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Research on Fuzzy Logic and Mathematics with Applications, 3rd Edition

Edited by
Saeid Jafari

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Guest Editor

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About the Editor

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Saeid Jafari is a Professor of Mathematics at the College of Vestsjaelland South as well as Director of the Mathematical and Physical Science Foundation (www.topositus.com). His research interests include topology, fuzzy, neutrosophic and digital topologies, graph theory and functional analysis, fractional differential equations, differential geometry, symplectic geometry, mathematical aspects of particle physics, the Philosophy of Science, and the philosophical aspects of quantum field theory. These areas form the foundation of the various projects and research endeavors he regularly undertakes, and he has published many research papers and book chapters on these topics.

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Editorial

Research on Fuzzy Logic and Mathematics with Applications, Third Edition

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Preface

The notion of the non-rigid fuzzy set was introduced by Lotfi A. Zadeh [1] in 1965 as a generalization of the ordinary sets. In effect, the theory of fuzzy sets sparked some new computing methods which have applications in decision making, identification, optimization, information, control, etc. To be more specific, Zadeh is also the founder of fuzzy logic [2–4]. Since the advent of the notion of the fuzzy set, Zadeh, his colleagues and followers have used this important and interesting set and established a great deal of important and interesting research in fuzzy logic, fuzzy topology, fuzzy arithmetic, etc. Indeed, the theory of fuzzy mathematics stems from the notion of graded membership which is allegedly motivated by the processes of human perception and cognition. It is well-known that the fuzzy set theory and fuzzy logic are used in different parts of engineering such as control systems, robotics, automation, consumer electronics, image processing, etc. This Special Issue deals with fuzzy logic and mathematics with applications in decision making, graph theory, soft set theory, fuzzy semantics, fuzzy and neutrosophic topologies, decision-making, theory of fuzzy graphic binary metroid and theory of neutrosophic rhotrices.

Twelve research articles appear in the Special Issue "Research on Fuzzy Logic and Mathematics with Applications, Third Edition" of the MDPI *Symmetry* journal. In what follows we will give a brief account of their results and findings.

In Contribution 1, "A Novel IVBPRT-ELECTRE III Algorithm Based on Bidirectional Projection and Its Application", the authors Juxiang Wang, Min Xu, Yanjun Wang and Ziqi Zhu present an extension of bidirectional projection to probabilistic linguistic term sets in order to preserve the completeness of information to the level it is possible. In fact, the authors extend, in a probabilistic linguistic environment, the large-scale group decision-making problem to limited interval values. Moreover, they proposed a new group decision-making method named IVBPRT-ELECTRE III algorithm (ELECTRE III based on bidirectional projection and regret theory under limited interval-valued probabilistic linguistic term set.

In Contribution 2, "The Neutrosophization of δ-Separation Axioms", the authors Ahu Açikgöz, Ferhat Esenbel, Abdulhamit Maman and Seher Zorlu offer the new notion of neutrosophic δ-open sets. They also propose the interesting and useful notions of neutrosophic δ-separation axioms, which stem from the concept of neutrosophic δ-open sets. Moreover, they study the interplay between these separation properties and their characteristics within subspaces.

In Contribution 3, "A New Graph Vulnerability Parameter: Fuzzy Node Integrity", the authors Ferhan Nihan Murater and Goksen Bacak-Turan introduce the useful parameter of fuzzy node integrity, which takes into account the number of disrupted elements and the strength of residual connections. They demonstrate the applicability of this parameter, by deriving general formulas for different types of symmetric and asymmetric fuzzy graph structures, including cycle graphs, wheel graphs, and star graphs. Their findings are important with respect to both theoretical research and practical improvements in transportation, disaster management, and communication networks.

In Contribution 4, "Some Properties of Boolean-like Laws in Fuzzy Logic", the focus of the authors, Sevilay Demir Sağlam and Gül Karadeniz Gözeri, is on the relationships between fuzzy logic and classical logic properties using fuzzy t-norms, t-conorms, and fuzzy implications. Indeed, the purpose is to extend the Boolean laws in classical logic to fuzzy logic. They establish the necessary and sufficient conditions for validating the generalizations of the proposed properties from classical to fuzzy logic. They also give some examples to demonstrate practical applicability and advantages of their approach.

In Contribution 5, "Characterization of Degree Energies and Bounds in Spectral Fuzzy Graphs", the authors Ruiqi Cai, Buvaneswari Rangasamy, Senbaga Priya Karuppusamy and Aysha Khan focus largely on the degree energy of fuzzy graphs to determine fundamental spectral bounds and characterize adjacency structures. In effect, they deduce the upper bounds on the sum of squared degree eigenvalues based on vertex degree distributions and establish constraints by utilizing the characteristic polynomial of the maximum degree matrix. Moreover, they showed that the average degree energy of a connected fuzzy graph remains strictly positive. They applied their proposed framework to protein–protein interaction networks to exhibit the critical proteins and to enhance the network resilience analysis.

In Contribution 6, "Some Characterizations of k-Fuzzy γ-Open Sets and Fuzzy γ-Continuity with Further Selected Topic", Fahad Alsharari, Hind Y. Saleh and Islam M. Taha introduced the notion of k-fuzzy γ-open (k-$\mathcal{F}$-γ-open) sets as a new generalized class of fuzzy open ($\mathcal{F}$-open) sets on fuzzy topological spaces in the sense of Šostak. By utilizing this notion, they offered several properties of new mappings, separation axioms and covering properties.

In Contribution 7, "Distance Measures of (m, a, n)-Fuzzy Neutrosophic Sets and Their Applications in Decision Making", Samajh Singh Thakur pointed out that an (m, a, n)-fuzzy neutrosophic set is more flexible and efficient than the existing extensions of neutrosophic sets when discussing the distance between multiple objects. In this regard, he not only gives ten distance measures for comparing (m, a, n)-fuzzy neutrosophic sets but also apply them in pattern classifications and multi-criteria decisions. Moreover, he provides some numerical examples to show the practical use and scientific applications of these distance measures by taking into account the classifying materials and investment problems. By utilizing the comparative analysis and graphical interpretations, he demonstrates the advantage and effectiveness of the proposed measures.

In Contribution 8, "Fuzzy Graphic Binary Matroid Approach to Power Grid Communication Network Analysis", the authors Jing Li, Buvaneswari Rangasamy, Saranya Shanmugavel and Aysha Khan give the conditions for fuzzy graphic binary matroids, and also propose a systematic method for analyzing the reliability of the power grid network. This framework ensures the identification of critical circuits and enhances the network performance and stability. The investigation presented in this paper exhibits that matroid-based analysis offers a structured approach to network reliability assessment.

In Contribution 9, "Belief Entropy-Based MAGDM Algorithm Under Double Hierarchy Quantum-like Bayesian Networks and Its Application to Wastewater Reuse", Juxiang Wang, Yaping Li, Xin Wang and Janjun Wang applied a multi-attribute quantum group decision-making method based on PLTS (probabilistic linguistic term set) and QLBN (quantum-like Bayesian network) in order to select the optimal wastewater reuse alternative. Among others, they proposed a new method for solving the quantum interference

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term. They are able to explain the uncertain belief state of the DM (decision maker) by applying quantum probability theory to solve the multi-attribute group decision-making problem. They also provide a new approach for multi-attribute group decision-making. The model is applied to the case of selecting the wastewater reuse alternative in Hefei City, and the comparative analysis demonstrates that the method is effective in simulating the disagreement, contradiction, and unpredictability inherent in the human decision-making process.

In Contribution 10, “m-Polar Picture Fuzzy Bi-Ideals and Their Applications in Semigroups”, the authors Warud Nakkhasen, Atthchai Chada and Teerapan Jodnok, utilizing the concept of m-PPFSs (m-polar picture fuzzy sets) for the study of semigroups, are able to introduce and explore fundamental algebraic structures such as m-PPFSubs (m-polar picture fuzzy subsemigroups), m-PPFLs (m-polar picture fuzzy left ideals), m-PPFRs (m-polar picture fuzzy right ideals), m-PPFIs (m-polar picture fuzzy ideals), m-PPFBs (m-polar picture fuzzy bi-ideals), and m-PPFGBs (m-polar picture fuzzy generalized bi-ideals) in semigroups. In this regard, they obtained several characterizations and fundamental properties of such sets. Furthermore, they studied the relationships among various m-PPFIs in regular semigroups. They also characterized the regular semigroups by their m-PPFSubs, m-PPFLs, m-PPFRs, m-PPFIs, m-PPFBs, and m-PPFGBs.

In Contribution 11, “Double-Framed Bipolar Fuzzy Soft Sets and Algorithmic Approaches with Symmetry for Multi-Criteria Decision-Making Under Uncertainty”, Shadya M. Mershkhan and Baravan A. Asaad define and analyze the fundamental concepts and operations of double-framed bipolar fuzzy soft sets (DFBFSSs). They also provide numerical examples in order to show the practicality and applicability of DFBFSSs in complex decision-making and information modeling scenarios. Indeed, the authors in this chapter establish a formally symmetric extension of bipolar fuzzy soft set theory, introducing a nontrivial automorphism-invariant structure that is not captured by existing fuzzy or bipolar models.

In Contribution 12, “On the Symmetric Structure of Neutrosophic Rhotrices: Index Characterization, Invertibility, and Decision-Making Applications”, Paulraj Gnanachandra, Saeid Jafari, Ananda Priya Baskar and Cheirmaraj Arunthathi investigated the structure of invertible refined neutrosophic rhotrices, thereby extending the classical theory of rhotrices into the domain of refined neutrosophics. Moreover, they not only analyze the index structures of rhotrices of small orders, which establishes the general characterization of the rhotrix index set and reveals an inherent symmetry of the rhotrix structure under the interchange of indices, but also propose a rhotrix-based representation of binary relations, which shows the broader applicability of refined neutrosophic rhotrices in modeling relational and algebraic structures. To demonstrate the practical relevance of the developed framework, the authors apply a refined neutrosophic decision-making model to the evaluation of selected women empowerment schemes. In effect, the proposed model provides a systematic mechanism for multi-criteria assessment and ranking of policy interventions.

Conflicts of Interest: The author declares no conflicts of interest.

List of Contributions:

1. Wang, J.; Xu, M.; Wang, Y.; Zhu, Z. A Novel IVBPRT-ELECTRE III Algorithm Based on Bidirectional Projection and Its Application. *Symmetry* **2025**, *17*, 26.
2. Açikgöz, A.; Esenbel, F.; Maman, A.; Zorlu, S. The Neutrosophization of δ-Separation Axioms. *Symmetry* **2025**, *17*, 271.
3. Murater, F.H.; Bacak-Turan, G. A New Graph Vulnerability Parameter: Fuzzy Node Integrity. *Symmetry* **2025**, *17*, 474.

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4. Sağlam, S.D.; Gözeri, G.K. Some Properties of Boolean-like Laws in Fuzzy Logic. *Symmetry* **2025**, *17*, 548.
5. Cai, R.; Rangasamy, B.; Karuppusamy, S.P.; Khan, A. Characterization of Degree Energies and Bounds in Spectral Fuzzy Graphs. *Symmetry* **2025**, *17*, 644.
6. Alsharari, F.; Saleh, H.Y.; Taha, I.M. Some Characterizations of k-Fuzzy γ-Open Sets and Fuzzy γ-Continuity with Further Selected Topic. *Symmetry* **2025**, *17*, 678.
7. Thakur, S.S. Distance Measures of (m,a,n)-Fuzzy Neutrosophic Sets and Their Applications in Decision Making. *Symmetry* **2025**, *17*, 939.
8. Li, J; Rangasamy, B.; Shanmugavel, S.; Khan, A. Fuzzy Graphic Binary Matroid Approach to Power Grid Communication Network Analysis. *Symmetry* **2025**, *17*, 1628.
9. Wang, J.; Li, Y.; Wang, X.; Wang, Y. Belief Entropy-Based MAGDM Algorithm Under Double Hierarchy Quantum-like Bayesian Networks and Its Application to Wastewater Reuse. *Symmetry* **2025**, *17*, 2013.
10. Nakkhasen, W.; Chada, A.; Jodnok, T. m-Polar Picture Fuzzy Bi-Ideals and Their Applications in Semigroups. *Symmetry* **2025**, *17*, 2051.
11. Mershkhan, S.M.; Asaad, B.A. Double-Framed Bipolar Fuzzy Soft Sets and Algorithmic Approaches with Symmetry for Multi-Criteria Decision-Making Under Uncertainty. *Symmetry* **2026**, *18*, 119.
12. Gnanachandra, P.; Jafari, S.; Baskar, A.P.; Arunthathi, C. On the Symmetric Structure of Neutrosophic Rhotrices: Index Characterization, Invertibility, and Decision-Making Applications. *Symmetry* **2026**, *18*, 785.

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Article

On the Symmetric Structure of Neutrosophic Rhotrices: Index Characterization, Invertibility, and Decision-Making Applications

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Abstract

The rhotrix, introduced as a rhomboidal extension of traditional matrix theory, offers a unique algebraic framework for the organization and manipulation of numerical arrays. The incorporation of refined neutrosophic concepts into rhotrix theory has enabled the modeling of uncertainty, indeterminacy, and inconsistency within algebraic frameworks. Motivated by these developments, this paper investigates the structure of invertible refined neutrosophic rhotrices, thereby extending the classical theory of rhotrices into the domain of refined neutrosophic. This study is carried out using heart-based multiplication, which acts as the fundamental operation governing interactions among rhotrix elements. This setting facilitates a rigorous analysis of rhotrix algebra, specifically identifying the conditions required for invertibility and exploring the structural response of refined neutrosophic elements to basic operations defined for rhotrices. In addition to the invertibility analysis, this study analyzes the index structures of rhotrices of small orders, establishes the general characterization of the rhotrix index set, and reveals an inherent symmetry of the rhotrix structure under the interchange of indices. It also proposes a rhotrix-based representation of binary relations, illustrating the broader applicability of refined neutrosophic rhotrices in modeling relational and algebraic structures. To demonstrate the practical relevance of the developed framework, a refined neutrosophic decision-making model is applied to the evaluation of selected women empowerment schemes. By incorporating real-valued performance measures together with indeterminacy components reflecting expert uncertainty, the proposed model provides a systematic mechanism for multi-criteria assessment and ranking of policy interventions. The results contribute to the advancement of rhotrix algebra in indeterminate environments and highlight its potential applications in mathematical modeling and decision-making.

Keywords: refined neutrosophic rhotrix; rhotrix; symmetry; binary relation; invertibility

1. Introduction

The concept of the rhotrix was first introduced by Ajibade [1] in 2003 as a distinctive framework for arranging numerical elements in a rhomboidal form and defined the structure of an n-dimensional rhotrix as an entity situated between an $(n-1) \times (n-1)$ matrix and an $n \times n$ matrix. Building on this foundation, Ajibade [2] later proposed the earliest multiplication procedure for rhotrices, known as heart-based multiplication. Shortly after, Sani [3] (2004) presented an alternative operation, referred to as row–column multiplication, which parallels the classical matrix multiplication process. These two approaches form the core methods for rhotrix multiplication, while other proposed techniques have generally not met the algebraic requirements needed for establishing formal structures.

Further progress was made when Sani [4] introduced the concept of a coupled matrix, providing a systematic means of converting a rhotrix into an equivalent matrix form. In 2008, Isere [5] advanced the theory by extending the framework to even-dimensional rhotrices, thereby overcoming the limitation inherent in Ajibade's original formulation, which was confined to odd dimensions. Since then, substantial developments have expanded the scope of rhotrices. These include the establishment of rhotrix groups [6,7], rhotrix rings [8], rhotrix vector spaces [9], and rhotrix modules [10], as well as the application of rhotrix-based models in fields such as cryptography [11,12].

In the 1990s, Florentin Smarandache [13,14] established the foundations of neutrosophic logic, effectively refining traditional and fuzzy systems by incorporating a tripartite valuation. This methodology utilizes truth, indeterminacy, and falsity parameters to navigate information that is inherently inconsistent or incomplete. This tripartite framework provides a robust means of addressing complex real-world problems characterized by incomplete, inconsistent, or uncertain information.

Building on this foundation, Kandasamy and Smarandache [15–17] developed neutrosophic algebraic structures that explicitly incorporate the indeterminacy parameter I. These generalizations have since been utilized to establish neutrosophic groups [18], modules [19], and vector spaces [20]. In subsequent developments, Abobala [21] examined the existence of inverses for refined neutrosophic matrices and later generalized this investigation to the case of n-refined neutrosophic matrices, thereby enriching the theoretical landscape of the neutrosophic structure.

Multi-criteria decision-making (MCDM) techniques have been widely applied in complex industrial decision environments, such as supplier selection in the textile industry [22] and strategic decision-making in the frozen shrimp sector [23]. Motivated by these applications, the present study employs an MCDM framework based on refined neutrosophic rhotrices to evaluate women-centric welfare schemes implemented by the Government of Tamil Nadu.

Let n be a positive odd integer. The standard configuration for a rhotrix of order n is defined as follows:

$$\left\langle \begin{array}{ccccccccc} & & & & x_1 & & & & \\ & & & x_2 & x_3 & x_4 & & & \\ & & .. & .. & .. & .. & .. & & \\ & .. & .. & .. & .. & .. & .. & .. & \\ x_{\{\frac{t+1}{2}\}-\frac{n}{2}} & .. & .. & .. & x_{\frac{t+1}{2}} & .. & .. & .. & x_{\{\frac{t+1}{2}\}+\frac{n}{2}} \\ & .. & .. & .. & .. & .. & .. & .. & \\ & & .. & .. & .. & .. & .. & & \\ & & & x_{t-3} & x_{t-2} & x_{t-1} & & & \\ & & & & x_t & & & & \end{array} \right\rangle,$$

where $t = \frac{n^2+1}{2}$. The entries $x_1, x_4, \cdots, x_{\{\frac{t+1}{2}\}+\frac{n}{2}}$ represent the first row, while the entries $x_1, x_2, \cdots, x_{\{\frac{t+1}{2}\}-\frac{n}{2}}$ represent the first column. The heart of a rhotrix $x_{\frac{t+1}{2}}$ is defined as the element positioned at the intersection of its major horizontal and vertical axes.

The computational framework for rhotrices relies on two primary operations: addition and heart-based multiplication. In heart-based multiplication, as formulated by Ajibade, the product is derived by scaling each entry of the first rhotrix by the central element (the "heart") of the second, then adding it to the product of the second rhotrix's entries and

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the heart of the first. Conversely, addition remains a straightforward, entry-wise process, where corresponding elements are summed directly.

The motivation behind this study is that the classical rhotrix theory deals with numerical entries defined over a field, which makes it suitable for deterministic mathematical modeling. However, many real-world problems involve uncertainty, vagueness, and incomplete information that cannot be adequately captured using precise numerical values. While fuzzy and intuitionistic fuzzy extensions address partial uncertainty, they do not explicitly incorporate indeterminacy. Although the neutrosophic rhotrix incorporates indeterminacy through a single neutrosophic component, it remains limited in its ability to represent multiple or layered sources of uncertainty. In many practical situations, indeterminacy may arise from different factors that cannot be adequately captured by a single indeterminate element.

To address this limitation, refined neutrosophic theory introduces multiple independent indeterminacy components, allowing a more granular and flexible representation of uncertain information. This motivates the development of refined neutrosophic rhotrices, in which each entry is expressed using multiple indeterminacy terms. Consequently, refined neutrosophic rhotrices provide a natural and necessary generalization of neutrosophic rhotrices, enhancing their applicability in complex decision-making and uncertain environments.

In this paper, Section 2 aims to advance rhotrix theory by investigating the fundamental properties of refined neutrosophic rhotrices, specifically focusing on their algebraic structure under heart-based multiplication. In Section 3, we examine the index structure of rhotrices and establish a general characterization of the rhotrix index set, highlighting its inherent symmetry under the interchange of indices. Beyond theoretical expansion, in Section 4, we demonstrate the practical utility of this developed framework through a refined neutrosophic decision-making model applied to the evaluation of women empowerment initiatives. By integrating real-valued performance metrics with indeterminacy components that reflect expert ambiguity, the proposed model provides a systematic mechanism for the multi-criteria assessment and ranking of complex policy interventions.

2. Invertibility of Refined Neutrosophic Rhotrices Under Heart-Based Multiplication

In this section, the inverse of a refined neutrosophic rhotrix is derived with respect to heart-based multiplication. It has been established in [24] that the operation is associative and that the corresponding multiplicative identity has been formulated under heart-based multiplication.

Definition 1. *A refined neutrosophic rhotrix is defined as a rhotrix with each of the entries from a refined neutrosophic field $(K(I_1, I_2), +, \cdot)$, where $I_1^2 = I_1, \quad I_2^2 = I_2, \quad I_1 I_2 = I_2 I_1 = I_1$.*

Suppose A is a rhotrix in $R_n(K)$, then the refined neutrosophic rhotrix is represented as

$$A_R = \left\langle \begin{array}{ccccccc}
 & & & \begin{matrix} a_1 + \\ k_1 I_1 + k_2 I_2 \end{matrix} & & & \\
 & & \begin{matrix} a_2 + \\ k_3 I_1 + k_4 I_2 \end{matrix} & \begin{matrix} a_3 + \\ k_5 I_1 + k_6 I_2 \end{matrix} & \begin{matrix} a_4 + \\ k_7 I_1 + k_8 I_2 \end{matrix} & & \\
 & .. & .. & .. & .. & .. & \\
 & .. & .. & .. & .. & .. & \\
\begin{matrix} a_{\{\frac{t+1}{2}\} - \frac{n}{2}} + \\ k_{t-n} I_1 + k_{t+1-n} I_2 \end{matrix} & .. & .. & \begin{matrix} a_{\frac{t+1}{2}} + \\ k_t I_1 + k_{t+1} I_2 \end{matrix} & .. & .. & \begin{matrix} a_{\{\frac{t+1}{2}\} + \frac{n}{2}} + \\ k_{t+n} I_1 + k_{t+1+n} I_2 \end{matrix} \\
 & .. & .. & .. & .. & .. & \\
 & .. & .. & .. & .. & .. & \\
 & & \begin{matrix} a_{t-3} + \\ k_{2t-7} I_1 + k_{2t-6} I_2 \end{matrix} & \begin{matrix} a_{t-2} + \\ k_{2t-5} I_1 + k_{2t-4} I_2 \end{matrix} & \begin{matrix} a_{t-1} + \\ k_{2t-3} I_1 + k_{2t-2} I_2 \end{matrix} & & \\
 & & & \begin{matrix} a_t + \\ k_{2t-1} I_1 + k_{2t} I_2 \end{matrix} & & &
\end{array} \right\rangle.$$

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And that can be re-written as

$$A_R = A + \left\langle \begin{array}{ccccccccc} & & & & k_1 & & & & \\ & & & k_3 & k_5 & k_7 & & & \\ & & .. & .. & .. & .. & .. & & \\ & .. & .. & .. & .. & .. & .. & .. & \\ k_{t-n} & .. & .. & .. & k_t & .. & .. & .. & k_{t+n} \\ & .. & .. & .. & .. & .. & .. & .. & \\ & & .. & .. & .. & .. & .. & & \\ & & & k_{2t-7} & k_{2t-5} & k_{2t-3} & & & \\ & & & & k_{2t-1} & & & & \end{array} \right\rangle I_1 +$$

$$\left\langle \begin{array}{ccccccccc} & & & & k_2 & & & & \\ & & & k_4 & k_6 & k_8 & & & \\ & & .. & .. & .. & .. & .. & & \\ & .. & .. & .. & .. & .. & .. & .. & \\ k_{t+1-n} & .. & .. & .. & k_{t+1} & .. & .. & .. & k_{t+1+n} \\ & .. & .. & .. & .. & .. & .. & .. & \\ & & .. & .. & .. & .. & .. & & \\ & & & k_{2t-6} & k_{2t-4} & k_{2t-2} & & & \\ & & & & k_{2t} & & & & \end{array} \right\rangle I_2, \text{where all } k_i \in K$$

$$A_R = A + K_{2i-1} I_1 + K_{2i} I_2,$$

where

$$K_{2i-1} = \left\langle \begin{array}{ccccccccc} & & & & k_1 & & & & \\ & & & k_3 & k_5 & k_7 & & & \\ & & .. & .. & .. & .. & .. & & \\ & .. & .. & .. & .. & .. & .. & .. & \\ k_{t-n} & .. & .. & .. & k_t & .. & .. & .. & k_{t+n} \\ & .. & .. & .. & .. & .. & .. & .. & \\ & & .. & .. & .. & .. & .. & & \\ & & & k_{2t-7} & k_{2t-5} & k_{2t-3} & & & \\ & & & & k_{2t-1} & & & & \end{array} \right\rangle \text{ and}$$

$$K_{2i} = \left\langle \begin{array}{ccccccccc} & & & & k_2 & & & & \\ & & & k_4 & k_6 & k_8 & & & \\ & & .. & .. & .. & .. & .. & & \\ & .. & .. & .. & .. & .. & .. & .. & \\ k_{t+1-n} & .. & .. & .. & k_{t+1} & .. & .. & .. & k_{t+1+n} \\ & .. & .. & .. & .. & .. & .. & .. & \\ & & .. & .. & .. & .. & .. & & \\ & & & k_{2t-6} & k_{2t-4} & k_{2t-2} & & & \\ & & & & k_{2t} & & & & \end{array} \right\rangle.$$

Example 1. *Consider the refined neutrosophic field $K(I_1, I_2)$, where $K = \mathbb{R}$ and the indeterminacy components satisfy $I_1^2 = I_1$, $I_2^2 = I_2$, and $I_1 I_2 = I_2 I_1 = I_1$.*

A refined neutrosophic rhotrix of order 3 can be written as

$$R = \left\langle \begin{array}{ccc} & 2 + I_1 + 0.2I_2 & \\ 1 + 0.2I_1 + I_2 & 3 + I_1 + I_2 & 4 + 0.3I_1 \\ & 5 + I_2 & \end{array} \right\rangle.$$

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Each entry of R is of the form $a + k_1 I_1 + k_2 I_2$, where $a, k_1, k_2 \in \mathbb{R}$. Hence, R satisfies the definition of a refined neutrosophic rhotrix.

Remark 1 (Distinction from Existing Neutrosophic Rhotrices)**.** *The concept of a neutrosophic rhotrix, as introduced in earlier studies, considers each entry in the rhotrix to be of the form $a + bI$, where I represents a single indeterminacy component satisfying $I^2 = I$. In contrast, the refined neutrosophic rhotrix defined in this work extends this representation by incorporating multiple independent indeterminacy components, namely I_1 and I_2, satisfying the relations $I_1^2 = I_1$, $I_2^2 = I_2$, and $I_1 I_2 = I_2 I_1 = I_1$.*

This refinement enables each entry of the rhotrix to be expressed in the form

$$a + k_1 I_1 + k_2 I_2,$$

thereby allowing a more detailed and flexible modeling of indeterminacy. Consequently, the proposed refined neutrosophic rhotrix generalizes the classical neutrosophic rhotrix by capturing multiple levels or sources of uncertainty, which cannot be represented within the single-indeterminacy framework.

Definition 2. *We define the operations on refined neutrosophic rhotrices as follows:*

(i) *The addition of two refined neutrosophic rhotrices is possible only if they are of the same size.*

An addition of two refined neutrosophic rhotrices A_R and A'_R is defined as

$$A_R + A'_R = A + (K_{2i-1} + K'_{2i-1})I_1 + (K_{2i} + K'_{2i})I_2.$$

That is, for rhotrices of size 3,

$$\begin{aligned} A_R &= \left\langle \begin{array}{ccc} & a_1 + k_1 I_1 + k_2 I_2 & \\ a_2 + k_3 I_1 + k_4 I_2 & a_3 + k_5 I_1 + k_6 I_2 & a_4 + k_7 I_1 + k_8 I_2 \\ & a_5 + k_9 I_1 + k_{10} I_2 & \end{array} \right\rangle \\ &= A + K_{2i-1} I_1 + K_{2i} I_2, \end{aligned}$$

and

$$\begin{aligned} A'_R &= \left\langle \begin{array}{ccc} & a'_1 + k'_1 I_1 + k'_2 I_2 & \\ a'_2 + k'_3 I_1 + k'_4 I_2 & a'_3 + k'_5 I_1 + k'_6 I_2 & a'_4 + k'_7 I_1 + k'_8 I_2 \\ & a'_5 + k'_9 I_1 + k'_{10} I_2 & \end{array} \right\rangle \\ &= A' + K'_{2i-1} I_1 + K'_{2i} I_2, \end{aligned}$$

of size 3,

$$A_R + A'_R = (A + A') + (K_{2i-1} + K'_{2i-1})I_1 + (K_{2i} + K'_{2i})I_2.$$

(ii) *The heart-based refined neutrosophic multiplication of two refined neutrosophic rhotrices is possible only if they are of the same size.*

Multiplication of two refined neutrosophic rhotrices A_R and A'_R is defined as

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$$
A_R \circ A'_R = \left\langle \begin{array}{ccccccc}
 & & & \begin{array}{c}[(a_1+k_1I_1+k_2I_2)\cdot \\ (a'_{\frac{t+1}{2}}+k'_tI_1+k'_{t+1}I_2)]+ \\ [(a'_1+k'_1I_1+k'_2I_2)\cdot \\ (a_{\frac{t+1}{2}}+k_tI_1+k_{t+1}I_2)]\end{array} & & & \\
 & & .. & .. & .. & & \\
 & .. & .. & .. & .. & .. & \\
 & .. & .. & .. & .. & .. & \\
.. & .. & .. & \begin{array}{c}(a_{\frac{t+1}{2}}+k_tI_1+k_{t+1}I_2)\cdot \\ (a'_{\frac{t+1}{2}}+k'_tI_1+k'_{t+1}I_2)\end{array} & .. & .. & .. \\
 & .. & .. & .. & .. & .. & \\
 & .. & .. & .. & .. & .. & \\
 & & .. & .. & .. & & \\
 & & & \begin{array}{c}[(a_t+k_{2t-1}I_1+k_{2t}I_2)\cdot \\ (a'_{\frac{t+1}{2}}+k'_tI_1+k'_{t+1}I_2)]+ \\ [(a'_t+k'_{2t-1}I_1+k'_{2t}I_2)\cdot \\ (a_{\frac{t+1}{2}}+k_tI_1+k_{t+1}I_2)]\end{array} & & &
\end{array} \right\rangle,
$$

where $\circ$ represents the multiplication among the refined neutrosophic numbers.

Definition 3. *Let* $R = (r_{ij}) = \left\langle \begin{array}{ccccccccc}
 & & & & r_{1,1} & & & & \\
 & & & r_{3,1} & r_{2,2} & r_{3,1} & & & \\
 & & r_{5,1} & r_{4,2} & r_{3,3} & r_{2,4} & r_{1,5} & & \\
 & .. & .. & .. & .. & .. & .. & .. & \\
r_{n,1} & .. & .. & .. & r_{\frac{n+1}{2},\frac{n+1}{2}} & .. & .. & .. & r_{1,n} \\
 & .. & .. & .. & .. & .. & .. & .. & \\
 & & .. & .. & .. & .. & .. & & \\
 & & & r_{n,n-2} & r_{n-1,n-1} & r_{n-2,n} & & & \\
 & & & & r_{n,n} & & & &
\end{array} \right\rangle$ *be a rhotrix of order $n = 2k+1$. The transpose of R, denoted by R^T, is defined by*

$$
(R^T)_{ij} = r_{ji}, \quad \textit{for all } (i,j)
$$

Definition 4. *A refined neutrosophic rhotrix R is said to be idempotent under heart-based multiplication if*

$$
R \circ R = R.
$$

Theorem 1. *Let R be an idempotent 2-refined neutrosophic rhotrix under heart-based multiplication. Then the heart element of* 2*-refined neutrosophic rhotrices under heart-based multiplication is finite and consists of exactly six elements.*

The six idempotent refined neutrosophic elements are given by

(i) $0 + 1I_1 + 0I_2$
(ii) $0 + 1I_1 + (-1)I_2$
(iii) $1 + 0I_1 + 0I_2$
(iv) $1 + (-1)I_1 + 0I_2$
(v) $1 + 0I_1 + (-1)I_2$
(vi) $1 + 1I_1 + (-1)I_2$

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Proof. Let $A_R = \left\langle \begin{array}{ccc} & a+k_1I_1+k_2I_2 & \\ b+k_3I_1+k_4I_2 & c+k_5I_1+k_6I_2 & d+k_7I_1+k_8I_2 \\ & e+k_9I_1+k_{10}I_2 & \end{array} \right\rangle$ be an idempotent 2-refined neutrosophic rhotrix.

Then $A_R \circ A_R = A_R$, and by heart-based multiplication, $h(A_R) \cdot h(A_R) = h(A_R)$. This implies

$$(c+k_5I_1+k_6I_2) \cdot (c+k_5I_1+k_6I_2) = c+k_5I_1+k_6I_2$$
$$c^2 + (2ck_5 + k_5^2 + 2k_5k_6)I_1 + (2ck_6 + k_6^2)I_2 = c + k_5I_1 + k_6I_2.$$

On comparing both sides, we get, $c^2 = c$, $2ck_5 + k_5^2 + 2k_5k_6 = k_5$ and $2ck_6 + k_6^2 = k_6$. Then $c^2 = c \implies c(c-1) = 0 \implies c = 0$ or $c = 1$.

Case (a): Let $c = 0$.
$2ck_5 + k_5^2 + 2k_5k_6 = k_5 \implies k_5 + 2k_6 = 1$ and
$2ck_6 + k_6^2 = k_6 \implies k_6^2 = k_6 \implies k_6(k_6 - 1) = 0 \implies k_6 = 0$ or 1.

If $k_6 = 0$, then $k_5 + 2k_6 = 1 \implies k_5 = 1$.
If $k_6 = 1$, then $k_5 + 2k_6 = 1 \implies k_5 = -1$.
Therefore, from case (a), the heart element in idempotent 2-refined neutrosophic rhotrix possibilities are $0 + 1I_1 + 0I_2$ and $0 + 1I_1 + (-1)I_2$.

Case (b): Let $c = 1$.
$2ck_5 + k_5^2 + 2k_5k_6 = k_5 \implies k_5^2 + k_5 + 2k_5k_6 = 0$ and
$2ck_6 + k_6^2 = k_6 \implies k_6^2 + k_6 = 0 \implies k_6(k_6 + 1) = 0 \implies k_6 = 0$ or -1.

If $k_6 = 0$, then $k_5^2 + k_5 + 2k_5k_6 = 0 \implies k_5 = 0$ or -1.
If $k_6 = 1$, then $k_5^2 + k_5 + 2k_5k_6 = 0 \implies k_5 = 0$ or 1.
Therefore, from case (b), the other possibilities of the heart element in an idempotent 2-refined neutrosophic rhotrix are $1 + 0I_1 + 0I_2$, $1 + (-1)I_1 + 0I_2$, $1 + 0I_1 + (-1)I_2$ and $1 + 1I_1 + (-1)I_2$. □

Corollary 1. *The identity rhotrix and zero rhotrix are the only rhotrices to be idempotent.*

Proof. From the above theorem, we get that the heart of the simple rhotrix is either 0 or 1. And the multiplication of two of the same rhotrices gives twice its heart multiplied by the rhotrix. That is, each entry in $A_R \circ A_R$ is $2h(A_R)a_{ij}$ for all entries except *heart*. Since $h(A_R) = 0$ or 1, $2a_{ij} = a_{ij} \implies a_{ij} = 0$. □

Lemma 1 ([24])**.** *The heart-based multiplication operation among any refined neutrosophic rhotrix $A_R = A + K_{2i-1}I_1 + K_{2i}I_2$ holds the following axioms:*

- $A_R \circ I_R = A_R$*, where*

$$I_R = \left\langle \begin{array}{ccccccccc} & & & & 0+0I_1+0I_2 & & & & \\ & & & 0+0I_1+0I_2 & 0+0I_1+0I_2 & 0+0I_1+0I_2 & & & \\ & & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & & \\ & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \\ 0+0I_1+0I_2 & \cdot\cdot & \cdot\cdot & \cdot\cdot & 1+0I_1+0I_2 & \cdot\cdot & \cdot\cdot & \cdot\cdot & 0+0I_1+0I_2 \\ & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \\ & & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & & \\ & & & 0+0I_1+0I_2+ & 0+0I_1+0I_2 & 0+0I_1+0I_2 & & & \\ & & & & 0+0I_1+0I_2 & & & & \end{array} \right\rangle,$$

(Multiplicative Identity) if A is invertible,

- $A_R \circ A'_R = A'_R \circ A_R$ *(Commutativity), where A_R, A'_N are any refined neutrosophic rhotrices*

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Theorem 2. *For any refined neutrosophic rhotrix A_R, the multiplicative inverse is*

$$A_R^{-1} = \left\langle \begin{array}{ccc} & \dfrac{-a_1}{a^2_{\frac{t+1}{2}}} + k'_1 I_1 + k'_2 I_2 & \\ \quad\ldots\ \ldots & \ldots & \ldots\ \ldots\quad \\ \ldots\ \ldots\ \ldots & \ldots & \ldots\ \ldots\ \ldots \\ \ldots\ \ldots\ \ldots & \dfrac{1}{a_{\frac{t+1}{2}}} + \dfrac{-k_t}{(a_{\frac{t+1}{2}} + k_t + k_{t+1})(a_{\frac{t+1}{2}} + k_{t+1})} I_1 + & \ldots\ \ldots\ \ldots \\ \ldots\ \ldots\ \ldots & \dfrac{-k_{t+1}}{a_{\frac{t+1}{2}}(a_{\frac{t+1}{2}} + k_{t+1})} I_2 & \ldots\ \ldots\ \ldots \\ \ldots\ \ldots\ \ldots & \ldots & \ldots\ \ldots\ \ldots \\ \quad\ldots\ \ldots & \ldots & \ldots\ \ldots\quad \\ & \dfrac{-a_t}{a^2_{\frac{t+1}{2}}} + k'_{2t-1} I_1 + k'_{2t} I_2 & \end{array} \right\rangle,$$

under the heart-based multiplication operation, and A is invertible.

Proof. Let us consider any two refined neutrosophic rhotrices A_R, A_R^{-1}.

Suppose $A_R = \left\langle \begin{array}{ccc} & a_1 + k_1 I_1 + k_2 I_2 & \\ a_2 + k_3 I_1 + k_4 I_2 & a_3 + k_5 I_1 + k_6 I_2 & a_4 + k_7 I_1 + k_8 I_2 \\ & a_5 + k_9 I_1 + k_{10} I_2 & \end{array} \right\rangle$ and $A_R^{-1} = \left\langle \begin{array}{ccc} & a'_1 + k'_1 I_1 + k'_2 I_2 & \\ a'_2 + k'_3 I_1 + k'_4 I_2 & a'_3 + k'_5 I_1 + k'_6 I_2 & a'_4 + k'_7 I_1 + k'_8 I_2 \\ & a'_5 + k'_9 I_1 + k'_{10} I_2 & \end{array} \right\rangle$ of size 3.

The 1st entry of

$$\begin{aligned} A_R \circ A_R^{-1} =& [(a_1 + k_1 I_1 + k_2 I_2) \cdot (a'_3 + k'_5 I_1 + k'_6 I_2)] + \\ & [(a'_1 + k'_1 I_1 + k'_2 I_2) \cdot (a_3 + k_5 I_1 + k_6 I_2)] \\ =& a_1 a'_3 + a'_1 a_3 + [k_1(a'_3 + k'_5 + k'_6) + k'_1(a_3 + k_5 + k_6) + k'_5(a_1 + k_2) + \\ & k_5(a'_1 + k'_2)] I_1 + [a_1 k'_6 + a'_1 k_6 + k_2(a'_3 + k'_6) + k'_2(a_3 + k_6)] I_2. \end{aligned}$$

The *heart* entry of

$$\begin{aligned} A_R \circ I_R =& (a_3 + k_5 I_1 + k_6 I_2) \cdot a'_3 + k'_5 I_1 + k'_6 I_2 \\ =& a_3 a'_3 + [k'_5(a_3 + k_6) + k_5(a'_3 + k'_5 + k'_6)] I_1 + [k_6 a'_3 + \\ & k'_6(a_3 + k_6)] I_2. \end{aligned}$$

For $A_R \circ A_R^{-1} = I_R$, by comparing the heart entry of $A_R \circ A_R^{-1}$ with the heart entry of I_R, we get $a_3 a'_3 = 1$, which implies

$$\boxed{a'_3 = \frac{1}{a_3}.}$$

Then, by comparing the I_2 coordinates of heart entries of $A_R \circ A_R^{-1}$ and I_R, we get $k_6 a'_3 + k_6 k'_6 + a_3 k'_6 = 0$. By applying the value of a'_3, we obtain

$$\boxed{k'_6 = \frac{-k_6}{a_3(a_3 + k_6)} = Heart[(A + K_{2i})^{-1} - A^{-1}].}$$

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Then, by comparing the I_1 coordinates of heart entries of $A_R \circ A_R^{-1}$ and I_R, we get $a_3k'_5 + k_5a'_3 + k_5k'_5 + k_5k'_6 + k_6k'_5 = 0$. By applying the value of a'_3 and k'_6, we obtain

$$\boxed{k'_5 = \frac{-k_5}{(a_3 + k_5 + k_6)(a_3 + k_6)} = Heart[(A + K_{2i-1} + K_{2i})^{-1} - (A + K_{2i})^{-1}].}$$

By comparing the 1st entry of $A_R \circ A_R^{-1}$ with the 1st entry of I_R, we get $a_1a'_3 + a'_1a_3 = 0$, which implies

$$\boxed{a'_1 = \frac{-a_1}{a_3^2}.}$$

By applying this strategy to the 2nd, 4th and 5th entry, we get

$$\boxed{a'_2 = \frac{-a_2}{a_3^2}}, \boxed{a'_4 = \frac{-a_4}{a_3^2}}, \boxed{a'_5 = \frac{-a_5}{a_3^2}}.$$

Then, by comparing the I_2 coordinates of the 1st entries of $A_R \circ A_R^{-1}$ and I_R, we get $a'_1k_6 + a_1k'_6 + k_2a'_3 + k'_2a_3 + k_2k'_6 + k'_2k'_6 = 0$. By applying the value of a'_3 and k'_6, we obtain

$$\boxed{k'_2 = \frac{a_1k_6^2 + 2a_1a_3k_6 - a_3^2k_2}{a_3^4 + a_3^3k_6 - a_3k_6}.}$$

Then, by comparing the I_1 coordinates of the 1st entries of $A_R \circ A_R^{-1}$ and I_R, we get $(a_1 + k_1 + k_2)k'_5 + (a'_3 + k'_6)k_1 - a'_1k_6 + (a_3 + k_5 + k_6)k'_1 = 0$. By applying the value of a'_1, a'_3, k'_5 and k'_6, we obtain

$$\boxed{k'_1 = \frac{1}{(a_3 + k_5 + k_6)}\left[\frac{1}{a_3 + k_6}k_1 + \left(\frac{-(a_1 + k_1 + k_2)}{(a_3 + k_5 + k_6)(a_3 + k_6)}\right)k_5 + \left(\frac{a_1}{a_3^2}\right)k_6\right].}$$

By using this similar strategy, we can find $k'_3, k'_4, k'_7, k'_8, k'_9, k'_{10}$. Therefore, in general, we get

$$A_R^{-1} = \left\langle \begin{array}{ccccccc} & & & \dfrac{-a_1}{a_{\frac{t+1}{2}}^2} + k'_1I_1 + k'_2I_2 & & & \\ & .. & .. & .. & .. & .. & \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & \dfrac{1}{a_{\frac{t+1}{2}}} + \dfrac{-k_t}{(a_{\frac{t+1}{2}} + k_t + k_{t+1})(a_{\frac{t+1}{2}} + k_{t+1})}I_1 + & .. & .. & .. \\ .. & .. & .. & \dfrac{-k_{t+1}}{a_{\frac{t+1}{2}}(a_{\frac{t+1}{2}} + k_{t+1})}I_2 & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ & .. & .. & .. & .. & .. & \\ & & & \dfrac{-a_t}{a_{\frac{t+1}{2}}^2} + k'_{2t-1}I_1 + k'_{2t}I_2 & & & \end{array} \right\rangle,$$

where k'_i can be derived using a'_i, k'_t, and k'_{t+1}. □

From the above theorem, we observe that invertibility under heart-based multiplication is governed by the invertibility of the associated component rhotrices, which implicitly determines the behavior of the heart element.

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Theorem 3. *Let $A_R = A + K_{2i-1}I_1 + K_{2i}I_2$ be any refined neutrosophic rhotrix. Then A_R is invertible if $A, A + K_{2i}$ and $A + K_{2i-1} + K_{2i}$ are invertible.*

Proof. We know that, for any rhotrix A to be invertible, *Heart* of A must not be zero. Suppose A_R is any invertible refined neutrosophic rhotrix. Then $Heart(A_R) \neq 0$. By using Theorem 2, we get $Heart(A) \neq 0, Heart(A + K_{2i}) \neq 0, Heart(A + K_{2i-1} + K_{2i}) \neq 0$. This implies that $A, A + K_{2i}$ and $A + K_{2i-1} + K_{2i}$ are invertible. By using Theorem 2, the converse part is also true. □

Theorem 4. *The set of all invertible refined neutrosophic rhotrices forms a group under heart-based multiplication.*

Proof. Since multiplication of two refined neutrosophic numbers is the refined neutrosophic number and multiplication of rhotrices is closed and associative under heart-based multiplication, closure and associative axioms hold. By directly applying the preceding Lemma 1, we obtain that identity with respect to heart-based multiplication exists and by Theorem 2, it proves that the inverse with respect to heart-based multiplication exists. Thus, it follows immediately that the collection of all refined neutrosophic rhotrices constitutes a group with respect to heart-based multiplication. □

3. On the Correlation Between Binary Relations and Symmetric Properties of Rhotrix Structures

3.1. Structural Symmetry of the Rhotrix Index Set

Building upon the indices of lower-order rhotrices, this section establishes the index set for a general n-order rhotrix. We show that the resulting index set follows a well-defined symmetric pattern, reinforcing the rhombic structural balance essential to rhotrix theory.

Theorem 5. *Let $n = 2k + 1$ be an odd integer. Then the index set of a rhotrix of order n is given by*

$$\mathcal{I}_r = \{(i, j) \mid 1 \leq i, j \leq n,\ |i - j| \in \{0, 2, 4, \ldots, 2k\}\}.$$

Proof. Consider the index structure of rhotrices of small orders.

For a rhotrix of order 3, the index pairs are

$$(1,1),\ (2,2),\ (3,3),\ (1,3),\ (3,1).$$

For a rhotrix of order 5, the index pairs are

$$(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (3,1), (1,5), (5,1), (2,4), (4,2), (3,5), (5,3).$$

For a rhotrix of order 7, the index pairs are

$$\begin{aligned}&(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (1,3), (3,1), (1,5), (5,1), (1,7), (7,1),\\ &(2,4), (4,2), (2,6), (6,2), (3,5), (5,3), (3,7), (7,3), (4,6), (6,4), (5,7), (7,5).\end{aligned}$$

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Let $R = (r_{ij}) = S = \left\langle \begin{array}{ccccccccc} & & & & r_{1,1} & & & & \\ & & & r_{3,1} & r_{2,2} & r_{3,1} & & & \\ & & r_{5,1} & r_{4,2} & r_{3,3} & r_{2,4} & r_{1,5} & & \\ & .. & .. & .. & .. & .. & .. & .. & \\ r_{n,1} & .. & .. & .. & r_{\frac{n+1}{2},\frac{n+1}{2}} & .. & .. & .. & r_{1,n} \\ & .. & .. & .. & .. & .. & .. & .. & \\ & & .. & .. & .. & .. & .. & & \\ & & & r_{n,n-2} & r_{n-1,n-1} & r_{n-2,n} & & & \\ & & & & r_{n,n} & & & & \end{array} \right\rangle$ be a rhotrix of order $n = 2k + 1$. For any entry (i, j) belonging to the rhotrix, we can observe that $|i - j|$ is an even integer. Furthermore, since the maximum possible difference between indices is $n - 1 = 2k$, it follows that

$$|i - j| \in \{0, 2, 4, \ldots, 2k\}.$$

Thus, the index set is given by

$$\mathcal{I}_r = \{(i, j) \mid 1 \leq i, j \leq n, \ |i - j| \in \{0, 2, 4, \ldots, 2k\}\}.$$

Hence, the pattern holds for all odd integers $n = 2k + 1$. □

Remark 2. *The transpose operation on rhotrices is well defined and closed within the set of rhotrices.*

Proof. Let $R = (r_{ij})$ be a rhotrix with index set $\mathcal{I}_r$. To show that the transpose is well defined, we must verify that for every $(i, j) \in \mathcal{I}_r$, the index (j, i) also belongs to $\mathcal{I}_r$.

Since $(i, j) \in \mathcal{I}_r$, we have

$$|i - j| \in \{0, 2, 4, \ldots, 2k\}.$$

But $|j - i| = |i - j|$, and hence,

$$|j - i| \in \{0, 2, 4, \ldots, 2k\}.$$

Therefore, $(j, i) \in \mathcal{I}_r$, showing that the transpose mapping is well defined.

To prove closure, observe that for every $(i, j) \in \mathcal{I}_r$, the entry $(R^T)_{ij} = r_{ji}$ exists and is defined within the same index set. Hence, R^T is also a rhotrix of order n. Thus, the transpose operation is both well defined and closed within the set of rhotrices. □

3.2. Rhotrix Representation of Binary Relations

We develop a rhotrix representation for binary relations on finite sets and identify the dimensional condition under which a bijective correspondence between relation entries and rhotrix cells exists, construct a canonical mapping for the admissible cases, and illustrate the procedure with a representative example.

The motivation for representing binary relations as rhotrices lies in the structure's inherent radial symmetry and the prioritization of the "heart" or central entry. Unlike standard $N \times N$ matrices that treat all relational pairs with equal spatial weight, the rhotrix framework is uniquely suited for modeling center periphery relations or hierarchical social networks where the strength of a relationship is tied to its Manhattan distance from a central authority. Potential applications include uncertainty-aware spatial modeling, where refined neutrosophic values can represent expert indeterminacy regarding the links within a multi-layered relational database.

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Let $X = \{x_1, x_2, \ldots, x_N\}$ be a finite set of cardinality N. A binary relation R on X is a subset of $X \times X$. Once an ordering of X is fixed, every relation can be represented by its Boolean incidence matrix

$$M_R = (m_{ij}) \in \{0,1\}^{N \times N},$$

where $m_{ij} = 1$ if and only if $(x_i, x_j) \in R$. The family of all binary relations on X, denoted by $\mathcal{B}_N$, forms a Boolean algebra under union and intersection and a semi ring under union and relational composition.

To acquire a compact geometric representation, we utilize rhotrices. A rhotrix of order $m = 2k - 1$, where $k \in \mathbb{Z}^+$, is defined over a diamond-shaped subset of an $m \times m$ grid. Let $c = k$ denote the center of the grid. The set of admissible rhotrix positions is given by

$$F(m) = \{(i,j) \in \{1, \ldots, m\}^2 : \ |i - c| + |j - c| \leq k - 1\}.$$

The total number of filled cells in the rhotrix is

$$|F(m)| = k^2 + (k-1)^2.$$

Definition 5. *Let $\varphi : \{1, \ldots, N\}^2 \to F(m)$ be a bijection. For a binary relation $R \subseteq X \times X$, the relation rhotrix associated with R, denoted by $\mathcal{R}(R)$, is defined by*

$$(\mathcal{R}(R))_{ij} = \begin{cases} 1, & \text{if } \varphi^{-1}(i,j) \in R, \\ 0, & \text{if } \varphi^{-1}(i,j) \notin R, \end{cases} \quad (i,j) \in F(m),$$

with all cells outside $F(m)$ left undefined.

For such a bijection to exist, the number of relation entries must coincide with the number of filled rhotrix cells. Hence, the necessary condition

$$N^2 = k^2 + (k-1)^2,$$

must be satisfied.

Rewriting the above equality yields

$$2k^2 - 2k + 1 = N^2.$$

Multiplying by 2 and completing the square gives

$$(2k-1)^2 - 2N^2 = -1.$$

Setting $x = 2k - 1$, we obtain the negative Pell equation

$$x^2 - 2N^2 = -1.$$

It is known that all positive integer solutions of this equation are given by

$$x_t + N_t\sqrt{2} = (1 + \sqrt{2})^{2t-1}, \quad t = 1, 2, 3, \ldots$$

From these solutions, we recover

$$k_t = \frac{x_t + 1}{2}, \qquad m_t = 2k_t - 1 = x_t.$$

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The first few admissible pairs are

$$(N_1, k_1) = (1, 1),\ (N_2, k_2) = (5, 4),\ (N_3, k_3) = (29, 21),\ (N_4, k_4) = (169, 120), \ldots$$

Thus, the equality $N^2 = k^2 + (k-1)^2$ holds precisely for this infinite family of Pell-type dimensions.

Remark 3. *For a general set size N, a bijective rhotrix representation of binary relations exists only for Pell-type dimensions, revealing a number-theoretic restriction inherent to the rhotrix framework.*

Canonical Bijection for Pell-Type Dimensions

Fix a Pell-type pair (N_t, k_t) and let $m_t = 2k_t - 1$ with center $c = k_t$. We construct a canonical bijection

$$\varphi_t : \{1, \ldots, N_t\} \times \{1, \ldots, N_t\} \to F(m_t),$$

as follows.

First, list the elements of $\{1, \ldots, N_t\}^2$ in lexicographic order:

$$(1,1), (1,2), \ldots, (1, N_t), (2,1), \ldots, (N_t, N_t).$$

Next, order the rhotrix cells by their Manhattan distance from the center,

$$r(i,j) = |i - c| + |j - c|,$$

for $r = 0, 1, \ldots, k_t - 1$. Within each layer of fixed r, the cells are arranged in counterclockwise order starting from the eastern position. The bijection φ_t is defined by matching the two ordered lists position by position.

Definition 6. *For a binary relation R on a set of size N_t, the rhotrix obtained via the bijection φ_t is called the canonical relation rhotrix of R.*

Example 2. *Consider the Pell-type cases $N = 5$, $k = 4$, and $m = 7$. Let*

$$X = \{1, 2, 3, 4, 5\},$$

and define the relation

$$R = \{(1,1), (1,3), (2,4), (3,2), (3,5), (4,1), (5,5)\}.$$

The 25 ordered pairs of $X \times X$ are listed lexicographically and mapped to the 25 filled cells of the rhotrix using the canonical bijection. Each pair belonging to R is assigned a value of 1 in its corresponding rhotrix cell, while all remaining filled cells are assigned 0. Cells outside the rhotrix domain remain undefined, yielding a faithful rhotrix representation of the relation.

Step 1—List relation pairs in lexicographic order.

There are $5^2 = 25$ pairs:

$$\begin{aligned}
&(1,1), (1,2), (1,3), (1,4), (1,5),\\
&(2,1), (2,2), (2,3), (2,4), (2,5),\\
&(3,1), (3,2), (3,3), (3,4), (3,5),\\
&(4,1), (4,2), (4,3), (4,4), (4,5),\\
&(5,1), (5,2), (5,3), (5,4), (5,5).
\end{aligned}$$

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Step 2—List filled cells of the rhotrix in radial order.

The filled cells $F(7)$ satisfy $|i-4|+|j-4| \le 3$. Order them by Manhattan distance r from the center $(4,4)$, and within each r order counter-clockwise starting east:

- $r = 0$: $(4,4)$
- $r = 1$: $(5,4)$, $(4,5)$, $(3,4)$, $(4,3)$
- $r = 2$: $(6,4)$, $(5,5)$, $(4,6)$, $(3,5)$, $(2,4)$, $(3,3)$, $(4,2)$, $(5,3)$
- $r = 3$: $(7,4)$, $(6,5)$, $(5,6)$, $(4,7)$, $(3,6)$, $(2,5)$, $(1,4)$, $(2,3)$, $(3,2)$, $(4,1)$, $(5,2)$, $(6,3)$

This gives exactly 25 cells, in the order L_{rhot}.

Step 3—Apply the bijection ϕ_2.

Map the ℓ-th pair in L_{rel} to the ℓ-th cell in L_{rhot}. For instance,

$$\begin{array}{lll} (1,1) \mapsto (4,4), & (1,2) \mapsto (5,4), & (1,3) \mapsto (4,5), \\ (1,4) \mapsto (3,4), & (1,5) \mapsto (4,3), & \ldots \end{array}$$

Step 4—Fill the rhotrix with 0/1 according to R.

For each relation pair in R, find its image under ϕ_2 and set that rhotrix cell to 1; all other filled cells get 0.

From R:

$$\begin{array}{lll} (1,1) \mapsto (4,4) = 1, & (1,3) \mapsto (4,5) = 1, & (2,4) \mapsto (3,5) = 1, \\ (3,2) \mapsto (3,2) = 1, & (3,5) \mapsto (6,5) = 1, & (4,1) \mapsto (5,6) = 1, \\ (5,5) \mapsto (6,3) = 1. & & \end{array}$$

All other filled cells are set to 0. Cells outside $F(7)$ are left blank.

Step 5—Write the full rhotrix array.

Arranging the values in a 7×7 grid (dots for non-filled cells) yields

$$R^r = \begin{pmatrix} \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 & 0 & \cdot & \cdot \\ \cdot & 0 & 0 & 0 & 1 & 0 & \cdot \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ \cdot & 0 & 0 & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & 1 & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \end{pmatrix}.$$

Every 1-entry of R^r corresponds uniquely to a pair in R, and vice versa. Thus, R is faithfully encoded in the rhotrix structure.

3.3. Non-Pell Dimensions

While the symmetric index property $|i-j| \equiv 0 \pmod 2$ holds for all odd n, a perfect bijective mapping between a set of size N and a rhotrix is mathematically restricted to the Pell-type family (where $N = 1, 5, 29, \ldots$). For all other dimensions, the framework requires the use of dummy elements or partial rhotrices. This number-theoretic restriction represents a structural limitation of the current model, as it necessitates additional computational overhead to embed non-Pell relations while maintaining the algebraic properties of heart-based multiplication.

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4. Multi-Criteria Evaluation of Women Empowerment Schemes Using 2-Refined Neutrosophic Rhotrices

An initial quantitative assessment [25] of the selected welfare schemes was carried out using the Simple Additive Weighting (SAW) method without incorporating refined neutrosophic values. This preliminary evaluation provides a baseline comparison of the relative effectiveness of the schemes prior to the application of the proposed refined neutrosophic rhotrix framework.

The initial evaluation using SAW scores reveals a wide performance spectrum among the first group of programs. The Magalir Urimai Thogai Scheme emerged as the clear leader in this set, securing a top score of 0.5298. In contrast, the E.V.R. Maniammaiyar Ninaivu Marriage Assistance Scheme also showed significant strength with a positive result of 0.375. Other initiatives in this category faced more challenges under the current metrics; for instance, the Pink Auto Scheme recorded a score of -0.1821, while both the Dr. Dharmambal Ammaiyar Ninaivu Widow Remarriage Assistance Scheme and the Government Working Women Hostel Scheme lagged behind with scores of -0.375 and -0.384, respectively.

A parallel assessment was conducted for a second suite of schemes focused on socio-economic development, yielding diverse outcomes. The Dr. Muthulakshmi Reddy Ninaivu Inter-Caste Marriage Assistance Scheme achieved a notable positive value of 0.375, matched in effectiveness by the Chief Minister's Girl Child Protection Scheme, which posted a score of 0.2723. Meanwhile, the Sathiyavanimuthu Ammaiyar Ninaivu Free Supply of Sewing Machine Scheme maintained a modest but positive standing at 0.087. On the lower end of the scale, the Moovalur Ramamirtham Ammaiyar Ninaivu Higher Education Assurance Scheme (-0.362) and the Annai Teresa Ninaivu Marriage Assistance Scheme (-0.375) showed lower performance levels within this specific analytical grouping.

Based on these computed SAW scores, a ranking procedure was implemented to identify the most promising alternatives for further investigation. From the six livelihood-oriented schemes included in the analysis, the three schemes with the highest SAW scores were selected as the most effective candidates. Similarly, among the four marriage assistance schemes considered in this study, the two schemes that attained the highest SAW scores were identified as the most suitable alternatives. These shortlisted schemes will subsequently be examined in greater detail using the refined neutrosophic rhotrix-based decision-making framework proposed in this study, which enables a more comprehensive treatment of uncertainty and expert indeterminacy in the evaluation process.

4.1. Algorithm for Evaluating Women-Centric Schemes Using a 2-Refined Neutrosophic Rhotrix

Let

$$R = \langle x_i \rangle_{n \times n},$$

be a 2-refined neutrosophic decision rhotrix where each element is defined as

$$x_i = (a_{ij} + k_i I_1 + k_{i+1} I_2),$$

where

- a_{ij} denotes the real-valued performance score of the i^{th} alternative with respect to the j^{th} criterion;
- $k_i I_1$ represents the indeterminacy value provided by Expert 1;
- $k_{i+1} I_2$ represents the indeterminacy value provided by Expert 2.

Step 1: Selection of Schemes and Evaluation Criteria

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The effectiveness of women empowerment initiatives can be assessed only through a systematic evaluation of relevant government schemes and the criteria that reflect their impact on women's lives.

Step 2: Construction of the Refined Neutrosophic Rhotrix

Construct the decision rhotrix

$$R = \langle x_{ij} \rangle_{m \times n},$$

where each entry is represented by a triple $(a_{ij} + k_i I_1 + k_{i+1} I_2)$.

Step 3: Determination of Criteria Weights

The relative importance of the evaluation criteria is quantified by assigning a weight to each criterion. Let the weight vector be defined as

$$w = (w_1, w_2, \ldots, w_n), \qquad \sum_{j=1}^{n} w_j = 1.$$

In this study, the weights are obtained using the Rank Order Centroid (ROC) method, which derives numerical weights based on the ranking of criteria according to their perceived importance.

If the criteria are arranged in decreasing order of importance, the weight corresponding to the j-th ranked criterion is calculated as

$$w_j = \frac{1}{n} \sum_{k=j}^{n} \frac{1}{k}, \qquad j = 1, 2, \ldots, n.$$

The ROC procedure provides a systematic mechanism for converting ordinal rankings of criteria into normalized weights, thereby reflecting their relative contribution in the overall evaluation process.

Justification of ROC Method. The Rank Order Centroid (ROC) method is employed in this study due to its simplicity and effectiveness in situations where only the relative importance (ranking) of criteria is available. Unlike methods such as the Analytic Hierarchy Process (AHP), which require pairwise comparisons and consistency checks, or entropy-based methods that rely on data variability, ROC provides a direct mechanism for deriving weights from ordinal rankings.

Furthermore, ROC weights are computationally efficient and avoid the complexity associated with subjective judgments or large datasets. This makes the method particularly suitable for decision-making problems under uncertainty, such as those modeled using refined neutrosophic rhotrices.

Hence, the ROC method is adopted as a practical and robust weighting technique in the proposed framework.

Sensitivity Analysis. To examine the robustness of the proposed model, a sensitivity analysis is performed by varying the weights obtained through the ROC method and observing the corresponding changes in the ranking of alternatives. Small perturbations are introduced to the weights while preserving their relative order.

The results indicate that the ranking of alternatives remains stable under minor variations in weights, demonstrating the robustness of the model. Additionally, variations in the indeterminacy components of the refined neutrosophic values were considered, and it was observed that the overall decision outcomes are not significantly affected.

This confirms that the proposed refined neutrosophic rhotrix-based decision-making approach is stable and reliable under reasonable variations in input parameters.

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Step 4: Define the Weighted Score Function

To evaluate each alternative under refined neutrosophic information, the following weighted score function is defined:

$$S(A_i) = \sum_{j=1}^{n} w_j \left[a_{ij} \left(1 - \frac{k_i + k_{i+1}}{2} \right) \right].$$

This formulation incorporates both the performance value and the associated indeterminacy levels. The real-valued score a_{ij} is penalized by the average indeterminacy contributed by the two experts, while the weights w_j regulate the influence of each criterion.

Step 5: Ranking of Alternatives

Compute the score values $S(A_i)$ for all alternatives. The alternatives are then ranked in descending order based on their score values. The alternative with the highest score is considered the most suitable option.

The overall procedure of the proposed refined neutrosophic rhotrix-based decision-making framework is summarized below. The sequence of steps involved in the evaluation process is illustrated through the flowchart (Figure 1).

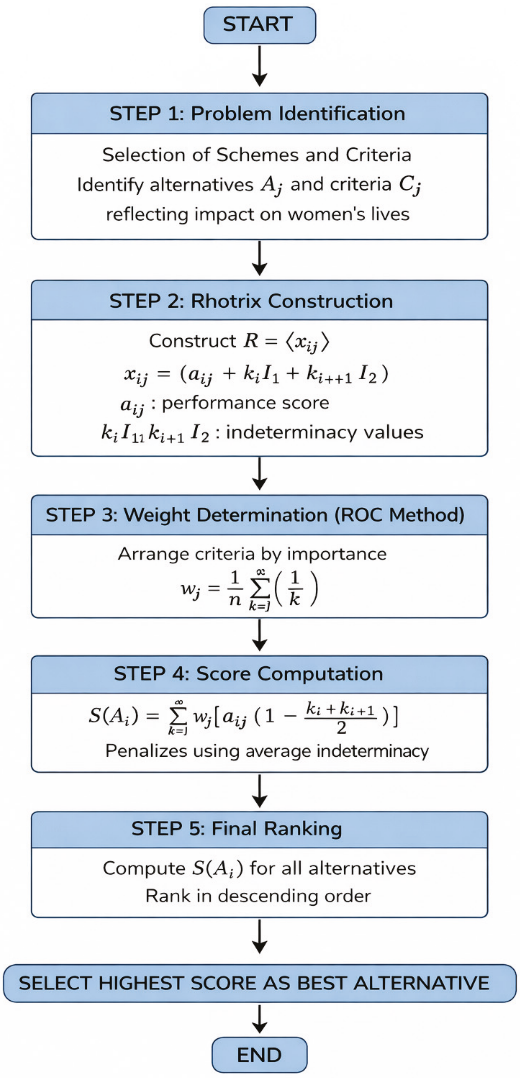

Figure 1. Flowchart for evaluating women-centric schemes using 2-refined neutrosophic rhotrix.

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4.2. Numerical Implementation for Scheme Evaluation

In order to demonstrate the applicability of the proposed decision-making framework, the algorithm is applied to the evaluation of selected women empowerment schemes implemented by the Government of Tamil Nadu. The schemes are assessed with respect to relevant socio-economic criteria using the proposed 2-refined neutrosophic model.

The evaluation information provided by experts is represented in the form of a 2-refined neutrosophic decision rhotrix, where each entry captures the real-valued performance score along with two indeterminacy values corresponding to the uncertainty in expert judgments.

Step 1: Selection of Schemes and Criteria

Scheme Selection: A set of women empowerment schemes is selected based on their performance in earlier evaluations using the Simple Additive Weighting (SAW) method reported in previous studies. Among the livelihood and educational schemes, the Magalir Urimai Thogai Scheme and Chief Minister's Girl Child Protection Scheme are chosen as they obtained the highest SAW scores in the first and second study. In addition, the Sathiyavanimuthu Ammaiyar Ninaivu Free Supply of Sewing Machine Scheme is considered from the group of six livelihood or educational schemes as it recorded a positive SAW score in the overall assessment.

With respect to marriage assistance schemes, the E.V.R. Maniammaiyar Ninaivu Marriage Assistance Scheme for Daughters of Poor Widows and Dr. Muthulakshmi Reddy Ninaivu Inter-Caste Marriage Assistance Scheme are selected from the first and second study. Both schemes are included since they achieved the highest SAW scores in their respective analyses.

These selected schemes are treated as the alternatives for demonstrating the applicability of the proposed 2-refined neutrosophic decision-making framework.

Evaluation Criteria Selection: For schemes that primarily focus on educational advancement and livelihood enhancement, namely the Sathiyavanimuthu Ammaiyar Ninaivu Free Supply of Sewing Machine Scheme, the Magalir Urimai Thogai Scheme, and the Chief Minister's Girl Child Protection Scheme, three evaluation criteria are considered.

1. Outcome efficiency: This criterion reflects the extent to which the scheme achieves its intended objectives, measured through the number of beneficiaries who receive support under the program.
2. Long-term viability: This criterion examines the durability and long-term impact of the scheme, including the duration and continuity of benefits provided to the beneficiaries.
3. Reachability: This criterion evaluates the ease with which eligible individuals can obtain the benefits of the scheme, taking into account the efficiency of administrative procedures and the time required for benefit delivery.

On the other hand, the E.V.R. Maniammaiyar Ninaivu Marriage Assistance Scheme for Daughters of Poor Widows and the Dr. Muthulakshmi Reddy Ninaivu Inter-Caste Marriage Assistance Scheme are designed to offer financial assistance related to marriage. Consequently, their evaluation is based on two criteria that reflect their socio-economic objectives.

1. Inclusive Social Development: This criterion measures the contribution of the schemes toward promoting social equity and supporting women belonging to marginalized or economically weaker sections of society.
2. Economic Empowerment: This criterion assesses the magnitude of financial support provided by the schemes and its role in alleviating the economic burden associated with marriage-related expenses.

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Step 2: Refined Neutrosophic Decision Rhotrix Construction

After identifying the relevant schemes and evaluation criteria, a decision rhotrix is constructed to represent the performance of each scheme with respect to the selected criteria. The decision rhotrix provides a structured framework in which rows correspond to the alternatives (schemes) and columns correspond to the evaluation criteria.

$$\left\langle \begin{array}{ccccc} & & (d_{11}^{E}, k_1, k_2) & & \\ & (d_{21}^{E}, k_3, k_4) & (d_{11}^{M}, k_5, k_6) & (d_{12}^{E}, k_7, k_8) & \\ (d_{31}^{E}, k_9, k_{10}) & (d_{21}^{M}, k_{11}, k_{12}) & (d_{22}^{E}, k_{13}, k_{14}) & (d_{12}^{M}, k_{15}, k_{16}) & (d_{13}^{E}, k_{17}, k_{18}) \\ & (d_{32}^{E}, k_{19}, k_{20}) & (d_{22}^{M}, k_{21}, k_{22}) & (d_{23}^{E}, k_{23}, k_{24}) & \\ & & (d_{33}^{E}, k_{25}, k_{26}) & & \end{array} \right\rangle,$$

where d_{ij}^{E} and d_{ij}^{m} represents the performance value of education or livelihood-related schemes and marriage schemes with respect to their criterion, for $i = 1, 2, 3$ and $j = 1, 2, 3$.

Each entry of the matrix reflects the evaluated score or rating of a scheme under the corresponding criterion. These values are obtained from Tamil Nadu Social Welfare Policy Notes.

$$\left\langle \begin{array}{ccccc} & & (11,400,000, 0.3, 0.5) & & \\ & (1,148,561, 0.4, 0.6) & (32,789, 0.2, 0.7) & (Max, 0.3, 0.4) & \\ (85,396, 0.5, 0.3) & (11,604, 0.4, 0.1) & (18, 0.8, 0.6) & (25,000-50,000, 0.4, 0.7) & (2, 0.2, 0.2) \\ & (Max, 0.6, 0.2) & (25,000-50,000, 0.4, 0.1) & (1, 0.3, 0.5) & \\ & & (3, 0.5, 0.5) & & \end{array} \right\rangle,$$

This decision matrix serves as the fundamental input for the subsequent stages of the analysis, including assignment of weights to criteria, and the application of the rhotrix-based decision-making approach.

Since the actual values of the refined neutrosophic value are represented in various units, the normalization of the actual values must be computed. So, the normalized actual component is computed as, if the criterion is of benefit-type (i.e., a larger value is preferred),

$$d_{ij}^{*} = \frac{d_{ij} - \min\limits_{j} d_{ij}}{\max\limits_{j} d_{ij} - \min\limits_{j} d_{ij}}.$$

If the criterion is of minimalization type (i.e., a smaller value is preferred like Reachability), the normalized value is given by

$$d_{ij}^{*} = \frac{\max\limits_{j} d_{ij} - d_{ij}}{\max\limits_{j} d_{ij} - \min\limits_{j} d_{ij}},$$

and the corresponding normalized refined neutrosophic value is

$$d_{ij}^{*} = \left(d_{ij}^{*}, k_i, k_{i+1} \right).$$

$$\left\langle \begin{array}{ccccc} & & (1, 0.3, 0.5) & & \\ & (0.094, 0.4, 0.6) & (1, 0.2, 0.7) & (1, 0.3, 0.4) & \\ (0, 0.5, 0.3) & (0, 0.4, 0.1) & (0, 0.8, 0.6) & (1, 0.4, 0.7) & (0.5, 0.2, 0.2) \\ & (1, 0.6, 0.2) & (1, 0.4, 0.1) & (1, 0.3, 0.5) & \\ & & (0, 0.5, 0.5) & & \end{array} \right\rangle.$$

Step 3: Determination of Criteria Weights using the ROC Method

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For the schemes related to education and livelihood support, three evaluation criteria are considered. Based on their relative significance, the criteria are ordered and the corresponding weights are obtained using the Rank Order Centroid (ROC) method.

For the criteria of livelihood or educational schemes,
Long-term Viability—Rank 1;
Outcome Efficiency—Rank 2;
Reachability—Rank 3.

Criterion	Weight
Long-Term Viability	0.611
Outcome Efficiency	0.278
Reachability	0.111

For the criteria of marriage assisting schemes,
Social Inclusion and Empowerment—Rank 1;
Economic Advancement—Rank 2.

Criterion	Weight
Social Inclusion and Empowerment	0.75
Economic Advancement	0.25

The assigned weight distribution reflects a stronger emphasis on long-term social empowerment outcomes compared to immediate financial assistance. Such prioritization is consistent with the broader objective of evaluating the transformative impact of women-oriented welfare initiatives.

Step 4: Score Computation, Result Analysis, and Interpretation

Using the proposed refined neutrosophic evaluation framework, the final scores of the selected schemes are obtained as follows:

Scheme	Score
Magalir Urimai Thogai Scheme	0.6084
Chief Minister's Girl Child Protection Scheme	0.0797
Sathiyavanimuthu Ammaiyar Ninaivu Free Supply of Sewing Machine Scheme	0.3666
E.V.R. Maniammaiyar Ninaivu Marriage Assistance Scheme for Daughters of Poor Widows	0.525
Dr. Muthulakshmi Reddy Ninaivu Inter-Caste Marriage Assistance Scheme	0.1875

The obtained scores reflect the overall performance of each scheme after incorporating the normalized performance values, the weights of the evaluation criteria, and the indeterminacy components derived from expert opinions.

The proposed framework is not limited to the present case study and can be extended to applications such as healthcare decision-making, resource allocation, and policy evaluation under uncertainty.

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5. Discussion

Among the evaluated livelihood and educational programs, the Magalir Urimai Thogai Scheme stands out with a leading score of 0.6084. This suggests superior performance in reach, long-term sustainability, and outcome efficiency compared to its peers. Meanwhile, the Sathiyavanimuthu Ammaiyar Ninaivu Free Supply of Sewing Machine Scheme earned a moderate 0.3666, reflecting a steady contribution to economic self-reliance. Conversely, the Chief Minister's Girl Child Protection Scheme received a lower score of 0.0797, pointing to the need for optimization under the specific metrics of this analytical framework.

Regarding marriage-related support, the E.V.R. Maniammaiyar Ninaivu Marriage Assistance Scheme for Daughters of Poor Widows achieved a significant score of 0.525, underscoring its efficacy in fostering social empowerment. In contrast, the Dr. Muthulakshmi Reddy Ninaivu Inter-Caste Marriage Assistance Scheme produced a score of 0.1875, suggesting that its impact remains moderate within the current evaluation criteria.

The rankings generated by this refined neutrosophic decision model effectively highlight which programs excel even when accounting for the inherent uncertainty of expert assessments. By providing a structured lens through which to view government welfare, this methodology serves as a robust tool for policymakers aiming to prioritize initiatives with the highest socio-economic returns.

The comparison in Table 1 highlights that refined neutrosophic rhotrices provide a more expressive and flexible framework for modeling uncertainty, while preserving the structural advantages of rhotrix theory. In particular, the rhomboidal structure of rhotrices allows the decomposition of alternatives into distinct groups that can be evaluated independently when the corresponding criteria are not interdependent. This facilitates parallel analysis and reduces computational complexity in multi-criteria decision-making problems.

Table 1. Comparison of matrices, rhotrices, and refined neutrosophic rhotrices.

Feature	**Matrix**	**Rhotrix**	**Refined Neutrosophic Rhotrix**
Structure	Rectangular array	Rhomboidal array	Rhomboidal array
Commutativity	Generally non-commutative	Commutative (under heart-based multiplication)	Commutative (under heart-based multiplication)
Multiplication Operation	Standard matrix multiplication	Two known operations (heart-based, row–column)	Heart-based multiplication (extended to refined neutrosophic entries)
Structure Alignment in MCDM	All alternatives and criteria are arranged in a single rectangular framework	Alternatives can be partitioned into substructures within the rhomboidal form, allowing independent evaluation	Partitioned evaluation with additional capability to model multi-level indeterminacy within each substructure
Indeterminacy Levels	Not considered	Not considered	Multiple indeterminacy components ($I_1, I_2, \ldots$)

Novelty, Limitations, and Future Research Directions

Novelty of the Proposed Work. The present study introduces a novel extension of rhotrix theory into the refined neutrosophic domain by incorporating multiple indeterminacy components within each entry. Unlike classical rhotrices and single-valued neutrosophic models, the proposed framework enables a more detailed representation of uncertainty.

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Furthermore, the work establishes conditions for invertibility under heart-based multiplication and investigates the associated algebraic structure. The integration of refined neutrosophic rhotrices into a multi-criteria decision-making framework, combined with the use of ROC-based weighting, provides a unified approach for handling uncertainty, indeterminacy, and structured aggregation.

Limitations of the Study. Despite its contributions, the proposed approach has certain limitations. The algebraic structure based on heart-based multiplication imposes restrictions on certain properties, such as the existence of non-trivial idempotent elements, which are more flexible in classical matrix theory.

Future Research Directions. Future work may focus on extending the algebraic analysis of refined neutrosophic rhotrices by exploring additional properties such as spectral characteristics, decomposition methods, and alternative multiplication schemes.

From an application perspective, the proposed model can be applied to more complex real-world decision-making problems, including large-scale group decision-making and dynamic systems. Further research may also investigate the use of alternative weighting techniques and compare their impact on decision outcomes.

In addition, developing efficient computational algorithms and software implementations for refined neutrosophic rhotrix operations would enhance the practical applicability of the proposed framework.

6. Conclusions

The theory of rhotrices advances by introducing invertible refined neutrosophic rhotrices specifically designed for heart-based multiplication. By embedding refined neutrosophic components into the rhotrix architecture, the model effectively represents complex numerical data alongside multiple layers of indeterminacy. Through this investigation, we established the necessary conditions for the existence of inverses and analyzed the structural behavior of these rhotrices under fundamental operations such as addition and heart-based multiplication. This study also establishes a general characterization of the rhotrix index set and demonstrates that it possesses an inherent symmetry under the interchange of indices. This structural symmetry provides a clearer understanding of the arrangement of rhotrix elements and contributes to the theoretical development of rhotrix algebra. Ultimately, these results bridge the gap between classical rhotrix algebra and real-world environments characterized by uncertainty and incomplete information.

To validate the practical utility of this mathematical framework, we applied it to a decision-making scenario involving several women empowerment initiatives. The methodology involved constructing a refined neutrosophic decision matrix, determining criteria weights via the Rank Order Centroid (ROC) method, and ranking the schemes through a specialized score function. The resulting analysis highlighted the Magalir Urimai Thogai Scheme (with a leading score of 0.6084) and the E.V.R. Maniammaiyar Ninaivu Marriage Assistance Scheme for Daughters of Poor Widows (scoring 0.525) as the most impactful programs in their respective categories. These findings confirm that the refined neutrosophic approach provides a more nuanced and reliable hierarchy for evaluating complex social welfare programs. Overall, the developed framework highlights the potential of refined neutrosophic rhotrices as an effective mathematical tool for modeling and evaluating complex decision-making problems characterized by indeterminacy, while simultaneously enriching the algebraic foundations of rhotrix theory.

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Double-Framed Bipolar Fuzzy Soft Sets and Algorithmic Approaches with Symmetry for Multi-Criteria Decision-Making Under Uncertainty

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Abstract

The bipolar fuzzy set and bipolar soft set have inspired the development of a new framework called double-framed bipolar fuzzy soft sets (DFBFSSs). This structure represents positive and negative membership information through ordered pairs, enabling a balanced treatment of uncertainty, imprecision, and bi-directional information in complex decision-making scenarios. The fundamental concepts and operations of DFBFSSs are rigorously defined and analyzed. The double-framed formulation is symmetric: exchanging the frames preserves the structure of DFBFSSs. This symmetry enables balanced handling of opposing or complementary information. The key properties of the proposed set show improved handling of uncertainty over existing fuzzy and soft set models. In addition, a decision-making algorithm based on DFBFSSs is developed and applied to a real-world problem to validate the framework's feasibility. Comparative analysis confirms the method's robustness and advantages in uncertain, dual-information settings.

Keywords: double-framed bipolar fuzzy soft set; bipolar fuzzy set; bipolar soft set; bipolar fuzzy soft set; symmetric operations; uncertainty modeling; multi-criteria decision-making; information systems

1. Introduction

In the social sciences, biological sciences, applied sciences, and physical sciences, there are numerous decision-making problems that frequently include datasets with imprecise and unclear information [1]. Fuzzy set theory was introduced by Zadeh in 1965 as mathematically rigorous approach to handling uncertainty and vagueness, properties that are inherent to everyday life [2]. In classical sets, an item can either belong to the set or not belong to the set, but fuzzy set can account for degrees of membership. They can be defined by degrees of membership between 0 and 1, which quantify the uncertainty or imprecision involved in belonging to a fuzzy sets. Given the lack of precise data in such domains, this flexibility is the reason why fuzzy sets have been widely utilized in a range of disciplines including decision-making [3,4], control system [5], pattern recognition [6,7], artificial intelligence [8,9], and supplier selection [10]. Additional resources are accessible for more research in a variety of fields, including decision-making, in [11–21].

Zhang [22] bipolarized fuzzy sets in 1998, and introduced so-called bipolar-valued fuzzy sets that are ale to model cases when an element can have both positive and negative characteristics. A bipolar-valued fuzzy set is used to describe the membership degrees of each element with the help of two values; one with the positive membership degree while the other with the negative membership degree. The dual structure in dual rating system handles both positive and negative ratings, which is suited to those applications where contradicting evaluation happens, such as sentiment classification or multiple factor risk assessment.

In 1999, Molodtsov introduced soft set theory as a mathematical framework for modeling uncertainty and vagueness [23]. Unlike fuzzy sets, which focus on partial membership, soft sets provide a parameterized description of objects, making them a foundational tool for decision-making under imprecision. This seminal work has inspired extensive research into both their theoretical structure and practical applications across various fields [24–30]. Soft sets handle uncertainty using parameters instead of membership degrees, making them ideal for situations with undefined criteria. They can be used independently or alongside models like fuzzy sets. The bipolar soft set extends this by integrating both positive and negative information, offering a balanced approach to uncertainty. This dual representation makes it a powerful tool for multi-criteria decision-making and handling inconsistent data [31,32].

Maji et al. [27] extended soft set theory to handle fuzzy membership degrees and led to fuzzy soft sets. The integration of these two models enable a more nuanced treatment of uncertainty, incorporating not only parameterized vagueness-seen in fuzzy logic- but also gradations of membership - evident in the second model. By showing how soft sets are closely related to fuzzy sets, this result highlighted soft sets' common ancestry with Zadeh's pioneering work.

In response to problems with bipolar uncertainty, where information consists of both positive and negative components (for example: pros and cons, agreement and disagreement, or risk and benefits), bipolar soft sets were introduced. The soft set and fuzzy soft set could hardly describe such two-sided data. Motivated by bipolar fuzzy sets [33], Abdullah et al. [34] proposed bipolar fuzzy soft sets, integrating the parameterized framework of soft sets and bipolarity to address complex, multidimensional uncertainty. It has been shown to be invaluable in areas such as medical diagnosis, investment analysis, and decision making, where both pro and con arguments need to be weighed at the same time. The bipolarism enables more complex problems to be solved by better modeling of bipolar information and has expanded the area of soft computing. Some recent work includes the development of multi-criteria decision-making frameworks using advanced bipolar fuzzy soft set extensions, which are available in [35–38].

Inspired by these concerns in this work, a new model called double-framed bipolar fuzzy soft set (DFBFSS) is defined as the combination of bipolar fuzzy set and bipolar soft set as its elements are represented by order pairs of positive and negative memberships grades with their negations. Fundamental ideas like subset relations, equality, complement, null, absolute, intersection and union operations are given in depth, along with numerical examples that help stake the principles. An idea of decision-making problem and general algorithm to solve it are introduced by an application of double-framed bipolar fuzzy soft sets along with a numerical example to demonstrate the applicability of the proposed model. A comparative study of the proposed approach with a few others existing is also given to validate the proposed decision-making process.

The proposed DFBFSS framework is a comprehensive generalization of established soft computing models, as it systematically reduces to them under specific constraints. By setting all negative and negation membership grades to zero, the DFBFSS simplifies to a standard fuzzy soft set (FSS), which models only positive membership. A further constraint, restricting these positive memberships to binary values $\{0, 1\}$, reduces it to the classical, crisp soft set (SS). Alternatively, by merely eliminating the negation frames while

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retaining positive and negative memberships, the model directly becomes a bipolar fuzzy soft set (BFSS). This nested relationship solidifies the DFBFSS as a flexible and unifying superset, backward compatible with prior theories while offering superior expressiveness for complex uncertainty. The following Figure 1 justifies this generalization as follows:

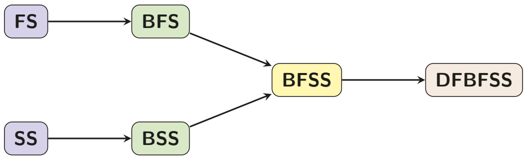

Figure 1. From variant structures to fuzzy and soft sets to arriving at DFBFSSs.

1.1. Objectives

- To model complex uncertainty by using double-framing to represent both positive and negative aspects of information separately.
- To create a richer data encoding that allows information to be viewed from two competing perspectives, improving the handling of incomplete or uncertain data.
- To enable balanced multi-criteria decisions (e.g., benefit vs. cost) by providing a framework to reconcile and jointly evaluate competing objectives.
- To establish a robust decision mechanism that systematically evaluates both the positive and negative parameters of a situation.
- To apply DFBFSSs to real-world scenarios characterized by contradictory facts or dual objectives, such as risk assessment.
- To enhance decision-making in dual-objective systems, like environmental assessments (economy vs. ecology) and financial planning (profit vs. sustainability).

1.2. Research Gap and Motivation

The inspiration for DFBFSSs arises from the need to model complex decisions with conflicting, uncertain information. Unlike models that only use positive and negative memberships, the key innovation of DFBFSSs is the explicit inclusion of the negations of these membership grades. This crucial addition captures a more complete spectrum of expert judgment—such as the distinction between “not good” and “bad”—dramatically enhancing the model’s expressiveness. By separately framing membership and its negation within a double-frame structure, DFBFSSs reduce information loss and ambiguity, enabling nuanced, balanced, and more reliable decision-making in areas like healthcare, finance, and resource management where subtle trade-offs are essential.

1.3. Symmetry in the Positive–Negative Dual Framework

The double-framed architecture of DFBFSSs inherently displays a structural symmetry arising from the two parallel components that encode positive and negative membership information. Swapping these components defines an involutive transformation that preserves the overall structural behavior of the model and the fundamental operations defined on it. This symmetry highlights the balanced role of the two frames and reinforces the suitability of DFBFSSs for representing dual, mutually reflective information within complex systems.

1.4. Main Contributions

The goal of this paper is to introduce DFBFSSs as a novel, generalized framework for improving multi-criteria decision-making (MCDM) under uncertainty and conflicting criteria.

- Generalization of Models: DFBFSSs unify and extend traditional soft set models by integrating bipolarity, fuzziness, and double-framing to capture graded positive and negative assessments.

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- Theoretical Foundation: We formally define DFBFSSs, establish their set-theoretic operations and algebraic properties, and build a theoretical basis for future work.
- Practical Application: We develop an MCDM algorithm based on DFBFSSs and demonstrate its effectiveness through a case study, proving its utility in real-world decision scenarios.
- Comparative Analysis: We compare the DFBFS model with existing approaches, highlighting its advantages and demonstrating its viability as a superior decision-making tool. Together, these contributions establish the DFBFS framework as a significant advancement for robust and nuanced MCDM in complex, ambiguous environments.

1.5. Structure of the Paper

This paper is structured into some sections. Section 2 is about the essential definitions of fuzzy sets, soft sets etc., and their generalized version in which we refer to some aspects related to DFBFSSs. A new framework of DFBFSSs, including its basic definitions and a few preliminary results, is presented in Section 3. This part also investigates set-theoretic operations and algebraic structures between DFBFSSs. Section 4 illustrates the use of DFBFSSs in MCDM through an algorithm used for the assessment and selection of best alternatives in decision problems. An application to model selection for searching a residential apartment to illustrate this algorithm is presented in Section 5. Results achieved by the two proposed algorithms and the reliability of results considering different techniques are compared and discussed in Section 6. Finally, Section 7 gives the summary and conclusions of the paper, including a discussion of DFBFS models' limitations, hints for future works and possible applications of the DFBFS model and their integration with other new suggested techniques.

This paper develops a theoretical model for double-framed bipolar fuzzy soft sets (DFBFSSs) and organizes its theoretical foundations, applications, and comparative studies in a comprehensive manner, highlighting its usefulness for multi-criteria decision-making (MCDM). As is common in foundational fuzzy-set research, the focus is on theoretical model development, and real-world datasets are not included at this stage; their applicability can be demonstrated in future work.

2. Preliminaries

In order to construct the desired model, it will be helpful to keep in mind the following definitions. keep in mind the following definitions. Λ represents the universal set, Υ an entire set of parameters and $\Gamma, \neg\Gamma \subseteq \Upsilon$ throughout this work.

Definition 1 ([39])**.** *Let $\Upsilon = \{\rho_1, \rho_2, \ldots, \rho_n\}$ be a set of parameters. Then the NOT set of Υ denoted by $\neg\Upsilon$ is defined by $\neg\Upsilon = \{\neg\rho_1, \neg\rho_2, \ldots, \neg\rho_n\}$ where $\neg\rho_i =$ not ρ_i for $i = 1, 2, \ldots, n$.*

Definition 2 ([27])**.** *A fuzzy soft set (FSS) is represented as a pair $(\eth, \Gamma)$, where $\eth : \Gamma \to FP(\Lambda)$ and $FP(\Lambda)$ is the collection of all fuzzy subsets of Λ.*

Definition 3 ([22])**.** *A bipolar fuzzy set (BFS) $\bar{\bigtriangledown}$ in Λ is an object having form $\bar{\bigtriangledown} = \{(\varsigma, \wp^+(\varsigma), \wp^-(\varsigma)) : \forall\varsigma \in \Lambda\}$ where $\wp^+(\varsigma) : \Lambda \to [0,1]$ and $\wp^-(\varsigma) : \Lambda \to [-1,0]$ are indicated by the positive and negative information, respectively.*

Definition 4 ([31])**.** *A triple $(\aleph, \breve{\aleph}, \Gamma)$ is called a bipolar soft set (BSS), where $\aleph : \Gamma \to P(\Lambda)$ and $\breve{\aleph} : \neg\Gamma \to P(\Lambda)$, such that $\aleph(\rho) \cap \breve{\aleph}(\neg\rho) = \emptyset$ for all $\rho \in \Gamma$ and $\neg\rho \in \neg\Gamma$.*

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Definition 5 ([40]). *A fuzzy bipolar soft set (FBSS) is represented as a triple triple* $(\eth, \breve{\eth}, \Gamma)$*, where* $\eth : \Gamma \to FP(\Lambda)$ *and* $\breve{\eth} : \neg\Gamma \to FP(\Lambda)$ *such that* $0 \leq \eth(\rho)(\varsigma) + \breve{\eth}(\neg\rho)(\varsigma) \leq 1$.

Definition 6 ([34]). *Define* $\vartheta : \Gamma \to BF(\Lambda)$*, where* $BF(\Lambda)$ *is the collection of all bipolar fuzzy subsets of* Λ*. Then a pair* (ϑ, Γ) *is said to be a bipolar fuzzy soft set (BFSS) and defined by* $(\vartheta, \Gamma) = \{(\rho, (\varsigma, \varrho^+(\varsigma), \varrho^-(\varsigma))) : \forall\rho \in \Gamma \text{and } \forall\varsigma \in \Lambda\}$ *where* $\varrho^+ : \Lambda \longrightarrow [0,1]$ *and* $\varrho^- : \Lambda \longrightarrow [-1,0]$ *are indicated by the positive and negative information, respectively.*

3. Double-Framed Bipolar Fuzzy Soft Sets (DFBFSS)

This section is focused on examining the DFBFSS concept and its basic operations as well as discussing its connection with DFBFSSs.

Definition 7. *A triple* (ϑ, ξ, Γ) *is called a DFBFSS over* Λ *where* $\vartheta : \Gamma \to BF(\Lambda)$, $\xi : \neg\Gamma \to BF(\Lambda)$*. A mathematical form of* (ϑ, ξ, Γ) *is defined by*

$$\begin{aligned}(\vartheta, \xi, \Gamma) &= \{(\rho, \vartheta(\rho), \xi(\neg\rho)) : \forall\rho \in \Gamma \text{ and } \forall\neg\rho \in \neg\Gamma\} \\ &= \Big\{ \Big(\rho, \{(\varsigma, \varrho^+(\varsigma), \varrho^-(\varsigma))\}, \{(\varsigma, \eta^+(\varsigma), \eta^-(\varsigma))\}\Big) : \forall\rho \in \Gamma \text{ and } \forall\varsigma \in \Lambda\Big\},\end{aligned}$$

such that

$$0 \leq \varrho^+(\varsigma) + \eta^+(\varsigma) \leq 1,$$

and

$$-1 \leq \varrho^-(\varsigma) + \eta^-(\varsigma) \leq 0,$$

where $\varrho^+, \eta^+ : \Lambda \longrightarrow [0,1]$ *and* $\varrho^-, \eta^- : \Lambda \longrightarrow [-1,0]$*. The degree of satisfaction of an element* ς *with the property corresponding to the bipolar fuzzy set* ϱ *is shown by the positive and negative memberships degree* $\varrho^+(\rho)(\varsigma)$ *and* $\varrho^+(\neg\rho)(\varsigma)$*, respectively, whereas the degree of satisfaction of* η *with an implicit feature of* η *is indicated by the positive and negative membership degree* $\eta^+(\rho)(\varsigma)$ *and* $\eta^-(\neg\rho)(\varsigma)$*, respectively, with respect to* ϑ *and* ξ*.*

Example 1. *Let* $\Lambda = \{\varsigma_1, \varsigma_2, \varsigma_3\}$ *be the set of three smart phones under consideration,* $\Upsilon = \{\rho_1 = \text{cheap}, \rho_2 = \text{good storage capacity}, \rho_3 = \text{high camera quality}\}$ *be the set of parameters,* $\neg\Upsilon = \{\neg\rho_1 = \text{expensive}, \neg\rho_2 = \text{poor storage capacity}, \neg\rho_3 = \text{low camera quality}\}$ *and* $\Gamma = \{\rho_1, \rho_2\} \subseteq \Upsilon$*. Then*

$$(\vartheta, \Gamma) = \begin{cases} \vartheta(\rho_1) = \begin{cases} (\varsigma_1, 0.6, -0.2) \\ (\varsigma_2, 0.3, -0.4) \\ (\varsigma_3, 0.8, -0.3) \end{cases} \\ \vartheta(\rho_2) = \begin{cases} (\varsigma_1, 0.3, -0.5) \\ (\varsigma_2, 0.7, -0.1) \\ (\varsigma_3, 0.2, -0.2), \end{cases} \end{cases}$$

and

$$(\xi, \neg\Gamma) = \begin{cases} \xi(\neg\rho_1) = \begin{cases} (\varsigma_1, 0.3, -0.6) \\ (\varsigma_2, 0.5, -0.2) \\ (\varsigma_3, 0.1, -0.6) \end{cases} \\ \xi(\neg\rho_2) = \begin{cases} (\varsigma_1, 0.5, -0.3) \\ (\varsigma_2, 0.2, -0.4) \\ (\varsigma_3, 0.4, -0.1). \end{cases} \end{cases}$$

A representative DFBFSS (ϑ, ξ, Γ) *is presented in Table 1.*

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Table 1. A DFBFSS (ϑ, ξ, Γ).

	(ϑ, Γ)		$(\xi, \neg\Gamma)$	
	$\vartheta(\rho_1)$	$\vartheta(\rho_2)$	$\xi(\neg\rho_1)$	$\xi(\neg\rho_2)$
ς_1	$(0.6, -0.2)$	$(0.3, -0.5)$	$(0.3, -0.6)$	$(0.5, -0.3)$
ς_2	$(0.3, -0.4)$	$(0.7, -0.1)$	$(0.5, -0.2)$	$(0.2, -0.4)$
ς_3	$(0.8, -0.3)$	$(0.2, -0.2)$	$(0.1, -0.6)$	$(0.4, -0.1)$

To formalize the symmetry introduced in Section 1.3, we define the frame-swap operator S acting on a DFBFSS $\mathcal{F}$ by interchanging its positive and negative components. For each $\varsigma \in \Lambda$,

$$S(\mathcal{F})(\varsigma) = (\text{negative component of } \mathcal{F}(\varsigma),\ \text{positive component of } \mathcal{F}(\varsigma)).$$

This operator is involutive, i.e., $S(S(\mathcal{F})) = \mathcal{F}$, and preserves the main operations of DFBFSSs, providing a formal tool to study the duality and symmetry inherent in this framework.

We emphasize that symmetry is not merely descriptive in the proposed framework, but is formalized as an algebraic invariance via a frame-swap automorphism acting on the class of DFBFSSs.

Definition 8. *Let $\mathcal{F} = (\vartheta, \xi, \Gamma)$ be a double-framed bipolar fuzzy soft set. The* frame-swap operator *S is defined by*

$$S(\mathcal{F}) = (\xi, \vartheta, \Gamma).$$

Proposition 1 (Symmetry Automorphism of DFBFSSs)**.** *The frame-swap operator S is an involutive automorphism on the class of DFBFSSs, that is,*

$$S^2 = \text{id}.$$

Moreover, S commutes with the basic operations of complement, union, and intersection.

Proof. By definition,

$$S(S(\vartheta, \xi, \Gamma)) = S(\xi, \vartheta, \Gamma) = (\vartheta, \xi, \Gamma),$$

hence $S^2 = \text{id}$. For each basic operation $\circ$, a direct verification shows that

$$S(\mathcal{F}_1 \circ \mathcal{F}_2) = S(\mathcal{F}_1) \circ S(\mathcal{F}_2),$$

which proves invariance under frame swapping.

Therefore, the collection of DFBFSSs together with the frame-swap operator S forms a structure invariant under a nontrivial involutive automorphism. □

Remark 1. *Neither BFSs nor BFSSs admit a frame-swap automorphism preserving all operations, since negation information is not explicitly encoded as an independent evaluative dimension.*

Definition 9. *An absolute DFBFSS, denoted by $(\ddot{\Lambda}, \ddot{\Phi}, \Gamma)$, is a DFBFSS (ϑ, ξ, Γ) such that $(\vartheta, \xi, \Gamma) = \Big\{ \Big(\rho, \{(\varsigma, 1, -1)\}, \{(\varsigma, 0, 0)\}\Big) : \forall \rho \in \Gamma \text{ and } \forall \varsigma \in \Lambda \Big\}$.*

Definition 10. *A null DFBFSS, denoted by $(\ddot{\Phi}, \ddot{\Lambda}, \Gamma)$, is a DFBFSS (ϑ, ξ, Γ) such that $(\vartheta, \xi, \Gamma) = \Big\{ \Big(\rho, \{(\varsigma, 0, 0)\}, \{(\varsigma, 1, -1)\}\Big) : \forall \rho \in \Gamma \text{ and } \forall \varsigma \in \Lambda \Big\}$.*

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Definition 11. *The DFBFS complement of* (ϑ, ξ, Γ)*, denoted by either* $(\vartheta, \xi, \Gamma)^c$ *or* $(\vartheta^c, \xi^c, \Gamma)$*, is defined by*

$$(\vartheta, \xi, \Gamma)^c = (\vartheta^c, \xi^c, \Gamma)$$
$$= \Big\{ \Big(\rho, \{(\varsigma, 1-\varrho^+(\varsigma), -1-\varrho^-(\varsigma))\}, \{(\varsigma, 1-\eta^+(\varsigma), -1-\eta^-(\varsigma))\}\Big) : \forall \rho \in \Gamma \text{ and } \forall \varsigma \in \Lambda \Big\}.$$

Example 2. *In Example 1, the DFBFS complement of DFBFSS* (ϑ, ξ, Γ) *is presented in Table 2.*

Table 2. A DFBFS complement $(\vartheta, \xi, \Gamma)^c$.

	$(\vartheta, \Gamma)^c$		$(\xi, \neg\Gamma)^c$	
	$\vartheta(\rho_1)^c$	$\vartheta(\rho_2)^c$	$\xi(\neg\rho_1)^c$	$\xi(\neg\rho_2)^c$
ς_1	$(0.4, -0.8)$	$(0.7, -0.5)$	$(0.7, -0.4)$	$(0.5, -0.7)$
ς_2	$(0.7, -0.6)$	$(0.3, -0.9)$	$(0.5, -0.8)$	$(0.8, -0.6)$
ς_3	$(0.2, -0.7)$	$(0.8, -0.8)$	$(0.9, -0.4)$	$(0.6, -0.9)$

Definition 12 (Subset)**.** *Let* (ϑ, ξ, Γ)*,* $(\vartheta_1, \xi_1, \text{ß}) \in$ *DFBFSSs. We say that* (ϑ, ξ, Γ) *is an DFBFS subset of* $(\vartheta_1, \xi_1, \text{ß})$*, denoted as* $(\vartheta, \xi, \Gamma) \ddot{\sqsubseteq} (\vartheta_1, \xi_1, \text{ß})$*, if*

1. $\Gamma \subseteq \text{ß}$.
2. $\vartheta(\rho) \subseteq \vartheta_1(\rho)$ *(i.e.,* $\varrho_\vartheta{}^+(\varsigma) \le \varrho_{\vartheta_1}^+(\varsigma), \varrho_\vartheta{}^-(\varsigma) \ge \varrho_{\vartheta_1}^-(\varsigma)$*) and* $\xi(\neg\rho) \supseteq \xi_1(\neg\rho)$ *(i.e.,* $\eta_\xi{}^+(\varsigma) \ge \eta_{\xi_1}^+(\varsigma), \eta_\xi{}^-(\varsigma) \le \eta_{\xi_1}^-(\varsigma)$*)* $\forall \rho \in \Gamma, \forall \varsigma \in \Lambda$ *and* $\forall \neg\rho \in \neg\Gamma$.

Example 3. *Let* $\Lambda = \{\varsigma_1, \varsigma_2, \varsigma_3\}$ *be the set of three horses under consideration* $\Upsilon = \{\rho_1 =$ *excellent health,* $\rho_2 =$ *fully trained,* $\rho_3 =$ *high price,* $\rho_4 =$ *high speed*$\}$ *be the set of parameters and* $\neg\Upsilon = \{\neg\rho_1 =$ *poor health,* $\neg\rho_2 =$ *untrained,* $\neg\rho_3 =$ *low price,* $\neg\rho_4 =$ *low speed*$\}$*. Suppose that* $\Gamma = \{\rho_1\}$ *and* $\text{ß} = \{\rho_1, \rho_2\}$ *are subsets of* Υ*. Then the representative DFBFSSs* (ϑ, ξ, Γ) *and* $(\vartheta_1, \xi_1, \text{ß})$ *are presented in Tables 3 and 4, respectively.*

Table 3. A DFBFSS (ϑ, ξ, Γ).

	(ϑ, Γ)	$(\xi, \neg\Gamma)$
	$\vartheta(\rho_1)$	$\xi(\neg\rho_1)$
ς_1	$(0.4, -0.1)$	$(0.3, -0.6)$
ς_2	$(0.3, -0.3)$	$(0.6, -0.2)$
ς_3	$(0.2, -0.2)$	$(0.1, -0.7)$

Table 4. A DFBFSS $(\vartheta_1, \xi_1, \text{ß})$.

	$(\vartheta_1, \text{ß})$		$(\xi_1, \neg\text{ß})$	
	$\vartheta_1(\rho_1)$	$\vartheta_1(\rho_2)$	$\xi_1(\neg\rho_1)$	$\xi_1(\neg\rho_2)$
ς_1	$(0.7, -0.2)$	$(0.3, -0.5)$	$(0.3, -0.6)$	$(0.5, -0.3)$
ς_2	$(0.4, -0.4)$	$(0.6, -0.1)$	$(0.5, -0.2)$	$(0.2, -0.4)$
ς_3	$(0.9, -0.5)$	$(0.1, -0.2)$	$(0.1, -0.6)$	$(0.4, -0.1)$

Then (1.) $\Gamma \subseteq \text{ß}$*, (2.)* $\forall \rho \in \Gamma, \forall \varsigma \in \Lambda$ *and* $\forall \neg\rho \in \neg\Gamma$*, we have* $\vartheta(\rho) \subseteq \vartheta_1(\rho)$ *and* $\xi(\neg\rho) \supseteq \xi_1(\neg\rho)$*. Therefore,* $(\vartheta, \xi, \Gamma) \ddot{\sqsubseteq} (\vartheta_1, \xi_1, \text{ß})$*.*

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Definition 13 (Equal)**.** *Two DFBFSSs* (ϑ, ξ, Γ) *and* $(\vartheta_1, \xi_1, \text{ß})$ *are said to be DFBFS equal if* $(\vartheta_1, \xi_1, \text{ß}) \ddot{\sqsubseteq} (\vartheta, \xi, \Gamma)$ *and* $(\vartheta, \xi, \Gamma) \ddot{\sqsubseteq} (\vartheta_1, \xi_1, \text{ß})$*. We write* $(\vartheta_1, \xi_1, \text{ß}) = (\vartheta, \xi, \Gamma)$*.*

4. Operations on DFBFSSs

Definition 14 (OR)**.** *The DFBFS OR-operation of* (ϑ, ξ, Γ) *and* $(\vartheta_1, \xi_1, \text{ß})$ *is a DFBFSS* $(\mho, \Omega, \hbar)$*, denoted by* $(\vartheta, \xi, \Gamma) \ddot{\vee} (\vartheta_1, \xi_1, \text{ß})$*, and is defined as* $(\vartheta, \xi, \Gamma) \ddot{\vee} (\vartheta_1, \xi_1, \text{ß}) = (\mho, \Omega, \Gamma \times \text{ß})$ *where* $\forall (\gamma, \jmath) \in \Gamma \times \text{ß}$*,* $(\neg\gamma, \neg\jmath) \in \neg\Gamma \times \neg\text{ß}$ *and* $\varsigma \in \Lambda$*,*

$$\begin{aligned} \mho(\gamma, \jmath) &= (\varrho_{\vartheta}{}^{+}(\varsigma) \vee \varrho_{\vartheta_1}^{+}(\varsigma), \varrho_{\vartheta}{}^{-}(\varsigma) \wedge \varrho_{\vartheta_1}^{-}(\varsigma)) \;\; \textit{and} \\ \Omega(\neg\gamma, \neg\jmath) &= (\eta_{\xi}{}^{+}(\varsigma) \wedge \eta_{\xi_1}^{+}(\varsigma), \eta_{\xi}{}^{-}(\varsigma) \vee \eta_{\xi_1}^{-}(\varsigma)). \end{aligned}$$

Example 4. *In Example 3,* $(\mho, \Omega, \Gamma \times \text{ß}) = (\vartheta, \xi, \Gamma) \ddot{\vee} (\vartheta_1, \xi_1, \text{ß})$*. The representative DFBFSS* $(\vartheta, \xi, \Gamma) \ddot{\vee} (\vartheta_1, \xi_1, \text{ß})$ *is presented in Table 5.*

Table 5. $(\vartheta, \xi, \Gamma) \ddot{\vee} (\vartheta_1, \xi_1, \text{ß})$.

	$\mho(\rho_1, \rho_1)$	$\mho(\rho_1, \rho_2)$	$\Omega(\neg\rho_1, \neg\rho_1)$	$\Omega(\neg\rho_1, \neg\rho_2)$
ς_1	(0.7, −0.2)	(0.4, −0.5)	(0.3, −0.6)	(0.3, −0.3)
ς_2	(0.4, −0.4)	(0.6, −0.3)	(0.5, −0.2)	(0.2, −0.2)
ς_3	(0.9, −0.5)	(0.2, −0.2)	(0.1, −0.6)	(0.1, −0.1)

Definition 15 (AND)**.** *The DFBFS AND-operation of* (ϑ, ξ, Γ) *and* $(\vartheta_1, \xi_1, \text{ß})$ *is a DFBFSS* $(\mho, \Omega, \hbar)$*, denoted by* $(\vartheta, \xi, \Gamma) \ddot{\wedge} (\vartheta_1, \xi_1, \text{ß})$*, and is defined by* $(\vartheta, \xi, \Gamma) \ddot{\wedge} (\vartheta_1, \xi_1, \text{ß}) = (\mho, \Omega, \Gamma \times \text{ß})$ *where* $\forall (\gamma, \jmath) \in \Gamma \times \text{ß}$*,* $(\neg\gamma, \neg\jmath) \in \neg\Gamma \times \neg\text{ß}$ *and* $\varsigma \in \Lambda$*,*

$$\begin{aligned} \mho(\gamma, \jmath) &= (\varrho_{\vartheta}{}^{+}(\varsigma) \wedge \varrho_{\vartheta_1}^{+}(\varsigma), \varrho_{\vartheta}{}^{-}(\varsigma) \vee \varrho_{\vartheta_1}^{-}(\varsigma)) \;\; \textit{and} \\ \Omega(\neg\gamma, \neg\jmath) &= (\eta_{\xi}{}^{+}(\varsigma) \vee \eta_{\xi_1}^{+}(\varsigma), \eta_{\xi}{}^{-}(\varsigma) \wedge \eta_{\xi_1}^{-}(\varsigma)). \end{aligned}$$

Example 5. *In Example 3,* $(\mho, \Omega, \Gamma \times \text{ß}) = (\vartheta, \xi, \Gamma) \ddot{\wedge} (\vartheta_1, \xi_1, \text{ß})$*. The representative DFBFSS* $(\vartheta, \xi, \Gamma) \ddot{\wedge} (\vartheta_1, \xi_1, \text{ß})$ *is presented in Table 6.*

Table 6. $(\vartheta, \xi, \Gamma) \ddot{\wedge} (\vartheta_1, \xi_1, \text{ß})$.

	$\mho(\rho_1, \rho_1)$	$\mho(\rho_1, \rho_2)$	$\Omega(\neg\rho_1, \neg\rho_1)$	$\Omega(\neg\rho_1, \neg\rho_2)$
ς_1	(0.4, −0.1)	(0.3, −0.1)	(0.3, −0.6)	(0.5, −0.6)
ς_2	(0.3, −0.3)	(0.3, −0.1)	(0.6, −0.2)	(0.6, −0.4)
ς_3	(0.2, −0.2)	(0.1, −0.2)	(0.1, −0.7)	(0.4, −0.7)

Proposition 2. *If* (ϑ, ξ, Γ) *and* $(\vartheta_1, \xi_1, \text{ß})$ *are two DFBFSSs over* Λ*, then*

1. $((\vartheta, \xi, \Gamma) \ddot{\vee} (\vartheta_1, \xi_1, \text{ß}))^c = (\vartheta, \xi, \Gamma)^c \ddot{\wedge} (\vartheta_1, \xi_1, \text{ß})^c$.
2. $((\vartheta, \xi, \Gamma) \ddot{\wedge} (\vartheta_1, \xi_1, \text{ß}))^c = (\vartheta, \xi, \Gamma)^c \ddot{\vee} (\vartheta_1, \xi_1, \text{ß})^c$.

Proof. 1. From Definitions 11 and 14, $((\vartheta, \xi, \Gamma) \ddot{\vee} (\vartheta_1, \xi_1, \text{ß}))^c = (\mho^c, \Omega^c, \Gamma \times \text{ß})$, where $\mho(\gamma, \jmath)^c = \vartheta(\gamma)^c \wedge \vartheta_1(\jmath)^c$ and $\Omega(\neg\gamma, \neg\jmath)^c = \xi(\neg\gamma)^c \vee \xi_1(\neg\jmath)^c$ for all $(\gamma, \jmath) \in \Gamma \times \text{ß}$ and $(\neg\gamma, \neg\jmath) \in \neg\Gamma \times \neg\text{ß}$.
Now, $(\vartheta, \xi, \Gamma)^c \ddot{\wedge} (\vartheta_1, \xi_1, \text{ß})^c = (\mho^c, \Omega^c, \Gamma \times \text{ß})$ where $\mho(\gamma, \jmath)^c = \vartheta(\gamma)^c \wedge \vartheta_1(\jmath)^c$ and $\Omega(\neg\gamma, \neg\jmath)^c = \xi(\neg\gamma)^c \vee \xi_1(\neg\jmath)^c$ for all $(\gamma, \jmath) \in \Gamma \times \text{ß}$ and $(\neg\gamma, \neg\jmath) \in \neg\Gamma \times \neg\text{ß}$. Thus, $((\vartheta, \xi, \Gamma) \ddot{\vee} (\vartheta_1, \xi_1, \text{ß}))^c = (\vartheta, \xi, \Gamma)^c \ddot{\wedge} (\vartheta_1, \xi_1, \text{ß})^c$.

2. Its proof is similar to part 1.

□

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Definition 16 (Extended union)**.** *The DFBFS extended union of* (ϑ, ξ, Γ) *and* $(\vartheta_1, \xi_1, ß)$*, denoted by* $(\vartheta, \xi, \Gamma) \ddot{\sqcup} (\vartheta_1, \xi_1, ß)$*, is a DFBFSS* $(\mho, \Omega, \hbar)$ *where* $\hbar = \Gamma \cup ß$ *and* $\forall \rho \in \hbar$*:*

$$\mho(\rho) = \begin{cases} \vartheta(\rho) & \text{if } \rho \in \Gamma \setminus ß \\ \vartheta_1(\rho) & \text{if } \rho \in ß \setminus \Gamma \\ \vartheta(\rho) \ddot{\vee} \vartheta_1(\rho) & \text{if } \rho \in \Gamma \cap ß \neq \emptyset, \end{cases}$$

and

$$\Omega(\neg\rho) = \begin{cases} \xi(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \setminus \negß \\ \xi_1(\neg\rho) & \text{if } \neg\rho \in \negß \setminus \neg\Gamma \\ \xi(\neg\rho) \ddot{\wedge} \xi_1(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \cap \negß \neq \emptyset. \end{cases}$$

Example 6. *In Example 3,* $(\mho, \Omega, \hbar) = (\vartheta, \xi, \Gamma) \ddot{\sqcup} (\vartheta_1, \xi_1, ß)$*, where* $\hbar = \Gamma \cup ß = \{\rho_1, \rho_2\}$*. The representative DFBFSS* $(\vartheta, \xi, \Gamma) \ddot{\sqcup} (\vartheta_1, \xi_1, ß)$ *is presented in Table 7.*

Table 7. $(\vartheta, \xi, \Gamma) \ddot{\sqcup} (\vartheta_1, \xi_1, ß)$.

