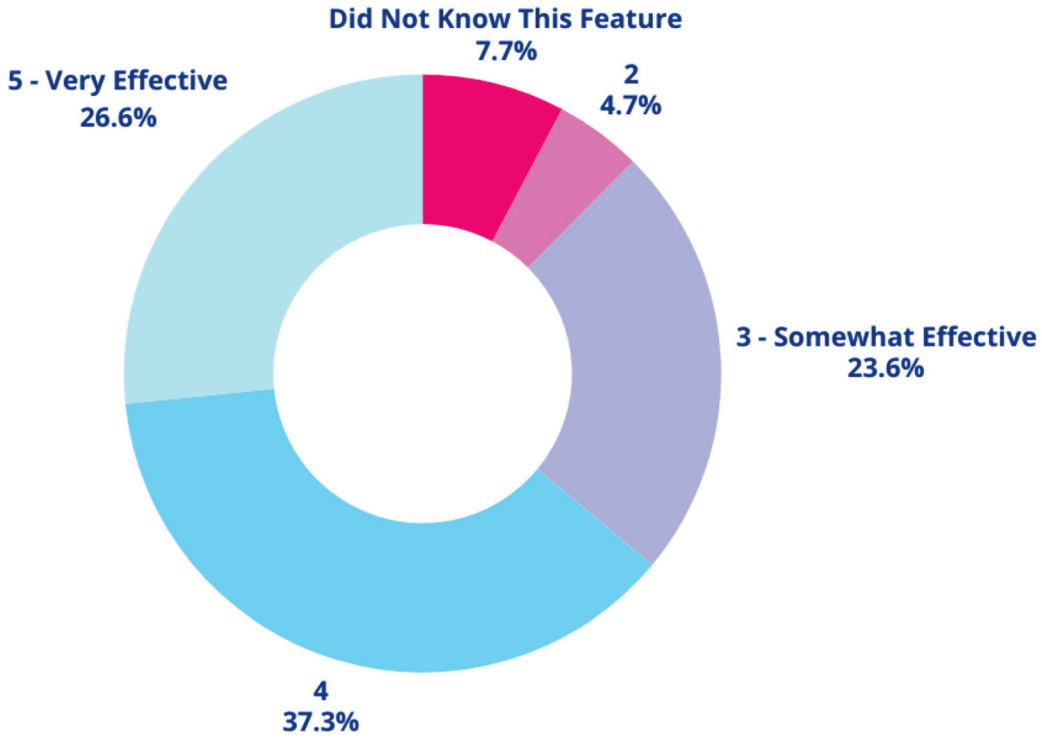


Figure 16. The effectiveness of the CMP STEM Problem format.

Our survey analysis found that teachers valued collaboration as a key component to student engagement and learning in math. Teachers appreciated that small-group and whole-group collaboration support was built into the curriculum program. Additionally, teachers appreciated the ability to use a problem-based curriculum that encourages collaboration. This is shown by a teacher saying, "There is nothing better for learning mathematics

than a problem-based program that encourages students to collaborate and build understanding". Specifically, teachers found that the curriculum design features and the various supports for students to demonstrate engagement and learning, such as matching, card sorts, games, and experiments, were beneficial. The survey results indicated the benefits of "problems with embedded card sorts, models, matching, games, and experiments" to be approximately 18% somewhat effective and approximately 65% effective and very effective. The problems that are in CMP4 allow students to engage with mathematics through a collaborative environment where students can see themselves reflected in the mathematics they are learning.

The survey analysis also indicated that CMP4 promotes student engagement in mathematics. A teacher stated, "students become better problem solvers and are truly engaged when they can collaborate and 'talk' math in real world scenarios". The survey results indicated that 40% of teachers (n = 92) found student engagement to be the same as last year using CMP4, while 55% of teachers (n = 129) found student engagement to be better than the previous year. It was noteworthy in our analysis that teachers' qualitative responses spoke directly to the distinction between engagement in class and engagement in the embedded mathematics. For example, the CMP4 Problems provide the opportunity to "engage students into math discussions" and "students have also displayed deeper conversations and collaboration through the use of this version".

While our analysis supported the finding that students' communication using mathematical language was the same or more effective than last year using CMP4, there were a few CMP4 teacher material features in which over 15% of teachers indicated they were unfamiliar with these features. For example, approximately 22% of teachers responded that they did not know there was a feature in CMP4 that attends "to Individual Learning Needs Framework with embedded supports in Launch—Explore—Summarize for diverse learners and language development" (see Figure 15). This is demonstrated by a teacher in the survey saying "Good curriculum—but I still need to supplement for more skill practice and for lower performing students. I especially had to supplement for our students with IEP math goals". This shows that this teacher needed additional supports that they had access to but were not aware existed. A teacher stated, "As a multilingual specialist who works in 7th grade as a co teacher, I would like to see more videos, visuals, and support for multilingual learners". Two additional features that almost 27% of teachers did not know about are "Overview videos discussing the mathematics of the Units" and "Welcome and general video recordings about CMP4". These features could be beneficial for teachers and/or students to have access to when they need to refer to overviews.

3.2. Result 2: CMP4 Problems Support Students in Solving Problems, Making Mathematical Connections to Real-Life Applications of Mathematics, and Sharing Multiple Solution Strategies

CMP4 Problems consist of contextualized problem situations that embed mathematics. This provides a storytelling aspect within the mathematics problems that provides an opportunity where students are "actively engaged in their learning and can talk about math". "Students participating in discourse" was rated to have approximately 40% effectiveness, which was the same as last year using CMP4, and approximately 54% effective or more effective than the previous year. When teachers were asked about what they would tell someone that was not familiar with CMP4, one teacher stated that "CMP4 is a highly rigorous curricular resource that promotes student engagement and discourse through story-based contexts and problems". This was furthered by another teacher who stated that CMP4 "leads to a mathematical discussion of deep value on a daily basis". A key aspect of CMP4 that teachers find particularly useful is the ability to promote mathematical discussion through "talking math". Using a Likert scale for measuring students' communication with mathematical language, as noticed by the teacher, our analysis showed that

teachers indicated that "students communicate using mathematical language". Nearly 45% of responses (n = 100) indicated teachers thought students' communication with mathematical language was the same as last year. Further, 51.50% (n = 120) rated it to be more effective than the previous year using CMP4, while a combined percentage of only 5.58% (n = 13) rated it as less effective. One teacher indicated that "CMP4 encourages student discussion and articulation of math ideas", while another teacher said, "students discuss math and relate ideas to each other". The survey data demonstrate how the problems in CMP4 promote student discussion and collaboration within the classroom, as reported by the teacher. CMP4 Problems provide students the opportunity to connect to the problems through storytelling while also pushing students to engage in mathematical discourse by communicating in mathematical language.

The survey analysis indicated that an important aspect of CMP4 is supporting students in solving problems and providing the opportunity for mathematical connections to real-life applications. An important part of mathematics learning in CMP classrooms is the ability to see how the mathematics in the classroom relates to everyday life. In the survey, a teacher shared that "CMP4 is an excellent way for students to not just 'do' math problems, but to analyze real life situations and apply math concepts in daily situations". Another teacher stated that CMP4 "allows students to take control of their learning and form a deeper understanding of mathematical content and its connection to the real world". A benefit of CMP4 that was noted by many teachers throughout the survey is that students were able to see themselves in mathematics, which allowed them to make connections to their daily lives. A teacher of CMP4 stated that the problems that comprise CMP4 "allows for students to see themselves in the mathematics and work on creating a sense of positive mathematics experiences for students to be successful in class". Therefore, CMP4 provides an opportunity not only for students to make connections with everyday life, but also to see themselves in the mathematics they are learning.

Additionally, while CMP4 allows students to make connections to math and their daily lives, it also allows them to see the connections between math concepts from multiple units. This connection is demonstrated by a teacher stating, "CMP4 has helped my students make authentic mathematical connections using real world applications". A different teacher said that their students "are able to collaborate with one another and see each other as contributors and doers of math. Students' ability to share their thinking, model, and discuss with each other has improved". CMP4 Problems connect school math and real-life math by allowing students to see themselves and their everyday lives in the mathematics that they are learning in school.

A characteristic of CMP4 Problems that resonate with teachers is the ability to make connections to prior concepts while continuing to build toward future concepts. CMP4's use of context problems creates a space where students are "challenged to think and discuss math in a way that has them remembering the context of unit problems all year long". This alignment is emphasized by a teacher stating that "CMP4 use of real-world problems through the initial challenge, exploration and summary is wonderful in connecting students with all levels of abilities to math concepts". The curriculum allows students to "access prior knowledge" while "building connections across content and units". Additionally, CMP4 Problems are "cyclical and continue to reinforce essential concepts throughout several of the units". It was stated that "CMP4 encourages students to think critically and approach math in multiple ways. . ..CMP4 promotes multiple strategies and connections between concepts rather than a one-and-done way to solve a problem". CMP4 Problems allow students to engage in mathematical discourse while making connections from past units and looking forward to future concepts.

4. Discussion

In this paper, we reported on the bi-directional relationship between instructional materials and the operational curriculum. We provide evidence for offering an alternative to the one-directional model proposed by Remillard and Heck (2014), which traditionally emphasizes the influence of materials on classroom practice without fully accounting for the reciprocal impact of classroom enactment on materials. As research has shown, teachers play a central role as mediators between curriculum materials and student learning (Ball & Cohen, 1996; Ben-Peretz, 1990; Pepin, 2019). Our study highlighted how feedback from field test teachers provides empirical evidence of a dynamic, reciprocal interaction between curriculum design and classroom enactment. While Remillard and Heck (2014) emphasize the influence of instructional materials on classroom practices, our study shows that teachers' adaptations and classroom practices also shape the curriculum materials. This aligns with findings from Stein and Kim (2009), who emphasized that teacher engagement with curriculum materials directly shapes how these materials are enacted in the classroom. What distinguishes this study from prior work is the use of design research methods to systematically collect and integrate teacher feedback throughout an extended development cycle. This represents a new contribution by demonstrating how iterative developer–teacher collaboration, embedded in a field test structure, informs curriculum refinement. Thus, our study reports evidence of a cyclical, iterative process, consistent with design research approaches (The Design-Based Research Collective, 2003; Edelson, 2002; Richey et al., 2004), where teacher feedback informs ongoing revisions to the curriculum before it is published. Teachers noted how certain problems and concepts were revisited across several units, reinforcing the evolving nature of the development of mathematical understanding embedded in the curriculum. This was particularly evident in the feedback about the CMP STEM Problem format, which teachers reported helped support student understanding across contexts. These findings highlight the importance of viewing the relationship between instructional materials and the operational curriculum as bi-directional, contributing to the improvement of both teaching practices and student learning outcomes.

Like Spagnolo et al. (2021), this paper reported on the design, field testing, and evaluation of the field test version of the fourth edition of a contextual problem-based curriculum, CMP4, which engaged over 500 teachers from the U.S. and internationally. While Spagnolo et al. (2021) used large-scale assessment as a framework and teacher beliefs to design a distance professional learning experience over one year culminating in a survey, the research reported in this study made use of a curriculum design and teacher enactment framework that guided the development, field testing, and evaluation of CMP4 over four years. The findings from the 2022–2023 survey highlight the effort to make CMP's STEM Problems both expansive and focused enough to foster a deep understanding of the mathematical learning goal. Additionally, the tensions in both curriculum design and teacher enactment, as outlined in the theoretical framework, are underscored in this study. Survey quotes suggest that students were able to make connections from mathematics to their daily lives, take more control of their learning, engage more effectively in mathematical discourse, draw connections both across units and within a unit, and see themselves as doers and contributors of mathematics. These responses align with the framework's emphasis on the importance of designing contextual mathematics problems that not only support procedural knowledge but also encourage deeper conceptual understanding (Edson et al., 2019; Lappan & Phillips, 2009). These data suggest that the CMP STEM Problem format in CMP4 was effective in promoting these outcomes. This study contributes new insights by systematically capturing teacher perceptions of a specific problem structure—CMP STEM Problem format—and its influence on student engagement and learning across a large and diverse sample of classrooms. These findings highlight the critical role of student involvement in

the development and enactment process. As teachers continue to adapt the curriculum and its design features to meet the evolving needs of students, it is essential to prioritize problems that balance both conceptual and procedural development while fostering deeper student engagement. This cyclical interaction between curriculum design, teacher enactment, and student outcomes reflects the evolving nature of problem-based learning, where both teachers and students actively contribute to the shaping of the curriculum.

Nonetheless, this study is not without limitations. In addition to the absence of student performance data, there were several other factors that may influence the interpretations of the findings. One key limitation is potential selection bias: many of the field test teachers were already familiar with CMP, particularly CMP3. Their prior experience and alignment with the CMP philosophy may have shaped their perception and feedback, potentially leading to more favorable evaluations of CMP4. While this group provides valuable insights into how CMP4 builds on its predecessors, future studies should intentionally sample teachers with no prior CMP experience.

While the data provided by the survey offer substantial support for the claims about the effectiveness of the CMP STEM Problem format, a key limitation is that it reflects only teachers' perspectives on classroom implementation of the print-based materials. The survey sought teachers' insights into the effectiveness of various paper-and-pencil curriculum features, the use of materials, and their perceptions of what they would share about CMP4. The respondents highlighted positive feedback, including praise for how the curriculum materials engage students, strengthen problem-solving and critical thinking skills, and enhance enjoyment of mathematics, with some noting improved student performance data and students seeing themselves as active participants in mathematical thinking. Specifically, the survey indicated that CMP4 Problems promote student engagement and learning in mathematics, as well as support students in solving problems, making mathematical connections to real-life applications, and sharing multiple solution strategies. However, the survey did not include classroom data or performance metrics and thus lacks student voice to further validate or triangulate these claims. Finally, although the findings are grounded in CMP4's development context, they have implications beyond this specific curriculum. The design research model described here—where teachers are active contributors to iterative design—offers a model that could be adapted by other curriculum development teams. Insights about the use of problem structure to foster mathematical reasoning, real-world connections, and multiple solution strategies may be generalizable to other mathematics curriculum materials aiming to increase student engagement and conceptual understanding. Future research could extend this study by directly examining student learning and engagement, tracking patterns of change over time, and focusing on the classroom enactment of the curriculum using the published version. Future research could provide a deeper understanding of how teacher support materials address enactment challenges, further refining the curriculum and its impact on students' mathematical development. Further studies could also explore how the principles of the CMP4 design model might inform curriculum development efforts in other subject areas or educational contexts. Future activities could also focus on students' mathematical understanding and problem-solving strategies—including those of special populations of students—by anticipating possible student responses and using these expectations to inform planning, teaching, assessing, and reflecting on student thinking.

5. Conclusions

Our study highlights the crucial role of teacher feedback in refining instructional materials, supporting the view that ongoing collaboration between educators and curriculum designers is essential for creating adaptable and effective resources (Ball & Cohen,

1996; Ben-Peretz, 1990; Jones & Pepin, 2016; Pepin, 2019; Stein & Kim, 2009). The survey of 233 teachers, coaches, and administrators in 2022–2023 found that CMP4 Problems promote student engagement, enhance problem-solving skills, and help students make real-world mathematical connections. Additionally, we challenged the one-directional model of curriculum development proposed by Remillard and Heck (2014), presenting evidence of a bi-directional, cyclical relationship between instructional materials and classroom enactment. Teachers' feedback played a pivotal role in shaping the curriculum, demonstrating the iterative process of refinement. A key limitation of this study is the reliance on teachers' perspectives without direct classroom data or student performance metrics. As a result, while the findings offer strong evidence of perceived effectiveness from the teacher's perspective, they do not allow for direct measurement of student learning gains. This limitation is important to consider when interpreting claims about the impact of CMP4 on student outcomes, as they are based solely on teacher-reported observations rather than empirical achievement data. Future research, from researchers outside the curriculum project, should investigate student feedback and assess learning outcomes to validate the curriculum's effectiveness. For curriculum developers, this study illustrates the value of design-based research in iteratively refining materials based on real-world classroom enactment. For teacher educators, CMP4 provides an example of how curriculum materials can also serve as educative tools, supporting teachers' professional learning through embedded structures and features of the curriculum materials. For policymakers, these findings reinforce the importance of long-term investments in curriculum development efforts that treat materials not just as static products but as evolving supports for both student and teacher learning.

Overall, the CMP4 field test materials offer a model of curriculum development that addresses both instructional design and learning simultaneously. The findings underscore the value of curriculum problems that are designed in collaboration with teachers to promote student engagement, support problem-solving, encourage connections to real-life applications, and foster the sharing of multiple solution strategies. These results affirm the critical role of teacher–curriculum interaction in shaping meaningful mathematical learning experiences across diverse educational settings.

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Institutional Review Board Statement: All rights and permissions for the CMP4 materials are owned by Michigan State University. The work reported in this paper involved feedback from participants regarding a commercial product. While names and location data were collected for identification and tracking purposes, they were not analyzed or used in the data analysis. The data were used solely for internal product development and to disseminate its results.

Informed Consent Statement: Teacher feedback was obtained from all teachers participating in the field test. Teachers applied to participate in the field testing, providing their feedback voluntarily. There was no cost to the teachers or schools for participating in the field test, as all testing materials were provided free of charge. Schools selected for participation primarily had previously used CMP materials or where affiliated with professional development providers using CMP materials. Additionally, participants received free professional learning as part of their involvement in the field testing.

Data Availability Statement: Data will be made available upon request.

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Conflicts of Interest: The authors declare no conflicts of interest.

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Article

Forces Influencing Technical Mathematics Curriculum Implementation: Departmental Heads' Understanding of Their Practices to Enact Roles and Responsibilities

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Abstract: This qualitative study explores forces influencing the practices of Departmental Heads (DHs) in enacting their roles in implementing and managing Technical Mathematics (TMAT) curriculum. TMAT was piloted in a few South African schools in 2016 and later scaled to others. Since its inception, learner performance has been uneven, raising questions about the processes of managing and implementing the curriculum. We use Samuel's Force Field Model to understand forces influencing DH practices in their quest to implement and manage the curriculum. Data were generated using one-on-one interviews and document analysis and thematically analysed using NVivo. The findings reveal that contextual and external forces are the main factors that influence DH practices when it comes to the implementation and management of the curriculum. These forces influence practices such that the roles and responsibilities are carried out mainly for compliance purposes. While in theory, DHs seem to believe in collaboration, they prefer working in silos and perceive that the success of the TMAT curriculum implementation should be at the hands of seasoned mathematics teachers. In addition, they seem to consider curriculum implementation and management to be solely about ensuring curriculum coverage. We argue that to ensure the sustainability and effectiveness of the TMAT curriculum, there is a need for the continuous professional development of DHs, such that they are able to balance external forces and internal forces.

Keywords: curriculum implementation; curriculum monitoring; curriculum sustainability; departmental head; technical mathematics

1. Introduction

The introduction of Technical Mathematics (TMAT) in South Africa has been one of the major developments in the mathematics curriculum. TMAT was implemented in technical schools in 2016, a decade after the introduction of Mathematical Literacy in 2006. TMAT focuses on the technical aspects of the Pure Mathematics curriculum, while Pure Mathematics covers abstract mathematics. The perennial question with these new developments is whether Departmental Heads (DHs) as curriculum leaders understand their practices enough to enact their roles and responsibilities effectively.

In the context of South Africa, as noted by the Department of Basic Education (DBE, 2009), the curriculum monitoring strategy should be regularly evaluated to determine whether intended outcomes have been achieved for learners and teachers. The monitoring strategy sought to curb shortcomings of Curriculum 2005 (C2005) and the National

Curriculum Statements (NCSs), such as the lack of clarity in learning outcomes, inconsistent implementation across schools, insufficient teacher training, and the overemphasis on outcomes-based education (OBE) without adequately addressing content knowledge, resulting in uneven learner performance and gaps in foundational skills. Despite policy documents providing broad guidelines and approaches to change, Maree (2016) argues that it is the teachers who are implementing policy changes in classrooms and, therefore, their understanding of their practices is crucial in the process of curriculum implementation. Extending this argument, Tapala (2019) stresses that DHs who have the dual role of teaching and curriculum monitoring should be accountable for their roles and responsibilities, such as evaluating, monitoring, and developing their teachers, and for this to happen, the DHs need to know what is expected of them, and their practices should be the driving force in ensuring the sustainability of the curriculum.

According to Msibi and Mchunu (2013), curriculum changes have been implemented without adequately addressing the issue of teacher professionalism. They contend that the DBE's fixation with the curriculum, rather than teachers and their abilities, is the main cause of the curriculum implementation failures in the South African educational system. Bansilal (2002) maintains that well-trained teachers are essential in the educational process, which means that the success or failure of an education system depends on the quality of its teachers. Many studies (Jaca, 2013; Malinga, 2016; Ogina, 2017; Seobi & Wood, 2016; Tapala, 2019) have pointed to a lack of training for DHs, which makes it difficult for them to enact their roles and responsibilities. Tapala (2019) contends that for DHs to carry out their assigned responsibilities, it is critical that they receive pre-training prior to taking on the post of DH, and that they continue to grow professionally after being appointed to the position.

Given the trends in learner performance in mathematics, we argue that for curriculum sustainability and success for TMAT, the execution of DHs' roles is an area that needs exploration in order to understand what they do, how they do it, and why they do what they do. Understanding these intricacies will assist in future planning. With this in mind, we set out to answer the following research questions: What forces influence DHs' practices of implementing and managing the TMAT curriculum? Also, how do DHs perceive their practices when enacting roles and responsibilities to implement and manage TMAT?

2. Literature Review

According to the DBE (2011), the daily preparation of instruction and learning and the actions that assist it constitute education/curriculum management. Middle managers in South Africa, known as DHs, oversee and manage the teachers in their departments to guarantee that teaching occurs and that the curriculum is followed (Shaked & Schechter, 2017; Tapala et al., 2022). The Education Labour Relations Council (ELRC, 1998, p. 66) states that DHs are to "develop curriculum-related policies, control the work of teachers and learners, appraise subordinates, and manage subject work schemes". The primary responsibilities of DHs, as spelled out in the DBE's Personnel Administrative Measures (PAM) document (DBE, 2016, 2022), are to engage in class teaching, oversee the effective functioning of the department, and organise relevant or related extracurricular activities to ensure that the subject, learning area or phase, and the education of the learners are promoted in a proper manner (DBE, 2016, 2022). The DBE (2016) defines DHs as school managers responsible for certain subject streams. The DBE's PAM document defines the core duties and responsibilities of DHs, which vary depending on the needs and approaches of individual schools and include teaching DBE (2016). Accordingly, as the intermediary between subject advisors and teachers, DHs must manage curricular instructions for teachers, which is the core duty separating them from ordinary teachers. Therefore, they

work hand in hand with the school principal and deputy principal to ensure the smooth running of the subject they manage.

To be promoted as a DH in South Africa, a teacher must have strong subject knowledge, at least three years of teaching experience, and a minimum matriculation qualification with a three-year teaching qualification. Additionally, understanding the school's specific context and conditions is essential (Christian, 2013). While education management qualifications are not required, teacher unions and School Governing Bodies (SGBs) often influence the appointment process. The unions often participate in interviews and provide recommendations, while the SGBs are heavily involved in decision-making, which could undermine merit (Christian, 2013).