	$(\mho, \hbar)$		$(\Omega, \neg\hbar)$	
	$\mho(\rho_1)$	$\mho(\rho_2)$	$\Omega(\neg\rho_1)$	$\Omega(\neg\rho_2)$
ς_1	$(0.7, -0.2)$	$(0.3, -0.5)$	$(0.3, -0.6)$	$(0.5, -0.3)$
ς_2	$(0.4, -0.4)$	$(0.6, -0.1)$	$(0.5, -0.2)$	$(0.2, -0.4)$
ς_3	$(0.9, -0.5)$	$(0.1, -0.2)$	$(0.1, -0.6)$	$(0.4, -0.1)$

Definition 17 (Extended intersection)**.** *The DFBFS extended intersection of* (ϑ, ξ, Γ) *and* $(\vartheta_1, \xi_1, ß)$*, denoted by* $(\mho, \Omega, \hbar) = (\vartheta, \xi, \Gamma) \ddot{\sqcap} (\vartheta_1, \xi_1, ß)$*, is a DFBFSS* $(\mho, \Omega, \hbar)$*, where* $\hbar = \Gamma \cup ß$ *and* $\forall \rho \in \hbar$*:*

$$\mho(\rho) = \begin{cases} \vartheta(\rho) & \text{if } \rho \in \Gamma \setminus ß \\ \vartheta_1(\rho) & \text{if } \rho \in ß \setminus \Gamma \\ \vartheta(\rho) \ddot{\wedge} \vartheta_1(\rho) & \text{if } \rho \in \Gamma \cap ß \neq \emptyset, \end{cases}$$

and

$$\Omega(\neg\rho) = \begin{cases} \xi(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \setminus \negß \\ \xi_1(\neg\rho) & \text{if } \neg\rho \in \negß \setminus \neg\Gamma \\ \xi(\neg\rho) \ddot{\vee} \xi_1(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \cap \negß \neq \emptyset. \end{cases}$$

Example 7. *In Example 3,* $(\mho, \Omega, \hbar) = (\vartheta, \xi, \Gamma) \ddot{\sqcap} (\vartheta_1, \xi_1, ß)$*, where* $\hbar = \Gamma \cup ß = \{\rho_1, \rho_2\}$*. The representative DFBFSS* $(\vartheta, \xi, \Gamma) \ddot{\sqcap} (\vartheta_1, \xi_1, ß)$ *is presented in Table 8.*

Table 8. $(\vartheta, \xi, \Gamma) \ddot{\sqcap} (\vartheta_1, \xi_1, ß)$.

	$(\mho, \hbar)$		$(\Omega, \neg\hbar)$	
	$\mho(\rho_1)$	$\mho(\rho_2)$	$\Omega(\neg\rho_1)$	$\Omega(\neg\rho_2)$
ς_1	$(0.4, -0.1)$	$(0.3, -0.5)$	$(0.3, -0.6)$	$(0.5, -0.3)$
ς_2	$(0.3, -0.3)$	$(0.6, -0.1)$	$(0.6, -0.2)$	$(0.2, -0.4)$
ς_3	$(0.2, -0.2)$	$(0.1, -0.2)$	$(0.1, -0.7)$	$(0.4, -0.1)$

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Definition 18 (Restricted union)**.** *The DFBFS restricted union of* (ϑ, ξ, Γ) *and* $(\vartheta_1, \xi_1, ß)$*, denoted by* $(\vartheta, \xi, \Gamma) \sqcup_{\Re} (\vartheta_1, \xi_1, ß)$*, is a DFBFSSs* $(\mho, \Omega, \hbar)$*, where* $\hbar = \Gamma \cap ß \neq \emptyset$ *and* $\forall \rho \in \hbar$*,*

$$\mho(\rho) = \vartheta(\rho) \ddot{\vee}\, \vartheta_1(\rho) \text{ and } \Omega(\neg\rho) = \xi(\neg\rho) \ddot{\wedge}\, \xi_1(\neg\rho).$$

Example 8. *In Example 3,* $(\mho, \Omega, \hbar) = (\vartheta, \xi, \Gamma) \sqcup_{\Re} (\vartheta_1, \xi_1, ß)$ *where* $\hbar = A \cap B = \{\rho_1\}$*. The representative DFBFSS* $(\vartheta, \xi, \Gamma) \sqcup_{\Re} (\vartheta_1, \xi_1, ß)$ *is presented in Table 9.*

Table 9. $(\vartheta, \xi, \Gamma) \sqcup_{\Re} (\vartheta_1, \xi_1, ß)$.

	$(\mho, \hbar)$	$(\Omega, \neg\hbar)$
	$\mho(\rho_1)$	$\Omega(\neg\rho_1)$
ς_1	$(0.7, -0.2)$	$(0.3, -0.6)$
ς_2	$(0.4, -0.4)$	$(0.5, -0.2)$
ς_3	$(0.9, -0.5)$	$(0.1, -0.6)$

Definition 19 (Restricted intersection)**.** *The DFBFS restricted intersection of* (ϑ, ξ, Γ) *and* $(\vartheta_1, \xi_1, ß)$*, denoted by* $(\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß)$*, is a DFBFSSs* $(\mho, \Omega, \hbar)$*, where* $\hbar = \Gamma \cap ß \neq \emptyset$ *and* $\forall \rho \in \hbar$*,*

$$\mho(\rho) = \vartheta(\rho) \ddot{\wedge}\, \vartheta_1(\rho) \text{ and } \Omega(\neg\rho) = \xi(\neg\rho) \ddot{\vee}\, \xi_1(\neg\rho).$$

Example 9. *In Example 3,* $(\mho, \Omega, \hbar) = (\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß)$*, where* $\hbar = A \cap B = \{\rho_1\}$*. The representative DFBFSS* $(\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß)$ *is presented in Table 10.*

Table 10. $(\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß)$.

	$(\mho, \hbar)$	$(\Omega, \neg\hbar)$
	$\mho(\rho_1)$	$\Omega(\neg\rho_1)$
ς_1	$(0.4, -0.1)$	$(0.3, -0.6)$
ς_2	$(0.3, -0.3)$	$(0.6, -0.2)$
ς_3	$(0.2, -0.2)$	$(0.1, -0.7)$

Proposition 3. *Let* (ϑ, ξ, Γ)*,* $(\vartheta_1, \xi_1, ß)$*,* $(\vartheta_2, \xi_2, \hbar) \in DFBFSS(\Lambda)$*. Then*

1. $(\ddot{\Lambda}, \ddot{\Phi}, \Gamma)^c = (\ddot{\Phi}, \ddot{\Lambda}, \Gamma)$ *and* $(\ddot{\Phi}, \ddot{\Lambda}, \Gamma)^c = (\ddot{\Lambda}, \ddot{\Phi}, \Gamma)$.
2. $(\ddot{\Phi}, \ddot{\Lambda}, \Gamma) \ddot{\sqcap} (\vartheta, \xi, \Gamma) = (\ddot{\Phi}, \ddot{\Lambda}, \Gamma)$ *and* $(\ddot{\Phi}, \ddot{\Lambda}, \Gamma) \ddot{\sqcup} (\vartheta, \xi, \Gamma) = (\vartheta, \xi, \Gamma)$.
3. $(\ddot{\Lambda}, \ddot{\Phi}, \Gamma) \sqcap_{\Re} (\vartheta, \xi, \Gamma) = (\ddot{\Phi}, \ddot{\Lambda}, \Gamma)$ *and* $(\ddot{\Phi}, \ddot{\Lambda}, \Gamma) \sqcup_{\Re} (\vartheta, \xi, \Gamma) = (\vartheta, \xi, \Gamma)$.
4. $(\ddot{\Phi}, \ddot{\Lambda}, \Gamma) \ddot{\sqsubseteq} (\vartheta, \xi, \Gamma)$.
5. $(\vartheta, \xi, \Gamma) \ddot{\sqsubseteq} (\ddot{\Lambda}, \ddot{\Phi}, \Gamma)$.
6. $((\vartheta, \xi, \Gamma)^c)^c = (\vartheta, \xi, \Gamma)$.
7. *If* $(\vartheta, \xi, \Gamma) \ddot{\sqsubseteq} (\vartheta_1, \xi_1, ß)$*, then* $(\vartheta_1, \xi_1, ß)^c \ddot{\sqsubseteq} (\vartheta, \xi, \Gamma)^c$.
8. *If* $(\vartheta, \xi, \Gamma) \ddot{\sqsubseteq} (\vartheta_1, \xi_1, ß)$*, then* $(\vartheta, \xi, \Gamma) \ddot{\sqcup} (\vartheta_1, \xi_1, ß) = (\vartheta_1, \xi_1, ß)$.
9. *If* $(\vartheta, \xi, \Gamma) \ddot{\sqsubseteq} (\vartheta_1, \xi_1, ß)$*, then* $(\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß) = (\vartheta, \xi, \Gamma)$.
10. *If* $(\vartheta, \xi, \Gamma) \ddot{\sqsubseteq} (\vartheta_1, \xi_1, ß)$ *and* $(\vartheta_1, \xi_1, ß) \ddot{\sqsubseteq} (\vartheta_2, \xi_2, \hbar)$*, then* $(\vartheta, \xi, \Gamma) \ddot{\sqsubseteq} (\vartheta_2, \xi_2, \hbar)$.

Proof. (1) Let, for simplicity $(\ddot{\Lambda}, \ddot{\Phi}, \Gamma) = \Big((1, -1), (0, 0)\Big)$. Then $(\ddot{\Lambda}, \ddot{\Phi}, \Gamma)^c = \Big((1-1, -1-(-1)), (1-0, -1-0)\Big) = \Big((0, 0), (1, -1)\Big) = (\ddot{\Phi}, \ddot{\Lambda}, \Gamma)$ and $(\ddot{\Phi}, \ddot{\Lambda}, \Gamma)^c = (\ddot{\Lambda}, \ddot{\Phi}, \Gamma)$ is similar to above.

The proofs of 2–5 can be directly proven from their definitions.

(6) Let, for simplicity, $(\vartheta, \xi, \Gamma) = \Big((\varrho^+, \varrho^-), (\eta^+, \eta^-)\Big)$. Then

$(\vartheta,\xi,\Gamma)^c = \Big((1-\varrho^+,-1-\varrho^-),(1-\eta^+,-1-\eta^-)\Big)$ and

$$\begin{aligned}((\vartheta,\xi,\Gamma)^c)^c &= \Big(1-(1-\varrho^+),-1-(-1-\varrho^-),1-(1-\eta^+),-1-(-1-\eta^-)\Big)\\ &= (\vartheta,\xi,\Gamma).\end{aligned}$$

(7) Since $(\vartheta,\xi,\Gamma) \ddot{\sqsubseteq} (\vartheta_1,\xi_1,ß)$, it follows that $\Gamma \subseteq ß$ and $\vartheta(\rho) \subseteq \vartheta_1(\rho)$ and $\xi(\neg\rho) \supseteq \xi_1(\neg\rho)$ $\forall\rho \in \Gamma$. Then, $({\vartheta_1}^c,{\xi_1}^c,ß)\ddot{\sqsubseteq}(\vartheta^c,\xi^c,\Gamma)$. Hence, $(\vartheta_1,\xi_1,ß)^c \ddot{\sqsubseteq} (\vartheta,\xi,\Gamma)^c$.
(8) Since $(\vartheta,\xi,\Gamma) \ddot{\sqsubseteq} (\vartheta_1,\xi_1,ß)$, it follows that $\Gamma \subseteq ß$ and $\vartheta(\rho) \subseteq \vartheta_1(\rho)$ and $\xi(\neg\rho) \supseteq \xi_1(\neg\rho)$ $\forall\rho \in \Gamma$. Put $(\mho,\Omega,\hbar) = (\vartheta,\xi,\Gamma) \ddot{\sqcup} (\vartheta_1,\xi_1,ß)$. Then $\hbar = \Gamma \cup ß$ and $\forall\rho \in \hbar = ß$,

$$\mho(\rho) = \begin{cases}\vartheta_1(\rho) & \text{if } \rho \in ß \setminus \Gamma\\ \vartheta(\rho)\ddot{\vee}\,\vartheta_1(\rho) & \text{if } \rho \in \Gamma \cap ß \neq \emptyset,\end{cases}$$

and

$$\Omega(\neg\rho) = \begin{cases}\xi_1(\neg\rho) & \text{if } \neg\rho \in \neg ß \setminus \neg\Gamma\\ \xi(\neg\rho)\ddot{\wedge}\,\xi_1(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \cap \neg ß \neq \emptyset.\end{cases}$$

Then $(\mho,\Omega,\hbar) = (\vartheta_1,\xi_1,ß)$. Thus $(\vartheta,\xi,\Gamma) \ddot{\sqcup} (\vartheta_1,\xi_1,ß) = (\vartheta_1,\xi_1,ß)$.
(9) Since $(\vartheta,\xi,\Gamma) \ddot{\sqsubseteq} (\vartheta_1,\xi_1,ß)$, it follows that $\Gamma \subseteq ß$, $\vartheta(\rho) \subseteq \vartheta_1(\rho)$ and $\xi(\neg\rho) \supseteq \xi_1(\neg\rho)$ $\forall\rho \in \Gamma$. Put $(\mho,\Omega,\hbar) = (\vartheta,\xi,\Gamma) \sqcap_{\Re} (\vartheta_1,\xi_1,ß)$. Then $\hbar = \Gamma \cap ß \neq \emptyset$ and $\forall\rho \in \hbar = \Gamma$,

$$\mho(\rho) = \vartheta(\rho) \text{ and } \Omega(\neg\rho) = \xi(\neg\rho).$$

Then $(\mho,\Omega,\hbar) = (\vartheta,\xi,\Gamma)$. Thus $(\vartheta,\xi,\Gamma) \sqcap_{\Re} (\vartheta_1,\xi_1,ß) = (\vartheta,\xi,\Gamma)$.
(10) Since $(\vartheta,\xi,\Gamma) \ddot{\sqsubseteq} (\vartheta_1,\xi_1,ß)$ and $(\vartheta_1,\xi_1,ß) \ddot{\sqsubseteq} (\vartheta_2,\xi_2,\hbar)$, it follows that $\Gamma \subseteq ß$, $ß \subseteq \hbar$ and $\vartheta(\rho) \subseteq \vartheta_1(\rho)$, $\vartheta_1(\rho) \subseteq \vartheta_2(\rho)$ and $\xi(\neg\rho) \supseteq \xi_1(\neg\rho)$, $\xi_1(\neg\rho) \supseteq \xi_2(\neg\rho)$ $\forall\rho \in \Gamma$. Therefore, $\Gamma \subseteq \hbar$, $\vartheta(\rho) \subseteq \vartheta_2(\rho)$ and $\xi(\neg\rho) \supseteq \xi_2(\neg\rho)$. Hence, $(\vartheta,\xi,\Gamma) \ddot{\sqsubseteq} (\vartheta_2,\xi_2,\hbar)$. □

Proposition 4. *Let (ϑ,ξ,Γ), $(\vartheta_1,\xi_1,ß) \in DFBFSS(\Lambda)$. Then the De-Morgan's law of (ϑ,ξ,Γ) and $(\vartheta_1,\xi_1,ß)$ is as follows:*

1. $((\vartheta,\xi,\Gamma) \ddot{\sqcup} (\vartheta_1,\xi_1,ß))^c = (\vartheta,\xi,\Gamma)^c \ddot{\sqcap} (\vartheta_1,\xi_1,ß)^c$.
2. $((\vartheta,\xi,\Gamma) \ddot{\sqcap} (\vartheta_1,\xi_1,ß))^c = (\vartheta,\xi,\Gamma)^c \ddot{\sqcup} (\vartheta_1,\xi_1,ß)^c$.
3. $((\vartheta,\xi,\Gamma) \sqcup_{\Re} (\vartheta_1,\xi_1,ß))^c = (\vartheta,\xi,\Gamma)^c \sqcap_{\Re} (\vartheta_1,\xi_1,ß)^c$.
4. $((\vartheta,\xi,\Gamma) \sqcap_{\Re} (\vartheta_1,\xi_1,ß))^c = (\vartheta,\xi,\Gamma)^c \sqcup_{\Re} (\vartheta_1,\xi_1,ß)^c$.

Proof. 1. By Definitions 11 and 16, we obtain $(\mho,\Omega,\hbar)^c = ((\vartheta,\xi,\Gamma) \ddot{\sqcup} (\vartheta_1,\xi_1,ß))^c$

$$\mho^c(\rho) = \begin{cases}\vartheta^c(\rho) & \text{if } \rho \in \Gamma \setminus ß\\ \vartheta_1^c(\rho) & \text{if } \rho \in ß \setminus \Gamma\\ \vartheta^c(\rho)\ddot{\wedge}\,\vartheta_1^c(\rho) & \text{if } \rho \in \Gamma \cap ß \neq \emptyset,\end{cases}$$

and

$$\Omega^c(\neg\rho) = \begin{cases}\xi^c(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \setminus \neg ß\\ \xi_1^c(\neg\rho) & \text{if } \neg\rho \in \neg ß \setminus \neg\Gamma\\ \xi^c(\neg\rho)\ddot{\vee}\,\xi_1^c(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \cap \neg ß \neq \emptyset.\end{cases}$$

Then $(\mho,\Omega,\hbar)^c = (\vartheta,\xi,\Gamma)^c\ddot{\sqcap}(\vartheta_1,\xi_1,ß)^c$. Thus $((\vartheta,\xi,\Gamma) \ddot{\sqcup} (\vartheta_1,\xi_1,ß))^c = (\vartheta,\xi,\Gamma)^c \ddot{\sqcap}$ $(\vartheta_1,\xi_1,ß)^c$.

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2. By Definitions 11 and 17, we obtain $(\mho, \Omega, \hbar)^c = ((\vartheta, \xi, \mathit{\Gamma}) \ddot{\sqcap} (\vartheta_1, \xi_1, ß))^c$

$$\mho^c(\rho) = \begin{cases} \vartheta^c(\rho) & \text{if } \rho \in \mathit{\Gamma} \setminus ß \\ \vartheta_1^c(\rho) & \text{if } \rho \in ß \setminus \mathit{\Gamma} \\ \vartheta^c(\rho) \ddot{\vee} \vartheta_1^c(\rho) & \text{if } \rho \in \mathit{\Gamma} \cap ß \neq \emptyset, \end{cases}$$

and

$$\Omega^c(\neg\rho) = \begin{cases} \xi^c(\neg\rho) & \text{if } \neg\rho \in \neg\mathit{\Gamma} \setminus \neg ß \\ \xi_1^c(\neg\rho) & \text{if } \neg\rho \in \neg ß \setminus \neg\mathit{\Gamma} \\ \xi^c(\neg\rho) \ddot{\wedge} \xi_1^c(\neg\rho) & \text{if } \neg\rho \in \neg\mathit{\Gamma} \cap \neg ß \neq \emptyset. \end{cases}$$

Then $(\mho, \Omega, \hbar)^c = (\vartheta, \xi, \mathit{\Gamma})^c \ddot{\sqcup} (\vartheta_1, \xi_1, ß)^c$. Thus $((\vartheta, \xi, \mathit{\Gamma}) \ddot{\sqcap} (\vartheta_1, \xi_1, ß))^c = (\vartheta, \xi, \mathit{\Gamma})^c \ddot{\sqcup} (\vartheta_1, \xi_1, ß)^c$.

3. By Definitions 11 and 16, we obtain $(\mho, \Omega, \hbar)^c = ((\vartheta, \xi, \mathit{\Gamma}) \sqcup_{\Re} (\vartheta_1, \xi_1, ß))^c$.

$$\mho^c(\rho) = \vartheta^c(\rho) \ddot{\wedge} \vartheta_1^c(\rho) \text{ and } \Omega^c(\neg\rho) = \xi^c(\neg\rho) \ddot{\vee} \xi_1^c(\neg\rho).$$

Then $(\mho, \Omega, \hbar)^c = (\vartheta, \xi, \mathit{\Gamma})^c \sqcap_{\Re} (\vartheta_1, \xi_1, ß)^c$. Thus $((\vartheta, \xi, \mathit{\Gamma}) \sqcup_{\Re} (\vartheta_1, \xi_1, ß))^c = (\vartheta, \xi, \mathit{\Gamma})^c \sqcap_{\Re} (\vartheta_1, \xi_1, ß)^c$.

The proof of part 4 is in a similar fashion of number 3.

□

Proposition 5. *Let* $(\vartheta, \xi, \mathit{\Gamma}) \in DFBFSS(\Lambda)$. *Then*

1. $(\vartheta, \xi, \mathit{\Gamma}) \sqcup_{\Re} (\vartheta, \xi, \mathit{\Gamma}) = (\vartheta, \xi, \mathit{\Gamma})$.
2. $(\vartheta, \xi, \mathit{\Gamma}) \sqcap_{\Re} (\vartheta, \xi, \mathit{\Gamma}) = (\vartheta, \xi, \mathit{\Gamma})$.
3. $(\vartheta, \xi, \mathit{\Gamma}) \ddot{\sqcup} (\vartheta, \xi, \mathit{\Gamma}) = (\vartheta, \xi, \mathit{\Gamma})$.
4. $(\vartheta, \xi, \mathit{\Gamma}) \ddot{\sqcap} (\vartheta, \xi, \mathit{\Gamma}) = (\vartheta, \xi, \mathit{\Gamma})$.

Proof. 1. Let $(\mho, \Omega, \hbar) = (\vartheta, \xi, \mathit{\Gamma}) \sqcup_{\Re} (\vartheta, \xi, \mathit{\Gamma})$, where $\hbar = \mathit{\Gamma} \cap \mathit{\Gamma} \neq \phi$ and $\forall \rho \in \hbar$. Then

$$\mho(\rho) = \vartheta(\rho) \ddot{\vee} \vartheta(\rho) = \vartheta(\rho) \text{ and } \Omega(\neg\rho) = \xi(\neg\rho) \ddot{\wedge} \xi(\neg\rho) = \xi(\neg\rho).$$

Hence, $(\vartheta, \xi, \mathit{\Gamma}) \sqcup_{\Re} (\vartheta, \xi, \mathit{\Gamma}) = (\vartheta, \xi, \mathit{\Gamma})$.

2. Let $(\mho, \Omega, \hbar) = (\vartheta, \xi, \mathit{\Gamma}) \sqcap_{\Re} (\vartheta, \xi, \mathit{\Gamma})$, where $\hbar = \mathit{\Gamma} \cap \mathit{\Gamma} \neq \phi$ and $\forall \rho \in \hbar = \mathit{\Gamma}$. Then

$$\mho(\rho) = \vartheta(\rho) \ddot{\wedge} \vartheta(\rho) = \vartheta(\rho) \text{ and } \Omega(\neg\rho) = \xi(\neg\rho) \ddot{\vee} \xi(\neg\rho) = \xi(\neg\rho).$$

Hence, $(\vartheta, \xi, \mathit{\Gamma}) \sqcap_{\Re} (\vartheta, \xi, \mathit{\Gamma}) = (\vartheta, \xi, \mathit{\Gamma})$.

3. Let $(\mho, \Omega, \hbar) = (\vartheta, \xi, \mathit{\Gamma}) \ddot{\sqcup} (\vartheta, \xi, \mathit{\Gamma})$, where $\hbar = \mathit{\Gamma} \cup \mathit{\Gamma}$ and $\forall \rho \in \hbar = \mathit{\Gamma}$, then

$$\mho(\rho) = \vartheta(\rho) \text{ and } \Omega(\neg\rho) = \xi(\neg\rho).$$

Hence, $(\vartheta, \xi, \mathit{\Gamma}) \ddot{\sqcup} (\vartheta, \xi, \mathit{\Gamma}) = (\vartheta, \xi, \mathit{\Gamma})$.

4. Let $(\mho, \Omega, \hbar) = (\vartheta, \xi, \mathit{\Gamma}) \ddot{\sqcap} (\vartheta, \xi, \mathit{\Gamma})$, where $\hbar = \mathit{\Gamma} \cup \mathit{\Gamma}$ and $\forall \rho \in \hbar = \mathit{\Gamma}$, then

$$\mho(\rho) = \vartheta(\rho) \text{ and } \Omega(\neg\rho) = \xi(\neg\rho).$$

Therefore, $(\vartheta, \xi, \mathit{\Gamma}) \ddot{\sqcap} (\vartheta, \xi, \mathit{\Gamma}) = (\vartheta, \xi, \mathit{\Gamma})$.

□

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Proposition 6. *Let* (ϑ, ξ, Γ), $(\vartheta_1, \xi_1, ß) \in DFBFSS(\Lambda)$. *Then the absorption of* (ϑ, ξ, Γ) *and* $(\vartheta_1, \xi_1, ß)$ *is as follows:*

1. $(\vartheta, \xi, \Gamma) \ddot{\sqcup} ((\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß)) = (\vartheta, \xi, \Gamma)$.
2. $(\vartheta, \xi, \Gamma) \ddot{\sqcap} ((\vartheta, \xi, \Gamma) \sqcup_{\Re} (\vartheta_1, \xi_1, ß)) = (\vartheta, \xi, \Gamma)$.
3. $(\vartheta, \xi, \Gamma) \sqcup_{\Re} ((\vartheta, \xi, \Gamma) \ddot{\sqcap} (\vartheta_1, \xi_1, ß)) = (\vartheta, \xi, \Gamma)$.
4. $(\vartheta, \xi, \Gamma) \sqcap_{\Re} ((\vartheta, \xi, \Gamma) \ddot{\sqcup} (\vartheta_1, \xi_1, ß)) = (\vartheta, \xi, \Gamma)$.

Proof. (1) Let $(\mho, \Omega, \hbar)$ be a DFBFS restricted intersection of (ϑ, ξ, Γ) and $(\vartheta_1, \xi_1, ß)$ where $\hbar = \Gamma \cap ß \neq \phi$. Then

$$(\mho, \Omega, \hbar) = (\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß),$$

where $\hbar = \Gamma \cap ß \neq \phi$. Define if $\rho \in \hbar = \Gamma \cap ß$, $\mho(\rho) = \vartheta(\rho) \ddot{\wedge} \vartheta_1(\rho)$ and $\Omega(\neg\rho) = \xi(\neg\rho) \ddot{\vee} \xi_1(\neg\rho)$.

Let $(\mathcal{Q}, \mathcal{K}, \mu)$ be a DFBFSS extended union of (ϑ, ξ, Γ) and $(\mho, \Omega, \hbar)$ which is

$$(\mathcal{Q}, \mathcal{K}, \mu) = (\vartheta, \xi, \Gamma) \ddot{\sqcup} (\mho, \Omega, \hbar),$$

defined by

$$\begin{aligned} \mathcal{Q}(\rho) &= \vartheta(\rho) && \text{if } \rho \in \Gamma \setminus \hbar \\ &= \mho(\rho) && \text{if } \rho \in \hbar \setminus \Gamma \\ &= \vartheta(\rho) \ddot{\vee} \mho(\rho) && \text{if } \rho \in \Gamma \cap \hbar \neq \emptyset, \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}(\neg\rho) &= \xi(\neg\rho) && \text{if } \neg\rho \in \neg\Gamma \setminus \neg\hbar \\ &= \Omega(\neg\rho) && \text{if } \neg\rho \in \neg\hbar \setminus \neg\Gamma \\ &= \xi(\neg\rho) \ddot{\wedge} \Omega(\neg\rho) && \text{if } \neg\rho \in \neg\Gamma \cap \neg\hbar \neq \emptyset. \end{aligned}$$

L.H.S: There are three cases:
Case 1: If $\rho \in \Gamma \setminus \hbar$, then

$$\begin{aligned} \mathcal{Q}(\rho) &= \vartheta(\rho) && \text{if } \rho \in \Gamma \setminus \hbar \\ &= \vartheta(\rho) && \text{if } \rho \in \Gamma \\ \mathcal{Q}(\rho) &= \vartheta(\rho), \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}(\neg\rho) &= \xi(\neg\rho) && \text{if } \neg\rho \in \neg\Gamma \setminus \neg\hbar \\ &= \xi(\neg\rho) && \text{if } \neg\rho \in \neg\hbar \setminus \neg\Gamma \\ \mathcal{K}(\neg\rho) &= \xi(\neg\rho). \end{aligned}$$

Thus $(\mathcal{Q}, \mathcal{K}, \mu) = (\vartheta, \xi, \Gamma)$.
Case 2: If $\rho \in \hbar \setminus \Gamma = \Gamma \cap ß \setminus \Gamma = \emptyset$, then

$$\begin{aligned} \mathcal{Q}(\rho) &= \vartheta(\rho) && \text{if } \rho \in \hbar \setminus \Gamma \\ &= \emptyset && \text{if } \rho \in \emptyset \\ \mathcal{Q}(\rho) &= \emptyset, \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}(\neg\rho) &= \xi(\neg\rho) && \text{if } \neg\rho \in \neg\hbar \setminus \neg\Gamma \\ &= \emptyset && \text{if } \neg\rho \in \emptyset \\ \mathcal{K}(\neg\rho) &= \emptyset. \end{aligned}$$

Thus $(\mathcal{Q}, \mathcal{K}, \mu) = (\ddot{\Phi}, \ddot{\Lambda}, \Gamma)$.
Case 3: If $\rho \in \Gamma \cap \hbar$, then

$$\begin{aligned}
\mathcal{Q}(\rho) &= \vartheta(\rho) \ddot{\vee}\, \mho(\rho) \quad \text{if } \rho \in \Gamma \cap \hbar \textit{ and } \hbar = \Gamma \cap ß \\
&= \vartheta(\rho) \ddot{\vee}\, (\vartheta(\rho) \ddot{\wedge}\, \xi(\rho)) \\
&= \vartheta(\rho) \text{ since } (\vartheta(\rho) \ddot{\wedge}\, \xi(\rho)) \subset \vartheta(\rho) \\
&= \vartheta(\rho) \\
\mathcal{Q}(\rho) &= \vartheta(\rho),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}(\neg\rho) &= \xi(\neg\rho) \ddot{\vee}\, \Omega(\neg\rho) \quad \text{if } \neg\rho \in \neg\Gamma \cap \neg\hbar \textit{ and } \neg\hbar = \neg\Gamma \cap \negß \\
&= \xi(\neg\rho) \ddot{\vee} (\xi(\neg\rho) \ddot{\wedge}\, \Omega(\neg\rho)) \\
&= \xi(\neg\rho) \text{ since } (\xi(\neg\rho) \ddot{\wedge}\, \Omega(\neg\rho)) \subset \xi(\neg\rho) \\
&= \xi(\neg\rho) \\
\mathcal{K}(\rho) &= \xi(\neg\rho).
\end{aligned}$$

Thus, $(\mathcal{Q}, \mathcal{K}, \mu) = (\vartheta, \xi, \Gamma)$.

In above three cases, it is satisfied. Consequently, $(\vartheta, \xi, \Gamma) \ddot{\sqcup} ((\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß)) = (\vartheta, \xi, \Gamma)$.

The proof of 2 is similar to the proof of 1.

(3) Let $(\mho, \Omega, \hbar)$ be a DFBFS extended intersection of (ϑ, ξ, Γ) and $(\vartheta_1, \xi_1, ß)$, where $\hbar = \Gamma \cup ß$. Then

$$(\mho, \Omega, \hbar) = (\vartheta, \xi, \Gamma) \ddot{\sqcap} (\vartheta_1, \xi_1, ß).$$

$$\mho(\rho) = \begin{cases} \vartheta(\rho) & \text{if } \rho \in \Gamma \setminus ß \\ \vartheta_1(\rho) & \text{if } \rho \in ß \setminus \Gamma \\ \vartheta(\rho) \ddot{\wedge}\, \vartheta_1(\rho) & \text{if } \rho \in \Gamma \cap ß \neq \emptyset, \end{cases}$$

and

$$\Omega(\neg\rho) = \begin{cases} \xi(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \setminus \negß \\ \xi_1(\neg\rho) & \text{if } \neg\rho \in \negß \setminus \neg\Gamma \\ \xi(\neg\rho) \ddot{\vee}\, \xi_1(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \cap \negß \neq \emptyset. \end{cases}$$

Let $(\mathcal{Q}, \mathcal{K}, \mu)$ be a DFBFSS restricted union of (ϑ, ξ, Γ) and $(\mho, \Omega, \hbar)$ which is

$$(\mathcal{Q}, \mathcal{K}, \mu) = (\vartheta, \xi, \Gamma) \sqcup_{\Re} (\mho, \Omega, \hbar).$$

Since $\mu = \Gamma \cap \hbar = \Gamma \cap (\Gamma \cup ß) = \Gamma$, we have $\mathcal{Q}(\rho) = \vartheta(\rho)$ and $\mathcal{K}(\neg\rho) = \xi(\neg\rho)$. Thus $(\mathcal{Q}, \mathcal{K}, \mu) = (\vartheta, \xi, \Gamma)$. Hence, $(\vartheta, \xi, \Gamma) \sqcup_{\Re} ((\vartheta, \xi, \Gamma) \ddot{\sqcap} (\vartheta_1, \xi_1, ß)) = (\vartheta, \xi, \Gamma)$.
The proof of 4 is similar to the proof of 3. □

Theorem 1. *Let (ϑ, ξ, Γ), $(\vartheta_1, \xi_1, ß) \in DFBFSS(\Lambda)$. Then the commutativity of (ϑ, ξ, Γ) and $(\vartheta_1, \xi_1, ß)$ is as follows:*

1. $(\vartheta, \xi, \Gamma) \ddot{\sqcup} (\vartheta_1, \xi_1, ß) = (\vartheta_1, \xi_1, ß) \ddot{\sqcup} (\vartheta, \xi, \Gamma)$.
2. $(\vartheta, \xi, \Gamma) \ddot{\sqcap} (\vartheta_1, \xi_1, ß) = (\vartheta_1, \xi_1, ß) \ddot{\sqcap} (\vartheta, \xi, \Gamma)$.
3. $(\vartheta, \xi, \Gamma) \sqcup_{\Re} (\vartheta_1, \xi_1, ß) = (\vartheta_1, \xi_1, ß) \sqcup_{\Re} (\vartheta, \xi, \Gamma)$.
4. $(\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß) = (\vartheta_1, \xi_1, ß) \sqcap_{\Re} (\vartheta, \xi, \Gamma)$.

Proof. (1) To prove that $(\vartheta, \xi, \Gamma) \ddot{\sqcup} (\vartheta_1, \xi_1, ß) = (\vartheta_1, \xi_1, ß) \ddot{\sqcup} (\vartheta, \xi, \Gamma)$. Let $(\mho, \Omega, \hbar)$ be a DFBFSS extended union of (ϑ, ξ, Γ) and $(\vartheta_1, \xi_1, ß)$, where $\hbar = \Gamma \cup ß$,. Then

$$(\mho, \Omega, \hbar) = (\vartheta, \xi, \Gamma) \ddot{\sqcup} (\vartheta_1, \xi_1, ß), \tag{1}$$

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defined by

$$\mho(\rho) = \vartheta(\rho) \quad \text{if } \rho \in \Gamma \setminus ß \tag{2}$$

$$= \vartheta_1(\rho) \qquad \text{if } \rho \in ß \setminus \Gamma \tag{3}$$

$$= \vartheta(\rho) \ddot{\vee} \vartheta_1(\rho) \quad \text{if } \rho \in \Gamma \cap ß \neq \emptyset, \tag{4}$$

and

$$\Omega(\neg\rho) = \xi(\neg\rho) \quad \text{if } \neg\rho \in \neg\Gamma \setminus \negß \tag{5}$$

$$= \xi_1(\neg\rho) \qquad \text{if } \neg\rho \in \negß \setminus \neg\Gamma \tag{6}$$

$$= \xi(\neg\rho) \ddot{\wedge} \xi_1(\neg\rho) \quad \text{if } \neg\rho \in \neg\Gamma \cap \negß \neq \emptyset. \tag{7}$$

There are three cases for that
Case 1: If $\rho \in \Gamma \setminus ß$, from Equations (2) and (5), we obtain

$$\begin{aligned} \mho(\rho) &= \vartheta(\rho) \quad \text{if } \rho \in \Gamma \setminus ß, \\ \Omega(\neg\rho) &= \xi(\neg\rho) \quad \text{if } \neg\rho \in \neg\Gamma \setminus \negß. \end{aligned} \tag{8}$$

Case 2: If $\rho \in ß \setminus \Gamma$, from Equations (3) and (6)

$$\begin{aligned} \mho(\rho) &= \vartheta_1(\rho) \quad \text{if } \rho \in ß \setminus \Gamma, \\ \Omega(\neg\rho) &= \xi_1(\neg\rho) \quad \text{if } \neg\rho \in \negß \setminus \neg\Gamma. \end{aligned} \tag{9}$$

Case 3: If $\rho \in \Gamma \cap ß$

$$\begin{aligned} \mho(\rho) &= \vartheta(\rho) \ddot{\vee} \vartheta_1(\rho) \quad \text{if } \rho \in \Gamma \cap ß = ß \cap \Gamma \\ &= \vartheta_1(\rho) \ddot{\vee} \vartheta(\rho), \end{aligned} \tag{10}$$

and

$$\begin{aligned} \Omega(\neg\rho) &= \xi(\neg\rho) \ddot{\wedge} \xi_1(\neg\rho) \quad \text{if } \neg\rho \in \neg\Gamma \cap \negß \textit{ and } \neg\hbar = \neg\Gamma \cap \negß \\ &= \xi_1(\neg\rho) \ddot{\wedge} \xi(\neg\rho). \end{aligned} \tag{11}$$

Based on the three situations mentioned above, it is determined that

$$\begin{aligned} \mho(\rho) &= \vartheta_1(\rho) & &\text{if } \rho \in ß \setminus \Gamma \\ &= \vartheta(\rho) & &\text{if } \rho \in \Gamma \setminus ß \\ &= \vartheta_1(\rho) \ddot{\vee} \vartheta(\rho) & &\text{if } \rho \in ß \cap \Gamma \neq \emptyset, \end{aligned}$$

and

$$\begin{aligned} \Omega(\neg\rho) &= \xi_1(\neg\rho) & &\text{if } \neg\rho \in \negß \setminus \neg\Gamma \\ &= \xi(\neg\rho) & &\text{if } \neg\rho \in \neg\Gamma \setminus \negß \\ &= \xi_1(\neg\rho) \ddot{\wedge} \xi(\neg\rho) & &\text{if } \neg\rho \in \negß \cap \neg\Gamma \neq \emptyset. \end{aligned}$$

Therefore, $(\mho, \Omega, \hbar) = (\vartheta_1, \xi_1, ß) \ddot{\sqcup} (\vartheta, \xi, \Gamma)$, where $\hbar = ß \cup \Gamma$. Hence,

$$(\vartheta, \xi, \Gamma) \ddot{\sqcup} (\vartheta_1, \xi_1, ß) = (\vartheta_1, \xi_1, ß) \ddot{\sqcup} (\vartheta, \xi, \Gamma).$$

The proof of 2 is similar to the proof of 1.
(3) Let $(\mho, \Omega, \hbar) = (\vartheta, \xi, \Gamma) \sqcup_{\Re} (\vartheta_1, \xi_1, ß)$, where $\hbar = \Gamma \cap ß \neq \phi$ and $\forall \rho \in \hbar$, we have

$$\mho(\rho) = \vartheta(\rho) \ddot{\vee} \vartheta_1(\rho) \text{ and } \Omega(\neg\rho) = \xi(\neg\rho) \ddot{\wedge} \xi_1(\neg\rho). \tag{12}$$

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Let $(\mathcal{Q}, \mathcal{K}, \mu) = (\vartheta_1, \xi_1, ß) \sqcup_{\Re} (\vartheta, \xi, \Gamma)$, where $\mu = \Gamma \cap ß \neq \phi$ and $\forall \rho \in \mu$, we have

$$\begin{aligned} \mathcal{Q}(\rho) &= \vartheta_1(\rho) \ddot{\vee}\, \vartheta(\rho) = \vartheta(\rho) \ddot{\vee}\, \vartheta_1(\rho) \\ \text{and } \mathcal{K}(\neg\rho) &= \xi_1(\neg\rho) \ddot{\wedge}\, \xi(\neg\rho) = \xi(\neg\rho) \ddot{\wedge}\, \xi_1(\neg\rho). \end{aligned} \tag{13}$$

from Equations (12) and (13), we obtain $(\vartheta, \xi, \Gamma) \sqcup_{\Re} (\vartheta_1, \xi_1, ß) = (\vartheta_1, \xi_1, ß) \sqcup_{\Re} (\vartheta, \xi, \Gamma)$.

The proof of 4 is similar to the proof of 3. □

Theorem 2. *Let (ϑ, ξ, Γ), $(\vartheta_1, \xi_1, ß)$, $(\vartheta_2, \xi_2, \hbar) \in DFBFSS(\Lambda)$. Then the associative property of (ϑ, ξ, Γ), $(\vartheta_1, \xi_1, ß)$ and $(\vartheta_2, \xi_2, \hbar)$ is as follows:*

1. $(\vartheta, \xi, \Gamma) \sqcap_{\Re} ((\vartheta_1, \xi_1, ß) \sqcap_{\Re} (\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß)) \sqcap_{\Re} (\vartheta_2, \xi_2, \hbar)$.
2. $(\vartheta, \xi, \Gamma) \sqcup_{\Re} ((\vartheta_1, \xi_1, ß) \sqcup_{\Re} (\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \Gamma) \sqcup_{\Re} (\vartheta_1, \xi_1, ß)) \sqcup_{\Re} (\vartheta_2, \xi_2, \hbar)$.
3. $(\vartheta, \xi, \Gamma) \ddot{\sqcap} ((\vartheta_1, \xi_1, ß) \ddot{\sqcap} (\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \Gamma) \ddot{\sqcap} (\vartheta_1, \xi_1, ß)) \ddot{\sqcap} (\vartheta_2, \xi_2, \hbar)$.
4. $(\vartheta, \xi, \Gamma) \ddot{\sqcup} ((\vartheta_1, \xi_1, ß) \ddot{\sqcup} (\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \Gamma) \ddot{\sqcup} (\vartheta_1, \xi_1, ß)) \ddot{\sqcup} (\vartheta_2, \xi_2, \hbar)$.

Proof. (1) Let $(\mho, \Omega, \mathcal{D})$ be a DFBFS restricted intersection of $(\vartheta_1, \xi_1, ß)$ and $(\vartheta_2, \xi_2, \hbar)$ such that $(\mho, \Omega, \mathcal{D}) = (\vartheta_1, \xi_1, ß) \sqcap_{\Re} (\vartheta_2, \xi_2, \hbar)$, where $\mathcal{D} = \Gamma \cap \hbar \neq \phi$ and $\forall \rho \in \hbar$ defined as:

$$\mho(\rho) = \vartheta_1(\rho) \ddot{\wedge} \vartheta_2(\rho) \text{ and } \Omega(\neg\rho) = \xi_1(\neg\rho) \ddot{\vee} \xi_2(\neg\rho). \tag{14}$$

Let $(\mathcal{Q}, \mathcal{K}, \mu)$ be a DFBFS restricted intersection of (ϑ, ξ, Γ) and $(\mho, \Omega, \mathcal{D})$ such that

$$(\mathcal{Q}, \mathcal{K}, \mu) = (\vartheta, \xi, \Gamma) \sqcap_{\Re} (\mho, \Omega, \mathcal{D}), \tag{15}$$

which is defined by

$$\mathcal{Q}(\rho) = \vartheta(\rho) \ddot{\wedge} \mho(\rho) \text{ and } \mathcal{K}(\neg\rho) = \xi(\neg\rho) \ddot{\vee} \Omega(\neg\rho),$$

where $\rho \in \mu = \Gamma \cap \hbar$ and $\neg\rho \in \neg\mu = \neg\Gamma \cap \neg\hbar$.

$$\begin{aligned} \mathcal{Q}(\rho) &= \vartheta(\rho) \ddot{\wedge}\, \mho(\rho) \\ &= \vartheta(\rho) \ddot{\wedge} (\vartheta_1(\rho) \wedge \vartheta_2(\rho)) \\ &= (\vartheta(\rho) \ddot{\wedge} \vartheta_1(\rho)) \wedge \vartheta_2(\rho), \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}(\neg\rho) &= \xi(\neg\rho) \ddot{\vee} \Omega(\neg\rho) \\ &= \xi(\neg\rho) \ddot{\vee} (\xi_1(\neg\rho) \vee \xi_2(\neg\rho)) \\ &= (\xi(\neg\rho) \ddot{\vee}\, \xi_1(\neg\rho)) \vee \xi_2(\neg\rho). \end{aligned}$$

Then,

$$(\vartheta, \xi, \Gamma) \ddot{\sqcap} (\mho, \Omega, \mathcal{D}) = ((\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß)) \sqcap_{\Re} (\vartheta_2, \xi_2, \hbar).$$

Hence,

$$(\vartheta, \xi, \Gamma) \sqcap_{\Re} ((\vartheta_1, \xi_1, ß) \sqcap_{\Re} (\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \Gamma) \sqcap_{\Re} (\vartheta_1, \xi_1, ß)) \sqcap_{\Re} (\vartheta_2, \xi_2, \hbar).$$

The proof of 2 is similar to the proof of 1.

(3) Suppose that $(\vartheta_1, \xi_1, ß) \ddot{\sqcap} (\vartheta_2, \xi_2, \hbar) = (\mho_1, \Omega_1, ß \cup \hbar)$. For all $\rho \in ß \cup \hbar$, we have the following.

$$\mho_1(\rho) = \begin{cases} \vartheta_1(\rho) & \text{if } \rho \in ß \setminus \hbar \\ \vartheta_2(\rho) & \text{if } \rho \in \hbar \setminus ß \\ \vartheta_1(\rho) \ddot{\wedge}\, \vartheta_2(\rho) & \text{if } \rho \in \hbar \cap ß \neq \emptyset, \end{cases}$$

and

$$\Omega_1(\neg\rho) = \begin{cases} \xi_1(\neg\rho) & \text{if } \neg\rho \in \neg\text{ß} \setminus \neg\hbar \\ \xi_2(\neg\rho) & \text{if } \neg\rho \in \neg\hbar \setminus \neg\text{ß} \\ \xi_1(\neg\rho) \ddot{\vee} \xi_2(\neg\rho) & \text{if } \neg\rho \in \neg\text{ß} \cap \neg\hbar \neq \emptyset. \end{cases}$$

Assume that $(\vartheta, \xi, \Gamma)\ddot{\sqcap}(\mho_1, \Omega_1, \text{ß} \cup \hbar) = (\mho_2, \Omega_2, \Gamma \cup (\text{ß} \cup \hbar)$ for all $\rho \in \Gamma \cup (\text{ß} \cup \hbar)$, we have the following.

$$\mho_2(\rho) = \begin{cases} \vartheta(\rho) & \text{if } \rho \in \Gamma \setminus \text{ß} \cup \hbar \\ \mho_1(\rho) & \text{if } \rho \in \text{ß} \cup \hbar \setminus \Gamma \\ \vartheta(\rho) \ddot{\wedge} \mho_1(\rho) & \text{if } \rho \in \Gamma \cap \text{ß} \cup \hbar \neq \emptyset, \end{cases}$$

and

$$\Omega_2(\neg\rho) = \begin{cases} \xi(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \setminus \neg(\text{ß} \cup \hbar) \\ \Omega_1(\neg\rho) & \text{if } \neg\rho \in \neg(\text{ß} \cup \hbar) \setminus \neg\Gamma \\ \xi(\neg\rho) \ddot{\vee} \Omega_1(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \cap \neg(\text{ß} \cup \hbar) \neq \emptyset. \end{cases}$$

On the other hand, let $((\vartheta, \xi, \Gamma)\ddot{\sqcap}(\vartheta_1, \xi_1, \text{ß})) = (\mho_3, \Omega_3, \Gamma \cup \text{ß})$

$$\mho_3(\rho) = \begin{cases} \vartheta(\rho) & \text{if } \rho \in \Gamma \setminus \text{ß} \\ \vartheta_1(\rho) & \text{if } \rho \in \text{ß} \setminus \Gamma \\ \vartheta(\rho) \ddot{\wedge} \vartheta_1(\rho) & \text{if } \rho \in \Gamma \cap \text{ß} \neq \emptyset, \end{cases}$$

and

$$\Omega_3(\neg\rho) = \begin{cases} \xi(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \setminus \neg\text{ß} \\ \xi_1(\neg\rho) & \text{if } \neg\rho \in \neg\text{ß} \setminus \neg\Gamma \\ \xi(\neg\rho) \ddot{\vee} \xi_1(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \cap \neg\text{ß} \neq \emptyset. \end{cases}$$

Suppose that $(\mho_3, \Omega_3, \Gamma \cup \text{ß})\ddot{\sqcap}(\vartheta_2, \xi_2, \hbar) = (\mho_4, \Omega_4, (\Gamma \cup \text{ß}) \cup \hbar) = (\mho_4, \Omega_4, \Gamma \cup (\text{ß} \cup \hbar))$. for all $\rho \in \Gamma \cup (\text{ß} \cup \hbar)$, we have the following.

$$\mho_4(\rho) = \begin{cases} \mho_3(\rho) & \text{if } \rho \in \Gamma \cup \text{ß} \setminus \hbar \\ \vartheta_2(\rho) & \text{if } \rho \in \hbar \setminus \Gamma \cup \text{ß} \\ \vartheta_2(\rho) \ddot{\wedge} \mho_3(\rho) & \text{if } \rho \in (\Gamma \cup \text{ß}) \cap \hbar \neq \emptyset, \end{cases}$$

and

$$\Omega_4(\neg\rho) = \begin{cases} \Omega_3(\neg\rho) & \text{if } \neg\rho \in \neg(\Gamma \cup \text{ß}) \setminus \neg\hbar \\ \xi_2(\neg\rho) & \text{if } \neg\rho \in \neg\hbar \setminus \neg(\Gamma \cup \text{ß}) \\ \xi_2(\neg\rho) \ddot{\vee} \Omega_3(\neg\rho) & \text{if } \neg\rho \in \neg(\Gamma \cup \text{ß}) \cap \neg\hbar \neq \emptyset. \end{cases}$$

This implies that

$$\mho_4(\rho) = \begin{cases} \vartheta(\rho) & \text{if } \rho \in \Gamma \setminus \text{ß} \cup \hbar \\ \mho_1(\rho) & \text{if } \rho \in \text{ß} \cup \hbar \setminus \Gamma \\ \vartheta(\rho) \ddot{\wedge} \mho_1(\rho) & \text{if } \rho \in \Gamma \cap (\text{ß} \cup \hbar) \neq \emptyset, \end{cases}$$

and

$$\Omega_4(\neg\rho) = \begin{cases} \xi(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \setminus \neg(\text{ß} \cup \hbar) \\ \Omega_1(\neg\rho) & \text{if } \neg\rho \in \neg(\text{ß} \cup \hbar) \setminus \neg\Gamma \\ \xi(\neg\rho) \ddot{\vee} \Omega_1(\neg\rho) & \text{if } \neg\rho \in \neg\Gamma \cap \neg(\text{ß} \cup \hbar) \neq \emptyset. \end{cases}$$

Since $(\mho_2, \Omega_2, \Gamma \cup (ß \cup \hbar)$ and $(\mho_4, \Omega_4, \Gamma \cup (ß \cup \hbar))$ are the same set-valued mapping for all $\rho \in \Gamma \cup ß \cup \hbar$, the proof is completed.

The proof of part 4 can be proved with the same method of part 3. □

Theorem 3. *Let* (ϑ, ξ, Γ), $(\vartheta_1, \xi_1, ß)$ *and* $(\vartheta_2, \xi_2, \hbar) \in DFBFSS$. *Then*

1. $(\vartheta, \xi, \Gamma) \ltimes ((\vartheta_1, \vartheta_1, ß) \leftthreetimes (\vartheta_2, \vartheta_2, \hbar)) = ((\vartheta, \xi, \Gamma) \ltimes (\vartheta_1, \xi_1, ß)) \leftthreetimes ((\vartheta, \xi, \Gamma) \ltimes (\vartheta_2, \xi_2, \hbar))$ *where* $\ltimes \neq \leftthreetimes$, $\ltimes \in \{\sqcap_{\Re}, \sqcup_{\Re}\}$ *and* $\leftthreetimes \in \{\sqcap_{\Re}, \ddot{\sqcap}, \ddot{\sqcup}, \sqcup_{\Re}\}$.
2. $(\vartheta, \xi, \Gamma)\ddot{\sqcup}((\vartheta_1, \xi_1, ß)\ddot{\sqcap}(\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \Gamma)\ddot{\sqcup}(\vartheta_1, \xi_1, ß))\ddot{\sqcap}((\vartheta, \xi, \Gamma)\ddot{\sqcup}(\vartheta_2, \xi_2, \hbar))$.
3. $(\vartheta, \xi, \Gamma)\ddot{\sqcup}((\vartheta_1, \xi_1, ß)\sqcup_{\Re}(\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \Gamma)\ddot{\sqcup}(\vartheta_1, \xi_1, ß))\sqcup_{\Re}((\vartheta, \xi, \Gamma)\ddot{\sqcup}(\vartheta_2, \xi_2, \hbar))$.
4. $(\vartheta, \xi, \Gamma)\ddot{\sqcup}((\vartheta_1, \xi_1, ß)\sqcap_{\Re}(\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \Gamma)\ddot{\sqcup}(\vartheta_1, \xi_1, ß))\sqcap_{\Re}((\vartheta, \xi, \Gamma)\ddot{\sqcup}(\vartheta_2, \xi_2, \hbar))$.
5. $(\vartheta, \xi, \Gamma)\ddot{\sqcap}((\vartheta_1, \xi_1, ß)\ddot{\sqcup}(\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \Gamma)\ddot{\sqcap}(\vartheta_1, \xi_1, ß))\ddot{\sqcup}((\vartheta, \xi, \Gamma)\ddot{\sqcap}(\vartheta_2, \xi_2, \hbar))$.
6. $(\vartheta, \xi, \Gamma)\ddot{\sqcap}((\vartheta_1, \xi_1, ß)\sqcup_{\Re}(\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \Gamma)\ddot{\sqcap}(\vartheta_1, \xi_1, ß))\sqcup_{\Re}((\vartheta, \xi, \Gamma)\ddot{\sqcap}(\vartheta_2, \xi_2, \hbar))$.
7. $(\vartheta, \xi, \Gamma)\ddot{\sqcup}((\vartheta_1, \xi_1, ß)\sqcup_{\Re}(\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \Gamma)\ddot{\sqcup}(\vartheta_1, \xi_1, ß))\sqcup_{\Re}((\vartheta, \xi, \Gamma)\ddot{\sqcup}(\vartheta_2, \xi_2, \hbar))$.

Proof. (1) Let $(\mho, \Omega, \mathcal{D})$ be a DFBFS extended union of $(\vartheta_1, \xi_1, ß)$ and $(\vartheta_2, \xi_2, \hbar)$. Then

$$(\mho, \Omega, \mathcal{D}) = (\vartheta_1, \xi_1, ß)\ddot{\sqcup}(\vartheta_2, \xi_2, \hbar), \tag{16}$$

where $\mathcal{D} = ß \cup \hbar$ and is defined by:

$$\begin{aligned} \mho(\rho) &= \vartheta_1(\rho) && \text{if } \rho \in ß \setminus \hbar \\ &= \vartheta_2(\rho) && \text{if } \rho \in \hbar \setminus ß \\ &= \vartheta_1(\rho)\ddot{\vee}\vartheta_2(\rho) && \text{if } \rho \in ß \cap \hbar \neq \emptyset, \end{aligned}$$

and

$$\begin{aligned} \Omega(\neg\rho) &= \xi_1(\neg\rho) && \text{if } \neg\rho \in \neg ß \setminus \neg\hbar \\ &= \xi_2(\neg\rho) && \text{if } \neg\rho \in \neg\hbar \setminus \neg ß \\ &= \xi_1(\neg\rho)\ddot{\wedge}\xi_2(\neg\rho) && \text{if } \neg\rho \in \neg ß \cap \neg\hbar \neq \emptyset. \end{aligned}$$

Let $(\mathcal{Q}, \mathcal{K}, \mu)$ be a DFBFS restricted intersection of (ϑ, ξ, Γ) and $(\mho, \Omega, \mathcal{D})$ which is

$$(\mathcal{Q}, \mathcal{K}, \mu) = (\vartheta, \xi, \Gamma)\sqcap_{\Re}(\mho, \Omega, \mathcal{D}), \tag{17}$$

defined by

$$\mathcal{Q}(\rho) = \vartheta(\rho)\ddot{\wedge}\,\mho(\rho) \text{ and } \mathcal{K}(\neg\rho) = \xi(\neg\rho)\ddot{\vee}\Omega(\neg\rho),$$

where $\rho \in \mu = \Gamma \cap \mathcal{D}$ and $\neg\rho \in \neg\mu = (\neg\Gamma) \cap (\neg\mathcal{D})$.

From left hand side:

$$\mathcal{Q}(\rho) = \vartheta(\rho)\ddot{\wedge}\,\mho(\rho) \text{ and } \mathcal{K}(\neg\rho) = \xi(\neg\rho)\ddot{\vee}\Omega(\neg\rho), \tag{18}$$

and this implies that $\rho \in \Gamma$ and $\rho \in \mathcal{D}$.

If $\rho \in \mathcal{D} = ß \cup \hbar$, then from (16), there are three cases:
Case 1: If $\rho \in ß \setminus \hbar$, then

$$\begin{aligned} \mho(\rho) &= \vartheta_1(\rho) && \text{if } \rho \in ß \setminus \hbar, \\ \Omega(\neg\rho) &= \xi_1(\neg\rho) && \text{if } \neg\rho \in \neg ß \setminus \neg\hbar. \end{aligned}$$

Cases 2: If $\rho \in \hbar \setminus ß$, then

$$\begin{aligned} \mho(\rho) &= \vartheta_2(\rho) && \text{if } \rho \in \hbar \setminus ß, \\ \Omega(\neg\rho) &= \xi_2(\neg\rho) && \text{if } \neg\rho \in \neg\hbar \setminus \neg ß. \end{aligned} \tag{19}$$

Cases 3: If $\rho \in \text{ß} \cap \hbar$, then

$$\begin{aligned} \mho(\rho) &= \vartheta_1(\rho) \ddot{\vee} \vartheta_2(\rho) && \text{if } \rho \in \text{ß} \cap \hbar, \\ \Omega(\neg\rho) &= \xi_1(\neg\rho) \tilde{\wedge} \xi_2(\neg\rho) && \text{if } \neg\rho \in \neg\text{ß} \cap \neg\hbar. \end{aligned} \tag{20}$$

Based on the above three cases, we obtain, from (18) that

$$\begin{aligned} \mathcal{Q}(\rho) &= \vartheta(\rho) \ddot{\wedge} \vartheta_1(\rho) && \text{if } \rho \in \mathit{\Gamma}, \rho \in \text{ß} \setminus \hbar \\ &= \vartheta(\rho) \ddot{\wedge} \vartheta_2(\rho) && \text{if } \rho \in \mathit{\Gamma}, \rho \in \hbar \setminus \text{ß} \\ &= \vartheta(\rho) \ddot{\wedge} (\vartheta_1(\rho) \ddot{\vee} \vartheta_2(\rho)) && \text{if } \rho \in \mathit{\Gamma}, \rho \in \text{ß} \cap \hbar \\ &= (\vartheta(\rho) \ddot{\wedge} \vartheta_1(\rho)) \ddot{\vee} (\vartheta(\rho) \ddot{\wedge} \vartheta_2(\rho)), \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}(\neg\rho) &= \xi(\neg\rho) \ddot{\vee} \xi_1(\neg\rho) && \text{if } \neg\rho \in \neg\mathit{\Gamma}, \neg\rho \in \neg\text{ß} \setminus \neg\hbar \\ &= \xi(\neg\rho) \ddot{\vee} \xi_2(\neg\rho) && \text{if } \neg\rho \in \neg\mathit{\Gamma}, \neg\rho \in \neg\hbar \setminus \neg\text{ß} \\ &= \xi(\neg\rho) \ddot{\vee} (\xi_1(\neg\rho) \ddot{\wedge} \xi_2(\neg\rho)) && \text{if } \neg\rho \in \neg\mathit{\Gamma}, \neg\rho \in \neg\text{ß} \cap \neg\hbar \neq \emptyset \\ &= (\xi(\neg\rho) \ddot{\vee} \xi_1(\neg\rho)) \ddot{\wedge} (\xi(\neg\rho) \ddot{\vee} \xi_2(\neg\rho)). \end{aligned}$$

As a result,

$$\mathcal{Q}(\rho) = (\vartheta(\rho) \ddot{\wedge} \vartheta_1(\rho)) \ddot{\vee} (\vartheta(\rho) \ddot{\wedge} \vartheta_2(\rho)),$$

and

$$\mathcal{K}(\neg\rho) = (\xi(\neg\rho) \ddot{\vee} \xi_1(\neg\rho)) \ddot{\wedge} (\xi(\neg\rho) \ddot{\vee} \xi_2(\neg\rho)).$$

Therefore,

$$(\mathcal{Q}, \mathcal{K}, \mu) = ((\vartheta, \xi, \mathit{\Gamma}) \sqcap_{\Re} (\vartheta_1, \xi_1, \text{ß})) \ddot{\sqcup} ((\vartheta, \xi, \mathit{\Gamma}) \sqcap_{\Re} (\vartheta_2, \xi_2, \hbar)). \tag{21}$$

Also,

$$(\vartheta, \xi, \mathit{\Gamma}) \sqcap_{\Re} (\mho, \Omega, \mathcal{D}) = ((\vartheta, \xi, \mathit{\Gamma}) \sqcap_{\Re} (\vartheta_1, \xi_1, \text{ß})) \ddot{\sqcup} ((\vartheta, \xi, \mathit{\Gamma}) \sqcap_{\Re} (\vartheta_2, \xi_2, \hbar)). \tag{22}$$

Thus,

$$(\vartheta, \xi, \mathit{\Gamma}) \sqcap_{\Re} ((\vartheta_1, \xi_1, \text{ß}) \ddot{\sqcup} (\vartheta_2, \xi_2, \hbar)) = ((\vartheta, \xi, \mathit{\Gamma}) \sqcap_{\Re} (\vartheta_1, \xi_1, \text{ß})) \ddot{\sqcup} ((\vartheta, \xi, \mathit{\Gamma}) \sqcap_{\Re} (\vartheta_2, \xi_2, \hbar).$$

(6) Suppose that $(\vartheta_1, \xi_1, \text{ß}) \sqcup_{\Re} (\vartheta_2, \xi_2, \hbar) = (\mho_1, \Omega_1, \text{ß} \cap \hbar)$. Forall $\rho \in \text{ß} \cap \hbar$, we have the following.

$$\mho_1(\rho) = \vartheta_1(\rho) \ddot{\vee} \vartheta_2(\rho) \text{ and } \Omega_1(\neg\rho) = \xi_1(\neg\rho) \ddot{\wedge} \xi_2(\neg\rho).$$

Assume that $(\vartheta, \xi, \mathit{\Gamma}) \ddot{\sqcap} (\mho_1, \Omega_1, \text{ß} \cap \hbar) = (\mho_2, \Omega_2, \mathit{\Gamma} \cup (\text{ß} \cap \hbar)) = (\mho_2, \Omega_2, \mu \cap \mathcal{D})$ where $\mu = \mathit{\Gamma} \cup \text{ß}$ and $\mathcal{D} = \mathit{\Gamma} \cup \hbar$. For all $\rho \in \mu \cap \mathcal{D}$, we have the following.

$$\mho_2(\rho) = \begin{cases} \vartheta(\rho) \ddot{\vee} \vartheta_1(\rho) & \text{if } \rho \in \mu \setminus \mathcal{D} \\ \vartheta(\rho) \ddot{\vee} \vartheta_2(\rho) & \text{if } \rho \in \mathcal{D} \setminus \mu \\ \vartheta(\rho) \ddot{\wedge} (\vartheta_1(\rho) \ddot{\vee} \vartheta_2(\rho)) & \text{if } \rho \in \mu \cap \mathcal{D} \neq \emptyset, \end{cases}$$

and

$$\Omega_2(\neg\rho) = \begin{cases} \xi(\neg\rho) \ddot{\wedge} \xi_1(\neg\rho) & \text{if } \neg\rho \in \neg\mu \setminus \neg\mathcal{D} \\ \xi(\neg\rho) \ddot{\wedge} \xi_2(\neg\rho) & \text{if } \neg\rho \in \neg\mathcal{D} \setminus \neg\mu \\ \xi(\neg\rho) \ddot{\vee} (\xi_1(\neg\rho) \ddot{\wedge} \xi_2(\neg\rho)) & \text{if } \neg\rho \in \neg\mu \cap \neg\mathcal{D} \neq \emptyset, \end{cases}$$

On the other hand, let $((\vartheta,\xi,\mathit{\Gamma})\ddot{\sqcap}(\vartheta_1,\xi_1,\text{ß}))=(\mho_3,\Omega_3,\mathit{\Gamma}\cup\text{ß})$

$$\mho_3(\rho)=\begin{cases}\vartheta(\rho) & \text{if } \rho\in\mathit{\Gamma}\setminus\text{ß}\\ \vartheta_1(\rho) & \text{if } \rho\in\text{ß}\setminus\mathit{\Gamma}\\ \vartheta(\rho)\ddot{\wedge}\,\vartheta_1(\rho) & \text{if } \rho\in\mathit{\Gamma}\cap\text{ß}\neq\emptyset,\end{cases}$$

and

$$\Omega_3(\neg\rho)=\begin{cases}\xi(\neg\rho) & \text{if } \neg\rho\in\neg\mathit{\Gamma}\setminus\neg\text{ß}\\ \xi_1(\neg\rho) & \text{if } \neg\rho\in\neg\text{ß}\setminus\neg\mathit{\Gamma}\\ \xi(\neg\rho)\ddot{\vee}\,\xi_1(\neg\rho) & \text{if } \neg\rho\in\neg\mathit{\Gamma}\cap\neg\text{ß}\neq\emptyset.\end{cases}$$

Let $(\vartheta,\xi,\mathit{\Gamma})\ddot{\sqcap}(\vartheta_2,\xi_2,\hbar)=(\mho_4,\Omega_4,\mathit{\Gamma}\cup\hbar)$.

$$\mho_4(\rho)=\begin{cases}\vartheta(\rho) & \text{if } \rho\in\mathit{\Gamma}\setminus\hbar\\ \vartheta_2(\rho) & \text{if } \rho\in\hbar\setminus\mathit{\Gamma}\\ \vartheta(\rho)\ddot{\wedge}\,\vartheta_2(\rho) & \text{if } \rho\in\mathit{\Gamma}\cap\hbar\neq\emptyset,\end{cases}$$

and

$$\Omega_4(\neg\rho)=\begin{cases}\xi(\neg\rho) & \text{if } \neg\rho\in\neg\mathit{\Gamma}\setminus\neg\hbar\\ \xi_2(\neg\rho) & \text{if } \neg\rho\in\neg\hbar\setminus\neg\mathit{\Gamma}\\ \xi(\neg\rho)\ddot{\vee}\,\xi_2(\neg\rho) & \text{if } \neg\rho\in\neg\mathit{\Gamma}\cap\neg\hbar\neq\emptyset.\end{cases}$$

Let $(\mho_3,\Omega_3,\mathit{\Gamma}\cup\text{ß})\sqcup_{\Re}(\mho_4,\Omega_4,\mathit{\Gamma}\cup\hbar)=(\mho_5,\Omega_5,\mu\cap\mathcal{D})$, where $\mu=\mathit{\Gamma}\cup\text{ß}$ and $\mathcal{D}=\mathit{\Gamma}\cup\hbar$.

For all $\rho\in\mu\cap\mathcal{D}$, we have the following.

$$\mho_5(\rho)=\mho_3(\rho)\ddot{\vee}\mho_4(\rho) \text{ and } \Omega_5(\neg\rho)=\Omega_3(\neg\rho)\ddot{\wedge}\Omega_4(\neg\rho).$$

In this case, we obtain the following:

$$\mho_5(\rho)=\begin{cases}\mho_3(\rho) & \text{if } \rho\in\mu\setminus\mathcal{D}\\ \mho_4(\rho) & \text{if } \rho\in\mathcal{D}\setminus\mu\\ \mho_3(\rho)\ddot{\vee}\mho_4(\rho) & \text{if } \rho\in\mu\cap\mathcal{D}\neq\emptyset,\end{cases}$$

and

$$\Omega_5(\neg\rho)=\begin{cases}\Omega_3(\neg\rho) & \text{if } \neg\rho\in\neg\mu\setminus\neg\mathcal{D}\\ \Omega_4(\neg\rho) & \text{if } \neg\rho\in\neg\mathcal{D}\setminus\neg\mu\\ \Omega_3(\neg\rho)\ddot{\wedge}\Omega_4(\neg\rho) & \text{if } \neg\rho\in\neg\mu\cap\neg\mathcal{D}\neq\emptyset,\end{cases}$$

then

$$\mho_5(\rho)=\begin{cases}\vartheta(\rho)\ddot{\vee}\vartheta_1(\rho) & \text{if } \rho\in\mu\setminus\mathcal{D}\\ \vartheta(\rho)\ddot{\vee}\vartheta_2(\rho) & \text{if } \rho\in\mathcal{D}\setminus\mu\\ ((\vartheta(\rho)\ddot{\wedge}\,\vartheta_1(\rho))\ddot{\vee}(\vartheta(\rho)\ddot{\wedge}\,\vartheta_2(\rho)) & \text{if } \rho\in\mu\cap\mathcal{D}\neq\emptyset,\end{cases}$$

and

$$\Omega_5(\neg\rho)=\begin{cases}\xi(\neg\rho)\ddot{\wedge}\xi_1(\neg\rho) & \text{if } \neg\rho\in\neg\mu\setminus\neg\mathcal{D}\\ \xi(\neg\rho)\ddot{\wedge}\xi_2(\neg\rho) & \text{if } \neg\rho\in\neg\mathcal{D}\setminus\neg\mu\\ (\xi(\neg\rho)\ddot{\vee}\,\xi_1(\neg\rho))\ddot{\wedge}(\xi(\neg\rho)\ddot{\vee}\,\xi_2(\neg\rho)) & \text{if } \neg\rho\in\neg\mu\cap\neg\mathcal{D}\neq\emptyset,\end{cases}$$

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Therefore,

$$\mho_5(\rho) = \begin{cases} \vartheta(\rho)\ddot{\vee}\vartheta_1(\rho) & \text{if } \rho \in \mu \setminus \mathcal{D} \\ \vartheta(\rho)\ddot{\vee}\vartheta_2(\rho) & \text{if } \rho \in \mathcal{D} \setminus \mu \\ \vartheta(\rho)\ddot{\wedge}(\vartheta_1(\rho)\ddot{\vee}\vartheta_2(\rho)) & \text{if } \rho \in \mu \cap \mathcal{D} \neq \emptyset, \end{cases}$$

and

$$\Omega_5(\neg\rho) = \begin{cases} \xi(\neg\rho)\ddot{\wedge}\xi_1(\neg\rho) & \text{if } \neg\rho \in \neg\mu \setminus \neg\mathcal{D} \\ \xi(\neg\rho)\ddot{\wedge}\xi_2(\neg\rho) & \text{if } \neg\rho \in \neg\mathcal{D} \setminus \neg\mu \\ \xi(\neg\rho)\ddot{\vee}(\xi_1(\neg\rho)\ddot{\wedge}\xi_2(\neg\rho)) & \text{if } \neg\rho \in \neg\mu \cap \neg\mathcal{D} \neq \emptyset. \end{cases}$$

Since $(\mho_2, \Omega_2, \mu \cap \mathcal{D})$ and $(\mho_5, \Omega_5, \mu \cap \mathcal{D})$ are the same set-valued mapping for all $\rho \in \mu \cap \mathcal{D}$, the proof is completed. The remaining parts can be proven with the same method. □

Definition 20 (Comparison Table)**.** *The comparison table of DFBFSS is a square table with an equal number of rows and columns (square table), both of which are labeled with the universe's object names (ρ_1, ρ_2, ρ_3,..., ρ_n) and the entries l_{ij}, where l_{ij} is the number of parameters for which*

- *the positive membership value of l_i is less than or equal to the positive membership value of l_j,*
- *the negative membership value of l_i is greater than or equal to the negative membership value of l_j,*
- *the negation of positive membership value of $\neg l_i$ is greater than or equal to the positive membership value of $\neg l_j$, and*
- *the negation of negative membership value of $\neg l_i$ is less than or equal to the negative membership value of $\neg l_j$.*

Definition 21. *Positive (P_i) and negative (N_i) information scores with their negations $(\neg P_i)$ and $(\neg N_i)$, respectively, are computed where*

$$P_i = r_i - c_i,$$

$$\neg P_i = (\neg r_i) - (\neg\hbar_i),$$

$$N_i = r'_i - c'_i,$$

and

$$\neg N_i = (\neg r'_i) - (\neg\hbar'_i).$$

Adjusted Positive (APS_i) and negative (ANS_i) information scores with their negations are computed where

$$APS_i = (P_i) - (\neg P_i),$$

and

$$ANS_i = (N_i) - (\neg N_i).$$

Adjusted final score (AFS_i) is computed where

$$AFS_i = APS_i - ANS_i.$$

5. Application

The proliferation of these rental apartments is a by-product of population growth that adds unprecedented bigger into urban areas.This increase in demand has made the apartment selection process more important. But families are increasingly getting pushed away, as the seek proximity to schools, access to public transportation and affordable rents.

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But when demand is high, finding apartments with the most sought-after factors car be difficult, leaving families to weigh compromises like higher rents or longer commutes. This discrepancy between demand and supply only magnifies the hunt for apartments that align with these key factors.

Algorithm. The algorithm and stepwise procedure of DFBFSS for choosing the best option is provided as:

Step 1: The algorithm begins by accepting a Double-Frame Bipolar Fuzzy Soft Set (DFBFSS) (ϑ, ξ, Γ), where (ϑ) is the set of alternatives, (ξ) the set of criteria, and (Γ) the mapping that assigns to each alternative-criterion pair a tuple of four membership values: positive membership ϱ^+, its negation η^+, negative membership ϱ^-, and its negation η^-. These values, derived from expert judgment or data, are then systematically organized into a decision matrix (tabular form), with rows representing alternatives, columns representing criteria, and each cell containing the corresponding four-dimensional membership tuple. This matrix serves as the structured input, clearly presenting both supporting and opposing evaluations alongside their respective negations for all considered factors.

Step 2: Normalizing is required if the criteria are not on the same scale. This involves normalizing the decision matrix based on the characteristics of each criterion by using the Definition 11, i.e., for each non-benefit criterion, the normalization operation applies the complement definition to each membership component—for example, transforming the original positive membership ϱ^+ into $1 - \varrho^+$ or a structurally equivalent complement, ensuring all criteria contribute uniformly within the bipolar fuzzy framework while preserving the relational logic between a membership and its negation. Benefit criteria remain unchanged, as their original values are already aligned with the desired direction of preference.

Step 3: In this step, comparison tabular for functions F and G is computed using Definition 20. The following three steps are found using Definition 21.

Step 4: Positive (P_i) and negative (N_i) information scores with their negations $(\neg P_i)$ and $(\neg N_i)$ are computed.

Step 5: Adjusted positive (APS_i) and negative (ANS_i) information scores with their negations are computed.

Step 6: Adjusted Final score (AFS_i) is computed.

Step 7: Find k where $AFS_k = \max_{1 \leq i \leq n} AFS_i$.

Step 8: For $k = 1, 2, \ldots, n$, AFS_k is the best choice to be chosen.

The overall structure of the proposed decision-making algorithm is illustrated in Figure 2.

The illustrative example used in this study is intentionally simplified to clearly demonstrate the operational mechanism of the proposed DFBFSS framework. Such synthetic examples are standard in the development of new fuzzy set models, where the primary objective is to validate structural behavior and decision consistency before large-scale empirical deployment.

The following section declares a numerical example on the above algorithm in order to verify the feasibility of the proposed method procedure.

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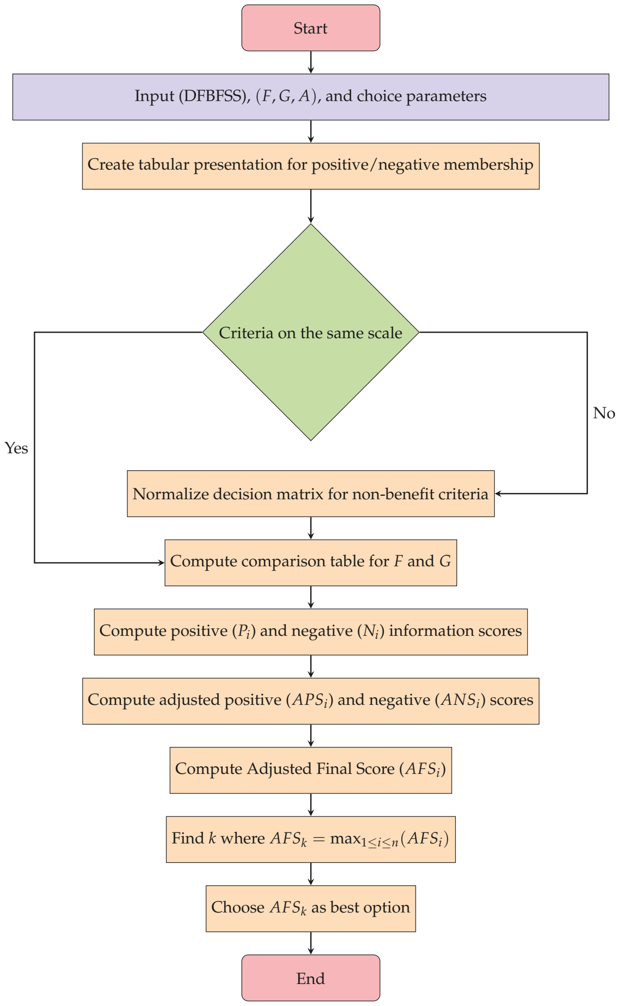

Figure 2. Flowchart.

6. Numerical Example

A family is searching for a residential apartment and assess their options based on some criteria. Let $\Lambda = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6\}$ be a set of six apartments. Consider $\Upsilon = \{\rho_1 =$ affordability monthly rent, $\rho_2 =$ accessibility to public transport, $\rho_3 =$ schools proximity and $\rho_4 =$ noise levels$\}$ be a set of parameters. Also, let the negation set of Υ be $\neg\Upsilon = \{\neg\rho_1 =$ unaffordability monthly rent, $\neg\rho_2 =$ no public transport, $\neg\rho_3 =$ school remoteness and $\neg\rho_4 =$ quietness levels$\}$. Following a thorough discussion, the family members determine that each apartment will be evaluated using a favorable subset $\Gamma = \{\rho_1, \rho_3, \rho_4\}$ of Υ.

Step 1 : Input $(DFBFSS)$ (ϑ, ξ, Γ) (see Table 11) and choice parameters for family (see Tables 12–15).

Table 11. Table of (ϑ, ξ, Γ).

	(ϑ, Γ)			$(\xi, \neg\Gamma)$		
	$\vartheta(\rho_1)$	$\vartheta(\rho_3)$	$\vartheta(\rho_4)$	$\xi(\neg\rho_1)$	$\xi(\neg\rho_3)$	$\xi(\rho_4)$
ς_1	$(0.35, -0.3)$	$(0.4, -0.25)$	$(0.5, -0.7)$	$(0.1, -0.4)$	$(0.6, -0.5)$	$(0.3, -0.23)$
ς_2	$(0.45, -0.2)$	$(0.65, -0.4)$	$(0.3, -0.8)$	$(0.37, -0.4)$	$(0.22, -0.6)$	$(0.64, -0.1)$
ς_3	$(0.45, -0.43)$	$(0.2, -0.7)$	$(0.8, -0.6)$	$(0.4, -0.15)$	$(0.5, -0.2)$	$(0.2, -0.3)$
ς_4	$(0.7, -0.5)$	$(0.8, -0.6)$	$(0.74, -0.31)$	$(0.25, -0.3)$	$(0.1, -0.4)$	$(0.1, -0.49)$
ς_5	$(0.74, -0.6)$	$(0.39, -0.7)$	$(0.7, -0.8)$	$(0.2, -0.4)$	$(0.5, -0.25)$	$(0.2, -0.1)$
ς_6	$(0.46, -0.1)$	$(0.6, -0.6)$	$(0.33, -0.8)$	$(0.1, -0.31)$	$(0.1, -0.3)$	$(0.52, -0.1)$

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Table 12. Tabular presentation for ϱ^{+}.

ϑ	ρ_1	ρ_3	ρ_4
ς_1	0.35	0.4	0.5
ς_2	0.45	0.65	0.3
ς_3	0.45	0.2	0.8
ς_4	0.7	0.8	0.74
ς_5	0.74	0.39	0.7
ς_6	0.46	0.6	0.33

Table 13. Tabular presentation for $\neg\eta^{+}$.

ξ	$\neg\rho_1$	$\neg\rho_3$	$\neg\rho_4$
ς_1	0.1	0.6	0.3
ς_2	0.37	0.22	0.65
ς_3	0.4	0.5	0.2
ς_4	0.25	0.1	0.1
ς_5	0.2	0.5	0.2
ς_6	0.1	0.1	0.52

Table 14. Tabular presentation for ϱ^{-}.

ϑ	ρ_1	ρ_3	ρ_4
ς_1	-0.3	-0.25	-0.7
ς_2	-0.2	-0.4	-0.8
ς_3	-0.43	-0.7	-0.6
ς_4	-0.5	-0.6	-0.31
ς_5	-0.6	-0.7	-0.8
ς_6	-0.1	-0.6	-0.8

Table 15. Tabular presentation for $\neg\eta^{-}$.

ξ	$\neg\rho_1$	$\neg\rho_3$	$\neg\rho_4$
ς_1	-0.4	-0.5	-0.23
ς_2	-0.4	-0.6	-0.1
ς_3	-0.15	-0.2	-0.3
ς_4	-0.3	-0.4	-0.49
ς_5	-0.4	-0.25	-0.1
ς_6	-0.31	-0.3	-0.1

Step 2: Normalization is needed in this step as the criteria are not in the same scale (see Tables 16–19).

Table 16. Normalized tabular presentation for ϱ^{+}.

ϑ	ρ_1	ρ_3	ρ_4
ς_1	0.35	0.6	0.5
ς_2	0.45	0.35	0.7
ς_3	0.45	0.8	0.2
ς_4	0.7	0.2	0.26
ς_5	0.74	0.61	0.3
ς_6	0.46	0.4	0.67

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Table 17. Normalized tabular presentation for $\neg\eta^{+}$.

ξ	$\neg\rho_1$	$\neg\rho_3$	$\neg\rho_4$
ς_1	0.9	0.6	0.3
ς_2	0.63	0.22	0.65
ς_3	0.6	0.5	0.2
ς_4	0.75	0.1	0.1
ς_5	0.8	0.5	0.2
ς_6	0.9	0.1	0.52

Table 18. Normalized tabular presentation for ϱ^{-}.

ϑ	ρ_1	ρ_3	ρ_4
ς_1	−0.3	−0.75	−0.3
ς_2	−0.2	−0.6	−0.2
ς_3	−0.43	−0.3	−0.4
ς_4	−0.5	−0.4	−0.69
ς_5	−0.6	−0.3	−0.2
ς_6	−0.1	−0.4	−0.2

Table 19. Normalized tabular presentation for $\neg\eta^{-}$.

ξ	$\neg\rho_1$	$\neg\rho_3$	$\neg\rho_4$
ς_1	−0.6	−0.5	−0.23
ς_2	−0.6	−0.6	−0.1
ς_3	−0.85	−0.2	−0.3
ς_4	−0.7	−0.4	−0.49
ς_5	−0.6	−0.25	−0.1
ς_6	−0.69	−0.3	−0.1

Step 3: In this step, comparison tabular for ϱ^{+}, ϱ^{-} with their negations $\neg\eta^{+}$ and $\neg\eta^{-}$, respectively, is used (see Tables 20–23).

Table 20. Comparison of normalized tabular presentation for ϱ^{+}.

	ς_1	ς_2	ς_3	ς_4	ς_5	ς_6
ς_1	3	1	1	2	1	1
ς_2	2	3	2	2	1	1
ς_3	2	2	3	1	1	1
ς_4	1	1	2	3	0	1
ς_5	2	2	2	3	3	2
ς_6	2	1	2	2	1	3

Table 21. Comparison of normalized tabular presentation for $\neg\eta^{+}$.

	ς_1	ς_2	ς_3	ς_4	ς_5	ς_6
ς_1	3	1	0	0	0	2
ς_2	2	3	1	1	2	1
ς_3	3	2	3	1	3	2
ς_4	3	2	2	3	3	3
ς_5	3	1	2	0	3	2
ς_6	2	2	1	1	1	3

Table 22. Comparison of normalized tabular presentation for ϱ^{-}.

	ς_1	ς_2	ς_3	ς_4	ς_5	ς_6
ς_1	3	3	1	1	2	3
ς_2	0	3	1	1	2	3
ς_3	2	2	3	0	2	2
ς_4	2	2	3	3	2	3
ς_5	1	2	2	1	3	2
ς_6	0	1	1	1	2	3

Table 23. Comparison of normalized tabular presentation for $\neg\eta^{-}$.

	ς_1	ς_2	ς_3	ς_4	ς_5	ς_6
ς_1	3	2	2	2	1	1
ς_2	2	3	2	2	2	2
ς_3	1	1	3	2	1	1
ς_4	1	1	1	3	0	0
ς_5	3	3	2	3	3	3
ς_6	2	2	2	3	1	3

Step 4: Positive and negative information scores with their negations are computed in this step (see Tables 24–27).

Table 24. Normalized tabular presentation score for ϱ^{+}.

	Row Sum: r_i	**Column Sum:** c_i	$P_i = r_i - c_i$
ς_1	9	12	−3
ς_2	11	10	1
ς_3	10	12	−2
ς_4	8	13	−5
ς_5	14	7	7
ς_6	11	9	2

Table 25. Normalized tabular presentation score for $\neg\eta^{+}$.

	Row Sum: $\neg r_i$	**Column Sum:** $\neg\hbar_i$	$\neg P_i = \neg r_i - \neg\hbar_i$
ς_1	6	16	−10
ς_2	10	11	−1
ς_3	14	9	5
ς_4	16	6	10
ς_5	11	12	−1
ς_6	10	13	−3

Table 26. Normalized tabular presentation score for ϱ^{-}.

	Row Sum: r'_i	**Column Sum:** c'_i	$N_i = r'_i - c'_i$
ς_1	13	8	5
ς_2	10	13	−3
ς_3	11	11	0
ς_4	15	7	8
ς_5	11	13	−2
ς_6	8	16	−8

Table 27. Normalized tabular presentation score for $\neg\eta^-$.

	Row Sum: $\neg{\prime}r_i$	**Column Sum:** $\neg\hbar'_i$	$\neg N_i = \neg r'_i - \neg c'_i$
ς_1	11	12	−2
ς_2	13	12	1
ς_3	9	12	−3
ς_4	6	15	−9
ς_5	17	8	9
ς_6	13	10	3

Step 5: Adjusted positive and negative information scores with their negations are computed in this step (see Tables 28 and 29).

Table 28. Adjusted Positive score(APS).

	P_i	$\neg P_i$	$APS_i = P_i - (\neg P_i)$
ς_1	−3	−10	7
ς_2	1	−1	2
ς_3	−2	5	−7
ς_4	−5	10	−15
ς_5	7	−1	8
ς_6	2	−3	5

Table 29. Adjusted negative score(ANS).

	N_i	$\neg N_i$	$ANS_i = N_i - (\neg N_i)$
ς_1	5	−2	7
ς_2	−3	1	−4
ς_3	0	−3	3
ς_4	8	−9	17
ς_5	−2	9	−11
ς_6	−8	3	−11

Step 6: Table for adjusted final score (AFS): (See Figure 3)

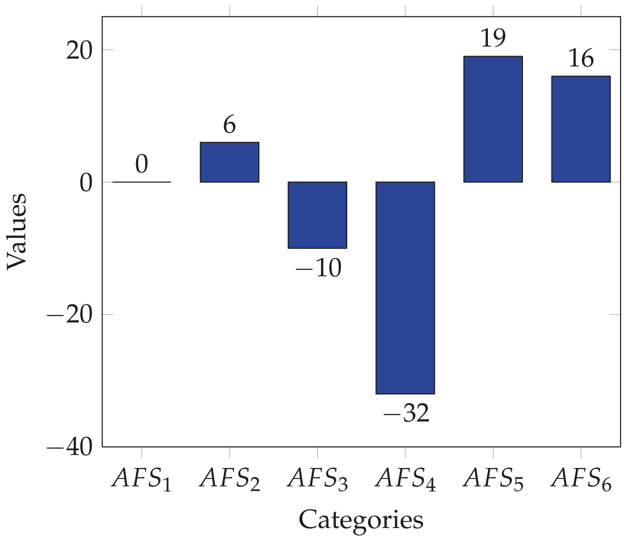

Figure 3. Adjusted final score.

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Step 7: From Table 30, fifth apartment AFS_5 is chosen to the best one among all.

Table 30. Adjusted final score (AFS).

	APS_i	ANS_i	$AFS_i = APS_i - ANS_i$
ς_1	7	7	0
ς_2	2	−4	6
ς_3	−7	3	−10
ς_4	−15	17	−32
ς_5	8	−11	19
ς_6	5	−11	16

7. Comparative Analysis

The DFBFSS seems to be an enhancement and an interconnection between bipolar fuzzy sets BFS and bipolar soft sets BSS, which serve as foundations for the proposed method. In this section, the suggested set is compared with certain existing ones in order to further confirm the correctness of the suggested approaches such as bipolar fuzzy soft set (BFSS) [34], fuzzy bipolar soft set (FBSS) [40] and fuzzy soft set (FSS) [27]. The details are demonstrated in Table 31 and Figure 4.

Table 31. Comparative analysis of DFBFSS with some existing sets.

Models	Ranking Results	Optimal Alternative
DFBFSS	$\varsigma_5 \succ \varsigma_6 \succ \varsigma_2 \succ \varsigma_1 \succ \varsigma_3 \succ \varsigma_4$	ς_5
FSS	$\varsigma_5 \succ \varsigma_6 \succ \varsigma_2 \succ \varsigma_3 \succ \varsigma_1 \succ \varsigma_4$	ς_5
FBSS	$\varsigma_5 \succ \varsigma_1 \succ \varsigma_6 \succ \varsigma_2 \succ \varsigma_3 \succ \varsigma_4$	ς_5
BFSS	$\varsigma_6 \succ \varsigma_5 \succ \varsigma_2 \succ \varsigma_3 \succ \varsigma_1 \succ \varsigma_4$	ς_6

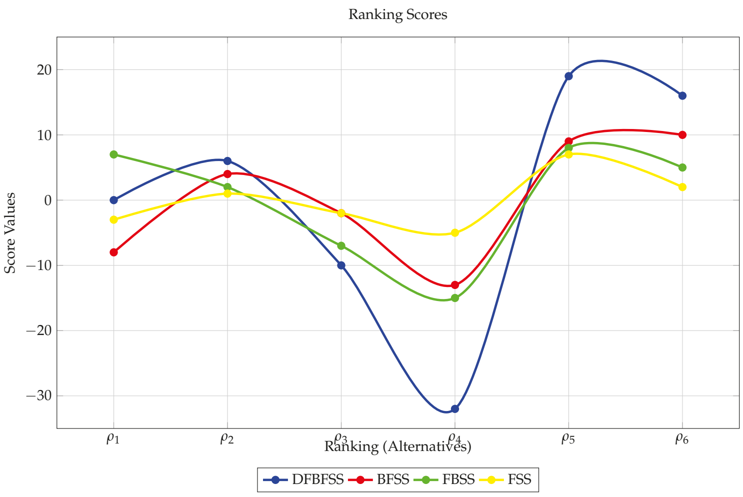

Figure 4. Comparing DFBFSS with FSS, FBSS and BFSS.

Unlike SS, FSS, and BFSS models, the proposed DFBFSS framework explicitly distinguishes between membership, counter-membership, and their negations, enabling symmetric handling of conflicting information.

The following comparative analysis in Table 32 elucidates the theoretical and practical advancements of the DFBFSS framework by benchmarking it against its direct predecessors, clearly demonstrating how its unique capacity to model four-dimensional judgment leads to more nuanced and robust decision outcomes.

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Table 32. Comparative Analysis of the Proposed DFBFSS Model with Related.

Aspect/Model	Soft Set (SS)	Fuzzy Soft Set (FSS)	Bipolar FSS (BFSS)	Proposed DFBFSS
Handles Vague Data	No (Crisp)	Yes	Yes	Yes
Models Bipolarity	No	No	Yes	Yes
Captures Negation	No	No	No	Yes
Complexity of Representation	Low	Medium	Medium	High (Four-dimensional)
Key Justification	Oversimplifies due to crisp logic	Misses conflicting criteria	Misses negation semantics	Captures complete expert judgment with dual-frame uncertainty

Table 32 provides a direct comparison between the proposed DFBFSS-based decision-making method and existing models such as SS, FSS, and BFSS. Unlike these approaches, which rely on single- or two-dimensional evaluations, the DFBFSS framework incorporates four complementary evaluative components: positive membership, negative membership, and their respective negations. This enriched structure allows the proposed method to preserve conflicting and incomplete information that is either ignored or collapsed in traditional models. As a result, the final ranking produced by the DFBFSS method reflects a more balanced and informative aggregation of expert opinions. This demonstrates that the advantage of the proposed approach lies not in numerical dominance but in its superior expressive power and its ability to model dual and contradictory assessments within a unified decision-making framework.

To clarify the distinctive features of the proposed DFBFSS framework and its advantages over existing models, a comparative analysis is summarized in Table 33.

Table 33. Comparison of DFBFSSs with Existing Fuzzy Soft Models.

Model	Positive Membership	Negative Membership	Negation	Bipolarity	Multi-Criteria Decision Support	Notes
SS [23]	✓	×	×	×	Limited	Binary or crisp values only
FSS [27]	✓	×	×	×	Moderate	Captures positive fuzziness only
BFSS [34]	✓	✓	×	✓	Moderate	No negation information
DFBFSS (Proposed)	✓	✓	✓	✓	High	Four-dimensional, dual-frame, handles negation

Table 33 clearly demonstrates that the proposed DFBFSS framework uniquely integrates bipolarity, negation semantics, and a dual-frame structure, which collectively enhance its suitability for complex multi-criteria decision-making problems.

8. Conclusions

The bipolar fuzzy set and bipolar soft set have evolved into robust frameworks, paving the way for the development of the theory of double-framed bipolar fuzzy soft sets (DFBFSSs). This innovative model provides a dual-layered structure for representing elements through ordered pairs of positive and negative membership grades, offering a nuanced perspective on uncertainty. Fundamental operations like subset relationships, equality, complement, intersection, and union have been rigorously defined and illustrated with numerical examples, demonstrating the practicality and applicability of DFBFSSs in complex decision-making and information modeling scenarios. In order to confirm the applicability of the suggested model, a numerical example has been provided along with an application of double-framed bipolar fuzzy soft sets to a decision-making problem and a general method to solve it. A comparison between the proposed strategy and some other existing ones has been given in order to verify the feasibility of the proposed decision-making process. This work establishes a formally symmetric extension of bipolar fuzzy soft set theory, introducing a nontrivial automorphism-invariant structure that is not captured by existing fuzzy or bipolar models.

Importantly, the symmetric structure of DFBFSSs preserves the inherent balance between positive and negative information, further enhancing their applicability in complex decision-making scenarios.

Future work will focus on both theoretical and practical extensions of the proposed framework. From a theoretical perspective, the DFBFSS model may be extended to more advanced settings such as intuitionistic, Pythagorean, and q-rung orthopair structures, as well as further investigations into its topological and functional properties. From a practical perspective, future studies will apply the DFBFSS framework to real-world datasets, including medical diagnosis and supplier selection problems, and will compare its performance with BFSS- and FSS-based decision-making models using expert-derived data and established evaluation criteria.

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m-Polar Picture Fuzzy Bi-Ideals and Their Applications in Semigroups

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Abstract: The concept of symmetry is fundamental to the study of algebra; it serves as the basis for a branch of group theory that is essential to abstract algebra. A semigroup is a structure that builds upon the concept of a group, similarly extending the idea of symmetry found within groups. In this study, we specifically focus on semigroups. The main objective of this research is to apply the notion of m-polar picture fuzzy sets (m-PPFSs), with m being a natural number, in investigations into semigroups, as this concept generalizes m-polar fuzzy sets (m-PFSs) and picture fuzzy sets (PFSs). This research introduces the concepts of m-polar picture fuzzy left ideals (m-PPFLs), m-polar picture fuzzy right ideals (m-PPFRs), m-polar picture fuzzy ideals (m-PPFIs), m-polar picture fuzzy bi-ideals (m-PPFBs), and m-polar picture fuzzy generalized bi-ideals (m-PPFGBs) in semigroups. This study examines the relationships between these concepts, showing that every m-PPFL (m-PPFR) in the semigroups is also an m-PPFB, and that every m-PPFB in the semigroups is an m-PPFGB. However, the opposite is not true. Additionally, we provide the characteristics of the m-PPFLs, m-PPFRs, m-PPFIs, m-PPFBs, and m-PPFGBs in semigroups. We further discuss the connections between the m-PPFLs (m-PPFIs) and the m-PPFBs within the framework of regular semigroups, and most importantly, we show that, if the semigroup is regular, then the m-PPFBs and m-PPFGBs are equal. Finally, we utilize the properties of the m-PPFLs, m-PPFRs, m-PPFIs, m-PPFBs, and m-PPFGBs within semigroups to explore the classifications of regular semigroups.

Keywords: m-polar picture fuzzy set; m-polar picture fuzzy ideal; m-polar picture fuzzy bi-ideal; regular semigroup

1. Introduction

Symmetry is a fundamental concept in algebra; it forms the basis of a branch of group theory that is essential to abstract algebra. Additionally, the role of symmetry in decision-making is significant both theoretically and in practice as it aids with systematically and equitably understanding the "structure" of choices, the "bias" of perception, and the "fairness" of judgments made regarding each choice. Zabzina et al. [1] presented a decision-making model in which symmetry-breaking is followed by a symmetry-restoring bifurcation, whereby enormous systems return to an even distribution of exploitation among options. Zhou et al. [2] explored the symmetry in distances to both positive and negative ideal solutions, similar to the TOPSIS methodology. Kumar et al. [3] analyzed the elements that substantially influence breaches in healthcare information security using a hybrid fuzzy-based symmetrical technique called AHP-TOPSIS. The concept of PFSs is included within decision-making, which depends on the principle of symmetry to assist with making decisions. The concept of PFSs was introduced by Cuong and Kreinovich [4] in 2013 as a generalization of the notions of fuzzy sets (FSs) [5] and intuitionistic fuzzy sets (IFSs) [6]. The PFS incorporates a third dimension, the neutral membership degree, together with the membership degree used in traditional fuzzy sets and the non-membership degree introduced by intuitionistic fuzzy sets. There are “yes” and “no” options as well as “abstain” or “undecided” options. This is a useful tool that can be used in many areas that involve difficult problems. The PFS notion has been studied in a variety of fields. For example, Thao et al. [7] introduced a divergence measure of picture fuzzy sets, developed a multi-criteria decision-making algorithm, and applied it to medical diagnosis and classification problems. Furthermore, Verma and Rohtagi [8] introduced novel similarity measures between two picture fuzzy sets, defining distance measures and weighted versions. The authors applied these measures in pattern recognition and medical diagnosis, demonstrating improved performance. Moreover, its applicability extends to algebra; Yiarayong [9] applied the concept of PFSs to semigroup theory. He identified various categories of regular semigroups, including intra-regular and semisimple types, by examining their fuzzy left and right ideals, known as picture fuzzy ideals. Subsequently, Nakkhasen employed the concept of PFSs to further characterize semigroups by exploring various types of picture fuzzy ideals [10,11]. Afterward, Kankaew et al. [12] introduced eight new concepts related to picture fuzzy sets and also discussed the connections between these concepts in UP-algebras. The references offer additional research on the notion of PFSs in algebraic structures [13–15].

In 2014, Chen et al. [16] introduced the notion of m-PFSs as a generalization of bipolar fuzzy sets (BFSs) [17], while the BFSs generalize the FSs. The primary advantage of m-PFSs is their capacity to concurrently manage multi-dimensional data, a capability that conventional fuzzy sets lack. For instance, a 4-PFS can immediately represent a decision regarding “technical, economic, environmental, and social aspects”. In 2018, Akram and Shahzadi [18] investigated the application of m-PFSs in hypergraphs, introducing regular and totally regular m-polar fuzzy hypergraphs. They examined their characteristics, applications in decision-making, and the creations of effective algorithms for addressing such issues. In 2019, Al-Masarwah and Ahmad [19] introduced the notion of m-polar (α, β)-fuzzy ideals in BCK/BCI-algebras, which generalizes fuzzy ideals, bipolar fuzzy ideals, and bipolar (α, β)-fuzzy ideals within these algebras. They characterized m-polar $(\in, \in \vee q)$-fuzzy ideals through level cut subsets and defined m-polar commutative ideals. In semigroups, Bashir et al. [20] came up with the idea of an m-PFS. Then, they considered generalized key results from BFSs to m-PFSs, focusing on m-polar fuzzy subsemigroups (ideals, generalized bi-ideals, bi-ideals, quasi-ideals, and interior ideals) in semigroups. They showed that every m-polar fuzzy bi-ideal of semigroups is the m-polar fuzzy generalized bi-ideal of semigroups, but this might not hold true the other way around. Thereafter, they explored the features of m-polar fuzzy ideals and used those features to describe regular and intra-regular semigroups. Additional investigations into m-PFSs have taken place, for instance [21–23]. Currently, there are still studies on the concept of different types of FSs to help address diverse data in different situations; for instance, El-Bably et al. [24] expanded traditional rough set theory through generalized rough set theory to improve decision-making under uncertainty. They introduced initial-minimal and initial-maximal neighborhoods, yielding eight new generalized rough approximations, which demonstrated accuracy rates up to 100% in experiments on integrated nano-topological structures for continuous uncertainty in COVID-19 diagnosis. Moreover, Osman et al. [25] demonstrated the applicability of multi-pretopological models in medical information systems.

They showed how multiset-based topology can effectively represent and analyze repetitive or uncertain medical data. Furthermore, Alshammari et al. [26] explored r-fuzzy soft δ-open sets within the context of fuzzy soft topological spaces. They also examined related concepts, including fuzzy soft δ-closure, δ-interior, and various types of fuzzy soft continuity, such as semi-continuous, pre-continuous, virtually continuous, and weakly continuous functions.

Building on the previously mentioned concepts of PFSs and m-PFSs, Doggra and Pal [27] introduced the concept of m-PPFSs within BCK-algebras as an extension of both PFSs and m-PFSs. They proved that an m-polar image fuzzy implicative ideal of a BCK-algebra is an m-polar picture fuzzy ideal. However, this is only true in implicative BCK-algebras. Research into m-PFSs in algebraic structures is a relatively new and growing concept, particularly with semigroups, which are popular structures in algebraic studies. Therefore, we are presented with the opportunity to apply the concept of m-PFSs to the study of semigroups. The objective of this research is to use the concept of m-PPFSs in an investigation into semigroups. The concepts of m-PPFLs, m-PPFRs, m-PPFIs, m-PPFBs, and m-PPFGBs are introduced, and the connections among these ideas in semigroups are investigated. Subsequently, we provide characterizations of each of the concepts that were discussed in semigroups. Finally, we focus on the properties of the m-PPFLs, m-PPFRs, m-PPFIs, m-PPFBs, and m-PPFGBs within semigroups to help with the characterization of regular semigroups.

2. Preliminaries

In this section, we will examine the fundamental concepts and characteristics that will be used in subsequent sections. A *semigroup* is a structure $(S, \cdot)$ consisting of a nonempty set S and a binary associative operation $\cdot$ on S. For each of the nonempty subsets A and B of semigroup S, we define $AB = \{ab \mid a \in A, b \in B\}$. Let S denote a semigroup. A nonempty subset T of S is defined as a *subsemigroup* of S if $TT \subseteq T$. A nonempty subset A of S is known as a *left ideal* (*right ideal*) of S if $SA \subseteq A$ ($AS \subseteq A$). A nonempty subset A of S is called an *ideal* of S if it is accepted as both a left ideal and a right ideal of S. A subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$. A nonempty subset G of S is known as a *generalized bi-ideal* of S if $GSG \subseteq G$.

A *fuzzy set* (FS) [5] μ of a nonempty set X is a function that maps elements from X to the closed interval $[0,1]$, denoted as $\mu : X \to [0,1]$. A *picture fuzzy set* (PFS) [4] $\mathcal{A}$ on a universe X is defined as follows, $\mathcal{A} = \Big\{ \Big(x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)\Big) \mid x \in X \Big\}$, where $\mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x) \in [0,1]$ denote the degree of positive membership, the degree of neutral membership, and the degree of negative membership, respectively, for each $x \in X$ to set $\mathcal{A}$, such that $\mu_{\mathcal{A}}$, $\eta_{\mathcal{A}}$, and $\nu_{\mathcal{A}}$ satisfy the following condition: the value of $0 \leqslant \mu_{\mathcal{A}}(x) + \eta_{\mathcal{A}}(x) + \nu_{\mathcal{A}}(x) \leqslant 1$ for all $x \in X$. An object of the form $\mathcal{A} = \Big\{ \Big(x, \mu^+_{\mathcal{A}}(x), \mu^-_{\mathcal{A}}(x)\Big) \mid x \in X \Big\}$ is called a *bipolar fuzzy set* (BFS) [17] of a nonempty set X. Here, $\mu^+_{\mathcal{A}} : X \to [0,1]$ and $\mu^-_{\mathcal{A}} : X \to [-1,0]$.

An *m-polar fuzzy set* (m-PFS) [16] $\mathcal{A}$ over the universe set X is defined as $\mathcal{A} = \Big\{ \Big(x, \mu_{\mathcal{A}}(x)\Big) \mid x \in X \Big\}$, where $\mu_{\mathcal{A}} : X \to [0,1]^m$ (with m representing a natural number) has been classified as an m-polar fuzzy set of X,

$$\mu_{\mathcal{A}}(x) = \Big(\Big(\pi_1 \circ \mu_{\mathcal{A}}\Big)(x), \Big(\pi_2 \circ \mu_{\mathcal{A}}\Big)(x), \ldots, \Big(\pi_m \circ \mu_{\mathcal{A}}\Big)(x) \Big).$$

In this regard, $\pi_i : [0,1]^m \to [0,1]$ is the i-th projection mapping, such that $(\pi_i \circ \mu_{\mathcal{A}})(x)$ represents the i-th component of $\mu_{\mathcal{A}}(x)$ for $i = 1, 2, \ldots, m$.

In addition, the poset with respect to partial order relation "$\preccurlyeq$" on $[0,1]^m$ is defined as for any $t = (t_1, t_2, \ldots, t_m), r = (r_1, r_2, \ldots, r_m) \in [0,1]^m$, $t \preccurlyeq r$ if and only if $t_i \leqslant r_i$ for all

$i = 1, 2, \ldots, m$. In addition, $t \prec r$ if and only if $t_i < r_i$ for all $i = 1, 2, \ldots, m$. So, in the case of $t \succcurlyeq r$ and $t \succ r$, this results in $r \preccurlyeq t$ and $r \prec t$, respectively. Furthermore, we say that $\mathbf{0} = (0, 0, \ldots, 0)$ (m-tuple) is *the smallest value* and $\mathbf{1} = (1, 1, \ldots, 1)$ (m-tuple) is *the largest value* in $[0, 1]^m$.

Let $\{n_i | i \in \Lambda\}$ be a family of real numbers. Subsequently, we denote

$$\bigwedge\{n_i \mid i \in \Lambda\} = \begin{cases} \min\{n_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{n_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases}$$

$$\bigvee\{n_i \mid i \in \Lambda\} = \begin{cases} \max\{n_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{n_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

If Λ is finite, we write $n_1 \wedge n_2 \wedge \cdots \wedge n_m$, instead of $\wedge\{n_1, n_2, \ldots, n_m\}$ and $n_1 \vee n_2 \vee \cdots \vee n_m$, instead of $\vee\{n_1, n_2, \ldots, n_m\}$. Let $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in [0, 1]^m$. We then denote the following:

(i) $u = v$ iff $u_i = v_i$ for all $i = 1, 2, \ldots, m$;
(ii) $u \curlywedge v$ iff $u_i \wedge v_i$ for all $i = 1, 2, \ldots, m$;
(iii) $u \curlyvee v$ iff $u_i \vee v_i$ for all $i = 1, 2, \ldots, m$.

In 2020, Dogra and Pal [27] introduced the notion of m-PPFS as a generalization of FSs, PFSs, BFSs, and m-PFSs.

Definition 1 ([27])**.** *Let X be a nonempty set and m be a natural number. An m-polar picture fuzzy set (m-PPFS) $\mathcal{A}$ over X is defined as an object of the following type:*

$$\mathcal{A} = \Big\{ \Big(x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)\Big) \mid x \in X \Big\},$$

where $\mu_{\mathcal{A}} : X \to [0, 1]^m, \eta : X \to [0, 1]^m$, and $\nu_{\mathcal{A}} : X \to [0, 1]^m$ with the condition $0 \leqslant \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) + \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) + \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \leqslant 1$ for all $x \in X$ and for all $i = 1, 2, \ldots, m$.

In this study, we will denote the m-PPFS $\mathcal{A} = \Big\{ \Big(x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)\Big) \mid x \in X \Big\}$ with the symbol $\mathcal{A} = \Big(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\Big)$. Let $\mathcal{A} = \Big(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\Big)$ and $\mathcal{B} = \Big(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}}\Big)$ be any two m-PPFSs on universe X. Then, we denote the following:

(i) $\mathcal{A} \sqsubseteq \mathcal{B}$ iff $\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \leqslant \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x)$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \geqslant \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x)$, and $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \geqslant \Big(\pi_i \circ \nu_{\mathcal{B}}\Big)(x)$ for all $x \in X$ and for all $i = 1, 2, \ldots, m$;
(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{B} \sqsubseteq \mathcal{A}$;
(iii) $\mathcal{A} \sqcap \mathcal{B} = \Big\{ \Big(x, \Big(\mu_{\mathcal{A}} \curlywedge \mu_{\mathcal{B}}\Big)(x), \Big(\eta_{\mathcal{A}} \curlyvee \eta_{\mathcal{B}}\Big)(x), \Big(\nu_{\mathcal{A}} \curlyvee \nu_{\mathcal{B}}\Big)(x)\Big) \mid x \in X \Big\}$, that is, $\mathcal{A} \sqcap \mathcal{B}$ iff $\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x)$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x)$, and $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \nu_{\mathcal{B}}\Big)(x)$ for all $x \in X$ and for all $i = 1, 2, \ldots, m$;
(iv) $\mathcal{A} \sqcup \mathcal{B} = \Big\{ (x, \Big(\mu_{\mathcal{A}} \curlyvee \mu_{\mathcal{B}}\Big)(x), \Big(\eta_{\mathcal{A}} \curlywedge \eta_{\mathcal{B}}\Big)(x), \Big(\nu_{\mathcal{A}} \curlywedge \nu_{\mathcal{B}}\Big)(x)) \mid x \in X \Big\}$, that is, $\mathcal{A} \sqcup \mathcal{B}$ iff $\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x)$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x)$, and $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \nu_{\mathcal{B}}\Big)(x)$ for all $x \in X$ and for all $i = 1, 2, \ldots, m$.

For any nonempty set X, we define an m-PPFS $\mathcal{S} = \Big(\mathbf{1}, \mathbf{0}, \mathbf{0}\Big)$ on X. It follows that $\mathcal{A} \sqsubseteq \mathcal{S}$ for all m-PPFS $\mathcal{A} = \Big(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\Big)$ on X.

Next, we define the concepts of m-PPFSubs, m-PPFLs, m-PPFRs, m-PPFIs, m-PPFBs, and m-PPFGBs in the semigroups as follows.

Definition 2. *Let S be a semigroup. An m-PPFS $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ over S is called an m-polar picture fuzzy subsemigroup (m-PPFSub) of S if for every $x, y \in S$,*

(i) $\mu_{\mathcal{A}}(xy) \succcurlyeq \mu_{\mathcal{A}}(x) \curlywedge \mu_{\mathcal{A}}(y)$;
(ii) $\eta_{\mathcal{A}}(xy) \preccurlyeq \eta_{\mathcal{A}}(x) \curlyvee \eta_{\mathcal{A}}(y)$;
(iii) $\nu_{\mathcal{A}}(xy) \preccurlyeq \nu_{\mathcal{A}}(x) \curlyvee \nu_{\mathcal{A}}(y)$;

that is,

$$\left(\pi_i \circ \mu_{\mathcal{A}}\right)(xy) \geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(y),$$
$$\left(\pi_i \circ \eta_{\mathcal{A}}\right)(xy) \leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(y),$$
$$\left(\pi_i \circ \nu_{\mathcal{A}}\right)(xy) \leqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{A}}\right)(y)$$

for all $i = 1, 2, \ldots, m$, respectively.

Definition 3. *Let $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ be an m-PPFS over semigroup S. Then,*

(i) *$\mathcal{A}$ is called an m-polar picture fuzzy left ideal (m-PPFL) of S if*

$$\mu_{\mathcal{A}}(xy) \succcurlyeq \mu_{\mathcal{A}}(y), \eta_{\mathcal{A}}(xy) \preccurlyeq \eta_{\mathcal{A}}(y), \text{and } \nu_{\mathcal{A}}(xy) \preccurlyeq \nu_{\mathcal{A}}(y),$$

that is, $\left(\pi_i \circ \mu_{\mathcal{A}}\right)(xy) \geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(y)$, $\left(\pi_i \circ \eta_{\mathcal{A}}\right)(xy) \leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(y)$, and $\left(\pi_i \circ \nu_{\mathcal{A}}\right)(xy) \leqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(y)$ for all $x, y \in S$ and for all $i = 1, 2, \ldots, m$;

(ii) *$\mathcal{A}$ is called an m-polar picture fuzzy right ideal (m-PPFR) of S if*

$$\mu_{\mathcal{A}}(xy) \succcurlyeq \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(xy) \preccurlyeq \eta_{\mathcal{A}}(x), \text{and } \nu_{\mathcal{A}}(xy) \preccurlyeq \nu_{\mathcal{A}}(x),$$

that is, $\left(\pi_i \circ \mu_{\mathcal{A}}\right)(xy) \geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x)$, $\left(\pi_i \circ \eta_{\mathcal{A}}\right)(xy) \leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x)$, and $\left(\pi_i \circ \nu_{\mathcal{A}}\right)(xy) \leqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x)$ for all $x, y \in S$ and for all $i = 1, 2, \ldots, m$;

(iii) *An m-PPFS $\mathcal{A}$ is called an m-polar picture fuzzy ideal (m-PPFI) of S if it is both an m-PPFL and an m-PPFR of S.*

Definition 4. *Let S be a semigroup. An m-PPFSub $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ of S is called an m-polar picture fuzzy bi-ideal (m-PPFB) of S if for any $x, y, z \in S$,*

(i) $\mu_{\mathcal{A}}(xyz) \succcurlyeq \mu_{\mathcal{A}}(x) \curlywedge \mu_{\mathcal{A}}(z)$;
(ii) $\eta_{\mathcal{A}}(xyz) \preccurlyeq \eta_{\mathcal{A}}(x) \curlyvee \eta_{\mathcal{A}}(z)$;
(iii) $\nu_{\mathcal{A}}(xyz) \preccurlyeq \nu_{\mathcal{A}}(x) \curlyvee \nu_{\mathcal{A}}(z)$;

that is,

$$\left(\pi_i \circ \mu_{\mathcal{A}}\right)(xyz) \geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(z),$$
$$\left(\pi_i \circ \eta_{\mathcal{A}}\right)(xyz) \leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(z),$$
$$\left(\pi_i \circ \nu_{\mathcal{A}}\right)(xyz) \leqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{A}}\right)(z)$$

for all $i = 1, 2, \ldots, m$, respectively.

Definition 5. *An m-PPFS* $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ *over semigroup S is said to be an m-polar picture fuzzy generalized bi-ideal (m-PPFGB) of S if for each* $x, y, z \in S$,

(i) $\mu_{\mathcal{A}}(xyz) \succcurlyeq \mu_{\mathcal{A}}(x) \curlywedge \mu_{\mathcal{A}}(z)$;
(ii) $\eta_{\mathcal{A}}(xyz) \preccurlyeq \eta_{\mathcal{A}}(x) \curlyvee \eta_{\mathcal{A}}(z)$;
(iii) $\nu_{\mathcal{A}}(xyz) \preccurlyeq \nu_{\mathcal{A}}(x) \curlyvee \nu_{\mathcal{A}}(z)$;

that is,

$$\left(\pi_i \circ \mu_{\mathcal{A}}\right)(xyz) \geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(z),$$
$$\left(\pi_i \circ \eta_{\mathcal{A}}\right)(xyz) \leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(z),$$
$$\left(\pi_i \circ \nu_{\mathcal{A}}\right)(xyz) \leqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{A}}\right)(z)$$

for all $i = 1, 2, \ldots, m$, *respectively.*

Proposition 1. *Every m-PPFL (m-PPFR) of semigroup S is also an m-PPFB of S.*

Proof. Let $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ be any m-PPFL of semigroup S. Let $x, y, z \in S$ and $i = 1, 2, \ldots, m$. Then, we have the following:

$$\left(\pi_i \circ \mu_{\mathcal{A}}\right)(xy) \geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(y) \geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(y),$$
$$\left(\pi_i \circ \eta_{\mathcal{A}}\right)(xy) \leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(y) \leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(y).$$

Similarly, we can prove that $\left(\pi_i \circ \nu_{\mathcal{A}}\right)(xy) \leqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{A}}\right)(y)$. Hence, $\mu_{\mathcal{A}}(xy) \succcurlyeq \mu_{\mathcal{A}}(x) \curlywedge \mu_{\mathcal{A}}(y), \eta_{\mathcal{A}}(xy) \preccurlyeq \eta_{\mathcal{A}}(x) \curlyvee \eta_{\mathcal{A}}(y)$, and $\nu_{\mathcal{A}}(xy) \preccurlyeq \nu_{\mathcal{A}}(x) \curlyvee \nu_{\mathcal{A}}(y)$. It follows that $\mathcal{A}$ is an m-PPFSub of S. Moreover,

$$\left(\pi_i \circ \mu_{\mathcal{A}}\right)(xyz) \geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(z) \geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(z),$$
$$\left(\pi_i \circ \eta_{\mathcal{A}}\right)(xyz) \leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(z) \leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(z).$$

Also, $\left(\pi_i \circ \nu_{\mathcal{A}}\right)(xyz) \leqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{A}}\right)(z)$. Thus, $\mu_{\mathcal{A}}(xyz) \succcurlyeq \mu_{\mathcal{A}}(x) \curlywedge \mu_{\mathcal{A}}(z), \eta_{\mathcal{A}}(xyz) \preccurlyeq \eta_{\mathcal{A}}(x) \curlyvee \eta_{\mathcal{A}}(z)$, and $\nu_{\mathcal{A}}(xyz) \preccurlyeq \nu_{\mathcal{A}}(x) \curlyvee \nu_{\mathcal{A}}(z)$. Therefore, $\mathcal{A}$ is an m-PPFB of S. For the m-PPFR $\mathcal{A}$ of S, we can prove similarly. □

The opposite of Proposition 1 is not necessarily true in general, as seen by the following example:

Example 1. *Consider semigroup* $S = \{a, b, c, d\}$, *as referenced in [28], presented in Table 1.*

Table 1. The operation "·" on S.

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Define a 4*-PPFS* $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ *on S as Table 2.*

Table 2. The representation of the value of a 4-PPFS $\mathcal{A}$ on S.

$\mathcal{A}$	$\mu_{\mathcal{A}}$	$\eta_{\mathcal{A}}$	$\nu_{\mathcal{A}}$
a	$\left(0.6, 0.5, 0.4, 0.7\right)$	$\left(0.1, 0, 0.1, 0\right)$	$\left(0, 0.1, 0.1, 0\right)$
b	$\left(0.4, 0.2, 0.2, 0.5\right)$	$\left(0.3, 0.2, 0.4, 0.2\right)$	$\left(0.3, 0.5, 0.4, 0.3\right)$
c	$\left(0.5, 0.4, 0.3, 0.6\right)$	$\left(0.2, 0.1, 0.3, 0.1\right)$	$\left(0.1, 0.3, 0.3, 0.2\right)$
d	$\left(0.1, 0.1, 0, 0.2\right)$	$\left(0.4, 0.3, 0.5, 0.3\right)$	$\left(0.5, 0.6, 0.5, 0.5\right)$

Following on, straightforward computations indicate that the 4*-PPFS* $\mathcal{A}$ *on* S *satisfies a* 4*-PPFB of* S*. Then, consider*

$$(\pi_2 \circ \mu_{\mathcal{A}})(dc) = (\pi_2 \circ \mu_{\mathcal{A}})(b) = 0.2 \ngeq 0.4 = (\pi_2 \circ \mu_{\mathcal{A}})(c),$$
$$(\pi_4 \circ \mu_{\mathcal{A}})(cd) = (\pi_4 \circ \mu_{\mathcal{A}})(b) = 0.5 \ngeq 0.6 = (\pi_4 \circ \mu_{\mathcal{A}})(c).$$

Here, we can see that $\mu_{\mathcal{A}}(dc) \not\succeq \mu_{\mathcal{A}}(c)$ *and* $\mu_{\mathcal{A}}(cd) \not\succeq \mu_{\mathcal{A}}(c)$*. Therefore, the* 4*-PPFB* $\mathcal{A}$ *is neither a* 4*-PPFL nor a* 4*-PPFR of* S*.*

Regarding Definition 4, it is not difficult to see that the m-PPFB $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ is also an m-PPFGB in semigroup S; however, the opposite is generally not valid, as seen in the following example:

Example 2. *Let* $S = \{a, b, c, d\}$ *denote a semigroup, as defined in [29], presented in Table 3.*

Table 3. The operation "$\cdot$" on S.

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Next, we define a 3*-PPFS* $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ *on* S *as Table 4.*

Table 4. The representation of the value of a 3-PPFS $\mathcal{A}$ on S.

$\mathcal{A}$	$\mu_{\mathcal{A}}$	$\eta_{\mathcal{A}}$	$\nu_{\mathcal{A}}$
a	$\left(0.5, 0.4, 0.6\right)$	$\left(0.1, 0.2, 0\right)$	$\left(0.2, 0, 0.1\right)$
b	$\left(0.1, 0, 0.2\right)$	$\left(0.3, 0.4, 0.3\right)$	$\left(0.6, 0.5, 0.5\right)$
c	$\left(0.5, 0.3, 0.4\right)$	$\left(0.2, 0.2, 0.1\right)$	$\left(0.3, 0.1, 0.2\right)$
d	$\left(0.4, 0.2, 0.3\right)$	$\left(0.2, 0.3, 0.3\right)$	$\left(0.4, 0.3, 0.4\right)$

Subsequent calculations verify that the 3*-PPFS* $\mathcal{A}$ *on* S *achieves the conditions of a* 3*-PPFGB of* S*. Now, we obtain*

$$\left(\pi_1 \circ \mu_{\mathcal{A}}\right)(dd) = \left(\pi_1 \circ \mu_{\mathcal{A}}\right)(b) = 0.1 \ngeq 0.4 = \left(\pi_1 \circ \mu_{\mathcal{A}}\right)(d) \wedge \left(\pi_1 \circ \mu_{\mathcal{A}}\right)(d).$$

Also, $\mu_{\mathcal{A}}(dd) \not\succeq \mu_{\mathcal{A}}(d) \curlywedge \mu_{\mathcal{A}}(d)$*. Hence,* $\mathcal{A}$ *is not a* 3*-PPFSub of* S*. This implies that the* 3*-PPFGB* $\mathcal{A}$ *is also not a* 3*-PPFB of* S*.*

Proposition 2. *Let $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ and $\mathcal{B} = \left(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}}\right)$ be any two m-PPFSs on semigroup S. Then, the following properties are true:*

(i) If $\mathcal{A}$ and $\mathcal{B}$ are m-PPFSubs of S, then $\mathcal{A} \sqcap \mathcal{B}$ is also an m-PPFSub of S;
(ii) If $\mathcal{A}$ and $\mathcal{B}$ are m-PPFLs of S, then $\mathcal{A} \sqcap \mathcal{B}$ is also an m-PPFL of S;
(iii) If $\mathcal{A}$ and $\mathcal{B}$ are m-PPFRs of S, then $\mathcal{A} \sqcap \mathcal{B}$ is also an m-PPFR of S;
(iv) If $\mathcal{A}$ and $\mathcal{B}$ are m-PPFIs of S, then $\mathcal{A} \sqcap \mathcal{B}$ is also an m-PPFI of S;
(v) If $\mathcal{A}$ and $\mathcal{B}$ are m-PPFGBs of S, then $\mathcal{A} \sqcap \mathcal{B}$ is also an m-PPFGB of S;
(vi) If $\mathcal{A}$ and $\mathcal{B}$ are m-PPFBs of S, then $\mathcal{A} \sqcap \mathcal{B}$ is also an m-PPFB of S.

Proof. (i) Assume that $\mathcal{A}$ and $\mathcal{B}$ are m-PPFSubs of S. For every $x, y \in S$ and for every $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
&\left(\pi_i \circ \mu_{\mathcal{A}}\right)(xy) \wedge \left(\pi_i \circ \mu_{\mathcal{B}}\right)(xy) \\
&\qquad \geqslant \left[\left(\pi_i \circ \mu_{\mathcal{A}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(y)\right] \wedge \left[\left(\pi_i \circ \mu_{\mathcal{B}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{B}}\right)(y)\right] \\
&\qquad = \left[\left(\pi_i \circ \mu_{\mathcal{A}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{B}}\right)(x)\right] \wedge \left[\left(\pi_i \circ \mu_{\mathcal{A}}\right)(y) \wedge \left(\pi_i \circ \mu_{\mathcal{B}}\right)(y)\right], \\
&\left(\pi_i \circ \eta_{\mathcal{A}}\right)(xy) \vee \left(\pi_i \circ \eta_{\mathcal{B}}\right)(xy) \\
&\qquad \leqslant \left[\left(\pi_i \circ \eta_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(y)\right] \vee \left[\left(\pi_i \circ \eta_{\mathcal{B}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{B}}\right)(y)\right] \\
&\qquad = \left[\left(\pi_i \circ \eta_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{B}}\right)(x)\right] \vee \left[\left(\pi_i \circ \eta_{\mathcal{A}}\right)(y) \vee \left(\pi_i \circ \eta_{\mathcal{B}}\right)(y)\right].
\end{aligned}$$

Similarly, it follows that $\left(\pi_i \circ \nu_{\mathcal{A}}\right)(xy) \vee \left(\pi_i \circ \nu_{\mathcal{B}}\right)(xy) \leqslant \left[\left(\pi_i \circ \nu_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{B}}\right)(x)\right] \vee \left[\left(\pi_i \circ \nu_{\mathcal{A}}\right)(y) \vee \left(\pi_i \circ \nu_{\mathcal{B}}\right)(y)\right]$. This means that

$$\begin{aligned}
\left(\mu_{\mathcal{A}} \curlywedge \mu_{\mathcal{B}}\right)(xy) &\succcurlyeq \left(\mu_{\mathcal{A}} \curlywedge \mu_{\mathcal{B}}\right)(x) \curlywedge \left(\mu_{\mathcal{A}} \curlywedge \mu_{\mathcal{B}}\right)(y), \\
\left(\eta_{\mathcal{A}} \curlyvee \eta_{\mathcal{B}}\right)(xy) &\preccurlyeq \left(\eta_{\mathcal{A}} \curlyvee \eta_{\mathcal{B}}\right)(x) \curlyvee \left(\eta_{\mathcal{A}} \curlyvee \eta_{\mathcal{B}}\right)(y), \\
\left(\nu_{\mathcal{A}} \curlyvee \nu_{\mathcal{B}}\right)(xy) &\preccurlyeq \left(\nu_{\mathcal{A}} \curlyvee \nu_{\mathcal{B}}\right)(x) \curlyvee \left(\nu_{\mathcal{A}} \curlyvee \nu_{\mathcal{B}}\right)(y).
\end{aligned}$$

Hence, $\mathcal{A} \sqcap \mathcal{B}$ is an m-PPFSub of S.

(ii) Let $\mathcal{A}$ and $\mathcal{B}$ be m-PPFLs of S. For each $x, y \in S$ and for each $i = 1, 2, \ldots, m$, we obtain the following:

$$\begin{aligned}
\left(\pi_i \circ \mu_{\mathcal{A}}\right)(xy) \wedge \left(\pi_i \circ \mu_{\mathcal{B}}\right)(xy) &\geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(y) \wedge \left(\pi_i \circ \mu_{\mathcal{B}}\right)(y), \\
\left(\pi_i \circ \eta_{\mathcal{A}}\right)(xy) \vee \left(\pi_i \circ \eta_{\mathcal{B}}\right)(xy) &\leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(y) \vee \left(\pi_i \circ \eta_{\mathcal{B}}\right)(y), \\
\left(\pi_i \circ \nu_{\mathcal{A}}\right)(xy) \vee \left(\pi_i \circ \nu_{\mathcal{B}}\right)(xy) &\leqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(y) \vee \left(\pi_i \circ \nu_{\mathcal{B}}\right)(y).
\end{aligned}$$

It follows that $\left(\mu_{\mathcal{A}} \curlywedge \mu_{\mathcal{B}}\right)(xy) \succcurlyeq \left(\mu_{\mathcal{A}} \curlywedge \mu_{\mathcal{B}}\right)(y)$, $\left(\eta_{\mathcal{A}} \curlyvee \eta_{\mathcal{B}}\right)(xy) \preccurlyeq \left(\eta_{\mathcal{A}} \curlyvee \eta_{\mathcal{B}}\right)(y)$, and $\left(\nu_{\mathcal{A}} \curlyvee \nu_{\mathcal{B}}\right)(xy) \preccurlyeq \left(\nu_{\mathcal{A}} \curlyvee \nu_{\mathcal{B}}\right)(y)$. We find that $\mathcal{A} \sqcap \mathcal{B}$ is an m-PPFL of S.

(iii) The proof is similar to (ii).

(iv) It is followed by (ii) and (iii).

(v) Assume that $\mathcal{A}$ and $\mathcal{B}$ are m-PPFGBs of S. For any $x, y, z \in S$ and for any $i = 1, 2, \ldots, m$, we obtain

$$\begin{aligned}
&\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(xyz) \wedge \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(xyz) \\
&\qquad \geqslant \Big[\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z)\Big] \wedge \Big[\Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(z)\Big] \\
&\qquad = \Big[\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x)\Big] \wedge \Big[\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z) \wedge \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(z)\Big], \\
&\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(xyz) \vee \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(xyz) \\
&\qquad \leqslant \Big[\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z)\Big] \vee \Big[\Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(z)\Big] \\
&\qquad = \Big[\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x)\Big] \vee \Big[\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z) \vee \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(z)\Big].
\end{aligned}$$

Proving it in the same way as the previous case, we have that $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(xyz) \vee \Big(\pi_i \circ \nu_{\mathcal{B}}\Big)(xyz) \leqslant \Big[\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \nu_{\mathcal{B}}\Big)(x)\Big] \vee \Big[\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(z) \vee \Big(\pi_i \circ \nu_{\mathcal{B}}\Big)(z)\Big]$. Thus,

$$\begin{aligned}
\Big(\mu_{\mathcal{A}} \curlywedge \mu_{\mathcal{B}}\Big)(xyz) &\succcurlyeq \Big(\mu_{\mathcal{A}} \curlywedge \mu_{\mathcal{B}}\Big)(x) \curlywedge \Big(\mu_{\mathcal{A}} \curlywedge \mu_{\mathcal{B}}\Big)(z), \\
\Big(\eta_{\mathcal{A}} \curlyvee \eta_{\mathcal{B}}\Big)(xyz) &\preccurlyeq \Big(\eta_{\mathcal{A}} \curlyvee \eta_{\mathcal{B}}\Big)(x) \curlyvee \Big(\eta_{\mathcal{A}} \curlyvee \eta_{\mathcal{B}}\Big)(z), \\
\Big(\nu_{\mathcal{A}} \curlyvee \nu_{\mathcal{B}}\Big)(xyz) &\preccurlyeq \Big(\nu_{\mathcal{A}} \curlyvee \nu_{\mathcal{B}}\Big)(x) \curlyvee \Big(\nu_{\mathcal{A}} \curlyvee \nu_{\mathcal{B}}\Big)(z).
\end{aligned}$$

Therefore, $\mathcal{A} \sqcap \mathcal{B}$ is an m-PPFGB of S.

(vi) The proof is obtained by (i) and (v). □

In general, the union of m-PPFSubs (m-PPFLs, m-PPFRs, m-PPFIs, m-PPFGBs, and m-PPFBs) in semigroups does not need to be the same, as in the following example.

Example 3. *Let semigroup $S = \{a, b, c, d\}$, as cited in [30], be given in Table 5.*

Table 5. The operation "·" on S.

·	***a***	***b***	***c***	***d***
a	a	a	a	a
b	a	b	c	b
c	a	a	a	a
d	a	a	b	d

Let us now explore two 3-PPFSs, $\mathcal{A} = \Big(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\Big)$ and $\mathcal{B} = \Big(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}}\Big)$, on S as seen in Table 6 and Table 7, respectively.

Table 6. The representation of the value of a 3-PPFS $\mathcal{A}$ on S.

$\mathcal{A}$	$\mu_{\mathcal{A}}$	$\eta_{\mathcal{A}}$	$\nu_{\mathcal{A}}$
a	$\Big(0.7, 0.5, 0.4\Big)$	$\Big(0.1, 0.2, 0.2\Big)$	$\Big(0, 0.1, 0\Big)$
b	$\Big(0.1, 0.2, 0.1\Big)$	$\Big(0.3, 0.4, 0.5\Big)$	$\Big(0.4, 0.3, 0.4\Big)$
c	$\Big(0.7, 0.4, 0.3\Big)$	$\Big(0.2, 0.3, 0.4\Big)$	$\Big(0.1, 0.2, 0.3\Big)$
d	$\Big(0.1, 0.2, 0.1\Big)$	$\Big(0.3, 0.4, 0.5\Big)$	$\Big(0.4, 0.3, 0.4\Big)$

Table 7. The representation of the value of a 3-PPFS $\mathcal{B}$ on S.

$\mathcal{B}$	$\mu_{\mathcal{B}}$	$\eta_{\mathcal{B}}$	$\nu_{\mathcal{B}}$
a	$\Big(0.5, 0.6, 0.3\Big)$	$\Big(0.2, 0.2, 0.4\Big)$	$\Big(0.1, 0, 0\Big)$
b	$\Big(0.1, 0.3, 0.1\Big)$	$\Big(0.4, 0.4, 0.7\Big)$	$\Big(0.5, 0.3, 0.2\Big)$
c	$\Big(0.1, 0.3, 0.1\Big)$	$\Big(0.4, 0.4, 0.7\Big)$	$\Big(0.5, 0.3, 0.2\Big)$
d	$\Big(0.4, 0.5, 0.2\Big)$	$\Big(0.3, 0.3, 0.5\Big)$	$\Big(0.3, 0.1, 0.1\Big)$

Through routine calculations, we determine that $\mathcal{A}$ and $\mathcal{B}$ can be considered as 3*-PPFSubs of S. We now investigate*

$$\begin{aligned}
\Big(\pi_2 \circ \mu_{\mathcal{A}}\Big)(dc) \vee \Big(\pi_2 \circ \mu_{\mathcal{B}}\Big)(dc) &= \Big(\pi_2 \circ \mu_{\mathcal{A}}\Big)(b) \vee \Big(\pi_2 \circ \mu_{\mathcal{B}}\Big)(b) \\
&= 0.2 \vee 0.3 = 0.3 \\
&\ngeq 0.4 = \Big(0.2 \vee 0.5\Big) \wedge \Big(0.4 \vee 0.3\Big) \\
&= \Big[\Big(\pi_2 \circ \mu_{\mathcal{A}}\Big)(d) \vee \Big(\pi_2 \circ \mu_{\mathcal{B}}\Big)(d)\Big] \wedge \Big[\Big(\pi_2 \circ \mu_{\mathcal{A}}\Big)(c) \vee \Big(\pi_2 \circ \mu_{\mathcal{B}}\Big)(c)\Big].
\end{aligned}$$

It can be seen that $(\mu_{\mathcal{A}} \curlyvee \mu_{\mathcal{B}})(dc) \not\succeq (\mu_{\mathcal{A}} \curlyvee \mu_{\mathcal{B}})(d) \curlywedge (\mu_{\mathcal{A}} \curlyvee \mu_{\mathcal{B}})(c)$. Consequently, $\mathcal{A} \sqcup \mathcal{B}$ cannot satisfy the requirements to be identified as a 3*-PPFSub of S.*

3. Characterizations of Many Types of m-PPFIs

In this section, we will explore the characterizations of different types of m-PPFIs of semigroups using some newly developed concepts and some existing ones, the properties of which will guide this study into the next section.

For any subset A of a nonempty set S, we denote by $\mathcal{C}_A = \Big(\mu_{\mathcal{C}_A}, \eta_{\mathcal{C}_A}, \nu_{\mathcal{C}_A}\Big)$ the *m-polar picture characteristic function* (m-PPCF) of S, where

$$\mathcal{C}_A(x) = \begin{cases} \Big(\mathbf{1}, \mathbf{0}, \mathbf{0}\Big) & \text{if } x \in A, \\ \Big(\mathbf{0}, \mathbf{0}, \mathbf{1}\Big) & \text{otherwise.} \end{cases}$$

This means that

$$\mu_{\mathcal{C}_A}(x) = \begin{cases} \mathbf{1} & \text{if } x \in A, \\ \mathbf{0} & \text{otherwise,} \end{cases} \eta_{\mathcal{C}_A}(x) = \mathbf{0} \text{ for all } x \in X, \nu_{\mathcal{C}_A}(x) = \begin{cases} \mathbf{0} & \text{if } x \in A, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Observe that, if $A = S$, then $\mathcal{C}_A = \mathcal{S}$.

Let S be a semigroup and $\mathcal{A} = \Big(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\Big)$ and $\mathcal{B} = \Big(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}}\Big)$ be m-PPFSs over S. The *product* of $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\mathcal{A} \diamond \mathcal{B} = \Big(\mu_{\mathcal{A}} \diamond \mu_{\mathcal{B}}, \eta_{\mathcal{A}} \diamond \eta_{\mathcal{B}}, \nu_{\mathcal{A}} \diamond \nu_{\mathcal{B}}\Big),$$

where

$$\begin{aligned}
\mu_{\mathcal{A}} \diamond \mu_{\mathcal{B}} &= \Big(\Big(\pi_1 \circ \mu_{\mathcal{A}}\Big) \diamond \Big(\pi_1 \circ \mu_{\mathcal{B}}\Big), \ldots, \Big(\pi_m \circ \mu_{\mathcal{A}}\Big) \diamond \Big(\pi_m \circ \mu_{\mathcal{B}}\Big)\Big), \\
\eta_{\mathcal{A}} \diamond \eta_{\mathcal{B}} &= \Big(\Big(\pi_1 \circ \eta_{\mathcal{A}}\Big) \diamond \Big(\pi_1 \circ \eta_{\mathcal{B}}\Big), \ldots, \Big(\pi_m \circ \eta_{\mathcal{A}}\Big) \diamond \Big(\pi_m \circ \eta_{\mathcal{B}}\Big)\Big), \\
\nu_{\mathcal{A}} \diamond \nu_{\mathcal{B}} &= \Big(\Big(\pi_1 \circ \nu_{\mathcal{A}}\Big) \diamond \Big(\pi_1 \circ \nu_{\mathcal{B}}\Big), \ldots, \Big(\pi_m \circ \nu_{\mathcal{A}}\Big) \diamond \Big(\pi_m \circ \nu_{\mathcal{B}}\Big)\Big),
\end{aligned}$$

and for any $i = 1, 2, \ldots, m$,

$$\left[\left(\pi_i \circ \mu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{B}}\right)\right](x) = \begin{cases} \bigvee\limits_{x=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{A}}\right)(a) \wedge \left(\pi_i \circ \mu_{\mathcal{B}}\right)(b)\right\} & \text{if } x \in S^2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\left[\left(\pi_i \circ \eta_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{B}}\right)\right](x) = \begin{cases} \bigwedge\limits_{x=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{A}}\right)(a) \vee \left(\pi_i \circ \eta_{\mathcal{B}}\right)(b)\right\} & \text{if } x \in S^2, \\ 1 & \text{otherwise,} \end{cases}$$

$$\left[\left(\pi_i \circ \nu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{B}}\right)\right](x) = \begin{cases} \bigwedge\limits_{x=ab} \left\{\left(\pi_i \circ \nu_{\mathcal{A}}\right)(a) \vee \left(\pi_i \circ \nu_{\mathcal{B}}\right)(b)\right\} & \text{if } x \in S^2, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 1. *Let A and B be nonempty subsets of semigroup S. Then, the following properties achieve*

(i) $\mathcal{C}_A \sqcap \mathcal{C}_B = \mathcal{C}_{A \cap B}$;

(ii) $\mathcal{C}_A \diamond \mathcal{C}_B = \mathcal{C}_{AB}$.

Proof. (i) Let $x \in S$. If $x \in A \cap B$, then $x \in A$ and $x \in B$. For every $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} \left(\pi_i \circ \mu_{\mathcal{C}_A}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{C}_B}\right)(x) &= 1 \wedge 1 = 1 = (\pi_i \circ \mu_{\mathcal{C}_{A\cap B}})(x), \\ \left(\pi_i \circ \eta_{\mathcal{C}_A}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{C}_B}\right)(x) &= 0 \vee 0 = 0 = \left(\pi_i \circ \eta_{\mathcal{C}_{A\cap B}}\right)(x), \\ \left(\pi_i \circ \nu_{\mathcal{C}_A}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{C}_B}\right)(x) &= 0 \vee 0 = 0 = \left(\pi_i \circ \nu_{\mathcal{C}_{A\cap B}}\right)(x). \end{aligned}$$

On the other hand, if $x \notin A \cap B$, then $x \notin A$ or $x \notin B$. For each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} \left(\pi_i \circ \mu_{\mathcal{C}_A}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{C}_B}\right)(x) &= 0 = \left(\pi_i \circ \mu_{\mathcal{C}_{A\cap B}}\right)(x), \\ \left(\pi_i \circ \eta_{\mathcal{C}_A}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{C}_B}\right)(x) &= 0 = \left(\pi_i \circ \eta_{\mathcal{C}_{A\cap B}}\right)(x), \\ \left(\pi_i \circ \nu_{\mathcal{C}_A}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{C}_B}\right)(x) &= 1 = \left(\pi_i \circ \nu_{\mathcal{C}_{A\cap B}}\right)(x). \end{aligned}$$

We conclude that $\mathcal{C}_A \sqcap \mathcal{C}_B = \mathcal{C}_{A \cap B}$.

(ii) Let $x \in S$. Case 1: $x \in AB$. Then, $x = ab$ for some $a \in A$ and $b \in B$. For any $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} \left[\left(\pi_i \circ \mu_{\mathcal{C}_A}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{C}_B}\right)\right](x) &= \bigvee_{x=uv} \left\{\left(\pi_i \circ \mu_{\mathcal{C}_A}\right)(u) \wedge \left(\pi_i \circ \mu_{\mathcal{C}_B}\right)(v)\right\} \\ &\geqslant \left(\pi_i \circ \mu_{\mathcal{C}_A}\right)(a) \wedge \left(\pi_i \circ \mu_{\mathcal{C}_B}\right)(b) \\ &= 1 = \left(\pi_i \circ \mu_{\mathcal{C}_{AB}}\right)(x), \end{aligned}$$

$$\begin{aligned} \left[\left(\pi_i \circ \eta_{\mathcal{C}_A}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{C}_B}\right)\right](x) &= \bigwedge_{x=uv} \left\{\left(\pi_i \circ \eta_{\mathcal{C}_A}\right)(u) \vee \left(\pi_i \circ \eta_{\mathcal{C}_B}\right)(v)\right\} \\ &\leqslant \left(\pi_i \circ \eta_{\mathcal{C}_A}\right)(a) \vee \left(\pi_i \circ \eta_{\mathcal{C}_B}\right)(b) \\ &= 0 = \left(\pi_i \circ \eta_{\mathcal{C}_{AB}}\right)(x), \end{aligned}$$

$$\begin{aligned}\left[\left(\pi_i \circ \nu_{\mathcal{C}_A}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{C}_B}\right)\right](x) &= \bigwedge_{x=uv} \left\{\left(\pi_i \circ \nu_{\mathcal{C}_A}\right)(u) \vee \left(\pi_i \circ \nu_{\mathcal{C}_B}\right)(v)\right\} \\ &\leqslant \left(\pi_i \circ \nu_{\mathcal{C}_A}\right)(a) \vee \left(\pi_i \circ \nu_{\mathcal{C}_B}\right)(b) \\ &= 0 = \left(\pi_i \circ \nu_{\mathcal{C}_{AB}}\right)(x).\end{aligned}$$

Also,

$$\begin{aligned}\left[\left(\pi_i \circ \mu_{\mathcal{C}_A}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{C}_B}\right)\right](x) &= \left(\pi_i \circ \mu_{\mathcal{C}_{AB}}\right)(x), \\ \left[\left(\pi_i \circ \eta_{\mathcal{C}_A}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{C}_B}\right)\right](x) &= \left(\pi_i \circ \eta_{\mathcal{C}_{AB}}\right)(x), \\ \left[\left(\pi_i \circ \nu_{\mathcal{C}_A}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{C}_B}\right)\right](x) &= \left(\pi_i \circ \nu_{\mathcal{C}_{AB}}\right)(x)\end{aligned}$$

for all $i = 1, 2, \ldots, m$. This proves that $\mathcal{C}_A \diamond \mathcal{C}_B = \mathcal{C}_{AB}$.

Case 2: $x \notin AB$. Then, $x \neq ab$ for all $a \notin A$ or $b \notin B$. For each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\left[\left(\pi_i \circ \mu_{\mathcal{C}_A}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{C}_B}\right)\right](x) &= \bigvee_{x=uv} \left\{\left(\pi_i \circ \mu_{\mathcal{C}_A}\right)(u) \wedge \left(\pi_i \circ \mu_{\mathcal{C}_B}\right)(v)\right\} \\ &= 0 = \left(\pi_i \circ \mu_{\mathcal{C}_{AB}}\right)(x),\end{aligned}$$

$$\begin{aligned}\left[\left(\pi_i \circ \eta_{\mathcal{C}_A}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{C}_B}\right)\right](x) &= \bigwedge_{x=uv} \left\{\left(\pi_i \circ \eta_{\mathcal{C}_A}\right)(u) \vee \left(\pi_i \circ \eta_{\mathcal{C}_B}\right)(v)\right\} \\ &= 0 = \left(\pi_i \circ \eta_{\mathcal{C}_{AB}}\right)(x),\end{aligned}$$

$$\begin{aligned}\left[\left(\pi_i \circ \nu_{\mathcal{C}_A}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{C}_B}\right)\right](x) &= \bigwedge_{x=uv} \left\{\left(\pi_i \circ \nu_{\mathcal{C}_A}\right)(u) \vee \left(\pi_i \circ \nu_{\mathcal{C}_B}\right)(v)\right\} \\ &= 1 = \left(\pi_i \circ \nu_{\mathcal{C}_{AB}}\right)(x).\end{aligned}$$

Therefore, $\mathcal{C}_A \diamond \mathcal{C}_B = \mathcal{C}_{AB}$. □

Lemma 2. *Let $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right), \mathcal{B} = \left(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}}\right)$, $\mathcal{C} = \left(\mu_{\mathcal{C}}, \eta_{\mathcal{C}}, \nu_{\mathcal{C}}\right)$, and $\mathcal{D} = \left(\mu_{\mathcal{D}}, \eta_{\mathcal{D}}, \nu_{\mathcal{D}}\right)$ be m-PPFSs on semigroup S. If $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{C} \sqsubseteq \mathcal{D}$, then $\mathcal{A} \diamond \mathcal{C} \sqsubseteq \mathcal{B} \diamond \mathcal{D}$.*

Proof. Let $x \in S$. It is obvious in case $x \notin S^2$. If $x \in S^2$, then for each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\left[\left(\pi_i \circ \mu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{C}}\right)\right](x) &= \bigvee_{x=uv} \left\{\left(\pi_i \circ \mu_{\mathcal{A}}\right)(u) \wedge \left(\pi_i \circ \mu_{\mathcal{C}}\right)(v)\right\} \\ &\leqslant \bigvee_{x=uv} \left\{\left(\pi_i \circ \mu_{\mathcal{B}}\right)(u) \wedge \left(\pi_i \circ \mu_{\mathcal{D}}\right)(v)\right\} \\ &= \left[\left(\pi_i \circ \mu_{\mathcal{B}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{D}}\right)\right](x)\end{aligned}$$

and

$$\begin{aligned}\left[\left(\pi_i \circ \eta_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{C}}\right)\right](x) &= \bigwedge_{x=uv} \left\{\left(\pi_i \circ \eta_{\mathcal{A}}\right)(u) \vee \left(\pi_i \circ \eta_{\mathcal{C}}\right)(v)\right\} \\ &\geqslant \bigwedge_{x=uv} \left\{\left(\pi_i \circ \eta_{\mathcal{B}}\right)(u) \vee \left(\pi_i \circ \eta_{\mathcal{D}}\right)(v)\right\} \\ &= \left[\left(\pi_i \circ \eta_{\mathcal{B}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{D}}\right)\right](x).\end{aligned}$$

Similarly, we can show that $\left[\left(\pi_i \circ \nu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{C}}\right)\right](x) \geqslant \left[\left(\pi_i \circ \nu_{\mathcal{B}}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{D}}\right)\right](x)$. This implies that $\mathcal{A} \diamond \mathcal{C} \sqsubseteq \mathcal{B} \diamond \mathcal{D}$. □

Theorem 1. *Let S be a semigroup, A be a nonempty subset of S, and $\mathcal{C}_A = \left(\mu_{\mathcal{C}_A}, \eta_{\mathcal{C}_A}, \nu_{\mathcal{C}_A}\right)$ be an m-PPFS on S. Then, the following statements hold:*

(i) A is a subsemigroup of S if and only if $\mathcal{C}_A$ is an m-PPFSub of S;
(ii) A is a left ideal of S if and only if $\mathcal{C}_A$ is an m-PPFL of S;
(iii) A is a right ideal of S if and only if $\mathcal{C}_A$ is an m-PPFR of S;
(iv) A is an ideal of S if and only if $\mathcal{C}_A$ is an m-PPFI of S;
(v) A is a bi-ideal of S if and only if $\mathcal{C}_A$ is an m-PPFB of S;
(vi) A is a generalized bi-ideal of S if and only if $\mathcal{C}_A$ is an m-PPFGB of S.

Proof. (i) Assume that A is a subsemigroup of S. Let $x, y \in S$. If $xy \in A$, then

$$\begin{aligned} \mu_{\mathcal{C}_A}(xy) &= \mathbf{1} \succcurlyeq \mu_{\mathcal{C}_A}(x) \curlywedge \mu_{\mathcal{C}_A}(y), \\ \eta_{\mathcal{C}_A}(xy) &= \mathbf{0} \preccurlyeq \eta_{\mathcal{C}_A}(x) \curlyvee \eta_{\mathcal{C}_A}(y), \\ \nu_{\mathcal{C}_A}(xy) &= \mathbf{0} \preccurlyeq \nu_{\mathcal{C}_A}(x) \curlyvee \nu_{\mathcal{C}_A}(y). \end{aligned}$$

On the other hand, let $xy \notin A$. By assumption, $x \notin A$ or $y \notin A$. Thus, we have

$$\begin{aligned} \mu_{\mathcal{C}_A}(xy) &= \mathbf{0} = \mu_{\mathcal{C}_A}(x) \curlywedge \mu_{\mathcal{C}_A}(y), \\ \eta_{\mathcal{C}_A}(xy) &= \mathbf{0} = \eta_{\mathcal{C}_A}(x) \curlyvee \eta_{\mathcal{C}_A}(y), \\ \nu_{\mathcal{C}_A}(xy) &= \mathbf{1} = \nu_{\mathcal{C}_A}(x) \curlyvee \nu_{\mathcal{C}_A}(y). \end{aligned}$$

Hence, $\mathcal{C}_A$ is an m-PPFSub of S. Conversely, assume that $\mathcal{C}_A$ is an m-PPFSub of S. Let $a, b \in A$. Then, we have

$$\begin{aligned} \mu_{\mathcal{C}_A}(ab) &\succcurlyeq \mu_{\mathcal{C}_A}(a) \curlywedge \mu_{\mathcal{C}_A}(b) = \mathbf{1}, \\ \eta_{\mathcal{C}_A}(ab) &\preccurlyeq \eta_{\mathcal{C}_A}(a) \curlyvee \eta_{\mathcal{C}_A}(b) = \mathbf{0}, \\ \nu_{\mathcal{C}_A}(ab) &\preccurlyeq \nu_{\mathcal{C}_A}(a) \curlyvee \nu_{\mathcal{C}_A}(b) = \mathbf{0}. \end{aligned}$$

This implies that $\mu_{\mathcal{C}_A}(ab) = \mathbf{1}, \eta_{\mathcal{C}_A}(ab) = \mathbf{0}$, and $\nu_{\mathcal{C}_A}(ab) = \mathbf{0}$. So, $\mathcal{C}_A(ab) = (\mathbf{1}, \mathbf{0}, \mathbf{0})$, and it follows $ab \in A$. Therefore, A is a subsemigroup of S.

(ii) Assume that A is a left ideal of S. Let $x, y \in S$. If $xy \in A$, then

$$\mu_{\mathcal{C}_A}(xy) = \mathbf{1} \succcurlyeq \mu_{\mathcal{C}_A}(y), \eta_{\mathcal{C}_A}(xy) = \mathbf{0} = \eta_{\mathcal{C}_A}(y), \text{ and } \nu_{\mathcal{C}_A}(xy) = \mathbf{0} \preccurlyeq \nu_{\mathcal{C}_A}(y).$$

Otherwise, let $xy \notin A$. So, $y \notin A$. Then, we have

$$\mu_{\mathcal{C}_A}(xy) = \mathbf{0} = \mu_{\mathcal{C}_A}(y), \eta_{\mathcal{C}_A}(xy) = \mathbf{0} = \eta_{\mathcal{C}_A}(y), \text{ and } \nu_{\mathcal{C}_A}(xy) = \mathbf{1} = \nu_{\mathcal{C}_A}(y).$$

It turns out that $\mathcal{C}_A$ is an m-PPFL of S. Conversely, let $x \in S$ and $a \in A$. Then, we have

$$\mu_{\mathcal{C}_A}(xa) \succcurlyeq \mu_{\mathcal{C}_A}(a) = \mathbf{1}, \eta_{\mathcal{C}_A}(xa) \preccurlyeq \eta_{\mathcal{C}_A}(a) = \mathbf{0}, \text{ and } \nu_{\mathcal{C}_A}(xa) \preccurlyeq \nu_{\mathcal{C}_A}(a) = \mathbf{0}.$$

Also, $\mu_{\mathcal{C}_A}(xa) = \mathbf{1}, \eta_{\mathcal{C}_A}(xa) = \mathbf{0}$, and $\nu_{\mathcal{C}_A}(xa) = \mathbf{0}$. This means that $\mathcal{C}_A(xa) = (\mathbf{1}, \mathbf{0}, \mathbf{0})$, and then $xa \in A$. Hence, A is a left ideal of S.

(iii) The proof can be demonstrated using an approach similar to the one shown in (ii).

(iv) The proof is derived from (ii) and (iii).

(v) Assume that A is a generalized bi-ideal of S. Let $x, y, z \in S$. If $xyz \in A$, then

$$\mu_{\mathcal{C}_A}(xyz) = \mathbf{1} \succcurlyeq \mu_{\mathcal{C}_A}(x) \curlywedge \mu_{\mathcal{C}_A}(z),$$
$$\eta_{\mathcal{C}_A}(xyz) = \mathbf{0} \preccurlyeq \eta_{\mathcal{C}_A}(x) \curlyvee \eta_{\mathcal{C}_A}(z),$$
$$\nu_{\mathcal{C}_A}(xyz) = \mathbf{0} \preccurlyeq \nu_{\mathcal{C}_A}(x) \curlyvee \nu_{\mathcal{C}_A}(z).$$

Suppose that $xyz \notin A$. Then, $x \notin A$ or $z \notin A$. It follows that

$$\mu_{\mathcal{C}_A}(xyz) = \mathbf{0} = \mu_{\mathcal{C}_A}(x) \curlywedge \mu_{\mathcal{C}_A}(z),$$
$$\eta_{\mathcal{C}_A}(xyz) = \mathbf{0} = \eta_{\mathcal{C}_A}(x) \curlyvee \eta_{\mathcal{C}_A}(z),$$
$$\nu_{\mathcal{C}_A}(xyz) = \mathbf{1} = \nu_{\mathcal{C}_A}(x) \curlyvee \nu_{\mathcal{C}_A}(z).$$

Hence, $\mathcal{C}_A$ is an m-PPFGB of S. Conversely, let $x \in S$ and $a, b \in A$. Then, we have

$$\mu_{\mathcal{C}_A}(axb) \succcurlyeq \mu_{\mathcal{C}_A}(a) \curlywedge \mu_{\mathcal{C}_A}(b) = \mathbf{1},$$
$$\eta_{\mathcal{C}_A}(axb) \preccurlyeq \eta_{\mathcal{C}_A}(a) \curlyvee \eta_{\mathcal{C}_A}(b) = \mathbf{0},$$
$$\nu_{\mathcal{C}_A}(axb) \preccurlyeq \nu_{\mathcal{C}_A}(a) \curlyvee \nu_{\mathcal{C}_A}(b) = \mathbf{0}.$$

So, $\mu_{\mathcal{C}_A}(axb) = \mathbf{1}, \eta_{\mathcal{C}_A}(axb) = \mathbf{0}$, and $\nu_{\mathcal{C}_A}(axb) = \mathbf{0}$. This shows that $\mathcal{C}_A(axb) = (\mathbf{1}, \mathbf{0}, \mathbf{0})$. We find that $axb \in A$. Therefore, A is a generalized bi-ideal of S.

(vi) It follows from (i) and (v). □

The following definitions have been adapted from [27].

Definition 6. *Let $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ be an m-PPFS on a nonempty set, X, and let $t = (t_1, t_2, \ldots, t_m), r = (r_1, r_2, \ldots, r_m), s = (s_1, s_2, \ldots, s_m) \in [0,1]^m$. Set $\mathcal{A}_{(t,r,s)}$ is called a (t,r,s)-cut or (t,r,s)-level set of $\mathcal{A}$ over X and is defined by*

$$\mathcal{A}_{(t,r,s)} = \left\{x \in X \mid \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x) \geqslant t_i, \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x) \leqslant r_i, \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x) \leqslant s_i \ \forall i = 1, 2, \ldots, m\right\},$$

and it satisfies the property $0 \leqslant t_i + r_i + s_i \leqslant 1$ for all $i = 1, 2, \ldots, m$.

Theorem 2. *Let S be a semigroup and $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ be an m-PPFS on S. Then, the following properties hold:*

- *(i) $\mathcal{A}$ is an m-PPFSub of S if and only if for every $t, r, s \in [0,1]^m$, $\mathcal{A}_{(t,r,s)}$ is a subsemigroup of S when it is nonempty;*
- *(ii) $\mathcal{A}$ is an m-PPFL of S if and only if for every $t, r, s \in [0,1]^m$, $\mathcal{A}_{(t,r,s)}$ is a left ideal of S when it is nonempty;*
- *(iii) $\mathcal{A}$ is an m-PPFR of S if and only if for every $t, r, s \in [0,1]^m$, $\mathcal{A}_{(t,r,s)}$ is a right ideal of S when it is nonempty;*
- *(iv) $\mathcal{A}$ is an m-PPFI of S if and only if for every $t, r, s \in [0,1]^m$, $\mathcal{A}_{(t,r,s)}$ is an ideal of S when it is nonempty;*
- *(v) $\mathcal{A}$ is an m-PPFGB of S if and only if for every $t, r, s \in [0,1]^m$, $\mathcal{A}_{(t,r,s)}$ is a generalized bi-ideal of S when it is nonempty;*
- *(vi) $\mathcal{A}$ is an m-PPFB of S if and only if for every $t, r, s \in [0,1]^m$, $\mathcal{A}_{(t,r,s)}$ is a bi-ideal of S when it is nonempty.*

Proof. (i) Assume that $\mathcal{A}$ is an m-PPFSub of S. Let $t = (t_1, t_2, \ldots, t_m), r = (r_1, r_2, \ldots, r_m)$, $s = (s_1, s_2, \ldots, s_m) \in [0,1]^m$ be such that $\mathcal{A}_{(t,r,s)} \neq \emptyset$. Let $x, y \in \mathcal{A}_{(t,r,s)}$. Then, $\big(\pi_i \circ$

$\mu_{\mathcal{A}}\Big)(x) \geqslant t_i$, $\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(y) \geqslant t_i$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \leqslant r_i$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(y) \leqslant r_i$, $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \leqslant s_i$, and $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(y) \leqslant s_i$ for all $i = 1, 2, \ldots, m$. By assumption, we have

$$\begin{aligned}
\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(xy) &\geqslant \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(y) \geqslant t_i,\\
\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(xy) &\leqslant \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(y) \leqslant r_i,\\
\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(xy) &\leqslant \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(y) \leqslant s_i
\end{aligned}$$

for all $i = 1, 2, \ldots, m$. It follows that $xy \in \mathcal{A}_{(t,r,s)}$. Hence, $\mathcal{A}_{(t,r,s)}$ is a subsemigroup of S.

Conversely, assume that nonempty (t, r, s)-level set $\mathcal{A}_{(t,r,s)}$ is a subsemigroup of S for all $t, r, s \in [0, 1]^m$. Let $x, y \in S$. Take

$$\begin{aligned}
t'_i &= \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(y),\\
r'_i &= \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(y),\\
s'_i &= \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(y)
\end{aligned}$$

for some $t'_i, r'_i, s'_i \in [0, 1]$, where $i = 1, 2, \ldots, m$. It turns out that $\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \geqslant t_i$, $\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(y) \geqslant t_i$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \leqslant r_i$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(y) \leqslant r_i$, $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \leqslant s_i$, and $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(y) \leqslant s_i$ for all $i = 1, 2, \ldots, m$. Now, let $t' = (t'_1, t'_2, \ldots, t'_m)$, $r' = (r'_1, r'_2, \ldots, r'_m)$, and $s' = (s'_1, s'_2, \ldots, s'_m)$. We find that $x, y \in \mathcal{A}_{(t',r',s')}$, and so $\mathcal{A}_{(t',r',s')} \neq \emptyset$. By the given assumption, we find that $\mathcal{A}_{(t',r',s')}$ is a subsemigroup of S. Thus, $xy \in \mathcal{A}_{(t',r',s')}$. This implies that

$$\begin{aligned}
\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(xy) &\geqslant t'_i = \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(y),\\
\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(xy) &\leqslant r'_i = \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(y),\\
\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(xy) &\leqslant s'_i = \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(y)
\end{aligned}$$

for all $i = 1, 2, \ldots, m$. That is, $\mu_{\mathcal{A}}(xy) \succcurlyeq \mu_{\mathcal{A}}(x) \curlywedge \mu_{\mathcal{A}}(y)$, $\eta_{\mathcal{A}}(xy) \preccurlyeq \eta_{\mathcal{A}}(x) \curlyvee \eta_{\mathcal{A}}(y)$, and $\nu_{\mathcal{A}}(xy) \preccurlyeq \nu_{\mathcal{A}}(x) \curlyvee \nu_{\mathcal{A}}(y)$. Therefore, $\mathcal{A}$ is an m-PPFSub of S.

(ii) Assume that $\mathcal{A}$ is an m-PPFL of S. Let $t = (t_1, t_2, \ldots, t_m)$, $r = (r_1, r_2, \ldots, r_m)$, $s = (s_1, s_2, \ldots, s_m) \in [0, 1]^m$ be such that $\mathcal{A}_{(t,r,s)} \neq \emptyset$. Let $x \in S$ and $y \in \mathcal{A}_{(t,r,s)}$. Then, $\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(y) \geqslant t_i$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(y) \leqslant r_i$, and $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(y) \leqslant s_i$ for all $i = 1, 2, \ldots, m$. By assumption, we have $\mu_{\mathcal{A}}(xy) \succcurlyeq \mu_{\mathcal{A}}(y)$, $\eta_{\mathcal{A}}(xy) \preccurlyeq \eta_{\mathcal{A}}(y)$, and $\nu_{\mathcal{A}}(xy) \preccurlyeq \nu_{\mathcal{A}}(y)$. We obtain that

$$\begin{aligned}
\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(xy) &\geqslant \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(y) \geqslant t_i,\\
\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(xy) &\leqslant \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(y) \leqslant r_i,\\
\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(xy) &\leqslant \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(y) \leqslant s_i
\end{aligned}$$

for all $i = 1, 2, \ldots, m$. This implies that $xy \in \mathcal{A}_{(t,r,s)}$. Thus, $\mathcal{A}_{(t,r,s)}$ is a left ideal of S.

Conversely, assume that nonempty (t, r, s)-level set $\mathcal{A}_{(t,r,s)}$ is a left ideal of S for all $t, r, s \in [0, 1]^m$. Let $x, y \in S$. Choose $t'_i = \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(y)$, $r'_i = \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(y)$, $s'_i = \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(y)$ for some $t'_i, r'_i, s'_i \in [0, 1]$, where $i = 1, 2, \ldots, m$. Let $t' = (t'_1, t'_2, \ldots, t'_m)$, $r' = (r'_1, r'_2, \ldots, r'_m)$, and $s' = (s'_1, s'_2, \ldots, s'_m)$. We have that $y \in \mathcal{A}_{(t',r',s')}$. By assumption, we find that $\mathcal{A}_{(t',r',s')}$ is a left ideal of S. Thus, $xy \in \mathcal{A}_{(t',r',s')}$. This implies that

$$\begin{aligned}\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(xy) &\geqslant t'_i = \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(y),\\ \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(xy) &\leqslant r'_i = \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(y),\\ \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(xy) &\leqslant s'_i = \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(y)\end{aligned}$$

for all $i = 1, 2, \ldots, m$. Also, $\mu_{\mathcal{A}}(xy) \succcurlyeq \mu_{\mathcal{A}}(y)$, $\eta_{\mathcal{A}}(xy) \preccurlyeq \eta_{\mathcal{A}}(y)$, and $\nu_{\mathcal{A}}(xy) \preccurlyeq \nu_{\mathcal{A}}(y)$. Therefore, $\mathcal{A}$ is an m-PPFL of S.

(iii) The proof is similar to (ii).

(iv) It obvious by (ii) and (iii).

(v) Assume that $\mathcal{A}$ is an m-PPFGB of S. Let $t = (t_1, t_2, \ldots, t_m), r = (r_1, r_2, \ldots, r_m)$, $s = (s_1, s_2, \ldots, s_m) \in [0,1]^m$ be such that $\mathcal{A}_{(t,r,s)} \neq \emptyset$. Let $x, z \in \mathcal{A}_{(t,r,s)}$ and $y \in S$. Then, $\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \geqslant t_i$, $\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z) \geqslant t_i$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \leqslant r_i$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z) \leqslant r_i$, $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \leqslant s_i$, and $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(z) \leqslant s_i$, where $i = 1, 2, \ldots, m$. Given the hypothesis, we have $\mu_{\mathcal{A}}(xyz) \succcurlyeq \mu_{\mathcal{A}}(x) \curlywedge \mu_{\mathcal{A}}(z)$, $\eta_{\mathcal{A}}(xyz) \preccurlyeq \eta_{\mathcal{A}}(x) \curlyvee \eta_{\mathcal{A}}(z)$, and $\nu_{\mathcal{A}}(xyz) \preccurlyeq \nu_{\mathcal{A}}(x) \curlyvee \nu_{\mathcal{A}}(z)$. Thus, for any $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(xyz) &\geqslant \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z) \geqslant t_i,\\ \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(xyz) &\leqslant \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z) \leqslant r_i,\\ \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(xyz) &\leqslant \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(z) \leqslant s_i.\end{aligned}$$

This means that $xyz \in \mathcal{A}_{(t,r,s)}$. Hence, $\mathcal{A}_{(t,r,s)}$ is a generalized bi-ideal of S.

Conversely, let $x, y, z \in S$. Take

$$\begin{aligned}t'_i &= \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z),\\ r'_i &= \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z),\\ s'_i &= \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(z)\end{aligned}$$

for some $t'_i, r'_i, s'_i \in [0,1]$, where $i = 1, 2, \ldots, m$. It follows that $\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \geqslant t_i$, $\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z) \geqslant t_i$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \leqslant r_i$, $\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z) \leqslant r_i$, $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \leqslant s_i$, and $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(z) \leqslant s_i$ for any $i = 1, 2, \ldots, m$. Then, $x\,z \in \mathcal{A}_{(t',r',s')}$. Given the assumption, we have $xyz \in \mathcal{A}_{(t',r',s')}$. So, for any $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(xyz) &\geqslant t'_i = \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z),\\ \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(xyz) &\leqslant r'_i = \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z),\\ \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(xyz) &\leqslant s'_i = \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(z).\end{aligned}$$

That is, $\mu_{\mathcal{A}}(xyz) \succcurlyeq \mu_{\mathcal{A}}(x) \curlywedge \mu_{\mathcal{A}}(z)$, $\eta_{\mathcal{A}}(xyz) \preccurlyeq \eta_{\mathcal{A}}(x) \curlyvee \eta_{\mathcal{A}}(z)$, and $\nu_{\mathcal{A}}(xyz) \preccurlyeq \nu_{\mathcal{A}}(x) \curlyvee \nu_{\mathcal{A}}(z)$. Therefore, $\mathcal{A}$ is an m-PPFGB of S.

(vi) The proof is complete by (i) and (v). $\square$

Theorem 3. *Let S be a semigroup and $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ be an m-PPFS on S. Then, the following conditions are true:*

(i) $\mathcal{A}$ is an m-PPFSub of S if and only if $\mathcal{A} \diamond \mathcal{A} \sqsubseteq \mathcal{A}$;
(ii) $\mathcal{A}$ is an m-PPFL of S if and only if $\mathcal{S} \diamond \mathcal{A} \sqsubseteq \mathcal{A}$;
(iii) $\mathcal{A}$ is an m-PPFR of S if and only if $\mathcal{A} \diamond \mathcal{S} \sqsubseteq \mathcal{A}$;
(iv) $\mathcal{A}$ is an m-PPFI of S if and only if $\mathcal{S} \diamond \mathcal{A} \sqsubseteq \mathcal{A}$ and $\mathcal{A} \diamond \mathcal{S} \sqsubseteq \mathcal{A}$;
(v) $\mathcal{A}$ is an m-PPFGB of S if and only if $\mathcal{A} \diamond \mathcal{S} \diamond \mathcal{A} \sqsubseteq \mathcal{A}$;
(vi) $\mathcal{A}$ is an m-PPFB of S if and only if $\mathcal{A} \diamond \mathcal{A} \sqsubseteq \mathcal{A}$ and $\mathcal{A} \diamond \mathcal{S} \diamond \mathcal{A} \sqsubseteq \mathcal{A}$.

Proof. (i) Assume that $\mathcal{A}$ is an m-PPFSub of S. Let $x \in S$. If $x \notin S^2$, then $\mathcal{A} \diamond \mathcal{A} \sqsubseteq \mathcal{A}$. Suppose that $x = ab$ for some $a, b \in S$. Then, for any $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\left[\left(\pi_i \circ \mu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{A}}\right)\right](x) &= \bigvee_{x=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{A}}\right)(a) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(b)\right\} \\ &\leqslant \bigvee_{x=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{A}}\right)(ab)\right\} = \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x)\end{aligned}$$

and

$$\begin{aligned}\left[\left(\pi_i \circ \eta_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{A}}\right)\right](x) &= \bigwedge_{x=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{A}}\right)(a) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(b)\right\} \\ &\geqslant \bigwedge_{x=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{A}}\right)(ab)\right\} = \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x).\end{aligned}$$

Furthermore, we have that $\left[\left(\pi_i \circ \nu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{A}}\right)\right](x) \geqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x)$. This implies that $(\mu_{\mathcal{A}} \diamond \mu_{\mathcal{A}})(x) \preccurlyeq \mu_{\mathcal{A}}(x)$, $(\eta_{\mathcal{A}} \diamond \eta_{\mathcal{A}})(x) \succcurlyeq \eta_{\mathcal{A}}(x)$, and $(\nu_{\mathcal{A}} \diamond \nu_{\mathcal{A}})(x) \succcurlyeq \nu_{\mathcal{A}}(x)$. Thus, $\mathcal{A} \diamond \mathcal{A} \sqsubseteq \mathcal{A}$.

Conversely, let $x, y \in S$. By assumption, we have $\left(\mu_{\mathcal{A}} \diamond \mu_{\mathcal{A}}\right)(xy) \preccurlyeq \mu_{\mathcal{A}}(xy)$, $\left(\eta_{\mathcal{A}} \diamond \eta_{\mathcal{A}}\right)(xy) \succcurlyeq \eta_{\mathcal{A}}(xy)$, and $\left(\nu_{\mathcal{A}} \diamond \nu_{\mathcal{A}}\right)(xy) \succcurlyeq \nu_{\mathcal{A}}(xy)$. So, for every $i = 1, 2, \ldots, m$, it follows

$$\begin{aligned}\left(\pi_i \circ \mu_{\mathcal{A}}\right)(xy) &\geqslant \left[\left(\pi_i \circ \mu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{A}}\right)\right](xy) \\ &= \bigvee_{xy=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{A}}\right)(a) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(b)\right\} \\ &\geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(y)\end{aligned}$$

and

$$\begin{aligned}\left(\pi_i \circ \eta_{\mathcal{A}}\right)(xy) &\leqslant \left[\left(\pi_i \circ \eta_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{A}}\right)\right](xy) \\ &= \bigwedge_{xy=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{A}}\right)(a) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(b)\right\} \\ &\leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(y).\end{aligned}$$

Additionally, we find that $\left(\pi_i \circ \nu_{\mathcal{A}}\right)(xy) \leqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{A}}\right)(y)$. It turns out that $\mu_{\mathcal{A}}(xy) \succcurlyeq \mu_{\mathcal{A}}(x) \wedge \mu_{\mathcal{A}}(y)$, $\eta_{\mathcal{A}}(xy) \preccurlyeq \eta_{\mathcal{A}}(x) \vee \eta_{\mathcal{A}}(y)$, and $\nu_{\mathcal{A}}(xy) \preccurlyeq \nu_{\mathcal{A}}(x) \vee \nu_{\mathcal{A}}(y)$. Therefore, $\mathcal{A}$ is an m-PPFSub of S.

(ii) Assume that $\mathcal{A}$ is an m-PPFL of S. Let $x \in S$. It is clear that $\mathcal{S} \diamond \mathcal{A} \sqsubseteq \mathcal{A}$ in the case of $x \notin S^2$. Suppose that $a, b \in S$ exist, such that $x = ab$. Then, for each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\left[\left(\pi_i \circ \mu_{\mathcal{S}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{A}}\right)\right](x) &= \bigvee_{x=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{S}}\right)(a) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(b)\right\} \\
&\leqslant \bigvee_{x=ab} \left\{1 \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(ab)\right\} \\
&= \bigvee_{x=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{A}}\right)(ab)\right\} \\
&= \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x)
\end{aligned}$$

and

$$\begin{aligned}
\left[\left(\pi_i \circ \eta_{\mathcal{S}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{A}}\right)\right](x) &= \bigwedge_{x=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{S}}\right)(a) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(b)\right\} \\
&\geqslant \bigwedge_{x=ab} \left\{0 \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(ab)\right\} \\
&= \bigwedge_{x=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{A}}\right)(ab)\right\} \\
&= \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x).
\end{aligned}$$

Similarly, $\left[\left(\pi_i \circ \nu_{\mathcal{S}}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{A}}\right)\right](x) \geqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x)$. This shows that $\left(\mu_{\mathcal{S}} \diamond \mu_{\mathcal{A}}\right)(x) \preccurlyeq \mu_{\mathcal{A}}(x)$, $\left(\eta_{\mathcal{S}} \diamond \eta_{\mathcal{A}}\right)(x) \succcurlyeq \eta_{\mathcal{A}}(x)$, and $\left(\nu_{\mathcal{S}} \diamond \nu_{\mathcal{A}}\right)(x) \succcurlyeq \nu_{\mathcal{A}}(x)$. We find that $\mathcal{S} \diamond \mathcal{A} \sqsubseteq \mathcal{A}$.

Conversely, let $x, y \in S$. By assumption, we have $\left(\mu_{\mathcal{S}} \diamond \mu_{\mathcal{A}}\right)(xy) \preccurlyeq \mu_{\mathcal{A}}(xy)$, $\left(\eta_{\mathcal{S}} \diamond \eta_{\mathcal{A}}\right)(xy) \succcurlyeq \eta_{\mathcal{A}}(xy)$, and $\left(\nu_{\mathcal{S}} \diamond \nu_{\mathcal{A}}\right)(xy) \succcurlyeq \nu_{\mathcal{A}}(xy)$. For any $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\left(\pi_i \circ \mu_{\mathcal{A}}\right)(xy) &\geqslant \left[\left(\pi_i \circ \mu_{\mathcal{S}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{A}}\right)\right](xy) \\
&= \bigvee_{xy=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{S}}\right)(a) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(b)\right\} \\
&\geqslant \left(\pi_i \circ \mu_{\mathcal{S}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(y) \\
&= 1 \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(y) \\
&= \left(\pi_i \circ \mu_{\mathcal{A}}\right)(y)
\end{aligned}$$

and

$$\begin{aligned}
\left(\pi_i \circ \eta_{\mathcal{A}}\right)(xy) &\leqslant \left[\left(\pi_i \circ \eta_{\mathcal{S}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{A}}\right)\right](xy) \\
&= \bigwedge_{xy=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{S}}\right)(a) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(b)\right\} \\
&\leqslant \left(\pi_i \circ \eta_{\mathcal{S}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(y) \\
&= 0 \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(y) \\
&= \left(\pi_i \circ \eta_{\mathcal{A}}\right)(y).
\end{aligned}$$

Similarly, $\left(\pi_i \circ \nu_{\mathcal{A}}\right)(xy) \leqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(y)$. This means that $\mu_{\mathcal{A}}(xy) \succcurlyeq \mu_{\mathcal{A}}(y)$, $\eta_{\mathcal{A}}(xy) \preccurlyeq \eta_{\mathcal{A}}(y)$, and $\nu_{\mathcal{A}}(xy) \preccurlyeq \nu_{\mathcal{A}}(y)$. Therefore, $\mathcal{A}$ is an m-PPFL of S.

(iii) The proof is similar to the one presented in (ii).

(iv) It is achieved by (ii) and (iii).

(v) Assume that $\mathcal{A}$ is an m-PPFGB of S. Let $x \in S$. If $x \neq uvw$ for all $u, v, w \in S$, then the proof is complete. Suppose that $x = abc$ for some $a, b, c \in S$. Also, there exists $m = ab$ such that $x = mc$. For any $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
&\left[\left(\pi_i \circ \mu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{S}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{A}}\right)\right](x) \\
&\quad = \bigvee_{x=mc} \left\{\left[\left(\pi_i \circ \mu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{S}}\right)\right](m) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(c)\right\} \\
&\quad = \bigvee_{x=mc} \left\{\bigvee_{m=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{A}}\right)(a) \wedge \left(\pi_i \circ \mu_{\mathcal{S}}\right)(b)\right\} \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(c)\right\} \\
&\quad = \bigvee_{x=mc} \left\{\bigvee_{m=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{A}}\right)(a) \wedge 1\right\} \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(c)\right\} \\
&\quad = \bigvee_{x=mc} \left\{\bigvee_{m=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{A}}\right)(a) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(c)\right\}\right\} \\
&\quad \leqslant \bigvee_{x=mc} \left\{\bigvee_{m=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{A}}\right)(abc)\right\}\right\} \\
&\quad = \bigvee_{x=mc} \left\{\left(\pi_i \circ \mu_{\mathcal{A}}\right)(mc)\right\} \\
&\quad = \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x)
\end{aligned}$$

and

$$\begin{aligned}
&\left[\left(\pi_i \circ \eta_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{S}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{A}}\right)\right](x) \\
&\quad = \bigwedge_{x=mc} \left\{\left[\left(\pi_i \circ \eta_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{S}}\right)\right](m) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(c)\right\} \\
&\quad = \bigwedge_{x=mc} \left\{\bigwedge_{m=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{A}}\right)(a) \vee \left(\pi_i \circ \eta_{\mathcal{S}}\right)(b)\right\} \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(c)\right\} \\
&\quad = \bigwedge_{x=mc} \left\{\bigwedge_{m=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{A}}\right)(a) \vee 0\right\} \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(c)\right\} \\
&\quad = \bigwedge_{x=mc} \left\{\bigwedge_{m=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{A}}\right)(a) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(c)\right\}\right\} \\
&\quad \geqslant \bigwedge_{x=mc} \left\{\bigwedge_{m=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{A}}\right)(abc)\right\}\right\} \\
&\quad = \bigwedge_{x=mc} \left\{\left(\pi_i \circ \eta_{\mathcal{A}}\right)(mc)\right\} \\
&\quad = \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x).
\end{aligned}$$

Also, we find that $\left[\left(\pi_i \circ \nu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{S}}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{A}}\right)\right](x) \geqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x)$. Thus, $\left(\mu_{\mathcal{A}} \diamond \mu_{\mathcal{S}} \diamond \mu_{\mathcal{A}}\right)(x) \preccurlyeq \mu_{\mathcal{A}}(x)$, $\left(\eta_{\mathcal{A}} \diamond \eta_{\mathcal{S}} \diamond \eta_{\mathcal{A}}\right)(x) \succcurlyeq \eta_{\mathcal{A}}(x)$, and $\left(\nu_{\mathcal{A}} \diamond \nu_{\mathcal{S}} \diamond \nu_{\mathcal{A}}\right)(x) \succcurlyeq \nu_{\mathcal{A}}(x)$. Hence, $\mathcal{A} \diamond \mathcal{S} \diamond \mathcal{A} \sqsubseteq \mathcal{A}$.

Conversely, let $x, y, z \in S$. By assumption, we have $\Big(\mu_{\mathcal{A}} \diamond \mu_{\mathcal{S}} \diamond \mu_{\mathcal{A}}\Big)(xyz) \preccurlyeq \mu_{\mathcal{A}}(xyz)$, $\Big(\eta_{\mathcal{A}} \diamond \eta_{\mathcal{S}} \diamond \eta_{\mathcal{A}}\Big)(xyz) \succcurlyeq \eta_{\mathcal{A}}(xyz)$, and $\Big(\nu_{\mathcal{A}} \diamond \nu_{\mathcal{S}} \diamond \nu_{\mathcal{A}}\Big)(xyz) \succcurlyeq \nu_{\mathcal{A}}(xyz)$. For each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(xyz) &\geqslant \Big[\Big(\pi_i \circ \mu_{\mathcal{A}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{S}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)\Big](xyz) \\
& \bigvee_{xyz=uv} \Big\{\Big[\Big(\pi_i \circ \mu_{\mathcal{A}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{S}}\Big)\Big](u) \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(v)\Big\} \\
&\geqslant \Big[\Big(\pi_i \circ \mu_{\mathcal{A}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{S}}\Big)\Big](xy) \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z) \\
&= \bigvee_{xy=mn} \Big\{\Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(m) \wedge \Big(\pi_i \circ \mu_{\mathcal{S}}\Big)(n)\Big\} \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z) \\
&\geqslant \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{S}}\Big)(y) \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z) \\
&= \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge 1 \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z) \\
&= \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{A}}\Big)(z)
\end{aligned}$$

and

$$\begin{aligned}
\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(xyz) &\leqslant \Big[\Big(\pi_i \circ \eta_{\mathcal{A}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{S}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)\Big](xyz) \\
&= \bigwedge_{xyz=uv} \Big\{\Big[\Big(\pi_i \circ \eta_{\mathcal{A}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{S}}\Big)\Big](u) \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(v)\Big\} \\
&\leqslant \Big[\Big(\pi_i \circ \eta_{\mathcal{A}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{S}}\Big)\Big](xy) \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z) \\
&= \bigwedge_{xy=mn} \Big\{\Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(m) \vee \Big(\pi_i \circ \eta_{\mathcal{S}}\Big)(n)\Big\} \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z) \\
&\leqslant \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{S}}\Big)(y) \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z) \\
&= \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee 0 \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z) \\
&= \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{A}}\Big)(z).
\end{aligned}$$

Similarly, we can show that $\Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(xyz) \leqslant \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(x) \vee \Big(\pi_i \circ \nu_{\mathcal{A}}\Big)(z)$. That is, $\mu_{\mathcal{A}}(xyz) \succcurlyeq \mu_{\mathcal{A}}(x) \curlywedge \mu_{\mathcal{A}}(z)$, $\eta_{\mathcal{A}}(xyz) \preccurlyeq \eta_{\mathcal{A}}(x) \curlyvee \eta_{\mathcal{A}}(z)$, and $\nu_{\mathcal{A}}(xyz) \preccurlyeq \nu_{\mathcal{A}}(x) \curlyvee \nu_{\mathcal{A}}(z)$. Consequently, $\mathcal{A}$ is an m-PPFGB of S.

(vi) This result is obtained from conditions (i) and (v). □

4. Regular Semigroups Characterized by Their *m*-PPFBs

In this section, we examine the connections among m-PPFLs, m-PPFRs, m-PPFIs, m-PPFBs, and m-PPFGBs in regular semigroups. Afterwards, we characterize regular semigroups from the viewpoint of m-PPFLs, m-PPFRs, m-PPFIs, m-PPFBs, and m-PPFGBs of the semigroups.

Semigroup S is called *regular* [31] if, for each element a in S, $x \in S$ exists, such that $a = axa$.

Proposition 3. *In any regular semigroup S, m-PPFGBs and m-PPFBs coincide.*

Proof. Regarding Definition 4, it is sufficient to show that any m-PPFGB is an m-PPFB of S. Let $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ be an m-PPFGB of S and $a, b \in S$. Then, $x \in S$ exists, such that $b = bxb$. For each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\left(\pi_i \circ \mu_{\mathcal{A}}\right)(ab) &= \left(\pi_i \circ \mu_{\mathcal{A}}\right)(a(bx)b) \geqslant \left(\pi_i \circ \mu_{\mathcal{A}}\right)(a) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(b), \\
\left(\pi_i \circ \eta_{\mathcal{A}}\right)(ab) &= \left(\pi_i \circ \eta_{\mathcal{A}}\right)(a(bx)b) \leqslant \left(\pi_i \circ \eta_{\mathcal{A}}\right)(a) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(b), \\
\left(\pi_i \circ \nu_{\mathcal{A}}\right)(ab) &= \left(\pi_i \circ \nu_{\mathcal{A}}\right)(a(bx)b) \leqslant \left(\pi_i \circ \nu_{\mathcal{A}}\right)(a) \vee \left(\pi_i \circ \nu_{\mathcal{A}}\right)(b).
\end{aligned}$$

It follows that $\mu_{\mathcal{A}}(ab) \succcurlyeq \mu_{\mathcal{A}}(a) \curlywedge \mu_{\mathcal{A}}(b)$, $\eta_{\mathcal{A}}(ab) \preccurlyeq \eta_{\mathcal{A}}(a) \curlyvee \eta_{\mathcal{A}}(b)$, and $\nu_{\mathcal{A}}(ab) \preccurlyeq \nu_{\mathcal{A}}(a) \curlyvee \nu_{\mathcal{A}}(b)$. It turns out that $\mathcal{A}$ is an m-PPFB of S. □

Proposition 3 shows that, if semigroup S is regular, then m-PPBs and m-PPFGBs of S are the same. However, this relationship between m-PPFBs and both m-PPFLs and m-PPFRs in regular semigroups does not always hold. The following example illustrates that, even if semigroup S is regular, the m-PPFBs differ from either m-PPFLs or m-PPFRs.

Example 4. *Let $S = \{a, b, c, d, e\}$ represent a semigroup, according to [29], as seen in Table 8.*

Table 8. The operation "$\cdot$" on S.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

It is not difficult to verify that S is regular. Now, let us consider a 4-PPFS $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ on S defined by Table 9.

Table 9. The representation of the value of a 4-PPFS $\mathcal{A}$ on S.

$\mathcal{A}$	$\mu_{\mathcal{A}}$	$\eta_{\mathcal{A}}$	$\nu_{\mathcal{A}}$
a	$\left(0.4, 0.5, 0.6, 0.5\right)$	$\left(0.1, 0.1, 0, 0.2\right)$	$\left(0, 0.1, 0.1, 0\right)$
b	$\left(0.4, 0.3, 0.5, 0.3\right)$	$\left(0.2, 0.2, 0.1, 0.3\right)$	$\left(0.1, 0.3, 0.2, 0.1\right)$
c	$\left(0.3, 0.2, 0.4, 0.3\right)$	$\left(0.3, 0.3, 0.2, 0.4\right)$	$\left(0.2, 0.4, 0.4, 0.2\right)$
d	$\left(0, 0.1, 0.2, 0.2\right)$	$\left(0.4, 0.4, 0.3, 0.5\right)$	$\left(0.6, 0.5, 0.5, 0.3\right)$
e	$\left(0, 0.1, 0.2, 0.2\right)$	$\left(0.4, 0.4, 0.3, 0.5\right)$	$\left(0.6, 0.5, 0.5, 0.3\right)$

Further calculations confirm that the 4-PPFS $\mathcal{A}$ on S satisfies the requirements of a 4-PPFB of S. Then, we consider

$$\begin{aligned}
\left(\pi_1 \circ \mu_{\mathcal{A}}\right)(eb) &= \left(\pi_1 \circ \mu_{\mathcal{A}}\right)(d) = 0 \ngeqslant 0.4 = \left(\pi_1 \circ \mu_{\mathcal{A}}\right)(b), \\
\left(\pi_3 \circ \mu_{\mathcal{A}}\right)(be) &= \left(\pi_3 \circ \mu_{\mathcal{A}}\right)(c) = 0.4 \ngeqslant 0.5 = \left(\pi_3 \circ \mu_{\mathcal{A}}\right)(b).
\end{aligned}$$

This means that $\mu_{\mathcal{A}}(eb) \not\succcurlyeq \mu_{\mathcal{A}}(b)$ and $\mu_{\mathcal{A}}(be) \not\succcurlyeq \mu_{\mathcal{A}}(b)$. Therefore, the 4-PPFB $\mathcal{A}$ is neither a 4-PPFL nor a 4-PPFR of S.

Now, we recall the basic features of regular semigroup characterizations that are important to the investigation performed in this section.

Lemma 3 ([31])**.** *Let S be a semigroup. Then, S is regular if and only if $R \cap L \subseteq RL$ for every left ideal L and right ideal R of S.*

Lemma 4 ([31])**.** *Let S be a semigroup. Then, S is regular if and only if $B = BSB$ for every bi-ideal B of S.*

Theorem 4. *Let S be a semigroup. Then, S is regular if and only if $\mathcal{R} \sqcap \mathcal{L} = \mathcal{R} \diamond \mathcal{L}$, for all m-PPFL $\mathcal{L} = \left(\mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\right)$ and for all m-PPFR $\mathcal{R} = \left(\mu_{\mathcal{R}}, \eta_{\mathcal{R}}, \nu_{\mathcal{R}}\right)$ of S.*

Proof. Assume that S is regular. Let $\mathcal{L} = \left(\mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\right)$ and $\mathcal{R} = \left(\mu_{\mathcal{R}}, \eta_{\mathcal{R}}, \nu_{\mathcal{R}}\right)$ be an m-PPFL and an m-PPFR of S, respectively. Regarding Lemma 2 and Theorem 3, we have $\mathcal{R} \diamond \mathcal{L} \sqsubseteq \mathcal{S} \diamond \mathcal{L} \sqsubseteq \mathcal{L}$ and $\mathcal{R} \diamond \mathcal{L} \sqsubseteq \mathcal{R} \diamond \mathcal{S} \sqsubseteq \mathcal{R}$. This means that $\mathcal{R} \diamond \mathcal{L} \sqsubseteq \mathcal{R} \sqcap \mathcal{L}$. Next, let $x \in S$. Then, $y \in S$ exists, such that $x = xyx$. For every $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\left[\left(\pi_i \circ \mu_{\mathcal{R}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{L}}\right)\right](x) &= \bigvee_{x=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{R}}\right)(a) \wedge \left(\pi_i \circ \mu_{\mathcal{L}}\right)(b)\right\} \\
&\geqslant \left(\pi_i \circ \mu_{\mathcal{R}}\right)(xy) \wedge \left(\pi_i \circ \mu_{\mathcal{L}}\right)(x) \\
&\geqslant \left(\pi_i \circ \mu_{\mathcal{R}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{L}}\right)(x)
\end{aligned}$$

and

$$\begin{aligned}
\left[\left(\pi_i \circ \eta_{\mathcal{R}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{L}}\right)\right](x) &= \bigwedge_{x=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{R}}\right)(a) \vee \left(\pi_i \circ \eta_{\mathcal{L}}\right)(b)\right\} \\
&\leqslant \left(\pi_i \circ \eta_{\mathcal{R}}\right)(xy) \vee \left(\pi_i \circ \eta_{\mathcal{L}}\right)(x) \\
&\leqslant \left(\pi_i \circ \eta_{\mathcal{R}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{L}}\right)(x).
\end{aligned}$$

From a similar proof of the case above, it follows that $\left[\left(\pi_i \circ \nu_{\mathcal{R}}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{L}}\right)\right](x) \leqslant \left(\pi_i \circ \nu_{\mathcal{R}}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{L}}\right)(x)$. We find that $\mathcal{R} \sqcap \mathcal{L} \sqsubseteq \mathcal{R} \diamond \mathcal{L}$. Hence, $\mathcal{R} \sqcap \mathcal{L} = \mathcal{R} \diamond \mathcal{L}$.

Conversely, let L be a left ideal and R be a right ideal of S. Then, $RL \subseteq R \cap L$. Let $x \in R \cap L$. Regarding Theorem 1, we have $\mathcal{C}_L$ and $\mathcal{C}_R$ that are an m-PPFL and an m-PPFR of S, respectively. Regarding the given assumption and Lemma 1, we obtain $\mathcal{C}_{RL} = \mathcal{C}_R \diamond \mathcal{C}_L = \mathcal{C}_R \sqcap \mathcal{C}_L = \mathcal{C}_{R \cap L}$. So, $\mathcal{C}_{RL}(x) = \mathcal{C}_{R \cap L}(x) = (\mathbf{1}, \mathbf{0}, \mathbf{0})$; that is, $x \in RL$. This shows that $R \cap L \subseteq RL$. Thus, $RL = R \cap L$. Regarding Lemma 3, we find that S is regular. □

Theorem 5. *Let S be a semigroup. Then, S is regular if and only if $\mathcal{B} = \mathcal{B} \diamond \mathcal{S} \diamond \mathcal{B}$ for every m-PPFB $\mathcal{B} = \left(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}}\right)$ of S.*

Proof. Assume that S is regular. Let $\mathcal{B} = \left(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}}\right)$ be an m-PPFB of S. Regarding Theorem 3, we have $\mathcal{B} \diamond \mathcal{S} \diamond \mathcal{B} \sqsubseteq \mathcal{B}$. Next, let $x \in S$. Then, $y \in S$ exists, such that $x = xyx$. For any $i = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
&\Big[\Big(\pi_i \circ \mu_{\mathcal{B}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{S}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)\Big](x) \\
&\qquad = \bigvee_{x=ab} \Big\{\Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(a) \wedge \Big[\Big(\pi_i \circ \mu_{\mathcal{S}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)\Big](b)\Big\} \\
&\qquad \geqslant \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x) \wedge \Big[\Big(\pi_i \circ \mu_{\mathcal{S}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)\Big](yx) \\
&\qquad = \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x) \wedge \bigvee_{yx=cd} \Big\{\Big(\pi_i \circ \mu_{\mathcal{S}}\Big)(c) \wedge \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(d)\Big\} \\
&\qquad \geqslant \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x) \wedge \Big\{\Big(\pi_i \circ \mu_{\mathcal{S}}\Big)(y) \wedge \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x)\Big\} \\
&\qquad = \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x) \\
&\qquad = \Big(\pi_i \circ \mu_{\mathcal{B}}\Big)(x)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big[\Big(\pi_i \circ \eta_{\mathcal{B}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{S}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)\Big](x) \\
&\qquad = \bigwedge_{x=ab} \Big\{\Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(a) \vee \Big[\Big(\pi_i \circ \eta_{\mathcal{S}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)\Big](b)\Big\} \\
&\qquad \leqslant \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x) \vee \Big[\Big(\pi_i \circ \eta_{\mathcal{S}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)\Big](yx) \\
&\qquad = \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x) \vee \bigwedge_{yx=cd} \Big\{\Big(\pi_i \circ \eta_{\mathcal{S}}\Big)(c) \vee \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(d)\Big\} \\
&\qquad \leqslant \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x) \vee \Big\{\Big(\pi_i \circ \eta_{\mathcal{S}}\Big)(y) \vee \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x)\Big\} \\
&\qquad = \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x) \\
&\qquad = \Big(\pi_i \circ \eta_{\mathcal{B}}\Big)(x).
\end{aligned}
$$

Similarly, we find that $\Big[\Big(\pi_i \circ \nu_{\mathcal{B}}\Big) \diamond \Big(\pi_i \circ \nu_{\mathcal{S}}\Big) \diamond \Big(\pi_i \circ \nu_{\mathcal{B}}\Big)\Big](x) \leqslant \Big(\pi_i \circ \nu_{\mathcal{B}}\Big)(x)$. This shows that $\mathcal{B} \sqsubseteq \mathcal{B} \diamond \mathcal{S} \diamond \mathcal{B}$. Therefore, $\mathcal{B} = \mathcal{B} \diamond \mathcal{S} \diamond \mathcal{B}$.

Conversely, let A be a bi-ideal of S. Then, $A \subseteq ASA$ and also $\mathcal{C}_A$ is an m-PPFB of S. Next, let $x \in ASA$. Regarding Theorem 1 and the hypothesis, we have $\mathcal{C}_A = \mathcal{C}_A \diamond \mathcal{S} \diamond \mathcal{C}_A = \mathcal{C}_{ASA}$. Thus, $\mathcal{C}_A(x) = \mathcal{C}_{ASA}(x) = \Big(\mathbf{1}, \mathbf{0}, \mathbf{0}\Big)$. Also, $x \in A$. Hence, $ASA \subseteq A$. It follows that $A = ASA$. Regarding Lemma 4, we find that S is regular. $\square$

The following theorem is obtained by Proposition 1 and Theorem 5.

Theorem 6. *Let S be a semigroup. Then, S is regular if and only if $\mathcal{G} = \mathcal{G} \diamond \mathcal{S} \diamond \mathcal{G}$ for every m-PPFGB $\mathcal{G} = \Big(\mu_{\mathcal{G}}, \eta_{\mathcal{G}}, \nu_{\mathcal{G}}\Big)$ of S.*

Theorem 7. *In semigroup S, the following conditions are equivalent:*

(i) *S is regular;*

(ii) *$\mathcal{G} \sqcap \mathcal{A} = \mathcal{G} \diamond \mathcal{A} \diamond \mathcal{G}$ for each m-PPFI $\mathcal{A} = \Big(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\Big)$ and each m-PPFGB $\mathcal{G} = \Big(\mu_{\mathcal{G}}, \eta_{\mathcal{G}}, \nu_{\mathcal{G}}\Big)$ of S;*

(iii) *$\mathcal{B} \sqcap \mathcal{A} = \mathcal{B} \diamond \mathcal{A} \diamond \mathcal{B}$ for each m-PPFI $\mathcal{A} = \Big(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\Big)$ and each m-PPFB $\mathcal{B} = \Big(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}}\Big)$ of S.*

Proof. (i) $\Rightarrow$ (ii) Assume that S is regular. Let $\mathcal{A} = \left(\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}}\right)$ and $\mathcal{G} = \left(\mu_{\mathcal{G}}, \eta_{\mathcal{G}}, \nu_{\mathcal{G}}\right)$ be an m-PPFI and an m-PPFGB of S, respectively. Regarding Lemma 2 and Theorem 3, we have

$$\mathcal{G} \diamond \mathcal{A} \diamond \mathcal{G} \sqsubseteq \mathcal{G} \diamond \mathcal{S} \diamond \mathcal{G} \sqsubseteq \mathcal{G} \text{ and } \mathcal{G} \diamond \mathcal{A} \diamond \mathcal{G} \sqsubseteq (\mathcal{S} \diamond \mathcal{A}) \diamond \mathcal{S} \sqsubseteq \mathcal{A} \diamond \mathcal{S} \sqsubseteq \mathcal{A}.$$

That is, $\mathcal{G} \diamond \mathcal{A} \diamond \mathcal{G} \sqsubseteq \mathcal{G} \sqcap \mathcal{A}$. Now, let $x \in S$. Then, $y \in S$ exists, such that $x = xyx$, and so $x = xyxyx$. For any $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
&\left[\left(\pi_i \circ \mu_{\mathcal{G}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{G}}\right)\right](x) \\
&\quad = \bigvee_{x=ab} \left\{\left[\left(\pi_i \circ \mu_{\mathcal{G}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{A}}\right)\right](a) \wedge \left(\pi_i \circ \mu_{\mathcal{G}}\right)(b)\right\} \\
&\quad \geqslant \left[\left(\pi_i \circ \mu_{\mathcal{G}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{A}}\right)\right](xyxy) \wedge \left(\pi_i \circ \mu_{\mathcal{G}}\right)(x) \\
&\quad = \bigvee_{xyxy=cd} \left\{\left(\pi_i \circ \mu_{\mathcal{G}}\right)(c) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(d)\right\} \wedge \left(\pi_i \circ \mu_{\mathcal{G}}\right)(x) \\
&\quad = \left\{\left(\pi_i \circ \mu_{\mathcal{G}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(yxy)\right\} \wedge \left(\pi_i \circ \mu_{\mathcal{G}}\right)(x) \\
&\quad \geqslant \left\{\left(\pi_i \circ \mu_{\mathcal{G}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x)\right\} \wedge \left(\pi_i \circ \mu_{\mathcal{G}}\right)(x) \\
&\quad = \left(\pi_i \circ \mu_{\mathcal{G}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{A}}\right)(x)
\end{aligned}$$

and

$$\begin{aligned}
&\left[\left(\pi_i \circ \eta_{\mathcal{G}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{G}}\right)\right](x) \\
&\quad = \bigwedge_{x=ab} \left\{\left[\left(\pi_i \circ \eta_{\mathcal{G}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{A}}\right)\right](a) \vee \left(\pi_i \circ \eta_{\mathcal{G}}\right)(b)\right\} \\
&\quad \leqslant \left[\left(\pi_i \circ \eta_{\mathcal{G}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{A}}\right)\right](xyxy) \vee \left(\pi_i \circ \eta_{\mathcal{G}}\right)(x) \\
&\quad = \bigwedge_{xyxy=cd} \left\{\left(\pi_i \circ \eta_{\mathcal{G}}\right)(c) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(d)\right\} \vee \left(\pi_i \circ \eta_{\mathcal{G}}\right)(x) \\
&\quad = \left\{\left(\pi_i \circ \eta_{\mathcal{G}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(yxy)\right\} \vee \left(\pi_i \circ \eta_{\mathcal{G}}\right)(x) \\
&\quad \leqslant \left\{\left(\pi_i \circ \eta_{\mathcal{G}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x)\right\} \vee \left(\pi_i \circ \eta_{\mathcal{G}}\right)(x) \\
&\quad = \left(\pi_i \circ \eta_{\mathcal{G}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{A}}\right)(x).
\end{aligned}$$

Similarly, we can show that $\left[\left(\pi_i \circ \nu_{\mathcal{G}}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{A}}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{G}}\right)\right](x) \leqslant \left(\pi_i \circ \nu_{\mathcal{G}}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{A}}\right)(x)$. This shows that $\mathcal{G} \sqcap \mathcal{A} \sqsubseteq \mathcal{G} \diamond \mathcal{A} \diamond \mathcal{G}$. Therefore, $\mathcal{G} \sqcap \mathcal{A} = \mathcal{G} \diamond \mathcal{A} \diamond \mathcal{G}$.

(ii) $\Rightarrow$ (iii) Since every m-PPFB is also an m-PPFGB of S, we can conclude that condition (iii) has been completed.

(iii) $\Rightarrow$ (i) Let $\mathcal{B} = \left(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}}\right)$ be any m-PPFB of S. The m-PPFS $\mathcal{S}$ is an m-PPFI of S, and, by assumption, we have $\mathcal{B} = \mathcal{B} \sqcap \mathcal{S} = \mathcal{B} \diamond \mathcal{S} \diamond \mathcal{B}$. Consequently, for Theorem 5, S is regular. □

Theorem 8. *The following statements are equivalent in semigroup S:*

(i) *S is regular;*

(ii) *$\mathcal{G} \sqcap \mathcal{L} \sqsubseteq \mathcal{G} \diamond \mathcal{L}$ for every m-PPFL $\mathcal{L} = \left(\mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\right)$ and every m-PPFGB $\mathcal{G} = \left(\mu_{\mathcal{G}}, \eta_{\mathcal{G}}, \nu_{\mathcal{G}}\right)$ of S;*

(iii) $\mathcal{B} \sqcap \mathcal{L} \sqsubseteq \mathcal{B} \diamond \mathcal{L}$ *for every m-PPFL* $\mathcal{L} = \left(\mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\right)$ *and every m-PPFB* $\mathcal{B} = \left(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}}\right)$ *of S.*

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{L} = \left(\mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\right)$ and $\mathcal{G} = \left(\mu_{\mathcal{G}}, \eta_{\mathcal{G}}, \nu_{\mathcal{G}}\right)$ be an m-PPFL and an m-PPFGB of S, respectively. Let $x \in S$. Then, $y \in S$ exists, such that $x = xyx$. For each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\left[\left(\pi_i \circ \mu_{\mathcal{G}}\right) \diamond \left(\pi_i \circ \mu_{\mathcal{L}}\right)\right](x) &= \bigvee_{x=ab} \left\{\left(\pi_i \circ \mu_{\mathcal{G}}\right)(a) \wedge \left(\pi_i \circ \mu_{\mathcal{L}}\right)(b)\right\} \\ &\geqslant \left(\pi_i \circ \mu_{\mathcal{G}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{L}}\right)(yx) \\ &\geqslant \left(\pi_i \circ \mu_{\mathcal{G}}\right)(x) \wedge \left(\pi_i \circ \mu_{\mathcal{L}}\right)(x)\end{aligned}$$

and

$$\begin{aligned}\left[\left(\pi_i \circ \eta_{\mathcal{G}}\right) \diamond \left(\pi_i \circ \eta_{\mathcal{L}}\right)\right](x) &= \bigwedge_{x=ab} \left\{\left(\pi_i \circ \eta_{\mathcal{G}}\right)(a) \vee \left(\pi_i \circ \eta_{\mathcal{L}}\right)(b)\right\} \\ &\leqslant \left(\pi_i \circ \eta_{\mathcal{G}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{L}}\right)(yx) \\ &\leqslant \left(\pi_i \circ \eta_{\mathcal{G}}\right)(x) \vee \left(\pi_i \circ \eta_{\mathcal{L}}\right)(x).\end{aligned}$$

Given a similar proof of the case above, we can show that $\left[\left(\pi_i \circ \nu_{\mathcal{G}}\right) \diamond \left(\pi_i \circ \nu_{\mathcal{L}}\right)\right](x) \leqslant \left(\pi_i \circ \nu_{\mathcal{G}}\right)(x) \vee \left(\pi_i \circ \nu_{\mathcal{L}}\right)(x)$. Thus, $\mathcal{G} \sqcap \mathcal{L} \sqsubseteq \mathcal{G} \diamond \mathcal{L}$.

(ii) $\Rightarrow$ (iii) It is clear.

(iii) $\Rightarrow$ (i) Let $\mathcal{L} = \left(\mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\right)$ and $\mathcal{R} = \left(\mu_{\mathcal{R}}, \eta_{\mathcal{R}}, \nu_{\mathcal{R}}\right)$ be an m-PPFL and an m-PPFR of S, respectively. Next, the m-PPFR $\mathcal{R}$ is also an m-PPFB of S. Regarding the hypothesis, we have $\mathcal{R} \sqcap \mathcal{L} \sqsubseteq \mathcal{R} \diamond \mathcal{L}$. On the other hand, $\mathcal{R} \diamond \mathcal{L} \sqsubseteq \mathcal{R} \sqcap \mathcal{L}$ always. This implies that $\mathcal{R} \sqcap \mathcal{L} = \mathcal{R} \diamond \mathcal{L}$. Consequently, for Theorem 4, S is regular. $\square$

The following theorem can be presented similarly to Theorem 8.

Theorem 9. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is regular;*

(ii) $\mathcal{R} \sqcap \mathcal{G} \sqsubseteq \mathcal{R} \diamond \mathcal{G}$ *for every m-PPFR* $\mathcal{R} = \left(\mu_{\mathcal{R}}, \eta_{\mathcal{R}}, \nu_{\mathcal{R}}\right)$ *and every m-PPFGB* $\mathcal{G} = \left(\mu_{\mathcal{G}}, \eta_{\mathcal{G}}, \nu_{\mathcal{G}}\right)$ *of S;*

(iii) $\mathcal{R} \sqcap \mathcal{B} \sqsubseteq \mathcal{R} \diamond \mathcal{B}$ *for every m-PPFR* $\mathcal{R} = \left(\mu_{\mathcal{R}}, \eta_{\mathcal{R}}, \nu_{\mathcal{R}}\right)$ *and every m-PPFB* $\mathcal{B} = \left(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}}\right)$ *of S.*

Theorem 10. *In semigroup S, the following conditions are equivalent:*

(i) *S is regular;*

(ii) $\mathcal{R} \sqcap \mathcal{G} \sqcap \mathcal{L} \sqsubseteq \mathcal{R} \diamond \mathcal{G} \diamond \mathcal{L}$ *for each m-PPFL* $\mathcal{L} = \left(\mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\right)$*, each m-PPFR* $\mathcal{R} = \left(\mu_{\mathcal{R}}, \eta_{\mathcal{R}}, \nu_{\mathcal{R}}\right)$*, and each m-PPFGB* $\mathcal{G} = \left(\mu_{\mathcal{G}}, \eta_{\mathcal{G}}, \nu_{\mathcal{G}}\right)$ *of S;*

(iii) $\mathcal{R} \sqcap \mathcal{B} \sqcap \mathcal{L} \sqsubseteq \mathcal{R} \diamond \mathcal{B} \diamond \mathcal{L}$ *for each m-PPFL* $\mathcal{L} = \left(\mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\right)$*, each m-PPFR* $\mathcal{R} = \left(\mu_{\mathcal{R}}, \eta_{\mathcal{R}}, \nu_{\mathcal{R}}\right)$*, and each m-PPFB* $\mathcal{B} = \left(\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{G}}\right)$ *of S.*

Proof. (i) $\Rightarrow$ (ii) Assume that S is regular. Let $\mathcal{L} = \left(\mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\right)$, $\mathcal{R} = \left(\mu_{\mathcal{R}}, \eta_{\mathcal{R}}, \nu_{\mathcal{R}}\right)$, and $\mathcal{G} = \left(\mu_{\mathcal{G}}, \eta_{\mathcal{G}}, \nu_{\mathcal{G}}\right)$ be an m-PPFL, an m-PPFR, and an m-PPFGB of S, respectively. For any

$x \in S$, we have $x = xyx$ for some $y \in S$. Also, $x = xyxyxyx$. For every $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
&\Big[\Big(\pi_i \circ \mu_{\mathcal{R}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{G}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{L}}\Big)\Big](x) \\
&\quad = \bigvee_{x=ab} \Big\{\Big[\Big(\pi_i \circ \mu_{\mathcal{R}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{G}}\Big)\Big](a) \wedge \Big(\pi_i \circ \mu_{\mathcal{L}}\Big)(b)\Big\} \\
&\quad \geqslant \Big[\Big(\pi_i \circ \mu_{\mathcal{R}}\Big) \diamond \Big(\pi_i \circ \mu_{\mathcal{G}}\Big)\Big](xyxyx) \wedge \Big(\pi_i \circ \mu_{\mathcal{L}}\Big)(yx) \\
&\quad \geqslant \bigvee_{xyxyx=cd} \Big\{\Big(\pi_i \circ \mu_{\mathcal{R}}\Big)(c) \wedge \Big(\pi_i \circ \mu_{\mathcal{G}}\Big)(d)\Big\} \wedge \Big(\pi_i \circ \mu_{\mathcal{L}}\Big)(x) \\
&\quad \geqslant \Big\{\Big(\pi_i \circ \mu_{\mathcal{R}}\Big)(xy) \wedge \Big(\pi_i \circ \mu_{\mathcal{G}}\Big)(xyx)\Big\} \wedge \Big(\pi_i \circ \mu_{\mathcal{L}}\Big)(x) \\
&\quad \geqslant \Big(\pi_i \circ \mu_{\mathcal{R}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{G}}\Big)(x) \wedge \Big(\pi_i \circ \mu_{\mathcal{L}}\Big)(x)
\end{aligned}$$

and

$$\begin{aligned}
&\Big[\Big(\pi_i \circ \eta_{\mathcal{R}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{G}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{L}}\Big)\Big](x) \\
&\quad = \bigwedge_{x=ab} \Big\{\Big[\Big(\pi_i \circ \eta_{\mathcal{R}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{G}}\Big)\Big](a) \vee \Big(\pi_i \circ \eta_{\mathcal{L}}\Big)(b)\Big\} \\
&\quad \leqslant \Big[\Big(\pi_i \circ \eta_{\mathcal{R}}\Big) \diamond \Big(\pi_i \circ \eta_{\mathcal{G}}\Big)\Big](xyxyx) \vee \Big(\pi_i \circ \eta_{\mathcal{L}}\Big)(yx) \\
&\quad \leqslant \bigvee_{xyxyx=cd} \Big\{\Big(\pi_i \circ \eta_{\mathcal{R}}\Big)(c) \vee \Big(\pi_i \circ \eta_{\mathcal{G}}\Big)(d)\Big\} \vee \Big(\pi_i \circ \eta_{\mathcal{L}}\Big)(x) \\
&\quad \leqslant \Big\{\Big(\pi_i \circ \eta_{\mathcal{R}}\Big)(xy) \vee \Big(\pi_i \circ \eta_{\mathcal{G}}\Big)(xyx)\Big\} \vee \Big(\pi_i \circ \eta_{\mathcal{L}}\Big)(x) \\
&\quad \leqslant \Big(\pi_i \circ \eta_{\mathcal{R}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{G}}\Big)(x) \vee \Big(\pi_i \circ \eta_{\mathcal{L}}\Big)(x).
\end{aligned}$$

Similarly, we can show that $\Big[\Big(\pi_i \circ \nu_{\mathcal{R}}\Big) \diamond \Big(\pi_i \circ \nu_{\mathcal{G}}\Big) \diamond \Big(\pi_i \circ \nu_{\mathcal{L}}\Big)\Big](x) \leqslant \Big(\pi_i \circ \nu_{\mathcal{R}}\Big)(x) \vee \Big(\pi_i \circ \nu_{\mathcal{G}}\Big)(x) \vee \Big(\pi_i \circ \nu_{\mathcal{L}}\Big)(x)$. It follows that $\mathcal{R} \sqcap \mathcal{G} \sqcap \mathcal{L} \sqsubseteq \mathcal{R} \diamond \mathcal{G} \diamond \mathcal{L}$.

(ii) $\Rightarrow$ (iii) Straightforward.

(iii) $\Rightarrow$ (i) Let $\mathcal{L} = \Big(\mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\Big)$ and $\mathcal{R} = \Big(\mu_{\mathcal{R}}, \eta_{\mathcal{R}}, \nu_{\mathcal{R}}\Big)$ be an m-PPFL and an m-PPFR of S, respectively. Since the m-PPFS $\mathcal{S}$ itself is an m-PPFB of S and regarding the hypothesis, we have $\mathcal{R} \sqcap \mathcal{L} = \mathcal{R} \sqcap \mathcal{S} \sqcap \mathcal{L} \sqsubseteq \mathcal{R} \diamond \mathcal{S} \diamond \mathcal{L} \sqsubseteq \mathcal{R} \diamond \mathcal{L}$. On the other hand, $\mathcal{R} \diamond \mathcal{L} \sqsubseteq \mathcal{R} \sqcap \mathcal{L}$. Hence, $\mathcal{R} \diamond \mathcal{L} = \mathcal{R} \sqcap \mathcal{L}$. We conclude that for Theorem 4, S is regular. $\square$

5. Conclusions

In this study, we successfully applied the concept of m-PPFSs to the study of semigroups, introducing and investigating fundamental algebraic structures such as m-PPFSubs, m-PPFLs, m-PPFRs, m-PPFIs, m-PPFBs, and m-PPFGBs in semigroups. Our findings demonstrate that every m-PPFL (m-PPFR) of semigroups is also an m-PPFB, though the opposite does not always hold. Moreover, we established that every m-PPFB is also an m-PPFGB in semigroups, but the reverse is not necessarily true, as shown in Examples 1 and 2, respectively. Additionally, we investigated the characterizations of different kinds of m-PPFIs in semigroups related to particular m-PPFS features. Given the fact that the concept of m-PPFSs is an extension of the concept of PFSs, it can be seen that the results obtained from this research are comprehensive in terms of the results of the classification of regular semigroups by PFSs in reference [9]. Furthermore, we studied the connections between various m-PPFIs in regular semigroups. Ultimately, the regular semigroups are characterized by their m-PPFSubs, m-PPFLs, m-PPFRs, m-PPFIs, m-PPFBs, and m-PPFGBs. In future work, we intend to indicate various types of regularities in semigroups, particularly focusing on intra-regular and weakly regular semigroups. We aim to achieve these objectives by utilizing different forms of m-PPFIs or by investigating these concepts within other algebraic structures, such as ordered semigroups, Γ-semigroups, or semihypergroups.

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Abbreviations

The following abbreviations are used in this research:

FS	Fuzzy set
IFS	Intuitionistic fuzzy set
PFS	Picture fuzzy set
BFS	Bipolar fuzzy set
m-PFS	m-Polar fuzzy set
m-PPFS	m-Polar picture fuzzy set
m-PPCF	m-Polar picture characteristic function
m-PPFL	m-Polar picture fuzzy left ideal
m-PPFR	m-Polar picture fuzzy right ideal
m-PPFI	m-Polar picture fuzzy ideal
m-PPFB	m-Polar picture fuzzy bi-ideal
m-PPFGB	m-Polar picture fuzzy generalized bi-ideal

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Article

Belief Entropy-Based MAGDM Algorithm Under Double Hierarchy Quantum-like Bayesian Networks and Its Application to Wastewater Reuse

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Abstract: The traditional multi-attribute group decision-making (MAGDM) method easily ignores the interference effect among decision-makers (DMs), while quantum theory can effectively portray the uncertainty in the decision-making process and quantify the preference interference among DMs. The asymmetry of evaluation information in social networks can have a significant impact on decision-making. In this paper, a quantum MAGDM algorithm based on probabilistic linguistic term sets (PLTSs) and a quantum-like Bayesian network (QLBN) is proposed (PL-QLBN), utilizing quantum theory and social network concepts and introducing a novel method for calculating interference effects based on belief entropy. Firstly, a complete trust network is constructed based on the probabilistic linguistic trust transfer operator and the minimum path method. A trust aggregation method, considering interference effects, is proposed for the QLBN to determine the DM weights. Next, the attribute weights are calculated based on the entropy weight method. Then, a probabilistic linguistic MAGDM considering interference effects is proposed based on the QLBN. Finally, the feasibility and validity of the provided method are verified through Hefei City's selection of wastewater reuse alternatives.

Keywords: quantum-like Bayesian network; belief entropy; social network; probabilistic linguistic term sets; multi-attribute group decision-making

1. Introduction

With the rapid development of the social economy and science and technology, the modern decision-making environment has become increasingly complex, and many problems involve interdisciplinary knowledge, requiring multiple DMs to conduct systematic evaluations based on multi-attribute criteria. Multi-attribute group decision-making methods have been widely used in engineering management, healthcare, finance, and other fields, such as risk assessment [1], emergency decision-making [2], and investment selection [3], by integrating experts' opinions and constructing a normalized evaluation model to achieve solution preference. Against this backdrop, Wang et al. [4] proposed the GDMD-PROMETHEE method for large-scale MAGDM based on a generalized extended hybrid distance measure between PLTSs. Wang et al. [5] proposed novel operations for PLTSs based on Dombi's T-conorm and T-norm and developed a new MAGDM method based on the TODIM. Wang et al. [6] extended the large-scale group decision-making problem under probabilistic linguistic environments to finite interval values and proposed a novel extended ELECTRE III method based on bidirectional projection and regret theory. These studies provide practical tools and critical theoretical references for addressing MAGDM problems under fuzzy and uncertain environments.

Qualitative language is often employed to convey nuanced evaluations in complex decision-making processes. Zadeh [7] proposed a fuzzy linguistic approach to describe uncertainty. Pang et al. [8] further proposed PLTS to characterize decision-making preferences more accurately through multiple terms and their associated probabilities. PLTS have been widely applied to MAGDM problems in various domains such as healthcare decision-making [9], donation channel selection [10], and site planning [11]. Trust interactions among DMs also affect decision outcomes. Gao et al. [12] proposed a probabilistic linguistic trust function (PLTF) to construct a comprehensive social trust network through the t-norm trust propagation operator and path-weighted averaging operator, thereby quantitatively analyzing the impact of trust behavior on decision-making.

However, the majority of MAGDM methods treat DMs as independent individuals without taking into account how they interact with one another. Sociologists contend that human decision-making is not totally logical. People's decision-making behaviors interfere with one another when decision paths are not observed [13]. Quantum probability theory (QPT) is an emerging framework for constructing probabilistic systems [14]. It effectively captures uncertainty in human decision-making and explains cognitive paradoxes that classical probability theory (CPT) struggles to address, such as the conjunction fallacy [15] and the sequence effect [16]. Building upon this foundation, scholars have conducted numerous extension studies: Tucci [17] proposed replacing classical probabilities with probability amplitudes, thereby extending classical Bayesian networks into quantum-like Bayesian networks; Moreira and Wichert [18] introduced a trigonometric similarity heuristic to compute interference effects in QLBN; Huang et al. [19] adopted an uncertainty perspective, proposing a method for calculating interference effects based on belief entropy; She et al. [20] considered interference effects between attributes within the QLBN framework and computed them using belief entropy. In recent years, multiple scholars have applied quantum probability theory to develop decision-making frameworks, yielding reliable research outcomes [21,22].

According to the analysis above, there are still gaps in the current research. Firstly, DMs find it difficult to conduct assessments using precise numerical data. Secondly, trust interference caused by the subjectivity of various DMs can damage the trust relationship between DMs, which calls for more research. Additionally, the subjective beliefs of DMs may interfere with one another during the decision-making process, potentially influencing the final decision. On this basis, a PL-QLBN algorithm to solve the MAGDM problems is proposed. The main contributions of this paper are as follows:

- Constructing a DM trust network based on PLTF, a trust aggregation method based on QLBN is proposed in the process of trust propagation, which considers the interference of trust transmission and makes the determination of DM weights more reasonable.
- Integrating evaluation information for DMs based on QLBN, which can fully consider the interference between DMs and make the decision-making results more reasonable and real.
- A new computational method based on belief entropy for calculating interference effects is proposed. Instead of looking for an overall uncertainty between DMs, the belief entropy and the interference term are calculated one-to-one, so that the computed results will be more accurate.

The rest of the paper is structured as follows: Section 2 gives some basic knowledge about quantum probability theory, confidence entropy, social networks, and PLTSs. Section 3 constructs a PL-QLBN algorithm. Section 4 presents an arithmetic example of wastewater reuse alternative selection, accompanied by a comparative analysis that illustrates the effectiveness and innovation of the method presented in this paper. Section 5 concludes the paper.

2. Preliminaries

This section briefly introduces the core concepts and definitions of this paper.

2.1. Hilbert Space and Quantum Probability Theory

In quantum probability theory, the Hilbert space is defined as a complex vector space whose structure is determined by a complete set of orthonormal basis vectors. Rather than being modeled as sets, events are conceptualized in terms of subspaces within this Hilbert space. Each such event is mathematically expressed using complex numbers, with every event characterized by two components corresponding to its real and imaginary parts. These events may exhibit mutual exclusivity or may correspond to intersections and unions of underlying fundamental states [23].

Figure 1a is a two-dimensional Hilbert space in QPT, which takes the circle on the coordinate system as the space, and the two basic states are defined as follows:

1. fully believe in B (noted as $|B\rangle$);
2. fully believe in G (noted as $|G\rangle$).

where $|B\rangle = (1,0), |G\rangle = (0,1)$.

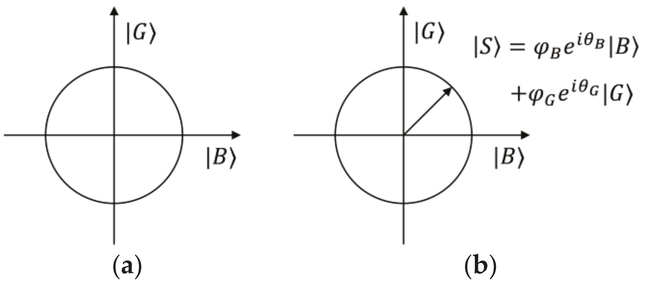


Figure 1. Two-dimensional Hilbert space: (**a**) The Hilbert space in QPT; (**b**) The two possible states of the superposition state in Hilbert space.

An event in a two-dimensional Hilbert space of quantum probability is depicted in Figure 1b as a superposition of fundamental events. Then $|S\rangle$ is represented as

$$|S\rangle = \varphi_B e^{i\theta_B}|B\rangle + \varphi_G e^{i\theta_G}|G\rangle, \tag{1}$$

where $\varphi_B e^{i\theta_B}$ and $\varphi_G e^{i\theta_G}$ represent the probability amplitudes of the fundamental states $|B\rangle$ and $|G\rangle$, respectively, while $e^{i\theta}$ denotes the phase of the amplitude.

A relationship between classical probabilities and probability amplitudes is provided by Born's rule [24], and a probability can be interpreted as the squared value of the probability amplitude [25]. Assume that there are n events $A_i(i = 1, 2, \cdots, n)$. Each event's probability is $p(A_i)$, which can be expressed as follows:

$$p(A_i) = \left|\varphi_{A_i} e^{i\theta_{A_i}}\right|^2 = \varphi_{A_i} e^{i\theta_{A_i}} \cdot \varphi_{A_i} e^{-i\theta_{A_i}} = \varphi_{A_i}^2 (i = 1, 2, \cdots, n), \tag{2}$$

where $\varphi_{A_i} e^{i\theta_{A_i}}$ is the probability amplitude associated with state $|A_i\rangle$.

The sum of the squares of the probability amplitudes of all events is equal to 1, as follows:

$$\sum_{i=1}^{n} p(A_i) = \sum_{i=1}^{n} \left|\varphi_{A_i} e^{i\theta_{A_i}}\right|^2 = \sum_{i=1}^{n} \varphi_{A_i}^2 = 1. \tag{3}$$

2.2. Quantum-like Bayesian Network

The classical Bayesian framework uses a Markov process to represent the decision-making process [26], as seen in Figure 2a. The probability of the final decision state of an event is the sum of the probabilities of all possible paths, in which the opinions of all DMs are independent of each other and do not influence each other. Figure 2a shows that there are two ways to get from A to D: via B or C. The full probability rule states that the probability of D being selected is

$$p(D) = p(D|B)p(B) + p(D|C)p(C), \tag{4}$$

where $p(B) + p(C) = 1$.

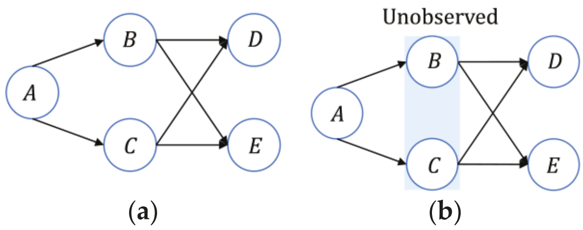


Figure 2. Decision-making in a Markov and quantum process: (**a**) Markov process; (**b**) Quantum process.

Quantum-like Bayesian networks are probabilistic inference models that extend classical Bayesian networks by employing complex probability amplitudes for representation, with their structure depicted in Figure 2b. In this framework, the system state is described by a superposition of fundamental states, represented by wave functions, and all events are modeled as subspaces of a Hilbert space. The probability of event D being selected is given by

$$\begin{aligned} p(D) &= \left|\varphi_B e^{i\theta_B}\varphi_{D\mid B}e^{i\theta_{D\mid B}} + \varphi_C e^{i\theta_C}\varphi_{D\mid C}e^{i\theta_{D\mid C}}\right|^2 \\ &= \left|\varphi_B\varphi_{D\mid B}\right|^2 + \left|\varphi_C\varphi_{D\mid C}\right|^2 + 2\left|\varphi_B\varphi_{D|B}\right|\left|\varphi_C\varphi_{D|C}\right|\cos(\theta_{12}), \end{aligned} \tag{5}$$

where $\theta_1 = \theta_B + \theta_{D|B}$, $\theta_2 = \theta_C + \theta_{D|C}$, $\cos(\theta_{12}) = \frac{e^{i(\theta_1-\theta_2)}+e^{i(\theta_2-\theta_1)}}{2}$.

The term $2\left|\varphi_B\varphi_{D|B}\right|\left|\varphi_C\varphi_{D|C}\right|\cos(\theta_{12})$ is called the interference term in the QLBN. It comes from the interaction between two pathways, with $\cos(\theta_{12}) \in [-1, 1]$. The value of this cosine determines the nature of the interference: when $\cos(\theta_{12}) \in [-1, 0)$, it indicates a negative interference effect; when $\cos(\theta_{12}) \in (0, 1]$, it indicates a positive interference effect; when $\cos(\theta_{12}) = 0$, the quantum-like Bayesian network degenerates into classical Bayesian network.

2.3. Belief Entropy and the Maximum Deng Entropy

Belief entropy serves as a measure for quantifying the uncertainty associated with information. Its value increases with the level of uncertainty in a system. Within this framework, Deng entropy is recognized as a specific form of belief entropy.

Definition 1. ([27]) *Deng entropy can be expressed as*

$$E_d(M) = -\sum\nolimits_{C\subseteq X} M(C)log_2\frac{M(C)}{2^{|C|}-1}, \tag{6}$$

where X denotes the universal set of all possibilities, C is a specific subset (or event) within X, and $M(C)$ is the mass function representing the degree of evidential support for C. The cardinality of C is denoted by $|C|$, and the term $2^{|C|}-1$ gives the number of its potential non-empty states.

Following further research on Deng entropy, Kang and Deng [28] defined its maximum value as follows:

Definition 2. ([28]) *The maximum Deng entropy is achieved if and only if the basic probability assignment (BPA) for any focal element C_i satisfies the following condition:*

$$M(C_i) = \frac{2^{|C_i|}-1}{\sum_i 2^{|C_i|}-1}, i = 1, 2, \cdots, 2^{|X|}-1, \tag{7}$$

where $|X|$ is the cardinality of X. The corresponding value of the maximum Deng entropy is given by

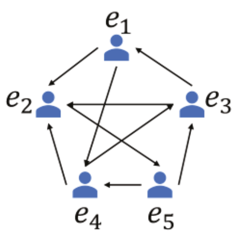
$$E_d^* = -\sum_{i=1}^{2^{|X|}-1} M(C_i) log_2 \frac{M(C_i)}{2^{|C_i|}-1} = log_2 \sum_{i=1}^{2^{|X|}-1} \left(2^{|C_i|}-1\right). \tag{8}$$

2.4. Social Trust Network

The social trust network (STN), as demonstrated by trust propagation and aggregation, is the main focus of current research on social network (SN) completeness analysis. Table 1 illustrates the three representation schemes that STN typically uses [29].

1. Sociometric representation: each element of the matrix corresponds to a trust relationship between the members, where 1 means complete trust and 0 means complete distrust.
2. Graph theoretic representation: the DMs are represented by the nodes in the graph, and the trust relationship between the DMs is represented by the connections between the nodes.
3. Algebraic representation: a number of different relations that represent combinations of relations are identified.

Table 1. Different representation schemes in social network analysis.