2.1. Curriculum Management

The DBE (2022) has highlighted key elements of curriculum management that DHs are expected to execute, which include curriculum supervision, evaluation and curriculum monitoring, staff support and resources, capacitating staff with the required skills, the creation of relevant learning activities, and providing quality assurance relating to learning and assessment. DHs play an important role in the assessment and development of teachers in accordance with the teacher evaluation system known as the Quality Management System (QMS). QMS is an essential tool used to measure the development and performance of teachers. In Switzerland, Shaked and Schechter (2017), affirms that DHs are middle managers responsible for the supervision and control of the teachers in their departments to ensure that teaching and learning take place and that the curriculum is implemented. DHs observe teachers at work and provide them with feedback after visiting their classrooms. Effective curriculum monitoring requires DHs to be knowledgeable about the teaching and learning of the areas they oversee; therefore, a candidate for the position of DH must possess subject matter competence (Tapala, 2019; Ogina, 2017; Seobi & Wood, 2016; Mampane, 2018). TMAT focuses on the hands-on application of mathematics concepts, requiring a different teaching strategy. If DHs lack the knowledge and expertise to implement and lead their departments, teachers will be left uncertain about implementing the necessary changes envisaged by the DBE. Hence, in the South African context, the DBE creates the curriculum, offers pacing recommendations via the Annual Teaching Plan (ATP), and supplies learning resources (textbooks), teaching resources (lesson plans), assessments, a programme of assessment, and tools for curriculum monitoring (school-based assessment). After the design is complete, schools are tasked with implementing it. It is, therefore, essential that DHs understand the full extent of their roles and responsibilities. As Leithwood (2016) alluded to, subject matter expertise, managerial prowess, and potent interpersonal abilities are essential managerial practices for DHs. According to Mthiyane et al. (2019), DHs must play a significant part in the development of teachers by serving as change agents and providing instructional leadership.

2.2. Curriculum Implementation

There is a significant body of international studies on curriculum implementation and management. In the United States, Elmore (2016) distinguishes between "implementation" as acting on established knowledge and "learning" as navigating the unknown. He critiques policy-driven reforms for prioritising uniformity, treating diversity as a problematic exception. Research by Miedijensky and Abramovich (2019) in Israel supports Elmore (2016) as they revealed that an exemplary school's implementation of the new curriculum was a gradual and structured process that executed many actions at each stage, and where the principal, DHs, and teachers were well qualified and fully committed. In the United Arab Emirates, Al-Husseini (2016) noted that frequent classroom observations and feedback

from DHs improved teachers' classroom practices and learners' performance. Similarly, in Australia, Roberts-Hull et al. (2015) reported that high-performing education systems emphasise content knowledge and subject-specific expertise.

In their study, du Plessis and Eberlein (2018) found that DHs of multi-subject departments face difficulties because they must be responsible for subjects that they may not have formal training in and find themselves burdened with a heavier workload than those in departments focusing on a single subject. On the other hand, Stephenson (2010) in New Zealand cited the lack of professional development initiatives intended to help DHs serve as curriculum leaders as an impending factor in curriculum implementation. In the United Kingdom, Brown et al. (2000) identified lack of time, space, specialist teachers, personnel management, staff morale, and homework policy as key challenges to curriculum management.

3. Theoretical Framework for the Study: Samuel's Force Field Model

In this study, we use Samuel's (2008) Force Field Model to understand how DHs reconcile the forces that operate in the sphere of implementation and teaching. The Force Field Model is used to understand how DH autonomy prevails as they enact their roles within the prescriptive Curriculum Assessment Policy Statements (CAPS). The theoretical framework provides a lens for understanding forces that enable or hinder DHs when enacting their roles and responsibilities. The success of curriculum management and the implementation of TMAT requires DHs to mediate the institutional and contextual forces within their schools, and their agency is important in knowing what to adapt, adopt, or neglect when these forces are in play. Figure 1 shows the constructs of the Force Field Model, which is modified for this study.

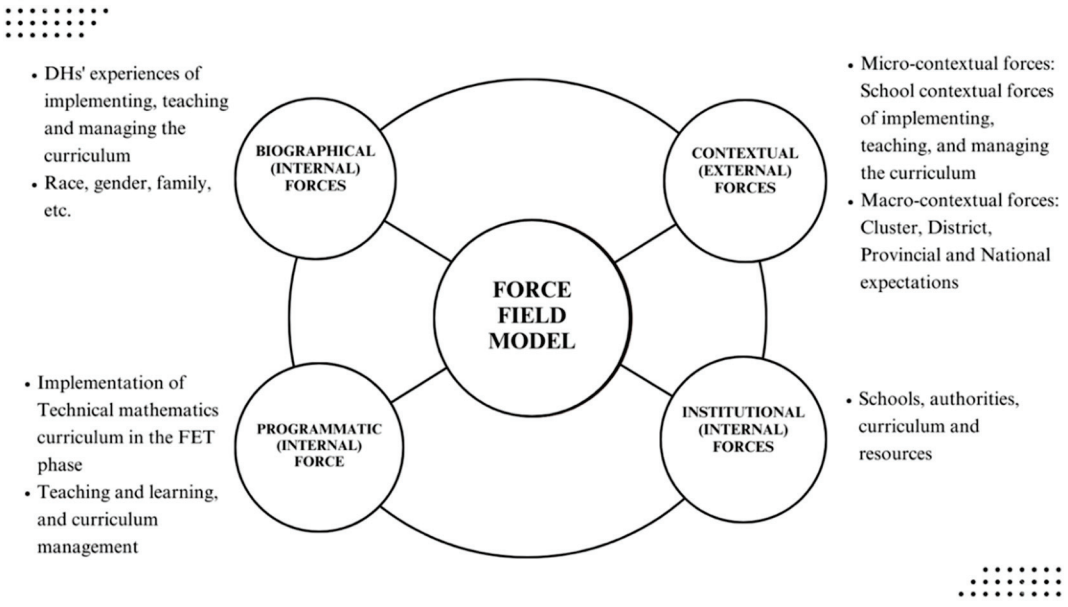


Figure 1. Modification of Samuel's Force Field Model to understand DHs' knowledge and practices in implementing and managing Technical Mathematics.

Samuel (2008) posits that teachers' behaviours are influenced by their unique life experiences and the contextual forces of their school environments. De Villiers (2021) emphasises the significance of teachers' biographies as internal authentic energy sources. According to Samuel (2008), teachers are products and processors of their history, affected by institutional forces and the broader socio-political context. These forces shape the ethos of institutions and impact teaching and learning quality. This means that characteristic

conceptions of professional teachers and DHs in the context of this study are the result of the force of their institutional expectations. To be specific, DHs' values and goals concerning the pedagogy and implementation of the Technical Mathematics curriculum are most likely to be shaped by the very prescriptive CAPS curriculum that they are required to implement.

Programmatic forces, such as curriculum interventions, also shape teaching expertise over time. The Force Field Model helps understand how various forces interact and influence DHs' roles and responsibilities and explains their dynamic interplay of influences. For this study, all forces—biographical, contextual, institutional, and programmatic—are crucial in comprehending DHs' enactment of their roles.

4. Methods

The aim of this study was to understand how and why DHs enact their roles and responsibilities to implement and manage the TMAT curriculum. In this study, we sought to understand DHs' interpretations of their world from both inside and outside (Jackson & Bazeley, 2019). This was a qualitative study in which we used semi-structured interviews to gain a deeper understanding of the phenomenon (Maree, 2016).

TMAT was integrated into the South African curriculum in 2016. A pilot phase was conducted in several schools before TMAT was expanded to all technical schools that offer technical subjects. Participants were chosen because of their involvement in the inception of TMAT, which suggests they possess greater knowledge of and expertise in implementing and managing TMAT. The sampled DHs were purposively selected in relation to their management position, their teaching role, and their experience in implementing Technical Mathematics. In the Pinetown District of KwaZulu-Natal, where the study was conducted, three schools offer TMAT, and hence, three DHs, one from each of the schools, were purposefully selected as study participants. Biographic information of the research participants is presented in Table 1, including age, teaching experience, experience as a DH, and teaching qualification/s.

Table 1. Participants' bibliographical information.

	Participant A	**Participant B**	**Participant C**
Participant (pseudonyms)	Mr Alpha	Mrs Beta	Mr Gamma
Age (years)	52	45	49
No. of years as DH	14	7	9
No. of years teaching TMAT	6	6	5

TMAT was initially piloted in Pinetown District in 2016. We obtained ethical clearance from the KwaZulu-Natal Department of Education and the University of KwaZulu-Natal Ethics Committee (HSSREC/00003217/2021) to carry out the study. Permission was secured from school principals and DHs, with informed consent from learners, and parental consent for their children's participation. Ethical considerations such as learner protection, voluntary participation, informed consent, anonymity, confidentiality, and transparency were addressed. Pseudonyms were used for school names, DHs, and learners to ensure anonymity.

This study was conducted by interviewing three DHs and analysing documents. Document analysis examined tools used by DHs for curriculum guidance and monitoring, such as files and ATPs. We used the computer-assisted qualitative data analysis software NVivo to assist with the coding and transcription of the data, providing an efficient and organised approach to managing the qualitative dataset (Jackson & Bazeley, 2019). Braun and Clarke (2006) outline that the analysis involved familiarising ourselves with the data, generating

initial codes, searching for themes, reviewing and defining them, and writing the report. Using NVivo ensured a rigorous and transparent procedure for organising, retrieving, and analysing data. This approach allowed us to identify and interpret key patterns and themes grounded in the dataset. Transcribed data were analysed to ensure alignment with interviews, maintaining authenticity. NVivo aided in coding transcripts and identifying themes from DHs' shared experiences, which were cross-checked by researchers for consistency. In order to gain a thorough understanding of DHs' practices and actions, it was necessary to triangulate their statements and documented practices. Themes that emerged were DHs' practices of teaching, managing, and monitoring Technical Mathematics curriculum; DHs' management of Technical Mathematics; DHs' implementation of Technical Mathematics; DHs' management of human resources (teachers); DHs' enactment of roles and responsibilities; and reasons influencing DHs to enact the roles in the way they do. During the coding process, we collapsed nodes and expanded them to ensure they were grounding transcripts from interviews and document analysis. DHs' commonly used words in the transcripts guided our initial node and code creation. This preserves participants' key themes and perspectives by keeping the codes accurate and close to the original transcripts.

5. Results

To respond to our research questions about how DHs enact their roles and responsibilities to implement and manage Technical Mathematics, why they enact their roles in the way they do, and the underlying reasons for how DHs carry out their roles and responsibilities, the results are presented under three broad themes that emerged from the data: managing TMAT curriculum, implementing TMAT curriculum, and reasons underlying reasons for how DHs carry out their roles and responsibilities.

5.1. Departmental Heads' Management of the Technical Mathematics Curriculum

To ensure the successful implementation of the envisaged curriculum, DHs should be clear about their roles and responsibilities and what is expected of them. The participants alluded to several expectations that they are aware of and managing, including monitoring curriculum coverage through checking teachers' daily/weekly progress on completing topics and keeping track of their ATP. When asked how they ensure effective management of the curriculum, they had the following to say:

> Mr Alpha: The expectations are quite high. As part of my responsibility, I need to ensure that I follow what is stipulated in the ATP. That is in terms of expectations by the DBE, we need to complete. We continuously have CASS (Continuous assessment) moderation where we have to take our portfolios together with our ATP. Our portfolios in this right here [showing his file], where we have to do all assessment tasks that are related to School Based Assessment (SBA). So I do the same to monitor the other teachers to ensure the curriculum is completed because that is what DBE wants.

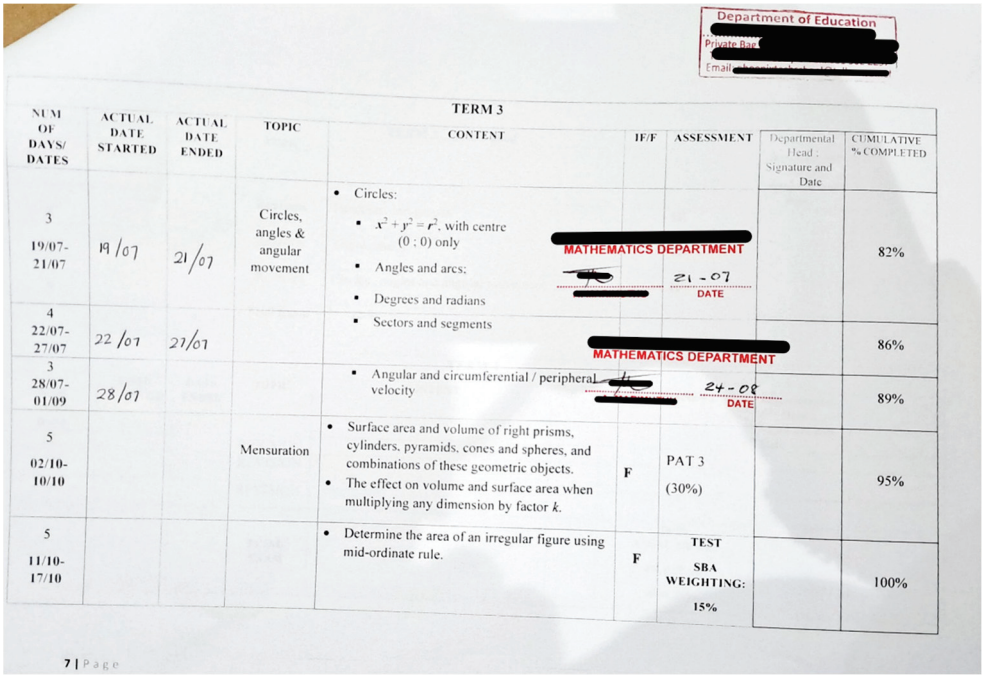
As noted from the above, Mr Alpha's focus is on curriculum coverage and meeting the expectations of the DBE. Mr Gamma and Mrs Beta were of the same view and mooted the following:

> Mr Gamma: Managing means checking that the ATP is followed, CAPS curriculum is followed or confirmed to see that assessments are done on time, and moderation of assessments are done in keeping with the CAPS document.

> Mrs Beta: The Department is expecting reports on the subject, like when we go for moderations our work is checked on a regular basis. The DBE expects learners

to do well in the subject so they can get access to tertiary institutions, specifically in Universities of Technology and Technikons.

From the above responses, it appears that external forces are driving the DHs' management of the curriculum. Knowledge of their practices is not coming from within but rather is driven by the need to meet external expectations. This is similar to the findings of Mkhwanazi et al. (2018), namely that the DH monitoring of curriculum coverage is mainly carried out for compliance purposes, and thus becomes just a tick-box exercise. Similarly, the data drawn from document analysis revealed that DHs' practices in managing the curriculum implementation are driven by the need to comply rather than monitoring the actual classroom practices, as shown in the extracts from the ATP in Figure 2.


Department of Education
[redacted]
Email [redacted]

TERM 3

NUM OF DAYS/ DATES	ACTUAL DATE STARTED	ACTUAL DATE ENDED	TOPIC	CONTENT	IF/F	ASSESSMENT	Departmental Head : Signature and Date	CUMULATIVE % COMPLETED
3 19/07-21/07	19/07	21/07	Circles, angles & angular movement	• Circles: • $x^2 + y^2 = r^2$, with centre (0 ; 0) only • Angles and arcs; • Degrees and radians		MATHEMATICS DEPARTMENT [redacted] 21-07 DATE		82%
4 22/07-27/07	22/07	27/07		• Sectors and segments		MATHEMATICS DEPARTMENT [redacted]		86%
3 28/07-01/09	28/07			• Angular and circumferential / peripheral velocity		24-08 DATE		89%
5 02/10-10/10			Mensuration	• Surface area and volume of right prisms, cylinders, pyramids, cones and spheres, and combinations of these geometric objects. • The effect on volume and surface area when multiplying any dimension by factor *k*.	F	PAT 3 (30%)		95%
5 11/10-17/10				• Determine the area of an irregular figure using mid-ordinate rule.	F	TEST SBA WEIGHTING: 15%		100%

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Figure 2. DBE TMAT Annual Teaching Plan for 2023.

Whereas in Figure 2 there is a stamp and signature in certain places, the signing dates are not consistent with the dates of completion. For example, in the first instance, the signing date corresponds with the date of completion. While it could be argued that monitoring and teaching should take place simultaneously, the evaluation of the implementation should happen after the teaching has taken place to ensure the integrity of the process. In other instances, as shown in the extract in Figure 2 above, there is no evidence of dates of monitoring and evaluation. Meanwhile, DHs in this study mooted that they use the ATP to manage curriculum implementation; evidence in Figure 2 indicates that this was not carried out for all sections, thus showing inconsistencies in ensuring curriculum implementation. The findings from document analysis indicate a gap between what the DHs perceived and actual practices when it comes to the implementation and management of TMAT.

In the Force Field Model, Samuel (2008) refers to various external forces, one of those being macro contextual forces where cluster, district, provincial, and national expectations influence the process. Drawing from the above responses, the findings showed that the DHs' practices in managing the TMAT curriculum are influenced by external expectations.

It is true that DHs as curriculum managers within the school have an obligation to external stakeholders; however, should internal forces, that is, the institutional needs, not be at the forefront for the efficacy and sustainability of the curriculum? DHs seem to put

more emphasis on the need to comply, which suggests that external forces are defining the process, devaluing the autonomy of an individual DH in the school setting. This was evident in the DHs' responses:

> Mr Gamma: DBE set dates for moderation. I think monitoring and management is also done on a term-wise basis, when educators go for moderation from Grade 10, 11 and 12. Moderation of past exam papers or if any assessments are done; together with content workshops that are done on a regular basis for the subject teachers.
>
> Mrs Beta: The Department is expecting reports on the subject, like when we go for moderations our work is checked on regular basis.
>
> Mr Alpha: We are all expected to do moderation, the DBE set dates, and I make sure by the time moderation comes the work is covered.

While the DHs allude to the processes of the DBE, none of them mention internal processes carried out to ensure that the TMAT curriculum is managed effectively.

5.2. Departmental Heads' Implementation of Technical Mathematics Curriculum

The DHs' agency in curriculum implementation when exposed to different factors imposed by the DBE and their school context was examined. Schools had to meet certain criteria so that TMAT and other technical subjects were populated in their schools. The DBE sets norms and standards aimed at the uniform implementation of TMAT across all technical schools in South Africa. The following responses reflect the DHs' views about the changes that came with the implementation of the TMAT curriculum:

> Mrs Beta: The changes are not dramatic; they are changes that one is able to handle. The content is mostly still the same, the changes that are there are the documentation, the types of documents that we use to record, and some of the things that are required by the SASMS (South Africa School Administration and Management System).
>
> Mr Alpha: To be honest to you, the implementation of the Technical Maths meant to revisit some topics that were no longer taught. They brought sections back that was taught previously. The newer educators who are joining the system would find a challenge, unfortunate for those educators in those schools. But fortunately for me, the educators that we have on the system have either been trained with those sections or they have done those sections. Like if you take an era back where learners were doing Mathematics at school, some didn't do geometry. So now with the introduction of geometry back into the syllabus, those teachers who did not do it will find it challenging to teach it. But if you take someone who did Mathematics and did geometry at school and university, they will manage. However, we have lots of workshops that take place on an ongoing basis for Mathematics. We have quite a bit of support.
>
> Mr Gamma: The challenges that we have encountered is the type of learners that we receive or that choose to do Technical Mathematics—some of the learners think TMAT is easier than Pure Mathematics. When you look at it carefully, it is not actually easier. The learners that take the subject with the aim of an easy pass, you find that they are struggling. They find that it is the same thing that is done in Pure Mathematics.

Mrs Beta considers the implementation of TMAT to be no different to that of Mathematics, while Mr Alpha and Mr Gamma alluded to the fact that there are some changes in the content that need to be taught and referred to the calibre

> of learners doing TMAT. In thinking about those changes, the DHs are drawing from internal forces in terms of biographical forces and programmatic forces. For example, Mrs Beta and Mr Alpha are seasoned Mathematics teachers, and seem to experience no challenge with implementing TMAT in their schools. However, Mr Alpha is cognizant that new teachers might find it challenging, since there are topics they did not learn while at school. While Mr Gamma considers the programme structure to be no different to Pure Mathematics, he is wary that the calibre of learners plays a crucial role in its implementation, and thus institutional forces need to come into play to prepare learners accordingly.

Implementation at the classroom level requires teachers and DHs to have adequate knowledge of the subject so they can teach learners effectively. DHs are a crucial component in managing and implementing the developments regarding curricular changes (Tapala, 2019). Tapala (2019, p. 73) argues that "The issue is that the DHs like all educators are not trained on the new curriculum developments, rendering them helpless when it comes to training and developing their own staff". Regarding their practices in terms of implementing the TMAT curriculum, the DHs had the following to say:

> Mr Gamma: I've got two seasoned educators that teach Technical Mathematics. So they know what needs to be done.
>
> Mr Alpha: Most of the educators in this department right now are seasoned educators, who taught Mathematics; they are capable of implementing Technical Maths curriculum.
>
> Mrs Beta: The teachers that I have are experienced in teaching Mathematics and fit in very quickly; teachers that struggle a bit are teachers that are teaching for the first time.

According to the DHs in this study, seasoned teachers do not need professional development; therefore, in the process of implementing the curriculum, it seems to be up to the teachers to decide what they do in their classroom. While this practice might be considered appropriate in allowing teachers autonomy, it is the DHs who have a responsibility to manage the curriculum. The question is, how are they able to manage it when they are not involved in the implementation? For effective and sustainable curriculum implementation, all stakeholders need to work together; however, it seems that the DHs participating in this study leave teachers to work in silos. In addition, the DHs seem to equate teachers' experience in the field to their competence in the subject matter and pedagogical knowledge. They posit that they only worry about new teachers in the field, suggesting that they only see the need to monitor the implementation of the TMAT curriculum when it is taught by a novice teacher.

The challenge with this notion is that TMAT is grounded more in application, while Pure Mathematics is grounded in the abstract—thus, all teachers, whether seasoned or novice, need to be supported for the effective implementation of the TMAT curriculum.

5.3. Underlying Reasons for How DHs Carry Out Their Roles and Responsibilities

The findings of this study suggest that DHs use different management styles when enacting their roles, which are influenced by the school contexts that they operate in. Staff support and expectations from the school principals, subject advisors, and the DBE inform the management style they adopt in their schools. In essence, the DHs were found to be using a top-down management style and shared instructional leadership (distributive leadership management), trying to find the balance between the two leadership styles, with different forces pushing and pulling them in different directions.

Mr Alpha uses a top-down management style, where he solely manages TMAT in his school. Rudhumbu (2015, p. 106) in Botswana attests that "Traditionally, the role of the academic middle manager has been viewed as transmitters of top management views to the lower echelons of the organization". He enacts his roles and responsibilities by interpreting the curricular information to subject teachers he manages:

> Mr Alpha: I'm also teaching the Grade 11 and 12. So, I'm monitoring Grade 10 and that is the only area that I was teaching, which I'm not teaching right now, that was due to my loading. So therefore, I have to find myself knowing and being hands-on in terms of checking what they are doing. I did the analysis for terms one and two, so I know the results, but that doesn't necessarily speak to their content in terms of the syllabus.

The influence of the considerable pressure brought by the CAPS policy makers, school context, and expected roles of the DH was found to be forces that inevitably shaped Mr Alpha's enactment of an instructional leadership style. He argues that the management of TMAT is not that challenging, but ensuring that learners are performing well in the subject is:

> Mr Alpha: Managing it is not challenging, but what is challenging is to get interventions with educators for learners to pass.