Sociometric	Graph	Algebraic
$TR = \begin{bmatrix} - & 1 & 0 & 1 & 0 \\ 0 & - & 0 & 0 & 1 \\ 1 & 1 & - & 0 & 0 \\ 0 & 1 & 1 & - & 0 \\ 0 & 0 & 1 & 1 & - \end{bmatrix}$		$e_1Re_2, e_1Re_4, e_2Re_5,$ $e_3Re_1, e_3Re_2, e_4Re_2,$ $e_4Re_3, e_5Re_3, e_5Re_4$

2.5. Probabilistic Linguistic Term Set and Probabilistic Linguistic Trust Function

The probabilistic language term set (PLTS) builds upon the linguistic term set (LTS) to capture DMs' qualitative assessments and preferences. Each possible term within this set is assigned an occurrence probability—thereby quantitatively expressing the likelihood of each qualitative assessment.

Definition 3. ([8]) *Let $S = \{s_\alpha | \alpha = -\tau, \cdots, -1, 0, 1, \cdots, \tau\}$ be a LTS, a PLTS is defined as*

$$L(p) = \left\{ s_{(k)}\left(p_{(k)}\right) \middle| s_{(k)} \in S, p_{(k)} \geqslant 0, k = 1, 2, \cdots, \#L(p), \sum_{k=1}^{\#L(p)} p_{(k)} \leq 1 \right\}, \tag{9}$$

where $s_{(k)}$ represents the kth linguistic term, $p_{(k)}$ represents its corresponding probability, and $\#L(p)$ is the total number of terms in $L(p)$.

If a PLTS $L(p)$ with $\sum_{k=1}^{\#L(p)} p_{(k)} < 1$, it can be normalized using Equation (10):

$$\hat{L}(p) = \left\{ s_{(k)}\left(\hat{p}_{(k)}\right) \middle| s_{(k)} \in S, \hat{p}_{(k)} \geqslant 0, k = 1, 2, \cdots, \#L(p), \sum_{k=1}^{\#L(p)} \hat{p}_{(k)} = 1 \right\}, \quad (10)$$

where $\hat{p}_{(k)} = p_{(k)} / \sum_{k=1}^{\#L(p)} p_{(k)}, k = 1, 2, \cdots, \#L(p)$.

Wu et al. [30] introduced the following definition of the expectation function in order to accomplish the sorting of PLTS:

Definition 4. ([30]) *Let $L(p) = \left\{ s_{(k)}\left(p_{(k)}\right) \middle| s_{(k)} \in S, p_{(k)} \geqslant 0, k = 1, 2, \cdots, \#L(p), \sum_{k=1}^{\#L(p)} p_{(k)} \leq 1 \right\}$ be a PLTS, r_k is the subscript of the $s_{(k)}$, the expectation of $L(p)$ is defined as*

$$E(L(p)) = \sum_{k=1}^{\#L(p)} \left[\frac{r_k + \tau}{2\tau} p_{(k)} / \sum_{k=1}^{\#L(p)} p_{(k)} \right]. \quad (11)$$

Based on Definition 4, the comparative ranking of different PLTSs is conducted using the $E(L(p))$. This implies that a PLTS with a higher $E(L(p))$ is considered larger.

On this basis, Gou and Xu [31] postulated equivalent transformation functions g and g^{-1} to enable seamless conversion between linguistic variables and their numerically equivalent membership degrees. For a linguistic term set LTS $S = \{ s_\alpha | \alpha = -\tau, \cdots, -1, 0, 1, \cdots, \tau \}$ and a PLTS $L(p)$, the transformation between any linguistic term s_α and its corresponding membership degree γ is defined by the following equations:

$$g : [-\tau, \tau] \to [0, 1], g(s_\alpha) = \frac{\alpha}{2\tau} + \frac{1}{2} = \gamma, \quad (12)$$

$$g^{-1} : [0, 1] \to [-\tau, \tau], g^{-1}(\gamma) = s_{(2\gamma - 1)\tau} = s_\alpha. \quad (13)$$

Let $L(p) = \left\{ s_{(k)}\left(p_{(k)}\right) \middle| s_{(k)} \in S, p_{(k)} \geqslant 0, k = 1, 2, \cdots, \#L(p), \sum_{k=1}^{\#L(p)} p_{(k)} = 1 \right\}$, $L_1(p) = \left\{ s_{(k_1)}\left(p_{(k_1)}\right) \middle| s_{(k_1)} \in S, p_{(k_1)} \geqslant 0, k_1 = 1, 2, \cdots, \#L_1(p), \sum_{k_1=1}^{\#L_1(p)} p_{(k_1)} = 1 \right\}$ and $L_2(p) = \left\{ s_{(k_2)}\left(p_{(k_2)}\right) \middle| s_{(k_2)} \in S, p_{(k_2)} \geqslant 0, k_2 = 1, 2, \cdots, \#L_2(p), \sum_{k_2=1}^{\#L_2(p)} p_{(k_2)} = 1 \right\}$ be three normalized PLTSs, μ be a positive real number. Using the equivalent transformation function g and g^{-1} given by Equations (12) and (13), the operational laws of PLTS can be defined as follows:

1. $L_1(p) \bigoplus L_2(p) = g^{-1}(\bigcup_{\eta_{k_1}^1 \in g(L_1), \eta_{k_2}^2 \in g(L_2)} \left\{ \left(\eta_{k_1}^1 + \eta_{k_2}^2 - \eta_{k_1}^1 \eta_{k_2}^2 \right) \left(p_{k_1}^1 p_{k_2}^2 \right) \right\});$
2. $L_1(p) \otimes L_2(p) = g^{-1}(\bigcup_{\eta_{k_1}^1 \in g(L_1), \eta_{k_2}^2 \in g(L_2)} \left\{ \left(\eta_{k_1}^1 \eta_{k_2}^2 \right) \left(p_{k_1}^1 p_{k_2}^2 \right) \right\});$
3. $\mu L(p) = g^{-1}(\bigcup_{\eta_k \in g(L)} \{ (1 - (1 - \eta_k)^\mu)(p_k) \})$.

Pang et al. [8] proposed the probabilistic linguistic weighted averaging (PLWA) operator to improve the use of PLTSs in decision-making, which is defined as follows:

Definition 5. ([8]) *Given n normalized PLTSs $L_i(p) = \left\{ s_{(k_i)}\left(p_{(k_i)}\right) \middle| k_i = 1, 2, \cdots, \#L_i(p), \sum_{k_i=1}^{\#L_i(p)} p_{(k_i)} = 1 \right\}$, where $s_{(k_i)}$ and $p_{(k_i)}$ denote the kth linguistic term and its probability separately in $L_i(p)$. $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vectors of $L_i(p) (i = 1, 2 \cdots n)$, $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$. Then PLWA operator can be defined, such that*

$$PLWA(L_1(p), L_2(p), \cdots L_i(p)) = \omega_1 L_1(p) \bigoplus \omega_2 L_2(p) \bigoplus \cdots \bigoplus \omega_i L_i(p). \quad (14)$$

Wu et al. [29] pointed out that human expressions of trust are not merely binary (trust or distrust) but often involve gradations such as high, middle, or low, alongside other preference information. Building on this understanding, Gao et al. [12] introduced the probabilistic linguistic trust function (PLTF) to quantitatively represent these nuanced trust evaluations, which is defined as follows:

Definition 6. ([12]) *Let $H = \{h_\beta | \beta = -\sigma, \cdots, -1, 0, 1, \cdots, \sigma\}$ be an LTS. Then the PLTF is defined as*

$$T(q) = \left\{ h_{(l)}\left(q_{(l)}\right) \middle| h_{(l)} \in H, q_{(l)} \geqslant 0, l = 1, 2, \cdots, \#T(q), \sum_{l=1}^{\#T(q)} q_{(l)} \leqslant 1 \right\}, \tag{15}$$

where $h_{(l)}(q_{(l)})$ is the lth linguistic term $h_{(l)}$ with the probability $q_{(l)}$, and $\#T(q)$ is the number of all linguistic terms in $T(q)$.

If a PLTF $T(q)$with $\sum_{l=1}^{\#T(q)} q_{(l)} < 1$, it can be normalized using Equation (16):

$$\hat{T}(q) = \left\{ h_{(l)}\left(\hat{q}_{(l)}\right) \middle| h_{(l)} \in H, \hat{q}_{(l)} \geqslant 0, l = 1, 2, \cdots, \#T(q), \sum_{l=1}^{\#T(q)} \hat{q}_{(l)} = 1 \right\}, \tag{16}$$

where $\hat{q}_{(l)} = q_{(l)} / \sum_{l=1}^{\#T(q)} q_{(l)}, l = 1, 2, \cdots, \#T(q)$.

Definition 7. ([12]) *Let $T_N(q) = \left\{ h_{(l)}\left(q_{N(l)}\right) \middle| l = 1, 2, \cdots, \#T(q), \sum_{l=1}^{\#T(q)} q_{N(l)} = 1 \right\}$ be a normalized PLTF and m_l be the subscript of the linguistic term $h_{(l)}$. The expectation of $T_N(q)$ is defined as*

$$E(T_N(q)) = \sum_{l=1}^{\#T(q)} \left[\frac{m_l + \sigma}{2\sigma} q_{N(l)} / \sum_{k=1}^{\#T(q)} q_{N(l)} \right]. \tag{17}$$

Some DMs might not directly give a specific DM a trust value in a social trust network. In this instance, the trust relationship between group DMs inside the social network is incomplete. To construct a comprehensive trust network, Gao et al. [12] proposed the probabilistic linguistic trust propagation operator based on t-norm.

Let $e_s \xrightarrow{1} e_{\delta(1)} \xrightarrow{2} e_{\delta(2)} \xrightarrow{3} e_{\delta(3)} \xrightarrow{4} \cdots \xrightarrow{z} e_{\delta(z)} \xrightarrow{z+1} e_t$ be a path for passing trust value from e_s to e_t, where e denotes DM; $z (z \in N^+)$ denotes the number of intermediate nodes of the trust path, the length of the path is equal to $z+1$. Let

$$T_{s,\delta(1)}(q) = \{h_{s,\delta(1)(l_1)}(q_{s,\delta(1)(l_1)}) | h_{s,\delta(1)(l_1)} \in H, q_{s,\delta(1)(l_1)} \geqslant 0, l_1 = 1,2,\cdots,\#T_{s,\delta(1)}(q), \sum_{l_1=1}^{\#T_{s,\delta(1)}(q)} q_{s,\delta(1)(l_1)} \leq 1\},\ T_{\delta(1),\delta(2)}(q) = \{h_{\delta(1),\delta(2)(l_2)}(q_{\delta(1),\delta(2)(l_2)}) | h_{\delta(1),\delta(2)(l_2)} \in H, q_{\delta(1),\delta(2)(l_2)} \geqslant 0, l_2 = 1,2,\cdots,\#T_{\delta(1),\delta(2)}(q), \sum_{l_2=1}^{\#T_{\delta(1),\delta(2)}(q)} q_{\delta(1),\delta(2)(l_2)} \leq 1\},\ \cdots,\ T_{\delta(z),t}(q) = \{h_{\delta(z),t(l_{z+1})}(q_{\delta(z),t(l_{z+1})}) | h_{\delta(z),t(l_{z+1})} \in H, q_{\delta(z),t(l_{z+1})} \geqslant 0, l_{z+1} = 1,2,,\cdots,T_{\delta(z),t}(q), \sum_{l_{z+1}=1}^{\#T_{\delta(z),t}(q)} q_{\delta(z),t(l_{z+1})} \leq 1\}$$

be $z+1$ PLTFs. The probabilistic linguistic trust propagation operator is defined as follows:

Definition 8. ([12]) *The probabilistic linguistic trust value passed from e_s to e_t is:*

$$\begin{aligned} T_{st}(q) &= T_{s,\delta(1)}(q) \otimes T_{\delta(1),\delta(2)}(q) \otimes \cdots \otimes T_{\delta(z),t}(q) \\ &= g^{-1}\left(\bigcup_{\gamma^1_{l_1} \in g(T_{s,\delta(1)}), \gamma^2_{l_2} \in g(T_{(\delta(1),\delta(2))}), \cdots, \gamma^{z+1}_{l_{z+1}} \in g(T_{\delta(z),t})} \left(\gamma^1_{l_1} \times \gamma^2_{l_2} \times \cdots \times \gamma^{z+1}_{l_{z+1}}\right)\left(q^1_{l_1} \times q^2_{l_2} \times \cdots \times q^{z+1}_{l_{z+1}}\right) \right), \end{aligned} \tag{18}$$

where $l_1 = 1, 2, \cdots, \#T_{s,\delta(1)}(q); l_2 = 1, 2, \cdots, \#T_{\delta(1),\delta(2)}(q); \cdots; l_{z+1} = 1, 2, \#T_{\delta(z),t}(q)$, $\gamma^1_{l_1}$, $\gamma^2_{l_2}, \cdots, \gamma^{z+1}_{l_{z+1}}$ can be calculated by Equation (12).

3. Multi-Attribute Quantum Group Decision-Making Algorithm Based on Probabilistic Language and Quantum-like Bayesian Network

In this section, an innovative PL-QLBN algorithm based on quantum MAGDM problem is proposed.

3.1. Problem Description

A representative probabilistic linguistic MAGDM framework comprises three core components: a set of alternatives $A_i (i = 1, 2, \cdots, m)$, a set of attributes $c_j (j = 1, 2, \cdots, n)$, and a group of DMs $e_k (k = 1, 2, \cdots, r)$. Each DM evaluates every alternative against all attributes using PLTSs. Let $S = \{s_\alpha | \alpha = -\tau, \cdots, -1, 0, 1, \cdots, \tau\}$ and $H = \{h_\beta | \beta = -\sigma, \cdots, -1, 0, 1, \cdots, \sigma\}$ be two finite LTSs. Based on LTS S, the probabilistic linguistic evaluation provided by e_k for alternative A_i under attribute c_j is denoted as $L^k_{ij}(p)$, forming an individual decision matrix $A^k = \left[L^k_{ij}(p)\right]_{m \times n}$. The corresponding normalized decision matrices are denoted as $\hat{A}^k = \left[\hat{L}^k_{ij}(p)\right]_{m \times n}$. Furthermore, based on LTS H, the DMs provide mutual trust ratings to form an incomplete trust network matrix $T = (T_{st}(q))_{r \times r}$, where $T_{st}(q)$ represents the trust degree from e_s to e_t. The normalized form of the trust matrix is denoted as $\hat{T} = (\hat{T}_{st}(q))_{r \times r}$.

3.2. Determination of DM Weights

This section determines the DM weights based on the social trust network (STN).

3.2.1. Trust Propagation for Probabilistic Linguistic Trust

A trust network matrix is created when each DM gives probabilistic linguistic trust values to other DMs. The probabilistic linguistic trust transfer operator described in Section 2.5 determines the indirect trust value between DMs in the absence of a direct trust value. If there exist y trust propagation paths between e_s and e_t, z denotes the number of intermediate nodes of the trust path, then the shortest path principle is adopted, i.e., the probabilistic linguistic trust propagation path with the smallest number of intermediate nodes is selected. If there are multiple shortest paths, the average of the expectation of the PLTF that is the largest among the paths is selected, defined as follows:

(1) $z_\alpha = min(z_1, z_2, \cdots, z_y) \Rightarrow T_{st}(q) = T^\alpha_{st}(q)$,

(2) $z_\alpha = z_\beta \wedge PQ_\alpha > PQ_\beta \Rightarrow T_{st}(q) = T^\alpha_{st}(q)$.

Where PQ denotes the average of the expectation of the PLTF in the trust path, the larger the value indicates the stronger the trust relationship of the trust path, and PQ is defined as follows:

$$PQ = \frac{1}{z+1}\left[E\left(T_{s,\delta(1)}(q)\right) + E\left(T_{\delta(1),\delta(2)}(q)\right) + \cdots + E\left(T_{\delta(z),t}(q)\right)\right], \tag{19}$$

where $E\left(T_{s,\delta(1)}(q)\right)$ denotes the expectation of PLTF, which can be calculated according to Equation (17).

3.2.2. Construction of QLBN in STN

Because of the cognitive uncertainty, DMs are vulnerable to outside influences and could be disturbed by others when they give trust values to strangers. In addition, current methods are unable to adequately capture the interaction of various consciousnesses within the human brain since human consciousness is not observable. In contrast, quantum theory may imitate state-to-state interference by superimposing wave functions and use wave functions to represent different states. Consequently, interference in trust aggregation is modeled using quantum theory. As seen in Figure 3, the QLBN will be constructed using the quantum probability amplitudes of the trustee and the trustor, which are determined independently.

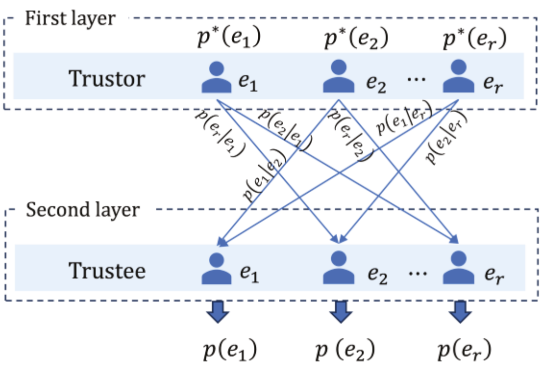


Figure 3. Quantum-like Bayesian network under STN.

Let $p^*(e_h)$ represent the prior probability of e_h, using it as the first layer of a QLBN. It can be calculated based on probabilistic linguistic trust expectations between DMs, i.e.,

$$p^*(e_h) = \frac{\sum_{k=1,k\neq h}^{r} t_{kh}}{\sum_{h=1}^{r}\sum_{k=1,k\neq h}^{r} t_{kh}}, h = 1, 2, \cdots, r, \tag{20}$$

where $t_{kh} = E[T_{kh}(q)]$, $E[T_{kh}(q)]$ is calculated according to Equation (17).

Let $p(e_k|e_h)$ represent the relative trusted degree of the e_h to the e_k, which will be used as the second layer of the QLBN, the calculation is as follows:

$$p(e_k|e_h) = \frac{t_{hk}}{\sum_{k=1,k\neq h}^{r} t_{hk}}, h = 1, 2, \cdots, r. \tag{21}$$

Let $p'(e_k)$ represent the total trusted probability of trustee e_k for other DMs. According to quantum probability theory, this can be calculated by taking the square of the sum of the amplitudes from all possible state transition paths, i.e.,

$$\begin{aligned} p'(e_k) &= \left|\sum_{h=1,h\neq k}^{r} \varphi_{e_h} e^{i\theta_{e_h}} \varphi_{e_k|e_h} e^{i\theta_{e_k|e_h}}\right|^2 = \sum_{h=1,h\neq k}^{r} p^*(e_h)p(e_k|e_h) \\ &\quad + 2\sum_{h=1,h\neq k}^{r-1}\sum_{u=h+1}^{r} \sqrt{p^*(e_h)p(e_k|e_h)p^*(e_u)p(e_k|e_u)}\cos(\theta_{kh}-\theta_{ku}) \\ &= \sum_{h=1,h\neq k}^{r} p^*(e_h)p(e_k|e_h) + \sum_{w=1}^{\frac{(r-1)(r-2)}{2}} R_{kw} \\ &= \sum_{h=1,h\neq k}^{r} p^*(e_h)p(e_k|e_h) + R_k, \end{aligned} \tag{22}$$

where $(\theta_{kh} - \theta_{ku})$ is the phase angle formed by two different paths between e_h and e_u; $\frac{(r-1)(r-2)}{2}$ is the total number of pairs of correspondences without duplicates between $r-1$ DMs, R_k is the interference term. This paper assumes the number of DMs r with $r \geq 3$ since interference effects can only happen when there are at least two DMs and a DM's confidence value in themselves is not taken into account.

The DM trusted probability is normalized using Equation (23) to guarantee that the total of the final probabilities equals 1:

$$p(e_k) = p'(e_k) / \sum_{k=1}^{r} p(e_k). \tag{23}$$

The trusted probability reflects the richness of the DM's interaction in the group; the higher the trust, the richer the communication and interaction between the DM and others, and it can be regarded as a higher status and importance of position in the group. Therefore, this paper takes the trust probability as the weight of the DM in MAGDM.

3.2.3. Determination of the Interference Term on the Trusted Degree

Phase angle measurement in real world scenarios is frequently challenging because it depends on DMs' subjective judgments. These opinions are susceptible to mutual interference and environmental influences, leading to inaccurate outcomes. In contrast, belief entropy offers a significant advantage for processing and fusing such uncertain information, enabling more robust judgment in MAGDM problems. This method possesses greater generalizability than the similarity heuristic approach. The latter relies on subjective estimation, which not only introduces errors but also allows the measurer's preferences to bias the final decision and subsequent probability calculations [20].

The interaction between e_h and e_u regarding trust value is analyzed by calculating the variability between DMs, which is calculated using the logistic function [32], as shown in Equation (24):

$$f_{kw} = \frac{1}{1 + e^{\chi |\alpha_{kw} - \beta_{kw}|}}, \chi > 0, \tag{24}$$

where $\alpha_{kw} = p^*(e_h)p(e_k|e_h)$, $\beta_{kw} = p^*(e_u)p(e_k|e_u)$, $(k = 1, 2, ..., r)$; χ reflects the sensitivity of the level of trust between different DMs.

The cross-entropy loss function was used to determine χ:

$$\chi = -\alpha_{kw} \times log_2(\beta_{kw}) - \beta_{kw} \times log_2(\alpha_{kw}). \tag{25}$$

Figure 4 shows the effect of changes in the parameters χ and M on the logistic function f, where $M = |\alpha_{kw} - \beta_{kw}|$. When χ is a definite value, the value of the logistic function decreases as $|\alpha_{kw} - \beta_{kw}|$ increases. χ reflects the sensitivity of the DM between different events, and the value of the logistic function decreases as χ increases.

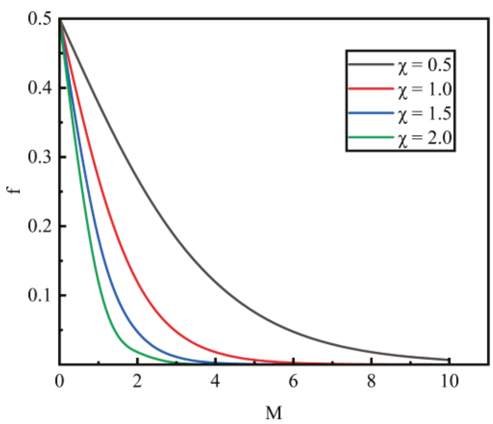


Figure 4. Effect of variation in parameters χ and M on the logistic function.

Since Deng entropy [27] measures the overall non-specificity and discord, it may be more useful in accounting for the uncertainty of BPA.

To generate the $BPAM_k$, M_{kw} $\left(w = 1, 2, \cdots \frac{(r-1)(r-2)}{2}\right)$ are obtained by normalizing f_{kw}, i.e.,

$$M_{kw} = f_{kw} / \sum_{w=1}^{\frac{(r-1)(r-2)}{2}} f_{kw}. \tag{26}$$

The belief entropy $E_d(M_{kw})$ and $cos(\theta_{kh} - \theta_{ku})$ of the two DM' trust values are calculated one to one, rather than seeking the overall uncertainty among DMs. Therefore, only the interferences between two elements are taken into account in Equation (6), where $C_w = \left\{\{\alpha_{kw}, \beta_{kw}\}, w = 1, 2, \cdots \frac{(r-1)(r-2)}{2}\right\}$, and use the Deng entropy to calculate the belief entropy between two DMs' trust values according to Equation (6), which is calculated as follows:

$$E_d(M_{kw}) = -M_{kw}(C_w) log_2 \frac{M_{kw}(C_w)}{2^{|C_w|} - 1}, \tag{27}$$

where $|C_w|$ is the number of focus elements in C_w.

The interference effect value lies within the interval $[-1, 1]$. A normalization method is applied to the entropy measurement value, as shown in Equation (28).

$$\begin{aligned} & cos(\theta_{kh} - \theta_{ku}) = E'_d(M_{kw}) \\ = & \frac{E_d(M_{kw}) - max\left(\overline{E}_d(M_k)\right)}{max\left(max\ (E_d(M_{kw})) - max\left(\overline{E}_d(M_k)\right), max\left(\overline{E}_d(M_k)\right) - min(E_d(M_{kw}))\right)}, \end{aligned} \tag{28}$$

where $max\left(\overline{E}_d(M_k)\right) = \frac{max\left(\overline{E}_d(M_k)\right)}{(r-1)(r-2)/2}$ is the average value of the maximum Deng entropy $max(E_d(M_k))$; $max\ (E_d(M_{kw}))$ and $min(E_d(M_{kw}))$ are the maximum and minimum values of $E_d(M_{kw})$. Because $0 \leq M_{kw}(C_w) \leq 1$ and $|C_w| = 2$, $min\ (E_d(M_{kw})) = 0$, $max\ (E_d(M_{kw})) = -log_2 \frac{1}{3} = log_2 3 = 1.5850$.

Since opinions from different DMs may interfere with one another, the final interference value is obtained by adding the interference value between each pair of DMs, which has a total of $\frac{(r-1)(r-2)}{2}$ non-empty elements. Consequently, the maximum Deng entropy $max(E_d(M_k))$ can be computed as follows using Equation (8):

$$\begin{aligned} max(E_d(M_k)) &= -\sum_{i=1}^{\frac{(r-1)(r-2)}{2}} M(C_i) log_2 \frac{M(C_i)}{2^{|C_i|} - 1} \\ &= log_2 \sum_{i=1}^{\frac{(r-1)(r-2)}{2}} 2^{|C_i|} - 1 = log_2(1.5(r-1)(r-2)) \end{aligned} \tag{29}$$

3.3. Determination of Attribute Weights

Different DMs may hold different opinions about attributes in real-world decision-making situations. Consequently, the attribute weights will be determined using the more objective entropy weighting method [33], which involves the following steps:

1. The individual normalized probabilistic linguistic decision matrix $\hat{A}^k = \left[\hat{L}_{ij}^{(k)}(p)\right]_{m \times n}$ is converted into a matrix with clear numbers $V_{ij}^{(k)} = \left[v_{ij}^{(k)}\right]_{m \times n}$, according to the expectation of the PLTS, calculated as follows:

$$v_{ij}^{(k)} = E\left(\hat{L}_{ij}^{(k)}(p)\right) / \sum_{i=1}^{m} E\left(\hat{L}_{ij}^{(k)}(p)\right). \tag{30}$$

2. Compute the entropy of the jth criterion by

$$en_j^{(k)} = -\frac{1}{lnm} \sum_{i=1}^{m} v_{ij}^{(k)} \ln v_{ij}^{(k)}, \tag{31}$$

where lnm is the normalization factor, in order to ensure that the value of $en_j^{(k)}$ is within the interval $[0, 1]$.

3. The weight of e_k with respect to the jth attribute is obtained by the following equation:

$$\omega_j^k = \left(1 - en_j^{(k)}\right) / \sum_{j=1}^{n} \left(1 - en_j^{(k)}\right). \tag{32}$$

Therefore, attribute weights can be calculated for each DM. Then, the obtained attribute weights are used as key parameters to calculate the comprehensive personal evaluation information.

3.4. Probabilistic Linguistic MAGDM Method Based on Quantum Theory

3.4.1. Aggregation of Individual Probabilistic Linguistic Evaluation

Before aggregating individual opinions, it is essential to obtain the collective decision matrix of each DM. The collective decision matrix for e_k is marked as

$$\tilde{A}^{k} = \left[M_i^{(k)}(p)\right]_{m\times r}, \tag{33}$$

where $M_i^{(k)}(p) = PLWA\left(\hat{L}_{i1}^{(k)}(p), \hat{L}_{i2}^{(k)}(p), \cdots, \hat{L}_{in}^{(k)}(p)\right) = \oplus_{j=1}^{n}\omega_j^k \cdot \hat{L}_{ij}^{(k)}(p)$, $PLWA$ operator is computed according to Equation (14); ω_j^k denote the weight of e_k with respect to the jth attribute, and $\hat{L}_{ij}^{(k)}(p)$ denote the normalized probabilistic linguistic evaluation information of e_k.

3.4.2. Construction of QLBN in MAGDM

Unobservable interference effects in MAGDM can arise from the opinions of various DMs interfering with one another or being influenced by the decision-making environment. The evaluation opinions are regarded as wave functions superimposed in the quantum model. This section offers a quantum framework to model the interference during the aggregation of individual DMs' opinions. Figure 5 illustrates how the MAGDM process can be described as a two-layer QLBN.

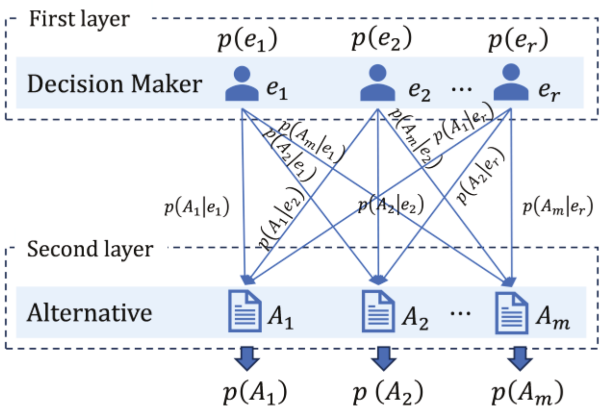


Figure 5. QLBN in MAGDM.

First, depending on their trust relationships, DMs are allocated different weights. Setting the DM weights $p(e_k)$ as the first layer of the QLBN is covered in detail in Section 3.2.

Second, conditional probability is needed to connect the alternative and DM layers. Individual collective evaluations are transformed into conditional probabilities by the expectation of PLTS. The relative rating $p(A_i|e_k)$ of the e_k on the alternative A_i is the second layer of the QLBN and is calculated as follows:

$$p(A_i|e_k) = E\left(M_i^{(k)}(p)\right) / \sum_{i=1}^{m} E\left(M_i^{(k)}(p)\right), \tag{34}$$

where $E\left(M_i^{(k)}(p)\right)$ can be calculated according to Equation (11).

The probability of a specific alternative is equal to the square of the sum of the amplitudes of all possible paths, according to quantum probability theory. Then, the quantum probability $p'(A_i)$ is

$$\begin{aligned} p'(A_i) &= \left|\sum_{k=1}^{r} \varphi_{e_k} e^{i\theta_{e_k}} \varphi_{A_i|e_k} e^{i\theta_{A_i|e_k}}\right|^2 = \sum_{k=1}^{r} p(e_k)p(A_i|e_k) \\ &+2\sum_{k=1}^{r-1} \sum_{k'=k+1}^{r} \sqrt{p(e_k)p(A_i|e_k)p(e_{k'})p(A_i|e_{k'})}\cos(\theta_{ik} - \theta_{ik'}) \\ &= \sum_{k=1}^{r} p(e_k)p(A_i|e_k) + \sum_{w=1}^{\frac{r(r-1)}{2}} R_{iw} \\ &= \sum_{k=1}^{r} p(e_k)p(A_i|e_k) + R_i, \end{aligned} \tag{35}$$

where $(\theta_{ik} - \theta_{ik'})$ is the phase angle formed by two different paths between e_h and e_u; $\frac{r(r-1)}{2}$ is the total number of pairs of correspondences without duplicates between r DMs; R_i is the interference term.

Each probability is normalized using Equation (36) so that the total of the alternative probabilities equals 1:

$$p(A_i) = p'(A_i)/\sum_{i=1}^{m} p'(A_i). \tag{36}$$

3.4.3. Determination of the Interference Term on Alternative Evaluation

The interaction between e_k and $e_{k'}$ regarding evaluation of alternatives are analyzed by calculating the variability between DMs, which is calculated using the logistic function, as shown in Equation (37):

$$f_{iw} = \frac{1}{1+e^{\chi|\alpha_{iw}-\beta_{iw}|}}, \chi > 0, \tag{37}$$

where $\alpha_{iw} = p(e_k)p(A_i|e_k)$, $\beta_{iw} = p(e_{k'})p(A_i|e_{k'})$, $(i = 1, 2, \ldots, m)$.

Similar to Equation (25), χ in Equation (37) can be determined as:

$$\chi = -\alpha_{iw} \times log_2(\beta_{iw}) - \beta_{iw} \times log_2(\alpha_{iw}). \tag{38}$$

To generate the $BPAM_i$, $M_{iw}\left(w = 1, 2, \cdots \frac{r(r-1)}{2}\right)$ is obtained by normalizing f_{iw}, i.e., normalization is performed:

$$M_{iw} = f_{iw}/\sum_{w=1}^{\frac{r(r-1)}{2}} f_{iw}. \tag{39}$$

The interferences between two elements are considered in Equation (6), i.e., $C = \left\{\{\alpha_{iw}, \beta_{iw}\}, w = 1, 2, \cdots \frac{r(r-1)}{2}\right\}$. The belief entropy of the evaluation of alternatives between two DMs is calculated using Deng entropy according to Equation (6):

$$E_d(M_{iw}) = -M_{iw}(C_w)log_2\frac{M_{iw}(C_w)}{2^{|C_w|}-1}, \tag{40}$$

Similar to Equation (28), a normalization method is applied to the entropy measurement, as shown in Equation (41):

$$\begin{gathered} \cos(\theta_{ik} - \theta_{ik'}) = E'_d(M_{iw}) \\ = \frac{E_d(M_{iw}) - max\left(\overline{E}_d(M_i)\right)}{max\left(maxE_d(M_{iw}) - max\left(\overline{E}_d(M_{iw})\right), max\left(\overline{E}_d(M_{iw})\right) - minE_d(M_{iw})\right)}, \end{gathered} \tag{41}$$

where $max\left(\overline{E}_d(M_i)\right) = \frac{max\left(\overline{E}_d(M_i)\right)}{r(r-1)/2}$ is the average value of the maximum Deng entropy $max(E_d(M_k))$; $max\ (E_d(M_{iw}))$ and $min(E_d(M_{iw}))$ are the maximum and the minimum value of $E_d(M_{iw})$, respectively. Because $0 \le M_{iw}(C_w) \le 1$ and $|C_w| = 2$,

$min\ (E_d(M_{kw})) = 0$, $max\ (E_d(M_{kw})) = log_2 3 = 1.5850$. According to Equation (29), $max(E_d(M_k)) = log_2(1.5r(r-1))$.

3.5. Steps of the Quantum MAGDM Algorithm Based on PLTS and QLBN

Based on the above analysis, a PL-QLBN algorithm is proposed. This algorithm introduces a novel method for calculating the interference effect based on belief entropy. Figure 6 illustrates the concrete process; the specific steps are as follows:

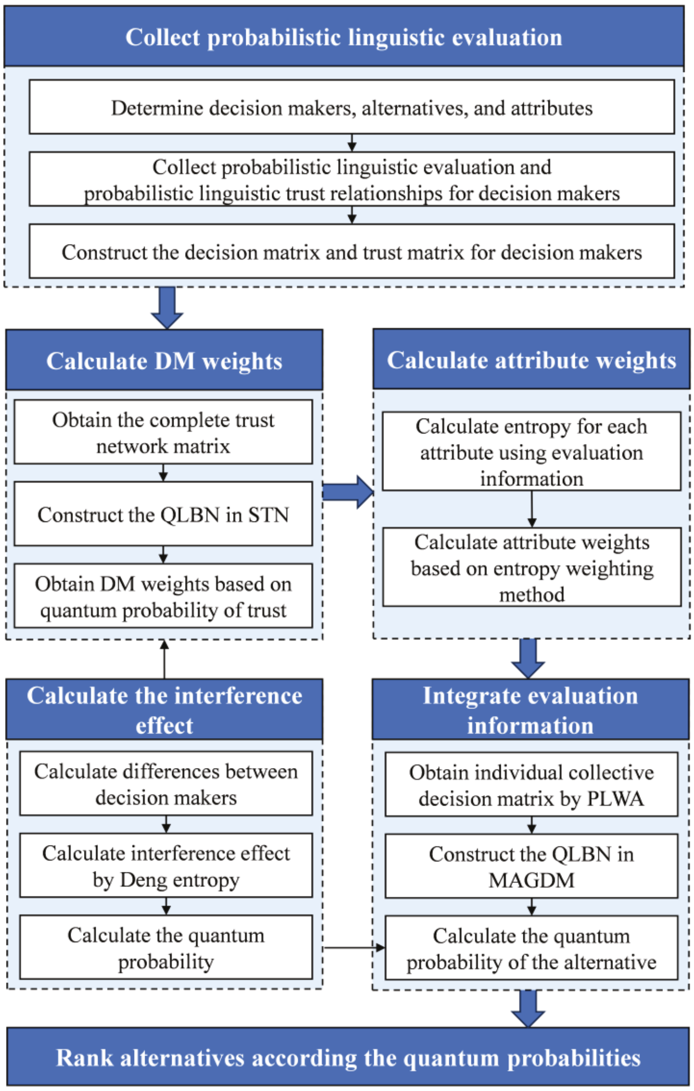


Figure 6. Flowchart of PL-QLBN algorithm.

Step 1: Collect normalized decision matrices and incomplete trust network matrix. The probabilistic linguistic evaluation given by DM $e_k(k = 1, 2, \cdots, r)$ to alternative $A_i(i = 1, 2, \cdots, m)$ under attribute $c_j(j = 1, 2, \cdots, n)$ is $L_{ij}^k(p)$, decision matrix is denoted as $A^k = \left[L_{ij}^k(p)\right]_{m\times n}$, which is normalized according to Equation (10) to obtain normalized decision matrix $\hat{A}^k = \left[\hat{L}_{ij}^k(p)\right]_{m\times n}$. The probabilistic linguistic trust value of DM $e_s(s = 1, 2, \cdots, r)$ to $e_t(t = 1, 2, \cdots, r, t \neq s)$ is marked as $T_{st}(q)$, probabilistic trust network matrix is denoted as $T = (T_{st}(q))_{r\times r}$, which is normalized according to Equation (16), thus obtaining the incomplete probabilistic trust network matrix $\hat{T} = (\hat{T}_{st}(q))_{r\times r}$.

Step 2: Obtain the complete trust network matrix. The indirect probabilistic linguistic trust values are calculated using the shortest path method, as outlined in Equations (18)–(19), and a complete trust network matrix $T^* = \left(\hat{T}_{st}(q)\right)_{r\times r}$ is built.

Step 3: Calculate the weights of DMs based on the trust network matrix. Construct the QLBN in the DM's STN, according to Equations (24)–(29), the interference effect on trusted degree can be quantified, the total trusted probability $p(e_k)$ is calculated, and the weight of DMs is determined, viewing in Equations (20)–(23).

Step 4: Calculate attribute weights based on the entropy weighting method. The attribute weights ω_j^k of the DM e_k are calculated based on the entropy weighting method, according to Equations (30)–(32).

Step 5: Obtain the individual collective decision matrix. Based on the attribute weights ω_j^k obtained in step 4, the individual collective decision matrix $\overset{\sim k}{A} = \left[M_i^{(k)}(p)\right]_{m\times r}$ of DM e_k for alternative A_i can be obtained using Equation (33).

Step 6: Calculate the quantum probability of the alternative. Construct the QLBN for the MAGDM problem and calculate the interference effect on alternative evaluation according to Equations (37)–(41) and calculate the quantum probability of the alternatives $p(A_i)$ according to Equations (34)–(36).

Step 7: Rank all the alternatives. The aforementioned analysis ranks the alternatives based on the obtained quantum probabilities of the alternatives $p(A_i)$.

4. Numerical Example

This section uses wastewater reuse alternative selection as an example to illustrate the effectiveness of the proposed methodology.

4.1. Background and Problem Description

Water is a vital resource for life, agriculture, and national development. Water resources in Anhui Province are unevenly distributed, and the contradiction between supply and demand is prominent. Hefei, as the provincial capital, is the economic, population and industrial core area of the province, and is also a representative city of water resource tension. To alleviate pressure on water resources and promote a circular economy, recycling wastewater is essential. Reclaimed water is used for a variety of purposes, and optimal utilization options need to be selected. In this paper, five DMs in the field of water resources $\{e_1, e_2, e_3, e_4, e_5\}$ are invited to rank five wastewater reuse alternatives $\{A_1, A_2, A_3, A_4, A_5\}$ to determine the optimal alternative. A_1 (urban reuse): reclaimed water is used for municipal greening, road washing, fire-fighting water supply, toilet flushing in buildings, etc., to relieve the pressure of urban freshwater demand. A_2 (agricultural reuse): treated wastewater is used to irrigate crops, gardens or pastures, to reduce the cost of water for agricultural use. A_3 (groundwater recharge): deeply treated reclaimed water is used to replenish aquifers, to prevent seawater intrusion. A_4 (industrial reuse): reclaimed water is used for industrial cooling and process water, etc., to reduce freshwater consumption and wastewater discharge. A_5 (environmental reuse): reclaimed water is used to restore wetlands, rivers, lakes, or to irrigate protected forests, to improve the ecology and regional climate.

The evaluation information for DMs in this paper is represented by a PLTS, given LTS $S = \{s_{-2} = \text{verypoor}; s_{-1} = poor; s_0 = medium; s_1 = good; s_2 = very\ good\}$. The attributes considered in decision-making are c_1 social, c_2 economic, c_3 environmental and c_4 technological [34,35]. Simultaneously, DMs also score the trust extent of other DMs in the team. Trust values are conveyed using an LTS $H = \{h_{-2} = very\ low; h_{-1} = low; h_0 = medium; h_1 = high; h_2 = very\ high\}$.

4.2. Decision-Making Steps

Step 1: Collect normalized decision matrices and incomplete trust network matrix.

Five DMs evaluate five wastewater reuse alternatives based on four attributes, and the probabilistic linguistic evaluation information given by DM e_k for alternative A_i under attribute c_j is $L_{ij}^k(p)$, which is normalized according to Equation (10), thus obtaining normalized probabilistic linguistic evaluation matrices $\hat{A}^k = \left[\hat{L}_{ij}^k(p)\right]_{5\times 4}$, as shown in Table 2. PLTF denotes the trust relationship among the five DMs, and this relationship is illustrated in Figure 7.

Table 2. The evaluation information for DMs.

DM	Alternative	c_1	c_2	c_3	c_4
e_1	A_1	$\{s_0(0.5), s_1(0.5)\}$	$\{s_{-1}(0.8), s_0(0.2)\}$	$\{s_{-1}(1)\}$	$\{s_{-2}(0.5), s_1(0.5)\}$
	A_2	$\{s_{-1}(1)\}$	$\{s_0(1)\}$	$\{s_1(1)\}$	$\{s_1(1)\}$
	A_3	$\{s_{-2}(0.5), s_{-1}(0.2), s_0(0.3)\}$	$\{s_0(1)\}$	$\{s_1(0.4), s_2(0.6)\}$	$\{s_1(1)\}$
	A_4	$\{s_0(1)\}$	$\{s_0(0.8), s_1(0.2)\}$	$\{s_0(1)\}$	$\{s_{-1}(0.8), s_0(0.2)\}$
	A_5	$\{s_0(1)\}$	$\{s_{-1}(0.4), s_0(0.6)\}$	$\{s_1(0.8), s_2(0.2)\}$	$\{s_1(1)\}$
e_2	A_1	$\{s_1(0.7), s_2(0.3)\}$	$\{s_{-1}(1)\}$	$\{s_{-1}(0.4), s_0(0.6)\}$	$\{s_{-1}(1)\}$
	A_2	$\{s_0(1)\}$	$\{s_0(0.8), s_1(0.2)\}$	$\{s_1(1)\}$	$\{s_1(1)\}$
	A_3	$\{s_1(1)\}$	$\{s_{-1}(1)\}$	$\{s_0(1)\}$	$\{s_0(0.7), s_1(0.3)\}$
	A_4	$\{s_0(0.3), s_1(0.7)\}$	$\{s_0(1)\}$	$\{s_0(0.7), s_1(0.3)\}$	$\{s_1(0.8), s_2(0.2)\}$
	A_5	$\{s_0(1)\}$	$\{s_0(0.4), s_1(0.6)\}$	$\{s_1(1)\}$	$\{s_0(1)\}$
e_3	A_1	$\{s_{-1}(1)\}$	$\{s_{-1}(0.6), s_0(0.4)\}$	$\{s_{-1}(0.3), s_0(0.4), s_1(0.3)\}$	$\{s_0(1)\}$
	A_2	$\{s_0(0.2), s_1(0.8)\}$	$\{s_{-1}(0.5), s_0(0.4), s_1(0.1)\}$	$\{s_2(1)\}$	$\{s_0(0.5), s_1(0.5)\}$
	A_3	$\{s_0(0.6), s_1(0.4)\}$	$\{s_1(1)\}$	$\{s_1(1)\}$	$\{s_{-1}(1)\}$
	A_4	$\{s_0(1)\}$	$\{s_{-1}(0.6), s_0(0.4)\}$	$\{s_1(0.3), s_2(0.7)\}$	$\{s_0(1)\}$
	A_5	$\{s_0(0.25), s_1(0.75)\}$	$\{s_0(1)\}$	$\{s_1(1)\}$	$\{s_0(1)\}$
e_4	A_1	$\{s_{-1}(0.9), s_2(0.1)\}$	$\{s_1(1)\}$	$\{s_1(0.9), s_2(0.1)\}$	$\{s_0(1)\}$
	A_2	$\{s_1(1)\}$	$\{s_{-2}(0.3), s_{-1}(0.7)\}$	$\{s_{-1}(1)\}$	$\{s_0(1)\}$
	A_3	$\{s_1(0.8), s_2(0.2)\}$	$\{s_0(1)\}$	$\{s_1(1)\}$	$\{s_1(0.5), s_2(0.5)\}$
	A_4	$\{s_1(1)\}$	$\{s_{-1}(1)\}$	$\{s_0(0.2), s_1(0.8)\}$	$\{s_0(0.4), s_1(0.6)\}$
	A_5	$\{s_1(1)\}$	$\{s_0(0.8), s_1(0.2)\}$	$\{s_0(0.5), s_1(0.5)\}$	$\{s_0(1)\}$
e_5	A_1	$\{s_0(1)\}$	$\{s_0(0.5), s_1(0.5)\}$	$\{s_0(0.4), s_1(0.6)\}$	$\{s_{-1}(1)\}$
	A_2	$\{s_{-1}(1)\}$	$\{s_{-1}(1)\}$	$\{s_0(0.75), s_1(0.25)\}$	$\{s_1(0.6), s_2(0.4)\}$
	A_3	$\{s_{-2}(0.6), s_0(0.4)\}$	$\{s_1(1)\}$	$\{s_{-1}(1)\}$	$\{s_1(1)\}$
	A_4	$\{s_0(1)\}$	$\{s_1(0.6), s_2(0.4)\}$	$\{s_1(1)\}$	$\{s_1(0.4), s_2(0.6)\}$
	A_5	$\{s_{-1}(0.6), s_0(0.4)\}$	$\{s_0(0.8), s_1(0.2)\}$	$\{s_1(1)\}$	$\{s_0(1)\}$

The probabilistic linguistic trust value of DM e_s to DM e_t is $T_{st}(q)$, and the incomplete probabilistic trust network matrix $\hat{T} = \left(\hat{T}_{st}(q)\right)_{5\times 5}$ obtained after normalization according to Equation (16) is shown below:

$$\hat{T} = \begin{bmatrix} - & - & - & \{h_{-1}(1)\} & \{h_1(1)\} \\ \{h_0(1)\} & - & \{h_0(0.4), h_1(0.6)\} & - & - \\ \{h_0(0.3), h_1(0.7)\} & - & - & \{h_{-1}(1)\} & \{h_0(1)\} \\ - & \{h_{-1}(1)\} & - & - & \{h_2(1)\} \\ \{h_1(1)\} & \{h_{-1}(0.5), h_0(0.5)\} & - & - & - \end{bmatrix}$$

Step 2: Obtain the complete probabilistic linguistic trust network matrix.

The indirect probabilistic linguistic trust values are calculated using the shortest paths and Equations (18) and (19), as shown in Table 3.

Table 3. Indirect probabilistic linguistic trust.

DM	Indirect Probabilistic Linguistic Trust
$T_{12}(q)$	$\{h_{-1.25}(0.5), h_{-0.5}(0.5)\}$
$T_{13}(q)$	$\{h_{-1.625}(0.2), h_{-1.438}(0.3), h_{-1.25}(0.2), h_{-0.875}(0.3)\}$
$T_{24}(q)$	$\{h_{-1.5}(0.4), h_{-1.25}(0.6)\}$
$T_{25}(q)$	$\{h_{-0.5}(1)\}$
$T_{32}(q)$	$\{h_{-1.5}(0.5), h_{-1}(0.5)\}$
$T_{41}(q)$	$\{h_{1}(1)\}$
$T_{43}(q)$	$\{h_{-1.5}(0.4), h_{-1.25}(0.6)\}$
$T_{53}(q)$	$\{h_{-1.5}(0.2), h_{-1.25}(0.3), h_{-1}(0.2), h_{-0.5}(0.3)\}$
$T_{54}(q)$	$\{h_{-1.25}(1)\}$

Then, the complete probabilistic linguistic trust network matrix $T^* = \left(\hat{T}_{st}(q)\right)_{r\times r}$ is built:

$$T^* = \begin{bmatrix} - & T_{12}(q) & T_{24}(q) & \{h_{-1}(1)\} & \{h_1(1)\} \\ \{h_0(1)\} & - & \{h_0(0.4)\,,\, h_1(0.6)\} & T_{24}(q) & T_{25}(q) \\ \{h_0(0.3),\, h_1(0.7)\} & T_{32}(q) & - & \{h_{-1}(1)\} & \{h_0(1)\} \\ T_{41}(q) & \{h_{-1}(1)\} & T_{43}(q) & - & \{h_2(1)\} \\ \{h_1(1)\} & \{h_{-1}(0.5)\,,\, h_0(0.5)\} & T_{53}(q) & T_{54}(q) & - \end{bmatrix}$$

Step 3: Calculate the weights of DMs based on the probabilistic linguistic trust network matrix.

Construct the QLBN in the DM's STN using the prior probability of the DM from Equation (20) to serve as the first layer. The results are as follows:

$p^*(e_1) = 0.3153, p^*(e_2) = 0.1289, p^*(e_3) = 0.1461, p^*(e_4) = 0.1002, p^*(e_5) = 0.3095.$

The relative trusted degree of the DMs is calculated according to Equation (21) as the second layer of QLBN, and the results are shown in Table 4.

Table 4. The relative trusted degree.

DM	e_1	e_2	e_3	e_4	e_5
e_1	0.0000	0.1921	0.1249	0.1708	0.5123
e_2	0.2963	0.0000	0.3852	0.0963	0.2222
e_3	0.4186	0.1163	0.0000	0.1550	0.3101
e_4	0.3468	0.1156	0.0751	0.0000	0.4624
e_5	0.4819	0.2410	0.1566	0.1205	0.0000

The interference term on the trusted degree is calculated by Equations (24)–(29):

$$R_1 = -0.1692,\ R_2 = 0.0084,\ R_3 = 0.1157,\ R_4 = -0.0172,\ R_5 = -0.1813.$$

Then, the trusted probabilities, i.e., DM weights, are calculated by Equations (22)–(29):

$$p(e_1) = 0.1507, p(e_2) = 0.2276, p(e_3) = 0.3447, p(e_4) = 0.1441, p(e_5) = 0.1329.$$

Step 4: Calculate attribute weights based on the entropy weighting method.

The attribute weights are calculated according to Equations (30)–(32), and the results are shown in Table 5.

Table 5. Weight matrix of attributes.

DM	c_1	c_2	c_3	c_4
e_1	0.3335	0.0852	0.3136	0.2678
e_2	0.1121	0.3888	0.1492	0.3499
e_3	0.3272	0.2965	0.1428	0.2335
e_4	0.1660	0.4813	0.2384	0.1143
e_5	0.2325	0.2481	0.2170	0.3024

Step 5: Obtain the individual collective decision matrix.

Calculate the individual collective decision matrix $\overset{\sim k}{A} = \left[M_i^{(k)}(p)\right]_{m\times r}$ according to Equation (33), as shown in Table 6.

Step 6: Calculate the quantum probability of the alternative.

Construct the QLBN for the MAGDM problem. The weights of the DMs calculated in Step 3 are used as the first layer of the QLBN, and the relative scores $p(A_i|e_k)$ of the DMs e_k on the alternatives A_i are used as the second layer of the QLBN, the relative scores $p(A_i|e_k)$ are calculated using Equation (34) and the result can be seen in Table 7.

According to Equations (37)–(41), the interference term on alternative between DMs can be calculated as

$$R_1 = 0.0031,\ R_2 = -0.1172,\ R_3 = 0.0283,\ R_4 = -0.0901,\ R_5 = -0.0501.$$

Then, the quantum probabilities of the alternatives are calculated as by Equations (22)–(29):

$$p(A_1) = 0.2240, p(A_2) = 0.1487, p(A_3) = 0.2418, p(A_4) = 0.1841, p(A_5) = 0.2014.$$

Step 7: Rank all the alternatives.

Based on the results of Step 6, it can be concluded that the ranking of the alternatives is: $A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$, and A_3 is the optimal alternative, i.e., among the five wastewater reuse alternatives, the groundwater recharge is the best alternative.

The ranking of alternatives derived from this study $A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$ indicates that A_3 (groundwater recharge) is the wastewater reuse alternative with the most optimal overall performance. This is not due to exceptional performance in any single attribute, but rather because it achieves the optimal balance among four potentially conflicting attributes (c_1 social, c_2 economic, c_3 environmental and c_4 technological), revealing the core trade-off mechanism in complex decision-making. When evaluating the optimal alternative A_3, positive interference emerged among DMs, indicating a latent consensus on core advantages of A_3 despite their diverse professional backgrounds. This clearly demonstrates the unique capability of QLBN to capture group psychological dynamics overlooked by traditional models. The trust network-based DM weight calculation revealed that e_3 possesses the highest weight. This stems not only from direct trust by other DMs but also from pivotal position of e_3 within the trust network, enabling the integration and dissemination of trust signals across multiple parties. Furthermore, the positive interference generated by DMs toward e_3 during trust value assessment indicates a shared recognition of social influence, validating that social trust relationships constitute an indispensable force shaping collective decision outcomes. In summary, this method's results transcend the single ranking provided by traditional methods, offering explanations for decision outcomes with profound behavioral and managerial significance.

Table 6. Collective evaluation matrix.

DM	Alternative	Collective Evaluation
e_1	A_1	$\{s_{-0.8304}(0.2), s_{0.0474}(0.2), s_{-0.7342}(0.05), s_{0.1137}(0.05), s_{-0.2462}(0.2), s_{0.4504}(0.2), s_{-0.1699}(0.05), s_{0.5030}(0.05)\}$
	A_2	$\{s_{0.4699}(1)\}$
	A_3	$\{s_{0.3159}(0.2), s_{0.4699}(0.08), s_{0.6635}(0.12), s_2(0.6)\}$
	A_4	$\{s_{-0.2292}(0.64), s_{-0.1014}(0.16), s_{0.0001}(0.16), s_{0.1148}(0.04)\}$
	A_5	$\{s_{0.6165}(0.32), s_{0.6635}(0.48), s_2(0.2)\}$
e_2	A_1	$\{s_{-0.6524}(0.28), s_{-0.4967}(0.42), s_2(0.3)\}$
	A_2	$\{s_{0.5849}(0.8), s_{0.9192}(0.2)\}$
	A_3	$\{s_{-0.1665}(0.7), s_{0.3001}(0.3)\}$
	A_4	$\{s_{0.4307}(0.168), s_{0.5849}(0.072), s_{0.548}(0.392), s_{0.6907}(0.168), s_2(0.2)\}$
	A_5	$\{s_{0.1965}(0.4), s_{0.6225}(0.6)\}$
e_3	A_1	$\{s_{-0.7290}(0.18), s_{-0.5755}(0.24), s_{-0.3328}(0.18), s_{-0.4199}(0.12), s_{-0.2837}(0.16), s_{-0.0685}(0.12)\}$
	A_2	$\{s_2(1)\}$
	A_3	$\{s_{0.3785}(0.6), s_{0.7076}(0.4)\}$
	A_4	$\{s_{-0.0429}(0.18), s_{0.1885}(0.12), s_2(0.7)\}$
	A_5	$\{s_{0.1885}(0.25), s_{0.5561}(0.75)\}$
e_4	A_1	$\{s_{0.7010}(0.81), s_2(0.19)\}$
	A_2	$\{s_{-0.7411}(0.375), s_{-0.3867}(0.625)\}$
	A_3	$\{s_{0.6040}(0.4), s_2(0.6)\}$
	A_4	$\{s_{-0.1668}(0.08), s_{-0.0017}(0.12), s_{0.1633}(0.32), s_{0.3032}(0.48)\}$
	A_5	$\{s_{0.2174}(0.4), s_{0.4889}(0.4), s_{0.7231}(0.1), s_{0.9176}(0.1)\}$
e_5	A_1	$\{s_{-0.2609}(0.2), s_{0.0548}(0.3), s_{0.0963}(0.2), s_{0.3622}(0.3)\}$
	A_2	$\{s_{0.0293}(0.45), s_{0.3045}(0.15), s_2(0.4)\}$
	A_3	$\{s_{0.1092}(0.48), s_{0.4079}(0.12), s_{0.2793}(0.32), s_{0.5512}(0.08)\}$
	A_4	$\{s_{0.8251}(0.24), s_2(0.76)\}$
	A_5	$\{s_{0.2096}(0.4), s_{0.4729}(0.4), s_{0.7345}(0.1), s_{0.9206}(0.1)\}$

Table 7. Alternative relative scores.

DM	A_1	A_2	A_3	A_4	A_5
e_1	0.1499	0.1980	0.2709	0.1476	0.2337
e_2	0.1821	0.2186	0.1627	0.2345	0.2021
e_3	0.1123	0.2866	0.1799	0.2447	0.1766
e_4	0.2358	0.1184	0.2753	0.1747	0.1957
e_5	0.1568	0.2143	0.1675	0.2787	0.1828

4.3. Sensitivity Analysis

This section conducts a sensitivity analysis on the parameter χ in the logical function to investigate its impact on decision outcomes. The results are presented in Table 8 and Figure 7.

χ reflects the DM's sensitivity to differences between events. As χ increases from 0.1 to 1.0, the decision system's discrimination capability significantly enhances. At low sensitivity (χ = 0.1–0.2), the ranking is $A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4$; at medium sensitivity (χ = 0.3–0.8), A_4 surpasses A_2; at high sensitivity ($\chi = 1.0$), A_5 surpasses A_1, resulting in the ranking $A_3 \succ A_5 \succ A_4 \succ A_1 \succ A_2$. This indicates that increasing χ amplify DMs' responsiveness to subtle differences among alternatives. A_3 consistently remains optimal, demonstrating its absolute advantage, while the shifting positions of A_1 and A_5 reveal the changing relative attractiveness of these two alternatives under highly sensitive conditions.

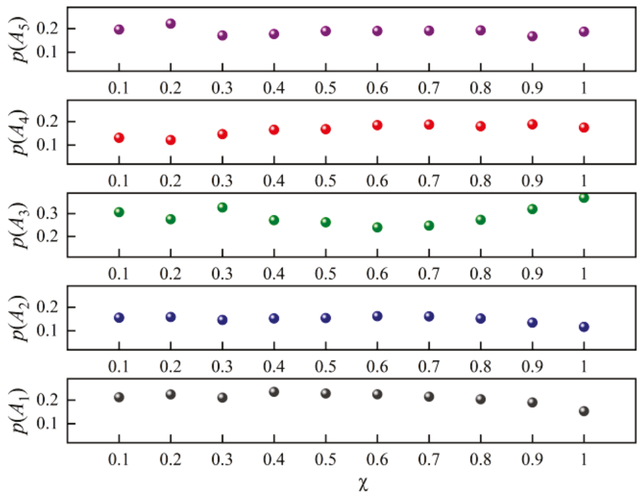


Figure 7. The quantum probabilities of alternatives under χ.

Table 8. The ranking results of alternatives under χ.

χ	A_1	A_2	A_3	A_4	A_5	Ranking Orders
0.1	0.2118	0.1555	0.3062	0.1309	0.1956	$A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4$
0.2	0.2237	0.1585	0.2753	0.1217	0.2208	$A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4$
0.3	0.2101	0.1458	0.3268	0.1464	0.1709	$A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$
0.4	0.2345	0.1524	0.2713	0.1649	0.1769	$A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$
0.5	0.2275	0.1539	0.2618	0.1676	0.1892	$A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$
0.6	0.2244	0.1619	0.2395	0.1843	0.1899	$A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$
0.7	0.2143	0.1608	0.2472	0.1869	0.1908	$A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$
0.8	0.2031	0.1519	0.2727	0.1801	0.1922	$A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$
0.9	0.1899	0.1345	0.3198	0.1881	0.1677	$A_3 \succ A_1 \succ A_4 \succ A_5 \succ A_2$
1.0	0.1529	0.1164	0.3695	0.1745	0.1867	$A_3 \succ A_5 \succ A_4 \succ A_1 \succ A_2$

4.4. Comparative Analysis

4.4.1. Comparative Analysis of Different Trust Propagation Methods

Compared with the model proposed by Liu et al. [36], whose all possible paths are considered in the trust propagation, a new mixed multi-path trust transfer aggregation operator that integrates both the path length and the path quality is defined, the ranking of the alternatives is: $A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$. In this paper, the shortest trust path is chosen for trust propagation, and the ranking of the alternatives is $A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$; both are the same. This is because the more DMs are involved in the trust propagation process, the greater the trust discount will be, ultimately leading to a negligible level of trust between DMs in this path. The proposed method produces ranking results that are consistent with Liu's method in the straightforward example given in this paper. However, the proposed method will show better operability and applicability in more complicated and general situations.

4.4.2. Comparative Analysis of Different Quantum Decision-Making Methods

Table 9 shows the differences between the proposed method and other reference methods, proving the superiority and rationality of the proposed method. The following summarizes the theoretical advantages of the proposed method:

(1) Evaluation Language: The first four decision-making methods require DMs to use precise numerical values for evaluation. However, in increasingly complex decision environments, the expression of human preferences often involves uncertainty. As a vital tool for expressing uncertain information, PLTS effectively captures DMs'

hesitation and exhibits good operational sensitivity. The methodology proposed in this paper constructs the evaluation process based on PLTS, making it more practical and operationally feasible.

(2) Weighting Method: Weighting is a critical step in enhancing the rationality of MAGDM results. Existing studies, such as Yan et al. [37] and Han and Liu [38], typically preset decision-maker weights, which are highly subjective. This paper determines weights through a decision-maker trust network and combines them with the entropy weight method to determine attribute weights, making the weight allocation more objective.

(3) Opinion Fusion: The classical Bayesian model fails to account for mutual interference among decision-makers. The similarity heuristic algorithm proposed by Moreira and Wichert [18] solves interference phase but relies heavily on prior knowledge, resulting in high subjectivity. Yan et al. [37] constructed an interference effect model based on minimum constraints and belief entropy, reducing subjectivity to some extent. However, it assumes identical interference effects for the same DM across different schemes, reflecting only overall uncertainty. The extended method proposed by Han and Liu [38], based on QLBN and belief entropy, does not account for the influence of decision-maker weights. This paper introduces a belief entropy-based interference effect calculation method. By computing the belief entropy for each pair of DMs regarding trust values or evaluations and summing these values, interference effects are obtained, enabling more precise probability calculations.

Table 9. Comparison of the proposed method with different methods.

Methods	Background	Weights		Interference
		Attribute	DM	
Bayesian network	Real number	-	-	-
Moreira and Wichert [18]	Real number	-	-	Similarity heuristic algorithm
Yan et al. [37]	Real number	Unknown	Known	Minimum constraint information entropy
Han and Liu [38]	Real number	Unknown	Known	Standardized information entropy
Proposed method	PLTS	Unknown	Unknown	Standardized entropy measurement

As shown in Table 10, the values of the interference term calculated by different methods vary significantly. The interference term obtained by the similarity heuristic algorithm proposed by Moreira and Wichert [18] relies on prior knowledge and exhibits a high degree of subjectivity, resulting in substantial discrepancies compared to our findings. The interference effect calculation method of Yan et al. [37] based on minimum constraints and belief entropy assumes identical interference effects for different schemes by the same DM,, reflecting only the overall uncertainty of the DM. The extended method proposed by Han and Liu [38], based on QLBN and belief entropy, does not account for the influence of DM weights.

Table 10. Comparison of interference term results with different methods.

Methods	R_1	R_2	R_3	R_4	R_5
Moreira and Wichert [18]	−0.0161	−0.0436	0.0181	0.0414	0.0237
Yan et al. [37]	−0.0076	−0.0177	0.0541	0.0370	−0.0103
Han and Liu [38]	−0.0242	−0.0780	0.0215	−0.0139	0.0292
Proposed method	0.0031	−0.1172	0.0283	−0.0901	−0.0501

The following conclusions can be drawn based on the results in Table 11 and Figure 8. The ranking results of the classical Bayesian network are inconsistent with those in this paper because the interference effects in the decision-making process are not considered. Moreira and Wichert [18] and Yan et al. [37] are inconsistent with the ranking results of this paper, as the former is based on a priori knowledge, which is somewhat subjective, and both assume that the same DM has the same interference effect for different alternatives. The optimal alternative proposed by Han and Liu [38] aligns with the results of the best alternative presented in this paper, which also employs the QLBN and considers quantum interference effects.

Table 11. Comparison of ranking results with different methods.

Methods	A_1	A_2	A_3	A_4	A_5	Ranking Orders
Bayesian network	0.1659	0.1560	0.2293	0.2419	0.2069	$A_4 \succ A_3 \succ A_5 \succ A_2 \succ A_1$
Moreira and Wichert [18]	0.1527	0.2199	0.2146	0.2086	0.2041	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$
Yan et al. [37]	0.1659	0.1560	0.2293	0.2419	0.2069	$A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$
Han and Liu [38]	0.1660	0.2005	0.2231	0.2128	0.1976	$A_3 \succ A_4 \succ A_2 \succ A_5 \succ A_1$
Proposed method	0.2240	0.1487	0.2418	0.1841	0.2014	$A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$

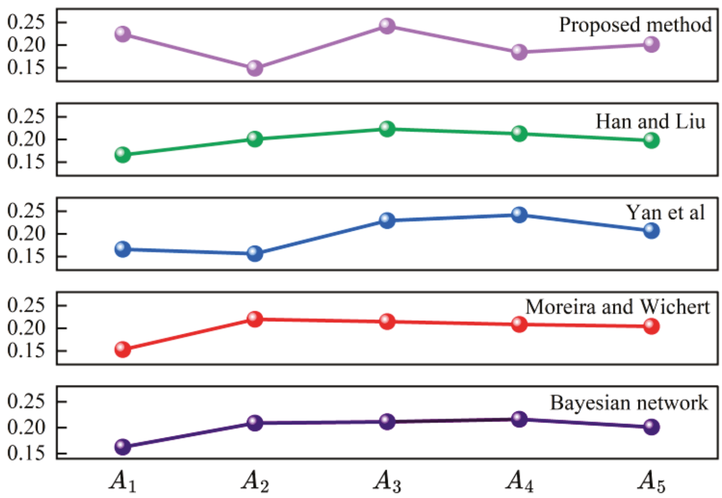


Figure 8. Ranking results of the different methods.

Compared with other methods, the advantages of this paper's method are:

- In the analysis of social trust network, a trust aggregation model based on QLBN is constructed, which fully takes into account the trust interference phenomenon driven by the subjectivity of DMs.
- Based on the evaluation information of different DMs, the entropy weighting method is used to determine the attribute weights of different DMs, and different DMs can give different weights to the same evaluation attributes, which fully takes into account the differences in the preferences of DMs.
- The evaluation information of multiple DMs is integrated using a QLBN, and a calculation method for the dynamic interference effect is proposed based on the belief entropy, which quantifies the degree of interference among DMs and fully takes into account the uncertainty in the decision-making process.

4.4.3. Comparative Analysis of Different MAGDM Methods

(1) A comparison with the classic ELECTRE method [39] is presented in Table 12. The ELECTRE method [39] determines DM weights by combining subjective weights and adjustment factors, while attribute weights are established through constructing a multi-objective programming model. As shown in Table 12, the rankings obtained by the ELECTRE method [39] align with those derived by the proposed method, fully demonstrating the universality of the proposed approach.
(2) A comparison with the classic PROMETHEE method [40] is presented in Table 13. The PROMETHEE method [40] employs a hybrid trust network to determine DM weights, combining public and expert-level information to establish attribute weights while ignoring the interference effects among DMs. As shown in Table 12, the rankings obtained by the PROMETHEE method [40] are largely consistent with those derived by the proposed method, fully demonstrating the latter's universality.
(3) A comparison with the classic TODIM method [41] is presented in Table 14. The TODIM method [41] presets DM weights and determines attribute weights through a combination of subjective and objective weighting methods, exhibiting a strong subjective nature. As shown in Table 14, the optimal and worst solutions obtained by the TODIM [41] method and the proposed method are consistent. However, the order of the second and third-ranked solutions is reversed. This discrepancy arises because the TODIM method [41] disregards interference effects in the decision-making process.

Table 12. Comparison of ranking results with ELECTRE method.

Methods	A_1	A_2	A_3	A_4	Ranking Orders
ELECTRE [39]	0.3200	0.8200	0.0000	0.1300	$A_2 \succ A_1 \succ A_4 \succ A_3$
Proposed method	0.2687	0.3059	0.1910	0.2344	$A_2 \succ A_1 \succ A_4 \succ A_3$

Table 13. Comparison of ranking results with PROMETHEE method.

Methods	A_1	A_2	A_3	A_4	A_5	Ranking Orders
PROMETHEE [40]	−0.2689	0.8571	0.2371	0.1010	−0.9263	$A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_5$
Proposed method	0.1154	0.3014	0.2641	0.1734	0.1457	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$

Table 14. Comparison of ranking results with TODIM method.

Methods	A_1	A_2	A_3	A_4	Ranking Orders
TODIM [41]	−1.4300	−0.3600	−1.9300	−1.2200	$A_2 \succ A_1 \succ A_3 \succ A_4$
Proposed method	0.1896	0.3146	0.2997	0.1961	$A_2 \succ A_3 \succ A_1 \succ A_4$

4.5. Application Guide

This section aims to elucidate the core application scenarios, target user groups, and the universal framework and value of this research model in addressing similar decision-making problems from a higher-level perspective, providing clear guidance for other researchers and managers applying this model.

Target Scenarios: This method is specifically designed for complex decision-making contexts where (1) DMs struggle to provide precise judgments; (2) complex interference exists among DMs; (3) decision-making groups exhibit divergent opinions and interactive dynamics.

Target Users: Primarily serves public management departments and corporate strategic teams engaged in scientific decision-making within such complex environments.

General Process: Application of this method follows a five-step framework: (1) Define decision objectives and alternatives; (2) Establish an attribute system and calculate weights; (3) Employ the probabilistic language for alternative evaluation and trust assessment; (4) Execute model calculations to quantify trust and interference effects; (5) Interpret group preference dynamics using quantum interference terms to inform final decisions. This framework ensures model outputs reveal not only rankings but also the underlying group cognitive dynamics driving decisions.

4.6. Discussion

In the preceding section, sensitivity analysis and comparative analysis demonstrated the feasibility and superiority of this model. These theoretical and numerical advantages were further validated through specific case applications.

The ranking results of this study $A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$ profoundly validate the theoretical framework's efficacy. The ultimate triumph of the optimal alternative A_3 (groundwater recharge) stems not from simple weighted calculations, but directly embodies the quantum interference effect within the theoretical model: DMs experienced positive mutual reinforcement in their evaluations across four attributes (c_1 social, c_2 economic, c_3 environmental and c_4 technological), generating positive interference that amplified the appeal of the proposal. Throughout this process, the PLTSs provided the foundation for DMs to articulate their uncertain evaluative preferences. Building upon this, the QLBN clearly reproduced the entire journey of collective belief—from its "superposition" state of uncertainty to its "collapse" into a final decision. Consequently, the results of this case study provide compelling evidence that the proposed PL-QLBN algorithm can more authentically simulate and explain complex decision-making behaviors.

More importantly, this paper provides a clear behavioral interpretation for abstract parameters in quantum theory. In the method proposed herein, we introduce a belief entropy-based method for quantifying interference effects, endowing quantum parameters with more explicit explanatory power for decision-making behavior. Specifically, the phase difference is no longer solely dependent on abstract mathematical constructs but is estimated through the information entropy difference between DMs' evaluation vectors. This reflects the uncertainty and divergence in opinions among different DMs. belief entropy is employed here to measure the uncertainty in DMs' evaluations. When DM evaluations exhibit high consistency, the phase difference approaches 0, yielding a positive interference term that produces constructive interference, reinforcing consensus. Conversely, when evaluations show significant divergence, the phase difference approaches π, resulting in a negative interference term that causes destructive interference, weakening the overall evaluation. This interference mechanism behaviorally corresponds to the non-independent decision-making process in reality, where DM opinions mutually influence and partially depend on one another.

This seamless integration from theory and validation to application ultimately highlights the profound practical value of this research. The practical impact of this research lies in providing a robust decision-making analysis tool. The proposed PL-QLBN algorithm precisely quantifies the interactive influences and uncertainties among DMs, offering a novel approach to addressing complex decision problems rife with contradictions and uncertainties, such as selecting a wastewater reuse alternative for Hefei City. The application of this model effectively assists government departments in discerning group decision dynamics within complex environments, enabling more scientific strategic choices that closely align with real human cognitive processes. It holds broad applicability and dissemination value.

5. Conclusions

In this paper, a multi-attribute quantum group decision-making method based on PLTS and QLBN is employed to select the optimal wastewater reuse alternative. A new method for solving the quantum interference term is also proposed. When the DM provides an evaluation, ambiguity and uncertainty may arise. In addition to the uncertain information expressed by DMs, trustworthy behavioral interactions among DMs may also have a significant impact on the decision results. Therefore, this paper introduces PLTSs and QLBN into the multi-attribute group decision-making model to quantify the interference effects in the decision-making process. By applying quantum probability theory to solve the multi-attribute group decision-making problem, the uncertain belief state of the DM can be well explained, and a new approach is provided for multi-attribute group decision-making. The model is applied to the case of selecting the wastewater reuse alternative in Hefei City, and the comparative analysis demonstrates that the method is effective in simulating the disagreement, contradiction, and unpredictability inherent in the human decision-making process.

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Fuzzy Graphic Binary Matroid Approach to Power Grid Communication Network Analysis

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Abstract: Matroid is a mathematical structure that extends the concept of independence. The fuzzy graphic binary matroid serves as a generalization of linear dependence, and its properties are examined. Power grid networks, which manage the generation, transmission, and distribution of electrical energy from power plants to consumers, are inherently a complex system. A key objective in analyzing these networks is to ensure a reliable and uninterrupted supply of electricity. However, several critical issues must be addressed, including uncertainty in communication links, detection of redundant or sensitive circuits, evaluation of network resilience under partial failures, and optimization of reliability in interconnected network systems. To support this goal, the concept of a fuzzy graphic binary matroid is applied in the analysis of power grid communication network, offering a framework that not only incorporates fuzziness and binary conditions but also enables systematic identification of weak circuits, redundancy planning, and reliability enhancement. This approach provides a more realistic representation of operational conditions, ensuring better fault tolerance and improved efficiency of the grid.

Keywords: fuzzy matroid; fuzzy graphic binary matroid; minimum reliablity; fuzzy incidence matrix

MSC: 05C62; 05C72; 05C70; 05C90; 94C15

1. Introduction

Graph theory provides a powerful framework for abstracting and solving complex problems through the analysis of connectivity and structure [1]. The field traces its origin to 1736, when Leonhard Euler resolved the renowned Konigsberg Bridge Problem, thereby establishing the foundation for modern network analysis [2]. The discipline's first formal textbook was published by Denes Konig in 1936. A significant advancement occurred in 1965 when Zadeh introduced fuzzy set theory in his seminal paper [3], addressing critical limitations in modeling real-world ambiguity by a mathematical bridge between human reasoning and computational systems. Fuzzy graphs extend classical graph structures by incorporating fuzzy set principles, allowing vertices and edges to possess membership values instead of strict binary relationships.

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A matroid is a mathematical structure that provides a framework for understanding linear independence, effectively bridging concepts from graph theory and linear algebra [4]. The theory of matroid was introduced by Hassler Whitney in 1935, who recognized that its axioms form a common abstraction of independence found in both graphs and matrices [5]. In 1936, Saunders Mac Lane explored connections between matroids and projective geometry. Van der Waerden explored analogies between algebraic dependence and linear dependence, contributing significantly to the algebraic foundation of the matroid theory in 1937. In 1940, Richard Rado expanded the scope of matroid theory by introducing independence systems, which generalized the idea of independent sets. The significant contributions from Tutte [6], who played a central role in expanding the field of matroid theory in 1950. Graphic matroids have practical applications in network reliability analysis and electrical grid design, where they are used to model and analyze the robustness and failure modes of networks [7]. Tutte was first to fully develop the theory of graphic matroids, defining a matroid in terms of the cycles of a graph and introducing the notion of dual matroids [8]. Pitsoulis [9] introduced two recognition algorithms for signed-graphic matroid, providing necessary and sufficient conditions for identifying a matroid in 2010.

The foundation of fuzzy graph [10] theory was first established by Kaufmann in 1973. Rosenfeld significantly advanced the field in 1975 by developing fuzzy graphs through the integration of fuzzy relations with fuzzy sets [11]. This innovative framework provided a more adaptable representation of relationships, overcoming the limitations of graph structures. Rosenfeld offered a significant advantages for modeling systems, including social networks, transportation networks, and biological network interactions.

Domination sets are vital in optimizing network design, particularly in ensuring that certain key nodes control or monitor the entire network [12]. The concept of domination sets, explored by Talebi and Rashmanlou in 2019 [12] presents opportunities for optimizing network design by ensuring that critical nodes efficiently control entire networks. Future developments in network analysis will likely focus on enhancing the handling of uncertainty and imprecision through advanced fuzzy and vague graph models [13,14]. Rashmanlou et al. (2018) introduced interval-valued fuzzy graphs, which offer potential for future exploration in uncertain environments [15]. Shoaib et al. (2020) contribute to this field with complex Pythagorean fuzzy graphs, offering a way to manage even more dimensions of uncertainty [16]. Future implications for energy management systems and sustainable infrastructure can be done by quantifying energy and network performance under various conditions [17].

Recent advancements in fuzzy graph theory and its applications to multi-attribute decision-making (MADM) problems highlight the growing relevance of uncertainty modeling in complex systems. The study by Rao et al. (2025) introduces a q-rung orthopair fuzzy Zagreb index, enhancing decision-making efficiency by incorporating flexible degrees of membership and non-membership, offering improved sensitivity in ranking alternatives under uncertainty [18]. Kosari et al. investigated topological indices in fuzzy graphs and their role in decision-making processes in 2024. Their analysis presents generalized measures that help quantify network attributes, thereby enabling better insight into system behavior and supporting optimal decision strategies in uncertain environments. This work builds a bridge between topological indices and fuzzy set theory for robust MADM frameworks [19].

Kosari et al. [20] examine vague fuzzy graphs to model the connectivity structure in university buildings, providing practical insight into infrastructure design and optimization in 2023. Expanding on the structural metrics of fuzzy graphs. He propose a novel formulation of perfectly regular fuzzy graphs and demonstrate their utility in psychological sciences, particularly in modeling cognitive stability and regular behavioral patterns [21].

This work emphasizes the capacity of fuzzy graphs to model imprecise human-centric data with high granularity. In 2024, Khan et al. explored interval-valued picture fuzzy hypergraphs, introducing a richer framework to represent complex, layered relationships in decision-making [22]. The integration of hypergraph structures with picture fuzzy sets allows for more expressive modeling of group preferences and interactions, accommodating multiple forms of indeterminacy. Shi et al. in 2024 utilize the connectivity index of cubic fuzzy graphs to identify potential tsunami danger zones [23].

In 1988, Goetschal and Voxman introduced the concept of fuzzy matroids, which integrate both graphical and algebraic structures through the use of membership degrees [24]. Building on this foundation, Sabana and Sameena [25] proposed a new class of matroid derived from fuzzy graphs in 2019. In 2021, Sabana and Sameena extended the theory by introducing the concept of representable fuzzy matroids, and developed a framework for constructing fuzzy matroid from fuzzy vector spaces [26]. Isomorphism and structural characteristics of fuzzy matroids was studied in [27], deepening the theoretical understanding of their properties. Binary and regular matroids were characterized by Anushka Murthy [28]. A matroid is termed binary if it admits a representation over the finite field [29], where the independent sets correspond to linearly independent subsets of matrix columns. A central structural property of binary matroids is the symmetric difference condition [30]. Rashmanlou et al. (2020) proposed a novel method for solving the shortest path problem using multi-valued logic and fuzzy set theory to account uncertainties in real-world networks [31].

1.1. Research Gap

Matroids provide a unifying framework for studying independence in various mathematical structures. Their fundamental properties have been studied since 1935, with several advancements explored in subsequent years. The concept of graphic matroids was introduced in 1965, and in 1988, the concept of fuzzy sets was integrated with matroids. In 2019, matroids were defined using fuzzy graphs. Building on these concepts, the fuzzy graphic matroid was introduced by incorporating the principles of fuzzy matroids and fuzzy graphs. Among the various extensions of fuzzy graphs, fuzzy graphic matroids have attracted significant research interest. This is primarily because they have been studied and applied in areas such as cryptography, network analysis, and optimization problems. However, there is limited research available on fuzzy graphic matroids in the existing literature. This motivates us to explore advanced aspects of fuzzy graphic matroids further.

1.2. Motivation

The communication network of the power grid is a critical component of modern electrical power systems, essential for enabling the secure and reliable exchange of information between various entities. As of today, societies become increasingly dependent on electricity, ensuring the continuous and efficient operation of these networks is important. Traditional methods often face challenges in modeling and optimizing the grid's performance, especially in scenarios involving unreliable connections or components. Fuzzy graphic matroid theory, with its versatile applications across multiple fields, offers a powerful approach to address these challenges. By modeling the power grid communication network structure as a fuzzy graphic binary matroid, it becomes easier to analyze and optimize its components, identify critical links, and enhance overall reliability.

1.3. Novelty and Key Contributions

Power grid network analysis is vital for the stable and efficient functioning of modern electrical systems. This framework models a novel contribution to the study of power grid network analysis that helps to optimize the flow of electricity, minimize energy losses, and ensure that the grid operates efficiently. This analysis, using fuzzy graphic binary matroids, identifies weak links in the system and helps improve the overall resilience of the network.

(1) The application of fuzzy graphic binary matroids to power grid networks provides the tools to identify critical links and ensures optimal network performance of the power grid communication network.
(2) The fuzzy graphic binary matroid helps to simplify complex systems into a manageable model for better decision-making and computational efficiency.

1.4. Organization of the Paper

The paper is organized as follows: Section 2 provides the necessary background and foundational concepts required for the paper. The detailed conditions for the formation and application of fuzzy graphic binary matroids over the finite field $\mathcal{F}_2$ are presented and defined in Section 3. Section 4 explores the network structure represented as a fuzzy graphic binary matroid. The reliability and performance of the communication links within the grid are analyzed and the critical components. Section 5 presents the findings derived from the application of fuzzy graphic binary matroid theory to the power grid network. The paper is concluded in Section 6.

2. Preliminaries

Definition 1 ([32])**.** *Let E be a ground set and I be the set of all independent sets. The pair $\mathcal{M} = (E, I)$ is called a matroid, with the following properties:*

(i) $\phi \in I$
(ii) *If $T_1 \in I, T_2 \subset T_1$, then $T_2 \in I$*
(iii) *If $T_1, T_2 \in I; |T_2| \geq |T_1|$, then there exists an element $x \in T_2 \backslash T_1$ such that $T_1 \cup \{x\} \in I$.*

Definition 2 ([33])**.** *Let $\sigma^* : V \longrightarrow [0,1]$ is a fuzzy vertex set of G and $\mu^* : V \times V \longrightarrow [0,1]$ is a fuzzy edge set of G where the membership value of each edge e_{ij} satisfies the condition $\mu^*(e_{ij}) \leq min(\sigma^*(e_i), \sigma^*(e_j))$ for every $e_i,\ e_j\ \in V$. The pair $G = (\sigma^*, \mu^*)$ is called a fuzzy graph.*

Let $\mathbb{I} : X \longrightarrow [0,1]$ be a fuzzy subset on $\mathbb{E}$. If $M, N \in \mathfrak{F}(\mathbb{E})$, then

(i) $supp(M) = \{e_{ij} \in \mathbb{E} \mid M(e_{ij}) > 0\}$, a crisp set
(ii) $m(M) = min\{M(e_{ij}) \mid e_{ij} \in supp(M)\}$
(iii) $M \cup N = max\{M(e_{ij}), N(e_{ij})\}, e_{ij} \in \mathbb{E}$
(iv) $M \cap N = min\{M(e_{ij}), N(e_{ij})\}, e_{ij} \in \mathbb{E}$.

Note: For the sake of convenience, different notations are used here.

Definition 3 ([34])**.** *Let $\mathbb{E}$ be a finite set and $\mathbb{I} \in \mathfrak{F}(\mathbb{E})$ be the family of independent sets. The pair $FM = (\mathbb{E}, \mathbb{I})$ is a fuzzy matroid if the following conditions are satisfied:*

(i) $\phi \in \mathbb{I}$
(ii) *$M \subset N$ and $N \in M$, where $M \subset N, M(e_{ij}) \leq N(e_{ij})$ forall $e_{ij} \in \mathbb{E}$*
(iii) *If $M, N \in \mathbb{I}$ with $|supp\ (M)| \leq |supp\ (N)|$, there exists $P \in \mathbb{I}$ such that*
    (a) $M \subset P \subseteq M \cup N$
    (b) $m(P) = min\ \{m(M), m(N)\}$.

Definition 4 ([35])**.** *A fuzzy graph $G(\sigma, \mu)$ that satisfies the independence property of a fuzzy matroid is defined to be a fuzzy graphic matroid. It is defined on the edge set of the fuzzy graph $G(\sigma, \mu)$ and it is denoted by $\mathfrak{M}(G)$.*

Definition 5 ([35])**.** *The dual of a fuzzy graphic matroid on $\mathbb{E}$ is $\mathfrak{M}^*(G)$ whose bases are the complements of the bases $\mathfrak{M}(G)$.*

Remark 1. *If $\mathfrak{M}(G)$ is a fuzzy graphic matroid, then the sub-bases are the spanning sets and the hypobases are the hyperplanes.*

Definition 6 ([35])**.** *A minor of a fuzzy graphic matroid $\mathfrak{M}(G)$ is formed by two operations:*

- *An edge e_{ij} is deleted from $\mathbb{E}$ without changing the independence or rank of $\mathfrak{M}(G)$.*
- *An edge e_{ij} is contracted from $\mathbb{E}$ and the rank of every fuzzy set is decreased.*

3. Fuzzy Graphic Binary Matroids

A $\mathfrak{M}(G)$ is said to be a fuzzy graphic binary matroid if it satisfies the following conditions:

(i) **Fuzzy Membership value:** The membership function $\mu_{ij} : \mathbb{E} \to [0,1]$ that assigns a membership value to each edge $e_{ij} \in \mathbb{E}$ such that each edge is either included or excluded from circuits. There exists a threshold value $\alpha \in [0,1]$ such that
   - If $e_{ij} \geq \alpha$, then the edge is considerd as 1.
   - If $e_{ij} < \alpha$, then the edge is considerd as 0.

(ii) **Totally Unimodular Matrix:** A matrix is called totally unimodular if every square submatrix has a determinant equal to either $0, 1, or -1$.

(iii) **Symmetric Difference Property:** For any two circuits $\mathbb{C}_1$ and $\mathbb{C}_2$ in $\mathfrak{M}(G)$, the symmetric difference $\mathbb{C}_1 \Delta \mathbb{C}_2$ is a union of disjoint circuits. The edges in $\mathbb{C}_1 \Delta \mathbb{C}_2$ form disjoint cycles or an empty set.

Example 1. *In Figure 1, $\mathbb{E} = \{e_{12}(0.4), e_{24}(0.8), e_{34})(0.7), e_{13}(0.3), e_{15}(0.5), e_{25}(0.6), e_{36}(0.2), e_{46}(0.3)$.*

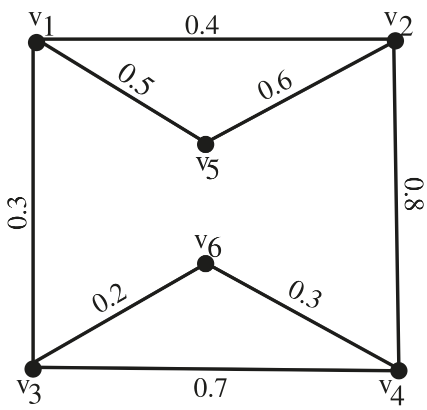

Figure 1. Fuzzy graph.

Let $\alpha = 0.4$, the incidence matrix obtained after thresholding is

	v_1	v_2	v_3	v_4	v_5	v_6
e_1	1	0	0	0	1	0
e_2	0	1	0	0	1	0
e_3	1	1	0	0	0	0
e_4	0	1	0	1	0	0
e_5	0	0	0	1	0	1
e_6	0	0	1	0	0	1
e_7	0	0	1	1	0	0

The fuzzy graph as shown in Figure 2 is obtained after applying the thershold value $\alpha = 0.4$.

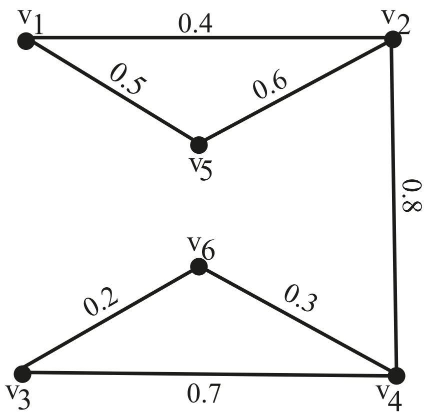

Figure 2. Fuzzy graph after thersholding.

The circuits $\mathbb{C}_1, \mathbb{C}_2 \in \mathbb{E}$.

$$\mathbb{C}_1 \Delta \mathbb{C}_2 = \{e_{12}, e_{25}, e_{15}\} \Delta \{e_{34}, e_{36}, e_{46}\} \tag{1}$$

Theorem 1. *A fuzzy graphic matroid is binary if and only if its family of circuits is closed under symmetric difference.*

Proof of Theorem 1. Let $\mathfrak{M}(G)$ be a fuzzy graphic matroid. If $\mathfrak{M}(G)$ is binary, then it is representable over a finite field $\mathcal{F}_2$.

Necessary Condition: If $\mathfrak{M}(G)$ is binary, then every circuit in $\mathfrak{M}(G)$ can be represented as a linear combination of other circuits over $\mathcal{F}_2$. Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be circuits in $\mathfrak{M}(G)$. Since $\mathfrak{M}(G)$ is binary, the circuit difference $\mathbb{C}_1 \backslash \mathbb{C}_2$ can be expressed as a linear combination of circuits in $\mathfrak{M}(G)$.

$$\mathbb{C}_1 \backslash \mathbb{C}_2 = \sum_{i=1}^{k} a_i C_i \tag{2}$$

$\mathbb{C}_i$ are the circuits in $\mathfrak{M}(G)$ and $a_i \in \mathcal{F}_2$. Hence, by the circuit difference property, the difference of any two circuits in $\mathfrak{M}(G)$ is expressible as a linear combination of other circuits. Therefore, $\mathfrak{M}(G)$ satisfies the circuit difference property.

Sufficient Condition: Suppose that $\mathfrak{M}(G)$ satisfies the condition of a fuzzy graphic binary matroid, which implies that any pair of circuits $\mathbb{C}_1$ and $\mathbb{C}_2$ in $\mathfrak{M}(G)$ can be expressed as linear combinations of other circuits in $\mathfrak{M}(G)$ over $\mathcal{F}_2$. The circuit difference property implies that every circuit can be expressed in terms of other circuits. The incidence matrix A of the fuzzy graph G encodes the relationship among the circuits. Since A is totally unimodular, the linear dependence among circuits is consistent with the constraints of total unimodularity. Therefore, $\mathfrak{M}(G)$ satisfies the circuit difference property. □

Theorem 2. *If two circuits share exactly an edge, then the symmetric difference between the circuits contains at least one disjoint circuit.*

Proof of Theorem 2. Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be the circuits in $\mathfrak{M}(G)$. If $\mathbb{C}_1$ and $\mathbb{C}_2$ shares exactly one edge then the removal of the edge e_{ij} from the symmetric difference leaving the set $(\mathbb{C}_1 \oplus \mathbb{C}_2) \backslash \{e_{ij}\}$. The symmetric difference of two circuits that share edges results in a new circuit or a union of disjoint circuits. If $\mathbb{C}_1$ and $\mathbb{C}_2$ share exactly one edge, then the remaining edges must form at least one disjoint circuit. The connected components of edges must be independent, and by the definition of a circuit, the set of remaining edges forms a minimal dependent set. Therefore, the symmetric difference of the circuits $\mathbb{C}_1$ and $\mathbb{C}_2$ contains at least one disjoint circuit in the fuzzy graphic matroid. □

Lemma 1. *If a fuzzy graphic matroid is binary, then it can be represented by a matrix A over $\mathcal{F}_2$ such that any columns corresponding to a circuit in $\mathfrak{M}(G)$ are dependent over $\mathcal{F}_2$.*

Proof of Lemma 1. If $\mathfrak{M}(G)$ is binary, then by the definition of a binary matroid there exists a matrix A over $\mathcal{F}_2$ representing the fuzzy graphic binary matroid such that the columns of $M(G)$ corresponds to a circuit of $\mathfrak{M}(G)$. Hence, the linear combination of columns in any circuit sums to zero in $\mathcal{F}_2$. □

Lemma 2. *If a matrix $M(G)$ over $\mathcal{F}_2$ represent a fuzzy graphic binary matroid, then determinant $det(M^{'})(G)$ for any square submatrix $M^{'}(G)$ in M is $0, 1$, and -1.*

Proof of Lemma 2. Let $\mathfrak{M}(G)$ be binary. The dependent sets holds over $\mathcal{F}_2$ and therefore, the determinant of any square submatrix $M^{'}(G)$ in A must be $0, 1$, and -1 for a binary matrix over $\mathcal{F}_2$. □

Lemma 3. *If a matrix $M(G)$ of a fuzzy graph G is totally unimodular, then the fuzzy graphic matroid is binary.*

Proof of Lemma 3. Let A be an $m \times n$ totally unimodular matrix, then every square submatrix of A has a determinant $\{0, \pm 1\}$. A fuzzy graphic matroid on a ground set $\mathbb{E}$ is binary if there exists a representation matrix over $\mathcal{F}_2$ such that the circuits of $G(\sigma, \mu, \mathfrak{M})$ correspond to the columns of the incidence matrix A. Since A is unimodular, every entry of A must be an integer, and every square submatrix $A^{'}(G)$ is totally unimodular. □

Theorem 3. *A fuzzy graphic matroid is binary if and only if the incidence matrix A of the fuzzy graph G is totally unimodular.*

Proof of Theorem 3. A fuzzy graphic matroid is binary if it can be represented over a finite field. The circuits in the fuzzy graphic matroid can be expressed as a linear combination of other circuits over $\mathcal{F}_2$.

Necessary Condition: Assume that $\mathfrak{M}(G)$ is binary. Every circuit in $\mathfrak{M}(G)$ can be expressed as a sum of other circuits. The incidence matrix A defines the relationship between the vertices and edges of the fuzzy graph G. Since the circuits in the fuzzy graph G are represented as a sum of other circuits, the matrix formed by these circuits also behaves as a linear combination over $\mathcal{F}_2$. The incidence matrix satisfies certain properties that ensure it is totally unimodular. Any linear combination of the rows of A corresponding to the circuits results in a matrix where the submatrix has the determinant $\{0, \pm 1\}$. Therefore, $\mathfrak{M}(G)$ is binary.

Sufficient Condition: If the incidence matrix is totally unimodular, then every submatrix of the circuits $\mathfrak{M}(G)$ remains within the set of values $\{0, 1\}$. All the circuits in $\mathfrak{M}(G)$ can be expressed as a sum of other circuits over $\mathcal{F}_2$. The total unimodularity of the incidence matrix implies that $\mathfrak{M}(G)$ can be represented over $\mathcal{F}_2$. □

Theorem 4. *If a fuzzy graphic matroid is binary, then every minor of $\mathfrak{M}(G)$ is binary.*

Proof of Theorem 4. Let G be a fuzzy graph and $\mathbb{E}$ be the ground set of a fuzzy graphic matroid. The fuzzy graphic matroid is represented by the incidence matrix $M(G)$. If $\mathfrak{M}(G)$ is binary, then the incidence matrix A is totally unimodular and it can be represented over $\mathcal{F}_2$. The fuzzy graphic matroid is binary, and every minor of $\mathfrak{M}(G)$ is binary. The minor of a fuzzy graphic matroid is obtained by a sequence of contraction and deletion of edges in the ground set of a fuzzy graph G. If $e_{ij} \in \mathbb{E}$ is an edge of G, then deletion of e_{ij} results in a fuzzy graph $G^{'}$. The incidence matrix of $G^{'}$ is $A(G^{'})$ obtained by removing the column corresponding to e_{ij} from A. Since the deletion of a column does not affect the total unimodularity of a matrix, $\mathfrak{M}'(G)$ is binary.

Similarly, the contraction of the edge $e_{ij} \in \mathbb{E}$ corresponds to the elementary row operation on A by removal of the row and column corresponding to the vertices and edges incident to the edge e_{ij}. The matrix formed after the removal of a row and column of an edge e_{ij} is totally unimodular. Since the fuzzy graphic matroid formed by contraction of an edge e_{ij} of a fuzzy graph G is binary. The deletion and contraction of an edge from a fuzzy graph G preserves the total unimodularity. Therefore, every minor of $\mathfrak{M}(G)$ obtained by contraction and deletion of an edge e_{ij} has a totally unimodular incidence matrix. Consequently, every minor of $\mathfrak{M}(G)$ can be represented over $\mathcal{F}_2$. □

Theorem 5. *$\mathfrak{M}(G)$ is represented over $\mathcal{F}_2$ if and only if its incidence matrix A can be transformed into a totally unimodular matrix.*

Proof of Theorem 5. Let G be a fuzzy graph, and the fuzzy graphic binary matroid is defined on the edge set $\mathbb{E}$, and its incidence matrix represents the relationship between the vertices and edges. The fuzzy graphic matroid $\mathfrak{M}(G)$ is said to be representable over the finite field $\mathcal{F}$ if there exists a matrix representation of the fuzzy graphic binary matroid whose entries belong to $\mathcal{F}$. This condition holds if and only if the incidence matrix is totally unimodular.

Necessary condition: Let $\mathfrak{M}(G)$ be representable over $\mathcal{F}_2$. The incidence matrix can be transformed into a matrix with entries in $\mathcal{F}_2$ by elementary row operations. Since $\mathfrak{M}(G)$ is binary, the matrix must be totally unimodular, which ensures that the fuzzy graphic matroid is binary. Therefore, any set of dependent edges can be expressed as a linear combination over $\mathcal{F}_2$.

Sufficient condition: If the incidence matrix of a fuzzy graph G is totally unimodular. A totally unimodular matrix has the property that every square submatrix has a determinant $\{0, \pm 1\}$, which implies that the matrix can be transformed into a matrix with entries in $\mathcal{F}_2$. Consequently, $\mathfrak{M}(G)$ can be represented over $\mathcal{F}_2$ as its circuits can be expressed as linear combinations over the finite field. □

Lemma 4. *The dual of a fuzzy graphic binary matroid is binary.*

Proof of Lemma 4. Let $\mathfrak{M}(G)$ be a fuzzy graphic matroid represented by an incidence matrix A. If $\mathfrak{M}(G)$ is binary, then the incidence matrix $M(G)$ is totally unimodular. The dual of $\mathfrak{M}(G)$ is represented by the transpose of the incidence matrix $A(G)^T$. Since the transpose of a totally unimodular matrix is totally unimodular. Therefore, the dual of a $\mathfrak{M}(G)$ is binary and it can be represented over $\mathcal{F}_2$ by a totally unimodular matrix $A(G)^T$. □

Theorem 6. *If $\mathfrak{M}(G)$ is a fuzzy graphic binary matroid, then adding an edge to the fuzzy graph G corresponding to a totally unimodular matrix is also binary.*

Proof of Theorem 6. Let $\mathfrak{M}(G)$ be a fuzzy graphic binary matroid and $\mathbb{E}$ be the edge set of a fuzzy graph G. The incidence matrix of fuzzy graph G is totally unimodular. The binary property implies that every submatrix of A has determinant $\{0, \pm 1\}$ and $\mathfrak{M}(G)$ becomes representable over a finite field $\mathcal{F}_2$. Let e_{ij} be an edge added to the fuzzy graph G and $e_{ij} \notin \mathbb{E}$, which results in a new fuzzy graph G'. The incidence matrix $A(G')$ is obtained by appending the incidence vector of e_{ij} to the matrix A. Since A is unimodular and e_{ij} corresponds to a totally unimodular vector, the augmented matrix $A(G')$ remains totally unimodular. The preservation of total unimodularity implies that every square submatrix of $A(G')$ has a determinant $\{0, \pm 1\}$. Therefore, adding an edge e_{ij} to a fuzzy graph G that corresponds to a totally unimodular matrix preserves the condition of the fuzzy graphic binary matroid, and $\mathfrak{M}(G)$ is representable over $\mathcal{F}_2$. □

Theorem 7. *A fuzzy graphic matroid is binary if it can be decomposed into a direct sum of binary matroids corresponding to a totally unimodular submatrix of the incidence matrix of G.*

Proof of Theorem 7. Let G be a fuzzy graph. The incidence matrix A of the fuzzy graph G encodes the relationships between the vertices and edges of a fuzzy graph G, where the independence corresponds to the acyclic subgraphs of G. If $\mathfrak{M}(G)$ can be decomposed into a direct sum of the fuzzy graphic matroid $\mathfrak{M}(G)_1, \mathfrak{M}(G)_2, \ldots, \mathfrak{M}(G)_k$ where each fuzzy graphic matroid $\mathfrak{M}(G)_k$ corresponds to a totally unimodular submatrix A_i of A. Since A_i is totally unimodular, the corresponding fuzzy graphic matroid is binary. If the incidence matrix of the direct sum of the diagonal and the diagonal matrices preserves total unimodularity, then the direct sum of binary fuzzy graphic matroids remains binary. Therefore, the fuzzy graphic matroid represented by the diagonal matrix formed by $\mathfrak{M}(G)_1, \mathfrak{M}(G)_2, \ldots, \mathfrak{M}(G)_k$ is binary.

Conversely, if $\mathfrak{M}(G)$ is binary, then its incidence matrix is totally unimodular, which implies that it can be decomposed into a direct sum of submatrices, each of which is totally unimodular. Therefore, $\mathfrak{M}(G)$ can be expressed as a direct sum of fuzzy graphic binary matroids. This completes the proof. □

Theorem 8. *A fuzzy graphic matroid is binary if and only if every circuit in $\mathfrak{M}(G)$ can be represented as a linear combination over $\mathcal{F}_2$ of other circuits in $\mathfrak{M}(G)$ that do not share edges with a fuzzy graph G.*

Proof of Theorem 8. Let $\mathfrak{M}(G)$ be a fuzzy graphic matroid and $\mathbb{E}$ be the ground set of $\mathfrak{M}(G)$ defined on the edge set of a fuzzy graph G. The circuits of the $\mathfrak{M}(G)$ correspond to the cycles of a fuzzy graph G.

Necessary Condition: Assume that $\mathfrak{M}(G)$ is binary. $\mathfrak{M}(G)$ can be represented as a linear combination over a finite field $\mathcal{F}_2$, and its incidence matrix A is totally unimodular. Consequently, any dependent edges between the cycles of a fuzzy graph G can be expressed as a linear combination over $\mathcal{F}_2$. Let $\mathbb{C} \subseteq \mathbb{E}$ be a circuit in $\mathfrak{M}(G)$. Since, $\mathfrak{M}(G)$ is binary, the circuits of $G(\sigma, \mu, \mathfrak{M})$ can be written as a linear combination over $\mathcal{F}_2$ of other circuits $\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_n$ as follows:

$$\mathbb{C} = \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \ldots \oplus \mathbb{C}_n \tag{3}$$

The symmetric difference ensures that if an edge appears an even number of times in the linear combination, then it is removed from the circuit. The circuits $\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_n$ must be disjoint sets in $\mathfrak{M}(G)$. Therefore, every circuit in $\mathfrak{M}(G)$ can be expressed as a linear combination of other circuits.

Sufficient Condition: Assume that every circuit in $\mathfrak{M}(G)$ can be expressed as a linear combination over $\mathcal{F}_2$ of other circuits in $\mathfrak{M}(G)$. For any circuit in $\mathfrak{M}(G)$, there exist $\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_n$ such that the circuits that do not share common edges in the fuzzy graph G. This implies that all the circuits in $\mathfrak{M}(G)$ satisfy the symmetric difference property. This completes the proof. □

4. Fuzzy Graphic Binary Matroid in Power Grid Network

In this section, a fuzzy graphic binary matroid is applied to model a power grid communication network of a region through the proposed algorithm. The algorithm helps in identifying weak links before complete failure; therefore, the network can achieve better reliability and reduce the risk of communication failures. The analysis of a power grid network using fuzzy graphic binary matroid offers a rigorous mathematical framework to model, evaluate, and optimize the reliability and fault tolerance of communication structures within power systems.

The analysis using fuzzy graphic binary matroid aids in the following:

- Identifying redundant paths to ensure communication continuity during partial failures.
- Detecting vulnerable components where failure risks are concentrated.
- Suggesting optimization strategies, such as upgrading low-reliability links or reinforcing specific circuits.

4.1. Problem Formulation

Let $\sigma^* : V \longrightarrow [0,1]$ and $\mu^* : \mathbb{E} \longrightarrow [0,1]$ be the membership function of vertices and edges of a fuzzy graph G. The fuzzy graphic binary matroid is defined from the edge set of a fuzzy graph G with $\mathbb{E}$ be the ground set and $\mathbb{I} = 2^n$ be the family of independent sets. Defining the fuzzy graphic binary matroid over G using the set of dependent edges.

- Each independent fuzzy set corresponds to an acyclic subgraph.
- Circuits in the fuzzy graphic matroid correspond to fundamental cycles in the fuzzy graph.

Using the fuzzy graphic matroid circuit property:

- If $\mathbb{C}$ is a minimal dependent set in $\mathbb{E}$, the symmetric difference property ensures redundancy and fault tolerance.

Reliability Analysis: For each cycle C, the cycle reliability is

$$R = min_{e_{ij} \in \mathbb{C}} \mu^*(e_{ij}) \tag{4}$$

The network reliability is determined by the reliability of the fundamental cycles as follows:

$$R_N = min_{e_{ij} \in \mathbb{C}} (R_C) \tag{5}$$

This identifies the weakest critical circuits affecting the reliability of the network.

4.2. Algorithm Description

For a fuzzy graphic matroid $\mathfrak{M}(\mathtt{G}) = (\mathbb{E}, \mathbb{I})$ having rank function $\mathbb{R}$, the ground set of a fuzzy graphic matroid $\mathbb{E}$ is the set of edges of a fuzzy graph G. The step-wise procedure to construct the power grid network is given below:

Step (i) **Construction of fuzzy graph:** Let G be a fuzzy graph having n vertices and m edges. G is a pair of functions $\sigma : V \longrightarrow [0,1]$ and $\mu : V \times V \longrightarrow [0,1]$. Here, the vertices represents the power grid substations and the inter-connections between the substations are represented by edges with fuzzy weights μ_{ij}.

Step (ii) **Defining the Fuzzy Graphic Matroid:** Let $\mathbb{E}$ be the ground set of a fuzzy graphic matroid and $\mathbb{I}$ be the independent edges of a fuzzy graph G.

Step (iii) **Identifying the fundamental circuits:** Cycles in the network indicate redundancy and alternative communication paths. Also, identifying the cycles assesses the network's resilience.

Step (iv) **Checking binary condition for redundancy:** The binary condition in a fuzzy graphic matroid requires that every cycle can be expressed as the symmetric difference of other cycles. This condition ensures that the network has sufficient redundancy, allowing alternative paths if certain links fail.

$$\text{Minimum reliability} = \min(\mu_{ij})$$

Step (v) **Analyzing network reliablity:** The overall network reliability is evaluated by the reliability of each cycle. The cycle reliability is determined by the minimum fuzzy membership value among its edges. This reflects the weakest link in the cycle which indictates the overall reliability of that communication path.

Step (vi) **Optimizing the network design:** Based on the reliability analysis, the weakest links can be identified and addressed to enhance the network's overall reliability.

5. Implementation, Results, and Discussion

In this study, a fuzzy graphic binary matroid approach is used to model a power grid communication network, which is crucial for capturing the inherent uncertainties in the transmission of data in the network. In this approach, the power grid is represented as a fuzzy graph, where vertices correspond to substations and edges represent communication links, each associated with a membership value that quantifies its reliability. The fuzzy graphic binary matroid is constructed by treating the edge set as the ground set and defining independent sets as those that form acyclic subgraphs, capturing the network's structural dependencies. Fundamental circuits in this matroid correspond to communication loops, which are essential for redundancy and fault tolerance. The binary nature of the matroid ensures that every cycle can be expressed as the symmetric difference of other cycles, supporting a resilient and reconfigurable network structure. The weakest link in each cycle determines its overall reliability and identifies critical failures and vulnerabilities in the network. The power grid comprises several substations and is interconnected through various communication links. The communication links between substations are categorized as follows based on their type, which influences their reliability:

- Fiber optic cables (Highly reliable, underground)
- Wireless links (Moderately reliable, susceptible to interference)
- Satellite links (Less reliable, affected by weather conditions)

The substations of the power grid network are considered as the vertices of the fuzzy graph G, and the communication link between the network is represented by the edges of a fuzzy graph G. The data considered in this study are not derived directly from real events.

In Table 1, the reliability of each edge connecting the zones of the city are given with membership value μ_{ij}. Let $\mathbb{E} = \{e_{AB}, e_{AC}, e_{AM}, e_{BD}, e_{AB}, e_{BC}, e_{BE}, e_{EC}, e_{CD}, e_{DM}, e_{CM}, e_{EF}, e_{FG}e_{GH}, e_{HM}, e_{HI}, e_{IM}, e_{IJ}, e_{JM}, e_{DF}, e_{CF}, e_{GM}, e_{EG}, e_{JH}, e_{CI}, e_{BI}\}$ be the edges shown in the Figure 3 and $\mathbb{I} = 2^n$ be the independent sets of a fuzzy graph G where n is number of edges.

Table 1. Zones of a city and their reliability values of the edges connecting the zones.

Edges	Substation	Substation	Type of link	Reliability (e_{ij})
e_{AB}	Zone A	Zone B	Fiber	0.7
e_{AC}	Zone A	Zone C	Fiber	0.8
e_{AM}	Zone A	Control Center	Fiber	0.6
e_{BD}	Zone B	Zone D	Fiber	0.7
e_{AB}	Zone A	Zone B	Fiber	0.7
e_{BC}	Zone B	Zone C	Wireless	0.6
e_{BE}	Zone B	Zone E	Fiber	0.7
e_{EC}	Zone E	Zone C	Fiber	0.8
e_{CD}	Zone C	Zone D	Wireless	0.8
e_{DM}	Zone D	Control Center	Satellite	0.7
e_{CM}	Zone C	Control Center	Satellite	0.8

Table 1. *Cont.*

Edges	Substation	Substation	Type of link	Reliability (e_{ij})
e_{EF}	Zone E	Zone F	Fiber	0.8
e_{FG}	Zone F	Zone G	Fiber	0.7
e_{GH}	Zone G	Zone H	Fiber	0.9
e_{HM}	Zone H	Control Center	Fiber	0.9
e_{HI}	Zone H	Zone I	wireless	0.8
e_{IM}	Zone I	Control Center	Satellite	0.7
e_{IJ}	Zone I	Zone J	Fiber	0.9
e_{JM}	Zone J	Control Center	Fiber	0.8
e_{DF}	Zone D	Zone F	Wireless	0.8
e_{CF}	Zone C	Zone F	Fiber	0.8
e_{GM}	Zone G	Control Center	Wireless	0.8
e_{EG}	Zone E	Zone G	Wireless	0.8
e_{JH}	Zone J	Zone H	Wireless	0.8
e_{CI}	Zone C	Zone I	Fiber	0.8
e_{BI}	Zone B	Zone I	Fiber	0.8

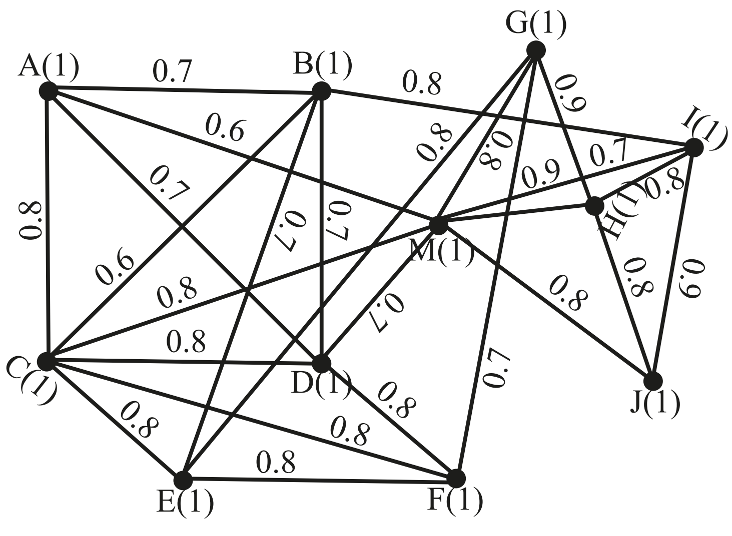

Figure 3. Fuzzy graph representation of a power grid network.

The incidence matrix for the $\mathfrak{M}(G)$ is given below:

$$A = \begin{vmatrix} 0.9 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.8 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.8 & 0.7 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0.8 & 0 & 0.8 & 1 & 0 & 0 & 0 \\ 0 & 0.8 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0.6 & 0.7 & 0 & 0.7 \end{vmatrix}$$

Setting the threshold value as $\alpha = 0.5$ ensures that only links with at least 50% reliability confidence contribute to the analysis, filtering out highly uncertain or poor-quality links without being overly restrictive.

$$A = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{vmatrix}$$

The incidence matrix of the fuzzy graph is totally unimodular. Hence, it satisfies the condition of a fuzzy graphic binary matroid.

The cycles in the network indicate redundancy and alternative communication paths. The cycles covering different parts of the network highlighting redundancy and multiple communication paths are given below:
Cycle 1: $C_1 = \{e_{AB}, e_{BE}, e_{EC}, e_{AC}\}$
Cycle 2: $C_2 = \{e_{BC}, e_{CF}, e_{EF}, e_{BE}\}$
Cycle 3: $C_3 = \{e_{CF}, e_{FG}, e_{GH}, e_{HM}, e_{CM}\}$
Cycle 4: $C_4 = \{e_{GH}, e_{EG}, e_{BE}, e_{BI}, e_{HI}\}$
Cycle 5: $C_5 = \{e_{IJ}, e_{JM}, e_{HM}, e_{HI}\}$
Cycle 6: $C_6 = \{e_{AM}, e_{AC}, e_{CM}\}$
Cycle 7: $C_7 = \{e_{CM}, e_{CF}, e_{DF}, e_{DM}\}$
Cycle 8: $C_8 = \{e_{CR}, e_{RM}, e_{MT}, e_{TC}\}$

The minimal dependent edges in these cycles forms a circuit in fuzzy graphic matroid $\mathfrak{M}(G)$.