> Mrs Beta's instructional leadership style sought to include TMAT teachers in the decision-making process, and they are active participants in management of the subjects in their department. The DHs' roles and responsibilities in her school were overwhelming, since she managed all Mathematics, Science, Technology, and technical subjects:

> Mrs Beta: In our department, Science, Mathematics, and Technology, we have what we call subject specialists or subject heads. So, I do not take care of all the needs for all the subjects in my department. We have subject heads for Mathematics, Physical Sciences, Life Sciences and so on. So, my job as the DH becomes easier because I have people that are supporting me in running of my roles, I don't just do it on my own.

She affirms that distributing her management role eases her own role and makes the teachers understand the expectations of the school and the demands of the curriculum. Both teachers and the DH work towards the common goal of meeting standards while narrowing their focus on learners' attainment of the learning goals. The DH posits that she shares responsibilities with other teachers and feels that this is helpful in the implementation of TMAT in her school.

> Mr Gamma used both the instructional leadership and shared instructional leadership styles. His management style did not fit into existing management styles, as he was found to be directive and also exercised some flexibility in terms of the extent to which teachers can execute roles in his department. He prefers to take ownership of his DH role, executing most of the management role and leaving out those roles that have no major or significant impact in the implementation of TMAT:

> Mr Gamma: We always talk about continuous professional development and keeping their best by reading up and always reflecting on their teaching methods, and so forth. Also working collaboratively with other educators. . . But at the end I need to ensure that everything is happening because I am accountable. Its my job that will be on the line. I am working closely with the Technical Mathematics subject advisor to ensure that we know what needs to be covered, what is not

> to be covered, and how they [teachers] cover the required content to the best of their ability. I delegate to them [teachers] smaller tasks.

While in theory, Mr Gamma seems to be keen on collaboration, he does not necessarily believe in delegating to teachers what he considers to be critical or what can jeopardise his job. Again, external forces seem to be influencing his decision-making when it comes to implementation and managing his partnership with teachers in ensuring that the curriculum is implemented effectively.

6. Discussion

The results of this study revealed how different contextual forces and programmatic as well as institutional forces influence DHs' practices in the management and implementation of the TMAT curriculum. While some encourage collaboration within their departments and are keen on distributing roles, they are pushed back by the need to secure their job and a lack of expertise in the subjects they manage in their departments.

When it comes to managing the curriculum implementation, the findings show that DHs' practices are influenced by programmatic forces, as they believe that seasoned teachers do not need support and that teachers who have been teaching Mathematics are capable of implementing the curriculum without support. Contradictory to the findings of Sengai (2021), that the DHs encourage teachers to work together in implementing the curriculum, in this study, it was found that the DHs are more inclined towards working in silos, believing that experienced teachers are capable of working on their own. In addition, in managing the TMAT curriculum implementation, the contextual forces are the pushing and pulling forces that influence DHs to execute the roles, as it was evident that they do not exercise autonomy and institutional forces but rather focus on meeting the expectations set by subject advisers. We therefore concluded that DHs' practices are driven by the need to comply rather than sustainability.

These findings resonate with those of Metcalfe (2015), who posits that in the quest to monitor curriculum coverage, DHs were mainly carrying out a tick-box exercise. Based on these findings, we concluded that while DHs are obligated to adhere to contextual forces; as immediate curriculum advisers in the school they need to draw more from internal forces, that is, institutional and biographical forces. They need to foreground working with teachers, both seasoned and novice, and utilise resources within their institution to ensure effective curriculum implementation.

7. Conclusions and Implications for TMAT Curriculum Sustainability

This study was conducted with three DHs who were among the first cohort that was involved in piloting the implementation of the TMAT curriculum. TMAT has now been scaled to other schools, highlighting the need for a larger study to understand the practices of DHs and the factors influencing these practices. Identifying areas where DHs require capacity development will enable the implementation of targeted interventions to support and ensure the sustainability of the TMAT curriculum.

To answer the research question about forces influencing DHs in enacting their roles and responsibilities to implement and manage TMAT, this study concludes that DHs' practices are influenced by external forces, the lack of confidence in novice teachers, and the belief that seasoned mathematics teachers are best suited to implement the curriculum. This suggests that for TMAT, there is a push for TMAT to be taught in the same abstract nature as Pure Mathematics, ignoring the application aspects. Being influenced by external forces suggests that DHs were not implementing the curriculum according to their school's needs, and their curriculum monitoring was mainly for compliance purposes, devaluing their autonomy and school setting. To answer the second research question of how they

perceive their practices, this study concludes that they perceive their practices to be about curriculum coverage and meeting the demands of external forces rather than effective curriculum oversight.

Drawing from the findings, we argue for the need for the continuous professional development of DHs when it comes to enacting the roles and responsibilities for effective curriculum implementation and management. Since TMAT has been rolled out and implemented in all the schools that offer technical subjects, it is imperative that the Department of Basic Education, Teacher Training Institution, and the schools that offer TMAT collaborate ensure that the existing DHs are continuously receiving professional development as well as in-service teachers teaching the subject. In addition, it is imperative that the graduates be inducted into the teaching of TMAT so that the DHs are not sceptical about their knowledge competencies of the subjects. Since TMAT was introduced six years ago, we, therefore, recommend a large-scale study to explore the evolvement of DHs' practices and also a comparative study with other countries to explore best practices to ensure the sustainability of the TMAT curriculum.

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Article

Teachers, Learners and Edu-Business Co-Constructing Mathematics Curriculum Implementation: An Insider's Lens in Cross-Phase Longitudinal Research

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Abstract: This paper draws on a five-component, large-scale, longitudinal and cross-phase mathematics curriculum implementation study in England from the vantage point of an insider to overlapping school, policy and edu-business actor communities. It probed those actors' emergent co-interpretation of, and response to, a new mathematics curriculum in England, analysing the ways in which edu-businesses, teachers and learners mediate mathematics curriculum policy documents through their own interpretations and schema. The combination of common 'classroom-close' research tools supported synergies of cross-phase and longitudinal lenses. The paper contributes an enhanced conceptualisation of inter-actor influence, a theorisation of learner as policy actor, and an understanding of constraints on mathematics policy-driven change at teacher and learner levels, including challenges to communication of intended curriculum policy, across phases of schooling. The approach appears fruitful for analysis of the experience and mediation of mathematics curriculum policy by key policy actors.

Keywords: mathematics curriculum; policy actor; edu-business; institutional ethnography; cross-phase; classroom-close research

1. Introduction

The study of education policy implementation has been of considerable interest in recent years, with a variety of theoretical and methodological approaches adopted. Policy communication takes shape via a nested range of levels, texts and actors, with 'iterative refraction' [1] of key messages at successive stages of communication. Here, I focus on (mathematics) curriculum policy, which is assumed to have the potential both directly and indirectly to shape learning outcomes. At each layer of communication from curriculum writer to learner in the classroom, there are complexities of relationships and meaning-making shaped by different curriculum actors [2,3]. As those meanings take shape, there is interaction between different actors, often mediated by curriculum-interpreting artefacts [4–6]. This paper focuses on evidence from the study of such phenomena, using approaches we frame as 'classroom-close'. The context is one where significant curriculum interpretation roles were also played by one 'edu-business', conceptualised as private for-profit and not-for-profit businesses with power and agency in the field of education [7]. Throughout the paper, we understand the school mathematics 'curriculum' to exist in at least three forms—the intended curriculum as captured in authoritative written documentation, implemented curriculum at successive levels of communication, and learner-attained curriculum—as adopted by Mullis et al. [8].

The curriculum policy literature addresses high-level inter-institutional or intra-school management policy enactments (e.g., [2,9] respectively). There are widespread small-scale classroom-focused policy implementation studies, but related large-scale studies usually stop short of a classroom or learner focus. However, the study of larger-scale, cross-phase curriculum policy enactment, at an individual teacher or student level—arguably the key intended objects of curriculum policy if that is to meet its goals—is under-represented in

the literature. This paper analyses, and argues for the validity and contribution of, the approaches we adopted in a five-component cross-phase, longitudinal implementation study as quantified in below. The context was the implementation of revised mathematics curricula for learners aged 5 to 18 rolled out in England from 2014, with first assessments in 2017 for learners aged 16, and in 2019 for learners then studying calculus-rich mathematics pathways pre-university. At all levels, there was an enhanced focus on mathematical reasoning, problem-solving and communication.

English school education is typically organised as primary (years 1–6, ages 5 to 11) and secondary (ages 11 to 18, or 11 to 16 plus 16–18, in institutions here generically named 'schools'). In-school 'mathematics leads (MLs)' are key school-level influencers. Associated with the official codification of intended curricula (the written 'National Curriculum' and A-Level 'specification') were a host of other artefacts reflecting curriculum interpretation. These include curriculum resources and those relating to assessments at ages 11, 16 ('GCSE Mathematics', offered in two overlapping 'tiers') and 18 ('A-Level Mathematics'), both entirely assessed by written timed papers. It is worth noting that OECD [10] characterises both GCSE and A-Level Mathematics as unusually demanding for the relevant cohorts. The wider context over recent years is one of marketised assessments at ages 16 and 18 that are 'high-stakes'—for learners, teachers and schools—and assessments at age 11 that are nationally produced and relatively high-stakes (for schools and teachers, but not learners). Assessments at 16 and 18 in England operate within a regulated but competitive marketplace; teacher and learner texts and other classroom-related materials are also marketised, but not regulated.

The focus-revised mathematics curricula are intended to support mathematics learning and progression via a renewed focus on deep conceptual fluency, on mathematical problem-solving and reasoning, and on rigorous communication of those processes. It is known that teaching for such aspirations is challenging (e.g., [11]). Associated policy discourses in England have privileged economic good—for individual and society—and employed discourses of equity of opportunity and international economic competitiveness. Such discourses underline the centrality of mathematics education in much education policy work across the globe. Education policy in England involves a wide range of policy actors, including teachers and a variety of edu-businesses, as analysed by Ball and Junemann [12]. Such policy roles exist in all education systems, but placements and profiles relative to central structures vary. Actor roles and interactions, and the use of power, in mathematics education networks remain oblique; this paper contributes to filling that gap.

Below, we discuss theoretical and methodological aspects of the study, and the revealed affordances of those. We draw on learner as well as teacher accounts and perspectives. Through much of the study, edu-business perspectives are largely inferred from those communicated in their curriculum and assessment materials; we show that these both directly and indirectly impact what mathematics is made available in the classroom. We also report edu-business responses to emerging research findings. We demonstrate that the adopted approaches can expose the policy roles played by learners aged 5 to 18, the relationships between learner and teacher perspectives, and the interplay of both with curriculum as interpreted by 'third-space' actors such as edu-businesses. Importantly, this study is able to illuminate how such phenomena might compare across phases of schooling. These approaches also contribute to the development of a theorisation of learner as policy actor. They also support additional evidence of the communication and validity challenges associated with mathematics education policy-driven change.

The underlying study was funded by one edu-business, nationally influential as the market leader in both mathematics curriculum materials and mathematics assessments (and related resources) at GCSE and A-Level. The edu-business also provides teacher development and is proactive in policy engagement. The funder's stated purpose, developed in conjunction with the national DfE and assessment bodies, was twofold. First, they targeted evidenced development of their curriculum and assessment materials for mathematics learners aged 5 to 18 so as to better support curriculum intentions for learning,

by identifying resource- and curriculum-specific issues as teachers and learners came to work with new curriculum expectations. Second, they sought a better grasp of curriculum reform processes, from policy statement to classroom implementation to learner learning, for both business and wider benefit. Initially, the five components of the study, together spanning mathematics education from 5 to 18, were undertaken and reported separately in, e.g., [13–16]. In then taking an overview in, for example, [17], we have started to interrogate the synergies of the five components. In this paper, we set out the overall study context and indicate the nature of some of the findings from cross-phase analysis; elsewhere, we begin to theorise some of the interactions in more detail [18].

Throughout, the author was employed to lead a team of external-to-funder subject phase-expert researchers and was responsible for all aspects of the research, including the framing of research questions from the broad goals communicated. For each component, we asked, '*How is the intended mathematics curriculum being enacted and experienced in the classroom, with what funder curriculum and assessment resources, and why? What is the impact on teacher and learner learning?*' This paper focuses on some of the answers to related questions the research team then asked when considering the study as a whole, catalysed by emerging data:

> *How do teachers, learners, and one edu-business (the funder) co-construct curriculum implementation? (and how does that vary across school phase?)*
>
> *In what ways, if at all, can learners be conceptualised as policy actors?*

Importantly, because of the market leadership of the funder's products in England, findings have broad applicability across English mathematics education, while the potential of the approaches generalises to mathematics curriculum implementation studies in any context, and possibly to work in other parts of the school curriculum. However, in England as elsewhere, mathematics remains a key curriculum focus for individuals, schools and nationally, with associated both affordances and constraints [19].

The study contributes conceptualisations, contrasts and synergies of teachers' work with curriculum and assessment resources in relation to curriculum planning and realisation. It also illuminates a range of curriculum-agentic roles of learners, and the interplay of those with teacher and funder resources.

2. Theoretical Approaches

Ball et al. [2] conceptualise in-school policy-related interactions as often 'complexly configured, contextually mediated, and institutionally rendered', and that was certainly our experience, as illustrated below. However, we sought emergent commonalities, as well as variations, across different school contexts. We used two complementary theoretical approaches to conceptualise cross-phase policy implementation processes, analysed below. These framed research design, analysis and interpretation, and together contributed a range of insights.

2.1. Policy Actors

We worked with the [20] extended version of Ball et al.'s [21] policy actor typology (Table 1, below, exemplified for this context). Such typologies support a shift away from conceptualisations of the teacher as a passive implementer of policy [22]. Related policy networks are complex, but our focus was on the edu-business and teacher, as well as potentially learner, as policy actors, with edu-business approaches mediated through their curriculum and assessment materials. All participants were working with those: the funder in their production, teachers usually employing those in their lesson planning and often also directly in their teaching, and learners in a variety of ways, usually including direct interaction with a subset of the curriculum materials. The funder had already interpreted the intended curriculum in important ways and so, at the classroom implementation stage, functioned as a curriculum 'narrator'—but also, in an ethnographic sense, as a curriculum-maker. Similarly, classroom teachers in their approaches to curriculum texts, and the

relation of those to curriculum planning and enactment, were also policy actors. They might potentially be active 'curriculum-makers' who are policy 'narrators' or 'enthusiasts', or might instead harness the received curriculum according to context: predominantly 'translators', or a variety of constructive 'critics' [20]; others might function as curriculum 'survivors' or 'receivers'.

Table 1. Policy actor typology, exemplified in this context (adapted from [2] p. 49); * extended in [20].

Policy Actors	Policy Work
Narrators	Interpretation, selection and enforcement of meanings, e.g., edu-business interpreting curriculum priorities to communicate those in teacher guides.
Entrepreneurs	Advocacy, creativity and integration, e.g., MLs suggesting ways in which curriculum materials can be adapted to build on a teacher's particular pedagogical strengths.
Outsiders	Entrepreneurship, partnership and monitoring, e.g., school governors interrogating classroom curriculum work in relation to attainment.
Transactors	Accounting, reporting, monitoring/supporting, facilitating, e.g., Headteachers critiquing impact of curriculum resources.
Enthusiasts	Investment, creativity, satisfaction and career, e.g., class teacher making substantial and sustained investment in getting to know curriculum resources.
Translators	Production of texts, artefacts and events, e.g., ML selecting from resources to craft a 'scheme of work', or class teachers planning the incorporation of published materials into their lessons.
Critics	Union representatives: monitoring of management, maintaining counter-talk OR * engaging in positive critique, enhancing enactment for the context, e.g., teacher analysing aspects of resources that do not work well for some learners, and re-crafting those.
Receivers	Coping, defending and dependency (may be inexperienced), e.g., time-pressed classroom teacher struggling to keep track of rapidly changing new assessments.
Survivors	* (May be experienced, weary). Dampening local policy aspirations, e.g., change-weary teacher looking to make minimal use of new materials while apparently conforming to school expectations.

In these 'policy actor' terms, it is tempting to identify learners as 'receivers' of curriculum, often coping and depending—on texts, including teacher narratives—and subject to popular hegemonies of the school mathematics experience as an imposed, and received, pathway to high-stakes assessment. But in the context of our emerging data, we moved to perceiving learners also as potentially 'survivors', or active 'translators', making new meanings underpinning their own sense of curriculum as experienced, or as a variety of constructive 'critics' or even 'enthusiasts'. Adopted curriculum roles can change over time, especially as teachers come to engage with a new curriculum; learners, potentially active policy actors as they engage with curriculum meaning-making with appropriation of authority and agency, may also change their adopted roles as they mature in their relationship with the curriculum and with their teachers.

There is, though, a challenge in accessing patterns of relationships between texts, the social dimensions of policy as practice, and institutional relationships and discourses, as well as their development and evolution: the study findings, consistent with much of the implementation literature, suggested that the associated roles and processes were highly contextualised and socialised, often at least partly bottom-up and emergent. They were frequently also apparently non-deterministic, reflecting the fundamental complexity of classroom work.

Working within our qualitative, exploratory and longitudinal approach, we found that further methodological conceptions were needed to address participants' apparently

idiosyncratic perceptions of the curriculum meaning-making and agency available to them, particularly in relation to impact of the variety of texts available. For these purposes, we adopted some of the conceptual tools and ontologies of institutional ethnography [23], which seeks to explain how social action in one 'world' (here, school, department or classroom) can be understood with the lens of another—eventually, that of policymakers at national level, but at an intermediate level, that of researchers or edu-businesses such as publishers.

2.2. Institutional Ethnography

In an institutional ethnography paradigm, interview/focus group data, but also accounts of classroom observations and of texts, are conceptualised as subjectively co-constructed by researchers with participants through interactions with and in the field. They address actions, processes, meanings, goals and contexts of everyday worlds *from the standpoint of the participant* [23]. This seemed particularly suitable for our goal of understanding subjective (co-)construction of enacted curriculum. Such research requires immersion in the focus context, and researchers' backgrounds significantly impinge on such co-constructions, even though efforts were made to 'step outside' when interviewing, to validate use of common semi-structured data collection methods within each component of the study, and to ground interpretation in the data. In this case, the author/lead researcher was very much an 'insider' to the overlapping communities of mathematics education policy as it is implemented from formalised curriculum to edu-business interpretation to classroom enactment: she was a very experienced classroom teacher formerly contributing to curriculum material and assessment production for the funder edu-business, and is still involved in central policy work. We argue that this is a considerable advantage for institutional ethnography work.

The institutional ethnographic approach progressively informed design and data collection decisions, and eventually, analysis of power and influence relationships within and beyond schools, including via Smith's [23] approaches to texts in action. Institutional ethnography aims to build accounts of ruling relationships 'from below' and 'look out beyond the everyday to discover how it came to happen as it does' [24], probing connections between individuals and social structures even as participants contribute to those structures. Importantly, though, it is also committed to going beyond any one individual's experience.

Within institutional ethnography, we asked in particular, the following questions: Which are the (policy, curriculum, assessment or professional discourse) texts that carry authority and meaning to teachers or learners, in relation to the mathematics curriculum? How are they selected, and their discursive and interpretive natures understood, harnessed and appropriated by actors in the school mathematics policy system? We included here the edu-business curriculum and assessment materials that were the focus of the studies, but also sought curriculum-related texts from other sources, for example, department-level 'schemes of work', school-level 'learning priority' documents, or social-media-recommended websites that learners harnessed—together with the discourses evidenced. The approach sought to explore the ways in which teachers and learners used these different resources for a variety of curriculum-related purposes, their perceptions of the related choices and agency available to them, and how that changed over time. In particular, Smith's [23] notion of 'inter-textual hierarchy' was used to tease out from our data which texts were privileged, for what purposes and when, and the surrounding discourse. We conceptualised teacher or learner interaction with text as agentic and mutually developmental with text, as in [25].

I outline below the tools developed to work within these theoretical paradigms. I then indicate the range and scope of findings, and especially the contributions in relation to the ways in which teachers, learners and edu-business act to co-construct curriculum.

3. The Study: Methods

We used a combination of well-established methods as outlined below: curriculum-related document analysis (representing edu-business interpretation of curriculum where appropriate), semi-structured interviews with teachers, semi-structured whole-lesson observations, learner focus groups, and surveys completed by teachers and by learners. These were adjusted as considered necessary, and ethical, during the global pandemic of 2020–2022, with learner focus groups and lesson observations replaced by online surveys of post-primary learners in 2021 and part of 2022. Sample research tools are given in the Supplementary Materials. We conceptualise our approach as 'classroom-close': it has much in common with the 'close-to-practice' research theorised by Wyse et al. [26], but the research questions were determined by the funder and central policy actors, rather than by teacher practitioners, and the impact was designed to be focused on understanding and development of support, including policy support, for valid curriculum enactment, rather than directly on teacher practice. Data, though, were derived from full-lesson observations, from materials closely related to classroom teaching and learning and its assessment, and from teacher and learner reflections on both those. Data were synthesised and applied in a novel way and at a novel scale for this field, supporting synergies of both cross-phase and longitudinal implementation research.

Research tools were edited iteratively, and for phase- and age-appropriateness, within a broadly common design for all five components of the study. The author led all fieldwork (by five phase-knowledgeable subject-expert researchers external to the funder), and also designed and piloted all aspects of the study. Data collection sampled year groups from all phases of English school education and is summarised in Table 2 below. Analysis supported the identification of similarities and differences within and across schools, as well as across phases. The reported five components A–E were longitudinal, each over at least two years of early curriculum implementation; researchers interacted with participants in each school at least termly, including, except during the pandemic, via full-day classroom-close visits each Spring term that included full lesson observations of participating classes. The study serves to evidence developing teacher and learner perceptions of the experienced curriculum, individual, social and institutional agency, authority, roles and responsibilities, and the meanings received from a range of curriculum-related texts.

Four funder-internal efficacy researchers supported the study administratively and carried out some (validated) analysis and writing. This 'internal–external' (to funder) researcher partnership supported access to detailed knowledge of resource and assessment material development, structure and content, as well as to additional documentation not in the public domain, although it also brought ethical tensions that are addressed below. Funder product teams were involved at least annually to consider and inform proposed research tools and emergent findings. We also engaged national Department for Education, inspector and Ofqual (national office for qualifications) personnel via summaries of annual findings, and they each contributed to further development of tools to ensure aspects of implementation of particular interest to them were probed. The study approach therefore offered symbiosis of a range of policy actor roles inside/outside policy-active institutions, bringing concomitant threats as well as advantages [27]. In particular, the inter-actor influences discussed below were also impinged on by relationships with the researchers. The main tool for addressing related tensions was a conscious and frequent application of reflexivity, in parallel with the above actions. However, the author's 'insider' roles also offered an existing community immersion that particularly supported institutional ethnographic lenses.

We sought a contextualised and embedded understanding of how the new curriculum was being realised and the critical influences on that. Standardised or national assessment quantitative learner progression data were used to locate the sample within national attainment and progression patterns, and identify any demographic or prior attainment links with apparent constraints on, or choice of, curriculum enactment. All interviews, observations and focus group conversations were semi-structured in order to optimise

balance between validity of response or observation to support grounded analysis and relevance to research foci. Our approach offers an opportunity to develop a grounded, rich understanding of edu-business, teacher and learner practice and perspectives, together with their inter-relationships, and to connect theory with both policy and practice.