The minimum reliability value of the circuit in $\mathfrak{M}(G)$ is given by
Cycle 1: $C_1 = \{e_{AB}(0.7), e_{BE}(0.7), e_{EC}(0.8), e_{AC}(0.8)\} = 0.7$
Cycle 2: $C_2 = \{e_{BC}(0.6), e_{CF}(0.8), e_{EF}(0.8), e_{BE}(0.7)\} = 0.6$
Cycle 3: $C_3 = \{e_{CF}(0.8), e_{FG}(0.7), e_{GH}(0.9), e_{HM}(0.9), e_{CM}(0.8)\} = 0.7$
Cycle 4: $C_4 = \{e_{GH}(0.9), e_{EG}(0.8), e_{BE}(0.7), e_{BI}(0.8), e_{HI}(0.8)\} = 0.7$
Cycle 5: $C_5 = \{e_{IJ}(0.9), e_{JM}(0.8), e_{HM}(0.9), e_{HI}(0.8)\} = 0.8$
Cycle 6: $C_6 = \{e_{AM}(0.6), e_{AC}(0.8), e_{CM}(0.8)\} = 0.6$
Cycle 7: $C_7 = \{e_{CM}(0.6), e_{CF}(0.8), e_{DF}(0.8), e_{DM}(0.7)\} = 0.6$
Cycle 8: $C_8 = \{e_{CR}(0.9), e_{RM}(0.8), e_{MT}(0.9), e_{TC}(0.7)\} = 0.7$

The overall network reliability is influenced by these cycle reliabilities. Cycles with lower reliability values indicate potential vulnerabilities where the network may fail if certain links within these cycles fail.

Overall Network Reliability Assessment:

- Highest Cycle Reliability: $C_5 = 0.8$
- Lowest Cycle Reliability: $C_2 = 0.6; C_6 = 0.6; C_7 = 0.6$

The weakest links to enhance the network's overall reliability. Based on the reliability analysis, the minimum reliability value (0.6) is addressed, and the cycles C_2, C_6, and C_7 are the weakest link among the overall network.

5.1. Comparision with the Coventional Methods

Graph-theoretic models encode interactions as crisp edges (i.e., either present or absent), failing to account for the varying degrees of interaction strength and reliability that exist in network systems. In contrast, fuzzy graph theory provides a more flexible model by representing interactions with membership values, typically in the range $[0, 1]$, which reflect the degree of interaction or confidence in the interaction between the substations. Traditional models for power grid reliability, such as crisp graph theory and probabilistic methods, have notable limitations. Crisp graphs represent connections as either present or absent, thereby neglecting intermediate reliability states that often exist in real systems. Similarly, a probabilistic approach is effective for smaller systems that will become computationally intensive in large grids and lack a structured mechanism to systematically identify weak regions.

In contrast, the fuzzy graphic binary matroid framework integrates fuzziness with binary conditions, offering a structured approach to reliability assessment. It captures uncertainty through fuzzy weights, ensures binary compatibility via thresholding, and identifies minimal dependent sets where failure significantly reduces the rank. This enables operators to detect weak circuits and prioritize maintenance, thereby providing both structural and functional reliability insights.

5.2. Advantages of the Study

1. The study aligns with modern smart grid requirements, where data transmission reliability is critical for automated control, fault detection, and load balancing.
2. The algorithm helps in planning future expansions of the network while maintaining reliability through optimal cycle formation.
3. The fuzzy graphic matroid captures independence and circuit properties that help identify minimal sets of communication links needed for connectivity.
4. Identifying critical failure points through circuit analysis helps prioritize maintenance and recovery efforts.

5.3. Limitations of the Study

1. Computing rank in large networks becomes computationally expensive since fuzzy independence must be checked for many subsets of edges.
2. Selecting an optimal fuzzy basis becomes NP-hard, limiting practical applicability.
3. This study typically captures a snapshot of the network without accounting for temporal variations in link reliability, which are critical for accurate fault prediction and recovery.
4. The fuzzy graphic matroid framework focuses on connectivity and redundancy, but may not fully capture other interdependencies, such as communication protocol constraints and bandwidth limitations.
5. The computational cost for identifying all circuits and fuzzy independent sets in very large power grid networks can be high.

5.4. Results and Discussion

The power grid network under analysis consists of 10 zones. Each zone is connected to a central control unit in a city, which is modeled using a fuzzy graphic binary matroid. The critical circuit of the network is found by addressing the minimum reliability value. Communication between these zones plays a vital role in the overall reliability of the power grid. The network is modeled using cycles, where each cycle represents a different path of communication between substations. The cycles C_2, C_6, and C_7 exhibits the lowest reliability value of 0.6. The circuits with these cycles require significant maintenance and improvement. The satellite networks are more prone to weather-related disturbances and have lower bandwidth; their performance could degrade over time, leading to further issues. To enhance the network reliability, replacing satellite networks with fiber networks in these critical cycles is essential. Hence, the network could achieve better reliability and reduce the risk of communication failures.

6. Conclusions and Future Work

In this paper, the authors present the conditions for fuzzy graphic binary matroids and propose a systematic method for analyzing the reliability of the power grid network. The framework effectively identifies critical circuits that require greater attention for maintenance to enhance network performance and stability. By applying fuzzy graphic binary matroid principles, the study ensures both structural and functional reliability of the system. The identification of weak circuits allows operators to prioritize resources toward sensitive parts of the network. This contributes to improved decision-making in preventive and corrective maintenance. The findings demonstrate that matroid-based analysis offers a structured approach to network reliability assessment.

As future work, the determination of maintenance cost and optimal time allotment for addressing the identified critical circuits will be explored. This will enable prioritization of resources towards the sensitive sections of the network. Incorporating cost–time analysis will strengthen decision-making for preventive and corrective maintenance. This will provide a more practical and efficient framework for managing large-scale power grid networks.

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Distance Measures of (m,a,n)-Fuzzy Neutrosophic Sets and Their Applications in Decision Making

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Abstract: A neutrosophic set is an important tool for handling vagueness and impreciseness in real-world problems, and distance measures are also exhibited in neutrosophic set theory. The (m,a,n)-fuzzy neutrosophic set is more flexible and efficient than the existing extensions of neutrosophic sets when discussing the distance between multiple objects. This paper invents ten distance measures for comparing (m,a,n)-fuzzy neutrosophic sets. Moreover, the created distance measures are applied in pattern classifications and multi-criteria decisions. Additionally, numerical examples demonstrate these distance measures in practical and scientific applications involving classifying materials and investment problems. The comparative analysis, along with graphical interpretations, further illustrates the effectiveness and superiority of the proposed measures.

Keywords: (m,a,n)-fuzzy neutrosophic sets; distance measure; pattern recognition; multi-criteria decision making

1. Introduction

Distance measures are foundational concepts in mathematics and data analysis, playing a vital role in pattern recognition, image processing, and decision-making models. Symmetry [1,2] plays a significant role in defining and understanding distance measures. In the context of distance measures, symmetry ensures that the calculated distance between two points or objects remains the same, irrespective of their order. This property is essential for maintaining consistency in evaluating similarities or dissimilarities among data elements. Together, symmetry and distance measures contribute to the robustness and reliability of various computational techniques, particularly in fuzzy systems, clustering, and classification tasks. Distance measures serve as quantitative methods for determining how dissimilar or far apart two entities are. Distance metrics are essential for evaluating the level of variation between elements, thereby enabling effective identification of different patterns in real-world applications. Although the terms "difference" and "distance" are often used interchangeably, distance specifically represents a type of difference, typically reflecting the level of deviation or irregularity between two items. On the other hand, difference more broadly captures any form of contrast or divergence between objects based on multiple attributes. In essence, distance measures help assess how separate or distinct two objects are. This study seeks to deepen our grasp of this key concept and its applications, offering insights that bridge disciplines and foster innovations and practical outcomes.

1.1. Literature Review

The notion of FSs was invented by Zadeh [3] as a generalization of classical sets that handles vagueness and ambiguity in information. Many applications of FS theory in the fields of science, engineering, medicine, management, and social science have been considered by different authors. After the invention of FSs, many researchers such as Helgason and Jobe [4], Saha and Bandyopadhyay [5], Colliota and his coworkers [6], Dubois and Prade [7], Rosenfeld [8], Adlassnig [9], Gerla and Volpe [10], Liu [11], and others created various fuzzy DMs and applied them to developed methods and algorithms for decision making and object recognition.

In 1986, Atanassov [12] invented IFSs as an extension of FSs by considering the PMD and the NMD, whose sum belongs to [0, 1]. After the invention of IFSs, several generalizations of IFSs such as PFS [13], q-ROFS [14], FFS [15], and (m,n)-FS [16,17] appeared in the literature. Researchers like Szmidt and Kacprzyk [18], Wang and Xin [19], Lee, Pedrycz, and Sohn [20], Nguyen [21], Zeng, Li, and Yin [22], Hussian and Yang [23], Peng and his coworkers [24], Ejegwa and his coworkers [25], Aydın [26], Kirişci [27], Ashraf and his coworkers [28], Peng and Liu [29], Thakur and his coworkers [30,31], Sivdas and John [32], and others have proposed different distance measures between the above-mentioned classes of FSs and presented applications in the areas of image processing, medical diagnosis, pattern recognition, and decision making.

In 1995, Smarandache [33] established a mathematical notion called neutrosophic sets by considering the PMD, IMD, and NMD, whose sum belongs to [0, 3]. In the recent past, many researchers such as Biswas, Pramanik, and Giri [34], Zulqarnain and his coworkers [35], Ali, Hussain, and Yang [36], Xu, and Zhao [37], and others created various DMs for NSs and applied them in decision making. In the recent past, PyNS [38], FNS [39], and q-RONS [40,41] were invented to depict vagueness and impreciseness in real-world problems. Recently, Theresa, Puzhakkara, and Shiny [42], Bhuvaneshwari, Sweety, and Broumi [43], and Princy, Radha, and Broumi [44] created some distance measures for FNSs and applied them in clustering, crop forming, and detecting patterns of infection-induced fertility.

1.2. Motivation

In existing neutrosophic set extensions such as PyFSs, FSSs, and q-ROFSs, a symmetric approach is generally adopted when assigning powers to the Positive Membership Degree (PMD), Indeterminacy Membership Degree (IMD), and Negative Membership Degree (NMD) for any attribute in the universe of discourse. This uniform treatment restricts the ability to differentiate the relative importance of the PMD, IMD, and NMD, which becomes particularly evident in systems like PyNSs, FNSs, and q-RONSs. For example, in the q-RONS model, the sum of the q^{th} powers of the PMD and NMD must not exceed 1, while the q^{th} power of the IMD must remain within the interval [0, 1]. Suppose we have a triplet (0.7, 0.4, 0.9) representing the PMD, IMD, and NMD, respectively. If a decision-maker wants to assign different levels of priority—say, a power of 2 to the PMD, 6 to the IMD, and 10 to the NMD—this level of customization is not possible within the rigid structure of q-RONSs, which enforces the same power value for all three components.

To address this limitation, Thakur and Jafari [45] introduced the (m,a,n)-FNSs framework, which enables distinct powers to be assigned to the PMD, IMD, and NMD, offering greater modeling flexibility in uncertain and imprecise environments. One of the main objectives of this research is to explore this newly proposed class—(m,a,n)-FNSs—to formulate a comprehensive family of distance measures. These measures provide a more expressive representation of the PMD, IMD, and NMD compared to conventional q-RONS-based models. The flexibility of choosing different powers allows a more context-sensitive assessment of input data by assigning varying significance to the PMD, IMD, and NMD—an important requirement in multi-criteria decision-making (MCDM) problems. Unlike other generalizations of IFSs in a neutrosophic framework such as PyNSs (where power = 2), FNSs (power = 3), and q-RONSs (power = q), which enforce fixed exponents, the (m,a,n)-FNS model accommodates variable exponentiation for each component. This power function-based flexibility extends the applicability of neutrosophic systems to a broader range of decision-making scenarios. As a result, (m,a,n)-FNSs offer a more inclusive modeling structure, capturing uncertainty more effectively than traditional q-RONSs. In particular, (m,a,n)-FNSs, which utilize triple universes, prove to be more adaptive and accurate than their single-exponent counterparts like m-RONSs, a-RONSs, and n-RONSs when assessing distances among multiple objects. In summary, (m,a,n)-FNSs represent a generalization that not only encompasses q-RONS-based methods as special cases but also provides enhanced capabilities for addressing complex MCDM problems.

1.3. Objectives and Contributions

The objectives of this research work were as follows:

1. To create an axiomatic definition of an (m,a,n)-fuzzy neutrosophic distance measure and introduce a family of DMs for (m,a,n)-fuzzy neutrosophic sets which generalizes the existing distance measures in orthopair fuzzy sets in a neutrosophic environment.
2. To show the validity of the proposed (m,a,n)-fuzzy neutrosophic distance measures by establishing theorems, remarks, and examples.
3. To design an algorithm to solve pattern classification problems with the aid of (m,a,n)-fuzzy neutrosophic distance measures.
4. To design an algorithm for solving multi-criteria decision-making (MCDM) problems using the TOPSIS approach within the (m,a,n)-fuzzy neutrosophic environment.
5. To implement the designed algorithms in real-world problems relating to material classification and investment strategies.

This paper builds upon previous work by developing distance measures tailored for (m,a,n)-FNSs. These new measures allow for differentiated treatment of the PMD, IMD, and NMD through distinct power parameters, thereby enhancing the evaluation of input data with varying significance, an especially valuable feature in multi-criteria decision-making problems. By employing diverse power function scales, the proposed distance measures extend the applicability of (m,a,n)-FNSs to more complex and nuanced decision-making scenarios. The main contributions of this research are summarized as follows:

1. This study introduces distance measures between two (m,a,n)-fuzzy neutrosophic numbers ((m,a,n)-FNs) based on classical metrics such as the Hamming, Euclidean, and Hausdorff distances. Additionally, hybrid and arctangent-based distance formulations are proposed to enhance the precision and adaptability of the framework.
2. Utilizing the defined distance measures for (m,a,n)-FNSs, a new algorithm is constructed for solving pattern classification problems, enabling effective discrimination between categories under uncertainty and indeterminacy.
3. The proposed classification algorithm is applied to a material recognition problem. A detailed sensitivity analysis of its parameters is conducted, along with a comparative evaluation in comparison with existing methods to demonstrate its robustness and effectiveness.
4. The well-known TOPSIS technique is customized to handle multi-criteria decision-making (MCDM) problems within the (m,a,n)-fuzzy neutrosophic environment. This is achieved by employing the newly defined weighted distance measures, thus enhancing decision support under complex conditions.
5. A comprehensive numerical example is presented to showcase the implementation of the proposed TOPSIS algorithm. The results are then compared with existing methods based on aggregation operators in q-RONSs and (m,n)-FS frameworks to validate the efficiency and consistency of the proposed approach.

The remainder of this paper is structured as follows: Section 1 provides an overview of the study, including the background, motivation, and primary objectives. Section 2 introduces the essential concepts and preliminary definitions required to understand the framework of (m,a,n)-FNSs. Section 3 presents the newly developed distance measures tailored for (m,a,n)-FNSs. It also discusses their mathematical properties and offers a comparative analysis with existing distance measures within the neutrosophic environment. Section 4 introduces a pattern classification algorithm utilizing the proposed (m,a,n)-FNS-based distance measures. This section also includes its application to a material recognition problem, along with a discussion on parameter sensitivity and a comparative performance evaluation. Section 5 proposes a multi-criteria decision-making method based on the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), adapted to the (m,a,n)-FNS environment. An illustrative example from the domain of investment decision making is used to demonstrate the method's practical applicability. Section 6 concludes the paper by summarizing the key contributions and insights and offering some final remarks regarding future work.

2. Preliminaries

The present section revisits the existing concepts of orthopair fuzzy sets and their associated distance measures within the framework of the neutrosophic environment. Throughout this paper, $\mathbb{S}$ denotes the universe of discourse, and $\mathbb{N}$ refers to the set of all natural numbers. The abbreviations used in this paper are listed in Table 1.

Table 1. Abbreviations and their descriptions.

Abbreviation	Description	Abbreviation	Description
DM	Distance Measure	SM	Similarity Measure
PMD	Positive Membership Degree	$\mho_{\mathscr{G}}(s)$	PMD of s to $\mathscr{G}$
IMD	Indeterminacy Degree	$\varpi_{\mathscr{G}}(s)$	IMD of s to $\mathscr{G}$
NMD	Negative Membership Degree	$\Omega_{\mathscr{G}}(s)$	NMD of s to $\mathscr{G}$
FS	Fuzzy Set	IFS	Intuitionistic Fuzzy Set
PyFS	Pythagorean Fuzzy Set	FFS	Fermatean Fuzzy Set
q-ROFS	q-Rung Orthopair Fuzzy Set	(m,n)-FS	(m,n)-Fuzzy Set
NS	Neutrosophic Set	PyNS	Pythagorean Neutrosophic Set
FNS	Fermatean Neutrosophic Set	q-RONS	q-Rung Orthopair Neutrosophic Set
(m,a,n)-FNS	(m,a,n)-Fuzzy Neutrosophic Set	(m,a,n)-FNS($\mathbb{S}$)	Family of (m,a,n)-FNSs on $\mathbb{S}$
(m,a,n)-FNDM	(m,a,n)-Fuzzy Neutrosophic Distance Measure	(m,a,n)-FNWDM	Fuzzy Neutrosophic Weighted Distance Measure

2.1. Orthopair Fuzzy Structures and Their Extensions in a Neutrosophic Environment

Definition 1. *An object of the form*

$$\mathscr{G} = \{< s, \mho_{\mathscr{G}}(s), \Omega_{\mathscr{G}}(s) >: s \in \mathbb{S}\}$$

is called the following:

(a) *An IFS [12] in* $\mathbb{S}$ *if* $0 \leq \mho_{\mathscr{G}}(s) + \Omega_{\mathscr{G}}(s) \leq 1, \ \forall \ s \in \mathbb{S}$.
(b) *A PyFS [13] in* $\mathbb{S}$ *if* $0 \leq \mho^2_{\mathscr{G}}(s) + \Omega^2_{\mathscr{G}}(s) \leq 1, \ \forall \ s \in \mathbb{S}$.
(c) *An FFS [15] in* $\mathbb{S}$ *if* $0 \leq \mho^3_{\mathscr{G}}(s) + \Omega^3_{\mathscr{G}}(s) \leq 1, \ \forall \ s \in \mathbb{S}$.
(d) *A q-ROFS [14] in* $\mathbb{S}$ *if* $0 \leq \mho^q_{\mathscr{G}}(s) + \Omega^q_{\mathscr{G}}(s) \leq 1, \ \forall \ s \in \mathbb{S}$ *and* $q \in \mathbb{N}$.
(e) *An (m,n)-FS [16,17] in* $\mathbb{S}$ *if* $0 \leq \mho^m_{\mathscr{G}}(s) + \Omega^n_{\mathscr{G}}(s) \leq 1, \ \forall \ s \in \mathbb{S}$ *and* $m, n \in \mathbb{N}$.

where $\mho_{\mathscr{G}} : \mathbb{S} \to [0,1]$ *and* $\Omega_{\mathscr{G}} : \mathbb{S} \to [0,1]$.

Definition 2 ([33]). *An NS* $\mathscr{G}$ *in* $\mathbb{S}$ *is a structure*

$$\mathscr{G} = \{< s, \mho_{\mathscr{G}}(s), \varpi_{\mathscr{G}}(s), \Omega_{\mathscr{G}}(s) >: s \in \mathbb{S}\}$$

where $\mho_{\mathscr{G}} : \mathbb{S} \rightarrow]^-0, 1^+[$, $\varpi_{\mathscr{G}} : \mathbb{S} \rightarrow]^-0, 1^+[$, *and* $\Omega_{\mathscr{G}} : \mathbb{S} \rightarrow]^-0, 1^+[$ *denotes the PMD, IMD, and NMD of* $\mathscr{G}$*, which satisfies the condition* $^-0 \leq \mho_{\mathscr{G}}(s) + \varpi_{\mathscr{G}}(s) + \Omega_{\mathscr{G}}(s) \leq 3^+$, $\forall\, s \in \mathbb{S}$.

Definition 3. *An object of the form*

$$\mathscr{G} = \{< s, \mho_{\mathscr{G}}(s), \varpi_{\mathscr{G}}(s), \Omega_{\mathscr{G}}(s) >: s \in \mathbb{S}\}$$

is called the following:

- *(a)* *A PyNS [38] in* $\mathbb{S}$ *if* $0 \leq \mho^2_{\mathscr{G}}(s) + \Omega^2_{\mathscr{G}}(s) \leq 1$ *and* $0 \leq \varpi^2_{\mathscr{G}}(s) \leq 1$, $\forall\ s \in \mathbb{S}$.
- *(b)* *An FNS [39] in* $\mathbb{S}$ *if* $0 \leq \mho^3_{\mathscr{G}}(s) + \Omega^3_{\mathscr{G}}(s) \leq 1$ *and* $0 \leq \varpi^3_{\mathscr{G}}(s) \leq 1$, $\forall\ s \in \mathbb{S}$.
- *(c)* *A q-RONS [40] in* $\mathbb{S}$ *if* $0 \leq \mho^q_{\mathscr{G}}(s) + \Omega^q_{\mathscr{G}}(s) \leq 1$, $0 \leq \varpi^q_{\mathscr{G}}(s) \leq 1$, $\forall\, s \in \mathbb{S}$*, and* $q \in \mathbb{N}$.
- *(d)* *An ((m,a,n)-FNS [45] in* $\mathbb{S}$ *if* $0 \leq \mho^m_{\mathscr{G}}(s) + \Omega^n_{\mathscr{G}}(s) \leq 1$, $0 \leq \varpi^a_{\mathscr{G}}(s) \leq 1$, $\forall\, s \in \mathbb{S}$*, and* $m, a, n \in \mathbb{N}$.

where $\mho_{\mathscr{G}} : \mathbb{S} \rightarrow [0, 1]$, $\varpi_{\mathscr{G}} : \mathbb{S} \rightarrow [0, 1]$, *and* $\Omega_{\mathscr{G}} : \mathbb{S} \rightarrow [0, 1]$ *denote the PMD, IMD, and NMD of* $\mathscr{G}$*, respectively.*

For simplicity, an (m,a,n)-FNS $\mathscr{G} = \{< s, \mho_{\mathscr{G}}(s), \varpi_{\mathscr{G}}(s), \Omega_{\mathscr{G}}(s) >: s \in \mathbb{S}\}$ in $\mathbb{S}$ is referred to by $(\mho_{\mathscr{G}}, \varpi_{\mathscr{G}}, \Omega_{\mathscr{G}})$.

Proposition 1 ([45])**.** *If* $\mathscr{G} = \{< s, \mho_{\mathscr{G}}(s), \varpi_{\mathscr{G}}(s), \Omega_{\mathscr{G}}(s) >: s \in \mathbb{S}\}$ *is an (m,a,n)-FNS over* $\mathbb{S}$*, then* $0 \leq \mho^m_{\mathscr{G}}(s) + \varpi^a_{\mathscr{G}}(s) + \Omega^n_{\mathscr{G}}(s) \leq 2$, $\forall\, s \in \mathbb{S}$*, and* $m, a, n \in \mathbb{N}$.

Remark 1 ([45])**.** *The notion of a q-RONS (resp. FNS, PyNS) is a special case of an (m,a,n)-FNS for m=a=n=q (resp. m=a=n=3, m=a=n=2).*

Definition 4 ([45])**.** *Let* $\mathscr{G}, \mathscr{G}_1, \mathscr{G}_2 \in (m, a, n) - FNS(\mathbb{S})$*. Then, the following apply:*

- *(a)* $\mathscr{G}_1 \subset \mathscr{G}_2 \Leftrightarrow \mho_{\mathscr{G}_1} \leq \mho_{\mathscr{G}_2}$, $\varpi_{\mathscr{G}_1} \geq \varpi_{\mathscr{G}_2}$*, and* $\Omega_{\mathscr{G}_1} \geq \Omega_{\mathscr{G}_2}$.
- *(b)* $\mathscr{G}_1 = \mathscr{G}_2 \Leftrightarrow \mho_{\mathscr{G}_1} = \mho_{\mathscr{G}_2}$, $\varpi_{\mathscr{G}_1} = \varpi_{\mathscr{G}_2}$*, and* $\Omega_{\mathscr{G}_1} = \Omega_{\mathscr{G}_2}$.
- *(c)* $\mathscr{G}^c = (\Omega^{\frac{n}{m}}_{\mathscr{G}}, 1 - \varpi_{\mathscr{G}}, \mho^{\frac{m}{n}}_{\mathscr{G}})$.
- *(d)* $\mathscr{G}_1 \cup \mathscr{G}_2 = (max\{\mho_{\mathscr{G}_1}, \mho_{\mathscr{G}_2}\}, max\{\varpi_{\mathscr{G}_1}, \varpi_{\mathscr{G}_2}\}, min\{\Omega_{\mathscr{G}_1}, \Omega_{\mathscr{G}_2}\})$.
- *(e)* $\mathscr{G}_1 \cap \mathscr{G}_2 = (min\{\mho_{\mathscr{G}_1}, \mho_{\mathscr{G}_2}\}, min\{\varpi_{\mathscr{G}_1}, \varpi_{\mathscr{G}_2}\}, max\{\Omega_{\mathscr{G}_1}, \Omega_{\mathscr{G}_2}\})$.

2.2. Distance Measures for Orthopair Fuzzy Structures in a Neutrosophic Environment

Definition 5 ([46])**.** *Let* $\mathbb{S} = \{s_1, s_2, \ldots s_r.\}$ *be a universe of discourse and* $\mathscr{G}_1, \mathscr{G}_2 \in PyNS(\mathbb{S})$*. Assume that* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weighting vector of the elements* s_i *(i = 1, 2, ..., r), satisfying the conditions* $\sum_{i=1}^{r} \omega_i = 1$, $\forall\ \omega_i \in [0, 1]$*, and i = 1, 2, ..., r.*

- *(i)* *The weighted Hamming distance between* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *is defined as follows:*

$$d_1(\mathscr{G}_1, \mathscr{G}_2) = \frac{1}{3}\sum_{1}^{r} \omega_i \begin{pmatrix} |\mho^2_{\mathscr{G}_1}(s_i) - \mho^2_{\mathscr{G}_2}(s_i)| \\ +|\varpi^2_{\mathscr{G}_1}(s_i) - \varpi^2_{\mathscr{G}_2}(s_i)| \\ +|\Omega^2_{\mathscr{G}_1}(s_i) - \Omega^2_{\mathscr{G}_2}(s_i)| \end{pmatrix} \tag{1}$$

- *(ii)* *The weighted Euclidean distance between* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *is defined by the following expression:*

$$d_2(\mathscr{G}_1, \mathscr{G}_2) = \sqrt{\frac{1}{3}\sum_{1}^{r} \omega_i \begin{pmatrix} |\mho^2_{\mathscr{G}_1}(s_i) - \mho^2_{\mathscr{G}_2}(s_i)|^2 \\ +|\varpi^2_{\mathscr{G}_1}(s_i) - \varpi^2_{\mathscr{G}_2}(s_i)|^2 \\ +|\Omega^2_{\mathscr{G}_1}(s_i) - \Omega^2_{\mathscr{G}_2}(s_i)|^2 \end{pmatrix}} \tag{2}$$

(iii) *The generalized weighted distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined by the following expression:*

$$d_p(\mathscr{G}_1,\mathscr{G}_2) = \sqrt[p]{\frac{1}{3}\sum_1^r \omega_i \begin{pmatrix} |\mho^2_{\mathscr{G}_1}(s_i) - \mho^2_{\mathscr{G}_2}(s_i)|^p \\ +|\varpi^2_{\mathscr{G}_1}(s_i) - \varpi^2_{\mathscr{G}_2}(s_i)|^p \\ +|\Omega^2_{\mathscr{G}_1}(s_i) - \Omega^2_{\mathscr{G}_2}(s_i)|^p \end{pmatrix}} \tag{3}$$

Definition 6 ([47]). *Let $\mathbb{S} = \{s_1, s_2, \ldots .s_r.\}$ be a universe of discourse and $\mathscr{G}_1, \mathscr{G}_2 \in FNS(\mathbb{S})$. Assume that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the elements s_i (i = 1, 2, ..., r), satisfying the conditions $\sum_{i=1}^r \omega_i = 1$, $\forall\ \omega_i \in [0, 1]$, and i = 1, 2, ..., r. Then, the following apply:*

(i) *The weighted Hamming distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d_H(\mathscr{G}_1,\mathscr{G}_2) = \frac{1}{3}\sum_1^r \omega_i \begin{pmatrix} |\mho^3_{\mathscr{G}_1}(s_i) - \mho^3_{\mathscr{G}_2}(s_i)| \\ +|\varpi^3_{\mathscr{G}_1}(s_i) - \varpi^3_{\mathscr{G}_2}(s_i)| \\ +|\Omega^3_{\mathscr{G}_1}(s_i) - \Omega^3_{\mathscr{G}_2}(s_i) \end{pmatrix} \tag{4}$$

(ii) *The weighted Euclidean distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined by the following expression:*

$$d_{E_u}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt{\frac{1}{3}\sum_1^r \omega_i \begin{pmatrix} |\mho^3_{\mathscr{G}_1}(s_i) - \mho^3_{\mathscr{G}_2}(s_i)|^2 \\ +|\varpi^3_{\mathscr{G}_1}(s_i) - \varpi^3_{\mathscr{G}_2}(s_i)|^2 \\ +|\Omega^3_{\mathscr{G}_1}(s_i) - \Omega^3_{\mathscr{G}_2}(s_i)|^2 \end{pmatrix}} \tag{5}$$

(iii) *The generalized weighted distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined by the following expression:*

$$d_{F_n}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt[\lambda]{\frac{1}{3}\sum_1^r \omega_i \begin{pmatrix} |\mho^3_{\mathscr{G}_1}(s_i) - \mho^3_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^3_{\mathscr{G}_1}(s_i) - \varpi^3_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^3_{\mathscr{G}_1}(s_i) - \Omega^3_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} \tag{6}$$

3. Distance Measures Between (m,a,n)-Fuzzy Neutrosophic Sets

In this section, several novel distance measures for (m,a,n)-FNSs are introduced. Their fundamental properties are discussed, and a comparative analysis is carried out with existing distance measures for PyNSs and FNSs.

3.1. Definitions and Properties

Definition 7. *Let $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3 \in (m,a,n) - FNS(\mathbb{S})$. Then, the DM is a function $d : (m,a,n) - FNS(\mathbb{S}) \times (m,a,n) - FNS(\mathbb{S}) \to [0,1]$ that satisfies the following conditions:*

(D1) $0 \le d(\mathscr{G}_1,\mathscr{G}_2) \le 1$.

(D2) $d(\mathscr{G}_1,\mathscr{G}_2) \Leftrightarrow \mathscr{G}_1 = \mathscr{G}_2$.

(D3) $d(\mathscr{G}_1,\mathscr{G}_2) = d(\mathscr{G}_2,\mathscr{G}_1)$.

(D4) *If $\mathscr{G}_1 \Subset \mathscr{G}_2 \Subset \mathscr{G}_3$, then $d(\mathscr{G}_1,\mathscr{G}_2) \le d(\mathscr{G}_1,\mathscr{G}_3)$ and $d(\mathscr{G}_2,\mathscr{G}_3) \le d(\mathscr{G}_1,\mathscr{G}_3)$.*

Definition 8. *Let $\mathbb{S} = \{s_1, s_2, \ldots .s_r.\}$ be a universe of discourse and $\mathscr{G}_1, \mathscr{G}_2 \in (m,a,n) - FNS(\mathbb{S})$. Then, the following apply:*

(i) *The Hamming distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^H(\mathscr{G}_1,\mathscr{G}_2) = \frac{1}{3}\sum_1^r \begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)| \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)| \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i) \end{pmatrix} \tag{7}$$

(ii) *The Euclidean distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined by the following expression:*

$$d^E(\mathscr{G}_1, \mathscr{G}_2) = \sqrt{\frac{1}{3}\sum_1^r \begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^2 \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^2 \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^2 \end{pmatrix}} \tag{8}$$

(iii) *The generalized distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined by the following expression:*

$$d^G(\mathscr{G}_1, \mathscr{G}_2) = \sqrt[\lambda]{\frac{1}{3}\sum_1^r \begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} \tag{9}$$

Remark 2. *The measures defined in Equations (7), (8), and (9) are not valid DMs because they do not satisfy all the conditions of Definition 7.*

Example 1. *Let $\mathbb{S} = \{s_1, s_2, s_3, s_4\}$ and $\mathscr{G}_1$ and $\mathscr{G}_2$ be two (2,1,3)-FNSs over $(\mathbb{S})$, where*

$$\mathscr{G}_1 = \Big\{< s_1, 0.3, 0.2, 0.8 >, < s_2, 0.0, 0.0, 1.0 >, < s_3, 0.8, 0.1, 0.5 >, < s_4, 0.0.0.0, 1.0 >\Big\},$$
$$\mathscr{G}_2 = \Big\{< s_1, 0.0, 0.0, 1.0 >, < s_2, 0.8, 0.1, 0.4 >, < s_3, 0.0, 0.0, 1.0 >, < s_4, 0.6, 0.3, 0.6 >\Big\}.$$

Then, from the definitions of d^H, d^E, and d^G, we obtain that

$$d^H(\mathscr{G}_1, \mathscr{G}_2) = 1.7443, d^E(\mathscr{G}_1, \mathscr{G}_2) = 1.0522, and\ d^G(\mathscr{G}_1, \mathscr{G}_2) = 1.0522 (for \lambda = 2).$$

It is clear that the Hamming distance, Euclidean distance, and generalized distance do not satisfy condition (D_1) of Definition 7 .

Now, we define the (m,a,n)-FNDMs by considering the PMD, IMD, and NMD of the (m,a,n)-FNSs.

Definition 9. *Let $\mathbb{S} = \{s_1, s_2, \ldots .s_r.\}$ and $\mathscr{G}_1, \mathscr{G}_2 \in (m, a, n) - FNS(\mathbb{S})$. Then, the following apply:*

(i) *The normalized Hamming distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{NH}(\mathscr{G}_1, \mathscr{G}_2) = \frac{1}{3r}\sum_1^r \begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)| \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)| \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)| \end{pmatrix} \tag{10}$$

(ii) *The normalized Euclidean distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined by the following expression:*

$$d^{NE}(\mathscr{G}_1, \mathscr{G}_2) = \sqrt{\frac{1}{3r}\sum_1^r \begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^2 \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^2 \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^2 \end{pmatrix}} \tag{11}$$

(iii) *The generalized normalized distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined by the following expression:*

$$d^{GN}(\mathscr{G}_1, \mathscr{G}_2) = \sqrt[\lambda]{\frac{1}{3r}\sum_1^r \begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} \tag{12}$$

(iv) *The normalized Hamming–Hausdorff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{NHH}(\mathscr{G}_1,\mathscr{G}_2) = \frac{1}{r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)| \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)| \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)| \end{pmatrix} \tag{13}$$

(v) *The normalized Euclidean–Hausdorff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{NEH}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt{\frac{1}{r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^2 \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^2 \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^2 \end{pmatrix}} \tag{14}$$

(vi) *The generalized normalized Hausdorff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{GNH}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt[\lambda]{\frac{1}{r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} \tag{15}$$

(vii) *The hybrid normalized Hamming distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{HNH}(\mathscr{G}_1,\mathscr{G}_2) = \frac{1}{r}\sum_{1}^{r}\left(\frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)| \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)| \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)| \end{pmatrix}}{6} + \frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)| \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)| \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)| \end{pmatrix}}{2}\right) \tag{16}$$

(viii) *The hybrid normalized Euclidean distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{HNE}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt{\frac{1}{r}\sum_{1}^{r}\left(\frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^2 \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^2 \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^2 \end{pmatrix}}{6} + \frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^2 \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^2 \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^2 \end{pmatrix}}{2}\right)} \tag{17}$$

(ix) *The generalized hybrid normalized distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{GHN}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt[\lambda]{\frac{1}{r}\sum_{1}^{r}\left(\frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}}{6} + \frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}}{2}\right)} \tag{18}$$

(x) *The tangent inverse distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{TI}(\mathscr{G}_1,\mathscr{G}_2) = \frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |tan^{-1}\mho^m_{\mathscr{G}_1}(s_i) - tan^{-1}\mho^m_{\mathscr{G}_2}(s_i)| \\ +|tan^{-1}\varpi^a_{\mathscr{G}_1}(s_i) - tan^{-1}\varpi^a_{\mathscr{G}_2}(s_i)| \\ +|tan^{-1}\Omega^n_{\mathscr{G}_1}(s_i) - tan^{-1}\Omega^n_{\mathscr{G}_2}(s_i)| \end{pmatrix} \tag{19}$$

Remark 3. *For any two (m,a,n)-FNSs $\mathscr{G}_1$ and $\mathscr{G}_2$ over a universe of discourse $\mathbb{S} = \{s_1, s_2, ...s_r.\}$, the following apply:*

(i) *The generalized normalized distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is reduced to the normalized Hamming distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 1$ and the normalized Euclidean distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 2$.*

(ii) *The generalized normalized Hausdorff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is reduced to the normalized Hamming–Hausdorff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 1$ and the normalized Euclidean–Hausdorff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 2$.*

(iii) *The generalized hybrid normalized distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is reduced to the hybrid normalized Hamming distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 1$ and the hybrid normalized Euclidean distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 2$.*

Example 2. *Let $\mathbb{S} = \{s_1, s_2, s_3, s_4\}$ and $\mathscr{G}_1$ and $\mathscr{G}_2$ be two (2,3,4)-FNSs over $\mathbb{S}$, where*

$$\mathscr{G}_1 = \Big\{ < s_1, 0.8, 0.7, 0.6 >, < s_2, 0.6, 0.4, 0.8 >, < s_3, 0.7, 0.8, 0.7 >, < s_4, 0.9.0.5, 0, 6 > \Big\},$$
$$\mathscr{G}_2 = \Big\{ < s_1, 0.7, 0.5, 0.8 >, < s_2, 0.4, 0.6, 0.4 >, < s_3, 0.5, 0.4, 0.6 >, < s_4, 0.7, 0.4, 0.3 > \Big\}.$$

Then, from Definition 9, we obtain that

$$\begin{aligned}
&d^{NH}(\mathscr{G}_1, \mathscr{G}_2) = 0.2237,\ d^{NE}(\mathscr{G}_1, \mathscr{G}_2) = 0.2589,\\
&d^{GN}(\mathscr{G}_1, \mathscr{G}_2) = 0.2728(\lambda = 3),\ d^{NHH}(\mathscr{G}_1, \mathscr{G}_2) = 0.3580,\\
&d^{NEH}(\mathscr{G}_1, \mathscr{G}_2) = 0.3636,\ d^{GNH}(\mathscr{G}_1, \mathscr{G}_2) = 0.3691(\lambda = 3),\\
&d^{HNH}(\mathscr{G}_1, \mathscr{G}_2) = 0.2908,\ d^{HNE}(\mathscr{G}_1, \mathscr{G}_2) = 0.3120,\\
&d^{GHN}(\mathscr{G}_1, \mathscr{G}_2) = 0.3280(\lambda = 3),\ d^{TI}(\mathscr{G}_1, \mathscr{G}_2) = 0.2009.
\end{aligned}$$

Theorem 1. *Let $\mathbb{S} = \{s_1, s_2, \ldots..s_r.\}$ and $\mathscr{G}_1, \mathscr{G}_2 \in (m, a, n) - FNS(\mathbb{S})$. Then, $d^{GN}(\mathscr{G}_1, \mathscr{G}_2)$ is a DM.*

Proof. We prove that $d^{GN}(\mathscr{G}_1, \mathscr{G}_2)$ satisfies the axioms (D1)–(D4).

(D1) Since $\mho_{\mathscr{G}_1} \geq 0, \varpi_{\mathscr{G}_1} \geq 0, \Omega_{\mathscr{G}_1} \geq 0$, and $\mho_{\mathscr{G}_2} \geq 0, \varpi_{\mathscr{G}_2} \geq 0, \Omega_{\mathscr{G}_2} \geq 0$, we have $|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \geq 0$, $|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \geq 0$, and $|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \geq 0$. It follows that

$$d^{GN}(\mathscr{G}_1, \mathscr{G}_2) = \sqrt[\lambda]{\frac{1}{3r} \sum_1^r \begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} \geq 0.$$

Again, since $\mho_{\mathscr{G}_1} \leq 1, \varpi_{\mathscr{G}_1} \leq 1, \Omega_{\mathscr{G}_1} \leq 1$, and $\mho_{\mathscr{G}_2} \leq 1, \varpi_{\mathscr{G}_2} \leq 1, \Omega_{\mathscr{G}_2} \leq 1$, we have $|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \leq 1, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \leq 1, and\ |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \leq 1$.
It follows that

$$d^{GN}(\mathscr{G}_1, \mathscr{G}_2) = \sqrt[\lambda]{\frac{1}{3r} \sum_1^r \begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} \leq 1.$$

Consequently, $0 \leq d^{GN}(\mathscr{G}_1, \mathscr{G}_2) \leq 1$.

(D2) It follows that

$$d^{GN}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt[\lambda]{\frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}}$$
$$= \sqrt[\lambda]{\frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_2}(s_i) - \mho^m_{\mathscr{G}_1}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_2}(s_i) - \varpi^a_{\mathscr{G}_1}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_2}(s_i) - \Omega^n_{\mathscr{G}_1}(s_i)|^\lambda \end{pmatrix}}$$
$$= d^{GN}(\mathscr{G}_2,\mathscr{G}_1).$$

(D3) For any two (m,a,n)-FNSs $\mathscr{G}_1$ and $\mathscr{G}_2$, we have

$$\mathscr{G}_1 = \mathscr{G}_2 \Leftrightarrow \mho_{\mathscr{G}_1}(s_i) = \mho_{\mathscr{G}_2}(s_i), \varpi_{\mathscr{G}_1}(s_i) = \varpi_{\mathscr{G}_2}(s_i), \text{ and } \Omega_{\mathscr{G}_1}(s_i) = \Omega_{\mathscr{G}_2}(s_i).$$
$$\Leftrightarrow \sqrt[\lambda]{\frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} = 0$$
$$\Leftrightarrow d^{GN}(\mathscr{G}_1,\mathscr{G}_2) = 0.$$

(D4) Let $\mathscr{G}_3 = (\mho_{\mathscr{G}_3}, \varpi_{\mathscr{G}_3}, \Omega_{\mathscr{G}_3})$ be an (m,a,n)-FNS such that $\mathscr{G}_1 \Subset \mathscr{G}_2 \Subset \mathscr{G}_3$. Then, we have $\mathscr{G}_1 \Subset \mathscr{G}_2$, and $\mathscr{G}_1 \Subset \mathscr{G}_3$. It follows that $\mho_{\mathscr{G}_1} \le \mho_{\mathscr{G}_2}$, $\varpi_{\mathscr{G}_1} \le \varpi_{\mathscr{G}_2}$, $\Omega_{\mathscr{G}_1} \ge \Omega_{\mathscr{G}_2}$, $\mho_{\mathscr{G}_1} \le \mho_{\mathscr{G}_3}$, $\varpi_{\mathscr{G}_1} \le \varpi_{\mathscr{G}_3}$, and $\Omega_{\mathscr{G}_1} \ge \Omega_{\mathscr{G}_3}$. Therefore, for $\forall\ s_i \in \mathbb{S}(i = 1\ldots r.)$, we have

$$|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \le |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_3}(s_i)|^\lambda,$$
$$|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \le |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_3}(s_i)|^\lambda,$$
$$|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \le |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_3}(s_i)|^\lambda.$$

It implies that

$$\sqrt[\lambda]{\frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} \le \sqrt[\lambda]{\frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_3}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_3}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_3}(s_i)|^\lambda \end{pmatrix}}.$$

Consequently, $d^{GN}(\mathscr{G}_1,\mathscr{G}_3) \le d^{GN}(\mathscr{G}_1,\mathscr{G}_3)$.
Similarly, $d^{GN}(\mathscr{G}_2,\mathscr{G}_3) \le d^{GN}(\mathscr{G}_1,\mathscr{G}_3)$.

□

Theorem 2. *Let $\mathbb{S} = \{s_1, s_2, \ldots ..s_r.\}$ and $\mathscr{G}_1, (\mho_{\mathscr{G}_2} \in (m,a,n) - FNS(\mathbb{S})$. Then, $d^{NH}(\mathscr{G}_1,\mathscr{G}_2)$ and $d^{NE}(\mathscr{G}_1,\mathscr{G}_2)$ are DMs.*

Proof. Follows from Theorem 1 and Remark 3(i). □

Theorem 3. *Let $\mathbb{S} = \{s_1, s_2, \ldots ..s_r.\}$ and $\mathscr{G}_1, \mathscr{G}_2 \in (m,a,n) - FNS(\mathbb{S})$. Then, $d^{GNH}(\mathscr{G}_1,\mathscr{G}_2)$ is a DM.*

Proof. We prove that $d^{GNH}(\mathscr{G}_1,\mathscr{G}_2)$ satisfies axioms (D1)–(D4).

(D1) Since $\mho_{\mathscr{G}_1} \ge 0$, $\varpi_{\mathscr{G}_1} \ge 0, \Omega_{\mathscr{G}_1} \ge 0$, and $(\mho_{\mathscr{G}_2} \ge 0, \varpi_{\mathscr{G}_2} \ge 0, \Omega_{\mathscr{G}_2} \ge 0$, we have $|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \ge 0$, $|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \ge 0, and\ |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \ge 0.$

It follows that

$$d^{GNH}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt[\lambda]{\frac{1}{r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} \geq 0.$$

Again, since $\mho_{\mathscr{G}_1} \leq 1, \varpi_{\mathscr{G}_1} \leq 1, \Omega_{\mathscr{G}_1} \leq 1$, and $(\mho_{\mathscr{G}_2} \leq 1, \varpi_{\mathscr{G}_2} \leq 1, \Omega_{\mathscr{G}_2} \leq 1$, we have $|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \leq 1, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \leq 1, and\ |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \leq 1.$
It follows that

$$d^{GNH}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt[\lambda]{\frac{1}{r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} \leq 1.$$

Consequently, $0 \leq d^{GNH}(\mathscr{G}_1,\mathscr{G}_2) \leq 1$.

(D2) It follows that

$$\begin{aligned} d^{GNH}(\mathscr{G}_1,\mathscr{G}_2) &= \sqrt[\lambda]{\frac{1}{r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} \\ &= \sqrt[\lambda]{\frac{1}{r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_2}(s_i) - \mho^m_{\mathscr{G}_1}(s_i)|^\lambda \\ \vee\, |\varpi^a_{\mathscr{G}_2}(s_i) - \varpi^a_{\mathscr{G}_1}(s_i)|^\lambda \\ \vee\, |\Omega^n_{\mathscr{G}_2}(s_i) - \Omega^n_{\mathscr{G}_1}(s_i)|^\lambda \end{pmatrix}} \\ &= d^{GNH}(\mathscr{G}_2,\mathscr{G}_1). \end{aligned}$$

(D3) For any two (m,a,n)-FNSs $\mathscr{G}_1$ and $\mathscr{G}_2$, we have

$$\begin{aligned} &\mathscr{G}_1 = \mathscr{G}_2 \Leftrightarrow \mho_{\mathscr{G}_1}(s_i) = \mho_{\mathscr{G}_2}(s_i),\ \varpi_{\mathscr{G}_1}(s_i) = \varpi_{\mathscr{G}_2}(s_i),\ \text{and } \Omega_{\mathscr{G}_1}(s_i) = \Omega_{\mathscr{G}_2}(s_i) \\ &\Leftrightarrow \sqrt[\lambda]{\frac{1}{r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} = 0 \\ &\Leftrightarrow d^{GNH}(\mathscr{G}_1,\mathscr{G}_2) = 0. \end{aligned}$$

(D4) Let $\mathscr{G}_3 = (\mho_{\mathscr{G}_3}, \varpi_{\mathscr{G}_3}, \Omega_{\mathscr{G}_3})$ be an (m,a,n)-FNS such that $\mathscr{G}_1 \Subset \mathscr{G}_2 \Subset \mathscr{G}_3$. Then, we have $\mathscr{G}_1 \Subset \mathscr{G}_2$, and $\mathscr{G}_1 \Subset \mathscr{G}_3$. It follows that $\mho_{\mathscr{G}_1} \leq \mho_{\mathscr{G}_2}$, $\varpi_{\mathscr{G}_1} \leq \varpi_{\mathscr{G}_2}$,$\Omega_{\mathscr{G}_1} \geq \Omega_{\mathscr{G}_2}$, $\mho_{\mathscr{G}_1} \leq \mho_{\mathscr{G}_3}$, $\varpi_{\mathscr{G}_1} \leq \varpi_{\mathscr{G}_3}$, and $\Omega_{\mathscr{G}_1} \geq \Omega_{\mathscr{G}_3}$. Therefore, $\forall\ s_i \in \mathbb{S}\ (i = 1 \ldots r.)$, and we have

$$\begin{aligned} &|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \leq |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_3}(s_i)|^\lambda, \\ &|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \leq |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_3}(s_i)|^\lambda, \quad \text{and} \\ &|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \leq |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_3}(s_i)|^\lambda, \end{aligned}$$

It implies that

$$\sqrt[\lambda]{\frac{1}{r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}} \leq \sqrt[\lambda]{\frac{1}{r}\sum_{1}^{r}\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_3}(s_i)|^\lambda \\ \vee\, |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_3}(s_i)|^\lambda \\ \vee\, |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_3}(s_i)|^\lambda \end{pmatrix}}$$

Hence, $d^{GNH}(\mathscr{G}_1,\mathscr{G}_2) \leq d^{GNH}(\mathscr{G}_1,\mathscr{G}_3)$. Similarly, $d^{GNH}(\mathscr{G}_2,\mathscr{G}_3) \leq d^{GNH}(\mathscr{G}_1,\mathscr{G}_3)$.

□

Theorem 4. *Let $\mathscr{G}_1, \mathscr{G}_2 \in (m,a,n) - FNS(\mathbb{S})$. Then, $d^{NHH}(\mathscr{G}_1, \mathscr{G}_2)$ and $d^{NEH}(\mathscr{G}_1, \mathscr{G}_2)$ are valid DMs.*

Proof. Follows from Theorem 3 and Remark 3(ii). □

Theorem 5. *Let $\mathscr{G}_1, \mathscr{G}_2 \in (m,a,n) - FNS(\mathbb{S})$. Then, $d^{GHN}(\mathscr{G}_1, \mathscr{G}_2)$ is a valid distance measure.*

Proof. We prove that $d^{GHN}(\mathscr{G}_1, \mathscr{G}_2)$ satisfies the axioms (D1)–(D4).

(D1) Since $0 \leq \mho_{\mathscr{G}_1}, \mho_{\mathscr{G}_2} \leq 1$, we have $0 \leq |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \leq 1$, $\forall, s_i \in \mathbb{S}$, and i = 1…r. Similarly, $0 \leq |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \leq 1$ and $|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \geq 0$. It follows that

$$0 \leq \frac{|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda + |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda + |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda}{6} \leq \frac{1}{2},$$

and

$$0 \leq \frac{|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \bigvee |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \bigvee |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda}{2} \leq \frac{1}{2}.$$

Therefore,

$$\begin{aligned} 0 &\leq \sqrt[\lambda]{\frac{1}{r}\sum_1^r\left(\frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}}{6} + \frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \bigvee |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \bigvee |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}}{2}\right)} \\ &\leq \frac{1}{r}(\frac{r}{2} + \frac{r}{2}) \\ &= 1. \end{aligned}$$

Hence, $0 \leq d^{GHN}(\mathscr{G}_1, \mathscr{G}_2) \leq 1$.

(D2) It follows that

$$\begin{aligned} d^{GHN}(\mathscr{G}_1, \mathscr{G}_2) &= \sqrt[\lambda]{\frac{1}{r}\sum_1^r\left(\frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}}{6} + \frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \bigvee |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \bigvee |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}}{2}\right)} \\ &= \sqrt[\lambda]{\frac{1}{r}\sum_1^r\left(\frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_2}(s_i) - \mho^m_{\mathscr{G}_1}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_2}(s_i) - \varpi^a_{\mathscr{G}_1}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_2}(s_i) - \Omega^n_{\mathscr{G}_1}(s_i)|^\lambda \end{pmatrix}}{6} + \frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_2}(s_i) - \mho^m_{\mathscr{G}_1}(s_i)|^\lambda \\ \bigvee |\varpi^a_{\mathscr{G}_2}(s_i) - \varpi^a_{\mathscr{G}_1}(s_i)|^\lambda \\ \bigvee |\Omega^n_{\mathscr{G}_2}(s_i) - \Omega^n_{\mathscr{G}_1}(s_i)|^\lambda \end{pmatrix}}{2}\right)} \\ &= d^{GHN}(\mathscr{G}_2, \mathscr{G}_1). \end{aligned}$$

(D3) For any two (m,a,n)-FNSs $\mathscr{G}_1$ and $\mathscr{G}_2$, we have

$$\mathscr{G}_1 = \mathscr{G}_2 \Leftrightarrow \mho_{\mathscr{G}_1}(s_i) = \mho_{\mathscr{G}_2}(s_i), \varpi_{\mathscr{G}_1}(s_i) = \varpi_{\mathscr{G}_2}(s_i), \text{ and } \Omega_{\mathscr{G}_1}(s_i) = \Omega_{\mathscr{G}_2}(s_i).$$

$$\Leftrightarrow \sqrt[\lambda]{\frac{1}{r}\sum_1^r\left(\frac{\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda\end{pmatrix}}{6} + \frac{\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda\end{pmatrix}}{2}\right)} = 0.$$

$$\Leftrightarrow d^{GHN}(\mathscr{G}_1, \mathscr{G}_2) = 0.$$

(D4) Let $\mathscr{G}_3 = (\mho_{\mathscr{G}_3}, \varpi_{\mathscr{G}_3}, \Omega_{\mathscr{G}_3})$ be an (m,a,n)-FNS such that $\mathscr{G}_1 \Subset \mathscr{G}_2 \Subset \mathscr{G}_3$. Then, we have $\mathscr{G}_1 \Subset \mathscr{G}_2$ and $\mathscr{G}_1 \Subset \mathscr{G}_3$. It follows that $\mho_{\mathscr{G}_1} \leq \mho_{\mathscr{G}_2}$, $\varpi_{\mathscr{G}_1} \leq \varpi_{\mathscr{G}_2}$,$\Omega_{\mathscr{G}_1} \geq \Omega_{\mathscr{G}_2}$ and $\mho_{\mathscr{G}_1} \leq \mho_{\mathscr{G}_3}$, $\varpi_{\mathscr{G}_1} \leq \varpi_{\mathscr{G}_3}$,$\Omega_{\mathscr{G}_1} \geq \Omega_{\mathscr{G}_3}$. Therefore, $\forall\, s_i \in \mathbb{S}$ $(i = 1 \ldots r.)$, and we have

$$\begin{aligned}&|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \leq |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_3}(s_i)|^\lambda, \\ &|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \leq |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_3}(s_i)|^\lambda, \text{ and,} \\ &|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \leq |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_3}(s_i)|^\lambda.\end{aligned}$$

It implies that

$$\sqrt[\lambda]{\frac{1}{r}\sum_1^r\left(\frac{\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda\end{pmatrix}}{6} + \frac{\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \vee|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda\end{pmatrix}}{2}\right)}$$

$$\leq \sqrt[\lambda]{\frac{1}{r}\sum_1^r\left(\frac{\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_3}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_3}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_3}(s_i)|^\lambda\end{pmatrix}}{6} + \frac{\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_3}(s_i)|^\lambda \\ \vee|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_3}(s_i)|^\lambda \\ \vee|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_3}(s_i)|^\lambda\end{pmatrix}}{2}\right)}.$$

Consequently, $d^{GHN}(\mathscr{G}_1, \mathscr{G}_2) \leq d^{GhN}(\mathscr{G}_1, \mathscr{G}_3)$. Similarly , $d^{GHN}(\mathscr{G}_2, \mathscr{G}_3) \leq d^{GHN}(\mathscr{G}_1, \mathscr{G}_3)$. □

Theorem 6. *Let $\mathscr{G}_1, \mathscr{G}_2 \in (m, a, n) - FNS(\mathbb{S})$. Then, $d^{HNH}(\mathscr{G}_1, \mathscr{G}_2)$ and $d^{HNE}(\mathscr{G}_1, \mathscr{G}_2)$ are valid distance measures.*

Proof. Follows from Theorem 5 and Remark 3(iii). □

Theorem 7. *Let $\mathscr{G}_1, \mathscr{G}_2 \in (m, a, n) - FNS(\mathbb{S})$. Then, $d^{TI}(\mathscr{G}_1, \mathscr{G}_2)$ is a valid distance measure.*

Proof. We prove that $d^{TI}(\mathscr{G}_1, \mathscr{G}_2)$ satisfies the axioms (D1)–(D4).

(D1) Since $0 \leq \mho_{\mathscr{G}_1}(s_i)$, $\mho_{\mathscr{G}_2}(s_i) \leq 1$, for i…r, we have $0 \leq tan^{-1}\mho^m_{\mathscr{G}_1}(s_i)$, $tan^{-1}\mho^m_{\mathscr{G}_2}(s_i) \leq 1$, and thus $0 \leq |tan^{-1}\mho^m_{\mathscr{G}_1}(s_i) - tan^{-1}\mho^m_{\mathscr{G}_2}(s_i)| \leq 1$. Similarly, we have $0 \leq |tan^{-1}\varpi^a_{\mathscr{G}_1}(s_i) - tan^{-1}\varpi^a_{\mathscr{G}_2}(s_i)| \leq 1$ and $0 \leq |tan^{-1}\Omega^n_{\mathscr{G}_1}(s_i) - tan^{-1}\Omega^n_{\mathscr{G}_2}(s_i)| \leq 1$. Therefore, we obtain $0 \leq d^{TI}(\mathscr{G}_1, \mathscr{G}_2) \leq 1$.

(D2) It follows that

$$d^{TI}(\mathscr{G}_1,\mathscr{G}_2) = \frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |tan^{-1}\mho^m_{\mathscr{G}_1}(s_i) - tan^{-1}\mho^m_{\mathscr{G}_2}(s_i)| \\ +|tan^{-1}\varpi^a_{\mathscr{G}_1}(s_i) - tan^{-1}\varpi^a_{\mathscr{G}_2}(s_i)| \\ +|tan^{-1}\Omega^n_{\mathscr{G}_1}(s_i) - tan^{-1}\Omega^n_{\mathscr{G}_2}(s_i)| \end{pmatrix}$$
$$= \frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |tan^{-1}\mho^m_{\mathscr{G}_2}(s_i) - tan^{-1}\mho^m_{\mathscr{G}_1}(s_i)| \\ +|tan^{-1}\varpi^a_{\mathscr{G}_2}(s_i) - tan^{-1}\varpi^a_{\mathscr{G}_1}(s_i)| \\ +|tan^{-1}\Omega^n_{\mathscr{G}_2}(s_i) - tan^{-1}\Omega^n_{\mathscr{G}_1}(s_i)| \end{pmatrix}$$
$$= d^{TI}(\mathscr{G}_2,\mathscr{G}_1)$$

(D3) For any two (m,a,n)-FNSs $\mathscr{G}_1$ and $\mathscr{G}_2$, we have

$$\mathscr{G}_1 = \mathscr{G}_2 \Leftrightarrow \mho_{\mathscr{G}_1}(s_i) = \mho_{\mathscr{G}_2}(s_i), \varpi_{\mathscr{G}_1}(s_i) = \varpi_{\mathscr{G}_2}(s_i), \text{and } \Omega_{\mathscr{G}_1}(s_i) = \Omega_{\mathscr{G}_2}(s_i)$$
$$\Leftrightarrow \frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |tan^{-1}\mho^m_{\mathscr{G}_1}(s_i) - tan^{-1}\mho^m_{\mathscr{G}_2}(s_i)| \\ +|tan^{-1}\varpi^a_{\mathscr{G}_1}(s_i) - tan^{-1}\varpi^a_{\mathscr{G}_2}(s_i)| \\ +|tan^{-1}\Omega^n_{\mathscr{G}_1}(s_i) - tan^{-1}\Omega^n_{\mathscr{G}_2}(s_i)| \end{pmatrix}$$
$$= \frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |tan^{-1}\mho^m_{\mathscr{G}_2}(s_i) - tan^{-1}\mho^m_{\mathscr{G}_1}(s_i)| \\ +|tan^{-1}\varpi^a_{\mathscr{G}_2}(s_i) - tan^{-1}\varpi^a_{\mathscr{G}_1}(s_i)| \\ +|tan^{-1}\Omega^n_{\mathscr{G}_2}(s_i) - tan^{-1}\Omega^n_{\mathscr{G}_1}(s_i)| \end{pmatrix}$$
$$\Leftrightarrow d^{TI}(\mathscr{G}_1,\mathscr{G}_2) = 0.$$

(D4) Let $\mathscr{G}_3 = (\mho_{\mathscr{G}_3}, \varpi_{\mathscr{G}_3}, \Omega_{\mathscr{G}_3})$ be an (m,a,n)-FNS such that $\mathscr{G}_1 \Subset \mathscr{G}_2 \Subset \mathscr{G}_3$. Then, we have $\mathscr{G}_1 \Subset \mathscr{G}_2$ and $\mathscr{G}_1 \Subset \mathscr{G}_3$. It follows that $\mho_{\mathscr{G}_1} \leq \mho_{\mathscr{G}_2}$, $\varpi_{\mathscr{G}_1} \leq \varpi_{\mathscr{G}_2}$, $\Omega_{\mathscr{G}_1} \geq \Omega_{\mathscr{G}_2}$, $\mho_{\mathscr{G}_1} \leq \mho_{\mathscr{G}_3}$, $\varpi_{\mathscr{G}_1} \leq \varpi_{\mathscr{G}_3}$, and $\Omega_{\mathscr{G}_1} \geq \Omega_{\mathscr{G}_3}$. Therefore, $\forall\, s_i \in \mathbb{S}(i = 1 \ldots r.)$, and we have

$$|tan^{-1}\mho^m_{\mathscr{G}_1}(s_i) - tan^{-1}\mho^m_{\mathscr{G}_2}(s_i)| \leq |tan^{-1}\mho^m_{\mathscr{G}_1}(s_i) - tan^{-1}\mho^m_{\mathscr{G}_3}(s_i)|,$$
$$|tan^{-1}\varpi^a_{\mathscr{G}_1}(s_i) - tan^{-1}\varpi^a_{\mathscr{G}_2}(s_i)| \leq |tan^{-1}\varpi^a_{\mathscr{G}_1}(s_i) - tan^{-1}\varpi^a_{\mathscr{G}_3}(s_i)|, \text{and}$$
$$|tan^{-1}\Omega^n_{\mathscr{G}_1}(s_i) - tan^{-1}\Omega^n_{\mathscr{G}_2}(s_i)| \leq |tan^{-1}\Omega^n_{\mathscr{G}_1}(s_i) - tan^{-1}\Omega^n_{\mathscr{G}_3}(s_i)|.$$

Therefore,

$$d^{TI}(\mathscr{G}_1,\mathscr{G}_2) = \frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |tan^{-1}\mho^m_{\mathscr{G}_1}(s_i) - tan^{-1}\mho^m_{\mathscr{G}_2}(s_i)| \\ +|tan^{-1}\varpi^a_{\mathscr{G}_1}(s_i) - tan^{-1}\varpi^a_{\mathscr{G}_2}(s_i)| \\ +|tan^{-1}\Omega^n_{\mathscr{G}_1}(s_i) - tan^{-1}\Omega^n_{\mathscr{G}_2}(s_i)| \end{pmatrix}$$
$$\frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |tan^{-1}\mho^m_{\mathscr{G}_2}(s_i) - tan^{-1}\mho^m_{\mathscr{G}_3}(s_i)| \\ +|tan^{-1}\varpi^a_{\mathscr{G}_1}(s_i) - tan^{-1}\varpi^a_{\mathscr{G}_3}(s_i)| \\ +|tan^{-1}\Omega^n_{\mathscr{G}_1}(s_i) - tan^{-1}\Omega^n_{\mathscr{G}_3}(s_i)| \end{pmatrix}$$
$$= d^{TI}(\mathscr{G}_1,\mathscr{G}_3)$$

Thus, $d^{TI}(\mathscr{G}_1,\mathscr{G}_2) \leq d^{TI}(\mathscr{G}_1,\mathscr{G}_3)$. Similarly, we have $d^{TI}(\mathscr{G}_2,\mathscr{G}_3) \leq d^{TI}(\mathscr{G}_1,\mathscr{G}_3)$.

□

Now, we define the (m,a,n)-FNWDMs between two (m,a,n)-FNSs $\mathscr{G}_1$ and $\mathscr{G}_2$ by considering the weighting vector of the elements in the (m,a,n)-FNSs.

Definition 10. *Let $\mathbb{S} = \{s_1, s_2, \ldots\ldots s_r.\}$ and $\mathscr{G}_1, \mathscr{G}_2 \in (m,a,n) - FNS(\mathbb{S})$. Assume that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the elements s_i (i = 1, 2, ..., r), satisfying the conditions $\sum_{i=1}^{r} \omega_i = 1$, $\forall\, \omega_i \in$ [0, 1], and i = 1, 2, ..., r. Then, the following statements apply:*

(i) *The weighted normalized Hamming distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{WNH}(\mathscr{G}_1,\mathscr{G}_2) = \frac{1}{3}\sum_{1}^{r}\omega_i\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)| \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)| \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|\end{pmatrix} \tag{20}$$

(ii) *The weighted normalized Euclidean distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined by the following expression:*

$$d^{WNE}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt{\frac{1}{3}\sum_{1}^{r}\omega_i\begin{pmatrix}(\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i))^2 \\ +(\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i))^2 \\ +(\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i))^2+\end{pmatrix}} \tag{21}$$

(iii) *The generalized weighted normalized distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined by the following expression:*

$$d^{GWN}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt[\lambda]{\frac{1}{3}\sum_{1}^{r}\omega_i\begin{pmatrix}(\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i))^\lambda \\ +(\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i))^\lambda \\ +(\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i))^\lambda\end{pmatrix}} \tag{22}$$

(iv) *The weighted normalized Hamming–Hausdorff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{WNHH}(\mathscr{G}_1,\mathscr{G}_2) = \sum_{1}^{r}\omega_i\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)| \\ \bigvee|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)| \\ \bigvee|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|\end{pmatrix} \tag{23}$$

(v) *The weighted normalized Euclidean–Hausdorff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{WNEH}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt{\sum_{1}^{r}\omega_i\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^2 \\ \bigvee|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^2, \\ \bigvee|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^2\end{pmatrix}} \tag{24}$$

(vi) *The generalized weighted normalized Hausdorff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{GWNH}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt[\lambda]{\sum_{1}^{r}\omega_i\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \bigvee|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \bigvee|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda\end{pmatrix}} \tag{25}$$

(vii) *The hybrid weighted normalized Hamming distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{HWNH}(\mathscr{G}_1,\mathscr{G}_2) = \sum_{1}^{r}\omega_i\left(\frac{\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)| \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)| \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|\end{pmatrix}}{6} + \frac{\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)| \\ \bigvee|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)| \\ \bigvee|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|\end{pmatrix}}{2}\right) \tag{26}$$

(viii) *The hybrid weighted normalized Euclidean distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{HWNE}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt{\sum_{1}^{r}\omega_i\left(\frac{\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^2 \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^2 \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^2\end{pmatrix}}{6} + \frac{\begin{pmatrix}|\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^2 \\ \bigvee|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^2 \\ \bigvee|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^2\end{pmatrix}}{2}\right)} \tag{27}$$

(ix) *The generalized hybrid weighted normalized distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{GHWN}(\mathscr{G}_1,\mathscr{G}_2) = \sqrt[\lambda]{\sum_1^r \omega_i \left(\frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ +|\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}}{6} + \frac{\begin{pmatrix} |\mho^m_{\mathscr{G}_1}(s_i) - \mho^m_{\mathscr{G}_2}(s_i)|^\lambda \\ \bigvee |\varpi^a_{\mathscr{G}_1}(s_i) - \varpi^a_{\mathscr{G}_2}(s_i)|^\lambda \\ \bigvee |\Omega^n_{\mathscr{G}_1}(s_i) - \Omega^n_{\mathscr{G}_2}(s_i)|^\lambda \end{pmatrix}}{2} \right)} \tag{28}$$

(x) *The weighted tangent inverse distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is defined as follows:*

$$d^{WTI}(\mathscr{G}_1,\mathscr{G}_2) = \frac{1}{3}\sum_1^r \omega_i \begin{pmatrix} |tan^{-1}\mho^m_{\mathscr{G}_1}(s_i) - tan^{-1}\mho^m_{\mathscr{G}_2}(s_i)| \\ +|tan^{-1}\varpi^a_{\mathscr{G}_1}(s_i) - tan^{-1}\varpi^a_{\mathscr{G}_2}(s_i)| \\ +|tan^{-1}\Omega^n_{\mathscr{G}_1}(s_i) - tan^{-1}\Omega^n_{\mathscr{G}_2}(s_i)| \end{pmatrix} \tag{29}$$

Remark 4. *For any two (m,a,n)-FNSs $\mathscr{G}_1$ and $\mathscr{G}_2$ over a universe of discourse $\mathbb{S} = \{s_1, s_2, \ldots..s_r.\}$, the following apply:*

(i) *The generalized weighted normalized distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is reduced to the weighted normalized Hamming distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 1$ and the weighted normalized Euclidean distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 2$.*

(ii) *The generalized weighted normalized Hausdorff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is reduced to the weighted normalized Hamming–Hausdorff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 1$ and the weighted normalized Euclidean–Hausdroff distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 2$.*

(iii) *The generalized hybrid weighted normalized distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ is reduced to the hybrid weighted normalized Hamming distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 1$ and the hybrid weighted normalized Euclidean distance between $\mathscr{G}_1$ and $\mathscr{G}_2$ for $\lambda = 2$.*

Remark 5. *When we take the weighting vector $\omega = (\frac{1}{r}, \frac{1}{r}, \ldots, \frac{1}{r})^T$, then, in the (m,a,n)-FNWDM, the following occur:*

(i) $d^{WNH}(\mathscr{G}_1,\mathscr{G}_2)$ *reduce to* $d^{NH}(\mathscr{G}_1,\mathscr{G}_2)$.
(ii) $d^{WNE}(\mathscr{G}_1,\mathscr{G}_2)$ *reduce to* $d^{NE}(\mathscr{G}_1,\mathscr{G}_2)$.
(iii) $d^{GWN}(\mathscr{G}_1,\mathscr{G}_2)$ *reduce to* $d^{GN}(\mathscr{G}_1,\mathscr{G}_2)$.
(iv) $d^{WNHH}(\mathscr{G}_1,\mathscr{G}_2)$ *reduce to* $d^{NHH}(\mathscr{G}_1,\mathscr{G}_2)$.
(v) $d^{WNEH}(\mathscr{G}_1,\mathscr{G}_2)$ *reduce to* $d^{NEH}(\mathscr{G}_1,\mathscr{G}_2)$.
(vi) $d^{GWNH}(\mathscr{G}_1,\mathscr{G}_2)$ *reduce to* $d^{GNH}(\mathscr{G}_1,\mathscr{G}_2)$.
(vii) $d^{HWNH}(\mathscr{G}_1,\mathscr{G}_2)$ *reduce to* $d^{HNH}(\mathscr{G}_1,\mathscr{G}_2)$.
(viii) $d^{HWNE}(\mathscr{G}_1,\mathscr{G}_2)$ *reduce to* $d^{HNE}(\mathscr{G}_1,\mathscr{G}_2)$.
(ix) $d^{GHWN}(\mathscr{G}_1,\mathscr{G}_2)$ *reduce to* $d^{GHN}(\mathscr{G}_1,\mathscr{G}_2)$.
(x) $d^{WTI}(\mathscr{G}_1,\mathscr{G}_2)$ *reduce to* $d^{TI}(\mathscr{G}_1,\mathscr{G}_2)$.

Example 3. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two (m,a,n)-FNSs over $\mathbb{S}$, defined in Example 2, and consider the weighting vector ω = $(0.18, 0.25, 0.22, 0.35)^T$. Then,*

$$\begin{aligned}
&d^{WNH}(\mathscr{G}_1,\mathscr{G}_2) = 0.2173,\ d^{WNE}(\mathscr{G}_1,\mathscr{G}_2) = 0.2530,\\
&d^{GWN}(\mathscr{G}_1,\mathscr{G}_2) = 0.2695(\lambda = 3),\ d^{WNHH}(\mathscr{G}_1,\mathscr{G}_2) = 0.3569,\\
&d^{WNEH}(\mathscr{G}_1,\mathscr{G}_2) = 0.3618,\ d^{GWNH}(\mathscr{G}_1,\mathscr{G}_2) = 0.3668(\lambda = 3),\\
&d^{HWNH}(\mathscr{G}_1,\mathscr{G}_2) = 0.2871,\ d^{HWNE}(\mathscr{G}_1,\mathscr{G}_2) = 0.3093,\\
&d^{GHWN}(\mathscr{G}_1,\mathscr{G}_2) = 0.3254(\lambda = 3),\ d^{WTI}(\mathscr{G}_1,\mathscr{G}_2) = 0.1937.
\end{aligned}$$

Theorem 8. *Let $\mathbb{S} = \{s_1, s_2, \ldots\ldots s_r.\}$ and $\mathscr{G}_1, \mathscr{G}_2 \in (m,a,n) - FNS(\mathbb{S})$. Assume that ω = $(\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the elements s_i (i = 1,2,..., r), satisfying* *the conditions $\sum_{i=1}^{r} \omega_i = 1$, $\forall\ \omega_i \in [0,1]$, and $i = 1, 2, \ldots, r$. Then, the (m,a,n)-FNWDMs proposed in Definition 10 are the valid DMs for the (m,a,n)-FNSs.*

Proof. This follows by considering a process similar to that used in the corresponding cases of (m,a,n)-FNDMs. □

3.2. Comparison of the Proposed DMs with the Existing DMs in a Neutrosophic Environment

This section presents a comparative analysis of the proposed (m,a,n)-FNDMs and (m,a,n)-FNWDMs and the existing DMs for PyNSs and FNSs using remarks and numerical examples.

Remark 6. *The weighted DMs d_1, d_2, and d_p proposed by Rajan and Krishnaswamy [46] for PyNSs are, respectively, the special cases of d^{WNH}, d^{WNE}, and d^{GWN} for m = a = n = 2.*

Remark 7. *The weighted DMs d_H, d_{E_u}, and d_{F_n} proposed by Saeed and his coworkers [47] for FNSs are, respectively, the special cases of d^{WNH}, d^{WNE}, and d^{GWN} for m = a = n = 3.*

Remark 8. *The following example shows the that the weighted DMs d_1, d_2, and d_p proposed by Rajan and Krishnaswamy [46] for PyNSs and the weighted DMs d_H, d_{E_u}, and d_{F_n} proposed by Saeed and his coworkers [47] for FNSs are inconsistent and irrational.*

Example 4. *Let $\mathbb{S} = \{s_1, s_2\}$, and (m,a,n)-FNS $\mathscr{G}_i$, $\mathscr{H}_i$ (i=1..4) be defined as follows:*

$$\mathscr{G}_1 = \{(s_1, 0.6, 0.5, 0.4), (s_2, 0.4, 0.3, 0.6)\},\ \mathscr{H}_1 = \{(s_1, 0.7, 0.6, 0.8), (s_2, 0.36, 0.2, 0.8)\}$$
$$\mathscr{G}_2 = \{(s_1, 0.6, 0.5, 0.4), (s_2, 0.4, 0.3, 0.6)\},\ \mathscr{H}_2 = \{(s_1, 0.4, 0.35, 0.6), (s_2, 0.4, 0.7, 0.8)\}$$
$$\mathscr{G}_3 = \{(s_1, 0.5, 0.8, 0.6), (s_2, 0.7, 0.8, 0.2)\},\ \mathscr{H}_3 = \{(s_1, 0.4, 0.3, 0.3), (s_2, 0.7, 0.3, 0.2)\}$$
$$\mathscr{G}_4 = \{(s_1, 0.5, 0.8, 0.6), (s_2, 0.7, 0.8, 0.2)\},\ \mathscr{H}_4 = \{(s_1, 0.3, 0.4, 0.85), (s_2, 0.2, 0.7, 0.3)\}$$

Assume that $\omega_1 = \omega_2 = \frac{1}{2}$ is the weight for the elements of $\mathbb{S}$. Table 2 displays the established (m,a,n)-FNWDMs and the existing weighted DMs for PyNSs and FNSs. In the considered cases, we observe that $\mathscr{G}_1 = \mathscr{G}_2$, $\mathscr{H}_1 \neq \mathscr{H}_2$, $\mathscr{G}_3 = \mathscr{G}_4$, and $\mathscr{H}_1 \neq \mathscr{H}_2$. Upon examining Table 2, it becomes apparent that the results produced by the Fermatean neutrosophic weighted Hamming distance d_H for the pairs ($\mathscr{G}_1, 4\mathscr{H}_1$) and ($\mathscr{G}_2$ $\mathscr{H}_2$) is identical, which is contradictory and logically inconsistent given the differences in $\mathscr{H}_1$ and $\mathscr{H}_2$. A similar anomaly is observed in the case of the Fermatean neutrosophic weighted Euclidean distance d_{E_u}, where the distance values for the pairs ($\mathscr{G}_3, \mathscr{H}_3$) and ($\mathscr{G}_4, \mathscr{H}_4$) are also the same, despite the dissimilarity between $\mathscr{H}_3$ and $\mathscr{H}_3$. Furthermore, the distance measures d_1, d_2, and d_p fail to compute valid results for the pair ($\mathscr{G}_1$, $\mathscr{H}_1$) because the sum of the squares of the Positive Membership Degree (PMD) and the Negative Membership Degree (NMD) at point s_1 of $\mathscr{H}_1$ exceeds 1, violating the required constraints. These inconsistencies highlight the limitations of existing distance measures for orthopair fuzzy sets within the neutrosophic environment.

Remark 9. *The (m,a,n)-FNDMs proposed in Definition 9 will be reduced to DMs for q-RONSs if we take m = a = n = q.*

Remark 10. *The (m,a,n)-FNWDMs proposed in Definition 10 will be reduced to weighted DMs for q-RONSs if we take m = a = n = q.*

Remarks 6–10 and Examples 2–4 reveal that the proposed distance measures for (m,a,n)-FNSs exhibit enhanced sensitivity and discriminatory power, enabling more reliable and meaningful differentiation among data points. Moreover, the developed (m,a,n)-fuzzy neutrosophic-based distance measures (FNDMs) and weighted distance measures (FNWDMs) are versatile enough to be applied in decision-making problems involving

PyNSs, FNSs, and q-RONSs. In contrast, the distance measures formulated under these existing frameworks are generally not applicable to the broader structure of (m,a,n)-FNSs. It is worth noting that the effectiveness of the proposed methods may depend on the selection of the parameters m, a, and n, highlighting the need for a comprehensive analysis of their robustness across varying parameter values. Future research may explore the practical implementation of this framework in real-world decision-making scenarios to assess its applicability and effectiveness.

Table 2. Comparison of (m,a,n)-FNWDMs.

Weighted DMs	Environment	$(\mathscr{G}_1, \mathscr{H}_1)$	$(\mathscr{G}_2, \mathscr{H}_2)$	$(\mathscr{G}_3, \mathscr{H}_3)$	$(\mathscr{G}_4, \mathscr{H}_4)$
d^{NH}	(2,3,4)-FNS	0.1584	0.1664	0.1969	0.2731
d^{NE}	(2,3,4)-FNS	0.2051	0.1982	0.2867	0.2051
d^{GN}	(2,3,4)-FNS	0.2387	0.2192	0.3375	0.3469
d^{NHH}	(2,3,4)-FNS	0.3968	0.3048	0.4850	0.4490
d^{NEH}	(2,3,4)-FNS	0.3360	0.2644	0.4850	0.4490
d^{GNH}	(2,3,4)-FNS	0.3400	0.2704	0.4850	0.4490
d^{HNH}	(2,3,4)-FNS	0.2439	0.2108	0.3410	0.3600
d^{HNE}	(2,3,4)-FNS	0.2784	0.2337	0.3984	0.3896
d^{GHN}	(2,3,4)-FNS	0.2979	0.2475	0.4241	0.3935
d^{TI}	(2,3,4)-FNS	0.1450	0.1558	0.1833	0.2469
d_H [47]	FNS	0.1664	0.1664	0.2033	0.2445
d_{E_u} [47]	FNS	0.2285	0.2002	0.2915	0.2915
d_{F_n} [47]	FNS	0.2704	0.2213	0.3397	0.3199
d_1 [46]	PyNS	uncertain	0.2013	0.2433	0.2754
d_2 [46]	PyNS	uncertain	0.2362	0.3381	0.2385
d_p [46]	PyNS	uncertain	0.2588	0.3890	0.3499

4. (m,a,n)-FNDMs-Based Approach for Pattern Recognition

In pattern recognition, determining the degree of similarity or difference between data objects plays a pivotal role. Classical approaches often struggle to cope with ambiguous or imprecise data. To address this, (m,a,n)-FNDMs have been proposed as flexible alternatives that incorporate PMD, IMD, and NMD information. These measures are particularly advantageous in vague classification scenarios where patterns may not have clearly defined boundaries.

4.1. Problem Description and Solution Procedure

Let $\mathbb{V} = \{v_1, v_2, .., v_p\}$ be a universe of discourse and

$$\mathbb{P} = \{\mathscr{P}_1, \mathscr{P}_2, .., \mathscr{P}_k\}$$

be k patterns represented by (m,a,n)-FNSs denoted as

$$\mathscr{P}_j = \{< v_i, \mho_{\mathscr{P}_j}(v_i), \varpi_{\mathscr{P}_j}(v_i), \Omega_{\mathscr{P}_j}(v_i) >: v_i \in \mathbb{V}\} \ (j = 1, 2, .., k).$$

Then, form r test samples

$$\mathbb{S} = \{\mathscr{S}_1, \mathscr{S}_2, ..., \mathscr{S}_r\}$$

represented by (m,a,n)-FNSs as

$$\mathscr{S}_g = \{< v_i, \mho_{\mathscr{S}_g}(v_i), \varpi_{\mathscr{S}_g}(v_i), \Omega_{\mathscr{S}_g}(v_i) >: v_i \in \mathbb{V}\} \ (g = 1, 2, .., r).$$

The object is to assign the test sample to the correct category by following the given pattern, ensuring precise classification.

Step 1: Calculate the distance of S_g from each of $\mathscr{P}_1, \mathscr{P}_2, ..., \mathscr{P}_k$ via the proposed (m,a,n)-FNDMs.

Step 2: Determine which pattern produces the lowest value in Step 1.

Step 3: Classify the test sample accordingly for each proposed (m,a,n)-FNDM.

Step 4: Calculate the degree of confidence (DoC) for each (m,a,n)-FNDM using the following formula:

$$DoC = \sum_{j=1, j\neq j_0}^{k} |\mathbb{D}(\mathscr{P}_j, \mathscr{S}) - \mathbb{D}(\mathscr{P}_{j_0}, \mathscr{S})| \tag{30}$$

where $\mathscr{P}_{j_0}$ represents the identified pattern assigned to the test sample $\mathscr{S}$. A higher value of DoC clearly indicates a greater level of confidence in the prediction made by the corresponding (m,a,n)-FNDM.

Herein, we present Algorithm 1 to classify the material using the proposed (m, a, n)-FNDMs.

Algorithm 1 Pattern Classification

Input: $\mathbb{P} = \{\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_k\}, \mathbb{S} = \{\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_r\}\}$
Output: The classification of S_g

```
for g = 1 to r do
    /* Step 1 */
    for j = 1 to k do
        Calculate the distance D(P_j, S_g) using proposed (m,a,n)-FNDMs
    end for
    /* Step 2 */
    Choose the minimum distance D(P_α, S_g) = min_{1≤j≤k} D(P_j, S_g)
    /* Step 3 */
    Compute α = argmin_{1≤j≤k} D(P_j, S_g) , P_α → S_g.
end for
/* Step 4 */
for each (m,a,n)-FNDM do
    Compute DoC = Σ_{j=1,j≠j0}^{k} |D(P_j, S) − D(P_j0, S)|
end for
```

Example 5. *A company imported four types of materials, but the label identifying one batch was lost during transit. The goal is to determine which type of material this unlabeled batch belongs to. The four types of materials are represented by (m,a,n)-FNSs (m = 3, a = 4, n = 5), denoted as $\mathscr{M}_i$ for $i = 1, 2, 3, 4$ in the feature space $V = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$, which are depicted in Table 2. Now, the data for unknown material $\mathscr{M}$ is also listed in Table 3. The object is to identify which type the unknown material $\mathscr{M}$ belongs to.*

Table 3. (m,a,n)-FN data of materials.

Material	$\mathscr{M}_1$	$\mathscr{M}_2$	$\mathscr{M}_3$	$\mathscr{M}_4$	$\mathscr{M}$
s_1	(0.56,0.47,0.22)	(0.81,0.30,.037)	(0.43,0.43,0.55)	(0.57,0.51,0.39)	(0.34,0.56,0.78)
s_2	(0.11,0.11,0.11)	(0.59,0.66,0.66)	(0.91,0.34,0.68)	(0.56,0.76,0.31)	(0.47,0.38,0.84)
s_3	(0.35,0.45,0.61)	(0.42,0.56,0.71)	(0.81,0.41,0.35)	(0.27,0.59,0.72)	(0.55,0.44,0.65)
s_4	(0.33,0.54,0.31)	(0.59,0.45,0.90)	(0.44,0.55,0.77)	(0.46,0.46,0.45)	(0.76,0.46,0.85)
s_5	(0.35,0.20,0.64)	(0.16,0.33,0.42)	(0.55,0.44,0.29)	(0.57,0.66,0.91)	(0.13,0.35,0.57)
s_6	(0.47,0.37,0.68)	(0.68,0.46,0.88)	(0.47,0.66,0.75)	(0.41,0.73,0.41)	(0.24,0.54,0.46)
s_7	(0.78,0.55,0.03)	(0.49,0.54,0.39)	(0.58,0.34,0.23)	(0.21,0.43,0.13)	(0.82,0.46,0.69)

Step 1: *Established (m,a,n)-FNDMs are utilized to compute the distances between* $\mathscr{M}$ *and* $\mathscr{M}_j$. *The results are depicted in Table 4.*

Step 2: *The minimum distances between* $\mathscr{M}$ *and* $\mathscr{M}_j$ *are also depicted in Table 4.*

Step 3: *Assign the material to the category associated with the lowest value identified in Step 2. For each proposed (m,a,n)-FN DM, material* $\mathbb{M}_1$ *consistently has the lowest distance score across all ten proposed (m,a,n)-FNDMs, which shows that the unknown material* $\mathbb{M}_1$ *is most similar to unknown* $\mathbb{M}$.

Step 4: *Calculate the value of the DoC for each (m,a,n)-FNDM using Equation (30). It is evident that the highest* d^{NHH} *value corresponds to the highest confidence in the prediction across all proposed (m,a,n)-FNDMs.*

Table 4. (m,a,n)-FNDMs for data given in Table 3.

DMs	$(\mathscr{M}_1, \mathscr{M})$	$(\mathscr{M}_2, \mathscr{M})$	$(\mathscr{M}_3, \mathscr{M})$	$(\mathscr{M}_4, \mathscr{M})$	Alternative	DoC
d^{NH}	0.1295	0.1691	0.1691	0.2012	$\mathscr{M}_1$	0.1507
d^{NE}	0.1864	0.2299	0.2312	0.2629	$\mathscr{M}_1$	0.1649
d^{GN}	0.2318	0.2742	0.2835	0.3075	$\mathscr{M}_1$	0.1699
d^{NHH}	0.2284	0.2999	0.3349	0.3674	$\mathscr{M}_1$	0.3168
d^{NEH}	0.2699	0.3461	0.3661	0.3972	$\mathscr{M}_1$	0.2999
d^{GNH}	0.3008	0.3753	0.3978	0.4204	$\mathscr{M}_1$	0.2911
d^{HNH}	0.1790	0.2345	0.2520	0.2843	$\mathscr{M}_1$	0.2338
d^{HNE}	0.2505	0.2938	0.3347	0.3368	$\mathscr{M}_1$	0.1854
d^{GHN}	0.2707	0.3324	0.3500	0.3635	$\mathscr{M}_1$	0.2339
d^{TI}	0.1245	0.1579	0.1558	0.1904	$\mathscr{M}_1$	0.1305

4.2. Parameter Sensitivity Analysis

This subsection analyzes how changes in parameters influence the ranking results, aiming to validate the stability of the proposed (m,a,n)-FNDMs-based classification method. To this end, a sensitivity analysis was performed to evaluate the robustness of the pattern recognition approach across varying parameter configurations. This involved making small adjustments to the parameters m, a, and n, which are considered the core elements of the (m,a,n)-FNDMs used in the method. The analysis explores how these variations affect the ranking outcomes. Table 5 displays the results, showing how the materials are ranked under each of the (m,a,n)-FNDMs with different values of m, a, and n tested in a systematic manner. It is evident from Table 5 that the proposed (m,a,n)-FNDMs are not suitable for handling the data in this example when the combination of m, a, and n lies between 1 and 3. This is because the sum of the m^{th} power of the PMD and the n^{th} power of the NMD exceeds 1 for some attributes in the considered dataset. Consequently, the distance measures d_1, d_2, and d_p proposed by Rajan and Krishnaswamy [46] for PyNSs (with m = a = n = 2), as well as the distance measures d_H, d_E, d_{Eu}, and d_{F_n} proposed by Saeed and his coworkers [47] for FNSs (with m = a = n = 3), fail to classify the material accurately. It is evident from the table that the ranking and selection of the optimal material remain consistent across all combinations of m, a, and n, particularly when greater emphasis is placed on the PMD compared to the IMD and NMD. The ranking and optimal results may vary if greater priority is assigned to the NMD compared to the PMD and IMD. Overall, it can be concluded that the proposed method offers greater efficiency and flexibility compared to existing approaches in pattern recognition.

Table 5. Ranking and classification results for different values of m, a, and n.

m	a	n	DMs	Ranking	Alternative
$1 \leq m \leq 3$	$1 \leq a \leq 3$	$1 \leq n \leq 3$	d^{NH}	*Uncertain*	Unclassified
			d^{NE}	*Uncertain*	Unclassified
			d^{GN}	*Uncertain*	UNclassified
			d^{NHH}	*Uncertain*	Unclassified
			d^{NEH}	*Uncertain*	Unclassified
			d^{GNH}	*Uncertain*	Unclassified
			d^{HNH}	*Uncertain*	Unclassified
			d^{HNE}	*Uncertain*	Unclassified
			d^{GHN}	*Uncertain*	Unclassified
			d^{TI}	*Uncertain*	Unclassified
3	4	5	d^{NH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{NE}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{GN}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{NHH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{NEH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{GNH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{HNH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{HNE}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{GHN}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{TI}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
5	5	5	d^{NH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{NE}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{GN}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{NHH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{NEH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{GNH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{HNH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{HNE}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{GHN}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
			d^{TI}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}^1$
6	8	10	d^{NH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^4 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{NE}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^4 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{GN}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^4 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{NHH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^4 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{NEH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^4 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{GNH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^4 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{HNH}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^4 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{HNE}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^4 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{GHN}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^4 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{TI}	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^4 < \mathscr{M}^3$	$\mathscr{M}^1$
10	50	100	d^{NH}	$\mathscr{M}^1 < \mathscr{M}^4 < \mathscr{M}^2 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{NE}	$\mathscr{M}^1 < \mathscr{M}^4 < \mathscr{M}^2 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{GN}	$\mathscr{M}^1 < \mathscr{M}^4 < \mathscr{M}^2 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{NHH}	$\mathscr{M}^1 < \mathscr{M}^4 < \mathscr{M}^4 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{NEH}	$\mathscr{M}^1 < \mathscr{M}^4 < \mathscr{M}^2 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{GNH}	$\mathscr{M}^1 < \mathscr{M}^4 < \mathscr{M}^2 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{HNH}	$\mathscr{M}^1 < \mathscr{M}^4 < \mathscr{M}^2 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{HNE}	$\mathscr{M}^1 < \mathscr{M}^4 < \mathscr{M}^2 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{GHN}	$\mathscr{M}^1 < \mathscr{M}^4 < \mathscr{M}^2 < \mathscr{M}^3$	$\mathscr{M}^1$
			d^{TI}	$\mathscr{M}^1 < \mathscr{M}^4 < \mathscr{M}^2 < \mathscr{M}^3$	$\mathscr{M}^1$

4.3. Comparative Study

This subsection aims to compare the outcomes obtained using the proposed (m,a,n)-FNDMs with those derived from distance measures developed within other orthopair fuzzy and neutrosophic frameworks. As discussed in the previous section, the distance measures d_1, d_2, and d_p introduced by Rajan and Krishnaswamy [46] for PyNSs (with m = a = n = 2), along with the distance measures d_H, d_E, d_{Eu}, and d_{F_n} proposed by Saeed and his coworkers [47] for FNSs (with m = a = n), fail to accurately classify the material in the considered example. To enable a meaningful comparison, the (m,a,n)-FNSs are reduced to (m,n)-fuzzy sets by omitting the Indeterminacy Membership Degree (IMD) in the data. The resulting dataset is then evaluated using the distance measures proposed by Thakur and his associates [30,31] and Sivdas and John [32] for (m,n)-FSs in pattern recognition applications. These methods are implemented on the same dataset using m = 3 and n = 5. The resulting rankings and classification outcomes are presented in Table 6, allowing for a direct comparison between the proposed method and existing approaches based on (m,n)-FS distance measures. Table 6 indicates that the classification outcomes achieved with the suggested distance measures closely correspond to the results obtained by most of the distance measures for the (m,n)-FSs discussed in [30–32]. This confirms the viability, effectiveness, and reliability of the proposed distance measures in pattern recognition problems.

Figure 1 depicts a comparison of our study with others, and it also explains that our results are suitable and more accurate than those of previous work.

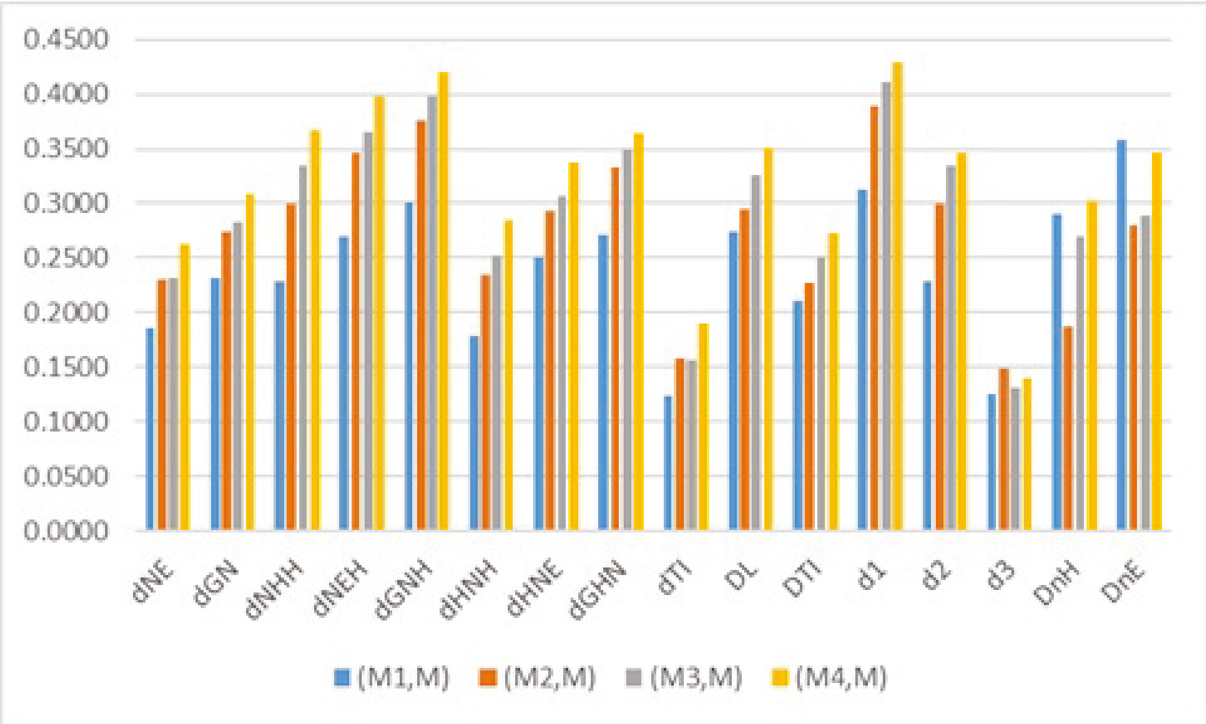

Figure 1. Comparison of current and existing methods.

Table 6. Comparison of current and existing methods.

Method	Environment	Ranking	Alternative
Proposed method based on d^{NH}	(3,4,5)-FNS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Proposed method based on d^{NE}	(3,4,5)-FNS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Proposed method based on d^{GN}	(3,4,5)-FNS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Proposed method based on d^{NHH}	(3,4,5)-FNS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Proposed method based on d^{NEH}	(3,4,5)-FNS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Proposed method based on d^{GNH}	(3,4,5)-FNS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Proposed method based on d^{HNH}	(3,4,5)-FNS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Proposed method based on d^{HNE}	(3,4,5)-FNS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Proposed method based on d^{GHN}	(3,4,5)-FNS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Proposed method based on d^{TI}	(3,4,5)-FNS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$

Table 6. *Cont.*

Method	Environment	Ranking	Alternative
Method based on D^L [31]	(3,5)-FS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Method based on d^{TI} [31]	(3,5)FS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Method based on d^1 [32]	(3,5)-FS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Method based on d^2 [32]	(3,5)-FS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Method based on d^3 [32]	(3,5)-FS	$\mathscr{M}^1 < \mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4$	$\mathscr{M}_1$
Method based on d^{nH} [30]	(3,5)-FS	$\mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^1 < \mathscr{M}^4$	$\mathscr{M}_2$
Method based on d^{dE} [30]	(3,5)-FS	$\mathscr{M}^2 < \mathscr{M}^3 < \mathscr{M}^4 < \mathscr{M}^1$	$\mathscr{M}_2$

5. Multi-Criteria Decision Making with TOPSIS Approach Under (m,a,n)-FNSs

Multi-criteria decision making (MCDM) is a widely recognized area within the field of decision analysis. It involves making optimal choices by assessing, ranking, or selecting among alternatives when multiple, often conflicting, criteria are involved. In this section, we develop a TOPSIS method for multi-criteria decision making in an (m,a,n)-fuzzy neutrosophic environment and demonstrate how the proposed (m,a,n)-FNWDMs effectively address complex MCDM problems involving uncertainty and ambiguity.

5.1. Problem Description and Solution Procedure

Let $\mathscr{C} = \{\mathscr{C}^1, \mathscr{C}^2, \ldots \mathscr{C}^q\}$ denote the set of q evaluation criteria, and let $\mathscr{A} = \{\mathscr{A}^1, \mathscr{A}^2, \ldots \mathscr{A}^p\}$ represent the set of p possible alternatives. According to the preferences provided by the decision-maker, each criterion $\mathscr{C}^j$ is assigned a corresponding weight ω_j, forming the weight vector $\omega = \{\omega_1, \omega_2, ..\omega_q\}$, where $\omega_j \in [0,1]$ for all j and $\Sigma_1^q \omega_j = 1$. Based on the evaluations provided by decision-makers using the (m,a,n)-FN framework, a decision matrix $\mathscr{D} = [d_{ij}]_{p\times q} = [(\mho_{ij}, \varpi_{ij}$ is constructed. In this matrix, $\mho_{ij}$ indicates the degree to which the alternative $\mathscr{A}^i$ satisfies criterion $\mathscr{C}^j$, φ_{ij} denotes the extent of neutral satisfaction of $\mathscr{A}^i$ with respect to $\mathscr{C}^j$, and Ω_{ij} reflects which alternative $\mathscr{A}^i$ does not satisfy criterion $\mathscr{C}^j$. These values must satisfy $0 \leq \mho_{ij}^m + \Omega_{ij}^n \leq 1$ and $0 \leq \varpi_{ij}^a \leq 1$ for $m, a, n \in \mathbb{N}$. The objective is to identify the most suitable alternative based on this structured evaluation.

Step 1: Construct the decision matrix $\mathscr{D} = [d_{ij}]_{p\times q} = [(\mho_{ij}, \varpi_{ij}, \Omega_{ij})]_{p\times q}$, where $\mho_{ij}$ encapsulates the assessment of each alternative with respect to each criterion. In this structure, $\mho_{ij}$ indicates how strongly the alternative $\mathscr{A}^i$ satisfies the criteria $\mathscr{C}^j$, ϖ_{ij} reflects the neutral satisfaction of the alternative $\mathscr{A}^i$ with respect to $\mathscr{C}^j$, and Ω_{ij} expresses the extent to which the alternative $\mathscr{A}^i$ does not meet criterion $\mathscr{C}^j$.

Step 2: Formulate the normalized decision matrix $\mathscr{D}' = [d'_{ij}]_{p\times q} = [(\mho'_{ij}, \varpi'_{ij}, \Omega'_{ij})]_{p\times q}$, where

$$(\mho'_{ij}, \varpi'_{ij}, \Omega'_{ij}) = \begin{cases} (\Omega_{ij}^{\frac{n}{m}}, 1-\varpi_{ij}, \mho_{ij}^{\frac{m}{n}}), & \text{if } \mathscr{C}^j \text{ is the cost criteria} \\ (\mho_{ij}, \varpi_{ij}, \Omega_{ij}), & \text{if } \mathscr{C}^j \text{ is the benefit criteria} \end{cases} \tag{31}$$

Step 3: Determine the (m,a,n)-FN positive ideal solution $\mathscr{P}^+$ and the (m,a,n)-FN negative ideal solution $\mathscr{P}^-$ as follows:

$$\mathscr{P}^+ = \{(\mho_1^+, \varpi_1^+, \Omega_1^+), (\mho_2^+, \varpi_2^+, \Omega_2^+), \ldots (\mho_q^+, \varpi_q^+, \Omega_q^+)\}$$

where $\mho_j^+ = \underset{i}{Max}\, \mho_{ij}^+$, $\varpi_j^+ = \underset{i}{Max}\, \varpi_{ij}^+$, and $\Omega_j^+ = \underset{i}{Min}\, \Omega_{ij}^+$ for j = 1, 2, ..., q.

$$\mathscr{P}^- = \{(\mho_1^-, \varpi_1^-, \Omega_1^-), (\mho_2^-, \varpi_2^-, \Omega_2^-), \ldots (\mho_q^-, \varpi_q^-, \Omega_q^-)\}$$

where $\mho_j^- = \underset{i}{Min}\, \mho_{ij}^-$, $\varpi_j^- = \underset{i}{Min}\, \varpi_{ij}^-$, and $\Omega_j^- = \underset{i}{Max}\, \Omega_{ij}^-$ for j = 1, 2, ..., q.

Step 4: Compute the separations $D(\mathscr{A}^i, \mathscr{P}^+)$ and $D(\mathscr{A}^i, \mathscr{P}^-)$ for each alternative $\mathscr{A}^i$ (i = 1, 2, … p) from the (m,a,n)-FN positive ideal solution $\mathscr{P}^+$ and the (m,a,n)-FN negative ideal solution $\mathscr{P}^-$ using the suggested (m,a,n)-FNWDMs.

Step 5: Compute the closeness index γ_i connected to the $\mathscr{A}^i$ (i = 1, 2, … p) as follows:

$$\gamma_i = \frac{D(\mathscr{A}^i, \mathscr{P}^-)}{D(\mathscr{A}^i, \mathscr{P}^+) + D(\mathscr{A}^i, \mathscr{P}^-)} \tag{32}$$

Step 6: Rank the alternatives according to their γ_i values in descending order. The alternative $\mathscr{A}^i$ with the highest value of γ_i is deemed the optimal alternative.

Herein, we present Algorithm 2 for the proposed TOPSIS method under the (m,a,n)-FNSs environment.

Algorithm 2 TOPSIS for MCDM under (m,a,n)-FNSs

Input: A set of alternatives $\mathscr{A} = \{\mathscr{A}^1, \mathscr{A}^2, \ldots, \mathscr{A}^p\}$,
A set of criteria $\mathscr{C} = \{\mathscr{C}^1, \mathscr{C}^2, \ldots, \mathscr{C}^q\}$,with Cost and benefit criteria,
(m,a,n)-FN decision matrix $\mathscr{D} = [d_{ij}]_{p\times q} = [(\mho_{ij}, \varpi_{ij}, \Omega_{ij})]_{p\times q}$,
Weight vector $\omega = \{\omega_1, \omega_2, \ldots, \omega_q\}$, with $\omega_j \in [0,1]$ and $\Sigma_1^q \omega_j = 1$.
Output: Ranking of alternatives.

- **/* Step 1: Construct (m,a,n)-FN normalized Decision Matrix $\mathscr{D}' = [d'_{ij}]_{p\times q}$*/**
- **for** each $i = 1$ to p **do**
  - **for** each $j = 1$ to q **do**
    - **if** $\mathscr{C}^j$ is cost criteria **then**
      - Compute $d'_{ij} = (\mho'_{ij}, \varpi'_{ij}, \Omega'_{ij}) = (\Omega_{ij}^{\frac{n}{m}}, 1 - \varpi_{ij}, \mho_{ij}^{\frac{m}{n}})$
    - **else**
      - Compute $d'_{ij} = (\mho'_{ij}, \varpi'_{ij}, \Omega'_{ij}) = (\mho_{ij}, \varpi_{ij}, \Omega_{ij})$
    - **end if**
  - **end for**
- **end for**
- **/* Step 2: Determine the Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS) */**
- **for** each criteria $\mathscr{C}^j$ **do**
  - Compute $\mathscr{P}_j^+ = (\mho_j^+, \varpi_j^+, \Omega_j^+) = (\underset{i}{Max}\, \mho_{ij}^+, \underset{i}{Max}\, \varpi_{ij}^+, \underset{i}{Min}\, \Omega_{ij}^+)$
  - Compute $\mathscr{P}_j^- = (\mho_j^-, \varpi_j^-, \Omega_j^-) = (\underset{i}{Min}\, \mho_{ij}^-, \underset{i}{Min}\, \varpi_{ij}^-, \underset{i}{Max}\, \Omega_{ij}^-)$
- **end for**
- **/* Step 3: Calculate Separation Measures */**
- **for** each alternative $\mathscr{A}^i$ **do**
  - Compute $D(\mathscr{A}^i, \mathscr{P}^+)$ distance from PIS $\mathscr{P}^+$ using the suggested (m,a,n)-FNWDMs.
  - Compute $D(\mathscr{A}^i, \mathscr{P}^-)$ distance from NIS $\mathscr{P}^-$ using the same (m,a,n)-FNWDMs metric.
- **end for**
- **/* Step 4: Calculate Closeness Coefficient */**
- **for** each alternative $\mathscr{A}_i$ **do**
  - $\gamma_i = \frac{D(\mathscr{A}^i, \mathscr{P}^-)}{D(\mathscr{A}^i, \mathscr{P}^+) + D(\mathscr{A}^i, \mathscr{P}^-)}$
- **end for**
- **/* Step 5: Rank Alternatives */**
- Rank the alternatives based on descending order of γ_i

Example 6. *Suppose a multinational corporation is preparing a financial plan for the upcoming year in alignment with its strategic goals. As part of this process, four clearly defined investment options have been identified and are denoted as $\mathscr{A}^1$: investment in the "Information Technology sector"; $\mathscr{A}^2$: investment in the "Energy sector"; $\mathscr{A}^3$: investment in the "Financial sector"; and $\mathscr{A}^4$: investment in the "Heathcare sector". Following an initial assessment, four key criteria have been selected for the evaluation process: $\mathscr{C}^1$, representing "growth", $\mathscr{C}^2$, denoting "risk analysis",*

$\mathscr{C}^3$, referring to "socio-political impact", and $\mathscr{C}^4$, covering "environmental and related factors". Based on the company's strategic financial priorities, the corresponding weight vector is defined as $\omega = (0.2, 0.3, 0.1, 0.4)^T$. For simplicity in this illustrative example, the parameter values are set as m = 2, a = 3, and n = 4. The following steps detail the computational procedure used to solve this problem using the proposed method.

***Step 1**: The information for the four investment alternatives, evaluated against the previously defined criteria in the (m,a,n)-FN framework with parameters m = 2, a = 3, and n = 4, are depicted in Table 7.*

Table 7. (m,a,n)-FN decision matrix.

	$\mathscr{C}^1$	$\mathscr{C}^2$	$\mathscr{C}^3$	$\mathscr{C}^4$
$\mathscr{A}^1$	(0.5,0.4,0.8)	(0.8,0.3,0.3)	(0.7,0.4,0.8)	(0.7,0.2,0.3)
$\mathscr{A}^2$	(0.7,0.1,0.4)	(0.5,0.6,0.6)	(0.9,0.3,0.4)	(0.6,0.6,0.8)
$\mathscr{A}^3$	(0.8,0.5,0.9)	(0.4,0.5,0.7)	(0.8,0.1,0.3)	(0.2,0.5,0.7)
$\mathscr{A}^4$	(0.7,0.4,0.9)	(0.5,0.4,0.9)	(0.4,0.2,0.7)	(0.6,0.4,0.4)

***Step 2**: Considering that $\mathscr{C}^2$ and $\mathscr{C}^3$ are cost criteria and $\mathscr{C}^1$ and $\mathscr{C}^4$ are benefit criteria, the normalized decision matrix is constructed using Equation (31). The resulting values are shown in Table 8.*

Table 8. (m,a,n)-FN normalized decision matrix for m,a,n = 2,3,4.

	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$
$\mathscr{A}^1$	(0.5,0.4,0.8)	(0.09,0.7,0.894)	(0.64,0.6,0.837)	(0.7,0.2,0.3)
$\mathscr{A}^2$	(0.7,0.1,0.4)	(0.36,0.4,0.707)	(0.16,0.7,0.949)	(0.6,0.6,0.8)
$\mathscr{A}^3$	(0.8,0.5,0.9)	(0.49,0.5,0.632)	(0.09,0.9,0.894)	(0.2,0.5,0.7)
$\mathscr{A}^4$	(0.7,0.4,0.9)	(0.81,0.6,0.707)	(0.49,0.8,0.632)	(0.6,0.4,0.4)

***Step 3**: Now, we will find the (m,a,n)-FN positive ideal solution $\mathscr{P}^+$ and (m,a,n)-FN negative ideal solution $\mathscr{P}^-$. These solutions are as follows:*

$$\mathscr{P}^+ = \{(0.8, 0.5, 0.4), (0.81, 0.7, 0.632), (0.64, 0.9, 0.632), (0.7, 0.6, 0.3)\}$$
$$\mathscr{P}^- = \{(0.5, 0.1, 0.9), (0.09, 0.4, 0.894), (0.09, 0.6, 0.949), (0.2, 0.2, 0.8)\}$$

***Step 4**: We use the proposed (m,a,n)-FNWDMs to compute the separation of each alternative $\mathscr{A}^i$ (i = 1, 2 ... p) between the (m,a,n)-FN positive ideal solution $\mathscr{P}^+$ and the (m,a,n)-FN negative ideal solution $\mathscr{P}^-$. The results are shown in Tables 9 and 10.*

Table 9. Separation of alternatives and positive ideal solution.

Weighted DMs	$D(\mathscr{A}^1, \mathscr{P}^+)$	$D(\mathscr{A}^2, \mathscr{P}^+)$	$D(\mathscr{A}^3, \mathscr{P}^+)$	$D(\mathscr{A}^4, \mathscr{P}^+)$
d^{WNH}	0.2302	0.2667	0.2454	0.1319
d^{WNE}	0.3236	0.3385	0.3190	0.1942
d^{GWN}	0.3743	0.3837	0.3606	0.2584
d^{WNHH}	0.4179	0.4425	0.4804	0.2462
d^{WNEH}	0.4532	0.4724	0.4860	0.3066
d^{GWNH}	0.4814	0.4941	0.4922	0.3662
d^{HWNH}	0.3240	0.3252	0.3163	0.1750
d^{HWNE}	0.3938	0.3856	0.3650	0.2401
d^{GHWN}	0.4344	0.4457	0.4363	0.3213
d^{WTI}	0.2031	0.2341	0.2226	0.1157

Table 10. Separation of alternatives and negative ideal solution.

Weighted DMs	$D(\mathscr{A}^1, \mathscr{P}^-)$	$D(\mathscr{A}^2, \mathscr{P}^-)$	$D(\mathscr{A}^3, \mathscr{P}^-)$	$D(\mathscr{A}^4, \mathscr{P}^-)$
d^{WNH}	0.1941	0.1576	0.1789	0.2924
d^{WNE}	0.2686	0.2412	0.2458	0.3525
d^{GWN}	0.3039	0.2955	0.2922	0.3964
d^{WNHH}	0.3600	0.3266	0.3622	0.4829
d^{WNEH}	0.3698	0.3812	0.4860	0.5102
d^{GWNH}	0.3786	0.4147	0.4113	0.5327
d^{HWNH}	0.2770	0.2003	0.2584	0.3364
d^{HWNE}	0.3232	0.2809	0.3222	0.3933
d^{GHWN}	0.3453	0.3648	0.3616	0.4743
d^{WTI}	0.1748	0.1438	0.1554	0.2622

Step 5: *Compute the closeness index γ_i connected to the $\mathscr{A}^i$ (i = 1, 2, . . . p) using Equation (32) and rank the alternatives according to their γ_i values in descending order. Choose the best alternative $\mathscr{A}^i$ with the highest value of γ_i, as depicted in Table 11.*

Table 11. Closeness index connected to the alternatives and their ranking.

Weighted DMs	γ_1	γ_2	γ_3	γ_4	Ranking	Alternative
d^{WNH}	0.4574	0.3715	0.4216	0.6892	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
d^{WNE}	0.4536	0.4161	0.4352	0.6448	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
d^{GWN}	0.4481	0.4350	0.4476	0.6053	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
d^{WNHH}	0.4628	0.4246	0.4298	0.6624	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
d^{WNEH}	0.4493	0.4465	0.5000	0.6246	$\mathscr{A}^4 > \mathscr{A}^3 > \mathscr{A}^1 > \mathscr{A}^2$	$\mathscr{A}^4$
d^{GWNH}	0.4402	0.4563	0.4553	0.5926	$\mathscr{A}^4 > \mathscr{A}^2 > \mathscr{A}^3 > \mathscr{A}^1$	$\mathscr{A}^4$
d^{HWNH}	0.4609	0.3812	0.4496	0.6578	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
d^{HWNE}	0.4508	0.4214	0.4689	0.6209	$\mathscr{A}^4 > \mathscr{A}^3 > \mathscr{A}^1 > \mathscr{A}^2$	$\mathscr{A}^4$
d^{GHWN}	0.4428	0.4501	0.4532	0.5961	$\mathscr{A}^4 > \mathscr{A}^3 > \mathscr{A}^2 > \mathscr{A}^1$	$\mathscr{A}^4$
d^{WTI}	0.4626	0.3805	0.4111	0.6938	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$

As shown in Table 11, although the rankings produced by different (m,a,n)-FNWDMs may differ, the top-ranked alternative is consistently $\mathscr{A}^4$. Therefore, it can be concluded that $\mathscr{A}^4$ represents the most suitable choice across all proposed (m,a,n)-FNWDM approaches. This consistency highlights the reliability of our proposed methods in delivering stable results when applied to multi-criteria decision-making problems.

5.2. Comparative Study

In this section, we compare the proposed MCDM method with existing decision-making methods within the framework of q-RNSs and (m,n)-FSs to validate its effectiveness. For the comparative analysis, the information and data of alternatives and criteria in numerical Example 6 are considered. This comparison allows for an evaluation of the effectiveness and consistency of the proposed method relative to existing techniques within the reduced neutrosophic and fuzzy environment.

5.2.1. Comparison with q-RSNWAO- and q-RSNWGO-Based Approaches [41]

In this section, we compare the proposed MCDM method with approaches based on the q-RSNWAO and q-RSNWGO aggregation operators under the framework of q-RONSs. To facilitate this comparison, the (m,a,n)-FNSs are transformed into q-RONSs by taking the maximum value among m, a, and n for each component. Al-Quran, Al-Sharqi, and Djaouti [41] proposed the q-RSNWAO- and q-RSNWGO-based methods to address MCDM problems. We implement their approach on the considered dataset for q = 4, and the procedural steps are outlined as follows:

Step 1: The decision matrix of q-RONNs based on alternative and criteria information is depicted in Table 7.

Step 2: Create the normalized decision matrix by taking the complements of cost criteria $\mathscr{C}^2$ and $\mathscr{C}^2$. The resulting normalized decision matrix is shown in Table 12.

Table 12. 4-RONS normalized decision matrix.

Alternative	$\mathscr{C}^1$	$\mathscr{C}^2$	$\mathscr{C}^3$	$\mathscr{C}^4$
$\mathscr{A}^1$	(0.5,0.4,0.8)	(0.3,0.7,0.8)	(0.8,0.6,0.7)	(0.7,0.2,0.3)
$\mathscr{A}^2$	(0.7,0.1,0.4)	(0.6,0.4,0.5)	(0.4,0.7,0.9)	(0.6,0.6,0.8)
$\mathscr{A}^3$	(0.8,0.5,0.9)	(0.7,0.5,0.4)	(0.3,0.9,0.8)	(0.2,0.5,0.7)
$\mathscr{A}^4$	(0.7,0.4,0.9)	(0.9,0.6,0.5)	(0.7,0.8,0.4)	(0.6,0.4,0.4)

Step 3: Compute the aggregated values of each alternative using q-RSNWAO and q-RSNWGO. These values are depicted in Tables 13 and 14, respectively.

Step 4: Calculate the score value for the aggregated values computed in Step 3 of each alternative, which are also shown in Tables 13 and 14, respectively.

Table 13. Aggregated and score values of alternatives using q-RSNWAO for q = 4.

Alternative	Aggregated Value	Score Value
$\mathscr{A}^1$	(0.6353,0.3734,0.5332)	0.5696
$\mathscr{A}^2$	(0.5551,0.2430,0.6120)	0.5706
$\mathscr{A}^3$	(0.6444,0.5303,0.6307)	0.4946
$\mathscr{A}^4$	(0.7786,0.4842,0.5030)	0.6064

Table 14. Aggregated and score values of alternatives using q-RSNWGO for q = 4.

Alternative	Aggregated Value	Score Value
$\mathscr{A}^1$	(0.5144, 0.4749, 0.7105)	0.4467
$\mathscr{A}^2$	(0.3829, 0.3189, 0.7336)	0.4710
$\mathscr{A}^3$	(0.4002, 0.5743, 0.7514)	0.3775
$\mathscr{A}^4$	(0.7097, 0.5240, 0.6832)	0.5039

Step 5: Rank the alternative corresponding to the obtained score values in descending order. The ranking of the alternatives is stated below:

$$\textbf{q-RSNWGO}: \mathscr{A}^4 > \mathscr{A}^2 > \mathscr{A}^1 > \mathscr{A}^3.$$
$$\textbf{q-RSNWGO}: \mathscr{A}^4 > \mathscr{A}^2 > \mathscr{A}^1 > \mathscr{A}^3.$$

In both methods, the optimum alternative with the highest ranking is $\mathscr{A}^4$.

5.2.2. Comparison with (m,n)-FWPA- and (m,n)-FWPG-Operators-Based Approaches [17]

The (m, a, n)-FNSs are reduced to (m, n)-fuzzy sets by assuming the Indeterminacy Membership Degree (IMD) to be zero. By neglecting the neutral component, the data from Example 6 is simplified to the (m, n)-FS framework. Under this setting, a comparative analysis is conducted of the proposed MCDM method and existing approaches based on the (m, n)-FWPA and (m, n)-FWPG aggregation operators introduced by Al-Shami and Mhemdi [17]. The procedural steps involved in implementing the (m, n)-FWPA- and (m, n)-FWPG-based algorithms [17] are outlined below:

Step 1: Create the decision matrix of (m,n)-fuzzy data by neglecting the IMD based on alternatives and criteria information. The decision matrix is depicted in Table 15.

Table 15. (m,n)-fuzzy decision matrix for m = 2, n = 4.

	$\mathscr{C}^1$	$\mathscr{C}^2$	$\mathscr{C}^3$	$\mathscr{C}^4$
$\mathscr{A}^1$	(0.5,0.8)	(0.8,0.3)	(0.7,0.8)	(0.7,0.3)
$\mathscr{A}^2$	(0.7,0.4)	(0.5,0.6)	(0.9,0.4)	(0.6,0.8)
$\mathscr{A}^3$	(0.8,0.9)	(0.4,0.7)	(0.8,0.3)	(0.2,0.7)
$\mathscr{A}^4$	(0.7,0.9)	(0.5,0.9)	(0.4,0.7)	(0.6,0.4)

Step-2: The normalized decision matrix is constructed by considering the (m,n)-fuzzy complement in cost criteria $\mathscr{C}^2$ and $\mathscr{C}^3$. The resulting values are shown in Table 16.

Table 16. (m,n)-fuzzy normalized decision matrix for m = 2, n = 4.

	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$
$\mathscr{A}^1$	(0.5,0.8)	(0.09,0.894)	(0.64,0.837)	(0.7,0.3)
$\mathscr{A}^2$	(0.7,0.4)	(0.36,0.707)	(0.16,0.949)	(0.6,0.8)
$\mathscr{A}^3$	(0.8,0.9)	(0.49,0.632)	(0.09,0.894)	(0.2,0.7)
$\mathscr{A}^4$	(0.7,0.9)	(0.81,0.707)	(0.49,0.632)	(0.6,0.4)

Step 3: Compute the aggregated values of each alternative using (m,n)-FWPA and (m,n)-FWPG operators. These values are depicted in Tables 17 and 18, respectively.

Step 4: Calculate the score value for the aggregated values computed in Step 3 of each alternative, which are also shown in Tables 17 and 18, respectively.

Table 17. Aggregated and score values of alternatives using (m,n)-FWPA for m = 2, n = 4.

Alternative	Aggregated Value	Score Value
$\mathscr{A}^1$	(0.5379, 0.7536)	−0.0331
$\mathscr{A}^2$	(0.3943, 0.7551)	−0.1696
$\mathscr{A}^3$	(0.4657, 0.7887)	−0.1222
$\mathscr{A}^4$	(0.6803, 0.6943)	0.2305

Table 18. Aggregated and score values of alternatives using (m,n)-FWPG for m = 2, n = 4.

Alternative	Aggregated Value	Score Value
$\mathscr{A}^1$	(0.5635, 0.7833)	−0.0589
$\mathscr{A}^2$	(0.4211, 0.7822)	−0.1969
$\mathscr{A}^3$	(0.5119, 0.7849)	−0.1175
$\mathscr{A}^4$	(0.6954, 0.7269)	0.2043

Step 5: Based on the obtained score values, the alternatives are ranked in descending order of their performance. The ranking of the alternatives is stated below:

$$\textbf{(m,n)-FWPA} : \mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2.$$
$$\textbf{(m,n)-FWPG} : \mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2.$$

In both methods, the optimum alternative with the highest ranking is clearly identified as $\mathscr{A}^4$.

The rankings of the alternatives according to the proposed and compared MCDM methods are presented in Table 19. Furthermore, the ranking outcomes are depicted graphically in Figure 2 for visual comparison. Considering all the methods under comparison, the optimum alternative is identified as $\mathscr{A}^4$. The consistency in rankings across the different methods, along with the agreement on the top-ranked alternative, demonstrates the robustness and reliability of the proposed method.

Table 19. Comparative analysis.

Method	Environment	Ranking	Alternative
Proposed method based on d^{WNH}	(2,3,4)-FNS	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
Proposed method based on d^{WNE}	(2,3,4)-FNS	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
Proposed method based on d^{GWN}	(2,3,4)-FNS	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
Proposed method based on d^{WNHH}	(2,3,4)-FNS	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
Proposed method based on d^{WNEH}	(2,3,4)-FNS	$\mathscr{A}^4 > \mathscr{A}^3 > \mathscr{A}^1 > \mathscr{A}^2$	$\mathscr{A}^4$
Proposed method based on d^{GWNH}	(2,3,4)-FNS	$\mathscr{A}^4 > \mathscr{A}^2 > \mathscr{A}^3 > \mathscr{A}^1$	$\mathscr{A}^4$
Proposed method based on d^{HWNH}	(2,3,4)-FNS	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
Proposed method based on d^{HWNE}	(2,3,4)-FNS	$\mathscr{A}^4 > \mathscr{A}^3 > \mathscr{A}^1 > \mathscr{A}^2$	$\mathscr{A}^4$
Proposed method based on d^{GHWN}	(2,3,4)-FNS	$\mathscr{A}^4 > \mathscr{A}^3 > \mathscr{A}^2 > \mathscr{A}^1$	$\mathscr{A}^4$
Proposed method based on d^{WTI}	(2,3,4)-FNS	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
Method based on $q - RSNWAO$ [41]	4-RNS	$\mathscr{A}^4 > \mathscr{A}^2 > \mathscr{A}^1 > \mathscr{A}^3$	$\mathscr{A}^4$
Method based on $q - RSNWGO$ [41]	4-RNS	$\mathscr{A}^4 > \mathscr{A}^2 > \mathscr{A}^1 > \mathscr{A}^3$	$\mathscr{A}^4$
Method based on $(m,n) - FWPAO$ [17]	(2,4)-FS	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$
Method based on $(m,n) - FWPGO$ [17]	(2,4)-FS	$\mathscr{A}^4 > \mathscr{A}^1 > \mathscr{A}^3 > \mathscr{A}^2$	$\mathscr{A}^4$

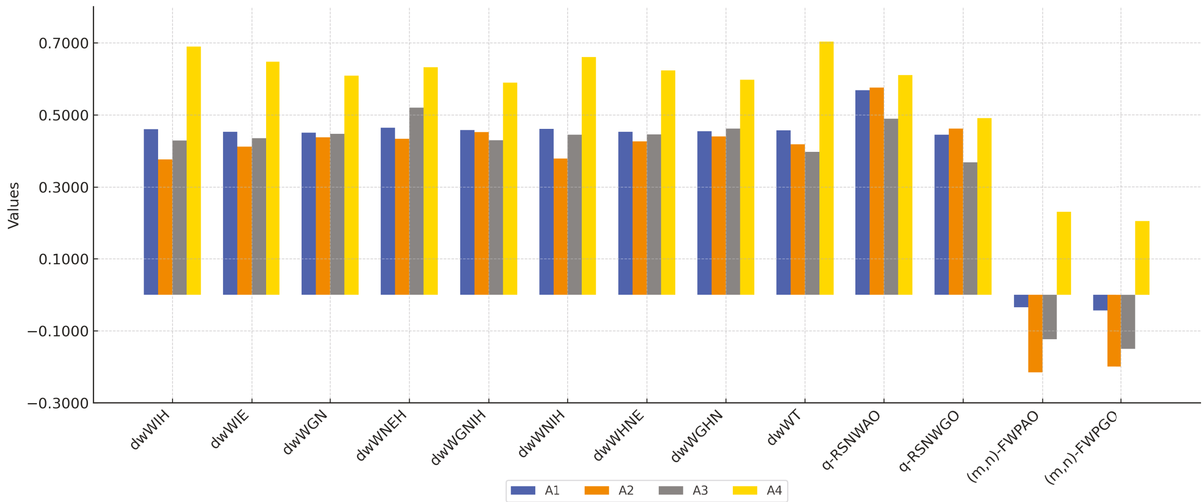

Figure 2. Graphical representation of the ranking of alternatives

6. Conclusions

This study explored distance measures for (m,a,n)-FNSs, which generalize multiple forms of neutrosophic set theories. By introducing three distinct functions, the PMD, IMD, and NMD, the proposed framework offers a versatile approach for managing uncertainty and ambiguity in decision-making contexts. The novel DMs developed for(m,a,n)-FNSs serve as effective tools for quantifying differences between sets, thereby improving the precision of decision-making models. Using illustrative examples such as material classification and investment strategy selection, the proposed measures were shown to be consistent and high-performing. The comparative analysis, along with graphical interpretations, further illustrates the effectiveness and superiority of the proposed measures. In summary, this research advances the field of distance measures by presenting new methodologies and insights for addressing complex decision-making challenges. The approaches outlined here have broad applicability and can support more informed and dependable decisions across a range of disciplines. In the future, researchers can study aggregation operators, graph structure, similarity measures, divergence measures, entropy, inclusion measures, and knowledge measures for (m,a,n)-fuzzy neutrosophic sets and employ them in MCDM methodologies like TOPSIS, VIKOR, and SAR. We hope that this paper will be a reference point for the researchers who study decision-making.

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Some Characterizations of k-Fuzzy γ-Open Sets and Fuzzy γ-Continuity with Further Selected Topics

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Abstract: In the present paper, we first introduced the notion of k-fuzzy γ-open (k-$\mathcal{F}$-γ-open) sets as a generalized novel class of fuzzy open ($\mathcal{F}$-open) sets on fuzzy topological spaces ($\mathcal{FTS}s$) in the sense of Šostak. The class of k-$\mathcal{F}$-γ-open sets is contained in the class of k-$\mathcal{F}$-β-open sets and contains all k-$\mathcal{F}$-semi-open and k-$\mathcal{F}$-pre-open sets. Also, we introduced the closure and interior operators with respect to the classes of k-$\mathcal{F}$-γ-closed and k-$\mathcal{F}$-γ-open sets and discussed some of their properties. After that, we defined and studied the notions of $\mathcal{F}$-γ-continuous (resp. $\mathcal{F}$-γ-irresolute) functions between $\mathcal{FTS}s$ $(M, \Im)$ and $(N, \digamma)$. However, we displayed and investigated the notions of $\mathcal{F}$-almost (resp. $\mathcal{F}$-weakly) γ-continuous functions, which are weaker forms of $\mathcal{F}$-γ-continuous functions. Next, we presented and characterized some new $\mathcal{F}$-functions via k-$\mathcal{F}$-γ-open and k-$\mathcal{F}$-γ-closed sets, called $\mathcal{F}$-γ-open (resp. $\mathcal{F}$-γ-irresolute open, $\mathcal{F}$-γ-closed, $\mathcal{F}$-γ-irresolute closed, and $\mathcal{F}$-γ-irresolute homeomorphism) functions. The relationships between these classes of functions were investigated with the help of some examples. We also introduced some new types of $\mathcal{F}$-separation axioms called k-$\mathcal{F}$-γ-regular (resp. k-$\mathcal{F}$-γ-normal) spaces via k-$\mathcal{F}$-γ-closed sets and discussed some properties of them. Lastly, we explored and studied some new types of $\mathcal{F}$-compactness called k-$\mathcal{F}$-almost (resp. k-$\mathcal{F}$-nearly) γ-compact sets.

Keywords: $\mathcal{F}$-topology; k-$\mathcal{F}$-γ-open set; $\mathcal{F}$-γ-closure operator; $\mathcal{F}$-γ-continuity; $\mathcal{F}$-γ-irresoluteness; k-$\mathcal{F}$-γ-compact set; k-$\mathcal{F}$-almost γ-compact set; k-$\mathcal{F}$-γ-regular space; k-$\mathcal{F}$-γ-normal space

MSC: 54A05; 54A40; 54C05; 54C08; 54D15

1. Introduction

The need for theories that cope with uncertainty emerges from daily experiences with complicated challenges requiring ambiguous facts. In 1965, the theory of fuzzy sets ($\mathcal{F}$-sets) was first defined by Zadeh [1] as a suitable approach to address uncertainty cases that cannot be efficiently managed using classical techniques. The concept of an $\mathcal{F}$-set of a nonempty set M is a mapping $\mathcal{D} : M \to I$ (where $I = [0, 1]$). Over the last decades, the research on fuzzy sets has had a vital role in mathematics and applied sciences and garnered significant attention due to its ability to handle uncertain and vague information in various real-life applications such as artificial intelligence [2,3], control systems [4,5], decision making [6,7], image processing [8,9], etc. The integration between $\mathcal{F}$-sets and some uncertainty approaches such as soft sets ($\mathcal{S}$-sets) and rough sets ($\mathcal{R}$-sets) has been discussed in [10–12]. The concept of a fuzzy topology ($\mathcal{F}$-topology) was presented in 1968 by Chang [13]. Several authors have successfully generalized the theory of general topology to the fuzzy setting with crisp methods. According to Šostak [14], the notion of an $\mathcal{F}$-topology being a crisp subclass of the class of $\mathcal{F}$-sets and fuzziness in the notion of openness of an $\mathcal{F}$-set have not been considered, which seems to be a drawback in the process of fuzzification of a topological space. Therefore, Šostak [14] defined a novel definition of an $\mathcal{F}$-topology as the concept of openness of $\mathcal{F}$-sets. It is an extension of an $\mathcal{F}$-topology introduced by Chang [13]. Thereafter, many researchers (see [15–23]) have redefined the same notion and studied $\mathcal{FTS}s$, being unaware of Šostak's work.

The generalizations of $\mathcal{F}$-open sets play an effective role in an $\mathcal{F}$-topology through their ability to improve on many results and to open the door to display and investigate several fuzzy topological notions such as $\mathcal{F}$-continuity [15,16], $\mathcal{F}$-connectedness [16], $\mathcal{F}$-compactness [16,17], etc. Overall, the notions of k-$\mathcal{F}$-semi-open, k-$\mathcal{F}$-pre-open, k-$\mathcal{F}$-α-open, and k-$\mathcal{F}$-β-open sets were defined and studied by the authors of [20,22] on $\mathcal{FTS}s$ in the sense of Šostak [14]. Also, Kim et al. [20] defined and discussed some weaker forms of $\mathcal{F}$-continuity called $\mathcal{F}$-semi-continuity (resp. $\mathcal{F}$-pre-continuity and $\mathcal{F}$-α-continuity) between $\mathcal{FTS}s$ in the sense of Šostak. Furthermore, Abbas [22] explored and characterized the notions of $\mathcal{F}$-β-continuous (resp. $\mathcal{F}$-β-irresolute) functions between $\mathcal{FTS}s$ in the sense of Šostak [14]. Additionally, Kim and Abbas [23] introduced some types of k-$\mathcal{F}$-compactness on $\mathcal{FTS}s$ in the sense of Šostak.

The notion of fuzzy soft sets ($\mathcal{FS}$-sets) was first presented in 2001 by the author of [24], which combines the $\mathcal{S}$-set [25] and $\mathcal{F}$-set [1]. Thereafter, the notion of an $\mathcal{FS}$-topology was defined, and many of its properties such as $\mathcal{FS}$-continuity, $\mathcal{FS}$-closure operators, $\mathcal{FS}$-interior operators, and $\mathcal{FS}$-subspaces were introduced in [26,27]. Moreover, the notions of k-$\mathcal{FS}$-regularly open, k-$\mathcal{FS}$-pre-open, and k-$\mathcal{FS}$-β-open sets were introduced by the authors of [28,29] based on the approach developed by Aygünoğlu et al. [26]. Overall, Taha [30] introduced and discussed the notions of $\mathcal{FS}$-almost (resp. $\mathcal{FS}$-weakly) k-minimal continuity, which are weaker forms of $\mathcal{FS}$-k-minimal continuity [29] based on the approach developed by Aygünoğlu et al. [26].

We lay out the remainder of this manuscript as follows:

- Section 2 contains some basic definitions that help in understanding the obtained results.
- In Section 3, we define a novel class of $\mathcal{F}$-open sets called k-$\mathcal{F}$-γ-open sets on $\mathcal{FTS}s$ in the sense of Šostak [14]. This class is contained in the class of k-$\mathcal{F}$-β-open sets and contains all k-$\mathcal{F}$-α-open, k-$\mathcal{F}$-pre-open, and k-$\mathcal{F}$-semi-open sets. Some properties of k-$\mathcal{F}$-γ-open sets, along with their mutual relationships, are discussed with the help of some illustrative examples. After that, we define the concepts of $\mathcal{F}$-γ-closure and $\mathcal{F}$-γ-interior operators and study some of their properties.
- In Section 4, we explore and investigate the concepts of $\mathcal{F}$-γ-continuous (resp. $\mathcal{F}$-γ-irresolute) functions between $\mathcal{FTS}s$ $(M, \Im)$ and $(N, \digamma)$. Moreover, we define and study the concepts of $\mathcal{F}$-almost (resp. $\mathcal{F}$-weakly) γ-continuous functions, which are weaker forms of $\mathcal{F}$-γ-continuous functions.
- In Section 5, we introduce and discuss some novel $\mathcal{F}$-functions using k-$\mathcal{F}$-γ-open and k-$\mathcal{F}$-γ-closed sets called $\mathcal{F}$-γ-open (resp. $\mathcal{F}$-γ-irresolute open, $\mathcal{F}$-γ-closed, $\mathcal{F}$-γ-irresolute closed, and $\mathcal{F}$-γ-irresolute homeomorphism) functions. Also, we present and investigate some new types of $\mathcal{F}$-separation axioms called k-$\mathcal{F}$-γ-regular (resp. k-$\mathcal{F}$-γ-normal) spaces. However, we display and discuss some new types of $\mathcal{F}$-compactness called k-$\mathcal{F}$-almost (resp. k-$\mathcal{F}$-nearly) γ-compact sets.
- In Section 6, we close this paper with conclusions and propose future papers.

2. Preliminaries

In this manuscript, nonempty sets will be denoted by M, N, W, etc. On M, I^M is the class of all $\mathcal{F}$-sets. For $\mathcal{D} \in I^M$, $\mathcal{D}^c(m) = 1 - \mathcal{D}(m)$ for each $m \in M$. Also, for $\sigma \in I$, $\underline{\sigma}(m) = \sigma$ for each $m \in M$.

An $\mathcal{F}$-point m_σ on M is an $\mathcal{F}$-set and is defined as follows: $m_\sigma(u) = \sigma$ if $u = m$, and $m_\sigma(u) = 0$ for any $u \in M - \{m\}$. Moreover, we say that m_σ belongs to $\mathcal{D} \in I^M$ ($m_\sigma \in \mathcal{D}$) if $\sigma \leq \mathcal{D}(m)$. On M, $P_\sigma(M)$ is the class of all $\mathcal{F}$-points.

On M, an $\mathcal{F}$-set $\mathcal{D} \in I^M$ is quasi-coincident with $\mathcal{P} \in I^M$ ($\mathcal{D}\ q\ \mathcal{P}$) if there is $m \in M$, with $\mathcal{D}(m) + \mathcal{P}(m) > 1$. Otherwise, $\mathcal{D}$ is not quasi-coincident with $\mathcal{P}$ ($\mathcal{D}\ \overline{q}\ \mathcal{P}$). The following results and notations will be used in the sequel.

Lemma 1 ([31]). *Let $\mathcal{D}, \mathcal{P} \in I^M$. Thus, we have the following:*

(1) $\mathcal{D}\ q\ \mathcal{P}$ iff there is $m_\sigma \in \mathcal{D}$ such that $m_\sigma\ q\ \mathcal{P}$.

(2) If $\mathcal{D}\ q\ \mathcal{P}$, then $\mathcal{D} \wedge \mathcal{P} \neq \underline{0}$.

(3) $\mathcal{D}\ \overline{q}\ \mathcal{P}$ iff $\mathcal{D} \leq \mathcal{P}^c$.

(4) $\mathcal{D} \leq \mathcal{P}$ iff $m_\sigma \in \mathcal{D}$ implies $m_\sigma \in \mathcal{P}$ iff $m_\sigma\ q\ \mathcal{D}$ implies $m_\sigma\ q\ \mathcal{P}$ iff $m_\sigma\ \overline{q}\ \mathcal{P}$ implies $m_\sigma\ \overline{q}\ \mathcal{D}$.

(5) $m_\sigma\ \overline{q}\ \bigvee_{j \in \Omega} \mathcal{P}_j$ iff there is $j_\circ \in \Omega$ such that $m_\sigma\ \overline{q}\ \mathcal{P}_{j_\circ}$.

Definition 1 ([14,15]). *A mapping $\Im : I^M \longrightarrow I$ is said to be a fuzzy topology on M if it satisfies the following conditions:*

(1) $\Im(\underline{1}) = \Im(\underline{0}) = 1$.

(2) $\Im(\mathcal{D} \wedge \mathcal{P}) \geq \Im(\mathcal{D}) \wedge \Im(\mathcal{P})$, for each $\mathcal{D}$, and $\mathcal{P} \in I^M$.

(3) $\Im(\bigvee_{j \in \Omega} \mathcal{D}_j) \geq \bigwedge_{j \in \Omega} \Im(\mathcal{D}_j)$ for each $\mathcal{D}_j \in I^M$.

Thus, $(M, \Im)$ is said to be a fuzzy topological space ($\mathcal{FTS}$) in the sense of Šostak.

Definition 2 ([16,19]). *In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D} \in I^M$ and $k \in I_\circ$ (where $I_\circ = (0, 1]$), we define $\mathcal{F}$-closure and $\mathcal{F}$-interior operators $C_\Im$ and $I_\Im : I^M \times I_\circ \to I^M$ as follows:*

$$C_\Im(\mathcal{D}, k) = \bigwedge \{\mathcal{P} \in I^M : \mathcal{D} \leq \mathcal{P},\ \Im(\mathcal{P}^c) \geq k\}.$$

$$I_\Im(\mathcal{D}, k) = \bigvee \{\mathcal{P} \in I^M : \mathcal{P} \leq \mathcal{D},\ \Im(\mathcal{P}) \geq k\}.$$

Definition 3 ([20,22]). *Let $(M, \Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and $k \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is said to be k-$\mathcal{F}$-regularly open (resp. k-$\mathcal{F}$-pre-open, k-$\mathcal{F}$-β-open, k-$\mathcal{F}$-semi-open, k-$\mathcal{F}$-α-open, and k-$\mathcal{F}$-open) if $\mathcal{D} = I_\Im(C_\Im(\mathcal{D}, k), k)$ (resp. $\mathcal{D} \leq I_\Im(C_\Im(\mathcal{D}, k), k)$, $\mathcal{D} \leq C_\Im(I_\Im(C_\Im(\mathcal{D}, k), k), k)$, $\mathcal{D} \leq C_\Im(I_\Im(\mathcal{D}, k), k)$, $\mathcal{D} \leq I_\Im(C_\Im(I_\Im(\mathcal{D}, k), k), k)$, and $\mathcal{D} \leq I_\Im(\mathcal{D}, k)$).*

Definition 4 ([17,23]). *Let $(M, \Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and $k \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is said to be k-$\mathcal{F}$-compact (resp. k-$\mathcal{F}$-nearly compact and k-$\mathcal{F}$-almost compact) iff for every family $\{\mathcal{P}_j \in I^M \mid \Im(\mathcal{P}_j) \geq k\}_{j \in \Omega}$, with $\mathcal{D} \leq \bigvee_{j \in \Omega} \mathcal{P}_j$, there is a finite subset $\Omega_\circ$ of Ω, with $\mathcal{D} \leq \bigvee_{j \in \Omega_\circ} \mathcal{P}_j$ (resp. $\mathcal{D} \leq \bigvee_{j \in \Omega_\circ} I_\Im(C_\Im(\mathcal{P}_j, k), k)$ and $\mathcal{D} \leq \bigvee_{j \in \Omega_\circ} C_\Im(\mathcal{P}_j, k)$).*

Definition 5 ([15,20]). *Let $(M, \Im)$ and $(N, \digamma)$ be $\mathcal{FTS}$s. An $\mathcal{F}$-function $h :\ I^M \longrightarrow I^N$ is said to be defined as follows:*

(1) $\mathcal{F}$-continuous if $\Im(h^{-1}(\mathcal{P})) \geq \digamma(\mathcal{P})$ for every $\mathcal{P} \in I^N$.

(2) $\mathcal{F}$-open if $\digamma(h(\mathcal{D})) \geq \Im(\mathcal{D})$ for every $\mathcal{D} \in I^M$.

(3) $\mathcal{F}$-closed if $\digamma((h(\mathcal{D}))^c) \geq \Im(\mathcal{D}^c)$ for every $\mathcal{D} \in I^M$.

Definition 6 ([20,22]). *Let $(M, \Im)$ and $(N, \digamma)$ be $\mathcal{FTS}$s and $k \in I_\circ$. An $\mathcal{F}$-function h : $I^M \longrightarrow I^N$ is said to be $\mathcal{F}$-α-continuous (resp. $\mathcal{F}$-pre-continuous, $\mathcal{F}$-semi-continuous, and $\mathcal{F}$-β-*

continuous) if $h^{-1}(\mathcal{P})$ *is a k-*$\mathcal{F}$*-*α*-open (resp. k-*$\mathcal{F}$*-pre-open, k-*$\mathcal{F}$*-semi-open, and k-*$\mathcal{F}$*-*β*-open) set for every* $\mathcal{P} \in I^N$*, with* $\digamma(\mathcal{P}) \geq k$*.*

Some basic notations and results that we need in the sequel are found in [15–23].

3. Some Characterizations of k-Fuzzy γ-Open Sets

Here, we define and study a new class of $\mathcal{F}$-open sets called k-$\mathcal{F}$-γ-open sets on $\mathcal{FTS}$s in the sense of Šostak [14]. Also, we explore and investigate the concepts of $\mathcal{F}$-γ-closure and $\mathcal{F}$-γ-interior operators.

Definition 7. *Let* $(M, \Im)$ *be an* $\mathcal{FTS}$ *and* $k \in I_\circ$*. An* $\mathcal{F}$*-set* $\mathcal{D} \in I^M$ *is said to be as follows:*
(1) k-$\mathcal{F}$*-*γ*-open set if* $\mathcal{D} \leq C_\Im(I_\Im(\mathcal{D}, k), k) \vee I_\Im(C_\Im(\mathcal{D}, k), k)$*.*
(2) k-$\mathcal{F}$*-*γ*-closed set if* $\mathcal{D} \geq C_\Im(I_\Im(\mathcal{D}, k), k) \wedge I_\Im(C_\Im(\mathcal{D}, k), k)$*.*

Remark 1. *The complement of k-*$\mathcal{F}$*-*γ*-open sets (resp. k-*$\mathcal{F}$*-*γ*-closed sets) are k-*$\mathcal{F}$*-*γ*-closed sets (resp. k-*$\mathcal{F}$*-*γ*-open sets).*

Proposition 1. *In an* $\mathcal{FTS}$ $(M, \Im)$*, for each* $\mathcal{D} \in I^M$ *and* $k \in I_\circ$*, we have the following:*
(1) Every k-$\mathcal{F}$*-pre-open set is k-*$\mathcal{F}$*-*γ*-open.*
(2) Every k-$\mathcal{F}$*-*γ*-open set is k-*$\mathcal{F}$*-*β*-open.*
(3) Every k-$\mathcal{F}$*-semi-open set is k-*$\mathcal{F}$*-*γ*-open.*

Proof. (1) If $\mathcal{D}$ is a k-$\mathcal{F}$-pre-open set,

$$\mathcal{D} \leq I_\Im(C_\Im(\mathcal{D}, k), k) \leq I_\Im(C_\Im(\mathcal{D}, k), k) \vee I_\Im(\mathcal{D}, k) \leq I_\Im(C_\Im(\mathcal{D}, k), k) \vee C_\Im(I_\Im(\mathcal{D}, k), k).$$

Thus, $\mathcal{D}$ is a k-$\mathcal{F}$-γ-open set.
(2) If $\mathcal{D}$ is a k-$\mathcal{F}$-γ-open set,

$$\begin{aligned}\mathcal{D} \leq C_\Im(I_\Im(\mathcal{D}, k), k) \vee I_\Im(C_\Im(\mathcal{D}, k), k) &\leq C_\Im(I_\Im(C_\Im(\mathcal{D}, k), k), k) \vee I_\Im(C_\Im(\mathcal{D}, k), k) \\ &\leq C_\Im(I_\Im(C_\Im(\mathcal{D}, k), k), k).\end{aligned}$$

Thus, $\mathcal{D}$ is a k-$\mathcal{F}$-β-open set.
(3) If $\mathcal{D}$ is a k-$\mathcal{F}$-semi-open set,

$$\mathcal{D} \leq C_\Im(I_\Im(\mathcal{D}, k), k) \leq C_\Im(I_\Im(\mathcal{D}, k), k) \vee I_\Im(\mathcal{D}, k) \leq C_\Im(I_\Im(\mathcal{D}, k), k) \vee I_\Im(C_\Im(\mathcal{D}, k), k).$$

Thus, $\mathcal{D}$ is a k-$\mathcal{F}$-γ-open set. □

Remark 2. *From the previous discussions and definitions, we have the following diagram:*

$$\begin{tikzcd}
 & k\text{-}\mathcal{F}\text{-}\textbf{pre-open set} \arrow[dr] & & \\
k\text{-}\mathcal{F}\text{-}\textbf{open set} \arrow[r] & k\text{-}\mathcal{F}\text{-}\alpha\text{-}\textbf{open set} \arrow[u] \arrow[r] \arrow[d] & k\text{-}\mathcal{F}\text{-}\gamma\text{-}\textbf{open set} \arrow[r] & k\text{-}\mathcal{F}\text{-}\beta\text{-}\textbf{open set} \\
 & k\text{-}\mathcal{F}\text{-}\textbf{semi-open set} \arrow[ur] & &
\end{tikzcd}$$

Remark 3. The converse of the above diagram fails, as Examples 1, 2, and 3 will show.

Example 1. *Let $M = \{m_1, m_2\}$, and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.4}, \frac{m_2}{0.3}\}$, $\mathcal{P} = \{\frac{m_1}{0.2}, \frac{m_2}{0.6}\}$, and $\mathcal{V} = \{\frac{m_1}{0.5}, \frac{m_2}{0.7}\}$. Define $\Im : I^M \longrightarrow I$ as follows:*

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \mathcal{C} = \mathcal{P}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{D}, \\ \frac{2}{3}, & \text{if} \quad \mathcal{C} = \mathcal{P} \wedge \mathcal{D}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{P} \vee \mathcal{D}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{V}$ is an $\frac{1}{3}$-$\mathcal{F}$-γ-open set, but it is neither $\frac{1}{3}$-$\mathcal{F}$-pre-open nor $\frac{1}{3}$-$\mathcal{F}$-α-open.

Example 2. *Let $M = \{m_1, m_2\}$, and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}\}$, $\mathcal{P} = \{\frac{m_1}{0.7}, \frac{m_2}{0.8}\}$, and $\mathcal{V} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$. Define $\Im : I^M \longrightarrow I$ as follows:*

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{2}{3}, & \text{if} \quad \mathcal{C} = \mathcal{D}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{P}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{V}$ is an $\frac{1}{2}$-$\mathcal{F}$-γ-open set, but it is not $\frac{1}{2}$-$\mathcal{F}$-semi-open.

Example 3. *Let $M = \{m_1, m_2\}$, and define $\mathcal{D}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$, and $\mathcal{V} = \{\frac{m_1}{0.4}, \frac{m_2}{0.5}\}$. Define $\Im : I^M \longrightarrow I$ as follows:*

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{D}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{V}$ is an $\frac{1}{4}$-$\mathcal{F}$-β-open set, but it is not $\frac{1}{4}$-$\mathcal{F}$-γ-open.

Definition 8. *In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D} \in I^M$ and $k \in I_\circ$, we define an $\mathcal{F}$-γ-closure operator $\gamma C_\Im : I^M \times I_\circ \longrightarrow I^M$ as follows: $\gamma C_\Im(\mathcal{D}, k) = \bigwedge \{\mathcal{P} \in I^M : \mathcal{D} \leq \mathcal{P},\ \mathcal{P}$ is k-$\mathcal{F}$-γ-closed$\}$.*

Proposition 2. *In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D} \in I^M$ and $k \in I_\circ$, an $\mathcal{F}$-set $\mathcal{D}$ is k-$\mathcal{F}$-γ-closed iff $\gamma C_\Im(\mathcal{D}, k) = \mathcal{D}$.*

Proof. This is easily proved from Definition 8. □

Theorem 1. *In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D}, \mathcal{P} \in I^M$ and $k \in I_\circ$, an $\mathcal{F}$-operator $\gamma C_\Im : I^M \times I_\circ \longrightarrow I^M$ satisfies the following properties:*

(1) $\gamma C_\Im(\underline{0}, k) = \underline{0}$.
(2) $\mathcal{D} \leq \gamma C_\Im(\mathcal{D}, k) \leq C_\Im(\mathcal{D}, k)$.
(3) $\gamma C_\Im(\mathcal{D}, k) \leq \gamma C_\Im(\mathcal{P}, k)$ if $\mathcal{D} \leq \mathcal{P}$.
(4) $\gamma C_\Im(\gamma C_\Im(\mathcal{D}, k), k) = \gamma C_\Im(\mathcal{D}, k)$.
(5) $\gamma C_\Im(\mathcal{D} \vee \mathcal{P}, k) \geq \gamma C_\Im(\mathcal{D}, k) \vee \gamma C_\Im(\mathcal{P}, k)$.
(6) $\gamma C_\Im(C_\Im(\mathcal{D}, k), k) = C_\Im(\mathcal{D}, k)$.

Proof. Examples (1), (2), and (3) are easily proved by Definition 8:

(4) From (2) and (3), $\gamma C_{\Im}(\mathcal{D},k) \leq \gamma C_{\Im}(\gamma C_{\Im}(\mathcal{D},k),k)$. Now, we show that $\gamma C_{\Im}(\mathcal{D},k) \geq \gamma C_{\Im}(\gamma C_{\Im}(\mathcal{D},k),k)$. If $\gamma C_{\Im}(\mathcal{D},k)$ does not contain $\gamma C_{\Im}(\gamma C_{\Im}(\mathcal{D},k),k)$, there are $m \in M$ and $\sigma \in (0,1)$ with

$$\gamma C_{\Im}(\mathcal{D},k)(m) < \sigma < \gamma C_{\Im}(\gamma C_{\Im}(\mathcal{D},k),k)(m). \tag{1}$$

Since $\gamma C_{\Im}(\mathcal{D},k)(m) < \sigma$, by Definition 8, there are $\mathcal{V} \in I^M$ as a k-$\mathcal{F}$-γ-closed set and $\mathcal{D} \leq \mathcal{V}$ with $\gamma C_{\Im}(\mathcal{D},k)(m) \leq \mathcal{V}(m) < \sigma$. Since $\mathcal{D} \leq \mathcal{V}$, then $\gamma C_{\Im}(\mathcal{D},k) \leq \mathcal{V}$. Again, this is by the definition of $\gamma C_{\Im}$, $\gamma C_{\Im}(\gamma C_{\Im}(\mathcal{D},k),k) \leq \mathcal{V}$. Hence, $\gamma C_{\Im}(\gamma C_{\Im}(\mathcal{D},k),k)(m) \leq \mathcal{V}(m) < \sigma$, which is a contradiction for (Z). Thus, $\gamma C_{\Im}(\mathcal{D},k) \geq \gamma C_{\Im}(\gamma C_{\Im}(\mathcal{D},k),k)$. Therefore, $\gamma C_{\Im}(\gamma C_{\Im}(\mathcal{D},k),k) = \gamma C_{\Im}(\mathcal{D},k)$.

(5) Since $\mathcal{D} \leq \mathcal{D} \vee \mathcal{P}$ and $\mathcal{P} \leq \mathcal{D} \vee \mathcal{P}$, hence by (3), $\gamma C_{\Im}(\mathcal{D},k) \leq \gamma C_{\Im}(\mathcal{D} \vee \mathcal{P},k)$, and $\gamma C_{\Im}(\mathcal{P},k) \leq \gamma C_{\Im}(\mathcal{D} \vee \mathcal{P},k)$. Thus, $\gamma C_{\Im}(\mathcal{D} \vee \mathcal{P},k) \geq \gamma C_{\Im}(\mathcal{D},k) \vee \gamma C_{\Im}(\mathcal{P},k)$.

(6) From Proposition 2 and the fact that $C_{\Im}(\mathcal{D},k)$ is a k-$\mathcal{F}$-γ-closed set, then

$$\gamma C_{\Im}(C_{\Im}(\mathcal{D},k),k) = C_{\Im}(\mathcal{D},k).$$

□

Definition 9. *In an $\mathcal{FTS}$ $(M,\Im)$, for each $\mathcal{D} \in I^M$ and $k \in I_\circ$, we define an $\mathcal{F}$-γ-interior operator $\gamma I_{\Im} : I^M \times I_\circ \longrightarrow I^M$ as follows: $\gamma I_{\Im}(\mathcal{D},k) = \bigvee \{\mathcal{P} \in I^M : \mathcal{P} \leq \mathcal{D},\ \mathcal{P}$ is k-$\mathcal{F}$-γ-open$\}$.*

Proposition 3. *Let $(M,\Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and let $k \in I_\circ$. Then, we have the following:*

(1) $\gamma C_{\Im}(\mathcal{D}^c,k) = (\gamma I_{\Im}(\mathcal{D},k))^c$;

(2) $\gamma I_{\Im}(\mathcal{D}^c,k) = (\gamma C_{\Im}(\mathcal{D},k))^c$.

Proof. (1) For each $\mathcal{D} \in I^M$ and $k \in I_\circ$, we have $\gamma C_{\Im}(\mathcal{D}^c,k) = \bigwedge\{\mathcal{P} \in I^M : \mathcal{D}^c \leq \mathcal{P},\ \mathcal{P}$ is k-$\mathcal{F}$-γ-closed$\}$ = $[\bigvee\{\mathcal{P}^c \in I^M : \mathcal{P}^c \leq \mathcal{D},\ \mathcal{P}^c$ is k-$\mathcal{F}$-γ-open$\}]^c$ = $(\gamma I_{\Im}(\mathcal{D},k))^c$.

(2) This is similar to that of (1). □

Proposition 4. *In an $\mathcal{FTS}$ $(M,\Im)$, for each $\mathcal{D} \in I^M$ and $k \in I_\circ$, an $\mathcal{F}$-set $\mathcal{D}$ is k-$\mathcal{F}$-γ-open iff $\gamma I_{\Im}(\mathcal{D},k) = \mathcal{D}$.*

Proof. This is easily proved from Definition 9. □

Theorem 2. *In an $\mathcal{FTS}$ $(M,\Im)$, for each $\mathcal{D},\mathcal{P} \in I^M$, and $k \in I_\circ$, an $\mathcal{F}$-operator $\gamma I_{\Im} : I^M \times I_\circ \longrightarrow I^M$ satisfies the following properties:*

(1) $\gamma I_{\Im}(\underline{1},k) = \underline{1}$.
(2) $I_{\Im}(\mathcal{D},k) \leq \gamma I_{\Im}(\mathcal{D},k) \leq \mathcal{D}$.
(3) $\gamma I_{\Im}(\mathcal{D},k) \leq \gamma I_{\Im}(\mathcal{P},k)$ if $\mathcal{D} \leq \mathcal{P}$.
(4) $\gamma I_{\Im}(\gamma I_{\Im}(\mathcal{D},k),k) = \gamma I_{\Im}(\mathcal{D},k)$.
(5) $\gamma I_{\Im}(\mathcal{D},k) \wedge \gamma I_{\Im}(\mathcal{P},k) \geq \gamma I_{\Im}(\mathcal{D} \wedge \mathcal{P},k)$.

Proof. The proof is similar to that of Theorem 1. □

4. On Fuzzy γ-Continuity and γ-Irresoluteness

Here, we define and discuss the concepts of $\mathcal{F}$-γ-continuous and $\mathcal{F}$-γ-irresolute functions between $\mathcal{FTS}$s $(M,\Im)$ and $(N,\digamma)$. We also define and study the concepts of $\mathcal{F}$-almost and $\mathcal{F}$-weakly γ-continuous functions, which are weaker forms of $\mathcal{F}$-γ-continuous functions.

Definition 10. *An $\mathcal{F}$-function $h: (M, \Im) \longrightarrow (N, F)$ is defined as follows:*

(1) $\mathcal{F}$-γ-continuous if $h^{-1}(\mathcal{D})$ is a k-$\mathcal{F}$-γ-open set for every $\mathcal{D} \in I^N$, with $F(\mathcal{D}) \geq k$ and $k \in I_\circ$;

(2) $\mathcal{F}$-γ-irresolute if $h^{-1}(\mathcal{D})$ is a k-$\mathcal{F}$-γ-open set for every k-$\mathcal{F}$-γ-open set $\mathcal{D} \in I^N$ and $k \in I_\circ$.

Remark 4. *From the previous definitions, we have the following diagram:*

$$\begin{tikzcd}
 & & & \mathcal{F}\textbf{-pre-continuity} \arrow[dr] & & & \\
\mathcal{F}\textbf{-continuity} \arrow[rr] & & \mathcal{F}\text{-}\alpha\textbf{-continuity} \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{F}\text{-}\gamma\textbf{-continuity} \arrow[rr] & & \mathcal{F}\text{-}\beta\textbf{-continuity} \\
 & & & \mathcal{F}\textbf{-semi-continuity} \arrow[ur] & & &
\end{tikzcd}$$

Remark 5. *The converse of the above diagram fails, as Examples 4–6 will show.*

Example 4. *Let $M = \{m_1, m_2\}$, and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.4}, \frac{m_2}{0.3}\}$, $\mathcal{P} = \{\frac{m_1}{0.2}, \frac{m_2}{0.6}\}$, $\mathcal{V} = \{\frac{m_1}{0.5}, \frac{m_2}{0.7}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:*

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \mathcal{C} = \mathcal{P}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{C} = \mathcal{D}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{P} \wedge \mathcal{D}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{P} \vee \mathcal{D}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{C} = \mathcal{V}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-γ-continuous, but it is neither $\mathcal{F}$-pre-continuous nor $\mathcal{F}$-α-continuous.

Example 5. *Let $M = \{m_1, m_2\}$, and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}\}$, $\mathcal{P} = \{\frac{m_1}{0.7}, \frac{m_2}{0.8}\}$, $\mathcal{V} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:*

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{P}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{C} = \mathcal{D}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{6}, & \text{if} \quad \mathcal{C} = \mathcal{V}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-γ-continuous, but it is not $\mathcal{F}$-semi-continuous.

Example 6. *Let $M = \{m_1, m_2\}$, and define $\mathcal{D}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$, $\mathcal{V} = \{\frac{m_1}{0.4}, \frac{m_2}{0.5}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:*

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{2}{3}, & \text{if} \quad \mathcal{C} = \mathcal{D}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{V}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-β-continuous, but it is not $\mathcal{F}$-γ-continuous.

Theorem 3. *An $\mathcal{F}$-function $h: (M, \Im) \longrightarrow (N, F)$ is $\mathcal{F}$-γ-continuous iff for any $m_\sigma \in P_\sigma(M)$ and any $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq k$ containing $h(m_\sigma)$, there is $\mathcal{A} \in I^M$ that is k-$\mathcal{F}$-γ-open containing m_σ, with $h(\mathcal{A}) \leq \mathcal{D}$ and $k \in I_\circ$.*

Proof. ($\Rightarrow$) Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$, with $F(\mathcal{D}) \geq k$ containing $h(m_\sigma)$, and then $h^{-1}(\mathcal{D}) \leq \gamma I_\Im(h^{-1}(\mathcal{D}), k)$. Since $m_\sigma \in h^{-1}(\mathcal{D})$, then we obtain $m_\sigma \in \gamma I_\Im(h^{-1}(\mathcal{D}), k) = \mathcal{A}$ (say). Hence, $\mathcal{A} \in I^M$ is k-$\mathcal{F}$-γ-open containing m_σ, with $h(\mathcal{A}) \leq \mathcal{D}$.

($\Leftarrow$) Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$, with $F(\mathcal{D}) \geq k$ and $m_\sigma \in h^{-1}(\mathcal{D})$. According to the assumption, there is $\mathcal{A} \in I^M$ that is k-$\mathcal{F}$-γ-open containing m_σ, with $h(\mathcal{A}) \leq \mathcal{D}$. Hence, $m_\sigma \in \mathcal{A} \leq h^{-1}(\mathcal{D})$, and $m_\sigma \in \gamma I_\Im(h^{-1}(\mathcal{D}), k)$. Thus, $h^{-1}(\mathcal{D}) \leq \gamma I_\Im(h^{-1}(\mathcal{D}), k)$, so $h^{-1}(\mathcal{D})$ is a k-$\mathcal{F}$-γ-open set. Then, h is $\mathcal{F}$-γ-continuous. $\square$

Theorem 4. *Let $h: (M, \Im) \longrightarrow (N, F)$ be an $\mathcal{F}$-function, and let $k \in I_\circ$. Then, the following statements are equivalent for every $\mathcal{P} \in I^M$ and $\mathcal{D} \in I^N$:*

(1) h is $\mathcal{F}$-γ-continuous.

(2) $h^{-1}(\mathcal{D})$ is k-$\mathcal{F}$-γ-closed for every $\mathcal{D} \in I^N$, with $F(\mathcal{D}^c) \geq k$.

(3) $h(\gamma C_\Im(\mathcal{P}, k)) \leq C_F(h(\mathcal{P}), k)$.

(4) $\gamma C_\Im(h^{-1}(\mathcal{D}), k) \leq h^{-1}(C_F(\mathcal{D}, k))$.

(5) $h^{-1}(I_F(\mathcal{D}, k)) \leq \gamma I_\Im(h^{-1}(\mathcal{D}), k)$.

Proof. (1) $\Leftrightarrow$ (2): The proof follows $h^{-1}(\mathcal{D}^c) = (h^{-1}(\mathcal{D}))^c$ and Definition 10.

(2) $\Rightarrow$ (3): Let $\mathcal{P} \in I^M$. By (2), we have that $h^{-1}(C_F(h(\mathcal{P}), k))$ is k-$\mathcal{F}$-γ-closed. Thus,

$$\gamma C_\Im(\mathcal{P}, k) \leq \gamma C_\Im(h^{-1}(h(\mathcal{P})), k) \leq \gamma C_\Im(h^{-1}(C_F(h(\mathcal{P}), k)), k) = h^{-1}(C_F(h(\mathcal{P}), k)).$$

Therefore, $h(\gamma C_\Im(\mathcal{P}, k)) \leq C_F(h(\mathcal{P}), k)$.

(3) $\Rightarrow$ (4): Let $\mathcal{D} \in I^N$. By (3), $h(\gamma C_\Im(h^{-1}(\mathcal{D}), k)) \leq C_F(h(h^{-1}(\mathcal{D})), k) \leq C_F(\mathcal{D}, k)$. Thus, $\gamma C_\Im(h^{-1}(\mathcal{D}), k) \leq h^{-1}(h(\gamma C_\Im(h^{-1}(\mathcal{D}), k))) \leq h^{-1}(C_F(\mathcal{D}, k))$.

(4) $\Leftrightarrow$ (5): The proof follows $h^{-1}(\mathcal{D}^c) = (h^{-1}(\mathcal{D}))^c$ and Proposition 3.

(5) $\Rightarrow$ (1): Let $\mathcal{D} \in I^N$, with $F(\mathcal{D}) \geq k$. By (5), we obtain $h^{-1}(\mathcal{D}) = h^{-1}(I_F(\mathcal{D}, k)) \leq \gamma I_\Im(h^{-1}(\mathcal{D}), k) \leq h^{-1}(\mathcal{D})$. Then, $\gamma I_\Im(h^{-1}(\mathcal{D}), k) = h^{-1}(\mathcal{D})$. Thus, $h^{-1}(\mathcal{D})$ is k-$\mathcal{F}$-γ-open, so h is $\mathcal{F}$-γ-continuous. $\square$

Lemma 2. *Every $\mathcal{F}$-γ-irresolute function is $\mathcal{F}$-γ-continuous.*

Proof. The proof follows Definition 10. $\square$

Remark 6. *The converse of Lemma 2 fails, as Example 7 will show.*

Example 7. *Let $M = \{m_1, m_2\}$, and define $\mathcal{D}, \mathcal{P} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.5}, \frac{m_2}{0.5}\}$, $\mathcal{P} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:*

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{C} = \mathcal{P}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if } \mathcal{C} = \mathcal{D}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-γ-continuous, but it is not $\mathcal{F}$-γ-irresolute.

Theorem 5. *Let $h: (M, \Im) \longrightarrow (N, F)$ be an $\mathcal{F}$-function, and let $k \in I_\circ$. Then, the following statements are equivalent for every $\mathcal{D} \in I^M$ and $\mathcal{P} \in I^N$:*

(1) h is $\mathcal{F}$-γ-irresolute.

(2) $h^{-1}(\mathcal{P})$ *is k-*$\mathcal{F}$*-*γ*-closed for every k-*$\mathcal{F}$*-*γ*-closed set* $\mathcal{P}$*.*
(3) $h(\gamma C_{\Im}(\mathcal{D},k)) \leq \gamma C_{F}(h(\mathcal{D}),k)$.
(4) $\gamma C_{\Im}(h^{-1}(\mathcal{P}),k) \leq h^{-1}(\gamma C_{F}(\mathcal{P},k))$.
(5) $h^{-1}(\gamma I_{F}(\mathcal{P},k)) \leq \gamma I_{\Im}(h^{-1}(\mathcal{P}),k)$.

Proof. (1) $\Leftrightarrow$ (2): The proof follows $h^{-1}(\mathcal{P}^c) = (h^{-1}(\mathcal{P}))^c$ and Definition 10.

(2) $\Rightarrow$ (3): Let $\mathcal{D} \in I^M$. By (2), we have that $h^{-1}(\gamma C_{F}(h(\mathcal{D}),k))$ is k-$\mathcal{F}$-γ-closed. Thus,

$$\gamma C_{\Im}(\mathcal{D},k) \leq \gamma C_{\Im}(h^{-1}(h(\mathcal{D})),k) \leq \gamma C_{\Im}(h^{-1}(\gamma C_{F}(h(\mathcal{D}),k)),k) = h^{-1}(\gamma C_{F}(h(\mathcal{D}),k)).$$

Therefore, $h(\gamma C_{\Im}(\mathcal{D},k)) \leq \gamma C_{F}(h(\mathcal{D}),k)$.

(3) $\Rightarrow$ (4): Let $\mathcal{P} \in I^N$. By (3), $h(\gamma C_{\Im}(h^{-1}(\mathcal{P}),k)) \leq \gamma C_{F}(h(h^{-1}(\mathcal{P})),k) \leq \gamma C_{F}(\mathcal{P},k)$. Thus, $\gamma C_{\Im}(h^{-1}(\mathcal{P}),k) \leq h^{-1}(h(\gamma C_{\Im}(h^{-1}(\mathcal{P}),k))) \leq h^{-1}(\gamma C_{F}(\mathcal{P},k))$.

(4) $\Leftrightarrow$ (5): The proof follows $h^{-1}(\mathcal{P}^c) = (h^{-1}(\mathcal{P}))^c$ and Proposition 3.

(5) $\Rightarrow$ (1): Let $\mathcal{P} \in I^N$ be a k-$\mathcal{F}$-γ-open set. By (5),

$$h^{-1}(\mathcal{P}) = h^{-1}(\gamma I_{F}(\mathcal{P},k)) \leq \gamma I_{\Im}(h^{-1}(\mathcal{P}),k) \leq h^{-1}(\mathcal{P}).$$

Thus, $\gamma I_{\Im}(h^{-1}(\mathcal{P}),k) = h^{-1}(\mathcal{P})$. Therefore, $h^{-1}(\mathcal{P})$ is k-$\mathcal{F}$-γ-open, so h is $\mathcal{F}$-γ-irresolute. □

Proposition 5. *Let* $(M,\Im)$*,* (W,η)*, let* (N,F) *be* $\mathcal{FTS}$*s, and let* $h : (M,\Im) \longrightarrow (W,\eta)$ *and* $f : (W,\eta) \longrightarrow (N,F)$ *be two* $\mathcal{F}$*-functions. Then, the composition* $f \circ h$ *is* $\mathcal{F}$*-*γ*-irresolute (resp.* $\mathcal{F}$*-*γ*-continuous) if* h *is* $\mathcal{F}$*-*γ*-irresolute, and* f *is* $\mathcal{F}$*-*γ*-irresolute (resp.* $\mathcal{F}$*-*γ*-continuous).*

Proof. The proof follows Definition 10. □

Definition 11. *An* $\mathcal{F}$*-function* $h : (M,\Im) \longrightarrow (N,F)$ *is called* $\mathcal{F}$*-almost* γ*-continuous if* $h^{-1}(\mathcal{D}) \leq \gamma I_{\Im}(h^{-1}(I_{F}(C_{F}(\mathcal{D},k),k)),k)$ *for every* $\mathcal{D} \in I^N$*, with* $F(\mathcal{D}) \geq k$ *and* $k \in I_{\circ}$*.*

Lemma 3. *Every* $\mathcal{F}$*-*γ*-continuous function is* $\mathcal{F}$*-almost* γ*-continuous.*

Proof. The proof follows Definitions 10 and 11. □

Remark 7. *The converse of Lemma 3 fails, as Example 8 will show.*

Example 8. *Let* $M = \{m_1, m_2, m_3\}$*, and define* $\mathcal{D},\mathcal{P},\mathcal{V} \in I^M$ *as follows:* $\mathcal{D} = \{\frac{m_1}{0.4}, \frac{m_2}{0.2}, \frac{m_3}{0.4}\}$*,* $\mathcal{P} = \{\frac{m_1}{0.5}, \frac{m_2}{0.5}, \frac{m_3}{0.4}\}$*,* $\mathcal{V} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}, \frac{m_3}{0.6}\}$*. Define* $\mathcal{F}$*-topologies* $\Im,\eta : I^M \longrightarrow I$ *as follows:*

$$\Im(\mathcal{U}) = \begin{cases} 1, & \text{if} \quad \mathcal{U} \in \{\underline{0}, \underline{1}\}, \\ \frac{2}{3}, & \text{if} \quad \mathcal{U} = \mathcal{D}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{U} = \mathcal{P}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{U}) = \begin{cases} 1, & \text{if} \quad \mathcal{U} \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{U} = \mathcal{V}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$*-function* $f : (M,\Im) \longrightarrow (M,\eta)$ *is* $\mathcal{F}$*-almost* γ*-continuous, but it is not* $\mathcal{F}$*-*γ*-continuous.*

Theorem 6. *An* $\mathcal{F}$*-function* $h : (M,\Im) \longrightarrow (N,F)$ *is* $\mathcal{F}$*-almost* γ*-continuous iff for any* $m_\sigma \in P_\sigma(M)$ *and any* $\mathcal{D} \in I^N$ *with* $F(\mathcal{D}) \geq k$ *containing* $h(m_\sigma)$*, there is* $\mathcal{A} \in I^M$ *that is k-*$\mathcal{F}$*-*γ*-open containing* m_σ*, with* $h(\mathcal{A}) \leq I_{F}(C_{F}(\mathcal{D},k),k)$ *and* $k \in I_{\circ}$*.*

Proof. ($\Rightarrow$): Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$, with $F(\mathcal{D}) \geq k$ containing $h(m_\sigma)$, and then $h^{-1}(\mathcal{D}) \leq \gamma I_{\Im}(h^{-1}(I_F(C_F(\mathcal{D},k),k)),k)$. Since $m_\sigma \in h^{-1}(\mathcal{D})$, then $m_\sigma \in \gamma I_{\Im}(h^{-1}(I_F(C_F(\mathcal{D},k),k)),k) = \mathcal{A}$ (say). Therefore, $\mathcal{A} \in I^M$ is k-$\mathcal{F}$-γ-open containing m_σ, with $h(\mathcal{A}) \leq I_F(C_F(\mathcal{D},k),k)$.

($\Leftarrow$): Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$, with $F(\mathcal{D}) \geq k$ such that $m_\sigma \in h^{-1}(\mathcal{D})$. According to the assumption, there is $\mathcal{A} \in I^M$ that is k-$\mathcal{F}$-γ-open containing m_σ, with $h(\mathcal{A}) \leq I_F(C_F(\mathcal{D},k),k)$. Hence, $m_\sigma \in \mathcal{A} \leq h^{-1}(I_F(C_F(\mathcal{D},k),k))$, and $m_\sigma \in \gamma I_{\Im}(h^{-1}(I_F(C_F(\mathcal{D},k),k)),k)$. Thus, $h^{-1}(\mathcal{D}) \leq \gamma I_{\Im}(h^{-1}(I_F(C_F(\mathcal{D},k),k)),k)$. Therefore, h is $\mathcal{F}$-almost γ-continuous. $\square$

Theorem 7. *Let $h : (M,\Im) \longrightarrow (N,F)$ be an $\mathcal{F}$-function, $\mathcal{P} \in I^N$, and let $k \in I_\circ$. Then, the following statements are equivalent:*

(1) h is $\mathcal{F}$-almost γ-continuous.

(2) $h^{-1}(\mathcal{P})$ is k-$\mathcal{F}$-γ-open for every k-$\mathcal{F}$-regularly open set $\mathcal{P}$.

(3) $h^{-1}(\mathcal{P})$ is k-$\mathcal{F}$-γ-closed for every k-$\mathcal{F}$-regularly closed set $\mathcal{P}$.

(4) $\gamma C_{\Im}(h^{-1}(\mathcal{P}),k) \leq h^{-1}(C_F(\mathcal{P},k))$ for every k-$\mathcal{F}$-γ-open set $\mathcal{P}$.

(5) $\gamma C_{\Im}(h^{-1}(\mathcal{P}),k) \leq h^{-1}(C_F(\mathcal{P},k))$ for every k-$\mathcal{F}$-semi-open set $\mathcal{P}$.

Proof. (1) $\Rightarrow$ (2): Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{P}$ be a k-$\mathcal{F}$-regularly open set, with $m_\sigma \in h^{-1}(\mathcal{P})$. Hence, by (1), there is $\mathcal{A} \in I^M$ that is a k-$\mathcal{F}$-γ-open set, with $m_\sigma \in \mathcal{A}$ and $h(\mathcal{A}) \leq I_F(C_F(\mathcal{P},k),k)$. Thus, $\mathcal{A} \leq h^{-1}(I_F(C_F(\mathcal{P},k),k)) = h^{-1}(\mathcal{P})$, and $m_\sigma \in \gamma I_{\Im}(h^{-1}(\mathcal{P}),k)$. Therefore, $h^{-1}(\mathcal{P}) \leq \gamma I_{\Im}(h^{-1}(\mathcal{P}),k)$, so $h^{-1}(\mathcal{P})$ is k-$\mathcal{F}$-γ-open.

(2) $\Rightarrow$ (3): If $\mathcal{P} \in I^N$ is k-$\mathcal{F}$-regularly closed, then by (2), $h^{-1}(\mathcal{P}^c) = (h^{-1}(\mathcal{P}))^c$ is k-$\mathcal{F}$-γ-open. Thus, $h^{-1}(\mathcal{P})$ is k-$\mathcal{F}$-γ-closed.

(3) $\Rightarrow$ (4): If $\mathcal{P} \in I^N$ is k-$\mathcal{F}$-b-open, and since $C_F(\mathcal{P},k)$ is k-$\mathcal{F}$-regularly closed, then by (3), $h^{-1}(C_F(\mathcal{P},k))$ is k-$\mathcal{F}$-γ-closed. Since $h^{-1}(\mathcal{P}) \leq h^{-1}(C_F(\mathcal{P},k))$, hence,

$$\gamma C_{\Im}(h^{-1}(\mathcal{P}),k) \leq h^{-1}(C_F(\mathcal{P},k)).$$

(4) $\Rightarrow$ (5): The proof follows from the fact that any k-$\mathcal{F}$-semi-open set is k-$\mathcal{F}$-γ-open.

(5) $\Rightarrow$ (3): If $\mathcal{P} \in I^N$ is k-$\mathcal{F}$-regularly closed, then $\mathcal{P}$ is k-$\mathcal{F}$-semi-open. By (5), $\gamma C_{\Im}(h^{-1}(\mathcal{P}),k) \leq h^{-1}(C_F(\mathcal{P},k)) = h^{-1}(\mathcal{P})$. Hence, $h^{-1}(\mathcal{P})$ is k-$\mathcal{F}$-b-closed.

(3) $\Rightarrow$ (1): If $m_\sigma \in P_\sigma(M)$ and $\mathcal{P} \in I^N$ with $F(\mathcal{P}) \geq k$ such that $m_\sigma \in h^{-1}(\mathcal{P})$, then $m_\sigma \in h^{-1}(I_F(C_F(\mathcal{P},k),k))$. Since $[I_F(C_F(\mathcal{P},k),k)]^c$ is k-$\mathcal{F}$-regularly closed, then by (3), we have that $h^{-1}([I_F(C_F(\mathcal{P},k),k)]^c)$ is k-$\mathcal{F}$-γ-closed. Hence, $h^{-1}(I_F(C_F(\mathcal{P}),k))$ is k-$\mathcal{F}$-γ-open, and

$$m_\sigma \in \gamma I_{\Im}(h^{-1}(I_F(C_F(\mathcal{P},k),k)),k).$$

Thus, $h^{-1}(\mathcal{P}) \leq \gamma I_{\Im}(h^{-1}(I_F(C_F(\mathcal{P},k),k)),k)$. Therefore, h is $\mathcal{F}$-almost γ-continuous. $\square$

Definition 12. *An $\mathcal{F}$-function $h : (M,\Im) \longrightarrow (N,F)$ is called $\mathcal{F}$-weakly γ-continuous if $h^{-1}(\mathcal{D}) \leq \gamma I_{\Im}(h^{-1}(C_F(\mathcal{D},k)),k)$ for every $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq k$ and if $k \in I_\circ$.*

Lemma 4. *Every $\mathcal{F}$-γ-continuous function is $\mathcal{F}$-weakly γ-continuous.*

Proof. The proof follows Definitions 10 and 12. $\square$

Remark 8. *The converse of Lemma 4 fails, as Example 9 will show.*

Example 9. *Let $M = \{m_1, m_2, m_3\}$, and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.4}, \frac{m_2}{0.2}, \frac{m_3}{0.4}\}$, $\mathcal{P} = \{\frac{m_1}{0.5}, \frac{m_2}{0.5}, \frac{m_3}{0.4}\}$, $\mathcal{V} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}, \frac{m_3}{0.6}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:*

$$\Im(\mathcal{U}) = \begin{cases} 1, & if \quad \mathcal{U} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & if \quad \mathcal{U} = \mathcal{D}, \\ \frac{1}{2}, & if \quad \mathcal{U} = \mathcal{P}, \\ 0, & otherwise, \end{cases} \qquad \eta(\mathcal{U}) = \begin{cases} 1, & if \quad \mathcal{U} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & if \quad \mathcal{U} = \mathcal{V}, \\ 0, & otherwise. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-weakly γ-continuous, but it is not $\mathcal{F}$-γ-continuous.

Theorem 8. *An $\mathcal{F}$-function $h : (M, \Im) \longrightarrow (N, F)$ is $\mathcal{F}$-weakly γ-continuous iff for any $m_\sigma \in P_\sigma(M)$ and any $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq k$ containing $h(m_\sigma)$, there is $\mathcal{A} \in I^M$ that is k-$\mathcal{F}$-γ-open containing m_σ, with $h(\mathcal{A}) \leq C_F(\mathcal{D}, k)$ and $k \in I_\circ$.*

Proof. ($\Rightarrow$): Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$, with $F(\mathcal{D}) \geq k$ containing $h(m_\sigma)$; then, $h^{-1}(\mathcal{D}) \leq \gamma I_\Im(h^{-1}(C_F(\mathcal{D}, k)), k)$. Since $m_\sigma \in h^{-1}(\mathcal{D})$, then $m_\sigma \in \gamma I_\Im(h^{-1}(C_F(\mathcal{D}, k)), k) = \mathcal{A}$ (say). Hence, $\mathcal{A} \in I^M$ is k-$\mathcal{F}$-γ-open containing m_σ, with $h(\mathcal{A}) \leq C_F(\mathcal{D}, k)$.

($\Leftarrow$): Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq k$ such that $m_\sigma \in h^{-1}(\mathcal{D})$. According to the assumption, there is $\mathcal{A} \in I^M$ that is k-$\mathcal{F}$-γ-open containing m_σ, with $h(\mathcal{A}) \leq C_F(\mathcal{D}, k)$. Hence, $m_\sigma \in \mathcal{A} \leq h^{-1}(C_F(\mathcal{D}, k))$, and $m_\sigma \in \gamma I_\Im(h^{-1}(C_F(\mathcal{D}, k)), k)$. Thus, $h^{-1}(\mathcal{D}) \leq \gamma I_\Im(h^{-1}(C_F(\mathcal{D}, k)), k)$. Therefore, h is $\mathcal{F}$-weakly γ-continuous. □

Theorem 9. *Let $h : (M, \Im) \longrightarrow (N, F)$ be an $\mathcal{F}$-function, and let $k \in I_\circ$. Then, the following statements are equivalent:*

(1) h is $\mathcal{F}$-weakly γ-continuous.
(2) $h^{-1}(\mathcal{P}) \geq \gamma C_\Im(h^{-1}(I_F(\mathcal{P}, k)), k)$ if $\mathcal{P} \in I^N$, with $F(\mathcal{P}^c) \geq k$.
(3) $\gamma I_\Im(h^{-1}(C_F(\mathcal{P}, k)), k) \geq h^{-1}(I_F(\mathcal{P}, k))$.
(4) $\gamma C_\Im(h^{-1}(I_F(\mathcal{P}, k)), k) \leq h^{-1}(C_F(\mathcal{P}, k))$.

Proof. (1) $\Leftrightarrow$ (2): The proof follows Proposition 3 and Definition 12.

(2) $\Rightarrow$ (3): Let $\mathcal{P} \in I^N$. Hence, by (2),

$$\gamma C_\Im(h^{-1}(I_F(C_F(\mathcal{P}^c, k), k)), k) \leq h^{-1}(C_F(\mathcal{P}^c, k)).$$

Thus, $h^{-1}(I_F(\mathcal{P}, k)) \leq \gamma I_\Im(h^{-1}(C_F(\mathcal{P}, k)), k)$.

(3) $\Leftrightarrow$ (4): The proof follows from Proposition 3.

(4) $\Rightarrow$ (1): Let $\mathcal{P} \in I^N$, with $F(\mathcal{P}) \geq k$. Hence, by (4), $\gamma C_\Im(h^{-1}(I_F(\mathcal{P}^c, k)), k) \leq h^{-1}(C_F(\mathcal{P}^c, k)) = h^{-1}(\mathcal{P}^c)$. Thus, $h^{-1}(\mathcal{P}) \leq \gamma I_\Im(h^{-1}(C_F(\mathcal{P}, k)), k)$, so h is $\mathcal{F}$-weakly γ-continuous. □

Lemma 5. *Every $\mathcal{F}$-almost γ-continuous function is $\mathcal{F}$-weakly γ-continuous.*

Proof. The proof follows Definitions 11 and 12. □

Remark 9. *The converse of Lemma 5 fails, as Example 10 will show.*

Example 10. *Let $M = \{m_1, m_2, m_3\}$, and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.6}, \frac{m_2}{0.2}, \frac{m_3}{0.4}\}$, $\mathcal{P} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}, \frac{m_3}{0.5}\}$, $\mathcal{V} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}, \frac{m_3}{0.4}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:*

$$\Im(\mathcal{U}) = \begin{cases} 1, & \text{if} \quad \mathcal{U} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{U} = \mathcal{D}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{U} = \mathcal{V}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{U}) = \begin{cases} 1, & \text{if} \quad \mathcal{U} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \mathcal{U} = \mathcal{P}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-weakly γ-continuous, but it is not $\mathcal{F}$-almost γ-continuous.

Remark 10. *From the previous discussions and definitions, we have the following diagram:*

$\mathcal{F}$-γ-**continuity** $\longrightarrow$ $\mathcal{F}$-**almost** γ-**continuity** $\longrightarrow$ $\mathcal{F}$-**weak** γ-**continuity**

Proposition 6. *Let $(M, \Im)$, (W, η), and $(N, \digamma)$ be $\mathcal{FTS}$s, and let $h : (M, \Im) \longrightarrow (W, \eta)$ and $g : (W, \eta) \longrightarrow (N, \digamma)$ be two $\mathcal{F}$-functions. Then, the composition $g \circ h$ is $\mathcal{F}$-almost γ-continuous if h is $\mathcal{F}$-γ-irresolute (resp. $\mathcal{F}$-γ-continuous) and g is $\mathcal{F}$-almost γ-continuous (resp. $\mathcal{F}$-continuous).*

Proof. The proof follows the previous definitions. □

5. Further Selected Topics

Here, we introduce and establish some new $\mathcal{F}$-functions using k-$\mathcal{F}$-γ-open and k-$\mathcal{F}$-γ-closed sets, which are called $\mathcal{F}$-γ-open (resp. $\mathcal{F}$-γ-irresolute open, $\mathcal{F}$-γ-closed, $\mathcal{F}$-γ-irresolute closed, and $\mathcal{F}$-γ-irresolute homeomorphism) functions. Also, we explore and study some new types of $\mathcal{F}$-compactness called k-$\mathcal{F}$-almost and k-$\mathcal{F}$-nearly γ-compact sets using k-$\mathcal{F}$-γ-open sets.

Definition 13. *An $\mathcal{F}$-function $h : (M, \Im) \longrightarrow (N, \digamma)$ is defined as follows:*
(1) $\mathcal{F}$-γ-open if $h(\mathcal{D})$ is a k-$\mathcal{F}$-γ-open set for every $\mathcal{D} \in I^M$ with $\Im(\mathcal{D}) \geq k$.
(2) $\mathcal{F}$-γ-closed if $h(\mathcal{D})$ is a k-$\mathcal{F}$-γ-closed set for every $\mathcal{D} \in I^M$ with $\Im(\mathcal{D}^c) \geq k$.
(3) $\mathcal{F}$-γ-irresolute open if $h(\mathcal{D})$ is a k-$\mathcal{F}$-γ-open set for every k-$\mathcal{F}$-γ-open set $\mathcal{D} \in I^M$.
(4) $\mathcal{F}$-γ-irresolute closed if $h(\mathcal{D})$ is a k-$\mathcal{F}$-γ-closed set for every k-$\mathcal{F}$-γ-closed set $\mathcal{D} \in I^M$.

Lemma 6. *(1): Each $\mathcal{F}$-γ-irresolute open function is $\mathcal{F}$-γ-open.*
(2): Each $\mathcal{F}$-γ-irresolute closed function is $\mathcal{F}$-γ-closed.

Proof. The proof follows Definition 13. □

Remark 11. *The converse of Lemma 6 fails, as Example 11 will show.*

Example 11. *Let $M = \{m_1, m_2\}$, and define $\mathcal{D}, \mathcal{P} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.5}, \frac{m_2}{0.5}\}$, $\mathcal{P} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$. Define $\mathcal{F}$-topologies $\Im$, $\eta : I^M \longrightarrow I$ as follows:*

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \mathcal{C} = \mathcal{D}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \mathcal{C} = \mathcal{P}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-γ-open, but it is not $\mathcal{F}$-γ-irresolute open.

Theorem 10. *Let $h : (M, \Im) \longrightarrow (N, \digamma)$ be an $\mathcal{F}$-function, and let $k \in I_\circ$. Then, the following statements are equivalent for every $\mathcal{A} \in I^M$ and $\mathcal{D} \in I^N$:*

(1) h is $\mathcal{F}$-γ-open.

(2) $h(I_{\Im}(\mathcal{A},k)) \leq \gamma I_{F}(h(\mathcal{A}),k)$.

(3) $I_{\Im}(h^{-1}(\mathcal{D}),k) \leq h^{-1}(\gamma I_{F}(\mathcal{D},k))$.

(4) For every $\mathcal{D}$ and every $\mathcal{A}$ with $\Im(\mathcal{A}^c) \geq k$ and $h^{-1}(\mathcal{D}) \leq \mathcal{A}$, there is $\mathcal{P} \in I^N$ is k-$\mathcal{F}$-γ-closed with $\mathcal{D} \leq \mathcal{P}$ and $h^{-1}(\mathcal{P}) \leq \mathcal{A}$.

Proof. (1) $\Rightarrow$ (2): Since $h(I_{\Im}(\mathcal{A},k)) \leq h(\mathcal{A})$, hence, by (1), $h(I_{\Im}(\mathcal{A},k))$ is k-$\mathcal{F}$-γ-open. Thus,

$$h(I_{\Im}(\mathcal{A},k)) \leq \gamma I_{F}(h(\mathcal{A}),k).$$

(2) $\Rightarrow$ (3): Set $\mathcal{A} = h^{-1}(\mathcal{D})$; hence, by (2), $h(I_{\Im}(h^{-1}(\mathcal{D}),k)) \leq \gamma I_{F}(h(h^{-1}(\mathcal{D})),k) \leq \gamma I_{F}(\mathcal{D},k)$. Thus, $I_{\Im}(h^{-1}(\mathcal{D}),k) \leq h^{-1}(\gamma I_{F}(\mathcal{D},k))$.

(3) $\Rightarrow$ (4): Let $\mathcal{D} \in I^N$ and $\mathcal{A} \in I^M$, with $\Im(\mathcal{A}^c) \geq k$ such that $h^{-1}(\mathcal{D}) \leq \mathcal{A}$. Since $\mathcal{A}^c \leq h^{-1}(\mathcal{D}^c)$, $\mathcal{A}^c = I_{\Im}(\mathcal{A}^c,k) \leq I_{\Im}(h^{-1}(\mathcal{D}^c),k)$. Hence, by (3), $\mathcal{A}^c \leq I_{\Im}(h^{-1}(\mathcal{D}^c),k) \leq h^{-1}(\gamma I_{F}(\mathcal{D}^c,k))$. Then, we have $\mathcal{A} \geq (h^{-1}(\gamma I_{F}(\mathcal{D}^c,k)))^c = h^{-1}(\gamma C_{F}(\mathcal{D},k))$. Thus, there is $\gamma C_{F}(\mathcal{D},k) \in I^N$ is k-$\mathcal{F}$-γ-closed with $\mathcal{D} \leq \gamma C_{F}(\mathcal{D},k)$ and $h^{-1}(\gamma C_{F}(\mathcal{D},k)) \leq \mathcal{A}$.

(4) $\Rightarrow$ (1): Let $\mathcal{B} \in I^M$, with $\Im(\mathcal{B}) \geq k$. Set $\mathcal{D} = (h(\mathcal{B}))^c$ and $\mathcal{A} = \mathcal{B}^c$; then, $h^{-1}(\mathcal{D}) = h^{-1}((h(\mathcal{B}))^c) \leq \mathcal{A}$. Hence, by (4), there is $\mathcal{P} \in I^N$ that is k-$\mathcal{F}$-γ-closed with $\mathcal{D} \leq \mathcal{P}$ and $h^{-1}(\mathcal{P}) \leq \mathcal{A} = \mathcal{B}^c$. Thus, $h(\mathcal{B}) \leq h(h^{-1}(\mathcal{P}^c)) \leq \mathcal{P}^c$. On the other hand, since $\mathcal{D} \leq \mathcal{P}$, $h(\mathcal{B}) = \mathcal{D}^c \geq \mathcal{P}^c$. Hence, $h(\mathcal{B}) = \mathcal{P}^c$, so $h(\mathcal{B})$ is a k-$\mathcal{F}$-γ-open set. Therefore, h is $\mathcal{F}$-γ-open. □

Theorem 11. *Let $h : (M,\Im) \longrightarrow (N,F)$ be an $\mathcal{F}$-function, and let $k \in I_{\circ}$. Then, the following statements are equivalent for every $\mathcal{B} \in I^M$ and $\mathcal{D} \in I^N$:*

(1) h is $\mathcal{F}$-γ-closed.

(2) $\gamma C_{F}(h(\mathcal{B}),k) \leq h(C_{\Im}(\mathcal{B},k))$.

(3) $h^{-1}(\gamma C_{F}(\mathcal{D},k)) \leq C_{\Im}(h^{-1}(\mathcal{D}),k)$.

(4) For every $\mathcal{D}$ and every $\mathcal{B}$ with $\Im(\mathcal{B}) \geq k$ and $h^{-1}(\mathcal{D}) \leq \mathcal{B}$, there is $\mathcal{P} \in I^N$ that is k-$\mathcal{F}$-γ-open with $\mathcal{D} \leq \mathcal{P}$ and $h^{-1}(\mathcal{P}) \leq \mathcal{B}$.

Proof. The proof is similar to that of Theorem 10. □

Theorem 12. *Let $h : (M,\Im) \longrightarrow (N,F)$ be an $\mathcal{F}$-function, and let $k \in I_{\circ}$. Then, the following statements are equivalent for every $\mathcal{B} \in I^M$ and $\mathcal{D} \in I^N$:*

(1) h is $\mathcal{F}$-γ-irresolute open.

(2) $h(\gamma I_{\Im}(\mathcal{B},k)) \leq \gamma I_{F}(h(\mathcal{B}),k)$.

(3) $\gamma I_{\Im}(h^{-1}(\mathcal{D}),k) \leq h^{-1}(\gamma I_{F}(\mathcal{D},k))$.

(4) For every $\mathcal{D}$ and every $\mathcal{B}$ that define a k-$\mathcal{F}$-γ-closed set with $h^{-1}(\mathcal{D}) \leq \mathcal{B}$, there is $\mathcal{P} \in I^N$ that is k-$\mathcal{F}$-γ-closed with $\mathcal{D} \leq \mathcal{P}$ and $h^{-1}(\mathcal{P}) \leq \mathcal{B}$.

Proof. The proof is similar to that of Theorem 10. □

Theorem 13. *Let $h : (M,\Im) \longrightarrow (N,F)$ be an $\mathcal{F}$-function, and let $k \in I_{\circ}$. Then, the following statements are equivalent for every $\mathcal{B} \in I^M$ and $\mathcal{D} \in I^N$:*

(1) h is $\mathcal{F}$-γ-irresolute closed.

(2) $\gamma C_{F}(h(\mathcal{B}),k) \leq h(\gamma C_{\Im}(\mathcal{B},k))$.

(3) $h^{-1}(\gamma C_{F}(\mathcal{D},k)) \leq \gamma C_{\Im}(h^{-1}(\mathcal{D}),k)$.

(4) For every $\mathcal{D}$ and every $\mathcal{B}$ that define a k-$\mathcal{F}$-γ-open set with $h^{-1}(\mathcal{D}) \leq \mathcal{B}$, there is $\mathcal{P} \in I^N$ that is k-$\mathcal{F}$-γ-open with $\mathcal{D} \leq \mathcal{P}$ and $h^{-1}(\mathcal{P}) \leq \mathcal{B}$.

Proof. The proof is similar to that of Theorem 10. □

Proposition 7. *Let $h : (M, \Im) \longrightarrow (N, F)$ be a bijective $\mathcal{F}$-function. Then, h is $\mathcal{F}$-γ-irresolute open iff h is $\mathcal{F}$-γ-irresolute closed.*

Proof. The proof follows from the following:

$$h^{-1}(\gamma C_F(\mathcal{V}, k)) \leq \gamma C_{\Im}(h^{-1}(\mathcal{V}), k) \iff h^{-1}(\gamma I_F(\mathcal{V}^c, k)) \leq \gamma I_{\Im}(h^{-1}(\mathcal{V}^c), k).$$

□

Definition 14. *A bijective $\mathcal{F}$-function $h : (M, \Im) \longrightarrow (N, F)$ is called $\mathcal{F}$-γ-irresolute homeomorphism if h^{-1} and h are $\mathcal{F}$-γ-irresolute.*

The proof of the following corollary is easy and so is omitted.

Corollary 1. Let $h : (M, \Im) \longrightarrow (N, F)$ be a bijective $\mathcal{F}$-function, and let $k \in I_\circ$. Then, the following statements are equivalent for every $\mathcal{P} \in I^M$ and $\mathcal{V} \in I^N$:

(1) h is an $\mathcal{F}$-γ-irresolute homeomorphism.
(2) h is $\mathcal{F}$-γ-irresolute closed and $\mathcal{F}$-γ-irresolute.
(3) h is $\mathcal{F}$-γ-irresolute open and $\mathcal{F}$-γ-irresolute.
(4) $h(\gamma I_{\Im}(\mathcal{P}, k)) = \gamma I_F(h(\mathcal{P}), k)$.
(5) $h(\gamma C_{\Im}(\mathcal{P}, k)) = \gamma C_F(h(\mathcal{P}), k)$.
(6) $\gamma I_{\Im}(h^{-1}(\mathcal{V}), k) = h^{-1}(\gamma I_F(\mathcal{V}, k))$.
(7) $\gamma C_{\Im}(h^{-1}(\mathcal{V}), k) = h^{-1}(\gamma C_F(\mathcal{V}, k))$.

Definition 15. *Let $m_\sigma \in P_\sigma(M)$, $\mathcal{A}, \mathcal{B} \in I^M$, and let $k \in I_\circ$. An $\mathcal{FTS}$ $(M, \Im)$ is defined as follows:*

(1) k-$\mathcal{F}$-γ-regular space if $m_\sigma \,\overline{q}\, \mathcal{A}$ for each k-$\mathcal{F}$-γ-closed set $\mathcal{A}$, there is $\mathcal{C}_j \in I^M$ with $\Im(\mathcal{C}_j) \geq k$ for $j \in \{1, 2\}$ such that $m_\sigma \in \mathcal{C}_1$, $\mathcal{A} \leq \mathcal{C}_2$, and $\mathcal{C}_1 \,\overline{q}\, \mathcal{C}_2$.

(2) k-$\mathcal{F}$-γ-normal space if $\mathcal{A} \,\overline{q}\, \mathcal{B}$ for each k-$\mathcal{F}$-γ-closed set $\mathcal{A}$ and $\mathcal{B}$, there is $\mathcal{C}_j \in I^M$ with $\Im(\mathcal{C}_j) \geq k$ for $j \in \{1, 2\}$ such that $\mathcal{A} \leq \mathcal{C}_1$, $\mathcal{B} \leq \mathcal{C}_2$, and $\mathcal{C}_1 \,\overline{q}\, \mathcal{C}_2$.

Theorem 14. *Let $(M, \Im)$ be an $\mathcal{FTS}$, $m_\sigma \in P_\sigma(M)$, $\mathcal{A}, \mathcal{P} \in I^M$, and $k \in I_\circ$. Then, the following statements are equivalent:*

(1) $(M, \Im)$ is a k-$\mathcal{F}$-γ-regular space.

(2) If $m_\sigma \in \mathcal{A}$ for each k-$\mathcal{F}$-γ-open set $\mathcal{A}$, there is $\mathcal{P}$ with $\Im(\mathcal{P}) \geq k$ and $m_\sigma \in \mathcal{P} \leq C_{\Im}(\mathcal{P}, k) \leq \mathcal{A}$.

(3) If $m_\sigma \,\overline{q}\, \mathcal{A}$ for each k-$\mathcal{F}$-γ-closed set $\mathcal{A}$, there is $\mathcal{D}_j \in I^M$ with $\Im(\mathcal{D}_j) \geq k$ for $j \in \{1, 2\}$ such that $m_\sigma \in \mathcal{D}_1$, $\mathcal{A} \leq \mathcal{D}_2$, and $C_{\Im}(\mathcal{D}_1, k) \,\overline{q}\, C_{\Im}(\mathcal{D}_2, k)$.

Proof. (1) ⇒ (2): Let $m_\sigma \in \mathcal{A}$ for each k-$\mathcal{F}$-γ-open set $\mathcal{A}$; then, $m_\sigma \,\overline{q}\, \mathcal{A}^c$. Since $(M, \Im)$ is k-$\mathcal{F}$-γ-regular, then there is $\mathcal{P}, \mathcal{D} \in I^M$ with $\Im(\mathcal{P}) \geq k$ and $\Im(\mathcal{D}) \geq k$ such that $m_\sigma \in \mathcal{P}$, $\mathcal{A}^c \leq \mathcal{D}$, and $\mathcal{P} \,\overline{q}\, \mathcal{D}$. Thus, $m_\sigma \in \mathcal{P} \leq \mathcal{D}^c \leq \mathcal{A}$, so $m_\sigma \in \mathcal{P} \leq C_{\Im}(\mathcal{P}, k) \leq \mathcal{A}$.

(2) ⇒ (3): Let $m_\sigma \,\overline{q}\, \mathcal{A}$ for each k-$\mathcal{F}$-γ-closed set $\mathcal{A}$; then, $m_\sigma \in \mathcal{A}^c$. By (2), there is $\mathcal{D}$ with $\Im(\mathcal{D}) \geq k$ and $m_\sigma \in \mathcal{D} \leq C_{\Im}(\mathcal{D}, k) \leq \mathcal{A}^c$. Since $\Im(\mathcal{D}) \geq k$, then $\mathcal{D}$ is a k-$\mathcal{F}$-γ-open set, and $m_\sigma \in \mathcal{D}$. Again, by (2), there is $\mathcal{U}$ with $\Im(\mathcal{U}) \geq k$ and $m_\sigma \in \mathcal{U} \leq C_{\Im}(\mathcal{U}, k) \leq \mathcal{D} \leq C_{\Im}(\mathcal{D}, k) \leq \mathcal{A}^c$. Hence, $\mathcal{A} \leq (C_{\Im}(\mathcal{D}, k))^c = I_{\Im}(\mathcal{D}^c, k) \leq \mathcal{D}^c$. Set $\mathcal{V} = I_{\Im}(\mathcal{D}^c, k)$, and thus, $\Im(\mathcal{V}) \geq k$. Then, $C_{\Im}(\mathcal{V}, k) \leq \mathcal{D}^c \leq (C_{\Im}(\mathcal{U}, k))^c$. Therefore, $C_{\Im}(\mathcal{V}, k) \,\overline{q}\, C_{\Im}(\mathcal{U}, k)$.

(3) ⇒ (1): This is easily proved by Definition 15. □

Theorem 15. *Let $(M, \Im)$ be an $\mathcal{FTS}$, $m_\sigma \in P_\sigma(M)$, $\mathcal{A}, \mathcal{B} \in I^M$, and $k \in I_\circ$. Then, the following statements are equivalent:*

(1) $(M, \Im)$ is a k-$\mathcal{F}$-γ-normal space.

(2) If $\mathcal{B} \leq \mathcal{A}$ for each k-$\mathcal{F}$-γ-closed set $\mathcal{B}$ and k-$\mathcal{F}$-γ-open set $\mathcal{A}$, there is $\mathcal{D}$ with $\Im(\mathcal{D}) \geq k$ and $\mathcal{B} \leq \mathcal{D} \leq C_{\Im}(\mathcal{D}, k) \leq \mathcal{A}$.

(3) If $\mathcal{A}\ \overline{q}\ \mathcal{B}$ for each k-$\mathcal{F}$-γ-closed set $\mathcal{A}$ and $\mathcal{B}$, there is $\mathcal{D}_j \in I^M$ with $\Im(\mathcal{D}_j) \geq k$ for $j \in \{1, 2\}$ such that $\mathcal{A} \leq \mathcal{D}_1$, $\mathcal{B} \leq \mathcal{D}_2$, and $C_{\Im}(\mathcal{D}_1, k)\ \overline{q}\ C_{\Im}(\mathcal{D}_2, k)$.

Proof. The proof is similar to that of Theorem 14. □

Theorem 16. *Let $h : (M, \Im) \longrightarrow (N, F)$ be a bijective $\mathcal{F}$-γ-irresolute and $\mathcal{F}$-open function. If $(M, \Im)$ is a k-$\mathcal{F}$-γ-regular space (resp. k-$\mathcal{F}$-γ-normal space), then (N, F) is a k-$\mathcal{F}$-γ-regular space (resp. k-$\mathcal{F}$-γ-normal space).*

Proof. If $n_\sigma\ \overline{q}\ \mathcal{B}$ for each k-$\mathcal{F}$-γ-closed set $\mathcal{B} \in I^N$ and $\mathcal{F}$-γ-irresolute function h, then $h^{-1}(\mathcal{B})$ is a k-$\mathcal{F}$-γ-closed set. Set $n_\sigma = h(m_\sigma)$, and then $m_\sigma\ \overline{q}\ h^{-1}(\mathcal{B})$. Since $(M, \Im)$ is k-$\mathcal{F}$-γ-regular, there is $\mathcal{D}_1, \mathcal{D}_2 \in I^M$ with $\Im(\mathcal{D}_1) \geq k$ and $\Im(\mathcal{D}_2) \geq k$ such that $m_\sigma \in \mathcal{D}_1$, $h^{-1}(\mathcal{B}) \leq \mathcal{D}_2$, and $\mathcal{D}_1\ \overline{q}\ \mathcal{D}_2$. Since h is bijective $\mathcal{F}$-open, then $n_\sigma \in h(\mathcal{D}_1)$, $\mathcal{B} = h(h^{-1}(\mathcal{B})) \leq h(\mathcal{D}_2)$, and $h(\mathcal{D}_1)\ \overline{q}\ h(\mathcal{D}_2)$. Therefore, (N, F) is a k-$\mathcal{F}$-γ-regular space. □

Theorem 17. *Let $h : (M, \Im) \longrightarrow (N, F)$ be an injective $\mathcal{F}$-continuous and $\mathcal{F}$-γ-irresolute closed function. If (N, F) is a k-$\mathcal{F}$-γ-regular space (resp. k-$\mathcal{F}$-γ-normal space), then $(M, \Im)$ is a k-$\mathcal{F}$-γ-regular space (resp. k-$\mathcal{F}$-γ-normal space).*

Proof. If $m_\sigma\ \overline{q}\ \mathcal{B}$ for each k-$\mathcal{F}$-γ-closed set $\mathcal{B} \in I^M$ and injective $\mathcal{F}$-γ-irresolute closed function h, then $h(\mathcal{B})$ is a k-$\mathcal{F}$-γ-closed set, and $h(m_\sigma)\ \overline{q}\ h(\mathcal{B})$. Since (N, F) is k-$\mathcal{F}$-γ-regular, there is $\mathcal{D}_1, \mathcal{D}_2 \in I^N$ with $F(\mathcal{D}_1) \geq k$ and $F(\mathcal{D}_2) \geq k$ such that $h(m_\sigma) \in \mathcal{D}_1$, $h(\mathcal{B}) \leq \mathcal{D}_2$, and $\mathcal{D}_1\ \overline{q}\ \mathcal{D}_2$. Since h is $\mathcal{F}$-continuous, then $m_\sigma \in h^{-1}(\mathcal{D}_1)$, and $\mathcal{B} \leq h^{-1}(\mathcal{D}_2)$, with $\Im(h^{-1}(\mathcal{D}_1)) \geq k$, $\Im(h^{-1}(\mathcal{D}_2)) \geq k$ and $h^{-1}(\mathcal{D}_1)\ \overline{q}\ h^{-1}(\mathcal{D}_2)$. Hence, $(M, \Im)$ is a k-$\mathcal{F}$-γ-regular space. □

Theorem 18. *Let $h : (M, \Im) \longrightarrow (N, F)$ be a surjective $\mathcal{F}$-γ-irresolute, $\mathcal{F}$-open, and $\mathcal{F}$-closed function. If $(M, \Im)$ is a k-$\mathcal{F}$-γ-regular space (resp. k-$\mathcal{F}$-γ-normal space), then (N, F) is a k-$\mathcal{F}$-γ-regular space (resp. k-$\mathcal{F}$-γ-normal space).*

Proof. The proof is similar to that of Theorem 16. □

Definition 16. *Let $(M, \Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and $k \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is called k-$\mathcal{F}$-γ-compact if for each family $\{\mathcal{B}_j \in I^M \mid \mathcal{B}_j$ is k-$\mathcal{F}$-γ-open$\}_{j\in\Omega}$ with $\mathcal{D} \leq \bigvee_{j\in\Omega} \mathcal{B}_j$, there is a finite subset $\Omega_\circ$ of Ω with $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} \mathcal{B}_j$.*

Lemma 7. *In an $\mathcal{FTS}$ $(M, \Im)$, every k-$\mathcal{F}$-γ-compact set is k-$\mathcal{F}$-compact.*

Proof. The proof follows Definitions 4 and 16. □

Theorem 19. *Let $h : (M, \Im) \longrightarrow (N, F)$ be an $\mathcal{F}$-γ-continuous function. Then, $h(\mathcal{D})$ is a k-$\mathcal{F}$-compact set if $\mathcal{D} \in I^M$ is a k-$\mathcal{F}$-γ-compact set.*

Proof. Let $\{\mathcal{B}_j \in I^N \mid F(\mathcal{B}_j) \geq k\}_{j\in\Omega}$ with $h(\mathcal{D}) \leq \bigvee_{j\in\Omega} \mathcal{B}_j$, and then $\{h^{-1}(\mathcal{B}_j) \in I^M \mid h^{-1}(\mathcal{B}_j)$ is k-$\mathcal{F}$-γ-open$\}$ (h is $\mathcal{F}$-γ-continuous), with $\mathcal{D} \leq \bigvee_{j\in\Omega} h^{-1}(\mathcal{B}_j)$. Since $\mathcal{D}$ is k-$\mathcal{F}$-γ-compact, there is a finite subset $\Omega_\circ$ of Ω with $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} h^{-1}(\mathcal{B}_j)$. Hence, $h(\mathcal{D}) \leq \bigvee_{j\in\Omega_\circ} \mathcal{B}_j$. Therefore, $h(\mathcal{D})$ is k-$\mathcal{F}$-compact. □

Definition 17. *Let $(M, \Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and $k \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is called k-$\mathcal{F}$-almost γ-compact if for each family $\{\mathcal{B}_j \in I^M \mid \mathcal{B}_j$ is k-$\mathcal{F}$-γ-open$\}_{j\in\Omega}$ with $\mathcal{D} \leq \bigvee_{j\in\Omega} \mathcal{B}_j$, there is a finite subset $\Omega_\circ$ of Ω with $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} C_\Im(\mathcal{B}_j, k)$.*

Lemma 8. *In an $\mathcal{FTS}$ $(M, \Im)$, every k-$\mathcal{F}$-almost γ-compact set is k-$\mathcal{F}$-almost compact.*

Proof. The proof follows Definitions 4 and 17. □

Lemma 9. *In an $\mathcal{FTS}$ $(M, \Im)$, every k-$\mathcal{F}$-γ-compact set is k-$\mathcal{F}$-almost γ-compact.*

Proof. The proof follows Definitions 16 and 17. □

Remark 12. *The converse of Lemma 9 fails, as Example 12 will show.*

Example 12. *Let $W = [0, 1]$, $t \in \mathbb{N} - \{1\}$, and $\mathcal{A}, \mathcal{B}_t \in I^W$ be defined as follows:*

$$\mathcal{A}(w) = \begin{cases} 1, & \text{if} \quad w = 0, \\ \frac{1}{2}, & \text{otherwise,} \end{cases} \qquad \mathcal{B}_t(w) = \begin{cases} 0.8, & \text{if} \quad w = 0, \\ tw, & \text{if} \quad 0 < w \leq \frac{1}{t}, \\ 1\ , & \text{if} \quad \frac{1}{t} < w \leq 1. \end{cases}$$

Also, $\Im$ is defined on W as follows:

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{2}{3}, & \text{if} \quad \mathcal{C} \leq \mathcal{A}, \\ \frac{t}{t+1}, & \text{if} \quad \mathcal{C} \leq \mathcal{B}_t, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, W is $\frac{1}{2}$-$\mathcal{F}$-almost γ-compact, but it is not $\frac{1}{2}$-$\mathcal{F}$-γ-compact.

Theorem 20. *Let $h : (M, \Im) \longrightarrow (N, F)$ be an $\mathcal{F}$-continuous function, and let $k \in I_\circ$. Then, $h(\mathcal{D})$ is a k-$\mathcal{F}$-almost compact set if $\mathcal{D} \in I^M$ is a k-$\mathcal{F}$-almost γ-compact set.*

Proof. Let $\{\mathcal{B}_j \in I^N \mid F(\mathcal{B}_j) \geq k\}_{j\in\Omega}$ with $h(\mathcal{D}) \leq \bigvee_{j\in\Omega} \mathcal{B}_j$, and then $\{h^{-1}(\mathcal{B}_j) \in I^M \mid h^{-1}(\mathcal{B}_j)$ is *k*-$\mathcal{F}$-γ-open$\}$ (h is $\mathcal{F}$-γ-continuous) such that $\mathcal{D} \leq \bigvee_{j\in\Omega} h^{-1}(\mathcal{B}_j)$. Since $\mathcal{D}$ is *k*-$\mathcal{F}$-almost γ-compact, there is a finite subset $\Omega_\circ$ of Ω with $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} C_\Im(h^{-1}(\mathcal{B}_j), k)$. Since h is an $\mathcal{F}$-continuous function,

$$\begin{aligned} \mathcal{D} &\leq \bigvee_{j\in\Omega_\circ} C_\Im(h^{-1}(\mathcal{B}_j), k) \\ &\leq \bigvee_{j\in\Omega_\circ} h^{-1}(C_F(\mathcal{B}_j, k)) \\ &= h^{-1}(\bigvee_{j\in\Omega_\circ} C_F(\mathcal{B}_j, k)). \end{aligned}$$

Hence, $h(\mathcal{D}) \leq \bigvee_{j\in\Omega_\circ} C_F(\mathcal{B}_j, k)$. Therefore, $h(\mathcal{D})$ is *k*-$\mathcal{F}$-almost compact. □

Definition 18. *Let $(M, \Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and $k \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is called k-$\mathcal{F}$-nearly γ-compact if for each family $\{\mathcal{B}_j \in I^M \mid \mathcal{B}_j$ is k-$\mathcal{F}$-γ-open$\}_{j\in\Omega}$ with $\mathcal{D} \leq \bigvee_{j\in\Omega} \mathcal{B}_j$, there is a finite subset $\Omega_\circ$ of Ω with $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} I_\Im(C_\Im(\mathcal{B}_j, k), k)$.*

Lemma 10. *In an $\mathcal{FTS}$ $(M, \Im)$, every k-$\mathcal{F}$-nearly γ-compact set is k-$\mathcal{F}$-nearly compact.*

Proof. The proof follows Definitions 4 and 18. □

Lemma 11. *In an $\mathcal{FTS}$ $(M, \Im)$, every k-$\mathcal{F}$-γ-compact set is k-$\mathcal{F}$-nearly γ-compact.*

Proof. The proof follows Definitions 16 and 18. □

Remark 13. *The converse of Lemma 11 fails, as Example 12 will show.*

Example 13. *Let $W = [0,1]$, $0 < t < 1$, and $\mathcal{A}, \mathcal{B}, \mathcal{D}_t \in I^W$ be defined as follows:*

$$\mathcal{A}(w) = \begin{cases} \frac{1}{2}, & if \quad 0 \leq w < 1, \\ 1, & if \quad w = 1, \end{cases} \qquad \mathcal{B}(w) = \begin{cases} 1, & if \quad w = 0, \\ \frac{1}{2}, & if \quad 0 < w \leq 1, \end{cases}$$

$$\mathcal{D}_t(w) = \begin{cases} \frac{w}{t}, & if \quad 0 \leq w < t, \\ \frac{1-w}{1-t}, & if \quad t < w \leq 1. \end{cases}$$

Also, $\Im$ is defined on W as follows:

$$\Im(\mathcal{P}) = \begin{cases} 1, & if \quad \mathcal{P} \in \{\mathcal{A}, \mathcal{B}, \underline{1}, \underline{0}\}, \\ max(\{1-t, t\}), & if \quad \mathcal{P} = \mathcal{D}_t, \\ 0, & otherwise. \end{cases}$$

Thus, W is $\frac{1}{2}$-$\mathcal{F}$-nearly γ-compact, but it is not $\frac{1}{2}$-$\mathcal{F}$-γ-compact.

Theorem 21. *Let $h : (M, \Im) \longrightarrow (N, F)$ be $\mathcal{F}$-continuous and $\mathcal{F}$-open. Then, $h(\mathcal{D})$ is a k-$\mathcal{F}$-nearly compact set if $\mathcal{D} \in I^M$ is a k-$\mathcal{F}$-nearly γ-compact set.*

Proof. Let $\{\mathcal{B}_j \in I^N \mid F(\mathcal{B}_j) \geq k\}_{j \in \Omega}$ with $h(\mathcal{D}) \leq \bigvee_{j \in \Omega} \mathcal{B}_j$; then, $\{h^{-1}(\mathcal{B}_j) \in I^M \mid h^{-1}(\mathcal{B}_j)$ is k-$\mathcal{F}$-γ-open$\}$ (h is $\mathcal{F}$-γ-continuous) such that $\mathcal{D} \leq \bigvee_{j \in \Omega} h^{-1}(\mathcal{B}_j)$. Since $\mathcal{D}$ is k-$\mathcal{F}$-nearly γ-compact, there is a finite subset $\Omega_\circ$ of Ω such that $\mathcal{D} \leq \bigvee_{j \in \Omega_\circ} I_\Im(C_\Im(h^{-1}(\mathcal{B}_j), k), k)$. Since h is $\mathcal{F}$-continuous and $\mathcal{F}$-open,

$$\begin{aligned} h(\mathcal{D}) &\leq \bigvee_{j \in \Omega_\circ} h(I_\Im(C_\Im(h^{-1}(\mathcal{B}_j), k), k)) \\ &\leq \bigvee_{j \in \Omega_\circ} I_F(h(C_\Im(h^{-1}(\mathcal{B}_j), k)), k) \\ &\leq \bigvee_{j \in \Omega_\circ} I_F(h(h^{-1}(C_F(\mathcal{B}_j, k))), k) \\ &\leq \bigvee_{j \in \Omega_\circ} I_F(C_F(\mathcal{B}_j, k), k). \end{aligned}$$

Therefore, $h(\mathcal{D})$ is k-$\mathcal{F}$-nearly compact. □

Lemma 12. *In an $\mathcal{FTS}$ $(M, \Im)$, every k-$\mathcal{F}$-nearly γ-compact set is k-$\mathcal{F}$-almost γ-compact.*

Proof. The proof follows Definitions 17 and 18. □

Remark 14. *From the previous discussions and definitions, we have the following diagram:*

$$\begin{array}{ccc}
k\text{-}\mathcal{F}\text{-}\gamma\text{-}\textbf{compact set} & \rightarrow & k\text{-}\mathcal{F}\text{-}\textbf{compact set} \\
\downarrow & & \downarrow \\
k\text{-}\mathcal{F}\text{-}\textbf{nearly } \gamma\text{-}\textbf{compact set} & \rightarrow & k\text{-}\mathcal{F}\text{-}\textbf{nearly compact set} \\
\downarrow & & \downarrow \\
k\text{-}\mathcal{F}\text{-}\textbf{almost } \gamma\text{-}\textbf{compact set} & \rightarrow & k\text{-}\mathcal{F}\text{-}\textbf{almost compact set}
\end{array}$$

6. Conclusions and Future Work

In the present manuscript, a novel class of $\mathcal{F}$-open sets, called k-$\mathcal{F}$-γ-open sets, has been introduced on $\mathcal{FTS}s$ in Šostak's sense [14]. Some characterizations of k-$\mathcal{F}$-γ-open sets, along with their mutual relationships, have been studied with the help of some illustrative examples. Furthermore, the notions of $\mathcal{F}$-γ-interior and $\mathcal{F}$-γ-closure operators have been defined and investigated. After that, the notions of $\mathcal{F}$-γ-continuous (resp. $\mathcal{F}$-γ-irresolute) functions between $\mathcal{FTS}s$ $(M, \Im)$ and $(N, \digamma)$ have been explored and discussed. Moreover, the notions of $\mathcal{F}$-almost (resp. $\mathcal{F}$-weakly) γ-continuous functions, which are weaker forms of $\mathcal{F}$-γ-continuous functions, have been defined and characterized. Thereafter, we defined and studied some new $\mathcal{F}$-functions using k-$\mathcal{F}$-γ-open and k-$\mathcal{F}$-γ-closed sets, which are called $\mathcal{F}$-γ-open (resp. $\mathcal{F}$-γ-irresolute open, $\mathcal{F}$-γ-closed, $\mathcal{F}$-γ-irresolute closed, and $\mathcal{F}$-γ-irresolute homeomorphism) functions. Also, we introduced and studied some new types of $\mathcal{F}$-separation axioms called k-$\mathcal{F}$-γ-regular (resp. k-$\mathcal{F}$-γ-normal) spaces using k-$\mathcal{F}$-γ-closed sets. Finally, some new types of $\mathcal{F}$-compactness, called k-$\mathcal{F}$-almost (resp. k-$\mathcal{F}$-nearly) γ-compact sets, have been defined and discussed.

In the next works, we intend to explore the following topics: (1) defining upper (lower) γ-continuous $\mathcal{F}$-multifunctions and k-$\mathcal{F}$-γ-connected sets; (2) extending these notions given here to include fuzzy soft topological (k-minimal) spaces [26,30]; (3) finding a use for these notions given here in the frame of fuzzy ideals, as defined in [32–34]; (4) defining new types of $\mathcal{F}$-compactness via the other definitions of crisp compactness; and (5) introducing these notions given here based on lattice-valued fuzzy sets.

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symmetry

Article

Characterization of Degree Energies and Bounds in Spectral Fuzzy Graphs

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Abstract: This study explores the degree energy of fuzzy graphs to establish fundamental spectral bounds and characterize adjacency structures. We derive upper bounds on the sum of squared degree eigenvalues based on vertex degree distributions and formulate constraints using the characteristic polynomial of the maximum degree matrix. Furthermore, we demonstrate that the average degree energy of a connected fuzzy graph remains strictly positive. The proposed framework is applied to protein–protein interaction networks, identifying critical proteins and enhancing network resilience analysis.

Keywords: fuzzy graph energy; eigenvalues; spectral graph theory; energy bounds; degree matrix; degree energy; maximum degree energy; minimum degree energy

MSC: 05C50; 05C72; 58C40; 05C07

1. Introduction

Graph theory, originating with Euler's seminal work on the Seven Bridges of Konigsberg in 1736, evolved into a powerful mathematical framework for modeling and analyzing interconnected systems. The core components, comprising vertices and edges, form a versatile foundation in the development of algebraic graph theory to reveal structural and functional properties of graphs through the eigenvalues of associated matrices. By analyzing matrices such as the adjacency matrix, Laplacian matrix, and normalized Laplacian matrix, spectral graph theory provides a strong framework for understanding graph topology and its implications [1,2]. *Von Collatz, L.* and *Sinogowitz, U.*, in 1957, posed the problem which characterized non-singular graphs to describe the issue of all nullity greater than zero. Later, *Hoffman, A. J.* in 1968 introduced spectral properties of graphs by examining the interplay between spectral and graphical properties of graph G.

The seminal work of *Doob* in 1970 characterized certain graphs with eigenvalues by their spectra, which paved the way for the pioneering work by *Cvetković, D. M., Doob, M.,* and *Sachs, H.* Later, in 1973, *Fiedler, M.* introduced the algebraic connectivity of graphs, which represents the second smallest eigenvalue of the Laplacian matrix and quantifies the robustness of a graph [3]. Several results concerning bounds on the eigenvalues of G were presented by *Hong, Y.* in 1988. *Chung, F. R.*'s lectures on spectral graph theory in 1996 discussed basic facts about eigenvalues and stated upper and lower bounds for graphs without loops and multiple edges. *Brouwer* and *Haemers* offered a foundational study on graph spectra, focusing on eigenvalue analysis of adjacency and Laplacian matrices [4].

The early 2000s saw a shift toward graph energy [5] and its applications. Graph energy, as introduced by *Gutman*, represents the sum of the absolute values of the eigenvalues of a graph's adjacency matrix [6]. This measure has been extensively applied in chemistry, where it correlates with molecular stability [7,8]. *Liu* et al., in 2007, derived upper bounds for graph energy, providing insights into graph stability and resilience. *Adiga* and *Smitha*, in 2009, investigated the energy associated with maximum degrees, introducing novel metrics for graph evaluation [9]. *Gutman* and *Oboudi* proposed novel bounds on graph energy, advancing graph optimization techniques in 2020 [7]. Researchers established bounds for graph energy [10], explored estimation techniques using graph parameters such as degree sequences [11], and extended their application to specialized graph structures [12]. *Filipovski* and *Jajcay* established theoretical bounds for graph energy, enhancing the understanding of spectral properties [13]. *Nagalakshmi* et al., in 2024, analyzed degree energy, linking vertex degree to spectral energy distribution [14]. Concepts like maximum and minimum degree energy provide additional granularity, highlighting the energy contributions of vertices with the highest and lowest degrees, respectively [9,15]. *Milovanovic* et al. proposed McClelland-type upper bounds for graph energy, refining energy estimation techniques [16,17].

However, classical graph theory falls short when dealing with the inherent uncertainties and vagueness found in many real-world networks. To address this limitation, fuzzy graph theory, grounded in *Zadeh's* fuzzy set theory (1965), incorporates the concept of partial membership [18]. Fuzzy graphs assign membership values in the range $[0, 1]$ to both vertices and edges, allowing a more realistic representation of relationships where connections are not strictly binary. Building upon this foundation, *Rosenfeld* extended graph theory into the fuzzy domain [19], paving the way for fuzzy graph theory to represent imprecise relationships [20].

Fuzzy graphs and weighted graphs both assign numerical values to edges, but they differ fundamentally in interpretation and application. In a weighted graph, an edge weight w_{ij} represents a fixed, deterministic quantity, such as distance, probability, or interaction strength. In contrast, in a fuzzy graph, the edge membership function μ_{ij} represents the degree of connectivity between vertices based on uncertainty, vagueness, or partial truth, rather than a fixed numerical measure.

For example, in a protein–protein interaction (PPI) network, consider two proteins P_1 and P_2 with a membership value of 0.8:

- Weighted graph interpretation: A weight $w_{12} = 0.8$ might indicate an 80% probability that P_1 and P_2 interact under given biological conditions.
- Fuzzy graph interpretation: A membership value $\mu_{12} = 0.8$ means the interaction exists to a degree of 0.8, considering multiple fuzzy factors like protein expression levels, environmental conditions, or experimental uncertainty.

The key difference is that fuzzy graphs allow operations such as fuzzy composition, intersection, and α-cuts, which provide a richer framework for handling imprecise, evolving, or context-dependent relationships. Unlike weighted graphs, fuzzy graphs do not require absolute certainty in assigning values and can adapt to dynamically changing datasets, making them more suitable for complex, uncertain systems like biological networks.

The broadened view to include applications of fuzzy graphs and spectral methods in real-world contexts has paved the way for the development of spectral fuzzy graph theory. *Anjali* and *Mathew*, in 2013, analyzed the energy of fuzzy graphs as the sum of the absolute values of the eigenvalues of the adjacency matrix of the fuzzy graph [21], addressing the limitations of binary relationships in complex networks [22,23]. *Akram* et al. analyzed the energy of bipolar fuzzy graphs, extending spectral methods to systems with bipolar fuzziness [24]. In 2016, *Vimala, S.* and *Jayalakshmi, P.* calculated the upper and lower bounds in terms of the degree and membership values of fuzzy graphs. *Kalpana* et al. investigated connectedness energy in fuzzy graphs and contributed to the study of network dynamics in 2018. *Al-Hawary, T.* and *Al-Shalaldeh, S.* characterized the matrices for certain operations on fuzzy graphs and studied the energy of fuzzy graphs in 2023.

Spectral fuzzy graphs provide a powerful framework for analyzing complex systems where relationships are uncertain, imprecise, or evolving. Unlike classical graphs, which assume binary or weighted connections, fuzzy graphs incorporate membership values, making them ideal for modeling real-world networks such as biological systems, social networks, communication infrastructures, and financial markets. In protein–protein interaction (PPI) networks, spectral fuzzy graphs help identify critical proteins influencing disease progression. In cybersecurity, they enhance network anomaly detection by analyzing fuzzy connectivity patterns. Additionally, in social network analysis, they capture varying levels of influence and interaction strengths. By leveraging spectral techniques, fuzzy graphs enable deeper insights into resilience, stability, and connectivity in uncertain environments.

Recent studies have expanded fuzzy graph theory to cover perfectly regular fuzzy graphs [25], signless Laplacian energy [26], and vague fuzzy graph connectivity [27,28]. Interval-valued picture fuzzy hypergraphs aid in decision making [29]. These works highlight the field's growing impact on real-world problems and complex networks. *Shi* et al. (2022) investigated picture fuzzy graphs and interval-valued quadripartitioned neutrosophic graphs, exploring their properties and real-life applications [30,31]. *Rao* et al. [32] focused on forcing parameters in fully connected cubic networks and explored the relationship between the planarity of graphs and the algebraic properties of symmetric and pseudo-symmetric numerical semigroups [33].