Table 2. Study timing, participants, and data collection (*: pandemic-constrained).

Study Component	Year Group/Age	Schools (2 Classes Each)	Focus Group Students	Teacher Interview Transcripts	Pre-/Post-Study Class Progression Data	Lesson Observation Notes	Useable End-of-Year Student Surveys	End-of-Year Teacher Surveys
A: Primary Cohort 1 (2016–18)	Y1–2 (Age 5–7)	9	70	68	17	18	-	-
A: Primary Cohort 2 (2016–18)	Y5–6 (Age 9–11)	9	72	68	18	18	-	-
B: Primary Cohort 3 (2019–21) *	Y1–2,3–4 (5–7,7–9)	24	216	96	54	54	-	96
B: Primary Cohort 4 (2019–21) *	Y5–6	21	168	84	42	42	-	84
B: Primary Cohort 5 (2021–22) *	Y2,4,6	9	86	19	-	18	-	18
C: Secondary Cohort 1 (2016–18)	Y7–9 (Age 11–14)	29	138	136	28	44	785	-
C: Secondary Cohort 2 (2016–18)	Y10–11 (Age 14–16)	31	164	144	34	48	845	-
D: Secondary Cohort 3 (2016–18)	Y11–12 (Age 15–17)	20	193	60	25	-	795	-
D: Secondary Cohort 4 (2017–19)	Y11–12 (Age 15–17)	21	192	61	23	-	807	-
E: A-level Cohort 1 (2017–19)	Y12–13 (Age 16–18)	11	156	50	50	22	432	50
E: A-level Cohort 2 (2018–20) *	Y12–13 (Age 16–18)	12	144	54	54	24	420	51
E: A-Level cohort 3 (2019–21) *	Y12–13 (Age 16–18)	12	71	-	22	14	640	65
E: A-level cohort 4 (2020–21) *	Y12 (Age 16–17)	12	-	-	13		394	43
Total		220	1670	840	380	302	5118	407

The samples recruited were stratified by several school characteristics known to impact teaching and learning, in proportion to their occurrence nationally: catchment area type (city, urban, urban fringe, rural), inspection assessment grade, level of historic learner performance, socio-economic and demographic composition of learners, and governance type. However, participants opted in, so were necessarily confident to welcome researchers into their lessons and sufficiently relaxed to devote time to research processes: they are therefore unlikely to be the most stressed in the system. This appeared not to fundamentally affect the measured representativeness of the sample, except post-16, where the achieved sample was skewed towards slightly higher-attaining school/college mathematics cohorts.

Small-scale pilots for the first four components listed were used to calibrate age-appropriate approaches to informed consent and to research tools. For other components, initial tools were piloted informally with a small group of learners and/or teachers comparable with those targeted. Overall, data collection was as in Table 2.

The main study for primary cohort 1, for example, followed classes of learners through years 1 and 2 and featured termly audio-recorded and transcribed teacher interviews, the second of which each year drew on the lesson observation made. As far as possible, the study followed a whole class and its mathematics teachers over at least two years. Class teachers and school maths leads were interviewed annually in the Autumn with a focus on individual, contextual and high-level implementation data collection. The requested focus for the Spring whole-lesson observation was a renewed focus of the new curriculum, such as the development of mathematical problem-solving or reasoning. This was potentially a 'telling' such opportunity [28], namely one which, while remaining deeply context-dependent, might most clearly expose to observation curriculum implementation challenges and opportunities. Teacher interviews probed the range of the research questions. These were revisited in successive interviews so as to discern development of approaches and attitudes over time, as participants adjusted to the new curriculum.

We used semi-structured tools, including observation schedules, iteratively developed through the study. Post-observation, four–six learners of mixed gender and mixed prior attainment, drawn from the class observed, took part in a focus group, also audio-recorded and transcribed, that probed the typicality of the lesson observed, the use of resources and teaching/learning activities seen, and other aspects of the received curriculum and their preferred approaches to learning. We also interviewed the class teacher, seeking reflection on key aspects of enactment.

Towards the end of each year, we either interviewed teachers again (where context supported that) or used an online survey to solicit their perspective on emerging implementation and resource/assessment use. We also surveyed all older classes at that stage. Kelley et al. [29] suggest that surveys, particularly online and of participants not well known to the researcher, can result in superficial responses, so we were cautious in their use. Quality of learner responses did indeed vary, with a ~2% wastage rate; however, many were mature, reflective and detailed. Where we used surveys for teachers, because of individual, school or wider constraints, we had already built professional relationships with them, and the typical depth and focus of response was judged high, although obviously lacking opportunity to probe. Additionally, online surveys respected the respondent by allowing choice in timing of completion. Integral response spreadsheets minimised the data handling required before analysis, with processing starting only at the cleaning of data stage: this approach was increasingly adopted through study components, as we gained confidence in the achieved quality of data, and as pandemic pressures shifted the practical and ethical balance of decisions.

Table 2 shows that similar approaches were used in all components, except for Secondary cohorts 3 and 4. For them, the focus was on learner- and teacher-reported experiences of, and approaches to, preparation for GCSE examinations at age 16, followed the next year by exploration of the post hoc reflections of learners on their GCSE experiences, together with their, and their teachers', emerging perceptions of the appropriateness of their mathematical preparation for the post-16 pathways on which they had by then embarked. The overall scale of the data collected from teachers and learners is as shown in the last line of Table 2.

3.1. Research Integrity

Any study of the efficacy of funder-published materials, with funder employees also contributing analysis, is susceptible to ethical issues relating to conflicts of interest, to criticality and to independence of findings, while also offering the advantages indicated above. Analysis and interpretation were achieved through discussion and validation by the range of ten researchers involved, and sometimes, participants too. Ethical approval was obtained from the author's university prior to all study components (RECs 836, 837, 855, 1019, 1108). All participants were repeatedly assured of a wish for direct and constructive critique of the focus funder curriculum, as well as curriculum and assessment resources, and the study has already resulted in significant developments being made to those. It has also supported the development of national assessment scrutiny and guidance. Throughout the studies, there were inevitably ethical judgments to be made—about how far to press teachers not responding to emails, when and how to communicate with schools once we were in lockdown, how to present the variety of teaching, and of the effectiveness of related learning, without judgement of teachers, etc. These have been addressed via reflexivity on the parts of the funder-internal and external leads.

Details of the study approach and, post-analysis, emergent findings were given external scrutiny by academic colleagues in local and national fora, by funder personnel including international senior staff, and by those in national policy fora and Ofqual/DfE teams considering reported outcomes.

Additional actions taken by the lead researcher to support research integrity included assumption of editing control of all surveys, synchronous data access with funder personnel, shadowing with the funder-internal lead of a stratified sample of fieldwork events

for consistency of approach and interpretation, validation of a random sample of transcriptions with recordings, and cross-validation of stratified samples of coding and of emergent sub-themes. High-level field researcher perceptions of implementation were triangulated against data and interpretation, with mismatches occasioning a revisit to the relevant data, or occasionally to one or more participants, for resolution. Similarly, all draft writing was validated by the lead researcher via stratified sampling of the range of data. Assessment data used to monitor the attainment and progression of participating learners, within national profiles, was via age-specific commercial tests standardised nationally, or (where available) national test outcomes at ages 7, 11, 16, 17 and 18: the slight inconsistency inherent in this approach was adopted in order to balance robustness of research outcomes against impact on participants. All interactions with learners due to take national examinations were completed the previous term, for similar reasons. In all reporting, care was taken to ensure typicality of the evidence cited, except where otherwise indicated.

Such research is not without further ethical tensions, for example, between the degree of contact necessary to establish trust and openness, and the disruption research participation can cause to school routines and core work. The intentions of edu-business in funding such research are assumed to be beneficent, on balance—though such research is also likely to increase their market share, and could also be argued to support a high-stakes marketised assessment regime that many would question. We in fact heard little overt criticism of the immense power of edu-business within that regime, but teachers and learners have de facto to work within that context. Although we took care to be as inclusive as possible of learners in our participant schools, we also note that the design of this study marginalises the standpoints of some teachers and learners: all participants were engaging in some way with written curriculum texts, largely from a single, if pervasive, publisher source, so were, for example, unlikely to be the most divergent of curriculum 'narrators'. Similarly, teachers opted in to the studies and with Headteacher consent, so were probably not among those most oppressed by the performative nature of much of England's education system. However, we reiterate that the achieved sample was, as above, broadly representative of the school population except in Study E, though at the school and student level that was largely representative of the (relatively high-attaining) A-Level Mathematics population.

3.2. Approach to Analysis and Interpretation

For the overall study, as reported elsewhere (for example, [30]), we adopted reflexive thematic analysis of the qualitative data [31], to which we then, for this paper, applied the above theoretical lenses so as to make contingent meaning. Consistent with Smith's [23] approach, our analysis for was initially thematic within research questions (here, interactions between any two of edu-business materials, teacher, and learner, and evidence of a learner interacting with policy, directly or indirectly). It also used elements of a constructivist grounded approach [32] to identify common sub-themes, while searching for meaning initially at an individual level. NVivo was used as an analytic tool that supported deeply structured understanding of all qualitative data, with emergent grounded 'child nodes' that offered transparency of structure and of analytic allocation for validation. Such nodes were followed through for organisation of internal reports to the funder. Given our theoretical and interpretative lenses, our analysis focused on depth and range of understanding, rather than a quantitative summary of essentially qualitative data, except for attainment progression data. This paper focuses on three of the emergent themes, as indicative of the potential for a cross-phase analysis: the learner as a policy actor or 'curriculum-maker', the interrelationships between teacher and learner responses to curriculum resources, and edu-business as a co-constructor with teachers and learners of the experienced curriculum. The first of course focuses on learners' perspectives, the second on both learners' and teachers', and the third on interactions of either with the funder edu-business, usually indirectly via their curriculum or assessment-related materials.

With a focus on texts, the study complements an existing international body of research that conceptualises, monitors and interprets evidence of the practice and impact of teachers'

and learners' use of curriculum and assessment-related resources, e.g., [19], as one aspect of curriculum implementation. It does so in collaboration with an influential edu-business, enabling exploration of curriculum actor relationships with texts (understood as including school- and classroom-level discourse) and exposure of a variety of power relations shaping local individual teacher and learner experience. Here, the funder edu-business as policy actor exerted power via both curriculum and assessment materials and surrounding texts—communications to teachers and learners, assessment 'surround' materials, messages conveyed in teacher professional development, etc.—and so they acted as one 'narrator' of curriculum in Ball, McGuire and Braun's [2] terms.

Oates [33] highlights the desirability of 'curriculum coherence': deep alignment of all parts of a curriculum system, including the written intended curriculum, the enacted curriculum, resources, teacher capacity and teaching approaches, assessment, etc. He identifies a particular role for curriculum materials in communicating and interpretating central curriculum intentions, for both teachers and learners, and the findings below point to a similar role for assessment materials, with the funder edu-business playing a key role in both. One obvious consequence of any lapses in coherence is that users of texts have to assign inter-textual hierarchy: examples of this are outlined below, though we also argue that an edu-business developing both curriculum and assessment materials is in a relatively strong position to support curriculum coherence.

4. Indicative Findings

To illustrate the contribution of the methodological approach to a policy field, some of the areas in which a distinctive contribution has been supported are outlined below. In quotations, 'By6L1', for example, refers to Study B year 6 learner 1, 'DT3' to study D teacher 3, 'Cy8LO' to study C year 8 lesson observation, etc. Almost all quotations are from mid-year focus groups (or, for teachers, interviews), since those often offered the most direct communication of participant views; other sources are indicated as 'LO' as above, or as 'survey', or with documentary source. We address each of the selected emergent themes in turn, referring to cross-phase evidence patterns as appropriate: there were a variety of cross-phase commonalities and differences identified, together with variation within and across schools. At a high level, we show how the longitudinal approach exposed issues not only for the inception, but also for the sustainability, of curriculum system coherence. We evidence ways in which the habits, skills and dispositions of teachers *and* learners interacted with the focus curriculum and assessment materials over time, including as those materials changed. Together, these framed the emergent meaning-making and practice at classroom level.

4.1. The Learner as Curriculum Policy Actor and Curriculum-Maker

School learners are commonly conceptualised, at least implicitly, as passive 'receivers' of curriculum, and our data supported that in some classrooms. However, the vast majority of our participant learners of all ages (at least 90%) appeared willing and constructive (though sometimes constructively critical) participants in their mathematics education. Learners very frequently made choices among the texts available to them at some point in the lesson observed. Even in primary classrooms, we almost always observed at least some learners choosing between, and selecting from, teacher or class discourse, front-of-class slides, textbook or practice book presentation, or peer learner, as their first external reference point for independent work. For almost all sample learners to around age 15, the dominant text, and source of mathematical authority, appeared to be teacher discourse rather than curriculum or assessment materials, though in at least 70% of focus groups across studies, learners identified particular features of written texts that spoke to, engaged and excited them:

> *I like how you can choose, . . .what method you want to do, and explain why. If someone (in the book) says something like this, and then someone says something like that, you can choose the easier method to work out the answer (By4L1);*

Always after the challenge there tends to be a Reflect. I think they're very useful, because once you've done your work, sometimes I feel like I need a bit more thinking about it so I don't just forget about it (By4L2);

I really like the Challenges. They're quite fun. They're like, although sometimes it's hard to explain things, I actually like quite a bit of a challenge (By4L3);

The Reflect is the hardest one so that's why we do it together (By2L1); And sometimes the Challenge is really hard too (By2L2); Yes, because Challenges are meant to be very hard (By2L3).

As learners approached GCSE examinations at age 16, though, and particularly in Spring of year 11, assessment texts and narratives came to dominate:

A small question with a lot of marks, so it gives you a hint it's going to be a difficult question (DL1);

So now that, because I'm getting 50% on most of (the papers), and that's a 4, so I know that if at least I've done half of it, then I've got a 4, which that means that that extra for a 5 is kind of just that 10% more (EL1);

I find I just like to build my confidence. Because if I'm struggling with one of these (early questions), I'd be thinking, how am I going to pass if I'm struggling with the first few? (CL1).

This was often abetted by teacher performativity [9] approaches, evidenced in both classroom observations and teacher interviews in both year 10 and year 11:

We think that's the way to go to just start preparing them a bit more for common questions (DT1);

I'm trying to get them to get the first 10 questions, I'm trying to get them to get them just 100%, they make some stupid mistakes, it's unbelievable, so I'm just trying to get them to try not to rush it, I'm still trying to do that mark a minute thing (DT2).

This led more reflective learners (at least half of focus group participants, across demographic groups) to adopt a 'curriculum narration' role, making active choices among a range of examination-focused resources—but also often adopting 'constructive critique' [20] of curriculum, resources and enactment:

I think the exam pressure sometimes gets to a learner, so if you kind of have things in bold, it. . . I've done many questions where I think I'm answering the question, and I look back at the question, and I've answered something completely different. So I'd have to cross it out and restart. So I feel like bold would draw my attention better to what I need to do (DL2);

If you want the maths paper to be like strictly about maths and learn how to do it, then bolding it. . . But if you want it to be more about problem solving and like picking up what you want to do then obviously not (DL3);

There's 160 people in our year, and they're putting us all under the same lot of pressure to do one style of exam. . .And everybody has different learning styles and they want 160 people to sit the exact same exam and get the grade. . .if they're sitting us all in the same exam people will thrive off certain questions (EL2).

There was, though, considerable resistance among mid-to-lower-attaining GCSE learners to some examination-linked experiences that were perceived as oppressive and 'miserable', particularly in retrospect:

I put so much pressure on myself to pass and I was just stressing the whole time. I need to pass, I need to pass. It's just not a nice situation (DL4);

Years 10 and 11 were just miserable: there is so much to know, and I wasn't succeeding with the papers, it got more and more stressful so in the end I tried to avoid them (DL5);

Some of the worded questions in the exams don't always relate to experiences I've had. I can do the maths in lessons, but some of the exam questions seems designed to wrap that up so that I stress about what maths to use and then I can't show what I can do (DL6).

A few made suggestions as to how to allow learners to show more of what they were capable of; one was to include course work *'maybe like having some coursework like added to the mark' (DL8)* or *'more multiple choice questions' (EL3).* About 50% of learners in affected cohorts could articulate advantages of engaging with authentic or familiar, realistic contexts, although a further about 25% were sceptical about such attempts:

Like the fish tank with the sand in it and then you have to convert the volume into meters, I think that's quite a good question to have in there. Because that's sort of almost a real-life situation, not many of the questions are (CL2);

Some of the questions are just silly, and that winds me up: whoever heard of someone buying 150 watermelons? (CL3).

A range of learners commented on the value to them of provided, or self-initiated, diagrams:

I think the diagrams help because, for me, if there's not a diagram I always draw . . . a diagram, or . . . a graph or something, so that . . . I understand it more than just the words. . . It's hard to visualise in your head without a diagram (CL4);

I think maybe for certain questions more diagrams could be useful because I know personally that sometimes I work better when I have the thing in front of me. For ones like probability where you have a certain amount of coins or balls . . . having that in front of me would definitely help. But for others, I think it's fine (CL5).

However, they were also impatient of diagrams that might mislead:

There was a question yesterday in our paper . . .it was like a diagram but, and the angles were different sizes, but the smaller angle was bigger, the big angle smaller. But it made me think I'd done it wrong, because on the sides of the diagram it didn't say whether it was drawn accurately or not (CL6).

Importantly, learners who had experienced, or were experiencing, a range of other approaches to summative assessment during the pandemic were quite sophisticated in their analysis of the affordances and constraints of those, again functioning as curriculum narrators:

Perhaps use a mix: exams are the only really fair way, but they could perhaps have a contribution from teachers, to incentivise hard work and give students confidence that not everything depends on that eon day (EL4);

(Centre-assessed grades were) completely unfair in some ways, there should have carried on with exams, in a socially distanced manner or invigilated online—grades have to be earned not given, if they're going to be respected (EL5).

Such issues are further explored in [30]. More broadly, the majority of 14–18-year-olds and up to about 30% of younger learners made relatively independent and selective use of the funder's (and a range of other) resources. These included freely available digital curriculum apps and online fora they chose for complementary mathematics support or enrichment:

Different online resources are useful for different topics, depending on how secure I am, I can choose what works best to improve (CL7);

Looking back, I'm now much more selective of whether I use apps or an online room: there are places that are good for consolidation and others that really help if you want to know a bit more, beyond what the exam needs (DL4);

I can rewind videos teachers record and learn at my own pace' (Ey12L6);

I created a study group where we would all join a Teams meeting and do maths questions and topics. . .. (That) helped all of us in revising and maintaining productivity (Ey13, L7).

These students, who crossed demographic and prior attainment boundaries although they were dominated by higher-attaining students, could be considered to act as 'narrators' of curriculum. Some students also functioned in a variety of other policy roles: the weakest-attaining quartile often functioned as 'survivors' or 'critics', but another (approximately) 10%, particularly those supported to attain highly at either 'tier' of entry, functioned as 'translators' or 'enthusiasts' in Ball et al.'s (2011) terms. Importantly, analysis of learner-sourced data in each element of the study led directly to changes in funder curriculum and assessment materials. It also fed into evidence underpinning central (Ofqual) guidance that developed over time and resulted in key changes to GCSE and A-Level mathematics papers. Details are given in a range of funder-internal reports, but changes included the use of more space and simpler language in younger learners' books; enhancement of the quality and quantity of problem-solving in resources for 11–14-year-olds; accessibility and structure of examination papers at age 16; and an enhanced range of support resources for A-Level Mathematics preparation. An additional, strategic and cross-cutting impact has included greater central exemplification of intended curriculum changes, and central addressing of specific teacher development needs identified [17]. Such changes are especially important in a system where direct use of student voice has historically been accorded only a very low profile.

4.2. Teacher and Learner Lenses Compared

Throughout phases, lesson observation, teacher interview and student focus group data taken together showed fairly close correlation between teacher discourse and that predominantly presented by their learners, although the ~5000 learner surveys showed small-scale (up to about 10%) contra-discourses evident in almost all study classes, underlining the importance of large-scale as well as more in-depth small-scale data collection, and possibly also privatised, as well as shared, narratives. Teachers who embraced curriculum intentions at the classroom level almost always sought resource support for doing so, but functioned as active agents in developing enactments for their classes:

(The teacher) took some of the structure out of the question, to enhance the challenge; she later explained she thought they could cope with a harder question than in the book, and wanted them to experience getting stuck—as well as having more than one possible solution (Cy8LO).

Their learners largely talked positively about challenge, identifying aspects of curriculum/assessment-related materials that supported that, and often offering proposals for further resource development:

The book is pretty good, but the question we had today was harder than what was already there. It took several of us to work out how to do it, but I think that's good—the book does encourage you to work together, but perhaps if there were starred questions that were particularly challenging, that would be even better. Those can be really rewarding, but it's also good if you feel OK if you can't do it' (Cy8L8).

Other teachers, in contrast, sought ways to 'survive' in the new curriculum and that too was largely mirrored in their learners' discourse—in all phases:

Since the new practice papers have been published, we don't do the reasoning questions in the book. In principle, I was in favour, but my students found them really hard, and if they're not going to be in the exam, it's not worth the effort (Cy10T1);

We don't do all the questions in the book, because the reasoning ones, and some of the problem-solving ones, are too hard (Cy10L9);

A lot of learners are finding it incredibly difficult to access the material. So I give them more structure and hints, and train them to recognize where they can pick up a mark or two—we don't attempt the real question (Dy11T3);

> *We're doing OK: Miss tells us what words to look out for, and where we can pick up enough marks. We can usually get enough (for the target grade) without having to go near the last few questions, provided we are careful to get what we can earlier in the paper (Dy11L5).*

Analysis of teacher and learner talk, as above, exposed the hegemony of assessment materials, particularly at upper secondary level, and it was here that the alignment of teacher and learner responses to curriculum change was most obvious. For secondary cohorts 3 and 4, if teachers adopted a procedural approach with learners expected to achieve only poorly on papers, this focus appeared transactional and their learners were generally presented as receivers, although there were exceptions evidenced in every study class: peers who were matched for prior attainment but entered instead for papers on which they realistically expected to perform well, who normatively appeared to thrive, seeking peer or material support and talking confidently about using mathematics post-16:

> *We shouldn't be entering them for papers they can't be really successful in, just so they might get a higher grade: . . . I think that's child abuse. I'd rather they got a better appreciation of the maths they can do and not the maths they can't (Dy12T4);*
>
> *I only did the Foundation papers, but I did quite well on those, and now I'm studying Core Maths, there are some new ideas, but that's OK, because they're quite useful for what I want to do, and I'm getting more confident I can make sense of some of the harder ideas (Dy12L1).*

At A-Level, surveys showed similar initial relationships between teacher and learner perceptions. This was exacerbated when successive iterations of the new assessments, especially around problem-solving, posed what several teachers called 'moving goal posts' [14]. Where (about half of) teachers were confident to attribute such change to the challenges of assessing new aspirations, their students of all demographics and previous attainment levels were generally phlegmatic about the varying demands they encountered in emerging papers:

> *It's a bit challenging, since we can't be too definite about expectations with the students. But I just tell them, focus on understanding the maths, and then show the examiners what you can with with whatever they choose to ask you (Ey13T3);*
>
> *They seem to be changing their minds about just how hard they're going to make the problem-solving, but that mirrors life and it's all good experience. In a sense it doesn't matter how difficult they make it on the live papers, we'll all be in it together and whatever, it does make you think with your maths (Ey13L26, same school, accompanied by nods from peers).*

Where, on the other hand, teachers reported, or communicated, feeling oppressed or overwhelmed by such demands, students usually seemed unconfident to deal with such variation: *'We now have no idea what to expect, which really doesn't seem reasonable, and it does mean it's not worth investing too much in getting to know the new specs' (Ey13T7); 'We're now really worried because we don't know what they'll be asking or if we've prepared in the right way' (Ey13L42, same school).* However, although again, there were exceptions to that (alignment, for example): *'The teachers aren't sure what will be wanted, but I think you just have to get on and deal with that uncertainty: we only get the one chance' (Ey13L45, same school).*

As the A-Level study continued, learners appeared to retain confidence in their teachers, but, over time, many also developed greater proactivity in independently selecting materials and strategies to match their emerging perceptions of assessment demands. They often framed such actions in terms of the 'exchange value' of the qualification, for example, its use in supporting access to their target university course, whereas teachers' concerns were more focused on learning the mathematics, including for future use, apparently less conscious of the qualification's role as gatekeeper to future career pathways. It would appear that while assessment systems in themselves might be insufficient to drive challeng-

ing curriculum reform within a high-stakes context, they are harnessed and influential in myriad ways.

Data showed that older learners were generally more confident, and better equipped, for proactivity in meeting learning needs via peers, older friends, siblings, and digital resources, and divergence from teacher recommendations. A-Level learners also showed other divergences from teacher standpoints. They perceived *mathematical* demands in their courses to progress fairly smoothly from pre-16 learning, but with a step change in *workload*; given revised curricula, their teachers in contrast were often still struggling to accommodate new expectations of the subject and subject pedagogical functioning.

Taking these two areas together, we see that many learners were themselves active in policy interpretation; they also significantly influenced both teacher and edu-business implementation: this study therefore supports a fairly nuanced conceptualisation of learner as mathematics curriculum policy actor.

4.3. Edu-Businesses as Co-Constructors of Curriculum with Teachers and Learners

The funder edu-business as policy actor influenced implementation in all study classrooms via their curriculum and assessment resources, and as above, those resources served as narrators, enthusiasts, and sometimes entrepreneurs of the mathematics curriculum. For both teachers and learners, the textual hierarchy, in this high-stakes assessment environment, was dominated by their published examination papers for ages 16 and 18 as classes approached those examinations. However, the cross-phase nature of the study revealed that for our sample, the dominant edu-business role for about 80% of our primary teachers—and at least half of primary learners—was as highly influential narrator, sometimes perceived as enthusiast, via their curriculum materials.

However, that did not mean that all teachers in a school responded to the new curriculum, or used the focus funder resources, in the same way. The school mathematics lead/Head of Department was observed very often to act as lead policy actor, whether as narrator or enthusiast or in a less positive role—and was typically influential on practice at least partly via the school mathematics 'scheme of work': *'So when I'm planning, I start from the scheme of work our maths coordinator put together, and I try to stick to that in principle because then the children get reasonably consistent experiences over time, but I tweak it to me, and the particular children in my class' (A17, T5).* However, analysis of lesson plans and observations, as well as teacher interviews, showed individual teachers within a school responding differentially, one with perhaps minimal compliance and another with deep investment in being able to understand, select from and adapt materials for their own learners. The longitudinal nature of the studies meant that we could observe teacher change in practice over time. Almost always, for the (approximately) 50% of teachers who showed significant progress towards the intended curriculum practice over the course of the studies, this appeared to require considerable and sustained (at least eighteen months') investment in rethinking practice with the support of at least the focus materials.

That was true in a fresh way during the pandemic, when the new constraints widely catalysed a re-thinking of primary teacher curriculum priorities and inter-dependencies in which curriculum materials, and particularly teacher guides, appeared to support transformational change in the subject and subject pedagogical knowledge and practice for about a quarter of participating primary teachers:

> *The teachers were saying how much they've gained from re-thinking their planning . . . for doing it online, using the Teacher Guides: they've come to look at things in new ways and had to sort out the main points, so they've now got a better structure, and better priorities, in their heads. And our formative assessment has become absolutely central to our practice: are the children bringing what they need? If not, then we need to nurture that before we can build on it (By4T1);*

> *I keep getting these 'lightbulb' moments where the maths suddenly makes more sense, because I've had to go back and sort out from the materials what are the absolute essentials,*

what are the building blocks for that, and how will I know whether the children have 'got it'—and my teaching is in a different place because of that (By6T2).

Longitudinal data showed that this was also true, if less prevalent, in more 'normal' times. In our sample, though, that happened only when teachers invested over a sustained period, and usually, collaboratively, in getting to know the curriculum materials, and particularly, the teacher guidance. In contrast, prevailing secondary curriculum—and assessment—materials did not seem to have that role: curriculum materials acted as narrator and translator of the intended curriculum, supporting teacher roles as narrator, receiver, and transactor—and sometimes receiver of policy—but even for teachers of lower secondary learners not teaching examination classes, we saw little evidence of curriculum materials catalysing significant, and certainly not transformational, subject knowledge or pedagogical quality change over the pandemic period. This was also true of assessment materials, though, as above, the ways in which, and the extent to which, they assessed the curriculum did frequently catalyse change in curriculum coverage and emphasis.

The influence, of course, occurred in both directions: funder-enhanced understanding of teacher and learner perceptions has already served to develop resources in ways that participants find helpful. In a marketised context for both curriculum and assessment materials, that is likely to support market share, as well as to inform edu-business interactions with policymakers.

5. Discussion

This study evidences conceptualisations, contrasts and synergies of teachers' work with curriculum and assessment resources in relation to curriculum planning and realisation, but also a range of curriculum-agentic roles adopted by learners, and the interplay of those with teacher and funder resources. It shows how many learners in this sample interacted very actively with their teachers and with curriculum resources to impact curriculum implementation. While teachers were clearly very influential on learner attitudes, in alignment with much other evidence [34,35], that relationship was often quite subtle and counter-discourses were evident in every study class. Both teachers and learners operated, differentially, with a hierarchy of texts often driven by agendas of performativity and accountability—and in which edu-businesses have to be complicit in order to thrive. However, this study also evidences, across phases of education, very constructive agency of some of each of edu-business, teacher and learner in forging meaningful and empowering interpretations of curriculum. In doing so, it exposes learners as potential policy actors in a range of proactive roles, and across phases. It therefore initiates a theorisation of learner as policy actor. It illuminates a range of curriculum-agentic roles of the sample learners, as well as the interplay of those with teachers and with the funder resources. Importantly, our data showed student policy role proactivity across demographic groups, although somewhat more common among students with previously relatively high attainment. Student proactivity was also somewhat associated with teacher proactive curriculum roles, across phases, as might be expected from some other evidence [35]. It is clear there is potential for learners to play a more active role in curriculum development at all levels, including in the design of curriculum resources and assessments, but also potentially in the development of the intended curriculum.

Our approach exposed an underlying and emergent (social) co-construction of meaning-making in relation to curriculum and related processes at the school level, as teachers and learners came to engage with the content, framing and structures of the variety of texts associated with the revised curriculum, with their implicit messages about agency and responsibility. Of course, researchers' own curriculum documentary analysis also drew on subjectively communicated tenets of policy, of policy development and of intended practice. When this was complemented by classroom observations, teacher interviews and learner focus groups, we were able to evidence how teachers and learners accounted the received meanings. Surveys for older learners then contextualised and further validated focus group and individual teacher responses, though offering co-constructed 'snapshots'

only. Those featured as key influencers often included the school mathematics lead for teachers and usually included the class teacher for learners. Those influencers appeared to vary in their policy-coherent capacity, beliefs and commitment to a solution focus.

The institutional ethnographic approach focuses on power relations, including inter-textual relations, and beyond our focus, participants of course described other sources of influence (and power), including school management and learners' parents, that we did not explore here. Funder resources exerted power in a number of ways and to a variety of extents, but in the vast majority of sample classrooms, they functioned as active social agents rather than static sources—in line with some other evidence in the field [25,36]. GCSE (age 16) and A-Level (age 18) assessment documentation almost always proved hegemonic in inter-textual hierarchies, even if that power was then mediated by teachers and/or learners: the meanings made by secondary teachers and learners were often dominated by those texts, leading to marginalisation of some aspects of other materials, where users perceived a limited coherence with either edu-business or their own assessment priorities. It is therefore critically important that assessment materials are designed so as to support the intended curriculum as far as possible. It is worth noting, also, that while study learners appeared to be significantly influenced by their teachers' communicated values, including being receptive to the roles of meaning-making, rigour and challenge in mathematics where their teacher promoted those, their own values did not always align with those of their teachers. For example, for a small minority of learners, achieving the grades they needed to meet university offers, even if that was for a mathematics-intense course, appeared more important than mastering the mathematics involved. Where teachers are aware of such tensions, they could perhaps harness those to achieve a synergy of their own and their learners' values. Similarly, it is useful for teachers to be aware that recently revised curricula are not necessarily an issue for learners, provided that teachers are supported, including via appropriate materials, to communicate confidence and clarity around the related changes: what can represent a challenging change for teachers is the only target curriculum familiar to learners.

The study approach exposed the fragility of curriculum policy coherence, adding to the existing substantial evidence in this area. Implementation was highly socialised and contextualised at several scales, as evidenced by the variation on individual, within-class, between-class and between-school scales, and also over time. Consequently, even our range of ecologically appropriate tools in a large-scale study only suggests subjective interpretations of this complex field. Observed patterns, though, can be helpful for practical purposes, such as developing materials to better support perceived need. Beyond highlighting threats to validity of implementation and pointing to apparently necessary characteristics of the curriculum system, if that is to be achieved, the deeply subjective, contextualised and idiosyncratic nature of individual standpoints precludes research identifying robust generalities, absolutes and deterministic features. The study does, though, offer good evidence that policy-coherent teacher-educative curriculum (and assessment) materials can, with investment of time and energy, support teachers to interpret, respond and enact in ways that align with (or even flesh out) the intended policy objectives. In this case, importantly, that was being achieved directly through the resources, without significant additional support. Policy churn and performativity considerations can, however, undermine that validity of enactment.

Reflection on the Methodology

In the reported study, academic–publisher research collaboration supported deeply informed, robust and credible research design. The study's longitudinal nature exposed aspects of reform implementation that would not have been clear from shorter exploration. Classroom closeness facilitated an ethnographic grasp, hidden in other large-scale curriculum implementation studies, of how participants came to act as they did, how that played out in the classroom, both constrained and enabled by other participants, and how institutional characteristics at a variety of scales further shaped that over time. In this

endeavour, though, it was limited as to exposure of power relations, since teachers and learners are also subject to a multitude of other influences not directly explored here. The use of phase- and subject-expert researchers in all fieldwork supported a good grasp of the social context being explored, though clearly, the nature of the research precluded a full immersion in those contexts.

Our data were able to expose roles for learners—in all phases, though overall, more commonly as they matured—as active 'policy actors', including as 'translators', 'constructive critics', and sometimes 'enthusiasts' for the intended curriculum, and the two theoretical frameworks used, of Ball et al.'s [21] policy actor typology, and of institutional ethnography [23,24], provided complementary lenses on that data. Together, these approaches begin to show just why curriculum system coherence is challenging to establish, and even more so, to maintain, perhaps especially in a high-stakes, marketised assessment regime; we suggest it is likely more challenging where, as in this case, those assessments are relatively demanding [10]. Such fragility also threatened to undermine teacher investment in professional development for achieving classroom coherence with the intended curriculum. However, there were also large numbers of classrooms (nearly half, across phases) where observations showed teachers making sustained progress towards valid, if variably effective, enactments of curriculum intentions over time, with curriculum, and sometimes assessment, resources supporting teachers to interpret, respond, and enact them in ways that align with (or even flesh out) the intended policy objectives. Comparable approaches would transfer to other education policy studies that are close to implementation at the policy's target sites.

However, importantly, analysis of these affordances also highlights some fundamental challenges associated with policy implementation research. In particular, teacher and learner implementation of intended curriculum change are deeply contextualised, contingent and idiosyncratic. There were (non-deterministic) associations between teacher attitudes to curriculum revision and their learners', as there were associations between teacher classroom implementation and maths lead intentions, approaches and strategies—but always also minority contra-discourses.

Finally, curriculum revision can create opportunities for reflection on, and development of, practice, but so can research participation, so there is necessarily an 'uncertainty principle' operating in relation to the reflections and enactments evidenced.

6. Conclusions

This paper outlines the affordances of harnessing two theoretical approaches to research methodology and conceptualisation of classroom-level curriculum policy enactment, in ways that give rise to complementary lenses on this complex field, enriched further by the possibility of looking at related evidence across phases of education. In doing so, we evidenced many commonalities of response to the curriculum across different phases—including variation in the validity of implementation within each phase (and some mathematically empowering and challenging implementation in each phase). However, performativity agendas often threatened to undermine moves towards validity of implementation. Additionally, research partnership between different policy 'actors', namely the funder edu-business and a lead researcher in a research-intensive university, supported access to information and research robustness not otherwise easily achievable. Application of such strategies to other areas of education policy has the potential to illuminate a comparable range of findings. In particular, we could analyse longitudinal and cross-phase as well as inter- and intra-school level implementation at a deeply informed level. Policy actor typologies supported identification of learners taking a range of policy-active roles, while institutional ethnographic approaches exposed inter-textual hierarchies and the sustained hegemony of texts associated with high-stakes assessments.

We evidenced the edu-business, teacher and learner taking a variety of policy actor roles, including very proactive ones, and mutually interacting to influence the roles of others. In our world of globalised multinationals, it seems likely that edu-businesses will

remain influential in education policy, including in high-stakes areas such as mathematics education. Understanding their roles and interactions with other stakeholders is therefore important. We also identified that while teacher and learner responses to policy are often inter-dependent, learners in all phases were capable of proactive independent assessment and re-imagining of curriculum implementation. We argue that there is considerable potential to harness that creativity better.

The specific findings of this study do not transfer elsewhere unproblematically, since English policy actors are working in a specific, high-stakes assessment context where a single edu-business dominates curriculum and assessment resources in mathematics education. Nevertheless, the methodology is transferable. It affords valuable insights into the complex dynamics of educational policy enactment, offering a fresh perspective that can contribute significantly to the field of education research. The analysis presented exposes a role for using a variety of conceptual and methodological approaches to education policy research, differentially selected and harnessed for complementary purposes, and for inter-actor research collaboration. Together, these support deep longitudinal, cross-phase study, enable comparison of learner and teacher standpoint, and expose possibilities for theorisation of learner policy roles. The paper's contribution is to focus on the intended object of curriculum policy—teachers and learners at the classroom level—highlighting for those purposes freshly developed and fruitful methodological and theoretical approaches for such study.

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Article

A Content Analysis of the Algebra Strand of Six Commercially Available U.S. High School Textbook Series

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Abstract: Algebra as a school subject is ill defined. Students experience algebra quite differently depending on the perspective of algebra taken by authors of the textbooks from which they learn. Through a content analysis of problems (n = 63,174) in the narrative and homework sections of six high school mathematics textbook series published in the U.S., we acquired systematic and reliable information about the algebra strand (i.e., symbolic algebra and functions) of each textbook series. We introduce plots to show the density, distribution, and sequencing of content, and present analyses of data for cognitive behavior, real-world context, technology, and manipulatives. Feedback on this study from an author of each textbook series is shared, and findings are discussed in terms of students' opportunities to learn.

Keywords: algebra; functions; intended curriculum; high school mathematics; textbook analysis; opportunity to learn

1. Introduction

There is tremendous variation in students' opportunities to learn the content and nature of algebraic activity in K-12 mathematics classes across educational jurisdictions in the United States (U.S.) and abroad [1]. Yerushalmy and Chazan [2] note that "School algebra is a complicated curricular arena to describe, one that is undergoing change" (p. 725). Kieran [3] stated that "Since the mid-1980s, the content of school algebra has been experiencing a tug of war between traditional and reformist views" (p. 709). Traditional (non-reform) approaches treat the concepts and skills of algebra in two separate year-long courses, usually separated by a year of deductive geometry. Textbooks aligned with this approach typically have a strong symbolic orientation. Although functions, along with their graphical, tabular, and letter-symbolic representations, are also included in textbooks with more traditional approaches, they are generally accorded a more minor role. In contrast to the traditional approach, reform approaches to algebra tend to give greater weight to functions, to various ways of representing functional situations, and to the solution of real-world problems using methods other than symbolic manipulation (e.g., using graphing calculators or computers). Textbooks aligned with this approach are often integrated, with the content strands of algebra, geometry, trigonometry, statistics, probability, and discrete mathematics appearing throughout each year of high school mathematics.

When asked, "What is algebra?" Hyman Bass [4] described both the state of the algebra landscape and the origins of the current "tug of war":

> The modern point of view . . . especially in the reform curricula has sort of taken the position [that] the world has changed a lot . . .. And so they've accordingly transformed the way they think of introducing algebra. So for a function, classically [it] was given by a formula, y = f (x), and then you'd try to graph it, etc. And now, a function is given by a collection of data points . . . you want to look for some pattern in them. So it's a kind of an homage, a bending in the direction of

> analysis of data and trying to fit data . . .. So, it leads to very different curricular trajectories and various different things that get foregrounded and backgrounded . . .. If you think of algebra as sort of situated on a big landscape or a big mountain . . . there's sort of different trails around the territory but they're not highly connected with each other. And in particular, the set of skills and knowledge that they cultivate by the time kids leave high school are very different, depending on which of these approaches you take . . .. The curriculum people sort of took the liberty to redefine algebra. Basically in terms of curriculum, they're actually talking about a different subject, to some extent, that doesn't merge with classical algebra until probably post high school . . .. So instead of clarifying the language, and explain what the two things are that people are engaged in, they're using the same name and doing a kind of colonial war about which one has the rightful claim to the territory . . .. It's not at all clear that there's a well-established consensus about which kind of treatment is most appropriate for all students.

This quote underscores the idea that algebra, as a school subject, is ill defined. Students experience algebra in quite different ways, depending on the perspective of the content taken by their teachers and the textbooks they use. It has been well established that textbooks are a strong determinant of what students have an opportunity to learn and what they learn [5–11]. Given the important role and influence of mathematics textbooks, together with lack of clarity as to the content of algebra in textbooks, it is important to understand what algebra is, as conveyed through textbooks, and thus, what students have an opportunity to learn.

The purpose of this research was to explore the content of the algebra strand of six high school textbook series (intended for students ages 15–18) in coherent, comprehensive, and commensurable ways. We defined the algebra strand to include symbolic algebra and functions. Three research questions guided this investigation:

1. What algebra content, including the breadth, sequence, and depth of topics covered, do students have opportunities to learn?
2. What sets of behaviors are expected of students as they engage with the content?
3. To what extent are problems set in real-world contexts, and to what extent are tools (technology and manipulatives) required to solve problems?

We investigated these questions through rigorous application of coding taxonomies to all items from the narrative and homework sections of six different textbook series. In addition, we engaged an author of each of the textbook series in a set of tasks to introduce them to our research methods and confirm that our application of codes reflected the objectives embedded in their textbooks. The goal of this study was to acquire systematic and reliable information about how algebra is represented in high school mathematics textbooks published in the U.S. In the sections that follow, we provide some background and framing for this study, which includes a discussion of the role of algebra in the high school curriculum, a rationale for this study, and a summary of prior content analyses.

1.1. Centrality of Algebra in the School Curriculum

Algebra plays a central role in the school mathematics curriculum. U.S. national curricular guidelines recommend that algebraic thinking be included in all grades, K–8, and that high school students study algebra for at least one year [12]. Knowledge of algebra is considered a gateway to future success in postsecondary endeavors and is increasingly being required for high school graduation. Gamoran and Hannigan [13] presented evidence that all students benefit from learning algebra—among students with very low prior achievement, the benefits are somewhat smaller, but algebra is nonetheless worthwhile for all students. Moses and Cobb believe that "math literacy—and algebra in particular—is the key to the future of disenfranchised communities" [14] (p. 5).

The importance of learning algebra is underscored in numerous documents. In a report from the Mathematical Association of America, an argument was made that algebra is "necessary for everyone desirous of participating in our democracy" and that "the

language of algebra has invaded virtually every discipline, including the social sciences and business" [15] (p. 6). In the report by the Committee on Prospering in the Global Economy of the 21st Century, the authors wrote that "students who choose not to or are unable to finish algebra 1 before 9th-grade—which is needed for them to proceed in high school to geometry, algebra 2, trigonometry, and precalculus—effectively shut themselves out of careers in the sciences" [16] (p. 102). As articulated in the report of the RAND Mathematics Study Panel [17], "We need systematic, reliable information on how algebra is actually represented in contemporary elementary and secondary curriculum materials, as designed and as enacted" (p. 49). Algebra is an area in critical need of further research.

1.2. Textbooks as Dominant Forces in the School Mathematics Curriculum

Textbooks and related ancillary materials used by teachers and students during instruction have a strong, perhaps unrivaled, influence on what and how mathematics is taught [18,19]. Reports from the National Assessment of Educational Progress indicate that mathematics teachers use textbooks as a major source for selecting content and instructional activities [20]. The report from the 2018 U.S. National Survey of Science and Mathematics Instruction states that "commercially published materials . . . exert substantial influence on instruction, from the frequency with which instruction is based on them to the ways teachers use them to plan for and organize instruction" [21] (p. 155). In this survey, it was found that 61% of U.S. high school mathematics teachers reported using commercially published textbooks as the basis of their instruction at least once a week. In addition to textbooks being used by teachers in planning and implementing lessons, textbooks serve as resources during teacher professional development and give policymakers insight into how recommendations from professional organizations and district/state mandates are implemented.

Figure 1 shows the theoretical framework that guided our work, adapted from Remillard and Heck [22], that situates the curriculum within a broad system of policy, design, and enactment that captures the relationships among different instantiations of the curriculum. The arrows in the diagram represent paths of likely influence as they are experienced in the U.S. system and illustrate that textbooks provide a critical link between the intended, enacted, and attained curriculum in school mathematics [23,24].

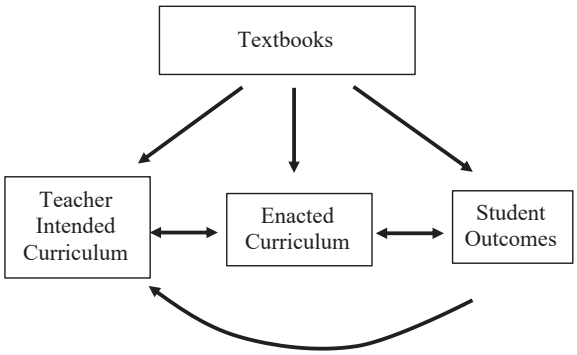

Figure 1. The influence of textbooks in the Tripartite Model of Curriculum. (Adapted from Remillard and Heck [22]).

Valverde et al. stated that "perhaps only students and teachers themselves are a more ubiquitous element of schooling than textbooks. As such a central facet of schooling, understanding textbooks is essential to understanding the learning opportunities provided in educational systems around the world" [11] (p. 2). As a result, scholars have called for mathematics education researchers to focus more attention on textbook analyses [25–31]. This has indeed been happening—in 2013, Fan [32] noted that "school textbooks have received increasing attention in the international research community of mathematics education over the last decades" (p. 765).