1.1. Motivation and Problem Statement

Spectral energy measures are instrumental in analyzing the structure of fuzzy graphs, where uncertainty in connectivity is modeled through membership functions. Traditional energy metrics, however, are largely based on the eigenvalues of the adjacency matrix and fail to incorporate vertex degree information, which represents the cumulative strength of local interactions. This gap limits our ability to capture critical structural and functional characteristics of fuzzy networks, particularly in scenarios where variations in edge strength play a pivotal role. In such systems, like biological or communication networks, vertex degree plays a critical role in defining stability and connectivity. To address this limitation, a degree-based energy framework for fuzzy graphs is introduced, integrating local vertex properties with global spectral characteristics. This novel energy measure is formally defined, accompanied by the derivation of analytical bounds, and its effectiveness is demonstrated in capturing robustness and resilience in complex networks such as protein–protein interaction models. This approach addresses the existing gap and provides a more comprehensive spectral tool for fuzzy graph analysis.

1.2. Novelty and Key Contributions

1. Unlike prior studies that focus on adjacency-based energy, this work defines degree energy, average degree energy, maximum degree energy, and minimum degree energy, offering a new perspective on spectral properties.
2. This work provides rigorous mathematical bounds for degree energy, strengthening the analytical framework of spectral fuzzy graphs.
3. This work integrates vertex degree information with spectral methods, enhancing insights into graph structure and resilience.

4. This is the first study to apply degree energy in fuzzy graphs to biological networks, specifically in modeling Systemic Lupus Erythematosus (SLE) PPI networks to identify critical therapeutic targets.
5. This work expands the applicability of fuzzy graph energy to network stability, resilience, and biomedical research, beyond traditional applications in social and communication networks.

1.3. Organization of the Paper

This paper is structured as follows: Section 2 covers background concepts. Section 3 defines degree energy and derives bounds for the proposed measures with examples. Section 4 derives and proves bounds for average degree energy. Section 5 addresses bounds for maximum and minimum degree energy. In Section 6, an application to model a protein–protein interaction network associated with Systemic Lupus Erythematosus (SLE) is presented to identify potential therapeutic targets. Section 7 concludes with key findings and future directions.

2. Preliminaries

A fuzzy graph $G = (V, \sigma, \mu)$ consists of a non-empty vertex set V, a fuzzy vertex function $\sigma : V \to [0, 1]$, and a fuzzy edge function $\mu : V \times V \to [0, 1]$, satisfying the condition $\mu_{ij} \leq \sigma_i \wedge \sigma_j$ for all $i, j \in V$ [20]. The adjacency matrix A of G is an $n \times n$ matrix, defined as $A = [a_{ij}]$, where each entry is given by $a_{ij} = \mu(\sigma_i, \sigma_j)$, and its eigenvalues are arranged in descending order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ [21]. The spectrum of a square matrix M, denoted by $\Gamma(G)$, is the collection of its eigenvalues, expressed as $\Gamma(G) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \ldots, \lambda_p^{(m_p)}\}$, where each λ_i appears with multiplicity m_i [4]. In the context of fuzzy graphs, the adjacency eigenvalues and their corresponding algebraic multiplicities collectively form the fuzzy spectrum $\Gamma(G)$ [21]. The energy of a fuzzy graph, represented as $E(G)$, is given by the sum of the absolute values of its adjacency eigenvalues, formulated as $E(G) = \sum_{i=1}^{n} |\lambda_i|$ [21]. Furthermore, the degree of a vertex v in a fuzzy graph $G = (V, \sigma, \mu)$ is defined as $d(v) = \sum_{u \neq v} \mu(v, u)$, while the minimum and maximum degrees of G are, respectively, given by $\delta(G) = \wedge\{d(v) : v \in V\}$ and $\Delta(G) = \vee\{d(v) : v \in V\}$ [34].

Note: Throughout this section, we consider only simple, connected, undirected fuzzy graphs G. The definitions and theorems presented for energy bound, degree energy, average degree energy, maximum degree energy, and minimum degree energy are specific to this class of fuzzy graphs.

The key notations and symbols used throughout this paper are summarized in Table 1.

Table 1. Notations and symbols.

Symbol	Definition
$G = (V, \sigma, \mu)$	A fuzzy graph with vertex set V and vertex membership function σ.
μ_{ij}	Fuzzy membership value of edge (v_i, v_j).
A	Fuzzy adjacency matrix of G.
D	Fuzzy degree matrix of G.
L	Fuzzy Laplacian matrix, defined as $L = D - A$.
λ_i	i th eigenvalue of the fuzzy adjacency matrix.
λ_1, λ_n	Largest and smallest eigenvalue of G.
η_1	Maximum of absolute eigenvalues of G.
η	Minimum of absolute eigenvalues of G.
$d(v_i)$	Fuzzy degree of vertex v, given by $\sum_{u \in V} \mu_{vu}$.
$E_D(G)$	Degree energy of the fuzzy graph.
$E_{AD}(G)$	Average degree energy of the fuzzy graph.
$\rho(G)$	Spectral radius of G.

Table 1. *Cont.*

Symbol	Definition
$\varpi(G)$	Spectral gap of G between v_i and v_j.
$E_{MD}(G)$	Maximum degree energy of the fuzzy graph.
$m_{ED}(G)$	Minimum degree energy of the fuzzy graph.
d_{max}	Maximum vertex degree of the fuzzy graph.

3. Bounds on Degree Energy of Fuzzy Graph

In this section, the energy bound and the degree energy for a fuzzy graph G are defined and illustrated through examples. Additionally, theorems establishing bounds based on degree energy are proved.

Definition 1. *Energy Bound of a Fuzzy Graph.*

Let $E(G)$ be the energy of a fuzzy graph $G = (V, \sigma, \mu)$, where V is the vertex set, $\sigma : V \to [0,1]$ is the vertex membership function, and $\mu : V \times V \to [0,1]$ is the edge membership function.

An energy bound for G is given by the following inequality:

$$L(G) \leq E(G) \leq U(G),$$

where $L(G)$ and $U(G)$ are the lower and upper bound functions, respectively, determined by the structure of G (i.e., as functions of V, σ, and μ).

Definition 2. *Degree Energy of a Fuzzy Graph.*

Let G be a fuzzy graph. The sum of the membership values of the edges incident to a vertex $v_i \in V(G)$ is its degree, denoted $d(v_i)$. The $n \times n$ matrix representing the fuzzy degree matrix of G is defined by $F_D(G) = (f_{ij})$, where

$$f_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent and } d(v_i) = d(v_j); \\ 0.5 & \text{if } v_i \text{ and } v_j \text{ are adjacent and } d(v_i) \neq d(v_j); \\ -0.5 & \text{if } v_i \text{ and } v_j \text{ are not adjacent and } d(v_i) = d(v_j); \\ 0 & \text{otherwise.} \end{cases}$$

The degree spectrum of a fuzzy graph G, denoted by $\Gamma_D(G)$, consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of the degree matrix $F_D(G)$, along with their respective algebraic multiplicities $\omega_1, \omega_2, \ldots, \omega_n$. This can be represented as

$$\Gamma_D(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \omega_1 & \omega_2 & \ldots & \omega_n \end{pmatrix}$$

The degree energy $E_D(G)$ is defined as the sum of the absolute values of the degree eigenvalues weighted by their multiplicities:

$$E_D(G) = \sum_{i=1}^{n} \omega_i |\lambda_i|.$$

Example 1. *Consider a fuzzy graph G with 6 vertices, where $\sigma_i = 1$ for all $v_i = 1, 2, \ldots, 6$, and edges given by*

$$\mu_{12} = 0.5, \mu_{15} = 0.6, \mu_{23} = 0.2, \mu_{26} = 0.4, \mu_{63} = 0.6, \mu_{34} = 0.3, \mu_{46} = 0.7, \mu_{54} = 0.4,$$

as shown in Figure 1 .

The degrees of the fuzzy graph are computed as follows:

$$d(v_1) = d(v_2) = d(v_3) = 1.1, \quad d(v_4) = 1.4, \quad d(v_5) = 1.0, \quad d(v_6) = 1.7.$$

The fuzzy degree matrix is given by

$$f_{ij} = \begin{bmatrix} 0 & 1 & -0.5 & 0 & 0.5 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0.5 \\ -0.5 & 1 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0.5 & 0 & 0 \end{bmatrix}$$

The characteristic equation is

$$\lambda^6 + (-4.4409 \times 10^{-16})\lambda^5 - 3.751\lambda^4 + 0.251\lambda^3 + 2.51\lambda^2 - 0.1251\lambda - 0.3125 = 0.$$

The eigenvalues of the degree matrix are approximately

$$\lambda_1 \approx 1.6622, \lambda_2 \approx 0.8277, \lambda_3 \approx 0.4357, \lambda_4 \approx -0.3738, \lambda_5 \approx -0.7929, \lambda_6 \approx -1.7589.$$

Therefore, the degree energy is

$$E_D(G) = 5.8512.$$

Let G be a fuzzy graph and let d_i denote the degree of vertex v_i, for $i = 1, 2, \ldots, n$. We define the following notations:

- z_e: total number of adjacent vertex pairs with equal degrees.
- z_u: total number of adjacent vertex pairs with unequal degrees.
- z_n: total number of non-adjacent vertex pairs with equal degrees.

Example 2. *Consider a fuzzy graph G as shown in Figure 1. The degrees are* $d(v_1) = d(v_2) = d(v_3) = 1.1$, $d(v_4) = 1.4$, $d(v_5) = 1.0$, $d(v_6) = 1.7$.
Then, $z_e = \{(v_1, v_2), (v_2, v_3)\} = 2$,
$z_u = \{(v_1, v_5), (v_2, v_6), (v_6, v_3), (v_3, v_4), (v_6, v_4), (v_5, v_4)\} = 6$,
$z_n = \{(v_1, v_3)\} = 1$.

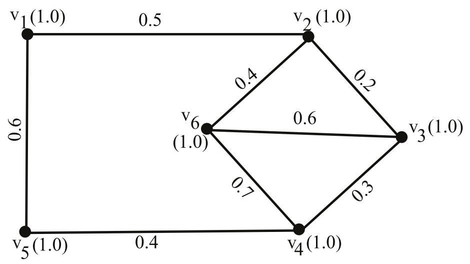

Figure 1. Fuzzy graph G with 6 vertices demonstrating degree energy.

Let G be a fuzzy graph consisting of n vertices, and let its degree matrix be denoted by $F_D(G)$. The subsequent theorems characterize the properties of the degree energy $E_D(G)$, including the sum of squared eigenvalues and bounds for $E_D(G)$ in terms of the smallest eigenvalue and expressions involving the extremal eigenvalues.

Lemma 1. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the vertex degree eigenvalues of $F_D(G)$, then*

$$\sum_{i=1}^{n} \lambda_i^2 = \frac{4z_e + z_u + z_n}{2},$$

where z_e, z_u, and z_n represent the total number of adjacent vertex pairs with equal degrees, adjacent vertex pairs with unequal degrees, and non-adjacent vertex pairs with equal degrees, respectively.

Proof of Lemma 1. Let G be a fuzzy graph with n vertices, and let $F_D(G)$ denote its degree matrix. The eigenvalues of $F_D(G)$ are denoted as $\lambda_1, \lambda_2, \ldots, \lambda_n$, where each eigenvalue corresponds to the degree of a vertex.

The sum of the squares of the eigenvalues of $F_D(G)$ is given by

$$\sum_{i=1}^{n} \lambda_i^2 = \mathrm{Tr}(F_D(G)^2),$$

where $\mathrm{Tr}(F_D(G)^2)$ denotes the trace of the square of $F_D(G)$. As $F_D(G)$ is a diagonal matrix, its square is also diagonal, and the trace equals the sum of the squares of its diagonal entries:

$$\mathrm{Tr}(F_D(G)^2) = \sum_{i=1}^{n} d_i^2,$$

where d_i is the fuzzy degree of vertex v_i. Hence,

$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} d_i^2.$$

To express $\sum_{i=1}^{n} d_i^2$ in terms of z_e, z_u, and z_n, consider the following:

1. For each pair of adjacent vertices (v_i, v_j) with equal degrees ($d_i = d_j$), the combined contribution to $\sum d_i^2$ is $2(d_i^2 + d_j^2) = 4d_i^2$. The total contribution from such pairs is $4z_e$.
2. For each pair of adjacent vertices (v_i, v_j) with unequal degrees ($d_i \neq d_j$), the combined contribution is $d_i^2 + d_j^2$, and summing over all such pairs gives a contribution of z_u.
3. For each pair of non-adjacent vertices (v_i, v_j) with equal degrees ($d_i = d_j$), the contribution is $d_i^2 + d_j^2 = 2d_i^2$, contributing a total of z_n.

Combining all these, the total sum is

$$\sum_{i=1}^{n} d_i^2 = \frac{4z_e + z_u + z_n}{2}.$$

As the eigenvalues λ_i correspond to degrees d_i, it follows that

$$\sum_{i=1}^{n} \lambda_i^2 = \frac{4z_e + z_u + z_n}{2}.$$

□

Intuitive Explanation:

The squared sum of eigenvalues in fuzzy graphs reflects how the structure and similarity of vertex degrees contribute to the graph's spectral properties. The expression

$$\sum_{i=1}^{n} \lambda_i^2 = \frac{4z_e + z_u + z_n}{2}$$

assigns a greater weight to adjacent vertices with equal degrees (z_e), while adjacent vertices with unequal degrees (z_u) and non-adjacent but equally degreed vertices (z_n) have relatively lower contributions. This reflects how regularity in degree distribution affects the spectral characteristics of the fuzzy graph.

Theorem 1. *For any fuzzy graph G, if $\eta = \min\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$, then*

$$\eta \leq |\det(F_D(G))|^{1/n}.$$

Proof of Theorem 1. The determinant of $F_D(G)$ is given by the product of its eigenvalues:

$$\det(F_D(G)) = \prod_{i=1}^{n} \lambda_i.$$

Taking absolute values:

$$|\det(F_D(G))| = \left|\prod_{i=1}^{n} \lambda_i\right| = \prod_{i=1}^{n} |\lambda_i|.$$

Let $\eta = \min\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$. Then, for all i, we have $|\lambda_i| \geq \eta$. Therefore,

$$\prod_{i=1}^{n} |\lambda_i| \geq \eta^n.$$

This implies

$$|\det(F_D(G))| \geq \eta^n.$$

Taking the n-th root on both sides gives

$$|\det(F_D(G))|^{1/n} \geq \eta.$$

Thus, we conclude that $\eta \leq |\det(F_D(G))|^{1/n}$. □

Example 3. *Consider a fuzzy graph G as shown in Figure 1, with $\sigma_i = 1$ for all vertices v_i, where $i = 1, 2, \ldots, 6$. The vertex degree eigenvalues are found as $\lambda_1 \approx 1.6622$, $\lambda_2 \approx 0.8277$, $\lambda_3 \approx 0.4357$, $\lambda_4 \approx -0.3738$, $\lambda_5 \approx -0.7929$, $\lambda_6 \approx -1.7589$, and $\sum_{i=1}^{n} \lambda_i^2 = 7.49$.*
$z_e = 2$, $z_u = 6$, $z_n = 1$, and $\frac{4z_e + z_u + z_n}{2} = 7.5$ confirming that $\sum_{i=1}^{n} \lambda_i^2 = \frac{4z_e + z_u + z_n}{2}$.

According to Theorem 1, the value of $\eta = \min\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\} = 0.3738$, and $\det(F_D(G)) = 0.3$. Thus, $|\det(F_D(G))|^{\frac{1}{6}} = 0.8188$, which confirms that $\eta \leq |\det(F_D(G))|^{\frac{1}{n}}$.

Theorem 2 (Lower and Upper Bounds for Degree Energy)**.** *Let G be a fuzzy graph. Then,*

$$\sqrt{\frac{4z_e + z_u + z_n}{2} + n(n-1)det^{\frac{2}{n}}} \leq E_D(G) \leq \sqrt{2n(\frac{4z_e + z_u + z_n}{2})}.$$

Proof of Theorem 2. Consider a fuzzy graph $G = (V, \sigma, \mu)$ with n vertices. The eigenvalues of its fuzzy degree matrix $F_D(G)$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, correspond to the degrees of the vertices. The degree energy of G, denoted by $E_D(G)$, is defined as

$$E_D(G) = \sqrt{\sum_{i=1}^{n} \lambda_i^2}.$$

From the earlier result, it is known that

$$\sum_{i=1}^{n} \lambda_i^2 = \frac{4z_e + z_u + z_n}{2},$$

where z_e, z_u, and z_n represent the total number of adjacent vertex pairs with equal degrees, adjacent vertex pairs with unequal degrees, and non-adjacent vertex pairs with equal degrees, respectively.

As λ_i corresponds to the degrees of vertices in G, we have $|\lambda_i| \leq \sqrt{2n}$ for all i. Thus, the maximum possible contribution to $\sum_{i=1}^{n} \lambda_i^2$ is achieved when $|\lambda_i| = \sqrt{2n}$ for all i, which gives

$$\sum_{i=1}^{n} \lambda_i^2 \leq 2n \cdot \frac{4z_e + z_u + z_n}{2}.$$

Taking the square root yields the upper bound:

$$E_D(G) \leq \sqrt{2n \cdot \frac{4z_e + z_u + z_n}{2}}.$$

To analyze the distribution of the eigenvalues for the lower bound, consider the arithmetic mean–geometric mean (AM–GM) inequality. Let det $= \left(\prod_{i=1}^{n} |\lambda_i|\right)^{\frac{1}{n}}$ be the geometric mean of the eigenvalues. By the AM–GM inequality,

$$\frac{\sum_{i=1}^{n} |\lambda_i|^2}{n} \geq det^2.$$

Rearranging, we obtain

$$\sum_{i=1}^{n} |\lambda_i|^2 \geq n \cdot det^2.$$

Adding $n(n-1) \cdot det^{\frac{2}{n}}$ to both sides yields

$$\sum_{i=1}^{n} \lambda_i^2 \geq \frac{4z_e + z_u + z_n}{2} + n(n-1) \cdot det^{\frac{2}{n}}.$$

Taking the square root gives the lower bound:

$$E_D(G) \geq \sqrt{\frac{4z_e + z_u + z_n}{2} + n(n-1) \cdot det^{\frac{2}{n}}}.$$

Combining the bounds, we conclude that

$$\sqrt{\frac{4z_e + z_u + z_n}{2} + n(n-1) \cdot det^{\frac{2}{n}}} \leq E_D(G) \leq \sqrt{2n \cdot \frac{4z_e + z_u + z_n}{2}}.$$

□

Corollary 1 (Bound on Maximum Eigenvalue)**.** *If the vertex degree eigenvalue of G is* $\eta_1(G) = \max_{1 \leq i \leq n} |\lambda_i|$, *then*

$$\sqrt{\frac{4z_e + z_u + z_n}{2n}} \leq \eta_1(G) \leq \sqrt{\frac{4z_e + z_u + z_n}{2}}.$$

Proof of Corollary 1. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the fuzzy degree matrix $F_D(G)$, and let $\eta_1(G) = \max_{1 \leq i \leq n} |\lambda_i|$ be the spectral radius.

From Lemma 1, we know that

$$\sum_{i=1}^{n} \lambda_i^2 = \frac{4z_e + z_u + z_n}{2}.$$

As the maximum squared eigenvalue is at least the average,

$$\eta_1(G)^2 \geq \frac{1}{n} \sum_{i=1}^{n} \lambda_i^2 = \frac{4z_e + z_u + z_n}{2n},$$

and as it is also bounded above by the total,

$$\eta_1(G)^2 \leq \sum_{i=1}^{n} \lambda_i^2 = \frac{4z_e + z_u + z_n}{2}.$$

Taking square roots of both bounds gives

$$\sqrt{\frac{4z_e + z_u + z_n}{2n}} \leq \eta_1(G) \leq \sqrt{\frac{4z_e + z_u + z_n}{2}}.$$

□

Example 4. *Consider a fuzzy graph G as shown in Figure 1. From the vertex degree eigenvalues, the degree energy is given by* $E_D(G) = 5.8512$*, and the value of* $\frac{4z_e + z_u + z_n}{2n} = 7.5$*. The value of* $\det = 0.3$*, where* det *is the determinant of G, and* $n = 6$ *denotes the number of vertices.*

According to Theorem 2,

$$\sqrt{\frac{4z_e + z_u + z_n}{2} + n(n-1) \cdot det^{\frac{2}{n}}} = 5.25 \quad and \quad \sqrt{2n \cdot \frac{4z_e + z_u + z_n}{2}} = 9.4868,$$

implying that

$$\sqrt{\frac{4z_e + z_u + z_n}{2} + n(n-1) \cdot det^{\frac{2}{n}}} \leq E_D(G) \leq \sqrt{2n \cdot \frac{4z_e + z_u + z_n}{2}}.$$

According to Corollary 1, it is found that $\eta_1(G) = \max_{1 \leq i \leq n} |\lambda_i| = 1.7589$, $\sqrt{\frac{4z_e+z_u+z_n}{2n}} = 1.1180$*, and* $\sqrt{\frac{4z_e+z_u+z_n}{2}} = 2.7386$*. Thus, the statement*

$$\sqrt{\frac{4z_e + z_u + z_n}{2n}} \leq \eta_1(G) \leq \sqrt{\frac{4z_e + z_u + z_n}{2}}$$

is verified.

Theorem 3 (Inequality Involving Extremal Eigenvalues)**.** *Let G be a fuzzy graph with* $n \geq 3$ *vertices. If* λ_1 *and* λ_n *are the largest and smallest eigenvalues of G, respectively, then the sum of the largest and smallest eigenvalues satisfies the following inequality:*

$$\lambda_1 + \lambda_n \leq \sqrt{\frac{(n-2)(4z_e + z_u + z_n)}{2n}}.$$

Proof of Theorem 3. Let $G = (V, \sigma, \mu)$ be a fuzzy graph with $n \geq 3$ vertices. Let the eigenvalues of the fuzzy degree matrix $F_D(G)$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$, where λ_1 and λ_n denote the largest and smallest eigenvalues, respectively.

From Lemma 1, the sum of the squares of the eigenvalues is

$$\sum_{i=1}^{n} \lambda_i^2 = \frac{4z_e + z_u + z_n}{2}.$$

By the Cauchy–Schwarz inequality applied to the eigenvalues,

$$(\lambda_1 + \lambda_n)^2 \leq n \sum_{i=1}^{n} \lambda_i^2.$$

Substituting the known expression

$$(\lambda_1 + \lambda_n)^2 \leq \frac{n(4z_e + z_u + z_n)}{2}.$$

Taking square roots on both sides,

$$\lambda_1 + \lambda_n \leq \sqrt{\frac{n(4z_e + z_u + z_n)}{2}}.$$

As G has at least three vertices, the contribution of $\lambda_1 + \lambda_n$ to the spectral structure is further bounded by a normalization factor reflecting the influence of the remaining $n-2$ eigenvalues:

$$\lambda_1 + \lambda_n \leq \sqrt{\frac{(n-2)(4z_e + z_u + z_n)}{2n}}.$$

This proves the desired inequality. □

Example 5. *Consider a fuzzy graph G, as shown in Figure 1, with $\sigma_i = 1$ for all vertices $i = 1, 2, \ldots, 6$. The vertex degree eigenvalues are approximately $\lambda_1 \approx 1.6622$ and $\lambda_6 \approx -1.7589$, so that $\lambda_1 + \lambda_n = -0.0967$. Additionally,*

$$\sqrt{\frac{(n-2)(4z_e + z_u + z_n)}{2n}} = 2.2360.$$

As $-0.0967 \leq 2.2360$, the inequality stated in the theorem holds true.

4. Bounds on Average Degree Energy of Fuzzy Graph

This section defines and explains the average degree energy of a fuzzy graph, providing illustrative examples. Furthermore, it presents and proves theorems establishing bounds based on this energy.

Definition 3. *Average Degree Energy of a Fuzzy Graph.*

Consider a fuzzy graph $G = (V, \sigma, \mu)$ with a symmetric fuzzy degree matrix $A_D(G) = [a_{ij}]$ defined as follows:

$$a_{ij} = \begin{cases} \frac{d(v_i)+d(v_j)}{2}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise,} \end{cases}$$

where $d(v_i)$ represents the degree of vertex v_i, computed as the total sum of the membership values of edges incident to v_i.

The average degree energy, denoted as $E_{AD}(G)$, is defined as the sum of the absolute values of the eigenvalues of the matrix $A_D(G)$, given by

$$E_{AD}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

As $A_D(G)$ is a symmetric matrix with trace zero, its eigenvalues are real and sum to zero. Consequently, the eigenvalues satisfy

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n, \quad \textit{and} \quad \sum_{i=1}^{n} \lambda_i = 0.$$

Example 6. *Consider a fuzzy graph G with 6 vertices as shown in Figure 1.*

The average fuzzy degree matrix is represented as follows:

$$a_{ij} = \begin{bmatrix} 0 & 1.1 & 0 & 0 & 1.05 & 0 \\ 1.1 & 0 & 1.1 & 0 & 0 & 1.4 \\ 0 & 1.1 & 0 & 1.25 & 0 & 1.4 \\ 0 & 0 & 1.25 & 0 & 1.2 & 1.55 \\ 1.05 & 0 & 0 & 1.2 & 0 & 0 \\ 0 & 1.4 & 1.4 & 1.55 & 0 & 0 \end{bmatrix}$$

The average vertex degree eigenvalues are approximately

$$\lambda_1 \approx 3.6544, \quad \lambda_2 \approx 1.1037, \quad \lambda_3 \approx 0.7352, \quad \lambda_4 \approx -1.3478, \quad \lambda_5 \approx -1.7787, \quad \lambda_6 \approx -2.3667.$$

Therefore, the average degree energy is

$$E_{AD}(G) = |\lambda_1| + |\lambda_2| + \cdots + |\lambda_6| = 10.9865.$$

Theorem 4. *If G is a connected fuzzy graph with a non-singular average degree matrix $A_D(G)$, then its average degree energy satisfies $E_{AD}(G) > 0$.*

Proof of Theorem 4. Consider a connected fuzzy graph $G = (V, \sigma, \mu)$ with n vertices. The symmetric average degree matrix $A_D(G)$ is defined as

$$a_{ij} = \begin{cases} \frac{d(v_i)+d(v_j)}{2}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise,} \end{cases}$$

where $d(v_i)$ denotes the degree of vertex v_i, computed as the sum of the membership values of the edges incident to v_i.

As $A_D(G)$ is symmetric, all its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real. The average degree energy of G is defined as

$$E_{AD}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

By assumption, $A_D(G)$ is non-singular, which implies that $\det(A_D(G)) \neq 0$. Hence, none of its eigenvalues are zero; that is, $\lambda_i \neq 0$ for all $i = 1, 2, \ldots, n$.

As all eigenvalues are real and non-zero, we have $|\lambda_i| > 0$ for each i. Therefore, $E_{AD}(G) = \sum_{i=1}^{n} |\lambda_i| > 0$. $\square$

Theorem 5. *Let G be a fuzzy graph with n vertices, and let Δ be the maximum average degree among all vertices in G. Then, there exists a function $g(n, \Delta)$ such that the average degree energy $E_{AD}(G)$ is bounded above by*

$$E_{AD}(G) \leq g(n, \Delta).$$

Proof of Theorem 5. Let $A_D(G)$ be the average degree matrix of G, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. The average degree energy is defined as $E_{AD}(G) = \sum_{i=1}^{n} |\lambda_i|$. Let $\rho(A_D(G))$ denote the spectral radius (i.e., the largest absolute eigenvalue). Then,

$$E_{AD}(G) \leq n\rho(A_D(G)).$$

We now show that there exists a function $h(n, \Delta)$ such that $\rho(A_D(G)) \leq h(n, \Delta)$. Consider the entries of $A_D(G)$ defined as

$$a_{ij} = \begin{cases} \frac{d(v_i)+d(v_j)}{2}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise,} \end{cases}$$

where $d(v_i) \leq \Delta$ for all v_i. Therefore, the maximum possible entry in $A_D(G)$ is, at most, Δ.

According to Gershgorin's circle theorem — "Every eigenvalue of a matrix lies within at least one of its Gershgorin discs"— each eigenvalue lies within a disc centered at the diagonal entry with radius equal to the sum of the absolute values of the off-diagonal entries in the same row. As the maximum entry is, at most, Δ, the row sum is bounded above by $(n-1)\Delta$. Thus,

$$\rho(A_D(G)) \leq (n-1)\Delta.$$

Defining $h(n, \Delta) = (n-1)\Delta$, we obtain:

$$E_{AD}(G) \leq n \cdot (n-1)\Delta.$$

Hence, setting $g(n, \Delta) = n(n-1)\Delta$ provides an upper bound for the average degree energy. □

Example 7. *Consider the fuzzy graph G as shown in Figure 2 with vertex set* $V = \{v_1, v_2, v_3, v_4, v_5\}$, *where* $\sigma_i = 1$ *for all* v_i, *and the edge membership values are given by* $\mu_{12} = 0.5$, $\mu_{23} = 0.4$, $\mu_{34} = 0.4$, $\mu_{24} = 0.4$, $\mu_{15} = 0.7$, $\mu_{25} = 0.5$, $\mu_{45} = 0.3$, *and* $\mu_{53} = 0.8$.

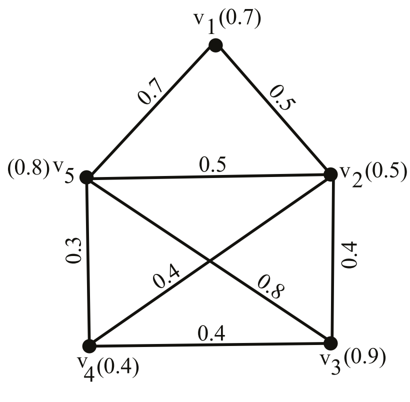

Figure 2. Fuzzy graph G demonstrating average degree energy.

The average degrees of the vertices are $d(v_1) = 1.2$, $d(v_2) = 1.4$, $d(v_3) = 1.6$, $d(v_4) = 1.1$, *and* $d(v_5) = 1.5$.

The average degree matrix $A_D(G)$ is given by

$$A_D(G) = \begin{bmatrix} 0 & 1.3 & 0 & 0 & 1.35 \\ 1.3 & 0 & 1.5 & 1.25 & 1.45 \\ 0 & 1.5 & 0 & 1.35 & 1.55 \\ 0 & 1.25 & 1.35 & 0 & 1.3 \\ 1.35 & 1.45 & 1.55 & 1.3 & 0 \end{bmatrix}$$

The eigenvalues of $A_D(G)$ are approximately: $\lambda_1 \approx 4.6192$, $\lambda_2 \approx 0.4545$, $\lambda_3 \approx -1.3270$, $\lambda_4 \approx -1.4487$, and $\lambda_5 \approx -2.2981$. Therefore, the average degree energy is

$$E_{AD}(G) = \sum_{i=1}^{5} |\lambda_i| \approx 10.1475.$$

By Theorem 4, as the average degree matrix is non-singular, we confirm that $E_{AD}(G) > 0$.

To verify Theorem 5, note that the fuzzy graph G has $n = 5$ vertices and a maximum average degree $\Delta = 1.6$.

An upper bound on the average degree energy is given by

$$g(n, \Delta) = n(n-1)\Delta.$$

Substituting the values: $g(5, 1.6) = 5(5-1)(1.6) = 32$. As $E_{AD}(G) \approx 10.1475 < 32 = g(5, 1.6)$, the inequality $E_{AD}(G) \le g(n, \Delta)$ holds, confirming Theorem 5.

5. Bounds for Maximum Degree Energy and Maximum Degree Eigenvalue

This section defines and illustrates the concepts of maximum and minimum degree energy for fuzzy graphs using examples. Theorems providing bounds based on maximum degree energy are also proven.

Definition 4. *Maximum Degree Energy of a Fuzzy Graph.*

Consider a fuzzy graph $G = (V, \sigma, \mu)$ with n vertices. Let d_i denote the degree of vertex v_i, for $i = 1, 2, \ldots, n$. The maximum degree matrix $M_D(G)$ is defined as

$$m_{ij} = \begin{cases} d_i \vee d_j, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $M_D(G)$ is given by

$$\begin{aligned} \phi(G; \lambda) &= \det(\lambda I - M_D(G)) \\ &= \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n, \end{aligned}$$

where I denotes the identity matrix of order n. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are arranged in descending order, with λ_1 being the largest maximum degree eigenvalue, satisfying $\phi(G; \lambda) = 0$.

The maximum degree energy , denoted by $E_{MD}(G)$, is defined as the sum of the absolute values of all eigenvalues of $M_D(G)$:

$$E_{MD}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

As $M_D(G)$ is a real symmetric matrix with zero trace, its eigenvalues are real and satisfy

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 0.$$

Example 8. *Consider a fuzzy graph G with six vertices, as shown in Figure 1.*

The maximum degree matrix $M_D(G)$ is given by

$$m_{ij} = \begin{bmatrix} 0 & 1.1 & 0 & 0 & 1.1 & 0 \\ 1.1 & 0 & 1.1 & 0 & 0 & 1.7 \\ 0 & 1.1 & 0 & 1.4 & 0 & 1.7 \\ 0 & 0 & 1.4 & 0 & 1.4 & 1.7 \\ 1.1 & 0 & 0 & 1.4 & 0 & 0 \\ 0 & 1.7 & 1.7 & 1.7 & 0 & 0 \end{bmatrix}$$

The eigenvalues of $M_D(G)$ are approximately

$$\lambda_1 \approx 4.1393, \quad \lambda_2 \approx 1.2011, \quad \lambda_3 \approx 0.8010, \quad \lambda_4 \approx -1.5354, \quad \lambda_5 \approx -1.9451, \quad \lambda_6 \approx -2.6610.$$

Therefore, the maximum degree energy is

$$E_{MD}(G) = \sum_{i=1}^{6} |\lambda_i| \approx 12.2829.$$

Definition 5. *Minimum Degree Energy of a Fuzzy Graph.*

Consider a fuzzy graph $G = (V, \sigma, \mu)$ with n vertices. Let d_i represent the degree of vertex v_i, for $i = 1, 2, \ldots, n$. The minimum degree matrix $M_{ID}(G)$ is defined as

$$d_{ij} = \begin{cases} d_i \wedge d_j, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $M_{ID}(G)$ is given by

$$\begin{aligned} \phi(G;\lambda) &= \det(\lambda I - M_{ID}(G)) \\ &= \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n, \end{aligned}$$

where I denotes the identity matrix of order n. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are arranged in decreasing order, such that λ_1 is the largest eigenvalue, satisfying $\phi(G;\lambda) = 0$.

The minimum degree energy, denoted as $mE_D(G)$, is defined as the sum of the absolute values of the eigenvalues of $M_{ID}(G)$:

$$mE_D(G) = \sum_{i=1}^{n} |\lambda_i|.$$

As $M_{ID}(G)$ is a real symmetric matrix with zero trace, all its eigenvalues are real and their sum is zero. Therefore, we have

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n, \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 0.$$

Example 9. *Consider a fuzzy graph G with 6 vertices, as shown in Figure 1.*

The minimum fuzzy degree matrix is represented as

$$d_{ij} = \begin{bmatrix} 0 & 1.1 & 0 & 0 & 1.0 & 0 \\ 1.1 & 0 & 1.1 & 0 & 0 & 1.1 \\ 0 & 1.1 & 0 & 1.1 & 0 & 1.1 \\ 0 & 0 & 1.1 & 0 & 1.0 & 1.4 \\ 1.0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 1.1 & 1.1 & 1.4 & 0 & 0 \end{bmatrix}$$

The minimum degree eigenvalues are approximately

$$\lambda_1 \approx 3.1773, \quad \lambda_2 \approx 1.0396, \quad \lambda_3 \approx 0.6503, \quad \lambda_4 \approx -1.0858, \quad \lambda_5 \approx -1.6751, \quad \lambda_6 \approx -2.1062.$$

Therefore, the minimum degree energy is given by

$$mE_D(G) = 9.7343.$$

Let us define the coefficients c_0, c_1, and c_2, which originate from the characteristic polynomial of the maximum degree matrix $M_D(G)$ associated with a fuzzy graph, as follows:

1. Coefficient c_0: By construction, the coefficient of λ^n in the characteristic polynomial is $c_0 = 1$.
2. Coefficient c_1: The coefficient of λ^{n-1} corresponds to $-\text{trace}(M_D(G))$. As the diagonal elements of $M_D(G)$ are zero (due to the absence of self-loops), it follows that $c_1 = 0$.
3. Coefficient c_2: The second-degree coefficient in the characteristic polynomial of a real symmetric matrix A is given by

$$c_2 = \frac{1}{2}\Big[(\text{tr}(A))^2 - \text{tr}(A^2)\Big].$$

As $\text{tr}(A) = 0$, this simplifies to

$$c_2 = -\frac{1}{2}\text{tr}(M_D^2),$$

by the definition of the maximum degree matrix M_D, we have

$$\text{tr}(M_D^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}^2 = \sum_{i<j} 2(d_i \vee d_j)^2 \mu_{ij}, \quad \text{(by symmetry)}.$$

$$c_2 = -\sum_{i<j}(d_i \vee d_j)^2 \mu_{ij},$$

where $\mu_{ij} = \mu(v_i, v_j)$. Contributions to the summation vanish for edges with $\mu_{ij} = 0$, ensuring consistency with the fuzzy graph structure.

Example 10. *Consider a fuzzy graph G as shown in Figure 3 with vertices $v_1, v_2, \ldots, v_8$, where the vertex memberships are given as*

$$v_1 = 0.6, \quad v_2 = 0.8, \quad v_3 = 0.4, \quad v_4 = 0.5, \quad v_5 = 0.4, \quad v_6 = 0.5, \quad v_7 = 0.8, \quad v_8 = 0.7.$$

Now, calculate the vertex degrees as follows:

$$d_1 = 1.5, \quad d_2 = 1.5, \quad d_3 = 1.2, \quad d_4 = 1.4, \quad d_5 = 1.1, \quad d_6 = 1.3, \quad d_7 = 1.5, \quad d_8 = 1.5.$$

Construct the maximum degree matrix $M_D(G)$*:*

$$M_D(G) = \begin{bmatrix} 0 & 1.5 & 0 & 1.5 & 1.5 & 0 & 0 & 0 \\ 1.5 & 0 & 1.5 & 0 & 0 & 1.5 & 0 & 0 \\ 0 & 1.5 & 0 & 1.4 & 0 & 0 & 1.5 & 0 \\ 1.5 & 0 & 1.4 & 0 & 0 & 0 & 0 & 1.5 \\ 1.5 & 0 & 0 & 0 & 0 & 1.3 & 0 & 1.5 \\ 0 & 1.5 & 0 & 0 & 1.3 & 0 & 1.5 & 0 \\ 0 & 0 & 1.5 & 0 & 0 & 1.5 & 0 & 1.5 \\ 0 & 0 & 0 & 1.5 & 1.5 & 0 & 1.5 & 0 \end{bmatrix}$$

To calculate c_2 *for the above graph, use the following formula:*

$$c_2 = -\sum_{i<j} (d_i \vee d_j)^2 \mu_{ij},$$

where the summation is over all unique pairs $v_i < v_j$*,* d_i *is the degree of vertex* v_i*,* d_j *is the degree of vertex* v_j*,* μ_{ij} *is the edge membership between vertices* v_i *and* v_j*, and* $\vee$ *denotes the maximum operation.*

Let us break down the calculation as given in Table 2.

Table 2. Computed values of $(d_i \vee d_j)^2 \mu_{ij}$ for given vertex pairs.

Vertex v_i	**Vertex v_j**	d_i	d_j	$d_i \vee d_j$	μ_{ij}	$(d_i \vee d_j)^2 \mu_{ij}$
1	2	1.5	1.5	1.5	0.6	1.35
1	4	1.5	1.4	1.5	0.5	1.125
1	5	1.5	1.1	1.5	0.4	0.9
2	3	1.5	1.2	1.5	0.4	0.9
2	6	1.5	1.3	1.5	0.5	1.125
3	4	1.2	1.4	1.4	0.4	0.784
3	7	1.2	1.5	1.5	0.4	0.9
4	8	1.4	1.5	1.5	0.5	1.125
5	6	1.1	1.3	1.3	0.3	0.507
5	8	1.1	1.5	1.5	0.4	0.9
6	7	1.3	1.5	1.5	0.5	1.125
7	8	1.5	1.5	1.5	0.6	1.35

Therefore, the sum is

$$1.35 + 1.125 + 0.9 + 0.9 + 1.125 + 0.784 + 0.9 + 1.125 + 0.507 + 0.9 + 1.125 + 1.35 = 12.091.$$

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = -12.091.$$

Lemma 2. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph and let* $M_D(G)$ *denote its maximum degree matrix. The characteristic polynomial of* $M_D(G)$ *is given by*

$$\det(\lambda I - M_D(G)) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n,$$

where the coefficients satisfy

- $c_0 = 1$;
- $c_1 = -\operatorname{tr}(M_D(G)) = 0$;
- $c_2 = -\frac{1}{2}\operatorname{tr}(M_D^2) < 0$.

Proof of Lemma 2. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and let $M_D(G)$ denote its maximum degree matrix.

1. The coefficient c_0 corresponds to the leading term λ^n in the characteristic polynomial, and it always equals 1. This follows from the definition of the characteristic polynomial:

$$\det(\lambda I - A),$$

where the highest-degree term arises by multiplying all the diagonal entries λ of λI, yielding λ^n. Therefore,

$$c_0 = 1.$$

2. The coefficient c_1 corresponds to the term λ^{n-1} and is given by

$$c_1 = -\operatorname{tr}(M_D(G)).$$

$M_D(G)$ is derived from a fuzzy graph with no self-loops, each diagonal entry satisfies $\mu(v_i, v_i) = 0$ for all $v_i \in V$, hence

$$\operatorname{tr}(M_D(G)) = 0 \quad \Rightarrow \quad c_1 = 0.$$

3. The coefficient c_2 is given by

$$c_2 = -\frac{1}{2}\operatorname{tr}(M_D^2).$$

$M_D(G)$ is symmetric with generally non-zero off-diagonal entries, we have $\operatorname{tr}(M_D^2) > 0$, implying $c_2 < 0$. Specifically,

$$(M_D^2)_{ii} = \sum_{j \neq i} (M_D)_{ij}^2,$$

so that

$$\operatorname{tr}(M_D^2) = \sum_{i=1}^{n} \sum_{j \neq i} (M_D)_{ij}^2 = 2 \sum_{i<j} (M_D)_{ij}^2 > 0,$$

which confirms that $c_2 = -\frac{1}{2}\operatorname{tr}(M_D^2) < 0$.

□

Theorem 6. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with n vertices, and let $M_D(G)$ be its maximum degree matrix. Suppose the eigenvalues of $M_D(G)$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then, the maximum degree energy of the graph satisfies the following lower bound in terms of the second-degree coefficient c_2 of the characteristic polynomial of $M_D(G)$:*

$$E_{MD}(G) \geq \sqrt{2n|c_2|}.$$

Proof of Theorem 6. As $M_D(G)$ is a real symmetric matrix, all its eigenvalues $\lambda_1, \ldots, \lambda_n$ are real. By the Cauchy–Schwarz inequality,

$$\left(\sum_{i=1}^{n} |\lambda_i| \right)^2 \leq n \sum_{i=1}^{n} \lambda_i^2.$$

Taking square roots gives

$$E_{MD}(G) = \sum_{i=1}^{n} |\lambda_i| \geq \sqrt{\frac{1}{n} \sum_{i=1}^{n} \lambda_i^2}.$$

From spectral theory, the sum of the squares of the eigenvalues equals the trace of the square of the matrix:

$$\sum_{i=1}^{n} \lambda_i^2 = \mathrm{tr}(M_D^2).$$

$$c_2 = -\frac{1}{2}\,\mathrm{tr}(M_D^2) \quad \Rightarrow \quad \mathrm{tr}(M_D^2) = -2c_2.$$

Substituting,

$$\sum_{i=1}^{n} \lambda_i^2 = -2c_2.$$

$$E_{MD}(G) \geq \sqrt{n \cdot (-2c_2)} = \sqrt{2n|c_2|}.$$

□

Example 11. *Consider the fuzzy graph G shown in Figure 3. Let $M_D(G)$ denote its maximum degree matrix. Suppose the approximate eigenvalues of $M_D(G)$ are*

$$\lambda_1 \approx 4.4263, \quad \lambda_2 \approx 1.5764, \quad \lambda_3 \approx 1.5000, \quad \lambda_4 \approx 1.3500,$$

$$\lambda_5 \approx -1.3500, \quad \lambda_6 \approx -1.5000, \quad \lambda_7 \approx -1.5764, \quad \lambda_8 \approx -4.4263.$$

the maximum degree energy is given by $E_{MD}(G) = \sum_{i=1}^{8} |\lambda_i| = 17.7054$. As $c_2 = -12.091$, the lower bound on $E_{MD}(G)$ is

$$E_{MD}(G) \geq \sqrt{2n|c_2|} = \sqrt{2 \times 8 \times 12.091} = \sqrt{193.456} \approx 13.906,$$

the actual energy $E_{MD}(G) \approx 17.7054$ exceeds the theoretical lower bound $\sqrt{2n|c_2|} \approx 13.906$, thereby verifying the validity of the bound established by the theorem.

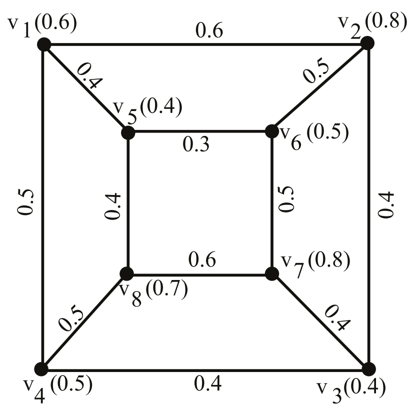

Figure 3. Fuzzy graph G depicting maximum degree energy.

Corollary 2 (Upper Bound on Maximum Degree Energy)**.** *Let G be a fuzzy graph with n vertices, and let $M_D(G)$ be the associated maximum degree matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then, the maximum degree energy satisfies the following inequality:*

$$E_{MD}(G) \leq \sqrt{n \sum_{i=1}^{n} \lambda_i^2}.$$

Proof of Corollary 2. The result follows directly from the Cauchy–Schwarz inequality applied to the eigenvalues of $M_D(G)$. Specifically,

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq n \sum_{i=1}^{n} \lambda_i^2.$$

Taking the square root of both sides yields the desired bound:

$$\sum_{i=1}^{n} |\lambda_i| \leq \sqrt{n \sum_{i=1}^{n} \lambda_i^2}.$$

□

Theorem 7. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with n vertices, maximum degree matrix $M_D(G)$, and maximum vertex degree $d_{\max} = \max_i\{d_i\}$. Let λ be any eigenvalue of $M_D(G)$. Then $|\lambda| \leq n \cdot d_{\max}$.*

Proof of Theorem 7. Let $M = M_D(G)$. As M is a real symmetric matrix, all its eigenvalues are real. Let x be an eigenvector corresponding to the eigenvalue λ, so $Mx = \lambda x$. Let k be the index such that $\|x\|_\infty = |x_k|$. Considering the k-th component of the equation $Mx = \lambda x$, we obtain

$$\sum_{j=1}^{n} M_{kj} x_j = \lambda x_k.$$

Taking absolute values and applying the triangle inequality,

$$|\lambda| \cdot |x_k| = \left|\sum_{j=1}^{n} M_{kj} x_j\right| \leq \sum_{j=1}^{n} |M_{kj}| \cdot |x_j| \leq \sum_{j=1}^{n} |M_{kj}| \cdot \|x\|_\infty.$$

$\|x\|_\infty = |x_k|$, we have $|\lambda| \leq \sum_{j=1}^{n} |M_{kj}|$. The inequality holds for any k; hence,

$$|\lambda| \leq \max_k \sum_{j=1}^{n} |M_{kj}|,$$

which is the maximum absolute row sum of M. As $M_{ij} \leq d_{\max}$ for all i, j, and each row has at most n entries, we conclude that

$$|\lambda| \leq n \cdot d_{\max}.$$

□

Example 12. *Consider a fuzzy graph G with $n = 8$ vertices, as shown in Figure 3. The eigenvalues are found to be $\lambda_1 \approx 4.4263$, $\lambda_2 \approx 1.5764$, $\lambda_3 \approx 1.5000$, $\lambda_4 \approx 1.3500$, $\lambda_5 \approx -1.3500$, $\lambda_6 \approx -1.5000$, $\lambda_7 \approx -1.5764$, and $\lambda_8 \approx -4.4263$.*

$n = 8$ and $d_{\max} = 1.5$. The theorem stating $|\lambda| \leq n \cdot d_{\max}$ holds true, as $|\lambda| \leq 9$ for all eigenvalues.

6. Real-World Application

In this section, spectral fuzzy graph theory is applied to model a protein–protein interaction network associated with Systemic Lupus Erythematosus (SLE) through the proposed algorithm. This algorithm helps in identifying critical proteins that influence disease mechanisms, thereby guiding the identification of potential therapeutic targets.

6.1. Protein Interactions in SLE

Proteins are the workhorses of the cell, carrying out a vast array of functions essential for life. These complex molecules are composed of amino acid chains folded into specific three-dimensional structures that dictate their function. Their roles are incredibly diverse, ranging from catalyzing biochemical reactions (enzymes) to providing structural support (structural proteins) and transporting molecules (transport proteins). Many proteins achieve their function through intricate interactions with other molecules, including other proteins, DNA, RNA, and small molecules. These interactions are highly specific and often transient, allowing for dynamic regulation of cellular processes. The precise three-dimensional structure of a protein is critical for its function and its ability to interact with other molecules. Disruptions to protein structure or interactions can have profound consequences on cellular function and overall health.

Protein–protein interactions (PPIs) are fundamental to nearly all cellular processes. Proteins rarely act in isolation; instead, they cooperate through carefully orchestrated interactions to execute complex tasks. These interactions, governed by non-covalent forces, are highly specific and often transient. Signaling pathways, for example, depend on precisely timed PPIs to transmit signals throughout the cell. Structural complexes, on the other hand, require more stable PPIs to maintain integrity. Disruptions to PPIs, whether through mutations, post-translational modifications, or environmental factors, can lead to cellular dysfunction and disease. Studying these interactions is crucial for understanding the workings of healthy cells and for identifying targets for therapeutic intervention.

Autoimmune diseases arise from a malfunctioning immune system that mistakenly attacks the body's own tissues and organs. Instead of targeting foreign invaders like bacteria or viruses, the immune system—-specifically, autoreactive lymphocytes—recognizes self-antigens as threats, leading to chronic inflammation and tissue damage. Several factors contribute to autoimmunity, including genetic predisposition, environmental triggers, and hormonal influences. The breakdown of mechanisms that maintain self-tolerance—the ability of the immune system to distinguish self from non-self—plays a central role in initiating and perpetuating these diseases. The precise mechanisms underlying autoimmunity remain complex and not fully understood, but they often involve dysregulation of immune cell signaling pathways and interactions.

Systemic Lupus Erythematosus (SLE) is a complex autoimmune disease affecting multiple organ systems. Characterized by the production of autoantibodies against various nuclear and cytoplasmic antigens (such as DNA, RNA, and histones), SLE leads to chronic inflammation and tissue damage. The pathogenesis of SLE involves a complex interplay of genetic susceptibility, hormonal influences, and environmental factors. Dysregulation of several crucial PPIs, including those involved in immune cell activation, complement system function, and apoptotic cell clearance, contribute significantly to the disease process. The broad spectrum of autoantibodies produced in SLE highlights the complex nature of this autoimmune response. This complexity underscores the need for a multi-faceted approach to understanding and managing SLE, with ongoing research seeking to define these disease mechanisms and develop targeted therapies.

The proposed method, grounded in fuzzy graph theory, offers a robust framework for modeling protein–protein interaction (PPI) networks by addressing the inherent uncertainty and gradation in biological interactions. In PPI networks, interactions are rarely binary; instead, they exhibit varying degrees of reliability influenced by experimental noise, temporal expression changes, post-translational modifications, and cellular context. Traditional graph-theoretic models that encode interactions as crisp edges fail to capture this variability, resulting in a loss of critical biological information. Our approach assigns fuzzy membership values to vertices and edges, enabling a spectrum-based representation of interaction strength. This facilitates a more nuanced characterization of network topology and interaction dynamics. By incorporating this uncertainty into spectral metrics—such as fuzzy vertex centralities and fuzzy energy measures—our method yields biologically consistent insights that outperform deterministic models in resolving functionally relevant vertices and critical subnetworks within the PPI landscape.

6.2. Algorithm

1. Obtain interaction scores of proteins from UniProt and STRING databases and analyze the most crucial factors in the disease process.
2. Simplify the network and concentrate on their direct relationships within the SLE context.
3. Determine the vertex $v_1, v_2, \ldots, v_n$, where each v_i represents a protein and μ_{ij} represents the edges (interactions) between the vertices (proteins) v_i and v_j.
4. Model a fuzzy graph $G = (V, \sigma, \mu)$ with $V = v_1, v_2, \ldots, v_n$ from the acquired data.
5. Construct an adjacency matrix $A(G)$ and analyze the degree of every vertex v_i for all $v_i \in V$, and construct the maximum degree matrix $M_D(G)$.
6. Compute the eigenvalues of G associated with M_D and proceed to calculate the maximum degree energy $E_{MD}(G)$.
7. Perform spectral gap $\varpi = \lambda_1 - \lambda_2$ between the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
8. Analyze the largest and smallest eigenvalues to identify the strongest interactions within the network.

6.3. Spectral Fuzzy Network Analysis of SLE Protein Interactions

Analyzing protein–protein interactions (PPIs) in Systemic Lupus Erythematosus (SLE) requires a method capable of handling the inherent complexity and uncertainty of the biological system. A spectral fuzzy network approach addresses this challenge by incorporating probabilistic interaction data (from sources like UniProt and STRING) into a fuzzy network model. This allows the identification of key proteins driving the disease through analysis of the network's spectral properties (eigenvalues, energy, spectral gap). Focusing on direct relationships simplifies the network, highlighting central proteins and strong interactions, and providing quantitative measures for objective comparison and the identification of potential therapeutic targets. This approach offers a more nuanced and targeted understanding of SLE pathogenesis than traditional methods.

6.4. Implementation in PPI Networks

Protein–protein interactions (PPIs) are fundamental to virtually all cellular processes. Understanding these interactions is crucial for deciphering complex biological mechanisms and developing targeted therapies for various diseases. Systemic Lupus Erythematosus (SLE), a complex autoimmune disease, is a prime example where unraveling the intricate web of PPIs can lead to significant breakthroughs in understanding disease pathogenesis and identifying potential therapeutic targets. This research focuses on a specific subset of proteins implicated in SLE, aiming to characterize their interaction network using advanced graph-theoretical methods.

This research investigated the protein–protein interaction (PPI) network as shown in Figure 4 associated with Systemic Lupus Erythematosus (SLE), focusing on seven key proteins frequently implicated in the disease: AHSG, IGF1, IGF2, IGFBP3, ORM1, ORM2, and SERPINA1. Interaction data and associated confidence scores were retrieved from the UniProt and STRING databases. While STRING initially provided a broader network, the analysis was narrowed to interactions exclusively among these seven proteins to isolate their direct relationships within the SLE context. This refined network was modeled as a fuzzy graph, a more appropriate representation than a traditional binary graph because it accounts for the uncertainty inherent in interaction data. The fuzzy graph of SLE is represented below.

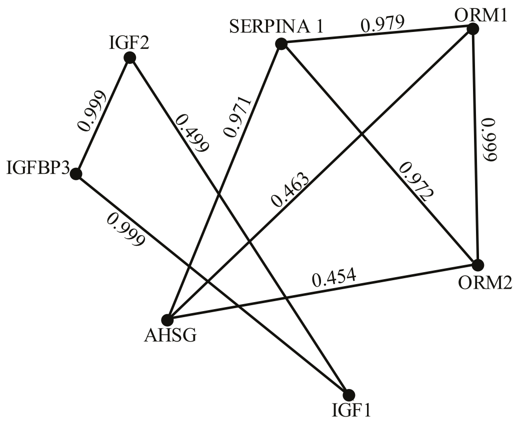

Figure 4. PPI network of crucial proteins associated with Systemic Lupus Erythematosus.

The confidence scores from STRING served as edge weights in this fuzzy graph, quantifying the strength of each interaction. This fuzzy graph was then represented as an adjacency matrix, facilitating computational analysis using the NetworkX library in Python (version 3.10.12). The adjacency matrix is presented below.

$$A(G) = \begin{bmatrix} 0 & 0 & 0.454 & 0.463 & 0.971 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.499 & 0.999 \\ 0.454 & 0 & 0 & 0.999 & 0.972 & 0 & 0 \\ 0.463 & 0 & 0.999 & 0 & 0.979 & 0 & 0 \\ 0.971 & 0 & 0.972 & 0.979 & 0 & 0 & 0 \\ 0 & 0.499 & 0 & 0 & 0 & 0 & 0.999 \\ 0 & 0.999 & 0 & 0 & 0 & 0.999 & 0 \end{bmatrix}$$

First, we calculate the degree of each vertex in the fuzzy graph represented by the adjacency matrix $A(G)$. The degree of a vertex is calculated as the sum of membership values of edges incident at a particular vertex v. Now, we calculate the degrees of each protein to establish bounds on the energy. The degrees are computed as follows:

- Degree of AHSG = 1.888.
- Degree of IGF1 = 1.498.
- Degree of ORM2 = 2.245.
- Degree of ORM1 = 2.441.
- Degree of SERPINA1 = 2.922.
- Degree of IGF2 = 1.498.
- Degree of IGFBP3 = 1.998.

We prefer the maximum degree energy (E_{MD}) over the degree energy (E_D) as the research focuses on identifying and analyzing the strongest interactions within the network, particularly when these interactions are hypothesized to be critical for network function and robustness to perturbations targeting high-degree vertices.

The maximum degree matrix is given by

$$M_D(G) = \begin{bmatrix} 0 & 0 & 2.425 & 2.441 & 2.922 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.498 & 1.998 \\ 2.425 & 0 & 0 & 2.441 & 2.922 & 0 & 0 \\ 2.441 & 0 & 2.441 & 0 & 2.922 & 0 & 0 \\ 2.922 & 0 & 2.922 & 2.922 & 0 & 0 & 0 \\ 0 & 1.498 & 0 & 0 & 0 & 0 & 1.998 \\ 0 & 1.998 & 0 & 0 & 0 & 1.998 & 0 \end{bmatrix}$$

The eigenvalues detailed below reveal key characteristics of the network's connectivity and the influence of individual proteins:

$$\lambda_1 \approx 8.0523,\ \lambda_2 \approx 3.6722,\ \lambda_3 \approx -1.4980,\ \lambda_4 \approx -2.1742,$$

$$\lambda_5 \approx -2.4250,\ \lambda_6 \approx -2.4463,\ \lambda_7 \approx -3.1810.$$

Crucially, this representation allows the calculation of maximum degree energy of the network, providing a novel metric for characterizing its structure and stability, as described in the previous sections. It is given by $E_{MD} = 23.449$. The spectral gap between two vertices v_i and v_j is found to be

$$\varpi_{12} = 4.3801,\ \varpi_{23} = 5.1702,\ \varpi_{34} = 0.6762,\ \varpi_{45} = 0.2508,\ \varpi_{56} = 0.0213,\ \varpi_{67} = 0.7347.$$

6.5. Comparison with Traditional Centrality Measures

To contextualize the significance of the proposed maximum degree energy (E_{MD}), we compare it against standard centrality metrics widely used in network analysis. These include degree centrality (C_D), betweenness centrality (C_B), and eigenvector centrality (C_E), each of which captures a different aspect of vertex importance.

The fuzzy PPI network associated with Systemic Lupus Erythematosus (SLE) was modeled as described in the previous sections. The confidence scores obtained from the STRING database were used as edge weights to construct the fuzzy adjacency matrix. Subsequently, we computed the degree of each protein based on these edge weights and calculated the corresponding E_{MD} values. Traditional centrality measures were also computed using the NetworkX library in Python (version 3.10.12).

The following are the mathematical formulations for standard centrality measures used in network analysis for a fuzzy graph G:

- Degree Centrality ($\boldsymbol{C_D}$): Degree centrality quantifies the relative importance of a vertex based on the number of direct connections it has to other vertices. It is defined as

$$\boldsymbol{C_D(v_i) = \frac{d(v_i)}{n-1}},$$

where $\boldsymbol{d(v_i) = \sum_{j=1}^{n} a_{ij}}$ denotes the degree of vertex $\boldsymbol{v_i}$, and $\boldsymbol{n}$ is the total number of vertices in the graph.

- Betweenness Centrality ($\boldsymbol{C_B}$): Betweenness centrality measures the extent to which a vertex lies on the shortest paths between other pairs of vertices. Formally, it is given by

$$\boldsymbol{C_B(v_i) = \sum_{\substack{s \neq v_i \neq t \\ s,t \in V}} \frac{\sigma_{st}(v_i)}{\sigma_{st}}},$$

where σ_{st} denotes the total number of shortest paths from vertex s to vertex t, and $\sigma_{st}(v_i)$ is the number of those paths that pass through vertex v_i.

- Eigenvector Centrality (C_E): Eigenvector centrality assigns influence scores to vertices based not only on their connections, but also on the importance of the vertices to which they are connected. It is defined as the solution to the eigenvector equation

$$C_E(v_i) = \frac{1}{\lambda} \sum_{j=1}^{n} a_{ij}\, C_E(v_j),$$

where λ is the largest eigenvalue of the adjacency matrix A, and the vector $\mathbf{C_E} = [C_E(v_1), \ldots, C_E(v_n)]^\top$ is the corresponding principal eigenvector.

The table below summarizes the comparative analysis.

As shown in Table 3, proteins such as SERPINA1, ORM1, and ORM2 exhibit consistently high values across all centrality metrics, including E_{MD}. This alignment validates the relevance of E_{MD} in capturing vertex influence. However, E_{MD} offers an added advantage: it incorporates fuzzy interaction strengths and leverages spectral properties, making it particularly well suited for biological networks characterized by uncertain or probabilistic edges.

Table 3. Comparison of E_{MD} with traditional centrality measures.

Protein	C_D	C_B	C_E	E_{MD}
AHSG	0.333	0.214	0.431	2.425
IGF1	0.333	0.000	0.294	1.498
IGF2	0.167	0.071	0.197	1.498
IGFBP3	0.333	0.214	0.392	1.998
ORM1	0.500	0.357	0.537	2.441
ORM2	0.500	0.429	0.519	2.245
SERPINA1	0.500	0.500	0.562	2.922

By comparing E_{MD} with established centrality measures, we reinforce its validity and underscore its potential as a robust metric for structural analysis in fuzzy-graph-based biological networks.

6.6. Advantage over Other Methods

1. Traditional degree centrality treats all connections equally, whereas E_{MD} gives higher importance to strong protein interactions. Stronger interactions often indicate more biologically significant relationships, such as stable protein complexes or key regulatory mechanisms.
2. SERPINA1 has the highest E_{MD}, which aligns with its known involvement in immune system regulation and inflammation in SLE.
3. Unlike betweenness or eigenvector centrality, which depend on path structures, E_{MD} is directly derived from interaction strengths.
4. E_{MD} correlates well with eigenvector centrality but focuses more on local structural strength, making it a better metric for identifying functionally relevant hubs rather than just influential vertices.
5. ORM1 and ORM2 have similar eigenvector centralities, but ORM1 has a slightly higher E_{MD}, highlighting its relatively stronger local interactions.
6. High E_{MD} proteins are likely to be crucial for network stability, meaning that their perturbation (mutation, inhibition) could significantly alter network dynamics. This

aligns with real-world biological findings, where proteins with strong interactions are often essential for disease progression and therapeutic targeting.

6.7. Limitations of the Study

1. By incorporating confidence scores as edge weights, E_{MD} ensures that stronger interactions contribute more to network energy, providing a more realistic biological insight.
2. The analysis only considered direct interactions between the selected proteins, ignoring potentially important indirect interactions.
3. The computational cost of applying this method to significantly larger PPI networks could be prohibitive, hindering its application to other complex biological systems.
4. As E_{MD} is derived from the spectral properties of the adjacency matrix, it provides a quantitative measure of structural stability, making it suitable for studying network resilience.

7. Results and Discussion

1. The metric (E_{MD}) maximum degree energy, is preferred because it directly reflects the energy associated with the maximum degree of vertices in a graph. In biological networks such as protein–protein interaction (PPI) networks, proteins with high degrees are often central to key processes, influencing the overall stability and function of the system.
2. The PPI network of seven key SLE-related proteins was modeled as a fuzzy graph. STRING confidence scores were used as edge weights to represent interaction strengths. The maximum degree energy (E_{MD}) quantified the influence of highly connected proteins.
3. The leading eigenvalue ($\lambda_1 \approx 8.0523$) indicated key protein dominance.
4. High-degree centrality proteins (SERPINA1, ORM1, ORM2) exhibited high E_{MD} values, confirming their structural importance.
5. IGF1 and IGF2 had lower betweenness centrality but comparable E_{MD}, revealing additional network insights.
6. The spectral gap (ϖ_{ij}) identified critical interaction thresholds, impacting network resilience.
7. In the context of disease, proteins associated with illness often show higher energy values due to a shift in their interaction patterns, indicating that their central roles in the network may be compromised.

8. Conclusions and Future Work

This work introduced degree-based energy measures for fuzzy graphs, including degree energy, average, maximum, and minimum degree energies, offering a refined spectral perspective by integrating vertex degree information. Sharp theoretical bounds were established and validated through rigorous proofs and illustrative examples. Application to a protein–protein interaction (PPI) network related to Systemic Lupus Erythematosus (SLE) demonstrated the practical relevance of these measures in uncovering structurally significant proteins.

One direction for future research involves extending the proposed degree-based energy measures to dynamic fuzzy graphs, thereby facilitating real-time analysis of evolving systems in biological and social networks. Another promising avenue lies in the development of efficient computational frameworks for analyzing large-scale fuzzy networks, enhancing their applicability to complex systems such as protein interactions and cybersecurity infrastructures.

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Some Properties of Boolean-like Laws in Fuzzy Logic

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Abstract: This article focuses on the relationships between fuzzy logic and classical logic properties using fuzzy t-norms, t-conorms, and fuzzy implications. It aims to contribute to fuzzy set theory by extending the Boolean laws in classical logic to fuzzy logic. We determine the necessary and sufficient conditions for validating the generalizations of the proposed properties from classical to fuzzy logic. Additionally, we provide examples demonstrating the practical applicability of this approach and its advantages over conventional methodologies, reinforcing its effectiveness.

Keywords: fuzzy logic; Boolean-like laws; t-norm; fuzzy implication

1. Introduction

Fuzzy logic has garnered considerable interest as a framework for handling vague information. Over the past few decades, various non-classical logic systems have been proposed and explored. Among them, Lukasiewicz–Zadeh's n-valued logic stands out as the most widely used, particularly in the fields of fuzzy sets and systems. In the 1930s, Lukasiewicz introduced n-valued logic, focusing predominantly on the logical operations of negation and implication. Zadeh later expanded on Lukasiewicz's work by introducing the concept of membership functions, which are crucial for defining fuzzy sets. This combination of Lukasiewicz's logical framework and Zadeh's fuzzy set theory laid the groundwork for modern fuzzy logic applications across various domains, including artificial intelligence and control systems [1].

Given the significance of T-norms in various fields, including information science, fuzzy logic, and fuzzy sets, these norms play a crucial role in many applications [2]. A T-norm, also known as a t-norm or triangular norm, is a type of binary operation used in fuzzy logic and multi-valued logic frameworks. The concept of a t-norm was first introduced by Menger [3] in 1942. He established a unique class of two-place functions on the unit square, referred to as triangular norms, to properly formulate the triangle inequality. Although Menger did not impose the requirement of associativity, the boundary conditions he established were inherently weaker than those for modern t-norms. In 1958, Schweizer and Sklar [4] formulated the current axioms for t-norms, as stated in Definition 1. These axioms include commutativity, associativity, monotonicity, and the existence of a neutral element, which is 1. Their definition of associativity was based on a correct formulation of the polygonal inequality.

In 1965, Zadeh proposed alternative methods for fuzzy intersection (and union) that employed the minimum (and maximum) functions [5]. Additionally, Dubois [6] documented the use of t-norms within fuzzy sets and fuzzy logic frameworks for the first time in 1980. These alternative methods provided new insights into fuzzy logic and expanded the applications of t-norms beyond their initial scope. Dubois's work in 1980 laid the foundation for the widespread use of t-norms in fuzzy sets and logic.

The dual operations known as T-conorms (also referred to as S-norms or triangular conorms) were introduced by Schweizer and Sklar in 1960 [7]. It is important to note that the term "s-norm" originates from the S in the original Schweizer–Sklar notation for t-conorms. Evidence suggests that both t-norms and t-conorms play a crucial role in the theory of fuzzy sets. Because they are dual to each other, any results derived for t-norms can be reformulated for t-conorms and vice versa. This duality allows for a deeper understanding of operations within fuzzy sets and provides flexibility in their application. However, it is possible to define t-conorms independently of t-norms using a similar set of axioms. The main distinction between the two lies in their neutral elements: 0 for t-conorms and 1 for t-norms. This distinction highlights the versatility of t-conorms within fuzzy set theory, enabling various interpretations and applications that differ from those of t-norms. Overall, the duality and independence of t-conorms contribute to the richness and complexity of operations in fuzzy set theory.

In fuzzy logic, fuzzy implications are essential for representing the relationship between input and output variables. By allowing for degrees of truth instead of relying solely on strict true or false values, fuzzy implications offer a more flexible and realistic framework for modeling uncertainty and imprecision across various domains. They generalize classical two-valued implications to a multi-valued context, proving significant in both theoretical and practical applications. This significance is evident in their use in multi-valued mathematical logic, fuzzy control systems, and data analysis. This discussion will focus on fuzzy implications from a theoretical perspective to highlight their fundamental principles.

Boolean laws, which are valid in classical logic, play a crucial role in fuzzy set theory and related Boolean-like principles through the use of t-norms, t-conorms, negation operators, and fuzzy implications. In the paper [8], the authors discuss certain Boolean-like laws that involve iterative variables in fuzzy logic. They explain that, in addition to the classical De Morgan triplets of connectives, which are defined by t-norms, t-conorms, and strong negations, there are fascinating models that can yield infinite solutions. Additionally, they highlight unexpected cases where no solutions exist. There has been considerable research on characterizing two types of fuzzy implication functions, specifically, R-implications and S-implications. However, there is limited knowledge about QM-implications. The papers [9,10] focus on studying certain characteristics of QM-implication operators. They examine the QM-implication operator both as an implication function and as a T-conditional function, providing useful tools for their characterization. In the paper [11], the author explores the well-known iterative Boolean-like law and discusses the property of approximation preservation related to the compositions of fuzzy implications. Finally, the necessary and sufficient conditions for the approximate equation of (S, N)-implications are provided.

The principle of distributivity, where disjunction distributes over conjunction, serves as a cornerstone in classical logic and forms the basis of algebraic structures defined by binary operators. In the context of fuzzy logic, this principle is extended through the use of t-norms and t-conorms, which provide a framework for generalizing classical properties. Several studies have focused on distributive equations for fuzzy implications, particularly about two aggregation functions, as outlined by Bala [12], Combs and Andrews [13], who pioneered the exploration of distributive equations by grounding their work in the logical validity within classical propositional logic. Their efforts aimed to address the complexity caused by the proliferation of combinatorial rules. Building on this, Trillas and Alsina [14] further investigated the general forms of these distributive equations. Moreover, other studies have identified solutions for fuzzy implications to these distributive equations for specific t-norms and t-conorms, as discussed by Pan [15]. This evolving discourse underscores the significance of distributivity in bridging classical and fuzzy logic, offering practical and theoretical insights into its application.

The ordering property (OP) and the exchange principle (EP) represent robust foundational principles in mathematics. Specifically, (OP) ensures a well-defined ordering of elements within a set, commonly known as the degree ranking property, particularly in the interval $[0, 1]$. Meanwhile, (EP) permits the rearrangement of elements without altering the intrinsic properties of the set, generalizing the classical exchange principle expressed as a tautology. In the realm of fuzzy logic, (S, N)-implications are a type of fuzzy implication that adheres to both the left neutrality property (NP) and the exchange principle (EP). However, it is worth noting that not all (S, N)-implications satisfy the identity principle (IP) or the ordering property (OP) [16].

Classical logic's application to fuzzy logic through equivalence showcases its practical utility, notably in preventing the explosion of combinatorial rules within fuzzy systems [17]. This approach effectively streamlines the complexity often encountered in such systems. Moreover, the research of Bala and Trillas [12,14] highlights the significance of the fuzzy logic translation of the distributivity functional equation. They successfully addressed this equation for a variety of implication functions derived from t-norms and t-conorms. These studies exemplify the seamless integration of classical logic principles into the fuzzy logic framework, offering both theoretical insights and practical benefits for handling complex logical structures.

In fuzzy logic, numerous laws of contraposition have been investigated, including the particular law of contraposition (CP), the law of left contraposition to N $(L - CP)$, and the law of right contraposition to N $(R - CP)$ [17,18]. These laws have been examined for their relevance and fundamental role within the principles of classical logic. When N is defined as a strong negation, these three laws exhibit equivalence, as their properties align seamlessly. This demonstrates the simplicity and coherence of the contraposition laws under strong negation. However, when N is defined simply as a fuzzy negation rather than a strong negation, the equivalence between these contraposition principles fails. In this case, each law may exhibit different properties under fuzzy negation [17]. This distinction underscores the significance of strong negation in preserving the harmony among contraposition laws.

The rest of this paper is structured as follows:

In Section 2, we introduce some fundamental concepts and characteristics of algebraic structures and fuzzy set theory. We review several key definitions and results related to fuzzy negations, t-norms, t-conorms, and the properties that connect these fuzzy operators.

Section 3 examines the key themes and properties, focusing on fuzzy t-norms, t-conorms, and fuzzy implications, as well as the connections between classical logic properties and their fuzzy counterparts. We identify the necessary and sufficient conditions to validate the generalizations of classical logic properties within fuzzy logic. Furthermore, an example is provided to illustrate the practical applications of these results.

2. Preliminaries

We assume that the reader is already familiar with the fundamental concepts of classical logic connectives. However, this section will summarize several key ideas that will be utilized later in the text to ensure it is self-contained. Additionally, we will briefly revisit some of the ideas and findings that will be employed in the subsequent sections. This overview will serve as a refresher for those already well versed in classical logic and provide essential background information for those new to the topic of fuzzy logic.

The core concept of fuzzy logic can be better grasped by contrasting it with classical binary logic. Suppose there is a set of n logical variables, denoted as $x_1, x_2, \ldots, x_n$. These n

variables can yield 2^n distinct combinations of truth values. As a result, the total count of logical functions that can define these variables is also 2^n. Logical functions can be broadly classified into two main categories:

(a) "negation", "and function", and "or function";
(b) "negation" and "the two implications".

Through either set of primary logical functions, any other logical functions can be generated using suitable algebraic expressions, referred to as logical formulas. This approach facilitates the creation of complex logical operations by combining simpler ones, offering a powerful method for designing and analyzing digital circuits. The three main logical functions—negation $(-)$, $(\wedge)$, and $(\vee)$—are combined in various types of logical formulas to create a logical function. It is important to specify the order of composition, for instance, by using parentheses to give a unique function. The logical expression

$$(x_1 \vee x_3) \wedge (\overline{x}_1 \vee x_2) \wedge (\overline{x}_2 \vee \overline{x}_3)$$

represents a distinct logical function involving three variables. The expression $a \equiv b$ is commonly used to describe the equivalence of two logical formulas, represented by the symbols a and b. For example, the following equivalency is evident:

$$(\overline{x}_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (\overline{x}_2 \wedge \overline{x}_3) \equiv (x_1 \wedge \overline{x}_2) \vee (\overline{x}_1 \wedge \overline{x}_3) \vee (x_2 \wedge x_3).$$

Boolean algebra may be used to verify that these expressions are equivalent with the variables x_1, x_2, and x_3 over all eight conceivable combinations of truth values. Boolean algebra is a binary logic structure with two values that bears the name of George Boole, a mathematician and logician in the 19th century. It provides a structured approach for manipulating logical expressions, facilitating the analysis and comparison of different formulations. By adhering to the principles of Boolean algebra, verifying the equivalence of logical expressions becomes a more streamlined and efficient process [19–22].

Boolean algebra consists of three fundamental logical operations: "negation $\neg$", "and $\wedge$", and "or $\vee$". By developing basic logical processes, Zadeh built on Lukasiewicz's logic and introduced an infinite-valued logical framework. For ease of algebraic manipulation, these operations are often represented with the symbols $-$, $\cdot$, and $+$, respectively. A truth table can be used to clearly define a Boolean algebraic formula by listing all the variables as inputs and indicating the resulting output for the entire formula. Boolean algebraic formulas can be constructed for any truth table and can often be simplified to a more straightforward, equivalent form using various useful properties of Boolean algebra.

Note that several Boolean algebraic rules, including $a \cdot 0 = 0$, $a \cdot 1 = a$, and $a + 0 = a$, are comparable to those of ordinary algebra. However, there are significant differences as well, for example, $a + 1 = 1$. Therefore, great care should be taken to avoid mistakenly applying standard algebraic principles to logical formulations, as this can lead to errors. Furthermore, even if Boolean algebra works well for streamlining logical formulas, when numerous variables are involved, the procedure can become very tiresome in real-world applications. By examining the connections between Boolean laws and fuzzy logic properties, readers can gain insight into the versatility and applicability of these concepts across different mathematical frameworks. For example, De Morgan's principles from classical logic can be applied to fuzzy logic by incorporating t-norms and t-conorms. This extension provides a deeper understanding of how fuzzy logic can more flexibly model uncertainty and vagueness compared to traditional Boolean logic.

We will review the main results regarding t-norms, t-conorms, fuzzy negation, and some concepts related to fuzzy implications that are referenced throughout this work [23,24].

Definition 1. *A* t-norm *is a function* $T : [0,1] \times [0,1] \to [0,1]$*, for all* $a, b, c, d \in [0,1]$ *that satisfies the following properties:*

(T1) (Commutativity): $T(a,b) = T(b,a)$,
(T2) (Associativity): $T\big(T(a,b), c\big) = T\big(a, T(b,c)\big)$,
(T3) (Monotonicity): $T(a,b) \leq T(c,d)$ *if* $a \leq c$ *and* $b \leq d$*, (*T *is non-decreasing in each argument),*
(T4) (1-identity): $T(a,1) = a$ *(The number* 1 *acts as identity element).*

Since a t-norm is a binary algebraic operation defined on the interval $[0,1]$, it can be represented using infix algebraic notation, where the t-norm is denoted by the symbol $*$. When examining the algebraic structure, the binary operation associated with the t-norm, represented in prefix notation as T, can also be expressed using the infix operator $*$. Consequently, the four axioms $(T1)$–$(T4)$ apply for every $a, b, c \in [0,1]$, as follows:

$$a * b = b * a$$

$$(a * b) * c = a * (b * c)$$

$$a * b \leq a * c, \text{ if } b \leq c$$

$$1 * a = a$$

Example 1. *The four most common t-norms are listed below:*

(1) Minimum t-norm: $T_{\min}(a,b) = \min\{a,b\}$*, also called the* Gödel t-norm.
(2) Product t-norm: $T_{prod}(a,b) = a \cdot b$ *(the ordinary product of real numbers).*
(3) Lukasiewicz t-norm: $T_{Luk}(a,b) = \max\big\{0, a + b - 1\big\}$.
(4) Drastic t-norm:

$$\top_{\mathrm{D}}(a,b) = \begin{cases} b, & \text{if } a = 1 \\ a, & \text{if } b = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1. *Considering the partial order on the family of all t-norms induced by the order on* $[0,1]$*,* $T_{min}(a,b)$ *is the greatest t-norm. Therefore, it is clear that for any t-norm* T

$$T(a,b) \leq T_{min}(a,b) \leq b, \text{ for all } a, b \in [0,1].$$

Remark 2. *The minimum t-norm is the sole idempotent t-norm, i.e.,* $T_{min}(a,a) = a$*, for all* $a \in [0,1]$.

Definition 2. T-conorms *(also known as S-norms) are dual to t-norms under the order-reversing operation that assigns* $1 - a$ *to* a *on* $[0,1]$*. The complementary conorm* S *is defined for a t-norm* T *by*

$$S(a,b) = 1 - T(1-a, 1-b).$$

As a result, for all $a, b, c \in [0,1]$*, a t-conorm meets the following criteria, which may be utilized for a similar axiomatic definition of t-conorms without regard to t-norms:*

(S1) (Commutativity): $S(a,b) = S(b,a)$
(S2) (Associativity): $S\big(S(a,b), c\big) = S\big(a, S(b,c)\big)$
(S3) (Monotonicity): $S(a,b) \leq S(c,d)$ *if* $a \leq c$ *and* $b \leq d$
(S4) (0-identity): $S(a,0) = a$ *(The number* 0 *acts as identity element).*

Example 2. *The following four t-conorms are common:*

(1) Maximum t-conorm: $S_{max}(a,b) = \max\{a,b\}$ *is dual to the minimum t-norm.*
(2) Probabilistic sum: $S_{sum}(a,b) = a + b - a \cdot b = 1 - (1-a) \cdot (1-b)$ *is dual to the product t-norm.*
(3) Bounded sum: $S_{Luk}(a,b) = \min\{a+b, 1\}$ *is dual to the Lukasiewicz t-norm.*
(4) Drastic t-conorm:

$$S_D(a,b) = \begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } b = 0 \\ 1, & \text{otherwise,} \end{cases}$$

is dual to the drastic t-norm.

Remark 3. *The least t-conorm is $S_{max}(a,b)$, taking into account the point-wise order on the family of all t-conorms induced from the order $[0,1]$. Consequently, given any t-conorm S,*

$$b \le S_{max}(a,b) \le S(a,b) \text{ for all } a,b \in [0,1].$$

Moreover, $S_{max}(a,1) = S_{max}(1,a) = 1$ is evident from the Maximum t-conorm definition, which implies that for every t-conorm S, $S(a,1) = S(1,a) = 1$, $a \in [0,1]$.

Definition 3. *For every $a,b \in [0,1]$, a* fuzzy negation (complement) *is a function $N : [0,1] \to [0,1]$ that meets the following properties:*

(N1) (Preservation of constants): $N(0) = 1$, $N(1) = 0$.
(N2) (Reversing of the order): $N(a) \le N(b)$ *iff* $b \le a$.

In addition, a fuzzy negation N is referred to as

(i) strict *if it is strictly decreasing and continuous;*
(ii) strong *if it is an involution, i.e., $N\big(N(a)\big) = a$, for all $a \in [0,1]$.*

Example 3. *Though there are several functions that can be employed in negation, the classical negation function $N(a) = 1 - a$, $a \in [0,1]$, is typically utilized in applications. The classical negation is a strong negation, $N(a) = 1 - a^2$ is merely strict, and the least and greatest fuzzy negations, N_{D1} and N_{D2}, are non-strict negations:*

$$N_{D1} = \begin{cases} 1, & \text{if } a = 0, \\ 0, & \text{if } a > 0. \end{cases} \quad , \qquad N_{D2} = \begin{cases} 1, & \text{if } a < 1, \\ 0, & \text{if } a = 1. \end{cases}$$

For every t-norm T, t-conorm S, and classical negation function N, it is easy to see that

$$S(a,b) = N(T(N(a), N(b))).$$

As an extension of the traditional implication operation, fuzzy implications were presented and examined in the literature in accordance with the specified truth values:

$$1 \Rightarrow 1 \equiv 1\,, \quad 1 \Rightarrow 0 \equiv 0\,, \quad 0 \Rightarrow 1 \equiv 1\,, \quad 0 \Rightarrow 0 \equiv 1.$$

It is well known that the implication $p \Rightarrow q$ is equivalent to $\neg p \vee q$. This is the standard definition of implication. In fuzzy logic, there are many definitions of fuzzy implications found in the literature, especially in the earlier sections. In this study, we will use Definition 4, which aligns with the definition proposed by Fodor and Roubens [25].

Definition 4. *A binary function $I : [0,1] \times [0,1] \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:*

(I1) *I is nonincreasing in the first place,*
(I2) *I is nondecreasing in the second one,*
(I3) *$I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.*

Note that, according to Definition 4, certain potential properties are permissible for specific fuzzy implications:

1. Boundary conditions: $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$;
2. Left antitonicity: if $a \le b$ then $I(b,c) \le I(a,c)$, for all $a,b,c \in [0,1]$;
3. Right isotonicity: if $a \le b$ then $I(c,a) \le I(c,b)$, for all $a,b,c \in [0,1]$;
4. Left boundary condition: $I(0,b) = 1$, for all $b \in [0,1]$;
5. Right boundary condition: $I(a,1) = 1$, for all $a \in [0,1]$;
6. Identity property: $I(a,a) = 1$, for all $a \in [0,1]$.

Not all fuzzy implications possess the desirable qualities of classical implications defined on $\{0,1\}^2$ mapping to $\{0,1\}$. While it is widely accepted that the fuzzy implication concept generalizes classical implication operations, several of these positive attributes have been assumed in prior definitions of fuzzy implications. Therefore, it is essential and intriguing to explore how these attributes interact. Numerous studies [18,26] have proposed various properties of fuzzy implications. For example, the exchange principle (EP) was included in the definition of a fuzzy implication in the paper [18] (see Definition 7 below). The most significant properties are listed below.

Fuzzy implications fall into three primary categories: $(S,N)-$, $R-$, and $QL-$ implications. From these, other classes can be derived. In the definition that follows, we will examine them.

Definition 5. *Let T be a t-norm, S a t-conorm and N a fuzzy negation, then:*

A function $I : [0,1]^2 \to [0,1]$ is called an (S,N)-implication (denoted by $I_{S,N}$) if

$$I(a,b) = S\big(N(a), b\big).$$

A function $I : [0,1]^2 \to [0,1]$ is called an R-implication (denoted by I_T) if

$$I(a,b) = \sup\Big\{t \in [0,1] : T(a,t) \le b\Big\}.$$

A function $I : [0,1]^2 \to [0,1]$ is called a QL-implication (denoted by $I_{S,N,T}$) if

$$I(a,b) = S\big(N(a), T(a,b)\big).$$

However, QL-implications in fuzzy logic have not been studied as extensively as (S,N)- and R-implications. One possible reason for this is that some members of this family do not satisfy the primary requirement for a fuzzy implication, which is left antitonicity. Additionally, earlier works were limited in terms of the class of operations from which QL-implications could be derived, as well as the properties that these implications satisfied. This was due to restrictions placed on the fuzzy logic operations used in the construction of QL-implications.

3. Proposed Properties and Boolean-like Laws

In fuzzy logic, traditional Boolean laws have been extended and investigated in the setting of functional equations or inequalities, which are known as Boolean-like laws. These laws are rarely found in standard structures such as $([0,1], T, S, N, I)$, where T represents a t-norm, S a t-conorm, N a fuzzy negation, and I a fuzzy implication. Many features that hold in classical logic may not hold in fuzzy logic when using t-norms, t-conorms, and strong negations to perform conjunctions, disjunctions, and negations, respectively. A popular area of research in fuzzy logic involves investigating which of these operators preserve certain features of classical logic. This approach is often employed to solve functional equations using various types of operators. Researchers have raised questions about the conditions under which a Boolean-like law is valid in this context.

The Law of the Excluded Middle (LEM) is a fundamental principle in classical logic, stating that for any proposition p, the expression $\neg p \vee p$ is always true. However, in the context of fuzzy logic, the Law of the Excluded Middle may not always apply due to the nature of fuzzy sets and the use of truth values that range between 0 and 1. Despite this, we can extend the principle of LEM within the framework of fuzzy logic.

Definition 6. *If S is a t-conorm and N is a fuzzy negation, the pair (S, N) satisfies the LEM iff*

$$S(N(a), a) = 1, \text{ for all } a \in [0,1].$$

The distributivity of disjunction over conjunction is a fundamental principle in classical logic. Its application to fuzzy logic considers t-norms and t-conorms:

A t-conorm S is said to be distributive over a t-norm T if

$$S(a, T(b,c)) = T(S(a,b), S(a,c)).$$

The following is an important finding regarding the distributive property in laws similar to Boolean algebra:

Theorem 1 ([23])**.** *If T is a t-norm and S is a t-conorm, then S is distributive over T iff $T = T_{min}$.*

Certain studies have concentrated on the distributive equations of fuzzy implications in relation to two aggregation functions [12]. The distributivity of one binary operator over another forms the fundamental structure of the algebra defined by these operators in classical logic. Overall, there are four classical distributive equations that involve implications, which are as follows:

$$\begin{aligned} (p \vee q) \Rightarrow r &\equiv (p \Rightarrow r) \wedge (q \Rightarrow r) \\ (p \wedge q) \Rightarrow r &\equiv (p \Rightarrow r) \vee (q \Rightarrow r) \\ p \Rightarrow (q \wedge r) &\equiv (p \Rightarrow q) \wedge (p \Rightarrow r) \\ p \Rightarrow (q \vee r) &\equiv (p \Rightarrow q) \vee (p \Rightarrow r) \end{aligned} \tag{1}$$

In classical logic, the equivalences discussed are considered tautologies. Recent research has focused on the distributivity of implication functions and aggregation functions, such as t-norms and t-conorms, which are essential in the framework of logical connectives. There is a significant body of work that examines the generalizations of these distributivity equations to fuzzy connectives, particularly those involving fuzzy implications.

Fuzzy systems are well known for their capability to approximate any continuous function with a high degree of precision. However, achieving this level of accuracy typically requires a large rule base. Many researchers have investigated distributive equations to mitigate the excessive expansion of combinatorial rules. Combs and Andrews [13] were the first to propose this concept, based on the logical validity of the second equation of (1) in classical propositional logic. In 2002, Trillas and Alsina [14] explored the general form

$$I(T(a,b),c) = S(I(a,c), I(b,c))$$

of the second equation of (1). Additionally, some studies have identified fuzzy implication solutions to the equations in (1) for specific t-norms and t-conorms [15].

Definition 7. (i) *Fuzzy implication I is considered to fulfill the left neutrality property if,*

$$I(1,b) = b\ , \quad b \in [0,1] \quad \textit{(NP)}$$

(ii) *Fuzzy implication I is said to satisfy the exchange principle, if*

$$I\big(a, I(b,c)\big) = I(b, I(a,c))\ , \quad a,b,c \in [0,1] \quad \textit{(EP)}$$

(iii) *Fuzzy implication I is said to satisfy the ordering property, if*

$$I(a,b) = 1 \Leftrightarrow a \leq b\ , \quad a,b \in [0,1] \quad \textit{(OP)}$$

The ordering property (OP) and the exchange principle (EP) are both strong conditions. The (OP) ensures that the elements in a set are well ordered, while the EP allows for the rearrangement of elements without changing the set's properties. These principles are fundamental in various mathematical concepts and proofs. The property (EP) generalizes the classical exchange principle, which is a tautology expressed as:

$$p \Rightarrow (q \Rightarrow r) \equiv q \Rightarrow (p \Rightarrow r).$$

The ordering property (OP), also known as the degree ranking property, establishes an ordering for the underlying set $[0,1]$.

Theorem 2 ([25]). *If a function* $I : [0,1]^2 \to [0,1]$ *satisfies* (EP) *and* (OP)*, then I satisfies* $(I1)$ *and* $(I3)$*.*

The above conclusion demonstrates that both the conditions (EP) and (OP) render any function $I : [0,1]^2 \to [0,1]$ essentially a fuzzy implication. The only property lacking from an I that fulfills (EP) and (OP) is $(I2)$.

All (S,N)-implications are fuzzy implications that meet the criteria (NP) and (EP). However, not every (S,N)-implication satisfies the identity principle (IP) or the ordering property (OP) [16]. The following solution presents an equivalence condition under which (S,N)-implications fulfill these properties.

Theorem 3 ([16]). *The following claims are equivalent for a fuzzy negation N and a t-conorm S:*

(i) (IP) *is fulfilled by the* (S,N)*-implication* $I_{S,N}$*.*
(ii) (LEM) *is satisfied by the pair* (S,N)*.*

It is clear that the QL-implication derived from the triple (S,N,T) does not satisfy (OP) if S is a positive t-conorm. A necessary condition for a QL-implication to fulfill (OP) is provided by the following result:

Theorem 4. *The negation N is strictly decreasing if a QL-implication* $I_{S,N,T}$ *derived from a non-positive t-conorm S satisfies* (OP)*.*

Proof. Assume that the property (OP) is satisfied by a QL-implication $I_{S,N,T}$ derived from a non-positive t-conorm S. Furthermore, let there exist $a, b \in [0,1]$ such that $N(a) = N(b)$ despite $a < b$. We can infer this using the ordering property (OP).

$$\begin{aligned} I_{S,N,T}(a,b) = 1 \quad &\Rightarrow \quad S\big(N(a)\ ,\ T(a,b)\big) = 1 \quad \big(\text{By def. of (QL)-implication}\big) \\ &\Rightarrow \quad S\big(N(a)\ ,\ T(b,a)\big) = 1 \quad \big(\text{T1-Commutativity}\big) \\ &\Rightarrow \quad S\big(N(b)\ ,\ T(b,a)\big) = 1 \quad \big(\text{By Assumption}\big) \\ &\Rightarrow \quad I_{S,N,T}(b,a) = 1 \quad \big(\text{By def. of (QL)-implication}\big) \\ &\Rightarrow \quad b \le a \quad \big(\text{By the ordering property (OP)}\big). \end{aligned}$$

However, since it contradicts itself, the negation N is strictly decreasing. □

Examples are given by the equivalence

$$(p \wedge q) \Rightarrow r \equiv (p \Rightarrow r) \vee (q \Rightarrow r) \tag{2}$$

which prevented combinatorial rule explosion in fuzzy systems [17]. Fuzzy logic translation of Equation (2) yields

$$I(T(a,b),c) = S(I(a,c), I(b,c)) \quad a,b,c \in [0,1] \tag{3}$$

where I is an implication function, T is a t-norm, and S is a t-conorm. In [12,14], this distributivity functional equation was solved for several types of implication functions that were obtained from t-norms and t-conorms.

The law of contraposition is one of the most important tautologies in classical two-valued logic, expressed as:

$$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$$

This law is essential for proving many results by contradiction. It is meaningful to generalize this concept to fuzzy logic, utilizing fuzzy negations and fuzzy implications. When working with fuzzy implications, the contrapositive symmetry of a fuzzy implication I concerning a fuzzy negation N is crucial in fuzzy logic.

Additionally, in classical logic, the following rules are also tautologies, as the classical negation adheres to the law of double negation:

$$\neg p \Rightarrow q \equiv \neg q \Rightarrow p$$

$$p \Rightarrow \neg q \equiv q \Rightarrow \neg p$$

As a result, we can investigate several laws of contraposition in the context of fuzzy logic.

Definition 8. *Let I be a fuzzy implication and N be a fuzzy negation.*

(i) *We say that I satisfies the law of contraposition (also known as the contrapositive symmetry) with respect to N, if*

$$I(a,b) = I(N(b), N(a)), a,b \in [0,1]. \quad (CP)$$

(ii) *We say that I satisfies the law of left contraposition with respect to N, if*

$$I(N(a),b) = I(N(b),a), a,b \in [0,1]. \quad (L-CP)$$

(iii) *We say that I satisfies the law of right contraposition with respect to N, if*

$$I(a, N(b)) = I(b, N(a)), a, b \in [0,1]. \quad (R - CP)$$

If I satisfies the (left, right) contrapositive symmetry with respect to N, then we also denote this by $CP(N)$ (respectively, by $L - CP(N)$, $R - CP(N)$).

When N is a strong negation, it is clear that all three qualities are equal. Many authors have explored the classic law of contraposition (CP) [17,18]. Generally speaking, N must be a strong negation, which means that it is unnecessary to consider three separate laws of contraposition. However, if N is merely a fuzzy negation without any additional assumptions, the various principles of contraposition may not be equivalent [17].

The well-known Boolean-law in Boolean logics

$$p \Rightarrow (q \Rightarrow p) \equiv 1 \tag{4}$$

is referred to as "Weakening", since it is said to be the weakening of "$p \Rightarrow p$". Similarly, assuming "$\Rightarrow$" as the material implication, we see that (4) is also a weakening formalization of Aristotle's Law of the Excluded Middle "$\neg p \vee p$" since, "$p \Rightarrow (q \Rightarrow p)$" would be equal to "$\neg p \vee (\neg q \vee p)$". It is possible to generalize that law (4) to the fuzzy context as

$$I(a, I(b, a)) = 1 \tag{5}$$

where I is a fuzzy implication. This law can be viewed as a fuzzy modification of the classic Weakening axiom from the perspective of formal logic [27,28].

One theoretical reason to examine the Equation (5) is that various definitions of fuzzy implications aim to capture the correct (common sense) understanding of what constitutes a logical implication. Consequently, exploring the framework of fuzzy implication classes can shed light on the meaning of implications in a fuzzy context. Additionally, the concept of implication across different logical systems provides a broad motivation for defining these classes. For instance, the implications found in classical and quantum logic are generalized by (S, N)-implication and QL-implication, respectively. Thus, characterizing approximate reasoning through each type of implication also benefits from analyzing the conditions under which logical laws are valid within each of these classes.

Next, we will present the necessary and sufficient conditions for the (S, N)-, R-, and QL-implications of the Boolean-like law (5) to hold. The following theorems will illustrate these fundamental properties.

Theorem 5. *Let T, N and $I_{S,N}$ be a t-norm, a fuzzy negation, and (S, N)-implication, respectively. An (S, N)-implication $I_{S,N}$ fulfills Equation (5), which extends the law (4) into the fuzzy context iff the pair (S, N) satisfies (LEM).*

Proof. Firstly, we assume that an (S, N)-implication $I_{S,N}$ satisfies (5). It is evident from the definition of $I_{S,N}$ that

$$I(1, a) = S\big(N(1), a\big) = S(0, a) = a$$

and from the Identity property

$$I\big(a, I(1, a)\big) = I(a, a) = 1.$$

Then, we obtain that

$$S\big(N(a), a\big) = S\big(N(a), I(1, a)\big) = I\big(a, I(1, a)\big) = 1 \quad \text{for all } a \in [0, 1].$$

Thus, the pair (S, N) satisfies (LEM).

We now suppose that the pair (S, N) satisfies (LEM), i.e., for all $a \in [0,1]$, $S(N(a), a) = 1$. Then, we derive that

$$\begin{aligned} I(a, I(b, a)) &= I\Big(a, S\big(N(b), a\big)\Big) = S\Big(N(a), S\big(N(b), a\big)\Big) \quad \text{(By def. of (S,N)-implication)} \\ &= S\Big(N(a), S\big(a, N(b)\big)\Big) \quad \text{(S1-Commutativity)} \\ &= S\Big(S\big(N(a), a\big), N(b)\Big) \quad \text{(S2-Associativity)} \\ &= S\big(1, N(b)\big) \quad \big(\text{(S,N) satisfies (LEM)}\big) \\ &= 1 \quad \text{(Remark 3).} \end{aligned}$$

Thus, it has been established that $(I_{S,N})$ satisfies Equation (5). □

The result outlined in Theorem 5 demonstrates and highlights the adaptability of classical logic principles when integrated into the framework of fuzzy logic, thereby enriching its theoretical depth. Moreover, the law (4) has been rewritten as follows:

$$p' \vee (q' \vee p) \equiv p' \vee (p \vee q') \equiv (p' \vee p) \vee q' \equiv 1 \vee q' \equiv 1 \tag{6}$$

By extending the law (6) to the fuzzy logic domain, we obtain results that align with the principles and properties inherent to fuzzy systems:

$$S\Big(N(a), S(a, N(b))\Big) = 1 \tag{7}$$

Corollary 1. *Let T, S, N be a t-norm, a t-conorm, and a fuzzy negation, respectively. If the pair (S, N) satisfies the (LEM) property, then Equation (7) is valid.*

Proof. Given that the pair (S, N) adheres to the conditions set forth by (LEM), it can be inferred that

$$\begin{aligned} S\Big(N(a), S(a, N(b))\Big) &= S\Big(S\big(N(a), a\big), N(b)\Big) \quad \text{(S2-Associativity)} \\ &= S\big(1, N(b)\big) \quad \big(\text{(S,N) satisfies (LEM)}\big) \\ &= 1 \quad \big(\text{Remark 3}\big). \end{aligned}$$

□

As a result, extending the law (6) into the fuzzy logic domain, we obtain results that align with the principles and properties inherent to fuzzy systems. This adaptation provides a theoretical framework that bridges classical logic with the nuances of fuzziness, ensuring consistency and expanding the applicability of the established law.

The following theorem demonstrates that Equation (5) is satisfied by every R-implication.

Theorem 6. *The Equation (5) holds for every R-implication.*

Proof. Let T be any t-norm and I_T an R-implication generated from it. By Remark 1, we know that fixing arbitrarily $a, b \in [0,1]$, $T(b, a) \leq \min\{b, a\} \leq a$. By definition of R-implication, the formula

$$I(b, a) = \sup\{t \in [0,1] : T(b, t) \leq a\}$$

holds that $a \leq I(b,a)$. Then, taking into account I-Right isotonicity, we have

$$I(c,a) \leq I\big(c, I(b,a)\big) \quad \text{for all} \;\; c \in [0,1]$$

or particularly, we can write that

$$I(a,a) \leq I\big(a, I(b,a)\big).$$

Since $I(a,a) = 1$, we obtain $I\big(a, I(b,a)\big) = 1$. It means that R-implication satisfies (5). □

Theorem 7. *If (S,N) satisfies (LEM) and $T = T_{min}$, then QL-implication $I_{S,N,T}$ satisfies (5).*

Proof. Assume that (S,N) satisfies (LEM) and $T = T_{min}$. By Lemma 1, S is distributive over T iff $T = T_{min}$. Therefore, for a QL-implication $I_{S,N,T}$, we write that

$$\begin{aligned}
I\big(a, I(b,a)\big) &= S\Big(N(a),\ T\big(a, I(b,a)\big)\Big) \\
&= S\Big(N(a),\ T\Big(a, S\big(N(b), T(b,a)\big)\Big)\Big) \quad \text{(By def. of QL-implication)} \\
&= S\Big(N(a)\ ,\ T\Big(a\ ,\ T\Big(S\big(N(b),b\big)\ ,\ S\big(N(b),a\big)\Big)\Big)\Big) \ \text{(S is distributive over T)} \\
&= S\Big(N(a)\ ,\ T\Big(a\ ,\ T\Big(1\ ,\ S\big(N(b),a\big)\Big)\Big)\Big) \quad \text{((S,N) satisfies (LEM))} \\
&= S\Big(N(a)\ ,\ T\Big(a\ ,\ S\big(N(b),a\big)\Big)\Big) \quad \text{(T4-identity)} \\
&= T\Big(S\big(N(a),a\big)\ ,\ S\Big(N(a)\ ,\ S\big(N(b),a\big)\Big)\Big) \quad \text{(S is distributive over T)} \\
&= T\Big(1\ ,\ S\Big(N(a)\ ,\ S\big(N(b),a\big)\Big)\Big) \quad \text{((S,N) satisfies (LEM))} \\
&= S\Big(N(a)\ ,\ S\big(N(b),a\big)\Big) \quad \text{(T4-identity)} \\
&= S\Big(N(a)\ ,\ S\big(a, N(b)\big)\Big) \quad \text{(S1-Commutativity)} \\
&= S\Big(S\big(N(a),a\big)\ , N(b)\Big) \quad \text{(S2-Associativity)} \\
&= S\big(1\ , N(b)\big) \quad \text{((S,N) satisfies (LEM))} \\
&= 1 \quad \text{(Remark 3)}.
\end{aligned}$$

Thus, we obtain the desired result. □

Theorem 8. *If the QL-implication $I_{S,N,T}$ is constructed using a strictly increasing t-conorm S on $[0,1]$, a t-norm T, and a fuzzy negation N, and it satisfies the condition (5), then it follows that the pair (S,N) fulfills (LEM), and $T = T_{min}$.*

Proof. Suppose that t-conorm S strictly increasing in $[0,1)$, a t-norm T, a fuzzy negation N and QL-implication $I_{S,N,T}$ satisfy the Equation (5). Therefore, for any $b \in [0,1]$, we can write that

$$\begin{aligned}
1 = I\big(1, I(b,1)\big) &= S\Big(N(1)\ ,\ T\big(1, I(b,1)\big)\Big) \\
&= S\Big(N(1)\ ,\ T\Big(1, S\big(N(b), T(b,1)\big)\Big)\Big) \ \text{(By def. of QL-implication)} \\
&= S\Big(0\ ,\ S\big(N(b),b\big)\Big) \quad \text{(N1-Preservation of constants and T4-identity)} \\
&= S\big(N(b),b\big) \quad \text{(S4-identity)}.
\end{aligned}$$

Thus, (S, N) satisfies (LEM). From assumption, since QL-implication $I_{S,N,T}$ satisfies the Equation (5), we have $I\big(a, I(1, a)\big) = 1$. Therefore, we deduce that

$$
\begin{aligned}
1 = I\big(a, I(1, a)\big) &= S\Big(N(a)\ ,\ T\big(a, I(1, a)\big)\Big) \\
&= S\Big(N(a)\ ,\ T\Big(a, S\big(N(1), T(1, a)\big)\Big)\Big) \ \text{(By def. of QL-implication)} \\
&= S\Big(N(a)\ ,\ T\Big(a, S\big(0, a\big)\Big)\Big) \quad \text{(N1 and T4-identity)} \\
&= S\big(N(a), T(a, a)\big) \quad \text{(S4-identity)}.
\end{aligned}
$$

Moreover, (S, N) satisfies (LEM), for any $a \in [0, 1]$, we have

$$S\big(N(a), T(a, a)\big) = 1 = S\big(N(a), a\big). \tag{8}$$

Now, for case $a = 1$ yields $T(a, a) = a$. And for case $a \in [0, 1)$, since S is strictly increasing t-conorm in $[0, 1)$

$$S\big(N(a), T(a, a)\big) = S\big(N(a), a\big)$$

implies $T(a, a) = a$. Consequently, we have proved for any $a \in [0, 1]$ that $T(a, a) = a$. It means that T is an idempotent t-norm. We know that T_{min} is the only idempotent t-norm. Thus, we obtain $T = T_{min}$. □

Theorem 9. *Let $I_{S,N,T}$ be a QL-implication generated by a strictly increasing t-conorm S in $[0, 1)$, a fuzzy negation N and a t-norm T. QL-implication $I_{S,N,T}$ satisfies (5) iff (S, N) satisfies (LEM) and $T = T_{min}$.*

Proof. Straightforward from Theorems 7 and 8. □

These results indicate that the generalization of a classical implication to a fuzzy context—a (S, N)-implication—satisfies Equation (5) if, and only if, (S, N) adheres to (LEM).

The following example illustrates that the conditions of the established theorems align with the well-known results of classical logic.

Example 4. *In classical logic, stating that for any proposition p and q, the following expression is a tautology*

$$\big[(p \Rightarrow q) \wedge q\,'\big] \Rightarrow p\,' \tag{9}$$

Indeed, according to the definition of implication, by employing the law of classical negation, the principle of distributivity, and Aristotle's Law of the Excluded Middle, this equivalence can be established:

$$
\begin{aligned}
\big[(p \Rightarrow q) \wedge q\,'\big] \Rightarrow p\,' &\equiv \big[(p' \vee q) \wedge q\,'\big]' \vee p\,' \equiv \big[(p \wedge q') \vee q\big] \vee p\,' \\
&\equiv \big[(p \vee q) \wedge (q' \vee q)\big] \vee p\,' \equiv \big[(p \vee q) \wedge 1\big] \vee p\,' \\
&\equiv (p \vee q) \vee p\,' \equiv (p \vee p') \vee\ q \\
&\equiv 1 \vee q \equiv 1
\end{aligned}
$$

Now, the conditions for translating Equation (9) to Formula (10) within the fuzzy logic framework can be established. These conditions will outline the necessary relationships and properties required for the translation to hold, ensuring consistency with the underlying principles of fuzzy systems.

Theorem 10. *Let T, S, N and $I_{S,N}$ be a t-norm, t-conorm, a fuzzy negation, and (S,N)-implication, respectively. If (S,N) fulfills (LEM) and $T = T_{min}$, then, the subsequent translation is valid:*

$$I\Big(T\Big(I(a,b), N(b)\Big), N(a)\Big) = 1 \quad , \; a, b \in [0,1] \tag{10}$$

Proof. To prove that, by using definition of (S,N)-implication, we conclude that

$$\begin{aligned}
I\Big(T\Big(I(a,b), N(b)\Big), N(a)\Big) &= I\Big(T\Big(S(N(a),b), N(b)\Big), N(a)\Big) \\
&= S\Big(N\Big(T(S(N(a),b), N(b))\Big), N(a)\Big) \\
&= S\Big(S\Big(T(a, N(b)), b\Big), N(a)\Big) \text{(By def. of classical negation)} \\
&= S\Big(T\Big(S(a,b), S(N(b),b)\Big), N(a)\Big) \text{(the distributive property)} \\
&= S\Big(T\Big(S(a,b), 1\Big), N(a)\Big) \quad \text{((S,N) satisfies (LEM))} \\
&= S\Big(S(a,b), N(a)\Big) \quad \text{(T4-identity)} \\
&= S\Big(a, S(b, N(a))\Big) \quad \text{(S2-Associativity)} \\
&= S\Big(a, S(N(a), b)\Big) \quad \text{(S1-Commutativity)} \\
&= S\Big(S(a, N(a)), b\Big) \quad \text{(S2-Associativity)} \\
&= S(1, b) \quad \text{((S,N) satisfies (LEM))} \\
&= 1 \quad \text{(Remark 3)}
\end{aligned}$$

□

4. Discussion

This paper delves into the connections between classical and fuzzy logic properties, offering a comprehensive framework for analysis. It demonstrates their application through t-norms, t-conorms, fuzzy negations, and implications. Key principles from classical logic—such as the Law of the Excluded Middle, the distributive property of disjunction over conjunction, and the law of contraposition—can be generalized by adapting the combinatorial rules of classical equations to a fuzzy logic context. We examine the well-known Boolean law (4) by converting it into a fuzzy context using Equation (5), addressing how specific classical properties may not hold equivalently when N is treated solely as a fuzzy negation without additional assumptions. We outline the conditions for various properties of this generalization to be valid within fuzzy logic. It allows for the extension of logical rules by reinterpreting classical combinatorial equations within fuzzy frameworks. Moreover, the provided examples underscore the practical advantages of this methodology and its impact across diverse applications. The study also highlights the broader potential of fuzzy logic to address complexities beyond the scope of traditional Boolean logic by providing a more in-depth explanation of the subject.

5. Conclusions

The study focuses on the critical role that Boolean laws play in connecting classical logic and fuzzy set theory. We establish the conditions under which these generalizations are valid within fuzzy logic by utilizing t-norms, t-conorms, negation operators, and fuzzy implications. By exploring Boolean laws alongside fuzzy logic properties, the study provides readers with valuable insights into the adaptability and scope of these concepts across various mathematical structures. We give examples to demonstrate the methodology's strengths and practical advantages over established techniques. For further research, it can be investigated whether some properties that are valid in classical logic can also be applied to fuzzy logic. If certain principles are incompatible with fuzzy logic, necessary and sufficient conditions can be developed to ensure the reliability of these generalizations.

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A New Graph Vulnerability Parameter: Fuzzy Node Integrity

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Abstract: Robustness in networks plays a vital role in mitigating the effects of failures caused by nodes or links, which can disrupt essential services. Among the various vulnerability parameters in graph theory, such as connectivity and integrity, their applications to fuzzy graphs remain underexplored, despite fuzzy graphs being a powerful tool for modeling uncertainty. In this paper, we introduce the parameter 'fuzzy node integrity', which considers both the number of disrupted elements and the strength of residual connections. We derive general formulas for different types of symmetric and asymmetric fuzzy graph structures, including cycle graphs, wheel graphs, and star graphs, to systematically demonstrate the utility of this parameter. The proposed parameter is then applied to a military logistics problem to gain insights into the identification of critical nodes and route optimization under uncertainty. This study bridges an important gap in fuzzy graph theory by redefining node integrity through the inclusion of connection strength, offering a promising tool for assessing network vulnerability. These findings lay the foundation not only for theoretical research but also for practical improvements in transportation, disaster management, and communication networks.

Keywords: fuzzy graphs; vulnerability of graphs; fuzzy node connectivity; fuzzy integrity

1. Introduction

Most modern systems consist of networks, including communication networks, transportation systems, and infrastructures for water distribution. In each of these, a failure in either a node or a link can have serious repercussions on their functions, underlining the importance of parameters related to their vulnerability in studies. Connectivity, integrity, tenacity, toughness, and rupture degree are some of the metrics for network vulnerability that are indispensable for understanding the resilience of a network to disruption and for developing effective measures to strengthen its resistance [1–5].

1.1. Literature Review

Fuzzy graphs were first proposed by Rosenfeld [6] and Yeh and Bang [7], based on the fuzzy sets and fuzzy relations introduced by Zadeh [8]. Thus, graph theory has been systematically extended to account for uncertainties in real-world systems. By allowing the relationships between nodes to be expressed as a degree of membership from 0 to 1, the representation using fuzzy graphs becomes seamless in modeling network dynamics. This flexibility makes fuzzy graphs even more useful for modeling systems where information is uncertain, imprecise, or incomplete.

Fuzzy graphs have applications in various fields, including research on directed networks, rough fuzzy digraphs, and toll road networks [9–11], as well as indices such as the Hyper–Wiener index and Zagreb indices, with applications in areas like the stock market, chemistry, and crime analysis [12–16]. Despite some advancements in fuzzy graph research, studies on its vulnerability parameters remain scarce compared to classical graph theory. Recent works investigated numerous methods for evaluating network vulnerabilities in the context of uncertainty, such as optimization-based methods [17]. Nevertheless, fuzzy graph-based methods remain critical in order to represent the gradual decline in network resilience. A few pioneering works have introduced new vulnerability concepts in fuzzy graphs, such as those by [18–21], which explore resilience against different types of structural attacks. Other advanced studies have examined various aspects of fuzzy connectivity, including generalized cycle connectivity [20], edge connectivity measures [21,22], and strong domination integrity [23]. Although these studies have significantly contributed to understanding fuzzy graph vulnerability, they fail to adequately capture the role of connection strength in maintaining network resilience.

1.2. Problem Statement

Network vulnerability has long been studied in classical graph theory, with integrity being one of the primary parameters used to measure network robustness. Integrity is typically defined as the sum of the number of disconnected components and the size of the largest remaining connected subgraph after disruption [2]. In fuzzy graphs, node failure does not necessarily lead to disconnection but rather to a reduction in the strength of connectedness between nodes. Existing definitions, such as those proposed by Saravanan et al. [24,25], fail to account for this and consider only the disconnection of graphs. Moreover, existing studies on fuzzy graph vulnerability do not explicitly address the role of network topology and redundancy in connectivity in resisting network failures.

1.3. Research Gap

Although integrity metrics have been extensively investigated in classical graphs, their extension to fuzzy graphs remains in its early stages. A fundamental gap exists in formulating a vulnerability index that captures the strength of connectedness between nodes rather than merely their disconnection. While some studies have explored connectivity measures in fuzzy graphs [26,27], they fail to account for the dynamic process of node failures and their impact on the residual network structure. Recent research on fuzzy graph-based vulnerability analysis [22,28] has proposed alternative approaches; however, these still lack an integrated framework that coherently incorporates both uncertainty and robustness.

1.4. Main Contributions

This study introduces a novel approach to measuring vulnerability, termed fuzzy node integrity. This method utilizes the strength of connectedness to assess network robustness under uncertainty. The key contributions of this study are as follows:

1. Redefining the integrity parameter of fuzzy graphs to incorporate the strength of connectedness, providing a more comprehensive measure of network vulnerability and extending earlier works [23,29].
2. Establishing general formulae for the fuzzy node integrity of various graph structures, including cycle graphs, wheel graphs, and star graphs, which exhibit different degrees of structural symmetry.
3. Demonstrating the utility of the proposed measure through its application to a real-world military logistics problem, where fuzzy node integrity is used for critical node detection and route optimization under uncertainty, building upon previous works [10,15].
4. Enhancing existing measures by introducing a parameter that balances both local and global structural resilience, capturing more subtle changes in network robustness than classical integrity measures. Unlike traditional integrity and connectivity metrics, the

proposed measure accounts for gradual degradations in network strength rather than treating failures as binary disconnections.

5. Providing a general and flexible framework that can be applied to a wide range of real-world problems, including transportation systems, communication networks, and cybersecurity, where network stability is influenced by uncertainty.