1.3. Analyses of the Algebra Strand in High School Textbooks

Other researchers have examined the content of the algebra strand of high school mathematics textbooks. Sherman et al. examined the author-recommended homework assignments in 16 textbook series that are used in Algebra 1 courses [33]. They replicated the result obtained by Nathan et al. [34] that a symbol precedence view (in which symbolic problems precede verbal problems) is more prevalent in textbooks than a verbal precedence view (in which concepts are first embedded in verbal situations).

Another study of the algebra strand of high school mathematics textbooks was conducted as part of the Comparing Options in Secondary Mathematics: Investigating Curriculum (COSMIC) project [35,36]. These researchers conducted textbook analyses investigating how linear functions were treated in an integrated textbook (Core-Plus Mathematics Course 1) and contrasted this with the approach to linear functions as developed in textbooks having a subject-specific organization (using Algebra 1 textbooks by various publishers). Using a table of contents analysis, they documented the extent to which the topic was present in each textbook. They used these data to develop assessments that reflected students' opportunities to learn linear functions using the different textbooks.

Two studies examined the content of algebra textbooks with the goal of informing adoption decisions. In one study, Project 2061 researchers at the American Association for the Advancement of Science [AAAS] worked with a team of teachers to analyze 12 Algebra 1 textbook series by comparing content and instruction related to specific learning goals from the domains of functions, operations, and variables [37]. The goals of this project were to assist adoption committees in making algebra textbook adoption decisions, help teachers revise existing materials to increase their effectiveness, guide developers in the creation of new materials, and contribute to the professional development of those who use their analysis procedure. In the other study, Rivers examined page 110 (or the nearest group of word problems) of five first-year algebra textbooks adopted for use in South Carolina in 1984, investigating the extent to which females and ethnic minorities are represented in word problems [38]. They also extended the research to examine the textbooks selected for the 1990 adoption process to investigate the extent to which changes had been made in response to the *Standards* [39].

The Mathematics Curriculum Study, conducted in conjunction with the 2005 U.S. National Assessment of Educational Progress High School Transcript Study, explored the relationship between course taking and achievement by examining the chapter review questions from approximately 120 Algebra 1, Geometry, and Integrated Mathematics textbooks to identify the mathematics topics covered and the complexity of the exercises (i.e., degree of cognitive challenge) [40]. Among the findings, researchers reported that core content made up about two-thirds of Algebra 1 courses, courses varied widely in the mathematics topics covered, and school course titles often overstated course content and challenge.

Across this collection of studies involving analyses of the algebra content strand within various textbooks, it is notable that the examination of textbooks by other authors was restricted to portions of textbooks and not entire textbooks. Some researchers' methods of analysis entailed close examinations of specific problem sets [33,40] or pages [38]. Other researchers focused on examining topics in tables of contents [35,36] or particular learning goals [37]. Of the studies reviewed here, the Brown et al. study [40] included cognitive challenge, the Rivers study [38] focused on racial and gender representation, and the Sherman et al. [33] and Nathan et al. [34] studies focused on mathematical representations. This study reported here presents a novel methodological approach: (a) We examined the entire algebra strand within multiple textbook series through problem-by-problem analyses; and (b) we focused our analysis beyond content alone, including cognitive behavior and tools (technology and manipulatives). Moreover, this study is unprecedented in that an author of each textbook series was invited to comment on our methodology and findings relative to their textbook series.

2. Materials and Methods

Our research design was informed by existing methodological approaches to textbook analyses [36,37,41,42] and by the 2004 report from a committee of the U.S. National Research Council (NRC) [27]. The following section includes an outline of our research design and methodology. Complete methodological details, including our rationale for choices regarding study design, are provided in Huntley et al. [43].

2.1. Textbooks and Unit of Analysis

The teacher's edition of six U.S. textbook series was analyzed (see Figure 2). The content of two textbook series was integrated: Core-Plus Mathematics Program (CPMP) [44] and Interactive Mathematics Program (IMP) [45–47]. Within these books, only units within the first three years that have a major focus on algebra were coded. These units were identified through discussions with the textbook series' authors. The other four series that were analyzed include content that is subject-specific. Of these, three series underwent extensive field-testing during development: Center for Mathematics Education (CME) [48,49], Discovering Mathematics (DM) [50,51], and University of Chicago School Mathematics Program (UCSMP) [52,53], and one series was commercially generated (Glencoe) [54,55]. For these four subject-specific textbook series, the entire Algebra 1 (CME, Glencoe)/Algebra (DM, UCSMP) and Algebra 2 (CME, Glencoe)/Advanced Algebra (DM, UCSMP) books were coded.

Textbook Series	Book Titles/Units	Publisher	Year
INTEGRATED			
Core-Plus Mathematics Program (CPMP)	Course 1 • Patterns of Change • Linear Functions • Exponential Functions • Quadratic Functions Course 2 • Functions, Equations, and Systems • Matrix Methods • Nonlinear Functions and Equations Course 3 • Reasoning and Proof • Inequalities and Linear Programming • Polynomial and Rational Functions • Inverse Functions	Glencoe/ McGraw-Hill	2008–2009
Interactive Mathematics Program (IMP)	Year 1 • Overland Trail • Patterns Year 2 • All About Alice • Cookies • Fireworks Year 3 • Meadows or Malls • Small World	Key Curriculum Press	2008–2009
SUBJECT-SPECIFIC – EXTENSIVE FIELD-TESTING DURING DEVELOPMENT			
Center for Mathematics Education (CME)	Algebra 1 Algebra 2	Pearson	2009
Discovering Mathematics (DM)	Algebra Advanced Algebra	Key Curriculum Press	2007, 2004
University of Chicago School Mathematics Program (UCSMP)	Algebra Advanced Algebra	Wright Group/ McGraw-Hill	2008–2009
SUBJECT-SPECIFIC – COMMERCIALLY GENERATED			
Glencoe	Algebra 1 Algebra 2	Glencoe/ McGraw-Hill	2008

Figure 2. Textbooks included in this study. (The subject-specific textbooks were coded in their entirety. The specific units that were coded within the integrated textbook series are indicated in this figure).

The unit of analysis in this study was an item. We assigned a separate set of codes to every part of a multi-part question (e.g., 1a, 1b, 1c). Every textbook item in the narrative (excluding worked-out examples) and exercises (homework problems) of each series was coded. This approach to textbook analysis—in which every item was examined in both the narrative and exercises portions of the books—is unprecedented. This time and labor-intensive process enabled us to capture the systematic sequencing of content, cognitive behaviors, and tools that authors have built into their textbooks.

2.2. Analytic Frameworks

To code the mathematical content of the textbooks, we considered several frameworks, including the algebra component of the TIMSS Advanced Mathematics Framework [56], but decided to use the Survey of Enacted Curriculum [SEC] K–12 Mathematics Taxonomy [57] because it offered a more granular look at the content. In Figure 3, the 16 main categories of this taxonomy are shown, together with the sub-topics for basic algebra, advanced algebra, and functions. The algebra and advanced algebra content areas represent a symbolic approach to algebra.

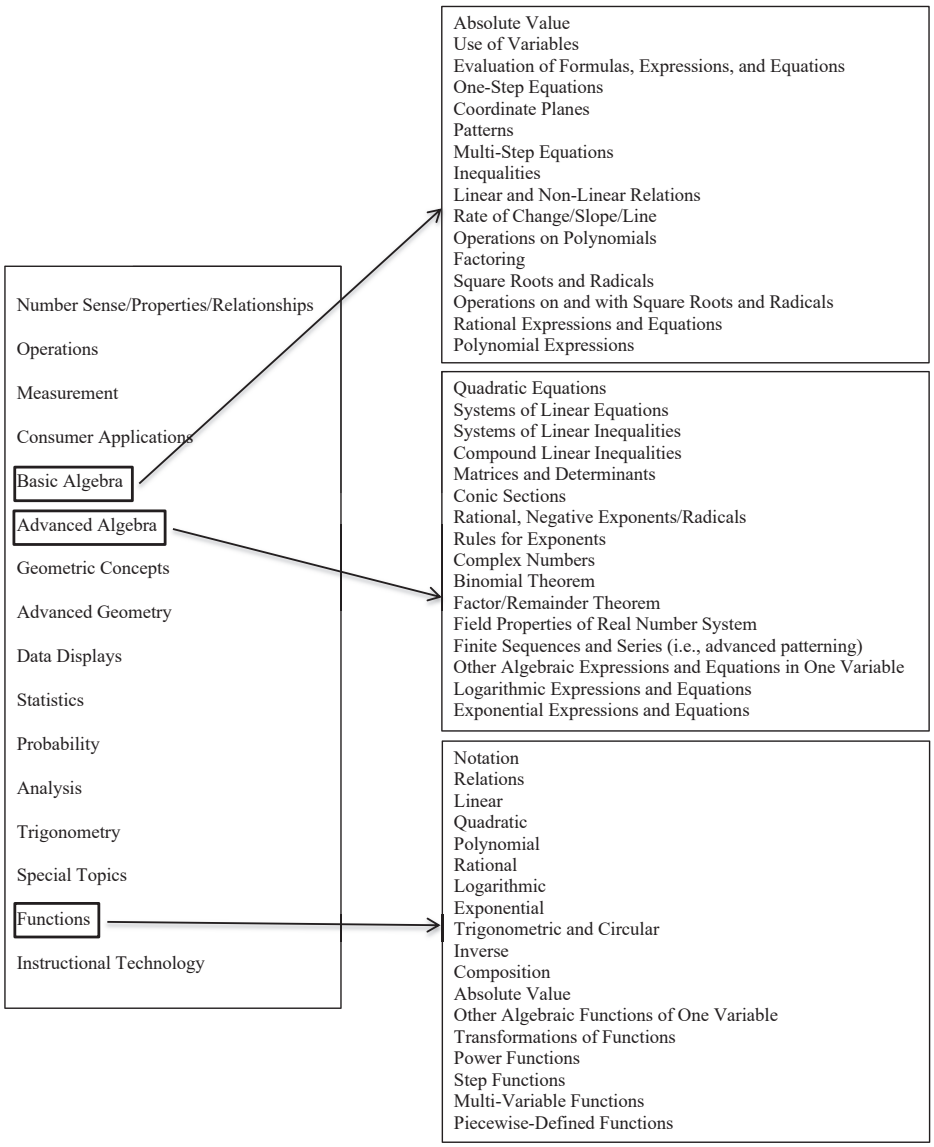

Figure 3. Coding categories in the content analysis taxonomy. (Adapted from the SEC K-12 Mathematics Taxonomy [57]. Reprinted with permission from IAP, August 2024).

We added extensive annotations to the coding taxonomies to illustrate how coders were to apply content codes to textbook items. The annotations consisted of specific mathematics problems (sometimes with illustrative examples from problems within the textbooks we were coding) and notes to coders (to facilitate consistent application of the content codes). For example, in Figure 4, we provide a portion of the original and annotated taxonomies corresponding to the content topic multi-step equations (code 507) within the algebra portion of the SEC Taxonomy [57]. Note that the content taxonomy is a coarse sieve. For example, with code 507 (multi-step equations), no distinction is made between evaluating, setting up, or solving multi-step equations.

Code	Portion of Original Content Taxonomy	Portion of Annotated Content Taxonomy
507	Multi-Step Equations	Multi-step Equations (solving multi-step equations OR setting up and solving multi-step equations) E.g., Solve for x: $5x + 3 = 7$. E.g., Solve for h: $A = \frac{1}{2}(b_1 + b_2)h$ (NOTE: This equation is linear in h)

Figure 4. A portion of the original and annotated content taxonomy. (Adapted from [57]).

The categories of cognitive processes from the TIMSS Advanced 2008 Assessment Framework [56] were used to code the cognitive behaviors expected of students as they engage with the mathematical content in the textbooks. As shown in Figure 5, this framework consists of three domains: knowing, applying, and reasoning. The cognitive process recount did not appear in the TIMSS Cognitive Behavior Taxonomy [56]. We added it to account for items that we were encountering in the textbooks that fell within the scope of the knowing domain but did not belong to the other subcategories (recall, recognize, compute, and retrieve).

	Knowing	Applying	Reasoning
Definition	The facts, procedures, and concepts students need to know.	The ability of students to make use of this knowledge to select or create models and solve problems	The ability to use analytical skills, generalize, and apply mathematics to unfamiliar or complex contexts
Cognitive Processes	Recall Recognize Compute Retrieve Recount	Select Represent Model Solve Routine Problems	Analyze Generalize Synthesize/Integrate Justify Solve Non-routine Problems

Figure 5. Overview of the TIMSS Cognitive Behavior Taxonomy [56]. (Reprinted with permission from IAP, August 2024).

In Figure 6, we provide an illustration of how the cognitive coding taxonomy [56] was used. In this example, there are problems that appear early in the UCSMP textbook series for each of the three cognitive domains: knowing, applying, and reasoning. The content code for each of these problems is 507 (multi-step equations).

In summary, the two primary curriculum variables that were examined in this study were content and cognitive behavior. The codes for content are on a numerical scale that has a partial ordering—the higher the code, the further in the school mathematics curriculum one expects to find the content. The cognitive behavior scale is a nominal scale in that the codes of knowing, applying, and reasoning are not ranked in a cognitive hierarchy. As a result of these choices, there were certain analyses that we could (and could not) perform with the data.

Knowing

Consider the steps used in the solution of $82n - 51 = 441$ below. What was done to go from the given equation to Step 1?

Given: $82n - 51 = 441$

Step 1 $82n = 492$

Step 2 $n = 6$

(UCSMP *Algebra*, p. 141)

Applying

Solve and check the equation $0.003 = 0.02y - 0.1$.

(UCSMP *Algebra*, p. 147)

Reasoning

Solve the equation $mx + (m + 1)x = 8mx + 4$ for x.

(UCSMP *Algebra*, p. 154)

Figure 6. Items coded within each cognitive domain. (UCSMP) [52,53].

2.3. Coding Procedure

Five people who have strong mathematical content knowledge and experience in grades K–12 classrooms coded the textbook series—three university faculty and two high school classroom teachers. A coding manual was used in a 2.5-day coder training session that was held before coding commenced. Each coding team consisted of a teacher and either a university mathematics educator or mathematician. Each coder worked separately to apply codes to items in the narrative and homework sections of a textbook. A textbook item was assigned up to three content codes (each consisting of a three- or four-digit number) and a single cognitive code K (knowing), A (applying), or R (reasoning). In addition to content and cognitive behavior, binary codes were used to indicate real-world context, technology, and manipulatives. Every effort was made to retain the intention of the codes in the original SEC and TIMSS taxonomies. At the same time, the taxonomies were treated as living documents, with updates being made as new issues arose. Codes were entered into a spreadsheet (which we call a "coding template"). Through a process of negotiation, the coding pair came to an agreement on the set of codes to apply to each item. Consistency in coding the textbooks was maintained throughout the coding process through frequent conversations across coding teams. When necessary, portions of textbooks were recoded when a clarification was made to a taxonomy that changed the meaning of a code.

The software package *Mathematica* (versions 8 and 9) was used to tabulate and explore the data [58,59]. Descriptive rather than inferential statistics were used in our reporting of the data in the sections that follow.

3. Results

In this section, data that were gathered from coding all items from the narrative and homework sections within the algebra strand of six different textbook series are presented. Findings from the content analysis are presented first, followed by findings from the cognitive behavior analysis. We conclude with an analysis of the extent to which problems were set in a real-world context and required the use of tools (technology and manipulatives). Our goal here is to describe and summarize the data, not to make inferences or judgments.

3.1. Findings from the Content Analysis

The first research question was: *What is the content, including the breadth, sequence, and depth of topics covered?* In this section, we address this question by sharing three aspects of the content analysis.

3.1.1. Number of Items Coded and Emphasis on Algebra, Advanced Algebra, and Functions

Altogether, 63,174 items were coded across the six textbook series. Although our analysis focused on codes from the basic algebra, advanced algebra, and functions categories of

the taxonomy, we did not restrict our coding to these three categories. The number of items that were coded in each textbook series is shown in Table 1. It may seem surprising that there is an order-of-magnitude difference between the number of items coded in the CPMP and IMP series and in the other four textbook series. We offer two explanations for this. First, recall that only selected units were coded from the first three years of the CPMP and IMP textbook series (those that the authors identified as having a major focus on algebra), whereas two entire books were coded for each of the other four series (CME, DM, Glencoe, and UCSMP). Second, the integrated textbook series (CPMP and IMP) are problem-based, with little or no narrative or explanatory text. By contrast, in the non-integrated series (CME, DM, Glencoe, and UCSMP), each section consists of narrative text that includes problems for students to solve, which we coded.

Table 1. Findings from the content analysis.

		Items Coded in the Algebra Strand		
Textbook Series	**Number of Items Coded**	**Algebra % (*n*)**	**Advanced Algebra % (*n*)**	**Functions (*n*)**
CME	12,752	35.3 (4501)	41.8 (5330)	33.7 (4297)
CPMP	7546	27.0 (2037)	27.8 (2098)	47.5 (3584)
DM	10,446	26.1 (2726)	23.9 (2496)	29.5(3082)
GLENCOE	17,020	41.2 (7012)	34.8 (5923)	21.3 (3625)
IMP	1735	39.2 (680)	32.6 (566)	34.2 (593)
UCSMP	13,675	35.5 (4855)	32.0 (4376)	27.0 (3692)

In addition to the number of items coded for each textbook, Table 1 also shows for each textbook series the percentage of items in which at least one content code from the algebra, advanced algebra, or functions portions of the taxonomy was used. In Table 1, the sum of each row does not add up to 100%. Recall that up to three content codes were assigned to each item, so an item could have been assigned a code from more than one of these three broad content domains (algebra, advanced algebra, functions) or none of these. According to the data in Table 1, the percentage of items coded as involving algebra ranged from 26.1% (DM) to 41.2% (Glencoe). The percentage of items coded as involving advanced algebra ranged from 23.9% (DM) to 41.8% (CME). The percentage of items coded as involving functions ranged from 21.3% (Glencoe) to 47.5% (CPMP).

These data provide a gross measure of the content emphasis within each textbook series, which is one portrayal of students' opportunity to learn. The two books that were coded from the CME textbook series provides the greatest opportunity for students to learn about content in advanced algebra. Moreover, CME may offer the most opportunities for students to learn across algebra, advanced algebra, and functions, as indicated by the most double or triple coding across the three broad content domains: algebra, advanced algebra, and functions. As another example of how the data suggests students' opportunity to learn, in the selected units of the CPMP textbook series that were coded, students have considerably more opportunity to learn about functions than they have an opportunity to learn about symbolic algebra (as indicated by the data in the columns labeled algebra and advanced algebra). Also, compared with the other five series that were examined, students using the DM textbooks have the fewest opportunities to learn about and practice problems involving symbolic algebra. In this series, 20.5% of the problems were not coded as belonging to the algebra, advanced algebra, or functions content domains.

3.1.2. Most Frequently Used Content Codes

Figure 7 contains a list of the ten most frequently used content codes for each textbook series, together with the percentage of items that were assigned that code. The data in Figure 7 span the entire 217 topic codes; that is, we did not limit these data to codes within the basic algebra, advanced algebra, and function categories of the content taxonomy.

CME	%
Finite Sequences & Series	6.8
Rate of Change/Slope/Line	6.7
Quadratic Equations	6.4
Complex Numbers	6.4
Factoring	5.3
Operations on Polynomials	5.3
Quadratic Functions	5.1
Matrices & Determinants	4.9
Multi-Step Equations	4.8
Transformations of Functions	4.2

CPMP	%
Linear Functions	12.9
Quadratic Functions	9.3
Exponential Functions	7.5
Matrices & Determinants	6.6
Iteration & Recursion	6.5
Operations on Polynomials	5.9
Quadratic Equations	5.5
Rate of Change/Slope/Line	4.3
Systems of Linear Equations	3.8
Power Functions	3.4

DM	%
Rate of Change/Slope/Line	9.5
Quadratic Functions	5.7
Transformations of Functions	4.5
Finite Sequences & Series	4.1
Iteration & Recursion	4.0
Systems of Linear Equations	3.7
Line/Curve of Best Fit	3.2
Coordinate Geometry	3.1
Mean, Median, & Mode	3.1
Area & Volume	3.0

Glencoe	%
Rate of Change/Slope/Line	6.8
Quadratic Equations	6.2
Multi-Step Equations	5.1
Operations on Polynomials	4.7
Rational Expressions & Eqns	4.6
Systems of Linear Equations	4.5
Inequalities	4.4
Quadratic Functions	4.2
Factoring	4.2
Matrices & Determinants	3.4

IMP	%
Quadratic Functions	10.9
Rate of Change/Slope/Line	9.9
Systems of Linear Equations	9.5
Linear Functions	6.8
Exponential Functions	6.6
Use of Variables	6.2
Function Notation	5.6
Operations on Polynomials	4.6
Linear Programming	4.4
Matrices & Determinants	4.3

UCSMP	%
Rate of Change/Slope/Line	8.4
Quadratic Functions	5.7
Operations on Polynomials	4.8
Matrices & Determinants	4.2
Finite Sequences & Series	3.8
Multi-Step Equations	3.7
Power Functions	3.6
Quadratic Equations	3.5
Systems of Linear Equations	3.4
Rules for Exponents	3.2

Figure 7. Percentage of items coded as involving the ten most frequently used content codes for each textbook series.

Two content topics appear in the top ten list of all six textbook series that were coded: rate of change/slope/line and quadratic functions. In fact, the code for rate of change/slope/line is the first or second most frequently used content code in five of the six textbook series. Three content topics appeared in the top ten list of five of the six textbook series: systems of linear equations, operations on polynomials, and matrices and determinants. CPMP and IMP, the two textbook series with integrated content, both have three function topics in their respective lists of top ten content codes: linear functions, quadratic functions, and exponential functions.

3.1.3. Density, Distribution, and Sequencing of Content

To examine the density, distribution, and sequencing of content across the textbook series, plots of topic codes as a function of "time" were generated (see Sections A and B). In these plots, time is represented as the sequentially coded items in the textbook. Each dot represents a content code applied to an item within the textbook. The diagrams use transparency; that is, lone dots are faint, and as they overlap, the image gets darker. It is not intended for one to see clearly the position of each dot. Rather, the aim is for these graphs to reveal differences in density, distribution, and sequencing of content topics.

Timeline dot plots for the four subject-specific textbook series (CME, DM, Glencoe, UCSMP) are shown in Appendix A. Reading each plot from left to right, the content in the textbooks progresses from the first page to the last; that is, the dots are at the exact places where that content occurs in the textbook. The categories of content (listed in Figure 3) appear along the left-hand margin of the plots, beginning with number sense/properties/relationships at the bottom and progressing upwards to instructional technology. The dashed vertical lines in each timeline plot indicate the textbook within each series from which the data came. For example, the CME series has textbooks entitled Algebra 1 and Algebra 2. The part of the graph to the left of the vertical dashed line contains data from the Algebra 1 book, and the part to the right contains data from the Algebra 2 book.

Three general conclusions are shared here regarding the timeline plots for the subject-specific textbook series. First, there is a higher density of dots for the content category algebra in the portion of the plots corresponding to the Algebra 1 (Algebra) books, and the density of dots in this content category decreases as one progresses to the Algebra 2 (Advanced Algebra) books. Similarly, there is a lower density of dots for the content category advanced algebra in the portion of the plots corresponding to the Algebra 1

(Algebra) books, and the density of dots in this category increases as one progresses to the Algebra 2 (Advanced Algebra) books. A second conclusion from the timeline plots concerns the place within the series where content related to functions begins. For CME and UCSMP, functions are introduced approximately halfway into the first book, whereas for DM and Glencoe, functions are introduced early in the first book. For all four subject-specific series, content related to functions is emphasized more heavily in the second book in the series, as evidenced by the heavier concentration of dots corresponding to functions to the right of the vertical dashed line. The third conclusion from the timeline plots concerns the emphasis on other content areas within each series. For example, CME more heavily emphasizes number (especially early in the series) compared with the other series. In the DM series, there is more emphasis on statistics and data displays as compared to the other series, and the density of dots in the three broad content domains, algebra, advanced algebra, and functions, is less as compared to the other series. This is consistent with the data for DM shown in Table 1. Topics in measurement are prominent in CME, Glencoe, and UCSMP.

Timeline dot plots for the two integrated textbook series (CPMP, IMP) are shown in Appendix B. Because only the problems from units that the authors identified as having a major emphasis on algebra were coded, large swaths of these textbooks were not coded. The units that were coded are displayed in the timeline plots, using the same ordering of the units as in the textbooks themselves. (The units that were not coded are not represented in the plots.)

Three general conclusions are shared here regarding the timeline plots for the integrated textbook series. First, as with DM and Glencoe, the timeline plots for CPMP and IMP illustrate that the concept of functions is introduced early in the first unit of these series. Moreover, there is a high density of codes related to functions throughout these two series. This is consistent with findings shown in Table 1, which indicates that within the CPMP units that were coded, 47.5% of the items were coded as having content related to functions, and within the IMP units that were coded, 34.2% of the items were coded as having content related to functions. This is also consistent with the data presented in Figure 7, in which four of the top ten codes for each of these two series are related to functions. Second, the low density of dots in the timeline plot for IMP reinforces data provided in Table 1; namely, there are considerably fewer items in the IMP series as compared to the others. The third conclusion regarding the timeline plots for the integrated series is that in addition to functions, the content areas of algebra and advanced algebra are also heavily represented. As with the subject-specific textbook series, there is a higher density of dots for algebra earlier in these plots (and hence the textbook series) and a higher density of dots for advanced algebra later on.

3.2. Findings from the Cognitive Behavior Analysis

The second research question was: *What sets of behaviors are expected of students as they engage with the content?* Recall that each item in the textbooks was assigned a cognitive code from one of three domains: knowing (the facts, procedures, and concepts students need to know), applying (the ability of students to make use of this knowledge to select or create models and solve problems), and reasoning (the ability to use analytical skills, generalize, and apply mathematics to unfamiliar or complex contexts).

In Figure 8, bar graphs represent the percentage of items in the six textbook series that were coded as belonging to each cognitive domain. The salient point is not the actual percentages of items that fall into each cognitive domain but rather the pattern of differences in the data.

In all six textbook series, more than half the items were coded as involving the cognitive domain applying. Glencoe had the most items coded as falling into the knowing domain, and IMP had the fewest. CPMP and IMP had the highest proportion of items coded as involving the reasoning domain, and DM, Glencoe, and UCSMP had the fewest.

We view the graphs as useful visualizations of the relative proportion of items coded in each of the three cognitive domains. Looking across these data, certain textbook series provide a more balanced opportunity to learn across the three knowledge domains (CPMP,

IMP, CME), whereas other series emphasize one or two knowledge domains over the other(s) (DM, Glencoe, UCSMP). The variation in these distributions of data conveys one characterization of how these cognitive domains are valued in each textbook series and thus provides a lens into the opportunities students are provided to experience the content.

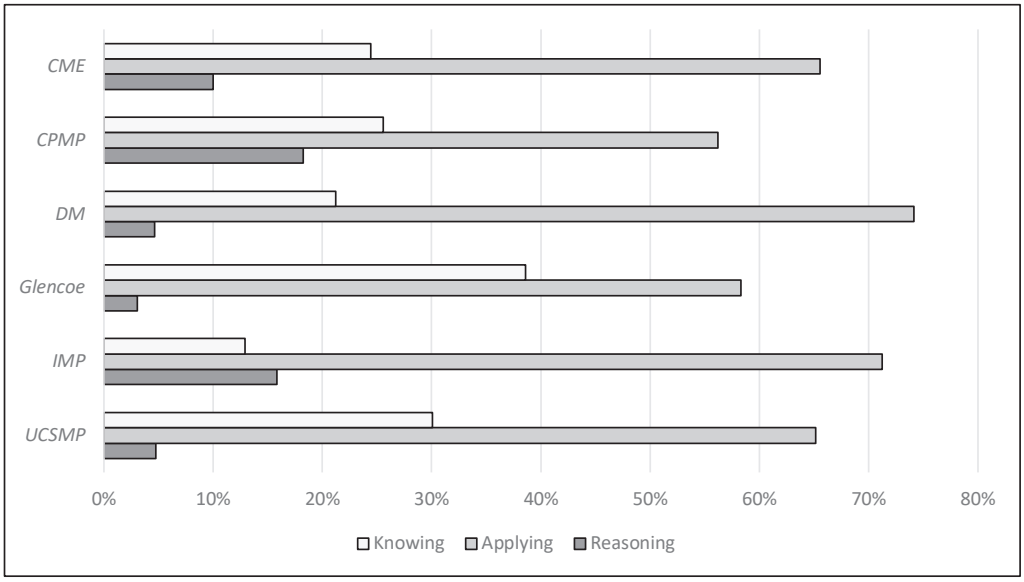

Figure 8. Cognitive behavior of problems in each textbook series.

The cognitive domains of knowing, applying, and reasoning are not ordered. Consistent with the view articulated in the TIMSS Advanced 2008 Assessment Framework [56] that "understanding a mathematics topic consists of having the ability to operate successfully in three cognitive domains" (p. 17), we believe that learning a concept involves students solving problems that correspond to all three cognitive domains—knowing, applying, and reasoning. As a result, we caution against trying to conclude that higher percentages of reasoning, applying, or knowing items are a priori better than another ordering of the percentages.

3.3. Other Findings

The third research question was: *To what extent are real-world context and tools (technology and manipulatives) included in problems in each textbook series?* Recall that the codes for each of these curriculum variables are binary—either the feature is present in the textbook item, or it is not. Data regarding real-world context and tools are reported in Table 2.

These data indicate that the extent of use of real-world contexts varies widely across the six textbook series that were coded. The algebra strand of the CME textbooks has the fewest items that are set in a real-world context, at 10.3%. The algebra strand of the Glencoe and UCSMP textbooks have 22.9% and 28.3% of items, respectively, set in a real-world context. The textbook series with the most items set in a real-world context are IMP, CPMP, and DM, which contain 54.3%, 44.9%, and 43.8%, respectively.

Compared with the percentage of items set in real-world contexts, the variation was smaller regarding the percentage of items that required the use of calculators or computers. The CME, IMP, and Glencoe textbooks contain the fewest items that required the use of calculators or computers, at 10.3%, 13.3%, and 13.5%, respectively. The DM, CPMP, and UCSMP textbooks contain the most items that required the use of calculators or computers, at 33.4%, 22.2%, and 19.9%.

Hands-on manipulatives were required for very few items in any of the six different textbook series. Across the series, the percentage of items that required the use of hands-on manipulatives ranged from 0.1% (CME) to 1.8% (IMP).

Table 2. Percentage of items coded as being set in a real-world context or involving the use of tools (technology or manipulatives) for each textbook series.

	Real-World Context	Calculator or Computer	Manipulatives
CME	10.3	10.3	0.1
CPMP	44.9	22.2	0.3
DM	43.8	33.4	1.3
Glencoe	22.9	13.5	0.9
IMP	54.3	13.3	1.8
UCSMP	28.3	19.9	0.2

4. Feedback from Textbook Series Authors

We conducted independent analyses of the textbooks. At the same time, we considered it important for the authors of the textbooks to understand the taxonomies and how they were applied to their textbooks, and to confirm that our application of codes reflected the content and behavioral objectives embedded in their respective textbooks. Thus, we invited an author of each of the six textbook series to engage in a series of coding exercises. Our goals in doing so were threefold: (1) to familiarize authors with our coding procedures, (2) to illustrate how the data were generated by having them code selected portions of their textbooks, and (3) to provide us with feedback regarding our coding process and our application of codes. An author of each textbook series accepted our invitation to collaborate with us (remotely, via e-mail) and was paid a stipend for doing so. As outlined in Figure 9, each author was asked to complete four tasks. In the sections that follow, we focus our discussion on the authors' written reflections (task 4). A comparative process of qualitative analysis was used to analyze the textbook authors' responses to the reflection task.

1. Background Reading: Read a seven-page document that outlined the research question and methodology of the study, and to become familiar with (a) the step-by-step coding guidelines outlined in the coding manual, (b) the content and cognitive taxonomies, and (c) the coding template.
2. Practice Coding: Do a practice coding exercise by using the content and cognitive behavior taxonomies to apply codes to the 39 items in the New York State 2010 Integrated Algebra Regents exam, and then compare their coding with the codes that we had assigned to these items.
3. Textbook Coding: Code a portion of their respective textbooks that focused on the following content:
   - Solving the linear equation $ax + b = c$
   - The slope of a line
   - The function with equation $y = ax^2$

   For each textbook series, we identified the specific lesson(s) that introduced these three content areas. Authors coded these specific lessons, and compared their codes with codes that we had negotiated and assigned to them.
4. Reflections: Respond to the following questions:
   a. To what extent do the codes we have applied to the portions of your textbook seem accurate?
   b. To what extent do you think this process of coding the textbooks will capture the essence of the book's content and the cognitive behaviors of students as they engage with the content?
   c. Do you have other ideas you would like to share regarding the coding process in which you have just engaged, or advice as we examine the data we have generated through our coding?

Figure 9. Tasks assigned to authors to obtain their feedback on this study.

4.1. Textbook Series Authors' Views on the Coding Taxonomies

There was considerable agreement between the codes that we applied to items in the authors' textbooks compared with the codes applied by the authors. Instances in which there were disagreements seemed to reflect the authors' dissatisfaction with the coding taxonomies rather than with our coding. They articulated this in various ways. For example, one author said the following:

> [I] did the best [I] could under the constraints of the allowable categories, but I have serious concerns about the categories themselves . . .. I found the categories quite unsatisfactory for describing the design of [my textbook] (or any curriculum, for that matter) . . . so many of the problems [that I coded in my textbook] are aimed at helping students develop the habit of abstracting a process from repeated numerical calculations. So, while a suite of problems may all look like computations, they are actually building to a punchline.

A different author expressed dissatisfaction with both the content and cognitive behavior taxonomies in the following way:

> Given the tool used, K-12 Math Taxonomy, the codes applied were as accurate as they could be. However, this tool seems too simple and limiting . . .. For example, One-Step Equations (solving one-step linear equations OR setting up and solving one-step equations) are coded as if these two types of tasks are at the same level of complexity. Most students can solve questions like $5x - 3 = 7$. Students can do the naked algebra. What they can't do is generate this simple type of equation from a situation. Solving an equation is not the same as setting up and solving the equation. If that were true, most students wouldn't dread the so-called word problems in a traditional algebra textbook. Another concern is that the codes and the domains can not and do not reflect the complexity of the task. For example [at this point the author provides a specific example from her/his textbook] this is the correct coding but this particular problem takes hours to solve and requires sophisticated ways of organizing the data, none of which you can pick up from a code and a domain. I realize . . . the categories of cognitive processes are supposed to help differentiate between knowing, applying, and reasoning with algebra. However, these two sets of tools can only give a glimpse of what the authors' goals and purposes are for students [sic] understanding and learning of algebra.

Some of the authors noted that the taxonomies do not take into consideration how teachers use textbooks. One author stated the following:

> One thorny issue here is that ostensibly routine curricular content can be taught richly in a way that fosters cognitive complexity and deep understanding, and, on the other hand, ostensibly rich curricular content can be taught in a way that routinizes it and makes it cognitively shallow. For example, we have had well-meaning teachers break down the [problems] into small-step worksheets so that they are easier for students (and teachers) but this then short circuits the thinking process and detracts from the deep-understanding goals. This can make the coding problematic . . .. Related to this is the issue of how much scaffolding is provided, how it's provided, and when. This is an adjustment that is made on the fly by good teachers in the classroom, but is very difficult to craft when writing a textbook. And it can make the coding problematic . . .. [In our textbook we] state the problem in a more open-ended way in the stem, then provide "hints", which are given as parts a-d of the problem. There are typically notes in the Teachers Edition that point out the options of less or more scaffolding. Of course, the students and teachers almost always go straight for the hints, parts a-d. However, if students did the problem in the less-scaffolded way, then the cognitive code becomes R, rather than K and A.

4.2. Textbook Series Authors' Views on the Methodology

In addition to commenting on their perceived limitations of the coding taxonomies, several authors commented on the methodology of this study. One author expressed concern that not all of the units of integrated textbooks were coded:

> It is very difficult to take an integrated curriculum where algebra is learned, developed, and used as a language over four years and compare it to a traditional algebra book. You would have to look at every problem in every unit [in an integrated program] over all four years to see all the ways algebra is learned and used.

Another author expressed concern regarding our decision not to code worked-out examples in the narrative portion of the textbooks:

> I can understand the concept of NOT coding the worked examples in the text, but in each lesson of [my textbook] you will often find open-ended questions that are there to at least prompt some thinking, and at best open a class discussion . . .. These questions can pop up anywhere including in the "solution" to a worked example.

This particular author, who has a background in statistics, was also critical of our coding of every item in the books, thinking that our coding fewer items would give sufficient information, saying, "Coding a dozen randomly selected sections will give you the cognitive information you wish." One author praised our coding of the narrative portion of lessons in the textbooks, saying the following:

> The inclusion of examples from the narrative of a lesson as coding items is a good idea. It is more reflective of the intent of [my] textbook, since the . . . author assumes a student would read the lesson at some point.

A different author pondered our expanding this study:

> It might be interesting to triangulate results from coding the student text, the teacher edition, and the provided assessment resources. What sort of content and cognitive behaviors are emphasized in the teacher edition? In the quizzes and tests, and other assessment resources? How do the expectations in the student edition, teacher edition, and assessments compare?

The various authors were not in agreement about whether the data we collected would accurately portray the material in their respective textbooks. An author said, "The process of coding content and cognitive behavior, applied consistently and accurately, can certainly be helpful in capturing what's presented in textbooks and, to some extent, what's expected of students." Consistent with this view, another author said, "You are certainly getting a great deal of information about the book. And I believe that for the most part you will get quite accurate information." However, this same author tempered this view by saying the following:

> What make[s] one text different from another is the developed progression of thought as much as the topics covered and depth of each. So content and cognitive behavior I would say are complete and fairly accurate. But a good text also must put these pieces in an order that makes sense to students so that ideas accumulate to reinforce one-another and connect between. How can this be measured or judged?

Other authors did not share these beliefs that the data we had collected would accurately portray the fundamental nature of their textbooks. One author explained their thinking as follows:

> I don't think this process will capture the essence of the book's content and cognitive behaviors of students as they engage with the content. . . The way you tackle a problem in [my textbook], [students are] taught to not take everything at face value. You need to understand why pi is pi, or why this formula is this formula, or why we are solving this problem this way. I'm not sure coding textbooks can capture this level of difference between textbooks. It is hard to capture how students are asked to create their own problems and discover relationships and generalizations.

Consistent with this view, the author of a different textbook said the following:

> Coding a textbook cannot reflect the essence of the book's content. How does one capture essence by coding? Any coding process is simply a compilation of check marks or instances of the inclusion of a content topic. Essence is much more encompassing than a quantity of check marks.

4.3. Advice and Other Comments from Textbook Series Authors

In their feedback to us, some of the authors made remarks about the vast amount of data we collected and provided advice, offered suggestions, and raised concerns about how to present it. One author wondered whether "any sort of training or other professional development [would be] needed or desired in order for someone to interpret coding results." Two authors noted that they found the process of coding the lessons assigned from their respective textbooks to be tedious. One said, "Although I found this work tedious, I'm glad I had the opportunity to do this", and another said, "The actual coding of [my] textbook was a tedious but enjoyable and reflective exercise."

In reflecting on this research project, one author offered the following insight about how different views of algebra, learning, and schooling lead authors to develop quite different textbooks:

> I think that there are very different beliefs about what algebra is, what is important for students to know and be able to do in algebra, what learning is, and what the purpose of school is. If you think the learning of algebra is primarily to pass on a certain body of mathematical knowledge, you develop a certain type of textbook. If you believe that students should learn to think and reason mathematically, and use algebra where appropriate, you develop a different type of textbook.

A summary comment by a different author offered praise: "[We] applaud the project team for attempting to analyze such a gargantuan set of data. We look forward to reading the analysis and conclusions."

5. Discussion

We begin this section by reflecting on and responding to some of the textbook authors' concerns. We next discuss our ideas for further investigations and conclude by revisiting ideas related to curricular choices and opportunity to learn.

5.1. Limitations of This Study

The textbook authors provided thoughtful feedback on many issues related to this study and articulated a number of limitations. Many of these limitations reflect budgetary constraints (e.g., our not coding all units of the integrated textbooks) and/or relevance to the research questions being investigated (e.g., our not triangulating results from coding the student text, teacher edition, and assessment resources).

Two comments from authors merit further discussion:

- "So while a suite of problems may all look like computations, they are actually building to a punchline;" and
- "What make[s] one text different from another is the developed progression of thought as much as the topics covered and depth of each . . . a good text also must put these pieces in an order that makes sense to students so that ideas accumulate to reinforce one-another and connect between."

Indeed, the *glue*, the *stuff* that connects the pieces together to make a textbook coherent, is not captured very well through the use of the content and cognitive behavior taxonomies chosen for this study. These taxonomies are coarse sieves, which means that some things have necessarily *fallen through the cracks*. Related to this, the results of this study are a direct reflection of our unit of analysis, namely, our choice to code at the level of individual items in the textbooks. We think that these factors—the choice of taxonomies and unit of analysis—do not capture a book's coherence very well, and we lend caution to overgeneralizing the

findings. Each textbook series had quite a different *feel* to it as we coded it, yet sometimes the coding and resulting data did not reflect these differences. For example, there are remarkable similarities in the data and graphs we presented for the Glencoe and UCSMP textbook series, yet the authors of these books seem to have quite different philosophies and intentions for using them. The use of different taxonomies and/or a different unit of analysis would likely lead to different findings and patterns in the data.

We appreciate one author saying that this project does "not take into consideration how teachers use textbooks". Indeed, this was beyond the scope of this project. However, some inferences about the intended use of the textbooks can be gleaned from the data. For example, as shown in Table 1, there is an order of magnitude difference in these data, suggesting that in one series, it is the authors' intention that students are engaged in sustained investigation of a few problems (IMP) in contrast to teachers selecting from a myriad of problems in a text that has many more (Glencoe).

5.2. Constructing Mosaics: Ideas for Further Investigations

The goal of this project was to acquire systematic and reliable information about how algebra is represented in six high school mathematics textbook series in the U.S. We coded several dimensions of every item in the narrative and exercise portions of the six textbook series. These dimensions include content, cognitive behavior, problem context, and tools (technology and manipulatives). This comprehensive approach resulted in an enormous dataset. Each element in the dataset is like a tile in a mosaic, with each tile conveying a small piece of information. In a mosaic, a picture emerges when all the tiles are put together. Likewise, as we look across the various dimensions of coding for each textbook series, we see the *portraits of algebra* that emerge from the data. Further discussion concerning the mosaics is in Huntley et al. [43].

There is much more that our data can tell mathematics educators, policymakers, and researchers about the algebra strand of the high school mathematics curriculum in the U.S., as conveyed through the six different textbook series. Further analyses of the data can help to refine the portraits of algebra in the textbook series. For example, to obtain more fine-grained information, we might look for patterns within the data for each textbook series and examine in greater depth particular algebra topics across the different series. For instance, numerous items were coded as linear functions together with the cognitive process applying, and by drilling down more deeply into items assigned these codes, we can examine the different ways that this coding pair plays out in the textbooks, thereby extending the research by Chávez et al. [35,36]. Another potentially fruitful investigation involves examining the ways in which some textbook series link algebra to arithmetic (i.e., the number and operation domains of the SEC taxonomy [57]), whereas others link symbolic algebra to functions. We also wonder how the different components of a textbook series interact (e.g., whether the cognitive behaviors of problems are the same across the narrative portion of a book and the homework problems) and whether there is a particular sequence (progression) of items within the homework exercises and in what ways the narrative supports this. We can examine progressions of content with a smaller scope (e.g., investigate the mathematical storyline within a textbook series and see how it is related to the co-development of equation solving and learning of functions). We are interested in examining the technology codes more closely within each textbook series (e.g., Where in each textbook series is technology use the most prevalent? Does the use of technology [about 20% for three series and about 10% for three other series] match our expectations regarding what is advocated for in the field, and what might this say with respect to opportunities to learn?).

Another line of inquiry involves our collaborating with a biometrician. Making sense of our data, which involves taxonomic units across sections and chapters in textbooks, shares many similarities with data analysis issues faced by ecologists examining species abundances at various locations. Both situations involve massively multivariate data with typically very sparse matrices. In ecology, only a few species from a species list are found at a particular location. Similarly, in our situation, only a few topics from the content

taxonomy are applied to problems in a chapter of a textbook. Highly specialized statistical software packages (e.g., Canoco 5.0 [60]) have been developed for ecologists, which we would like to use with the datasets we have generated. One issue this software may shed light on is determining whether coding fewer items will yield similar results. What we did in coding every item in the textbooks is the *gold standard* and is not feasible under typical circumstances (i.e., without funding and enormous amounts of time). We wish to investigate whether there is a way to determine how many and which sets of items to code within each chapter to minimize coder fatigue.

5.3. Curricular Choices and Opportunity to Learn

Revisiting the theoretical framework (Figure 1), we recognize that a textbook is just one factor that shapes students' learning. We acknowledge the broad interacting web of factors that ultimately shape students' understanding of algebra. Although the curriculum is instantiated through teacher and student engagement around various tools and artifacts, a textbook arguably provides a necessary and important starting point for making sense of the opportunities students have to learn mathematics. Indeed, as Porter said: "Knowing the content of the intended curriculum is important because the intended curriculum is the content target for the enacted curriculum" [61] (p. 141).

Textbook authors are influenced by their own deeply held beliefs about what features (curriculum variables) positively affect student learning. In this way, textbook writers' values and perspectives on school mathematics guide their writing and provide specific opportunities for student learning. For example, in discussing the design principles of CME, Cuoco outlines the role of technical fluency, saying, "... reasoning about calculations in abstract symbol systems is *useful* [emphasis in original] ... in the CME Project, we invite students to become fluent in algebraic calculations so that they can reason about them" [62] (p. 124). Our data are consistent with this perspective. As reported in Table 2, 10.3% of the items in the algebra strand of the CME textbook series are set in a real-world context. By contrast, 44.9% of problems in the CPMP units that have a major emphasis on algebra (as defined by the textbook authors) were coded as being set in a real-world context. This is consistent with one of the central tenets of the approach to algebra in CPMP: "The primary role of algebra at the school level is to provide effective models of numerical patterns and quantitative relations—in pure mathematics and in the many applications of mathematics in which numerical data are important" [63] (pp. 330–331).

As illustrated in these examples, based on textbook authors' values and perspectives on school mathematics, authors make reasoned decisions about specific content and processes to include and emphasize in their books. Each set of authors has a set of *best bets* about what curriculum variables will positively impact student learning. If teachers use textbooks as the authors intend, then these best bets translate into opportunities for student learning. In this way, understanding the content of textbooks is essential to understanding the learning opportunities provided to students. As Hy Bass remarked, answering the question "What is algebra?" is more than a matter of content and cognitive behaviors; it is also a statement of beliefs and values. Representing textbook series through descriptive statistics and visual depictions such as those offered in this research offers readers a snapshot of those values.

6. Conclusions

Through this project, we acquired systematic and detailed information about how algebra is represented in six high school mathematics textbook series in the U.S., and we have provided concrete portrayals of algebra in the textbooks. By looking at several curricular variables—content, cognitive behavior, real-world context, technology, and manipulatives—we were able to capture robust characterizations, beyond broad brushstrokes, of what algebra is in each textbook series. Our choice of analytic frameworks provided one way, but certainly not the only way, to characterize algebra in the textbook series.

As discussed in Huntley et al. [43], a major contribution of this study involves the methodology. Authors of prior content analyses of the algebra strand of textbooks have looked only at portions of textbooks or tables of contents. Although we acknowledge the limitations of our study, our research contributes unprecedented findings on the algebra strand of textbooks across integrated and traditional course sequences by taking a comprehensive and systematic approach to data collection and analysis.

Another contribution of this study involves the display of data in timeline plots, which is novel in capturing visualizations of how content is sequenced across units and textbooks. Such analyses help to tell a story of not only what content is introduced but also in what sequence and with what density. We can imagine how such techniques of visualizing large data sets could be applied in other domains of research (e.g., learning progressions across high school courses) and practice (e.g., trends in students' outcomes on standardized test measures).

Our methodology, namely, problem-by-problem analysis of all items within a textbook series, is the *gold standard* and cannot be replicated for every new textbook series that enters the market. However, one thing we learned is that our analysis confirms the statements of beliefs and values that some textbook authors have communicated in publications external to their textbooks. For this reason, we recommend that authors of future textbooks explicitly communicate their perspectives on how students learn algebra and their decisions about the order and emphasis of specific topics to help inform decision-makers, including teachers, who are tasked with textbook adoption decisions.

We reiterate the well-established axiom that students learn what they have an opportunity to learn. Our study sheds new light on the opportunities students are afforded to learn algebra when using specific textbooks but leaves open fundamental questions about what understanding and skill in algebra is most important for students to acquire from their school mathematics experience, and with what balance and sequencing of content and cognitive behavior.

Around the world, textbooks are important resources for teaching and learning mathematics. They play an integral role in defining mathematics as a school subject and shape the learning opportunities for students by teachers. As stated by Fan, "school textbooks have received increasing attention in the international research community of mathematics education over the last decades [yet this] field of research is still at an early stage of development" [32] (p. 765). Building on this study, together with a sustained inquiry into the data that were generated, will allow for more complete portraits of algebra to emerge in the intended U.S. mathematics curriculum.

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Appendix A

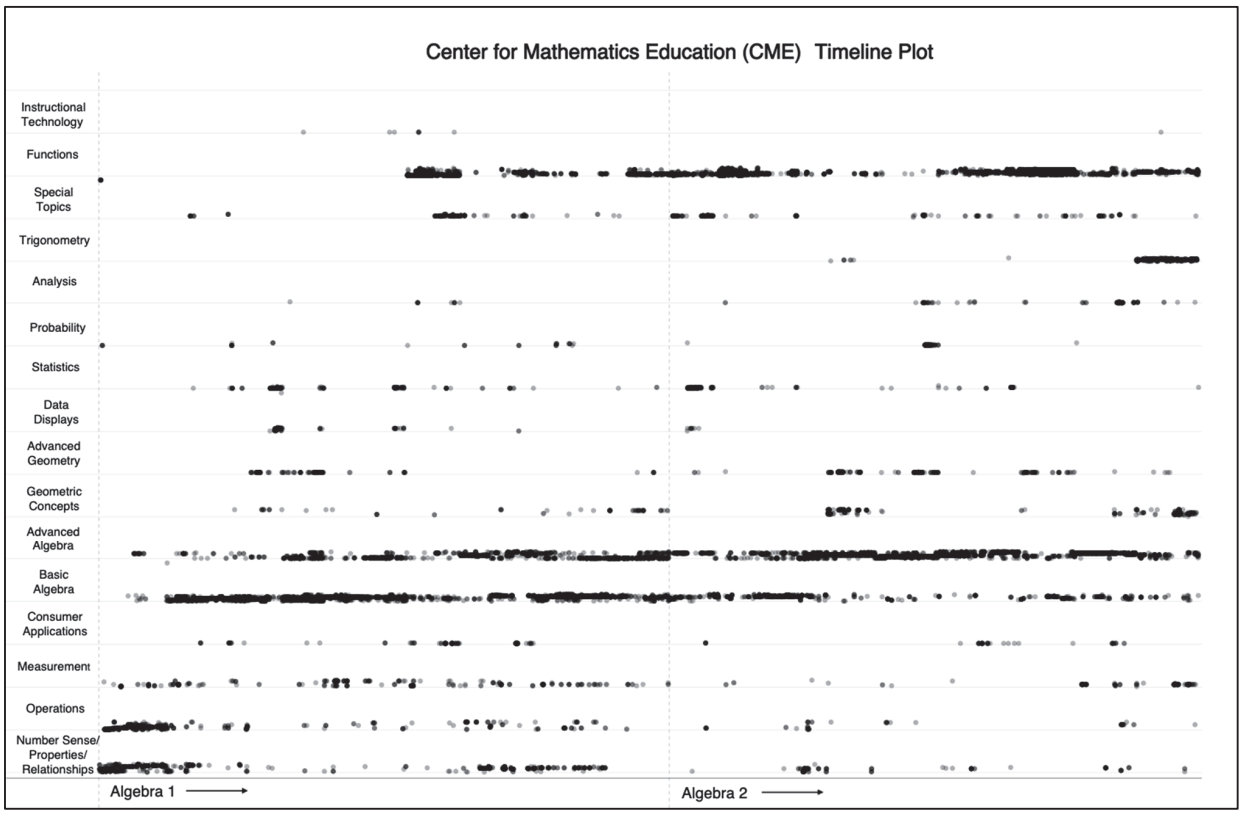

Figure A1. Timeline plot for CME. (Reprinted with permission from IAP, August 2024.).

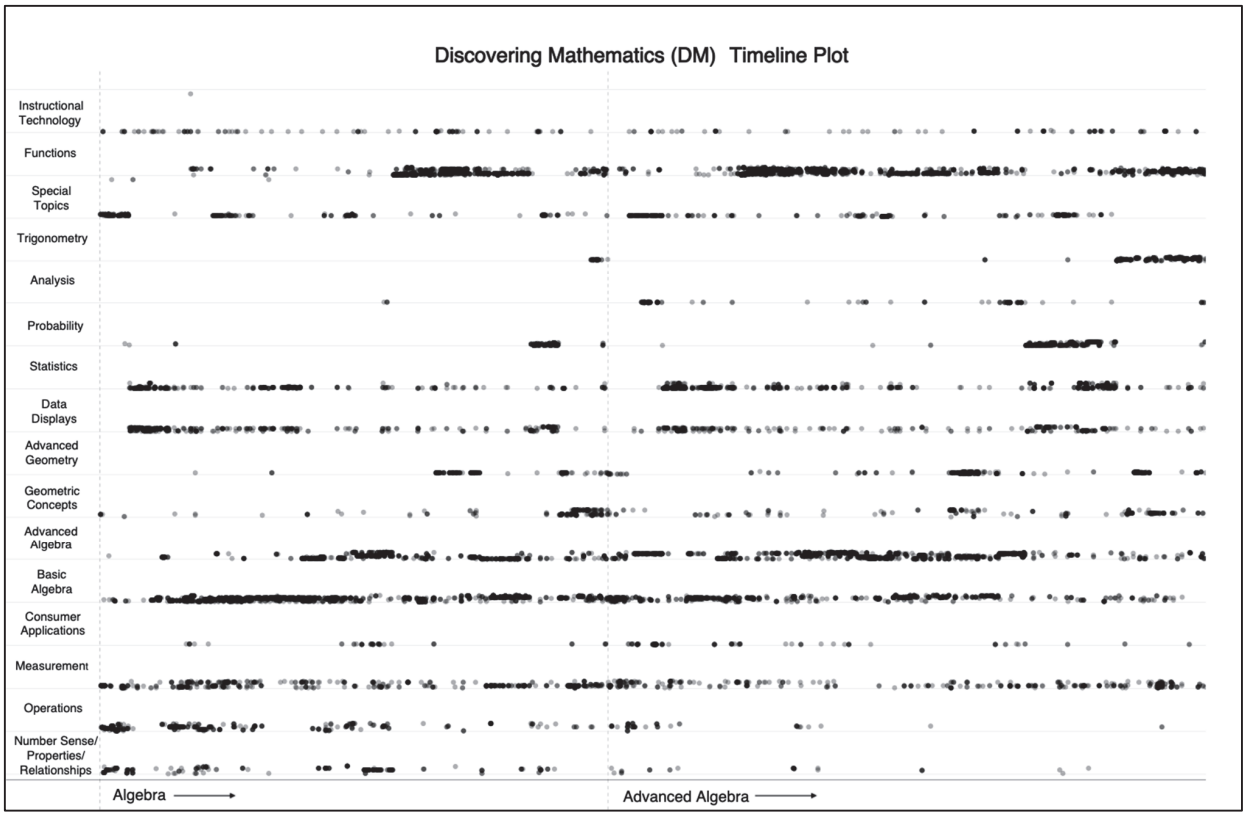

Figure A2. Timeline plot for Discovering Mathematics.

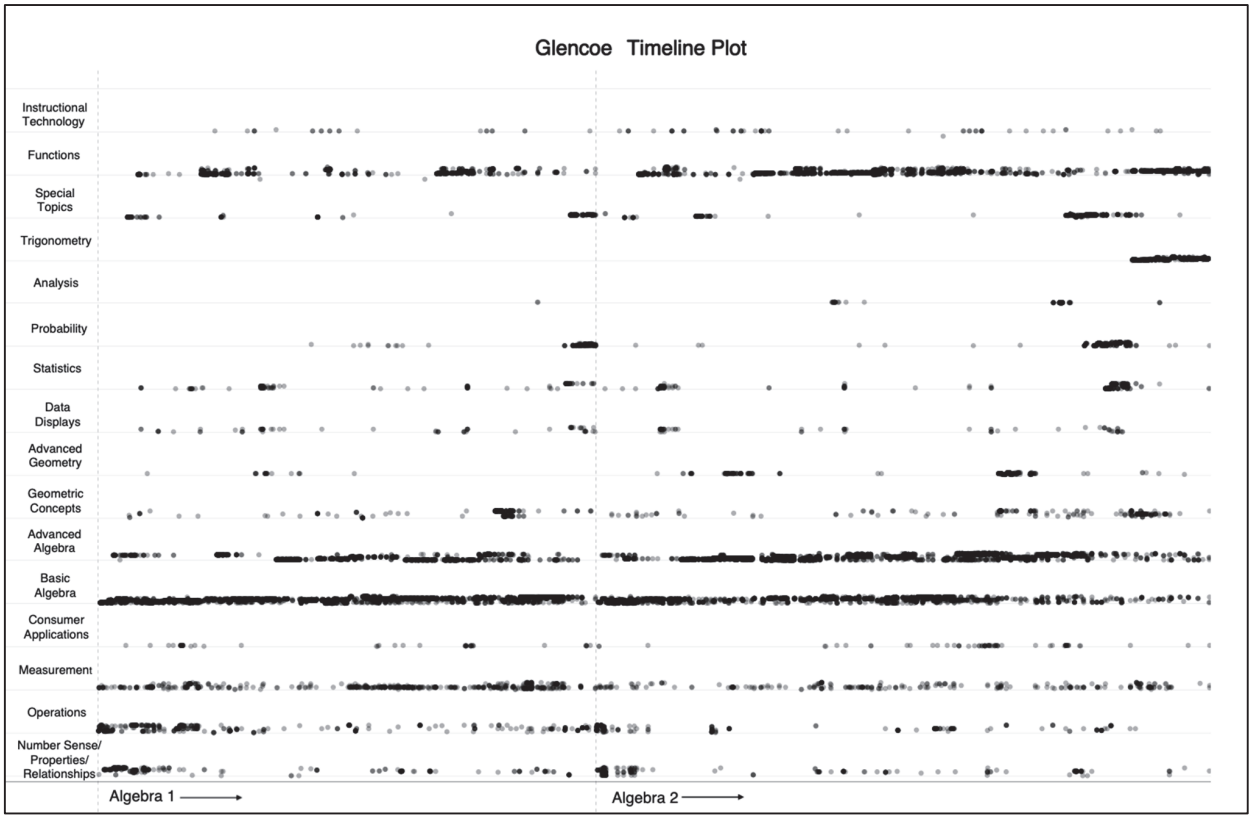

Figure A3. Timeline plot for Glencoe.

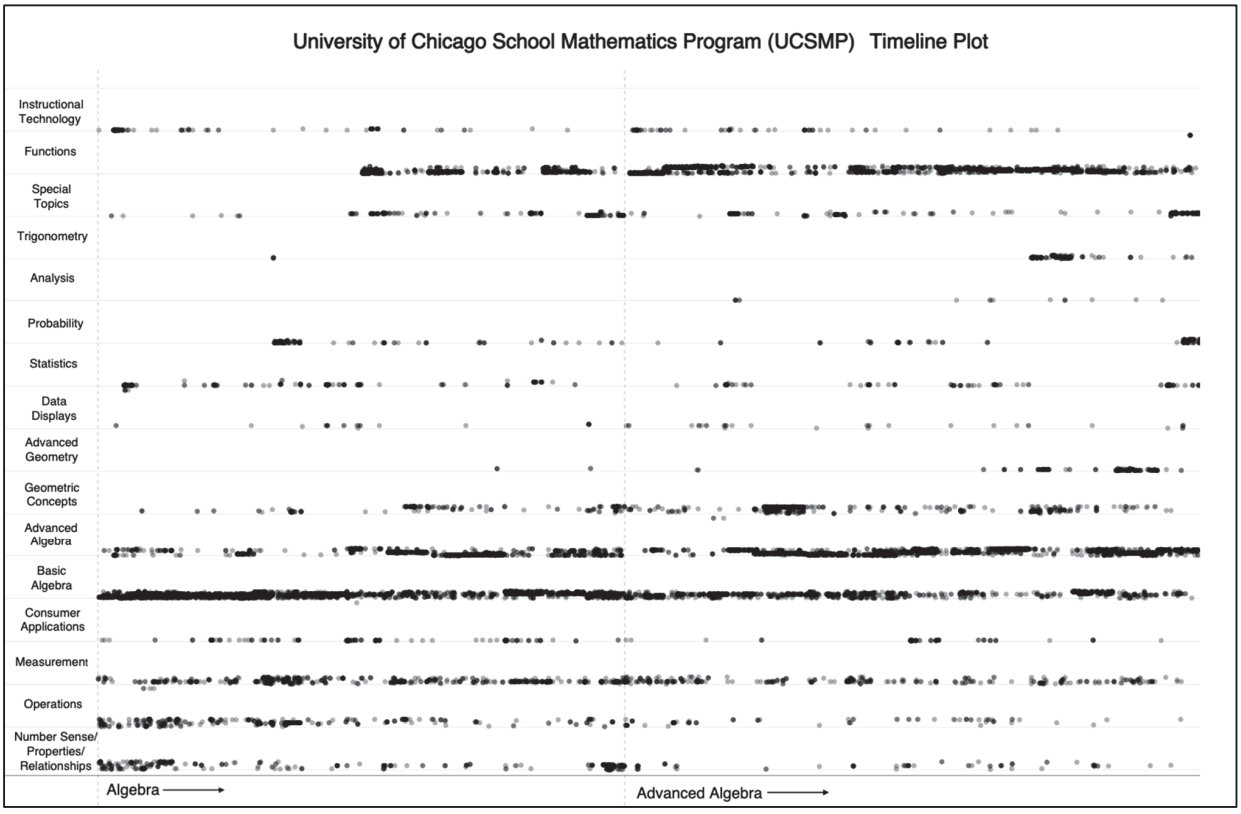

Figure A4. Timeline plot for UCSMP.

Appendix B

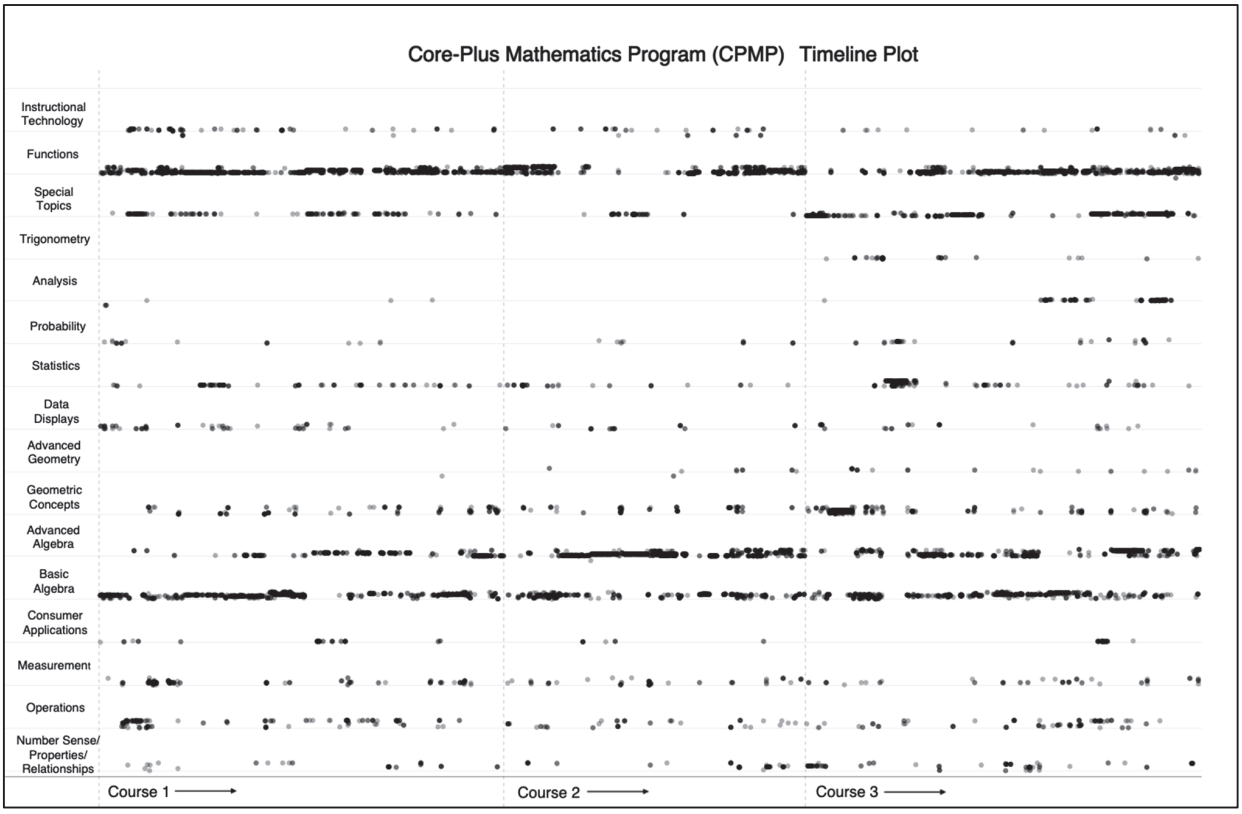

Figure A5. Timeline plot for CPMP. (Reprinted with permission from IAP, August 2024.).

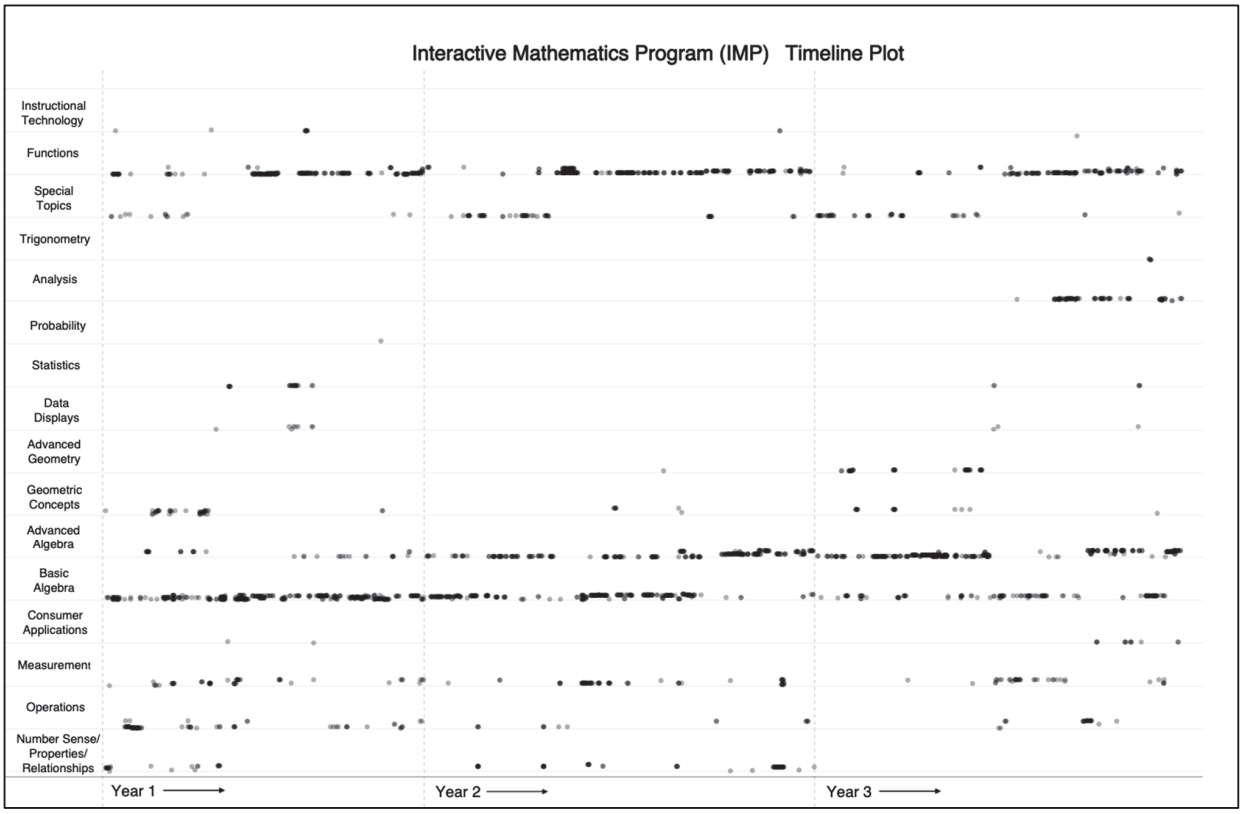

Figure A6. Timeline plot for IMP.

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