By addressing these limitations, this study contributes to the theoretical advancement of fuzzy graph vulnerability metrics and their application to real-world network systems. Future research will explore extensions of fuzzy node integrity to dynamic and time-evolving networks.

This research contributes both theoretically and practically by deriving general formulae for specific fuzzy graph structures and presenting an application to a military logistics case study, demonstrating its potential utility. Although this study highlights possible applications in fields such as disaster management and infrastructure planning, it serves as an initial step toward broader implementation. The role of symmetry in network failure scenarios can be further explored in future studies to refine these metrics. Future research should focus on further improving this metric, developing computationally efficient algorithms for large-scale networks, and exploring its applications in dynamic and evolving systems.

2. Preliminaries

Some fundamental definitions are provided in this section. Those not included here can be found in [26,30,31].

Definition 1 ([32])**.** *A fuzzy graph, referred to as an f-graph, is a structure $G = (V, \sigma, \mu)$, where V is the set of nodes, σ is a fuzzy subset of V, and μ is a fuzzy relation on σ, such that $\mu(u,v) \leq \sigma(u) \wedge \sigma(v), \quad \forall u,v \in V$.*

Definition 2 ([33])**.** *An arc with the least membership value is called the weakest arc. Let $d(\mu) = \wedge\{\mu(u,v) \mid (u,v) \in \mu^*\}$ and $h(\mu) = \vee\{\mu(u,v) \mid (u,v) \in \mu^*\}$. $d(\mu)$ is called the depth of μ and $h(\mu)$ is called the height of μ.*

Definition 3 ([32])**.** *A fuzzy cycle is referred to as a multimin cycle if it has more than one weakest arc. It is called a locamin cycle if every node is incident with a weakest arc.*

Definition 4 ([34])**.** *The strength of connectedness between two nodes, u and v, which is the maximum of the strengths of all paths between them, is represented by $CONN_G(u,v)$. A fuzzy graph $G = (V, \sigma, \mu)$ is connected if, for every u, v in σ^*, $CONN_G(u,v) > 0$.*
We accept that G is connected throughout this paper.

Definition 5 ([30])**.** *Let (u,v) be an arc of G. It is called α-strong, if $\mu(u,v) > CONN_{G-(u,v)}(u,v)$; if $\mu(u,v) = CONN_{G-(u,v)}(u,v)$, then it is β-strong, and if $\mu(u,v) < CONN_{G-(u,v)}(u,v)$, then it is called a δ-arc.*

Definition 6 ([32])**.** *If (u,v) is a strong arc, then u and v are strong neighbors, and an $u - v$ path P is called a strong path if all the arcs in P are strong.*

Definition 7 ([30])**.** *If the removal of an arc reduces the strength of connectedness between any pair of nodes, then that arc is called a fuzzy bridge. Accordingly, if the removal of a node results in a reduction in the strength of connectedness between any pair of nodes, then it is referred to as a fuzzy cut node. A connected fuzzy graph $G = (V, \sigma, \mu)$ is a block if G has no fuzzy cut nodes. If a node has at most one strong arc incident with it, it is called a fuzzy end node of G.*

Definition 8 ([35])**.** *Let $G = (V, \sigma, \mu)$ be a fuzzy graph where $|\mu^*| = m$. Then, $\{q_1, q_2, ..., q_{m-1}, q_m\}$ is called the arc-strength sequence of G with $q_1 \leq q_2 \leq q_2 \leq ... \leq q_{m-1} \leq q_m$ where q_i is the strength of arc i for $0 < q_i \leq 1$. Specifically, q_1 is the depth of μ and q_m is the height of μ.*

Definition 9 ([26])**.** *Let $G = (V, \sigma, \mu)$ be a connected fuzzy graph. Fuzzy node cut (FNC) is defined as a collection of nodes $S = \{v_1, v_2, ..., v_m\}$, for each pair of nodes $u, v \in \sigma^*$ and $u, v \neq v_i$ for $i = 1, 2, ..., m$, if either $CONN_{G-S}(u, v) < CONN_G(u, v)$ or $G - S$ is trivial.*
If there are m nodes in S, then S is called an $m - FNC$. Obviously, a $1 - FNC$ is a singleton set $S = \{u\}$, where u is a fuzzy cut node.

Definition 10 ([26])**.** *Let S be a fuzzy node cut in G. The strong weight of S, denoted by $s(S)$, is defined as $s(S) = \sum_{u \in S} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the weights of strong arc incident on u.*

Definition 11 ([26])**.** *The least strongest weight of fuzzy node cuts of a connected fuzzy graph G is called the fuzzy node connectivity of G. It is denoted by $\kappa(G)$.*

3. Fuzzy Node Integrity

The concept of integrity for fuzzy graphs, as initially defined by Saravanan et al. [24], is

$$\widetilde{I} = min\Big\{|S| + m(G - S) : S \in V\Big\} \tag{1}$$

where $|S|$ denotes the sum of membership values of nodes in S, and $m(G - S)$ is the total membership value of the largest connected component in $G - S$. However, this definition does not take into consideration the strength of connectedness, which is an indispensable characteristic for network vulnerability modeling under uncertainty.

To handle this limitation, we have introduced a new parameter, fuzzy node integrity, which includes the strength of connectedness and thereby provides a more refined degree of network robustness under uncertain conditions.

Definition 12. *The fuzzy node integrity of a connected fuzzy graph G is*

$$I_f(G) = \min\Big\{s(S) + m_f(G - S) : S \subseteq \sigma^*\Big\} \tag{2}$$

where

- *$s(S)$: The strong weight of the fuzzy node cut S, defined in [26], is calculated as the sum of membership values:*
$$s(S) = \sum_{u \in S} \mu(u, v) \tag{3}$$
where $\mu(u, v)$ is the minimum of the membership values of strong arcs incident on u.
- *$CONN_{G-S}(u, v)$: The strength of connectedness between nodes u and v in the residual graph $G - S$.*
- *$m_f(G - S)$: The residual strength of connectedness is the maximum of the strengths of all paths among the nodes with diminishing strength of connectedness in the residual graph, defined in [35] as*
$$m_f(G - S) = \max\Big\{CONN_{G-S}(u, v) : CONN_{G-S}(u, v) < CONN_G(u, v)\Big\} \tag{4}$$

where $u, v \in \sigma^* - S$. If $G - S$ is trivial, then $m_f(G - S) = 0$.
Any set S of nodes that realizes the integrity value is called an I_f-set.

The fuzzy node integrity parameter extends traditional metrics by incorporating the strength of connectedness into vulnerability analysis. It provides a systematic approach for evaluating network robustness in uncertain environments and lays the groundwork for further theoretical and practical developments.

In practical applications, membership values express the notion of reliability or strength of nodes and edges represented by $\sigma(u)$ and $\mu(u,v)$, respectively. These values will be determined based on real factors, such as terrain conditions, node reliability, traffic flow capacities, etc. In real-world applications, membership values may be estimated by expert judgment, data analysis, or simulations of one form or another, depending upon the system being modeled.

The newly defined parameter fuzzy node integrity value for a simple graph given in Figure 1 is evaluated in detail in the following example.

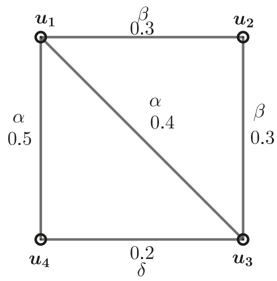

Figure 1. A fuzzy graph with fuzzy node integrity value $I_f(G) = 0.5$.

Example 1. *To compute the fuzzy node integrity $I_f(G)$ of the fuzzy graph G shown in Figure 1, we will analyze all possible node cuts S. The membership values and the types of arcs, α-strong, β-strong or δ-arc, are given in the figure. The strong weight of each node is obtained as follows: $s(u_1) = \min\{\mu(u_1,u_2), \mu(u_1,u_3), \mu(u_1,u_4)\} = 0.3$, $s(u_2) = \mu(u_1,u_2) = 0.3$, $s(u_3) = \min\{\mu(u_1,u_3), \mu(u_2,u_3)\} = 0.3$, $s(u_4) = \mu(u_1,u_4) = 0.5$.*

The calculations are summarized in Table 1, which displays key values such as the strong weight $s(S)$, the strength of connectedness $CONN_G(u,v)$, the residual strength of connectedness $m_f(G-S)$, and the resulting fuzzy node integrity $I_f(G)$.

Table 1. Summary of calculations for fuzzy node integrity of the graph in Figure 1.

S	$u-v$	$CONN_{G-S}(u,v)$	$CONN_G(u,v)$	$m_f(G-S)$	$s(S)$	$s(S)+m_f(G-S)$
$\{u_1\}$	u_2-u_3	0.3	0.3	–	0.3	–
$\{u_1\}$	u_2-u_4	0.2	0.3	0.2	0.3	0.5
$\{u_1\}$	u_3-u_4	0.2	0.4	0.2	0.3	0.5
$\{u_1,u_2\}$	u_3-u_4	0.2	0.4	0.2	0.6	0.8
$\{u_1,u_3\}$	u_2-u_4	0	0.3	0	0.6	0.6
$\{u_2,u_3\}$	u_1-u_4	0.5	0.5	–	0.6	0.6
$\{u_1,u_2,u_3\}$	u_4	–	–	0	0.9	0.9
$\{u_1,u_2,u_4\}$	u_3	–	–	0	1.1	1.1
$\{u_2,u_3,u_4\}$	u_1	–	–	0	1.1	1.1

From Table 1, we observe the following:

- *Node Cut* $S = \{u_1\}$*: The residual strength of connectedness is* $m_f(G-S) = \max\{CONN_{G-S}(u_2, u_4), CONN_{G-S}(u_3, u_4)\} = 0.2$*, and the strong weight of the node cut is* $s(S) = 0.3$*. Combining these values, the fuzzy node integrity is*

$$I_f(G) = s(S) + m_f(G-S) = 0.5. \tag{5}$$

- *Node Cut* $S = \{u_1, u_2\}$*: Removing* u_1 *and* u_2*, the residual strength of connectedness is* $m_f(G-S) = CONN_{G-S}(u_3, u_4) = 0.2$*, and the strong weight of this cut is* $s(S) = 0.6$*. Thus,*

$$I_f(G) = 0.6 + 0.2 = 0.8. \tag{6}$$

- *Node Cut* $S = \{u_1, u_2, u_3\}$*: Removing* u_1*,* u_2*, and* u_3*, only* u_4 *remains, leaving no connections in the residual graph. The strong weight is* $s(S) = 0.9$*, and* $m_f(G-S) = 0$*. Therefore,*

$$I_f(G) = 0.9. \tag{7}$$

By analyzing all possible cuts, the minimum fuzzy node integrity value of the fuzzy graph given in Figure 1 is

$$I_f(G) = \min\{0.5, 0.8, 0.9\} = 0.5. \tag{8}$$

The fuzzy node integrity value for a fuzzy wheel graph given in Figure 2 is evaluated in the following example.

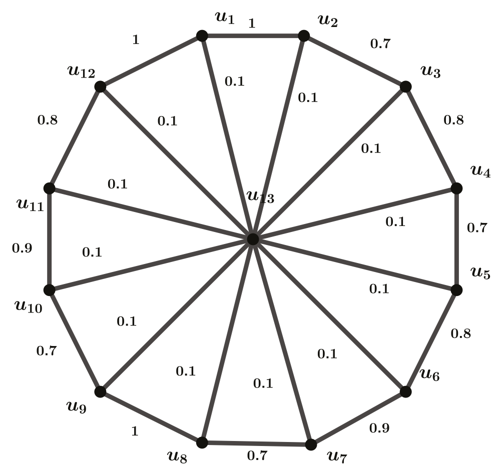

Figure 2. A fuzzy wheel graph G.

Example 2. *According to Figure 2, let us consider the possible fuzzy node cuts.*
For $S_1 = \{u_1\}$, $s(S_1) = 0.1$ *and*
$CONN_{G-S_1}(u_{12}, u_2) = 0.7 < CONN_G(u_{12}, u_2) = 1$,
and for $S_2 = \{u_6\}$, $s(S_2) = 0.1$ *and*
$CONN_{G-S_2}(u_5, u_7) = 0.7 < CONN_G(u_5, u_7) = 0.8$.
Also, $S_3 = \{u_7\}$, $S_4 = \{u_8\}$, $S_5 = \{u_{11}\}$, $S_6 = \{u_{12}\}$ *are the only 1-FNC of G. We obtain* $m_f(G-S) = 0.7$ *for 1-FNC of G by definition. So,* $I_f(G) \leq 0.8$.
For $S_7 = \{u_6, u_8\}$, $CONN_{G-S_7}(u_5, u_7) = 0.1 < CONN_G(u_5, u_7) = 0.8$, $s(S_7) = 0.2$ *and* $m_f(G-S_7) = 0.1$*. Hence,* $I_f(G) \leq 0.3$.
For $S_8 = \{u_7, u_8\}$, $CONN_{G-S_8}(u_6, u_9) = 0.7 < CONN_G(u_6, u_9) = 0.9$ *and* $s(S_8) = 0.2$*. Thus,* $m_f(G-S_8) = 0.7$ *and* $I_f(G) \leq 0.9$.
For $S_9 = \{u_2, u_8, u_{13}\}$, $CONN_{G-S_9}(u_7, u_9) = 0 < CONN_G(u_7, u_9) = 1$ *and* $s(S_9) = 0.3$*. Then,* $m_f(G-S_9) = 0$*. So,* $I_f(G) \leq 0.3$.

Among the fuzzy node cuts of G, S_7 and S_9 are I_f- sets of G. By taking the minimum, we obtain $I_f(G) = 0.3$, according to the definition of fuzzy node integrity.

To determine the fuzzy node connectivity of the fuzzy graph in Figure 2, the fuzzy node cut of G with the minimum strong weight is $S = \{u_1\}$, where $s(S) = 0.1$ and $CONN_{G-S}(u_{12}, u_2) = 0.7 < CONN_G(u_{12}, u_2) = 1$. Thus, $\kappa(G) = 0.1$. In addition, each of the nodes in $\{u_6, u_7, u_8, u_{11}, u_{12}\}$ provides the fuzzy node connectivity value since all the 1-FNC sets have the same strong weight value.

This example shows that the fuzzy node cuts may not be the same, which gives the fuzzy node connectivity and the fuzzy node integrity values for a fuzzy graph.

The general formulas for the fuzzy node integrity values of different types of fuzzy graphs are obtained and given below.

Theorem 1. *Let us consider the fuzzy wheel graph W_n, where node u_n is a fuzzy hub and the remaining nodes are located on the fuzzy cycle C_{n-1}. Moreover, assume that the membership values of the arcs between nodes u_n and u_i are no greater than that of the weakest arc on the cycle, $\mu(u_n, u_i) \leq \mu(e)$ for $e \in E(C_{n-1})$, where all of the membership values of the arcs on the cycle are distinct. Consequently, the fuzzy node integrity value is*

$$I_f(W_n) = \begin{cases} 2d(\mu), & \text{if there is a FCN;} \\ 3d(\mu), & \text{otherwise.} \end{cases} \tag{9}$$

Proof. All the paths on the fuzzy wheel graph W_n are strong paths. Since a strong path contains only α-strong or β-strong arcs, the weakest arcs of W_n are β-strong and the rest of the arcs are α-strong. Apart from $i = 1, ..., n-1$, the membership values of the arcs (u_n, u_i) and weakest arcs on the fuzzy cycle C_{n-1} are equal; thus, $\mu(u_n, u_i) = d(\mu)$. Moreover, all the nodes u_i for $i = 1, ..., n-1$ are adjacent to the fuzzy hub u_n. By the definition of the fuzzy wheel graph, we obtain $s(u_i) = d(\mu)$. Thus, there are two cases according to the existence of a fuzzy cut node of W_n since $\mu(u_n, u_i) \leq \mu(e)$ for $i = 1, ..., n-1$.

Case i. Let fuzzy wheel graph W_n contain a fuzzy cut node, e.g., x, and let S be a fuzzy node cut such that $S = \{x\}$. Fuzzy hub u_n is not a fuzzy cut node since all the (u_n, u_i) arcs are $\beta - strong$ arcs. Therefore, the fuzzy cut node x should lie on the fuzzy cycle C_{n-1}. A node on a fuzzy cycle is a fuzzy cut node if and only if the node is a common node of two fuzzy bridges. An arc of a fuzzy cycle is a fuzzy bridge if and only if the arc is $\alpha - strong$ [30].
In this case, the arcs (u, x) and (x, v) which are incident to the node x on fuzzy cycle C_{n-1} are $\alpha - strong$ arcs. If the x node is removed from the graph, then

$$CONN_{G-S}(u, v) = d(\mu) < CONN_G(u, v). \tag{10}$$

Since $s(S) = d(\mu)$ and $m_f(G - S) = d(\mu)$, the fuzzy node integrity is

$$I_f(W_n) = \min\{s(S) + m_f(G - S)\} = 2d(\mu) \tag{11}$$

Case ii. Let fuzzy wheel graph W_n not contain a fuzzy cut node. For any arc (u, v) of the wheel W_n that is not strong, each path between these nodes u and v is a strong path since every path has the same strength between u and v. Therefore, the number of internally disjoint strongest paths between u and v is at most three. Hence, the cardinality of the minimal $u - v$ strength reducing set is at most three according to Menger's theorem [36]. Thus, $|S| = |S_G(u, v)| = 3$. Since $S(u_i) = d(\mu)$ for all $i = 1, ..., n$, we obtain $s(S) = 3d(\mu)$. In this case, $CONN_{G-S}(u,v) = 0$ since W_n is disconnected after removing three nodes. Thus, $m_f(G-S) = 0$ and the fuzzy node integrity is

$$I_f(W_n) = \min\{s(S) + m_f(G-S)\} = 3d(\mu) \tag{12}$$

The proof is completed by (11) and (12). □

Theorem 2. *Let W_n be a fuzzy wheel graph where the node u_n is a fuzzy hub and the nodes $u_1, u_2, ..., u_{n-1}$ are on the fuzzy cycle C_{n-1}. Assume that $\mu(u_n, u_i) \geq \mu(e)$, where all $\mu(e)$ values, the membership values of the arcs lying on the cycle, are not equal for $e \in E(C_{n-1})$. Moreover, for $i = 1, ..., n-1$, all $\mu(u_n, u_i)$ are constant. Then, the fuzzy node integrity is*

$$I_f(W_n) = h(\mu) + q_{m-1}. \tag{13}$$

Proof. **Case i.** Let $\mu(u_n, u_i) > \mu(e)$ for $e \in E(C_{n-1})$ and $i = 1, ..., n-1$, and let $\mu(u_n, u_i)$ be constant. The nodes u_i for $i = 1, ..., n-1$ lying on the cycle are incident to at most one of the strongest arcs on the fuzzy wheel graph W_n. Thus, each node u_i is a fuzzy end node. Since a node should be incident to at least two strong arcs to be a fuzzy cut node, the nodes u_i are not fuzzy cut nodes [34]. However, the node u_n is a cut node since u_n is adjacent to all the nodes u_i and $\mu(u_n, u_i) > \mu(e)$. Hence, $s(u_i) = h(\mu) = q_m$ for $i = 1, ..., n$. Let S be a fuzzy cut set of W_n. If $|S| = 1$, then $s(S) = |S|.h(\mu)$ where $S = \{u_n\}$. For $(x, y) \in E(C_{n-1})$, $\mu(x, y) = \max\{\mu(e) : \forall e \in E(C_{n-1})\} = q_{m-1}$. Hence,

$$CONN_{G-S}(x,y) = \mu(x,y) = q_{m-1} < CONN_G(x,y) = h(\mu) = q_m \tag{14}$$

Therefore, $m_f(G-S) = q_{m-1}$ and

$$I_f(W_n) = \min\{s(S) + m_f(G-S)\} = q_m + q_{m-1} \tag{15}$$

If $|S| \geq 2$, then $s(S) \geq 2h(\mu) \geq 2q_m$ and $m_f(G-S) \geq 0$. Thus,

$$I_f(W_n) = \min\{s(S) + m_f(G-S)\} \geq 2q_m \tag{16}$$

Via (15) and (16), the fuzzy node integrity is

$$I_f(W_n) = \min\{s(S) + m_f(G-S)\} = q_m + q_{m-1} \tag{17}$$

Case ii. Let $\mu(u_n, u_i)$ be constant for $i = 1, ..., n-1$ and $\mu(u_n, u_i) \geq \mu(e)$ for $e \in E(C_{n-1})$, where at least one of the arcs on the fuzzy cycle has the same membership value with $\mu(u_n, u_i)$. All $u - u_n - v$ paths of length two where u_n is the fuzzy hub and $u, v \in V(C_{n-1})$ are the strongest paths. Thus, all (u_n, u_i) arcs are strong for $i = 1, ..., n-1$. The e arcs on the cycle for $e \in E(C_{n-1})$ are strong if $\mu(u_n, u_i) = \mu(e)$; otherwise, they would be δ-arcs. In this case, the node u_n is the only fuzzy cut node of W_n since $\mu(e)$ is different for each $e \in E(C_{n-1})$ by the assumption and $s(u_i) = h(\mu)$ for $i = 1, ..., n$ since u_n is adjacent to all u_i. Let S be a fuzzy cut set of W_n. If $|S| = 1$, then $s(S) = h(\mu)$ where $S = \{u_n\}$ for $(x, y) \in E(C_{n-1})$. $\mu(x, y) = \max\{\mu(e) : \forall e \in E(C_{n-1})\} = q_{m-1}$. Thus,

$$CONN_{G-S}(x,y) = \mu(x,y) = q_{m-1} < CONN_G(x,y) = h(\mu) = q_m \tag{18}$$

Therefore, $m_f(G-S) = \mu(x,y) = q_{m-1}$ and

$$I_f(W_n) = \min\{s(S) + m_f(G-S)\} = q_m + q_{m-1} \tag{19}$$

The proof is completed by Case i and Case ii. □

Theorem 3. *Let W_n be a fuzzy wheel graph where u_n is a fuzzy hub and the nodes $u_1, u_2, ..., u_{n-1}$ are on the fuzzy cycle C_{n-1}. If $\mu(u_n, u_i)$ values are constant for $i = 1, ..., n-1$ and $\mu(u_n, u_i) < \mu(e)$ for $e \in E(C_{n-1})$, then the fuzzy node integrity is*

$$I_f(W_n) = \min\{d(\mu) + q_2, 3d(\mu)\}. \tag{20}$$

Proof. Let W_n be a fuzzy wheel graph and let $\mu(u_n, u_i)$ values be constant for $i = 1, ..., n-1$ and for all $e \in E(C_{n-1})$, $\mu(u_n, u_i) < \mu(e)$. In this case, all the paths on the wheel W_n are strong paths. Since a strong path only contains $\alpha - strong$ or $\beta - strong$ arcs, W_n does not contain any $\delta-$arc. Since u_n is adjacent to all the nodes u_i and $\mu(u_n, u_i) < \mu(e)$ for $i = 1, ..., n-1$ and $e \in E(C_{n-1})$, all the arcs (u_n, u_i) are $\beta - strong$ arcs and $\mu(u_n, u_i) = d(\mu)$. Therefore, fuzzy hub u_n is not a fuzzy cut node. Thus, $s(u_i) = d(\mu)$ for $i = 1, ..., n$ and $s(S) = |S|d(\mu)$, where S is a fuzzy node cut of W_n.
There are two cases according to the existence of a fuzzy cut node of W_n.

Case i. Let W_n have a fuzzy cut node, e.g., x. Thus, x is a fuzzy cut node if and only if x is a common vertex of two fuzzy bridges. Moreover, an arc is a fuzzy bridge if and only if the arc is $\alpha - strong$. For this reason, x is incident to the $\alpha - strong$ arcs (y, x) and (x, z), where $y, z \in \sigma^*$.
If $|S| = 1$ where $S = \{x\}$, then $s(S) = d(\mu) = q_1$. Hence,

$$CONN_{G-S}(y, z) = q_2 < CONN_G(y, z) = q_i \tag{21}$$

for $i = 3, ..., m$ and $m_f(G - S) = q_2$. Thus, the fuzzy node integrity is

$$I_f(W_n) = \min\{s(S) + m_f(G - S)\} = d(\mu) + q_2 \tag{22}$$

If $|S| = 2$, then $s(S) = |S|d(\mu) = 2d(\mu)$. Let $S = \{u_1, u_3\}$ where $u_i \in V(C_{n-1})$ for $i = 1, ..., n-1$. Then,

$$CONN_{G-S}(u_2, u_4) = d(\mu) < CONN_G(u_2, u_4) = q_i \tag{23}$$

for $i = 2, ..., m$. Hence, $m_f(G - S) = d(\mu)$ and the fuzzy node integrity is

$$I_f(W_n) = \min\{s(S) + m_f(G - S)\} = 3d(\mu) \tag{24}$$

If $|S| \geq 3$, then $s(S) = |S|d(\mu) \geq 3d(\mu)$. In this case, $m_f(G - S) \geq 0$ and the fuzzy node integrity is

$$I_f(W_n) = \min\{s(S) + m_f(G - S)\} \geq 3d(\mu) \tag{25}$$

By (22), (24) and (25),

$$I_f(W_n) = \min\{d(\mu) + q_2, 3d(\mu)\} \tag{26}$$

Case ii. Let W_n not contain a fuzzy cut node. Therefore, $|S| \geq 2$. Let $S = \{u, v\}$ where $u, v \in \sigma^*$ and $(u, v) \notin E(W_n)$. Thus, $s(S) = |S|d(\mu) = 2d(\mu)$ and $m_f(G - S) = d(\mu)$. By definition,

$$I_f(W_n) = \min\{s(S) + m_f(G - S)\} = 3d(\mu) \tag{27}$$

If $|S| \geq 3$, then $s(S) = |S|d(\mu) \geq 3d(\mu)$ and $m_f(G - S) \geq 0$. Hence,

$$I_f(W_n) = \min\{s(S) + m_f(G - S)\} \geq 3d(\mu) \tag{28}$$

Via (27) and (28), we obtain

$$I_f(W_n) = \min\{s(S) + m_f(G - S)\} = 3d(\mu) \tag{29}$$

Thus, via (26) and (29), we obtain the result. □

Lemma 1. *Let $G = (V, \sigma, \mu)$ be a fuzzy cycle graph without a fuzzy cut node. Then,*

$$m_f(G - S) = 0. \tag{30}$$

Proof. Let S be a fuzzy node cut of a fuzzy cycle G. Since there is no fuzzy cut node, $|S| \geq 2$.
Case i. If the remaining graph contains only one component, then S should contain $n - 1$ nodes to be a fuzzy node cut. Thus, there is only one remaining node and $m_f(G - S) = 0$.
Case ii. If the remaining graph has two or more components, then $m_f(G - S) = 0$ since $CONN_{G-S}(u, v) = 0$ or $CONN_{G-S}(u, v) = CONN_G(u, v)$ for all $\forall u, v \in \sigma^* - S$. Therefore, $m_f(G - S) = 0$.
The proof is completed via Case i and Case ii. □

Theorem 4. *Let $G = (V, \sigma, \mu)$ be a locamin fuzzy cycle. Then, the fuzzy node integrity is*

$$I_f(G) = 2d(\mu) \tag{31}$$

Proof. Let S be a fuzzy cut set. $|S| \geq 2$ since G has no fuzzy cut node. Therefore, $s(S) \geq 2d(\mu)$ where $s(u_i) = d(\mu)$ for $i = 1, ..., n$. Moreover, via Lemma 1, $m_f(G - S) = 0$. Hence, the value of the of fuzzy node integrity is

$$I_f(G) = \min\{s(S) + m_f(G - S)\} = 2d(\mu) \tag{32}$$

□

Theorem 5. *Let $G = (V, \sigma, \mu)$ be a fuzzy cycle graph with only one $\alpha - strong$ arc. Then, the fuzzy node integrity is*

$$I_f(G) = 2d(\mu) \tag{33}$$

Proof. If G has only one $\alpha - strong$ arc, then G is a locamin cycle. Thus, via Theorem 4, the following result is obtained:

$$I_f(G) = 2d(\mu). \tag{34}$$

□

Theorem 6. *Let $G = (V, \sigma, \mu)$ be a self-centered fuzzy cycle graph with n nodes, and let the arcs be represented by $e_i = (u_i, u_{i+1})$ for $i = 1, ..., n - 1$ and $e_n = (u_n, u_1)$. If $0 < t < s \leq 1$*
Case i. *$I_f(G) = 2d(\mu)$ if one of the following holds;*
(1) *$\mu(e_i) = t$ for $i = 1, ..., n$;*
(2) *$\mu(e_{2i-1}) = t$ and $\mu(e_{2i}) = s$ for even integer n and $i = 1, 2, ..., \frac{n}{2}$;*
(3) *$\mu(e_{2i-1}) = \mu(e_n) = t$ and $\mu(e_{2i}) = s$ for $n = 4k + 1$ where $k = 1, 2, ...$ and $i = 1, 2, ..., \frac{n-1}{2}$.*
Case ii. *If $\mu(e_{2i-1}) = \mu(e_n) = s$ and $\mu(e_{2i}) = t$ for $n = 4k - 1$ where $k = 1, 2, 3, ...$ and $i = 1, 2, ..., \frac{n-1}{2}$, then*

$$I_f(G) = \min\{h(\mu) + d(\mu), 2d(\mu)\}. \tag{35}$$

Proof. Case i. If G is a self-centered fuzzy cycle graph satisfying one of the conditions $(1 - 3)$, G is a multimin cycle since it has more than one weak arc [34]. G is also a locamin cycle since every node is incident to a weak arc. A multimin cycle is a locamin cycle if and only if it has no fuzzy cut nodes [34]. Thus, G does not have a fuzzy cut node. Then, by Lemma 1, $m_f(G-S)=0$. Since G is a locamin fuzzy cycle, we obtain the following via Theorem 4:

$$I_f(G)=2d(\mu). \tag{36}$$

Case ii. Let G be a self-centered fuzzy cycle with $\mu(e_{2i-1})=\mu(e_n)=s$ and $\mu(e_{2i})=t$ where $n=4k-1$, $k=1,2,3,...$ and $i=1,2,...,\frac{n-1}{2}$. Then, G is a multimin cycle since it has more than one weak arc. The node u_1 is the only cut node because it is incident to two $\alpha-strong$ arcs where $\mu(e_1)=\mu(e_n)=h(\mu)$. Thus, $s(u_1)=h(\mu)$ and $s(u_i)=d(\mu)$ for $i=2,3,...,n$. Let S be a fuzzy cut set of G. If $|S|=1$, then $s(S)=h(\mu)$ where $S=\{u_1\}$. Therefore,

$$CONN_{G-S}(u_2,u_n) \quad = \quad d(\mu)<CONN_G(u_2,u_n)=h(\mu) \tag{37}$$

and $m_f(G-S)=d(\mu)$. Thus, by definition,

$$I_f(G)=\min\{s(S)+m_f(G-S)\}=h(\mu)+d(\mu) \tag{38}$$

If $|S|\geq 2$, then $m_f(G-S)=0$. Because $\forall u,v\in\sigma^*-S$, we obtain $CONN_{G-S}(u,v)=0$ or $CONN_{G-S}(u,v)=CONN_G(u,v)$ for $u_1\in S$, $s(S)\geq h(\mu)+(|S|-1)d(\mu)\geq h(\mu)+d(\mu)$. Hence,

$$I_f(G)=\min\{s(S)+m_f(G-S)\}\geq h(\mu)+d(\mu). \tag{39}$$

The equality holds for $|S|=2$. Thus,

$$I_f(G)=\min\{s(S)+m_f(G-S)\}=h(\mu)+d(\mu) \tag{40}$$

For $u_1\notin S$, $s(S)\geq |S|d(\mu)\geq 2d(\mu)$. Hence,

$$I_f(G)=\min\{s(S)+m_f(G-S)\}\geq 2d(\mu). \tag{41}$$

By the definition of fuzzy node integrity, the minimum value is satisfied when $|S|=2$. Therefore,

$$I_f(G)=\min\{s(S)+m_f(G-S)\}=2d(\mu)+0=2d(\mu) \tag{42}$$

By (38),(40) and (42), we obtain

$$I_f(G)=\min\{h(\mu)+d(\mu),2d(\mu)\}. \tag{43}$$

□

Theorem 7. *Let $G=(V,\sigma,\mu)$ be a regular fuzzy cycle graph. Then, the fuzzy node integrity value*

$$I_f(G)=2d(\mu) \tag{44}$$

Proof. If G is a regular fuzzy cycle, then either the membership values $\mu(e_i)$ of the arcs $e_i\in E(G)$ are equal or the membership values of the alternate arcs are equal [37]. Thus, there are two cases for the proof.

Case i. If the membership values of all arcs are equal, then all the arcs are weakest arcs. For this reason, all nodes are central nodes. Thus, G is a self-centered fuzzy cycle [38]. By Theorem 6, we obtain

$$I_f(G)=2d(\mu) \tag{45}$$

Case ii. If the membership values of the alternate arcs are equal, then G has an even number of nodes [37]. Thus, G is a self-centered fuzzy cycle, and by Theorem 6

$$I_f(G) = 2d(\mu). \tag{46}$$

□

Theorem 8. *Consider a fuzzy graph $G = (V, \sigma, \mu)$ with a single weakest arc and a cycle graph G^*. Then, the fuzzy node integrity value is*

$$I_f(G) = q_2 + d(\mu). \tag{47}$$

Proof. Let (u_1, u_2) be the weakest arc of the cycle. It can thus be concluded that the nodes u_1 and u_2 are fuzzy end nodes, while the nodes labeled u_i for $i = 3, ..., n$ are fuzzy cut nodes [33]. Hence, (u_1, u_2) is δ-arc and $\mu(u_1, u_2) = d(\mu) = q_1$. The other arcs are α-strong. Hence, $s(u_i) = q_j$ for $i = 1, ..., n$, $j = 2, ..., m$. Assume that S is a fuzzy cut set and that there are two cases according to the cardinality of S.

Case i. Let $|S| = 1$. Then, $S = \{u_i\}$ where $i = 3, ..., n$ and $s(S) = \min\{q_j \mid j = 2, ..., m\} = q_2$. For any $x, y \in \sigma^*$,

$$CONN_{G-S}(x, y) = d(\mu) < CONN_G(x, y) = q_j \tag{48}$$

for $j = 2, ..., m$ Hence, $m_f(G - S) = d(\mu) = q_1$ and

$$I_f(G) = \min\{s(S) + m_f(G - S)\} = q_2 + d(\mu) = q_2 + q_1 \tag{49}$$

Case ii. Let $|S| \geq 2$. Then, $s(S) \geq q_2 + q_3$ and $m_f(G - S) \geq 0$. Thus,

$$I_f(G) = \min\{s(S) + m_f(G - S)\} \geq q_2 + q_3 \tag{50}$$

Therefore, by taking the minimum of Equations (49) and (50), we obtain the following result:

$$I_f(G) = \min\{s(S) + m_f(G - S)\} = q_2 + d(\mu) = q_2 + q_1. \tag{51}$$

□

Theorem 9. *Let $G = (V, \sigma, \mu)$ be a fuzzy star graph. Then, the fuzzy node integrity value is*

$$I_f(G) = d(\mu). \tag{52}$$

Proof. Let $u_1, u_2, ..., u_{n+1}$ be the nodes of a fuzzy star, where u_{n+1} is connected to all other nodes. Then, all arcs $(u_i, u_{n+1}) \in E(G)$ are $\alpha - strong$ arcs. For $i = 1, ..., n$, the node u_{n+1} is a fuzzy cut node. Assume that S is a fuzzy cut set of G.

Case i. If $|S| = 1$, then $S = \{u_{n+1}\}$ and $s(S) = d(\mu) = q_1$. Hence, $CONN_{G-S}(u_i, u_j) = 0; i, j = 1, ..., n; i \neq j$ and $m_f(G - S) = 0$. Thus, by the definition

$$I_f(G) = \min\{s(S) + m_f(G - S)\} = d(\mu) = q_1 \tag{53}$$

Case ii. If $|S| \geq 2$, then $S = \{u_1, u_2, ..., u_n\}$ for $u_{n+1} \notin S$. Therefore, $s(S) = \sum_{i=1}^{n} s(u_i) = \sum_{j=1}^{m} q_j$. Since $G - S$ contains only the node u_{n+1}, $m_f(G - S) = 0$. Thus,

$$I_f(G) = \min\{s(S) + m_f(G - S)\} = \sum_{j=1}^{m} q_j + 0 = \sum_{j=1}^{m} q_j \tag{54}$$

If $u_{n+1} \in S$, then $s(S) \geq q_1 + q_2$ and $m_f(G - S) = 0$ since for any $x, y \in \sigma^*$

$$CONN_{G-S}(x, y) = 0 < CONN_G(x, y) = q_j \tag{55}$$

Hence, by definition,

$$I_f(G) = \min\{s(S) + m_f(G - S)\} \geq q_1 + q_2 \tag{56}$$

Via (53), (54) and (56), we obtain the following result:

$$I_f(G) = d(\mu) = q_1. \tag{57}$$

□

4. Real-World Case Study: Military Logistics

In military logistics, identifying critical nodes and optimizing routes are crucial for ensuring the smooth movement of supplies and troops, even under uncertain conditions. This section utilizes the aforementioned fuzzy node integrity parameter to enhance network resilience and strategic movement planning.

4.1. Problem Definition

Military logistics is not solely concerned with finding the shortest or fastest route but rather with identifying the most suitable route between predefined nodes in a battle environment. This involves determining the optimal path based on factors beyond distance and duration, such as slope gradients, route traversability, water barriers, and vegetative obstacles.

Lytvynenko and Lytvynenko addressed the problem of determining the most comfortable routes for military unit movements between two points using fuzzy graph models in [39]. In their study, they utilized data on slope steepness, water and vegetative barriers, and other terrain features obtained through GIS analysis to define membership values for routes [39]. The fuzzy graph they obtained is illustrated in Figure 3 [39]. In this graph, as the membership values of the arcs decrease, the difficulty of crossing the roads increases.

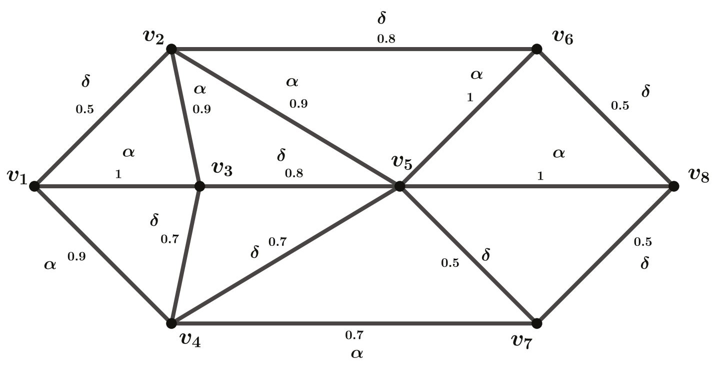

Figure 3. The fuzzy graph G obtained in [39].

One such scenario analyzed in this study involves identifying the most comfortable route between nodes v_1 and v_8 for troop movement, as shown in Figure 3. Lytvynenko and Lytvynenko determined the most comfortable route as $\{v_1, v_3, v_2, v_5, v_8\}$ in [39]. However, their proposed solution does not fully consider network resilience against node failures. This study further extends their work by incorporating fuzzy node integrity, which not only verifies the feasibility of existing routes but also assesses network resilience against node failures.

4.2. Application of Fuzzy Node Integrity to Military Logistics

The logistics network is modeled as a fuzzy graph G, given in Figure 3, where nodes represent key military locations, edges represent roads between locations, and membership values indicate road suitability.

The objective of this analysis is to evaluate network susceptibility and identify the critical nodes and roads in the event of disruptions. The proposed strategy consists of the following steps:

1. Finding the adjacency matrix of the fuzzy graph G, which is given in Figure 3.
2. Finding the matrix including strength of connectedness for all possible node pairs.
3. Determining the strong arcs.
4. Finding the strongest weights of all nodes.
5. Determining fuzzy cut nodes.
6. Computing fuzzy node connectivity to identify the network's weakest nodes and fuzzy node integrity to estimate the impact of node failures.

- **Step 1:** The adjacency matrix $A[v_i, v_j]$ represents the membership values of the arcs between the nodes v_i and v_j given in Table 2.

Table 2. Adjacency matrix A and strength of connectedness matrix C.

A	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	0	0.5	1.0	0.9	0.0	0.0	0.0	0.0
v_2	0.5	0	0.9	0.0	0.9	0.8	0.0	0.0
v_3	1.0	0.9	0	0.7	0.8	0.0	0.0	0.0
v_4	0.9	0.0	0.7	0	0.7	0.0	0.7	0.0
v_5	0.0	0.9	0.8	0.7	0	1.0	0.5	1.0
v_6	0.0	0.8	0.0	0.0	1.0	0	0.0	0.5
v_7	0.0	0.0	0.0	0.7	0.5	0.0	0	0.5
v_8	0.0	0.0	0.0	0.0	1.0	0.5	0.5	0
C	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	0	0.9	1.0	0.9	0.9	0.9	0.7	0.9
v_2	0.9	0	0.9	0.9	0.9	0.9	0.7	0.9
v_3	1.0	0.9	0	0.9	0.9	0.9	0.7	0.9
v_4	0.9	0.9	0.9	0	0.9	0.9	0.7	0.9
v_5	0.9	0.9	0.9	0.9	0	1.0	0.7	1.0
v_6	0.9	0.9	0.9	0.9	1.0	0	0.7	1.0
v_7	0.7	0.7	0.7	0.7	0.7	0.7	0	0.7
v_8	0.9	0.9	0.9	0.9	1.0	1.0	0.7	0

- **Step 2:** The strength of the connectedness matrix $C[v_i, v_j]$ represents the strength of connectedness between the nodes v_i and v_j, denoted by $CONN_G(v_i, v_j)$, which is the maximum of the strengths of all paths between these nodes given in Table 2.
- **Step 3:** If $\mu(v_i, v_j) \geq CONN_{G-(v_i,v_j)}(v_i, v_j)$, then the arc between the nodes v_i and v_j is called a strong arc; otherwise, it is a δ-arc. If the off-diagonal entries of the strength of the connectedness matrix are equal to the non-zero off-diagonal entries of the adjacency matrix, then the corresponding arcs are called strong arcs. Checking these matrices gives the set of strong arcs for the fuzzy graph G. $L =$

$\{(v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_5), (v_4, v_7), (v_5, v_6), (v_5, v_8)\}$. To decide if the arcs are α-strong or β-strong, we made a table to check if either $\mu(v_i, v_j) > CONN_{G-(v_i,v_j)}(v_i, v_j)$ for α-strong or $\mu(v_i, v_j) = CONN_{G-(v_i,v_j)}(v_i, v_j)$ for β-strong.

According to Table 3, all the arcs in this graph are either α-strong arcs or δ-arcs, as shown in Figure 3.

Table 3. Strong arcs of the fuzzy graph G.

$u-v$	$CONN_G(u,v)$	$CONN_{G-(u,v)}(u,v)$	**Strength**
$v_1 - v_3$	1	0.7	α-strong
$v_1 - v_4$	0.9	0.7	α-strong
$v_2 - v_3$	0.9	0.8	α-strong
$v_2 - v_5$	0.9	0.8	α-strong
$v_4 - v_7$	0.7	0.5	α-strong
$v_5 - v_6$	1	0.8	α-strong
$v_5 - v_8$	1	0.5	α-strong

- **Step 4:** The strong weight of a node v is the minimum of the weights of strong arcs incident on v. The strongest weight of each node of G is given in Table 4.

Table 4. The strongest weight of each node of G.

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$s(v)$	0.9	0.9	0.9	0.7	0.9	1	0.7	1

- **Step 5:** The nodes v_1, v_2, v_3, v_4, v_5 are fuzzy cut nodes since each one is a common node of at least two α-strong arcs [30].
- **Step 6:** The calculations are summarized in Table 5, which displays key values, such as fuzzy cut nodes; corresponding strong weights; the strength of connectedness $CONN_G(u,v)$ and $CONN_{G-S}(u,v)$, where the values decrease for those nodes after removing S; the residual strength of connectedness $m_f(G-S)$; and fuzzy node integrity values corresponding to the fuzzy cut nodes.

Table 5. Summary of calculations for fuzzy node connectivity and fuzzy node integrity of the fuzzy graph G in Figure 3.

S	$u-v$	$CONN_G(u,v)$	$CONN_{G-S}(u,v)$	$m_f(G-S)$	$s(S)$	$s(S)+m_f(G-S)$
$\{v_1\}$	$v_2 - v_4$	0.9	0.7	0.7	0.9	1.6
	$v_3 - v_4$	0.9	0.7			
	$v_4 - v_5$	0.9	0.7			
	$v_4 - v_6$	0.9	0.7			
	$v_4 - v_8$	0.9	0.7			
$\{v_2\}$	$v_1 - v_5$	0.9	0.8	0.8	0.9	1.7
	$v_1 - v_6$	0.9	0.8			
	$v_1 - v_8$	0.9	0.8			
	$v_3 - v_5$	0.9	0.8			
	$v_3 - v_6$	0.9	0.8			
	$v_3 - v_8$	0.9	0.8			
	$v_4 - v_5$	0.9	0.8			
	$v_4 - v_6$	0.9	0.8			
	$v_4 - v_8$	0.9	0.8			
$\{v_3\}$	$v_1 - v_2$	0.9	0.7	0.7	0.9	1.6
	$v_1 - v_5$	0.9	0.7			
	$v_1 - v_6$	0.9	0.7			
	$v_1 - v_8$	0.9	0.7			
	$v_2 - v_4$	0.9	0.7			
	$v_4 - v_5$	0.9	0.7			
	$v_4 - v_6$	0.9	0.7			
	$v_4 - v_8$	0.9	0.7			

Table 5. *Cont.*

S	$u-v$	$CONN_G(u,v)$	$CONN_{G-S}(u,v)$	$m_f(G-S)$	$s(S)$	$s(S)+m_f(G-S)$
$\{v_4\}$	v_1-v_7	0.7	0.5	0.5	0.7	1.2
	v_2-v_7	0.7	0.5			
	v_3-v_7	0.7	0.5			
	v_5-v_7	0.7	0.5			
	v_6-v_7	0.7	0.5			
	v_7-v_8	0.7	0.5			
$\{v_5\}$	v_1-v_6	0.9	0.8	0.8	0.9	1.7
	v_2-v_6	0.9	0.8			
	v_3-v_6	0.9	0.8			
	v_4-v_6	0.9	0.8			
	v_1-v_8	0.9	0.5			
	v_2-v_8	0.9	0.5			
	v_3-v_8	0.9	0.5			
	v_4-v_8	0.9	0.5			
	v_6-v_8	1.0	0.5			
	v_7-v_8	0.7	0.5			

Other fuzzy node cuts having two or more nodes were not considered since fuzzy node connectivity and fuzzy node integrity are equal to the minimum values.

Fuzzy node connectivity of this graph is $\kappa(G) = \min\{s(S)\} = 0.7$ and fuzzy node integrity is $I_f(G) = \min\{s(S) + m_f(G-S)\} = 1.2$.

The strength of connectedness between v_1 and v_8 is equal to 0.9, and the strongest path with the strength of connectedness value 0.9 is $\{v_1, v_3, v_2, v_5, v_8\}$, which is the same route as Lytvynenko and Lytvynenko found in their study [39].

In the event of any disruptions in the transport hubs during the transportation of vehicles and ammunition through routes, there may be a requirement to change the route or secure the concerned roads. Critical node identification is extremely necessary in order to counter the expected problem. Vulnerability parameters can be applied in the critical node identification. Applying the fuzzy node connectivity parameter to the graph yields $\kappa(G) = 0.7$, which aids in effectively identifying a different route. Specifically, v_4 is the cluster element that provides this value. This suggests that v_4 is a crucial node, and any malfunction in this node causes trouble when an army is transitioning from one point to another.

If a failure at node v_4 does occur, it is critical for alternative pathways to be identified. Therefore, when choosing a route, either this node should be excluded in the route or some necessary precautions should be assessed. The fuzzy node integrity parameter proposed in this paper provides a valuable tool for analyzing these issues. In that situation, graph analysis will give $I_f(G) = s(S) + m_f(G-S) = 1.2$, meaning that a failure at node v_4 will equate to a diversion in the route with passability being lowered to 0.5. Since this measure is influenced by the arc between nodes v_5 and v_7, a revalidation of road conditions is necessary. The combination of the two criteria indicates that the alternative route should try to avoid the v_4 node. If this is not feasible, either the road between the v_5 and v_7 nodes should be improved or the shipment should be modified to fit the requirements of the route.

If another fuzzy cut node would be chosen to be checked as a critical point, fuzzy node connectivity could not be utilized since four of the fuzzy cut nodes have the same strong weight. However, if fuzzy node integrity is used by performing a check, the minimum values of integrity state that the second critical node is v_3 and that it has a corresponding integrity value of 1.6. A failure at node v_3 will cause the route to be rerouted, with passability being reduced to 0.7. As this measure is influenced by the arc between nodes v_4 and v_5, a revalidation of road conditions is necessary. During the transport of vehicles and military equipment along routes, it may be necessary to change the route or secure specific roads in the case of disruptions at transfer centers.

4.3. Comparative Analysis

In order to better validate the performance of the introduced fuzzy node integrity, we are comparing it with existing network vulnerability parameters. The key performance distinctions in fuzzy node integrity in relation to fuzzy node connectivity [26] and integrity for fuzzy graphs [24] are discussed in this section.

The comparative analysis is based upon the following aspects:

- **Failure Impact:** The measure in which a method incorporates node failures and how they impact network connectivity.
- **Adaptability to Uncertainty:** Whether or not the measure uses fuzzy or probabilistic aspects to deal with uncertainty.
- **Resilience Estimation:** The ability of the method to quantify remaining connectivity following disturbances.
- **Alternative Path Planning:** If the method provides alternative route suggestions in cases of failures.

Table 6 summarizes how different methods compare based on these criteria.

Table 6. Comparative analysis of network vulnerability parameters.

Method	Failure Impact	Uncertainty Adaptation	Resilience Estimation	Alternative Path Planning	Numerical Value
Integrity	Binary Disconnection	No	No	No	Not Suitable for this Graph
Fuzzy Node Connectivity	Partial Failure Consideration	Yes	No	Limited	0.7
Fuzzy Node Integrity (Proposed)	Strength-Based Failure Consideration	Yes	Yes	Yes	1.2

This comparison highlights that fuzzy node integrity provides a more adaptive and realistic vulnerability measure than integrity and fuzzy node connectivity methods by incorporating partial degradation and a residual network strength assessment.

The fuzzy node connectivity parameter provides information about which node is critical, while the fuzzy node integrity parameter provides information not only about the critical node but also about which arcs need to be strengthened.

The concept of integrity for fuzzy graphs defined by Saravanan et al. in [24] only depends on the disconnection and the membership values of nodes. Therefore, this parameter cannot be evaluated for this type of fuzzy graph, which is given in Figure 3. It is not possible to accurately study the fuzziness of the situation as their definition relies on the disconnection. Consequently, their research motivated us to redefine the integrity parameter for fuzzy graphs.

5. Results and Discussion

The concept of 'fuzzy node integrity' was applied in a military logistics case study to measure its real-world efficacy. The implications and findings of this case study are presented in this section.

5.1. Key Insights from the Case Study

- **Identification of Critical Nodes:** The strategy was able to determine nodes whose removal would result in a substantial loss in the resilience of the network. This was especially true when taking into account the network's nodes of passage, which are the nodes with the least strong weight but the highest difficulty.

- **Scenario-based Analysis:** The following two main failure scenarios were analyzed:
  - **Targeted Node Failures:** The removal of nodes with least strong weight affected the strength of connectedness by reducing the overall efficiency of the logistics network. The outcome showed a drop in network connectivity when critical nodes were targeted.
  - **Random Node Failures:** The network could tolerate some node failures with the backup routes still being effective. The capability of the system to reroute logistics was still maintained with node failures.
- **Quantitative Performance Comparison:**
  - Integrity for fuzzy graph is useless for this problem. It cannot be evaluated in these kinds of graphs since the definition depends only on the disconnection and the membership values of nodes.
  - Fuzzy node connectivity shows enhanced flexibility compared to traditional approaches while still being short of full resilience.
  - Fuzzy node integrity proves its high resilience estimation performance by providing information not only about the critical node but also about which arcs need to be strengthened.
- **Real-Time Flexibility:** The study showed how, by recalculating fuzzy node integrity in real time, logistical operations can be reallocated dynamically to optimize routes in line with fluctuating conditions, such as unexpected disruptions or road obstructions.

5.2. Practical Implications

The findings of this study have practical implications for real-world applications, such as the following:

- **Transport Networks:** Enhancing the resilience of road and rail networks against disruptions.
- **Cybersecurity:** Applying fuzzy node integrity to assess network vulnerability in critical communication infrastructures.
- **Disaster Management:** Planning for emergency responses in dynamic environments.
- **Supply Chain Optimization:** Identifying vulnerabilities in supply chains to ensure efficient distribution of products.

5.3. Limitations and Future Work

Even though the proposed method is a notable improvement in network vulnerability estimation, certain limitations exist, such as the following:

- **Static Network Assumption:** We have assumed a static network in this model, but real-world military logistics networks are highly dynamic in nature. Future research studies should incorporate real-time adaptability in line with time-dependent fuzzy graphs.
- **Computational Complexity:** The calculation for fuzzy node integrity is becoming complex as network sizes expand. Future studies should concentrate on creating efficient techniques for dealing with real-time large-scale networks.
- **Limited Real-World Evaluation:** The method is established in a theoretical case study but is in need of empirical testing in real logistics networks in order to validate its effectiveness.

- **Interdependency Considerations:** Interdependent networks are normally engaged in logistics in armed conflicts, such as supply networks and communications networks. The application may be extended to cover multi-layer fuzzy networks.

Future research can be directed towards merging machine learning mechanisms for adapting path selection in real-time conditions in fuzzy networks. Predictive modeling based on past network outages can be utilized for network vulnerability estimation improvement. A computational structure for efficient solutions for large-scale problems and applications in multiple-layered fuzzy networks will continue to have increased real-world applications in dynamic networks.

One promising direction for further research is the extension of the fuzzy node integrity framework to neutrosophic graphs, which would contribute to even more effective vehicles for quantifying network susceptibility by incorporating a more complete representation of indeterminacy and uncertainty in complex networks.

6. Conclusions

This paper addresses an important challenge in network vulnerability studies and introduces the fuzzy node integrity parameter, a metric specifically devised for fuzzy graphs. Contrary to other integrity metrics, this parameter considers the strength of connectedness; hence, it offers a subtler and more realistic appraisal of the robustness of a network in uncertain environments. Therefore, with this contribution, new definitions were openly extended, and an important gap in the fuzzy graph theory was covered by an integrated framework covering both theoretical improvements and practical applications.

The theoretical contributions of this work are related to the derivation of general formulas for fuzzy graph structures, such as cycle graphs, wheel graphs, and star graphs, which can serve as basic tools for future research studies in the stated field. The applicability of the proposed parameter is assured by its practical relevance in a case study related to military logistics, to which it was applied with a view to finding out the most critical nodes and route optimization in uncertain conditions. This example illustrates the potential of the parameter for modeling real-world problems such as disrupted supply chains and vulnerabilities in infrastructures.

Despite these contributions, there are still some limitations, and there are avenues for future research. One limitation is the stationarity assumed in current modeling when real logistics networks are highly dynamic in nature. Real-time adaptive, time-varying fuzzy graph-based algorithms must be incorporated in future research to better capture this dynamism. Also, as the network size is increased, computational complexity in fuzzy node integrity calculations is increased as well, and efficient algorithms for larger networks must be derived. A second major area is empirical validation; although we carried out a test on a hypothetical case study in this paper, real-world validation is needed for measuring practical performance in different application domains. Also, in typical military logistics, there are multiple-layer networks like supply chains, transportation networks, and communications networks. The model could be extended to account for interdependencies across them, and this could greatly enhance its applicability and resilience.

Fuzzy node integrity methodology, as a network vulnerability testing tool, can be extended to incorporate advanced path optimization techniques and predictive analyses for predicting network vulnerabilities in complex networks. The method can be applied for enhancing network resilience in unpredictable and dynamic environments.

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The Neutrosophization of δ-Separation Axioms

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Abstract: Fuzzy topology has long been celebrated for its ability to address real-world challenges in areas such as information systems and decision making. However, with ongoing technological advancements and the increasing complexity of practical requirements, the focus has gradually shifted toward neutrosophic topology, a broader and more inclusive framework than fuzzy topology. While neutrosophic topology is primarily rooted in neutrosophic open sets, other related families, including neutrosophic pre-open sets, neutrosophic semi-open sets, and neutrosophic beta-open sets, have also proven instrumental in driving progress in this field. This study introduces neutrosophic δ-open sets as a significant enhancement to the current theoretical framework. In addition, we propose a novel category of separation axioms, termed neutrosophic δ-separation axioms, which are derived from the concept of neutrosophic δ-open sets. Moreover, we explore the interplay between these separation properties and their characteristics within subspaces. Our findings confirm that neutrosophic δ-separation axioms are reliably upheld in neutrosophic regular open subspaces.

Keywords: neutrosophic δ-open set; neutrosophic δ-interior point; neutrosophic δ-separation axioms; neutrosophic δ-continuous

MSC: 54A05; 54A40; 54A10; 54C08

1. Introduction

Fuzzy and soft sets, as in [1–5], have long been recognized as a valuable tool for reducing vagueness in data analysis, enabling decision makers to develop more precise methods that classical mathematics cannot achieve. However, the growing complexities of human life have rendered this concept insufficient, prompting intellectuals and researchers to devise various advanced mathematical approaches for data analysis.

In this context, Smarandache revolutionized the classical concept of fuzzy sets by introducing neutrosophic sets in his seminal work [6]. The concept of neutrosophic topology was later pioneered by Salama and Alblowi [7]. Since then, various scholars, including Acikgoz and Esenbel, have made substantial contributions to advancing the theory of neutrosophic sets and exploring their practical applications across multiple disciplines, as detailed in [8–20]. Moreover, as discussed in [21–32], this theory has emerged as a crucial component of scientific research in fields such as communication, engineering, education, epidemiology, pharmacy, medicine, e-learning, banking, marketing, and geography, profoundly impacting and reshaping numerous aspects of human activity.

In this paper, we introduce the notions of neutrosophic δ-closure and neutrosophic δ-interior for a neutrosophic set within a neutrosophic topological space. Building on these definitions, we develop a framework for separation axioms within neutrosophic δ-topological spaces, ensuring their logical consistency and alignment with the concept of neutrosophic δ-compactness. To achieve this, we define a novel category of separation axioms, referred to as neutrosophic δ-separation axioms, based on the concept of neutrosophic δ-open sets. Additionally, we investigate the interplay between these separation properties and subspaces, showing that neutrosophic δ-separation axioms are hereditary in neutrosophic regular open subspaces. Interestingly, these axioms exhibit unique characteristics compared to those found in general topology, adding further depth to their study.

2. Preliminaries

In this section, we provide the fundamental definitions associated with neutrosophic set theory.

Definition 1 ([6]). *A neutrosophic set E within the universal set U is described as*

$$E = \{\langle u, V_E(u), Y_E(u), W_E(u)\rangle : u \in U\},$$

where V, Y, W $: U \to]^-0, 1^+[$, *and* $^-0 \leq V_E(u) + Y_E(u) + W_E(u) \leq 3^+$.

From a scientific perspective, the membership, indeterminacy, and non-membership functions of a neutrosophic set take values from real standard or non-standard subsets of $]^-0, 1^+[$. *However, these subsets can sometimes be impractical for real-life applications, such as those in economics and engineering. To address this issue, we focus on neutrosophic sets whose membership, indeterminacy, and non-membership functions take values within the closed interval* $[0, 1]$.

Definition 2 ([33]). *Let U be a nonempty set. When m, n, and p are real standard or non-standard subsets of* $]^-0, 1^+[$, *the neutrosophic set* $u_{m,n,p}$ *is defined as a neutrosophic point in U, defined as*

$$u_{m,n,p}(u_j) = \begin{cases} (m, n, p), & \text{if } u = u_j \\ (0, 0, 1), & \text{if } u \neq u_j \end{cases}$$

For $u_j \in U$, *it is known as the support of* $u_{m,n,p}$, *where m stands for the degree of membership, n stands for the degree of indeterminacy, and p stands for the degree of non-membership of* $u_{m,n,p}$.

Definition 3 ([7]). *Consider a neutrosophic set E defined on the universe set U. The complement of E, denoted by* E^c, *is expressed as*

$$E^c = \{\langle u, W_E(u), 1 - Y_E(u), V_E(u)\rangle : u \in U\}.$$

It is clear that $[E^c]^c = E$.

Definition 4 ([7]). *Suppose that E and F are two neutrosophic sets over the universal set U. E is said to be a neutrosophic subset of F if, for every* $u \in U$, *the following conditions hold:* $V_E(u) \leq V_F(u)$, $Y_E(u) \leq Y_F(u)$, *and* $W_E(u) \geq W_F(u)$. *This is denoted by* $E \subseteq F$. *Additionally, E is considered neutrosophic equal to F if* $E \subseteq F$ *and* $F \subseteq E$. *This is represented by* $E = F$.

Definition 5 ([7]). *Assume that* W_1 *and* W_2 *are two neutrosophic sets defined on the universal set U. Their union, denoted by* $W_1 \cup W_2 = W_3$, *is defined as*

$$W_3 = \{\langle u, V_{W_3}(u), Y_{W_3}(u), W_{W_3}(u) : u \in U\rangle\},$$

where

$$V_{W_3(u)} = \max\{V_{W_1(u)}, V_{W_2}(u)\},$$

$$Y_{W_3(u)} = \max\{Y_{W_1(u)}, Y_{W_2}(u)\},$$

$$W_{W_3(u)} = \min\{W_{W_1(u)}, W_{W_2}(u)\}.$$

Definition 6 ([7]). *Assume that W_1 and W_2 are two neutrosophic sets defined on the universal set U. Their intersection, denoted by $W_1 \cap W_2 = W_4$, is defined as follows:*

$$W_4 = \{\langle u, V_{W_4}(u), Y_{W_4}(u), W_{W_4}(u) : u \in U\rangle\},$$

where

$$V_{W_4(u)} = \min\{V_{W_1(u)}, V_{W_2}(u)\},$$

$$Y_{W_4(u)} = \min\{Y_{W_1(u)}, Y_{W_2}(u)\},$$

$$W_{W_4(u)} = \max\{W_{W_1(u)}, W_{W_2}(u)\}.$$

Definition 7 ([7]). *A neutrosophic set W over the universal set U is considered a null neutrosophic set if for every $u \in U$, $V_W(u) = 0$, $Y_W(u) = 0$, and $W_W(u) = 1$. It is represented as 0_U.*

Definition 8 ([7]). *A neutrosophic set W on the universal set U is said to be an absolute neutrosophic set if for every $u \in U$, $V_W(u) = 1$, $Y_W(u) = 1$, and $W_W(u) = 0$. This is denoted as 1_U. It is evident that $0^c_U = 1_U$ and $1^c_U = 0_U$.*

Definition 9 ([7]). *Assume $NS(U)$ to be the collection of all neutrosophic sets defined on the set U, and let $\sigma \subset NS(U)$. Then, σ is said to form a neutrosophic topology on U if the following conditions are met:*

(1) Both 0_U and 1_U are elements of σ;
(2) The union of any finite or infinite number of neutrosophic soft sets in σ is contained within σ;
(3) The intersection of a finite number of neutrosophic soft sets in σ belongs to σ.

Thus, (U, σ) is referred to as a neutrosophic topological space on U. Every element of σ is called a neutrosophic open set [7].

Definition 10 ([7]). *Assume that (U, σ) is a neutrosophic topological space on U and W is a neutrosophic set defined on U. Then, W is called a neutrosophic closed set if and only if its complement is a neutrosophic open set.*

Definition 11 ([8]). *A neutrosophic point $u_{m,n,p}$ is referred to as neutrosophic quasi-coincident (or neutrosophic q-coincident for short) with W, symbolized as $u_{m,n,p}\ q\ W$, if and only if $u_{m,n,p}$ is not contained within W^c. If $u_{m,n,p}$ is not neutrosophic quasi-coincident with W, it is expressed as $u_{m,n,p}\ \tilde{q}\ W$.*

Definition 12 ([8]). *A neutrosophic set W in a neutrosophic topological space (U, σ) is called a neutrosophic q-neighborhood of a neutrosophic point $u_{m,n,p}$ if and only if there exists a neutrosophic open set Z such that $u_{m,n,p}\ q\ Z$ is contained in W.*

Definition 13 ([8]). *A neutrosophic set Z is considered neutrosophic quasi-coincident (or neutrosophic q-coincident for short) with W, denoted by Z q W, if and only if Z is not contained within W^c. If Z is not neutrosophic quasi-coincident with W, it is denoted as Z $\tilde{q}$ W.*

Definition 14 ([11]). *A neutrosophic point $u_{m,n,p}$ is referred to as a neutrosophic interior point of a neutrosophic set W if and only if there exists a neutrosophic open q-neighborhood Z of $u_{m,n,p}$ such that Z is contained in W. The collection of all neutrosophic interior points of W is called the neutrosophic interior of W, denoted by W°.*

Definition 15 ([8]). *A neutrosophic point $u_{m,n,p}$ is considered a neutrosophic cluster point of a neutrosophic set W if and only if every neutrosophic open q-neighborhood Z of $u_{m,n,p}$ is q-coincident with W. The collection of all neutrosophic cluster points of W is referred to as the neutrosophic closure of W and is denoted by $\overline{W}$.*

Definition 16 ([8]). *Assume that r is a function from U to Q, and let F be a neutrosophic set in Q with membership function $V_F(y)$, indeterminacy function $Y_F(v)$, and non-membership function $W_F(v)$. The inverse image of F under r, denoted by $r^{-1}(F)$, is a neutrosophic subset of U with its membership function, indeterminacy function, and non-membership function defined as follows: $V_{r^{-1}(F)}(u) = V_F(r(u))$, $Y_{r^{-1}(F)}(u) = Y_F(r(u))$, and $W_{r^{-1}(F)}(u) = W_F(r(u))$ for every $u \in U$.*

On the other hand, suppose that E is a neutrosophic set in U with a membership function $V_E(u)$, indeterminacy function $Y_E(u)$, and non-membership function $W_E(u)$. The image of E under the function r, denoted as $r(E)$, is a neutrosophic subset of Q whose membership function, indeterminacy function, and non-membership function are defined as follows:

$$V_{r(E)}(v) = \begin{cases} sup_{z \in r^{-1}(v)}\{V_E(z)\}, & if\ r^{-1}(v)\ is\ not\ empty, \\ 0, & if\ r^{-1}(v)\ is\ empty, \end{cases}$$

$$Y_{r(E)}(v) = \begin{cases} sup_{z \in r^{-1}(v)}\{Y_E(z)\}, & if\ r^{-1}(v)\ is\ not\ empty, \\ 0, & if\ r^{-1}(v)\ is\ empty, \end{cases}$$

$$W_{r(E)}(y) = \begin{cases} inf_{z \in r^{-1}(v)}\{W_E(z)\}, & if\ r^{-1}(v)\ is\ not\ empty, \\ 1, & if\ r^{-1}(v)\ is\ empty, \end{cases}$$

for each v in Q, where $r^{-1}(v) = u : r(u) = v$, correspondingly.

3. Some Definitions

The current part presents a series of new definitions that form the basis for the discussions in the subsequent sections.

Definition 17. *A neutrosophic set W in a neutrosophic topological space (U, σ) is said to be neutrosphic regular open if and only if $W = (\overline{W})^\circ$.*

Definition 18. *A neutrosophic point $u_{m,n,p}$ is said to be a neutrosophic δ-cluster point of a neutrosophic set E if and only if every neutrosophic regular open q-neighborhood T of $u_{m,n,p}$ is q-coincident with E. The set of all neutrosophic δ-cluster points of E is called the neutrosophic δ-closure of E and is denoted by $\overline{E}_\delta$.*

Remark 1. *For any neutrosophic set E in a neutrosophic topological space (U, σ), the δ-closure of E is represented as follows: $\overline{E}_\delta = \bigcap\{\overline{T} : E \subseteq \overline{T},\ U \in \ \sigma\}$*

Definition 19. *A neutrosophic set E is said to be a neutrosophic δ-neighborhood of a neutrosophic point $u_{m,n,p}$ if and only if there exists a neutrosophic regular open q-neighborhood N of $u_{m,n,p}$ such that $N \subseteq E$.*

Definition 20. *A neutrosophic set E is considered neutrosophic δ-closed if and only if $E = \overline{E}_\delta$. The complement of a neutrosophic δ-closed set is referred to as a neutrosophic δ-open set.*

As a δ-open set is the complement of a δ-closed set, Z is δ-open if and only if $Z = (Z)^\circ_\delta$. Additionally, it is known that $((E^c)^\circ_\delta)^c = \overline{E}_\delta$. A neutrosophic set E is considered neutrosophic δ-open in a neutrosophic topological space U if and only if, for every neutrosophic point $u_{m,n,p}$, such that $u_{m,n,p}qE$, E is a neutrosophic δ-neighborhood of $u_{m,n,p}$.

It can be easily demonstrated that $\overline{E} \subseteq \overline{E}_\delta \subseteq \overline{E}_\theta$ for any neutrosophic set E in a neutrosophic topological space U. On the other hand, for a neutrosophic open set E in a neutrosophic topological space (U, σ), we have $\overline{E} \subseteq \overline{E}_\delta$. Furthermore, it is evident that every regular open set is δ-open, and every δ-open set is open.

In any neutrosophic topological space (U, σ), the set $\overline{E}_\delta$ is a neutrosophic δ-closed set for any neutrosophic set E. In other words, $\overline{(\overline{E}_\delta)}_\delta = \overline{E}_\delta$.

Definition 21. *A function $f : X \to Y$ is called neutrosophic δ-continuous (abbreviated as n. δ.c.) if, for every neutrosophic point $u_{m,n,p}$ in X and any regular open q-neighborhood V of $f(u_{m,n,p})$ in Y, there exists a regular open q-neighborhood U of $u_{m,n,p}$ such that $f(U)$ is contained in V.*

Remark 2. *A function $r : U \to Q$ is considered neutrosophic δ-continuous if, for every neutrosophic δ-open set T in Q, the preimage $r^{-1}(T)$ is neutrosophic δ-open in U. Thus, the composition of two neutrosophic δ-continuous functions remains neutrosophic δ-continuous.*

Definition 22. *Suppose $r : U \to Q$ is a neutrosophic mapping.*

(1) *r is neutrosophic δ-open if, for every neutrosophic δ-open set E in U, the image $r(E)$ is neutrosophic δ-open in Q.*

(2) *r is considered neutrosophic δ-closed if, for every neutrosophic δ-closed set F in U, the image $r(F)$ is neutrosophic δ-closed in Q.*

Theorem 1. *Given that $r : (U, \sigma) \to (Q, \vartheta)$ is neutrosophic δ-continuous, the following conditions are interchangeable.*

(a) $r(\overline{E}_\delta) \subseteq \overline{(r(E))}_\delta$.

(b) $\overline{(r^{-1}(E))}_\delta \subseteq r^{-1}(\overline{E}_\delta)$.

(c) *Given any neutrosophic δ-closed set E in Q, the preimage $r^{-1}(E)$ is neutrosophic δ-closed in U.*

(d) *Given any neutrosophic δ-open set E in Q, the preimage $r^{-1}(E)$ is neutrosophic δ-open in U.*

Definition 23. *Assume that (U, σ) is a neutrosophic topological space and η is a neutrosophic set in (U, σ). Define $U^\eta = \{N \cap \eta : N$ is a neutrosophip subset in $(U, \sigma)\}$. The collection $\sigma^\eta = \{N \cap \eta : N \in \sigma\}$ is referred to as the neutrosophic η-topology induced by σ over η. The pair (η, σ^η) is called the subspace. The elements of σ^η are known as neutrosophic open sets in the subspace η. A set $N \in U^\eta$ is said to be neutrosophic closed in η when $\eta \cap N \in \sigma^\eta$.*

Definition 24. *Consider $u_{m,n,p}$ as a neutrosophic point within η, denoted by $u_{m,n,p} \in \eta$. A set $H \in U^\eta$ is called a neutrosophic neighborhood of $u_{m,n,p}$ in the subspace η whenever there exists a $T \in \sigma^\eta$ such that $u_{m,n,p}$ belongs to T and T is contained within H.*

Theorem 2. *Consider $T \in U^\eta$. The set U is a member of σ^η if and only if T functions as a neutrosophic neighborhood of every point $u_{m,n,p}$ that lies in T within η.*

Definition 25. *Consider $N \in U^{\eta}$. The interior of N in the subspace η is defined as the largest neutrosophic open set in η that is contained within N. In other words, $N^{\eta\circ} = sup\{T : T \subseteq N, T \in \sigma^{\eta}\}$. In a similar manner, the closure of N in η is defined as the smallest closed set in η that contains N. More precisely, $\overline{N}_{\eta} = inf\{\eta \cap T^c : N \subseteq \eta \cap T^c, T \in \sigma^{\eta}\}$.*

Definition 26. *Consider $E \in U^{\eta}$ and $u_{m,n,p} \in U$. The point $u_{m,n,p}$ is described as q-coincident with E in the subspace η when $m + V_E(u) > V_{\eta}(u)$ or $n + Y_E(u) > Y_{\eta}(u)$ or $p + W_E(u) < W_{\eta}(u)$. This relationship is denoted as $u_{m,n,p}q^{\eta}E$.*

Definition 27. *Consider $u_{m,n,p} \in \eta$. A set $H \in U^{\eta}$ is called a neutrosophic q-neighborhood of $u_{m,n,p}$ in η whenever there exists a $T \in \sigma^{\eta}$ such that $u_{m,n,p}q^{\eta}T$ and $T \subseteq H$.*

Remark 3. *Consider $T \in U^{\eta}$. The set T belongs to σ^{η} if and only if T serves as a q-neighborhood of every $u_{m,n,p}$ satisfying $u_{m,n,p}q^{\eta}T$ in η.*

Remark 4. *Consider (U, σ) as a neutrosophic topological space and (η, σ^{η}) as a neutrosophic subspace.*

(1) Given that $E \in U^{\eta}$, it follows that $E^{\circ} \subseteq E^{\eta\circ}$.
(2) In the case of any neutrosophic subset $E \in U^{\eta}$, E° is contained in $E^{\circ} \subseteq E^{\eta\circ} \cap \eta^{\circ}$.
(3) Given that $\eta \in \sigma$ and $E \in U^{\eta}$, it follows that $E^{\circ} = E^{\eta\circ}$.

4. Neutrosophic δ-Separation Axioms

In this section, we introduce new separation axioms by utilizing the concept of neutrosophic δ-open sets. Regarding neutrosophic disjointness, we know that $\eta_1 \cap \eta_2 = 0_U$ implies $\eta_1 \tilde{q} \eta_2$, but the converse does not necessarily hold. Based on this principle, we now define a new set of neutrosophic δ-separation axioms.

Definition 28. *A neutrosophic topological space U is called neutrosophic $\delta - T_0$ whenever, for any pair of neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in U, there exists a neutrosophic δ-open set T such that $j_{m,n,p} \subseteq T \subseteq k^c_{r,s,t}$ or $k_{r,s,t} \subseteq T \subseteq j^c_{m,n,p}$.*

The separation axiom defined here differs from the neutrosophic T_0 axiom, as demonstrated in the following example.

$$V_{T_\alpha}(u) = \begin{cases} 1 & if\, 0 \le u \le \alpha \\ 0 & if\, \alpha < u \le 1 \end{cases}, \quad Y_{T_\alpha}(u) = \begin{cases} 1 & if\, 0 \le u \le \alpha \\ 0 & if\, \alpha < u \le 1 \end{cases}, \quad W_{T_\alpha}(u) = \begin{cases} 0 & if\, 0 \le u \le \alpha \\ 1 & if\, \alpha < u \le 1 \end{cases}$$

Consider $\sigma = \{0_U, 1_U\} \bigcup \{T_\alpha : \alpha \in (0,1)\}$. It is clear that σ forms a neutrosophic topology on U, and the set of all neutrosophic δ-open sets in (U, σ) is $\{0_U, 1_U\}$. Therefore, for any two distinct neutrosophic points in (U, σ), there exists a neutrosophic open subset of U that contains one point but not the other. As a result, (U, σ) is neutrosophic T_0, but it is not neutrosophic $\delta - T_0$.

Theorem 3. *Given that $r : U \to Q$ is injective and neutrosophic δ-continuous, and that Q is neutrosophic $\delta - T_0$, it follows that U must also be neutrosophic $\delta - T_0$.*

Proof. Consider two points $j_{m,n,p}$ and $k_{r,s,t}$ in U with distinct supports. Since r is injective, $r(j_{m,n,p})$ and $r(k_{r,s,t})$ are two neutrosophic points with different supports in Q. Given that Q is neutrosophic $\delta - T_0$, there exists a neutrosophic δ-open set T, such that $r(j_{m,n,p}) \subseteq T \subseteq (r(k_{r,s,t}))^c$ or $r(k_{r,s,t}) \subseteq T \subseteq (r(j_{m,n,p}))^c$. Therefore, we have $j_{m,n,p} \subseteq r^{-1}(T) \subseteq k^c_{r,s,t}$ or $k_{r,s,t} \subseteq r^{-1}(T) \subseteq j^c_{m,n,p}$. Moreover, since r is neutrosophic δ-continuous, $r^{-1}(T)$ is a neutrosophic δ-open set. Hence, U is neutrosophic $\delta - T_0$. □

Definition 29. *In the case of a neutrosophic topological space U, it is termed neutrosophic $\delta - T_1$ whenever, for any two neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in U, there exist two neutrosophic δ-open sets T_1 and T_2 such that $j_{m,n,p} \subseteq T_1 \subseteq k^c_{r,s,t}$ and $k_{r,s,t} \subseteq T_2 \subseteq j^c_{m,n,p}$.*

It is evident that every neutrosophic $\delta - T_1$ space is also neutrosophic $\delta - T_0$.

Example 1. *Let $U = \{s,t\}$ and $\tau = \{t_{p,p,1-p} \mid p \in [0,1]\} \cup \{0_U, 1_U\}$, where $t_{p,p,1-p}$ is the neutrosophic point with membership value p, indeterminacy value p, and non-membership value $1-p$ at the support t. Then, clearly σ is a neutrosophic topology and all elements in σ are neutrosophic regular open, so they are neutrosophic δ-open. Take any two neutrosophic points $a_{m,m,1-m}$ and $b_{p,p,1-p}$ where r and p are nonzero. Then, there is a neutrosophic δ-open set $b_{p,p,1-p}$ such that $b_{p,p,1-p} \subseteq b_{p,p,1-p} \subseteq (a_{m,m,1-m})^c$, and 1_X is an only neutrosophic δ-open set with $a_{m,m,1-m} \subseteq 1_U$. Clearly, for any p, $1_U \nsubseteq (b_{p,p,1-p})^c$. Hence (U,σ) is neutrosophic $\delta - T_0$, but it is not neutrosophic $\delta - T_1$.*

Theorem 4. *A space U is neutrosophic $\delta - T_1$ if and only if every crisp neutrosophic point in U is neutrosophic δ-closed.*

Proof. Consider U as a neutrosophic $\delta - T_1$ space. For a crisp neutrosophic point $j_{m,n,p}$ in U, we aim to show that $j^c_{m,n,p}$ is neutrosophic δ-open. Choose a neutrosophic point $k_{r,s,t} \in j^c_{m,n,p}$ with a different support from $j_{m,n,p}$. Since U is neutrosophic $\delta - T_1$, there exists a neutrosophic δ-open set T such that $k_{r,s,t} \subseteq T \subseteq j^c_{m,n,p}$. Thus, we can express $j^c_{m,n,p}$ as $j^c_{m,n,p} = \bigcup_{k_{r,s,t} \in j^c_{m,n,p}} \{T : k_{r,s,t} \subseteq T \subseteq j^c_{m,n,p}, T \textit{ is neutrosophic } \delta - \textit{open in } U\}$. Since the union of these neutrosophic δ-open sets is neutrosophic δ-open, it follows that $j^c_{m,n,p}$ is neutrosophic δ-open. Therefore, $j_{m,n,p}$ is neutrosophic δ-closed. □

Corollary 1. *Let U be a neutrosophic topological space. U is neutrosophic $\delta - T_1$ if and only if $u_{m,n,p} = \bigcap\{\overline{T}_\delta : u_{m,n,p} \subseteq \overline{T}_\delta\}$.*

Theorem 5. *Consider $r : U \to Q$ as an injective and neutrosophic δ-continuous function. Provided that Q is neutrosophic $\delta - T_1$, it follows that U is also neutrosophic $\delta - T_1$.*

Proof. Consider two neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ in U with distinct supports. Since r is injective, $r(j_{m,n,p})$ and $r(k_{r,s,t})$ are neutrosophic points in Q that have different supports. Given that Q is neutrosophic $\delta - T_1$, there exist two neutrosophic δ-open sets T_1 and T_2, where $r(j_{m,n,p}) \subseteq T_1 \subseteq (r(k_{r,s,t}))^c$ and $r(k_{r,s,t}) \subseteq T_2 \subseteq (r(j_{m,n,p}))^c$. As a result, $r^{-1}(T_1)$ and $r^{-1}(T_2)$ are neutrosophic δ-open sets in U, and we have $j_{m,n,p} \subseteq r^{-1}(T_1) \subseteq k^c_{r,s,t}$ and $k_{r,s,t} \subseteq r^{-1}(T_2) \subseteq j^c_{m,n,p}$. This shows that U is neutrosophic $\delta - T_1$. □

Definition 30. *Given any two neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in U, a neutrosophic space U is called neutrosophic δ-Hausdorff or neutrosophic $\delta - T_2$ when there exist neutrosophic δ-open sets T_1 and T_2 such that $j_{m,n,p} \subseteq T_1 \subseteq k^c_{r,s,t}$, $k_{r,s,t} \subseteq T_2 \subseteq j^c_{m,n,p}$, and $T_1 \tilde{q} T_2$.*

It is clear that a neutrosophic $\delta - T_2$ space is also a neutrosophic $\delta - T_2$ space.

Theorem 6. *Assume that (U,σ) is a neutrosophic $\delta - T_1$ space. When the complement of each neutrosophic δ-open set is also neutrosophic δ-open, (U,σ) becomes a neutrosophic $\delta - T_2$ space.*

Proof. Consider two neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ in U with different supports. Since U is neutrosophic $\delta - T_1$, there exists a neutrosophic δ-open set T satisfying $j_{m,n,p} \subseteq T \subseteq k^c_{r,s,t}$ or $k_{r,s,t} \subseteq T \subseteq j^c_{m,n,p}$. Suppose $j_{m,n,p} \subseteq T \subseteq k^c_{r,s,t}$. Then, it follows that $k_{r,s,t} \subseteq T^c \subseteq j^c_{m,n,p}$. By the given condition, T^c is neutrosophic δ-open. Therefore, (U,σ) is neutrosophic $\delta - T_2$. □

Theorem 7. *When $u \in U$ satisfies that the crisp neutrosophic point $u_{1,1,0}$ is neutrosophic δ-open in (U, σ), it follows that (U, σ) is neutrosophic $\delta - T_2$.*

Proof. Consider two neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports. In this case, $j_{m,n,p} \subseteq j_{1,1,0} \subseteq k^c_{r,s,t}$ and $k_{r,s,t} \subseteq k_{1,1,0} \subseteq j^c_{m,n,p}$. It is evident that $j_{1,1,0} \subseteq (k_{1,1,0})^c$, and based on this assumption, $j_{1,1,0}$ and $k_{1,1,0}$ are neutrosophic δ-open. Therefore, (U, σ) is neutrosophic $\delta - T_2$. □

Theorem 8. *When $r : U \to Q$ is injective and neutrosophic δ-continuous, and Q is neutrosophic $\delta - T_2$, it implies that U is also neutrosophic $\delta - T_2$.*

Proof. Consider two points $j_{m,n,p}$ and $k_{r,s,t}$ in U with different supports. As f is injective, $r(j_{m,n,p})$ and $r(k_{r,s,t})$ are neutrosophic points in Q with distinct supports. Given that Q is neutrosophic $\delta - T_2$, there exist neutrosophic δ-open sets T_1 and T_2 such that $r(j_{m,n,p}) \subseteq T_1 \subseteq (r(k_{r,s,t}))^c$, $r(k_{r,s,t}) \subseteq T_2 \subseteq (r(j_{m,n,p}))^c$, and $T_1 \subseteq (T_2)^c$. Consequently, $r^{-1}(T_1)$ and $r^{-1}(T_2)$ are neutrosophic δ-open sets in U satisfying $j_{m,n,p} \subseteq r^{-1}(T_1) \subseteq k^c_{r,s,t}$, $k_{r,s,t} \subseteq r^{-1}(T_2) \subseteq j^c_{m,n,p}$, and $r^{-1}(T_1) \subseteq (r^{-1}(T_2))^c$. Thus, U is neutrosophic $\delta - T_2$. □

Definition 31. *In the context of neutrosophic topology, a space U is described as neutrosophic δ-regular when, given a neutrosophic point $j_{m,n,p}$ in U and a neutrosophic δ-closed set K such that $j_{m,n,p} \subseteq C^c$, there exist neutrosophic δ-open sets T_1 and T_2 satisfying $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$, and $T_1 \subseteq (T_2)^c$. Additionally, U is identified as neutrosophic $\delta - T_3$ when it meets the conditions of being both neutrosophic δ-regular and neutrosophic $\delta - T_1$.*

It is straightforward to demonstrate that any neutrosophic $\delta - T_3$ space is also neutrosophic $\delta - T_2$.

It is well established that for every neutrosophic closed set C, C° is a neutrosophic regular open set. Consequently, it is also neutrosophic δ-open. Hence, the following theorem holds true.

Theorem 9. *In a neutrosophic topological space (U, σ):*

(1) U is neutrosophic δ-regular.
(2) Given a neutrosophic point $j_{m,n,p}$ and a neutrosophic δ-open set N containing $j_{m,n,p}$, there exists a neutrosophic δ-open set T such that $j_{m,n,p} \subseteq T \subseteq \overline{T}_\delta \subseteq N$.
(3) Given a neutrosophic δ-closed set C and a neutrosophic point $j_{m,n,p}$ such that $j_{m,n,p} \subseteq C^c$, there exist neutrosophic δ-open sets T_1 and T_2 with $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$ and $\overline{(T_1)} \subseteq (\overline{T_2})^c$.
(4) Given a neutrosophic δ-closed set C and a neutrosophic point $j_{m,n,p}$ with $j_{m,n,p} \subseteq C^c$, there exist neutrosophic open sets T_1 and T_2 such that $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$ and $\overline{(T_1)} \subseteq (\overline{T_2})^c$.

Proof. (1) ⇒ (2) Consider a neutrosophic point set $j_{m,n,p}$ and a neutrosophic δ-open set N containing $j_{m,n,p}$. It follows that there exist neutrosophic δ-open sets T_1 and T_2 with $j_{m,n,p} \subseteq T_1$, $N^c \subseteq T_2$, and $T_1 \subseteq (T_2)^c$. Consequently, $j_{m,n,p} \subseteq T_1 \subseteq (T_2)^c \subseteq N$. Therefore, $j_{m,n,p} \subseteq T_1 \subseteq \overline{T_1}_\delta \subseteq \overline{((T_2)^c)}_\delta = (T_2)^c \subseteq N$.
(2) ⇒ (3) Consider C, a neutrosophic δ-closed subset of U, and $j_{m,n,p}$, a neutrosophic point set satisfying $j_{m,n,p} \subseteq C^c$. It follows that C^c is a neutrosophic δ-open set containing $j_{m,n,p}$. By (2), there exists a neutrosophic δ-open set T with $p \subseteq T \subseteq \overline{(T)}_\delta \subseteq C^c$. Since T is a neutrosophic δ-open set containing $j_{m,n,p}$, there exists a neutrosophic δ-open set N with $j_{m,n,p} \subseteq N \subseteq \overline{(N)}_\delta \subseteq T \subseteq \overline{(T)}_\delta \subseteq C^c$. Define $T_1 = N$ and $T_2 = (\overline{T}_\delta)^c$. It follows that T_1 and T_2 are neutrosophic δ-open sets with $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$. Furthermore, $\overline{T_2} \subseteq \overline{(\overline{T}_\delta)^c)} \subseteq \overline{(T^c)} = T^c$. Therefore, $\overline{(T_1)} = \overline{N} \subseteq T \subseteq (\overline{T_2})^c$.
(3) ⇒ (4) It is evident.
(4) ⇒ (1) Consider C as a neutrosophic δ-closed subset of U and $j_{m,n,p}$ as a neutrosophic point set where $j_{m,n,p} \subseteq c^c$. By (4), there exist neutrosophic open sets T and N where $j_{m,n,p} \subseteq T$, $C \subseteq N$, and $\overline{T} \subseteq (\overline{N})^c$. Since $j_{m,n,p} \subseteq T \subseteq \overline{T}$, it follows that $j_{m,n,p} \subseteq T^\circ =$

$T \subseteq (\overline{T})^{\circ}$. Define $T_1 = (\overline{T})^{\circ}$; consequently, T_1 is neutrosophic δ-open, and $j_{m,n,p} \subseteq T_1$. Since $C \subseteq N \subseteq \overline{N}$, we have $C \subseteq N^{\circ} = N \subseteq (\overline{N})^{\circ}$. Define $T_2 = (\overline{N})^{\circ}$; as a result, T_2 is neutrosophic δ-open, and $C \subseteq T_2$. Additionally, since $(\overline{T})^{\circ} \subseteq \overline{T} \subseteq (\overline{N})^c \subseteq ((\overline{N})^{\circ})^c$, we conclude that $T_1 \subseteq (T_2)^c$. □

Definition 32. *A neutrosophic δ-normal space is one where, for any pair of neutrosophic δ-closed sets C and S in U with $C \subseteq S^c$, there exist neutrosophic δ-open sets T_1 and T_2 such that $C \subseteq T_1$, $S \subseteq T_2$, and $T_1 \subseteq (T_2)^c$. A neutrosophic space U is called neutrosophic $\delta - T_4$ if it is both neutrosophic $\delta - T_1$ and neutrosophic δ-normal.*

Theorem 10. *Within a neutrosophic topological space (U, σ), the following statements are all true simultaneously:*

(1) U is neutrosophic δ-normal.
(2) Given a neutrosophic δ-closed set C and a neutrosophic δ-open set T containing C, there exists a neutrosophic δ-open set N with the properties $C \subseteq N \subseteq \overline{N} \subseteq T$.
(3) Given a neutrosophic δ-closed set C and a neutrosophic δ-open set T containing C, a neutrosophic open set N can be found with $C \subseteq N \subseteq \overline{N} \subseteq T$.
(4) Given a pair of neutrosophic δ-closed subsets C and S in U with $C \subseteq S^c$, neutrosophic open sets T_1 and T_2 exist, satisfying $C \subseteq T_1$, $S \subseteq T_2$, and $\overline{T_1} \subseteq (\overline{T_2})^c$.

Proof. (1) $\Rightarrow$ (2) Consider a neutrosophic δ-closed set C and a neutrosophic δ-open set T containing C. It follows that T^c is a neutrosophic δ-closed set, and $T^c \subseteq C^c$. As a result, there exist neutrosophic δ-open sets N_1 and N_2 where $C \subseteq N_1$, $T^c \subseteq N_2$, and $N_1 \subseteq (N_2)^c$. Consequently, we have $C \subseteq N_1 \subseteq (N_2)^c \subseteq T$ and $N_1 \subseteq \overline{N_1} \subseteq \overline{(N_2)^c} = (N_2)^c$. Hence, $C \subseteq N_1 \subseteq \overline{N_1} \subseteq \overline{N_1}_{\delta} \subseteq T$.
(2) $\Rightarrow$ (3) It is apparent.
(3) $\Rightarrow$ (4) Given that C and S are neutrosophic δ-closed subsets of U with $C \subseteq S^c$, it follows that S^c is a neutrosophic δ-open set containing C. By (3), there exists a neutrosophic open set N where $C \subseteq N \subseteq \overline{N} \subseteq S^c$. Since $\overline{N}$ is neutrosophic δ-closed and S^c is a neutrosophic δ-open set containing $\overline{N}$, a neutrosophic open set T must exist such that $C \subseteq N \subseteq \overline{N} \subseteq T \subseteq \overline{T} \subseteq S^c$. Define $T_1 = N$ and $T_2 = (\overline{T})^c$, so T_1 and T_2 are neutrosophic open sets where $C \subseteq T_1$ and $S \subseteq T_2$. Additionally, $\overline{(T_2)} = \overline{((\overline{T})^c)} \subseteq \overline{(T^c)} = T^c$. Therefore, $\overline{(T_1)} = \overline{N} \subseteq T \subseteq (\overline{(T_2)})^c$.
(4) $\Rightarrow$ (1) Given neutrosophic δ-closed subsets C and S of U with $C \subseteq S^c$, according to (4), there exist neutrosophic open sets T and N satisfying $C \subseteq T$, $S \subseteq N$, and $\overline{T} \subseteq (\overline{N})^c$. Additionally, since $C \subseteq T \subseteq \overline{T}$, it follows that $C \subseteq T = T^{\circ} \subseteq (\overline{T})^{\circ}$. Define T_1 as $(\overline{T})^{\circ}$, so T_1 is neutrosophic δ-open and $C \subseteq T_1$. Likewise, define T_2 as $(\overline{N})^{\circ}$, which makes T_2 neutrosophic δ-open and $S \subseteq T_2$. Furthermore, since $(\overline{T})^{\circ} \subseteq \overline{T} \subseteq (\overline{N})^c \subseteq ((\overline{N})^{\circ})^c$, we have $T_1 \subseteq (T_2)^c$. □

Example 2. *Let $U = [0,1]$ and, for each $\beta \in U$, $V_{T_\beta}(u) = \beta$, $Y_{T_\beta}(u) = \beta$, $W_{T_\beta}(u) = 1 - \beta$, for all $u \in U$. Meanwhile, let $\sigma = \{T_\beta : \beta \in U\}$. Then, σ is a neutrosophic topology and each T_β is neutrosophic δ-open. Therefore, σ is neutrosophic δ-normal and neutrosophic δ-regular. However, it is not neutrosophic $\delta - T_1$. So, it is neither neutrosophic $\delta - T_3$ nor neutrosophic $\delta - T_4$.*

Example 3. *Let $U = [0,1]$ and, for each $\beta \in U$,*

$$V_{T_\beta}(u) = \begin{cases} 1 & \textit{if } 0 \leq u \leq \beta \\ 0 & \textit{if } \beta < u \leq 1 \end{cases}, \; Y_{T_\beta}(u) = \begin{cases} 1 & \textit{if } 0 \leq u \leq \beta \\ 0 & \textit{if } \beta < u \leq 1 \end{cases}, \; W_{T_\beta}(u) = \begin{cases} 0 & \textit{if } 0 \leq u \leq \alpha \\ 1 & \textit{if } \beta < u \leq 1 \end{cases}$$

$$V_{N_\beta}(u) = \begin{cases} 0 & \text{if } 0 \le u \le \beta \\ 1 & \text{if } \beta < u \le 1 \end{cases}, \; Y_{N_\beta}(u) = \begin{cases} 0 & \text{if } 0 \le u \le \beta \\ 1 & \text{if } \beta < u \le 1 \end{cases}, \; W_{N_\beta}(u) = \begin{cases} 1 & \text{if } 0 \le u \le \beta \\ 0 & \text{if } \beta < u \le 1 \end{cases}$$

Let σ be a neutrosophic topology on U generated by the subbase $\{T_\beta : \beta \in U\} \cup \{N_\beta : \beta \in U\}$. Then, σ is a neutrosophic topology and $(\overline{(T_\beta)})^\circ = T_\beta$ for all $\beta \in U$. So, every T_β is neutrosophic δ-open. Similarly, every N_β is also neutrosophic δ-open. Therefore, (U, σ) is neutrosophic $\delta - T_4$ and also neutrosophic $\delta - T_3$.

5. Neutrosophic δ-Closure and δ-Interior in the Neutrosophic Subspace

Consider (U, σ) as a neutrosophic topological space and η as a neutrosophic subset of U. The neutrosophic subspace on η is denoted by (η, σ^η). If η is neutrosophic regular open (or regular closed) in X, then (η, σ^η) is referred to as a neutrosophic regular open (or regular closed) subspace, respectively.

Given any subset $Q \subseteq U$, suppose η is a neutrosophic subset defined as follows:

$$V_\eta(u) = \begin{cases} 1 & \text{if } u \in Q \\ 0 & \text{if } u \notin Q \end{cases}, \; Y_\eta(x) = \begin{cases} 1 & \text{if } u \in Q \\ 0 & \text{if } u \notin Q \end{cases}, \; W_\eta(x) = \begin{cases} 0 & \text{if } u \in Q \\ 1 & \text{if } u \notin Q \end{cases}$$

In that case, the neutrosophic subspace (η, σ^η) will be represented as χQ.

Definition 33. *Consider $E \in U^\eta$. We define E as neutrosophic regular open (or regular closed) in the subspace η, if $E = (\overline{E}^\eta)^{\eta\circ}$ (or $\overline{(E^{\eta\circ})}^\eta$.*

Definition 34. *Consider $E \in U^\eta$. A neutrosophic point $u_{m,n,p} \in \eta$ is defined as a neutrosophic δ-cluster point of E in η if and only if every neutrosophic regular open q-neighborhood T of $u_{m,n,p}$ in η is q-coincident with E in η. The set of all neutrosophic δ-cluster points of E in η is referred to as the neutrosophic δ-closure of E in η, denoted by $\overline{E}^\eta_\delta$.*

Theorem 11. *Consider $E \in U^\eta$ and $u_{m,n,p} \in \eta$. The element $u_{m,n,p}$ belongs to the set $\bigcap\{W \in U^\eta : E \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\}$ if and only if every neutrosophic regular open q-neighborhood T of $u_{m,n,p}$ in η is q-coincident with E in η.*

Proof. Consider H to be a neutrosophic regular open q-neighborhood of $u_{m,n,p}$ such that $H\tilde{q}E$. Then, H is a neutrosophic open set in η where $u_{m,n,p}qH \subseteq H$ and $E\tilde{q}H$ are satisfied. Since H^c is neutrosophic regular closed and $E \subseteq H^c$, it follows that $\bigcap\{W \in U^\eta : E \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\} \subseteq H^c$. Furthermore, because $u_{m,n,p} \notin H^c$, we conclude that $u_{m,n,p} \notin \bigcap\{W \in X^\eta : E \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\}$.

On the other hand, suppose $u_{m,n,p} \notin \bigcap\{W \in U^\eta : E \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\}$. In this case, there exists a neutrosophic regular closed set W such that $u_{m,n,p} \notin W$ and $E \subseteq W$. As a result, W^c is a neutrosophic regular open set where $u_{m,n,p}qW^c$ and $E\tilde{q}W^c$ hold. Thus, $u_{m,n,p}$ cannot be a neutrosophic δ-cluster point of E in η.

According to the aforementioned theorem, in a neutrosophic subspace (η, σ^η), we have $\overline{E}^\eta_\delta = \bigcap\{W \in X^\eta : E \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\}$ for any set $E \in U^\eta$. Next, we introduce the concept of the δ-interior in a subspace. □

Definition 35. *Assume $E \in U^\eta$. The δ-interior of E within η is defined in the following way:*

$$\begin{aligned}\overline{E}^{\eta}_{\delta} = \eta \cap (\overline{(\eta \cap E^c)}^{\eta}_{\delta})^c &= \eta \cap (\bigcap\{W \in U^\eta : \eta \cap E^c \subseteq W, W = \overline{(W^{\eta\circ})}^{\eta}\})^c \\ &= \eta \cap (\bigcup\{W^c : W \in U^\eta, \eta \cap E^c \subseteq W, W = \overline{(W^{\eta\circ})}^{\eta}\}) \\ &= \bigcup\{\eta \cap W^c : W \in U^\eta, \eta \cap E^c \subseteq W, W = \overline{(W^{\eta\circ})}^{\eta}\} \\ &= \bigcup\{\eta \cap W^c : F \in U^\eta, \eta \cap W^c \subseteq E, \eta \cap W^c = \eta \cap (\overline{(W^{\eta\circ})}^{\eta})^c\} \\ &= \bigcup\{T \in U^\eta, T \subseteq E, T = (\overline{T}^{\eta})^{\eta\circ}\}\end{aligned} \tag{1}$$

We aim to demonstrate that for any neutrosophic set E in a neutrosophic subspace (η, σ^η), the following holds: $\overline{E}^{\eta}_{\delta} = \bigcap\{\overline{T}^{\eta} : E \subseteq \overline{T}^{\eta}, T \in \sigma^\eta\}$. To accomplish this, we will first prove two lemmas.

Lemma 1. *Consider (η, σ^η) as a neutrosophic subspace. If $N \in \sigma^\eta$, then $\overline{N}^{\eta}$ is a neutrosophic regular closed set in η.*

Proof. Since $N \subseteq \overline{N}^{\eta}$, $N = N^{\eta\circ} \subseteq (\overline{N}^{\eta})^{\eta\circ}$ and hence $\overline{N}^{\eta} \subseteq \overline{((\overline{N}^{\eta})^{\eta\circ})}^{\eta}$.

On the other hand, since $\overline{((\overline{N}^{\eta})^{\eta\circ})}^{\eta} \subseteq \overline{N}^{\eta}$, $\overline{((\overline{N}^{\eta})^{\eta\circ})}^{\eta} \subseteq \overline{(\overline{N}^{\eta})}^{\eta} = \overline{N}^{\eta}$. Hence, $\overline{N}^{\eta} = \overline{((\overline{N}^{\eta})^{\eta\circ})}^{\eta}$. □

Lemma 2. *Consider (η, σ^η) as a neutrosophic subspace. In that case, $\{\overline{T}^{\eta} : T \in \sigma^\eta\} = \{W \in U^\eta : W$ is neutrosophic regular closed in $\eta\}$.*

Proof. It is known that for every neutrosophic open set T in η, $\overline{T}^{\eta}$ is neutrosophic regular closed in η.

On the other hand, consider any neutrosophic regular closed set W in η. In that case, $W = \overline{(W^{\eta\circ})}^{\eta} = \overline{(\bigcup\{T : T \subseteq W, T \in \sigma^\eta)}^{\eta} \in \{\overline{T}^{\eta} : T \in \sigma^\eta\}$.

There may be a challenge in determining the neutrosophic δ-closure of any neutrosophic set. However, based on the above lemmas, we have a hint on how to find it. □

Theorem 12. *Consider any neutrosophic set E in a neutrosophic subspace (η, σ^η), $\overline{E}^{\eta}_{\delta} = \bigcap\{\overline{T}^{\eta} : E \subseteq \overline{T}^{\eta}, T \in \sigma^\eta\}$.*

Proof. It is clear from Lemma 2. □

Furthermore, if η is neutrosophic open in U and if $(\eta, \sigma^\eta) = \chi Q$, $\overline{E}^{\eta}_{\delta} = \overline{E}_{\delta} \cap \eta$ for any neutrosophic subset E of η. At this point, we will demonstrate it.

Lemma 3. *Consider U as a neutrosophic topological space, $Q \subseteq U$, and η as a neutrosophic open subset of U, where $(\eta, \sigma^\eta) = \chi Q$. Assume $E \in U^\eta$. When E is neutrosophic regular open in U, it follows that E is also neutrosophic regular open in η.*

Proof. Consider any neutrosophic subset $E \in U^\eta$; the following holds: $(\overline{E}^{\eta})^{\eta\circ} = (\overline{E} \cap \eta)^{\eta\circ} = (\overline{E} \cap \eta)^{\circ} = (\overline{E})^{\circ} \cap \eta^{\circ} = (\overline{E})^{\circ} \cap \eta$. Hence, if $E = (\overline{E})^{\circ}$, then $(\overline{E}^{\eta})^{\eta\circ} = (\overline{E})^{\circ} \cap \eta = E \cap \eta = E$. □

Theorem 13. *Consider (U, σ) as a neutrosophic topological space and $Q \subseteq U$. Suppose that $(\eta, \sigma^\eta) = \chi Q$, and η is neutrosophic regular open in U. Under these conditions, for any neutrosophic subset $E \in U^\eta$, it follows that $\overline{E}^{\eta}_{\delta} = \overline{E}_{\delta} \cap \eta$.*

Proof. Suppose that $u_{m,n,p} \notin \overline{E}^{\eta}_{\delta}$. Then, there is a neutrosophic regular open q-neighborhood H of $u_{m,n,p}$ *in* η with $H\tilde{q}^{\eta} A$, i.e., $(\overline{H}^{\eta})^{\eta\circ} \subseteq E^c$. Since $u_{m,n,p} q^{\eta} H$, $u_{m,n,p} q H$ and

N are neutrosophic open in U, $H = H^{\circ} \subseteq (\overline{H})^{\circ}$. Note that $(\overline{H})^{\circ}$ is a neutrosophic regular open q-neighborhood of $u_{m,n,p}$ in U. Since $(\overline{H})^{\circ} = (\overline{(H \cap \eta)})^{\circ} = (\overline{H}^{\eta})^{\circ} \subseteq (\overline{H}^{\eta})^{\eta\circ} \tilde{q}^{\eta} A$, we have $(\overline{H})^{\circ} \tilde{q} E$. Thus, $u_{m,n,p} \notin \overline{E}_{\delta} \cap \eta$. Conversely, take $u_{m,n,p} \in \overline{E}^{\eta}_{\delta}$ and a neutrosophic regular open q-neighborhood H of $u_{m,n,p}$ in U. Then, $u_{m,n,p} \notin H^c$ and so $u_{m,n,p} \notin (H \cap \eta)$. Thus, $H \cap \eta$ is a neutrosophic regular open q-neighborhood of $u_{m,n,p}$ in U. By the above lemma, $H \cap \eta$ is also a neutrosophic regular open q-neighborhood of $u_{m,n,p}$ in η. Since $u_{m,n,p} \in \overline{E}^{\eta}_{\delta}$, $(H \cap \eta) q E$. Hence, HqE. Therefore, $u_{m,n,p}$ is a neutrosophic δ-cluster point of E in U.

A neutrosophic regular open set in η does not automatically qualify as neutrosophic regular open in U. However, when η is a neutrosophic regular open set in U, where $(\eta, \sigma^{\eta}) = \chi Q$, it follows that any neutrosophic δ-open set in U, contained within U^{η}, will also be considered neutrosophic δ-open in η. This is illustrated by the subsequent theorem. □

Theorem 14. *Consider (U, σ) as a neutrosophic topological space and $Q \subseteq U$. Suppose that η is neutrosophic regular open in U and $E \in U^{\eta}$, where $(\eta, \sigma^{\eta}) = \chi Q$. It follows that $E^{\eta\circ}_{\delta} = E^{\circ}_{\delta}$.*

Proof. $E^{\eta\circ}_{\delta} = \eta \cap (\overline{(\eta \cap E^c)}^{\eta}_{\delta})^c = \eta \cap (\overline{(\eta \cap E^c)}_{\delta} \cap \eta)^c = \eta \cap ((\overline{(\eta \cap E^c)}_{\delta})^c \cup \eta^c) =$
$\eta \cap (\overline{(\eta \cap E^c)}_{\delta})^c = \eta \cap ((\eta \cap E^c)^c)^{\circ}_{\delta} = (\eta \cap (\eta \cap E^c)^c)^{\circ}_{\delta} = E^{\circ}_{\delta}$ □

6. Neutrosophic δ-Separation Axioms in the Neutrosophic Subspace η

We proceed by defining the neutrosophic δ-separation axioms within neutrosophic subspaces. It should be noted that for any set $E \in U^{\eta}$, the complement of A, denoted as E^c, in the neutrosophic subspace η is equivalent to $\eta \cap E^c$.

Definition 36. *A neutrosophic subspace (η, σ^{η}) is called neutrosophic $\delta - T_0$ whenever, given any pair of neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in η, a neutrosophic δ-open set T in η exists, where $j_{m,n,p} \subseteq T \subseteq k^c_{r,s,t}$ or $k \subseteq T \subseteq j^c_{m,n,p}$.*

Definition 37. *Assume (U, σ) is a neutrosophic $\delta - T_0$ space, with $Q \subseteq U$ and η being a neutrosophic regular open set of U, where $(\eta, \sigma^{\eta}) = \chi Q$. It follows that χQ is neutrosophic $\delta - T_0$.*

Proof. Consider U as a neutrosophic $\delta - T_0$ space, and suppose η is a neutrosophic regular open subset of U, where $(\eta, \sigma^{\eta}) = \chi Q$. For neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with different supports in the subspace χQ, it follows that $j_{m,n,p}$ and $k_{r,s,t}$ also have different supports in the space U. Since U is neutrosophic $\delta - T_0$, a neutrosophic δ-open set $T \in U$ exists, satisfying $j_{m,n,p} \subseteq T \subseteq k^c_{r,s,t}$ or $k_{r,s,t} \subseteq T \subseteq j^c_{m,n,p}$. Furthermore, since $T \cap \eta$ is neutrosophic δ-open in U with $T \cap \eta \subseteq \eta$, $T \cap \eta$ is also neutrosophic δ-open in η. Additionally, either $j_{m,n,p} \subseteq T \cap \eta \subseteq (\eta \cap k^c_{r,s,t})$ or $k_{r,s,t} \subseteq T \cap \eta \subseteq (\eta \cap j^c_{m,n,p})$. Therefore, (η, σ^{η}) is neutrosophic $\delta - T_0$. □

Definition 38. *A neutrosophic subspace (η, σ^{η}) is termed neutrosophic $\delta - T_1$ when, given a pair of neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in η, two neutrosophic δ-open sets T_1 and T_2 exist in η such that $j_{m,n,p} \subseteq T_1 \subseteq k^c_{r,s,t}$ and $k_{r,s,t} \subseteq T_2 \subseteq j^c_{m,n,p}$.*

Theorem 15. *A neutrosophic subspace (η, τ^{η}) is referred to as neutrosophic $\delta - T_1$ whenever a pair of neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with different supports in η is neutrosophic δ-closed in η.*

Proof. Consider a crisp neutrosophic point $j_{m,n,p}$ in η. It will be shown that $\eta \cap j^c_{m,n,p}$ is neutrosophic δ-open in η. A neutrosophic point $k_{r,s,t} \in \eta \cap j^c_{m,n,p}$ with a different support from $j_{m,n,p}$ can be chosen. Since (η, σ^{η}) is neutrosophic $\delta - T_1$, there exists a neutrosophic δ-open set T in η such that $k_{r,s,t} \subseteq T \subseteq \eta \cap j^c_{m,n,p}$. Therefore, as $\eta \cap j^c_{m,n,p}$ can be expressed as $\bigcup_{k_{r,s,t} \in \eta \cap j^c_{m,n,p}} T : k_{r,s,t} \subseteq T \subseteq \eta \cap j^c_{m,n,p}$, U is neutrosophic δ-open in η. Since this union is neutrosophic δ-open in η, it follows that $\eta \cap j^c_{m,n,p}$ is neutrosophic δ-open in η. As a result, $j_{m,n,p}$ is neutrosophic δ-closed in η. □

Corollary 2. *Consider (η, σ^{η}) as a neutrosophic subspace. η is neutrosophic $\delta - T_1$ if and only if $u_{m,n,p} = \bigcap \overline{T}\delta : u_{m,n,p} \subseteq \overline{T}\delta$ holds when $u_{m,n,p} \in \eta$.*

Theorem 16. *Consider (U, σ) as a neutrosophic $\delta - T_1$ space, where $Q \subseteq U$ and η is neutrosophic regular open in U, with $(\eta, \sigma^{\eta}) = \chi Q$. It follows that χQ is neutrosophic $\delta - T_1$.*

Proof. This is obvious. □

Definition 39. *A neutrosophic subspace (η, σ^{η}) is termed neutrosophic δ-Hausdorff, or neutrosophic $\delta - T_2$, when, given any pair of neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in η, there exist neutrosophic δ-open sets T_1 and T_2 in η such that $j_{m,n,p} \subseteq T_1 \subseteq k^c_{r,s,t}$, $k_{r,s,t} \subseteq T_2 \subseteq j^c_{m,n,p}$, and $T_1 \subseteq (T_2)^c$.*

It is clear that every neutrosophic $\delta - T_2$ subspace is also a neutrosophic $\delta - T_1$ subspace. In addition, it is easily seen that since every open set is δ-open, every T_i space is also $\delta - T_i$ space for i = 0, 1, 2.

Theorem 17. *Given that U is a neutrosophic $\delta - T_2$ space and $Q \subseteq U$, with η being neutrosophic regular open in U, where $(\eta, \sigma^{\eta}) = \chi Q$, it follows that χQ is neutrosophic $\delta - T_2$.*

Proof. Consider neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with different supports in a subspace (η, σ^{η}). It follows that $j_{m,n,p}$ and $k_{r,s,t}$ are also neutrosophic points with different supports in the space U. Since U is neutrosophic $\delta - T_2$, neutrosophic δ-open sets T_1 and T_2 exist in U where $j_{m,n,p} \subseteq T_1 \subseteq k^c_{r,s,t}$, $k_{r,s,t} \subseteq T_2 \subseteq j^c_{m,n,p}$, and $T_1 \subseteq (T_2)^c$. Consequently, neutrosophic δ-open sets $T_1 \cap \eta$ and $T_2 \cap \eta$ exist in η, where $j_{m,n,p} \subseteq T_1 \cap \eta \subseteq k^c_{r,s,t} \cap \eta$, $j_{m,n,p} \subseteq T_2 \cap \eta \subseteq j^c_{m,n,p} \cap \eta$, and $T_1 \cap \eta \subseteq (T_2)^c \cap \eta$. Therefore, η is neutrosophic $\delta - T_2$. □

Definition 40. *A neutrosophic subspace (η, σ^{η}) is defined as neutrosophic δ-regular if, for every pair consisting of a neutrosophic point $j_{m,n,p}$ in η and a neutrosophic δ-closed set C in η such that $j_{m,n,p} \subseteq C^c$, there exist neutrosophic δ-open sets T_1 and T_2 in η with $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$, and $T_1 \subseteq (T_2)^c$. A neutrosophic subspace (η, σ^{η}) is referred to as neutrosophic $\delta - T_3$ if it is both neutrosophic δ-regular and neutrosophic $\delta - T_1$.*

It is straightforward to demonstrate that every neutrosophic $\delta - T_3$ subspace is also a neutrosophic $\delta - T_2$ subspace.

It is known that for any neutrosophic closed set C in η, $C^{\eta\circ}$ is neutrosophic regular open in η. As a result, it is also neutrosophic δ-open. This leads to the conclusion of the following theorem.

Theorem 18. *The following conditions are equivalent for a neutrosophic subspace (η, σ^{η}):*

(1) (η, σ^{η}) *is neutrosophic δ-regular.*

(2) *Given a neutrosophic point $j_{m,n,p}$ and a neutrosophic δ-open set N containing $j_{m,n,p}$ in (η, σ^{η}), there exists a neutrosophic δ-open set T in η such that $j_{m,n,p} \subseteq T \subseteq \overline{T}^{\eta}_{\delta} \subseteq N$.*

(3) *In (η, σ^{η}), consider a neutrosophic δ-closed set C and a neutrosophic point $j_{m,n,p}$ where $j_{m,n,p} \subseteq \eta \cap C^c$. There exist neutrosophic δ-open sets T_1 and T_2 in η, with $j_{m,n,p} \subseteq T_1$, $K \subseteq T_2$, and $\overline{T_1}^{\eta} \subseteq \eta \cap (\overline{T_2}^{\eta})^c$.*

(4) *In (η, σ^{η}), given a neutrosophic δ-closed set C and a neutrosophic point $j_{m,n,p}$ with $j_{m,n,p} \subseteq \eta \cap C^c$, there exist neutrosophic open sets T_1 and T_2 in (η, σ^{η}), where $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$, and $\overline{T_1}^{\eta} \subseteq \eta \cap (\overline{T_2}^{\eta})^c$.*

Proof. (1) $\Rightarrow$ (2) Let $j_{m,n,p}$ be a neutrosophic point set in η and N be a neutrosophic δ-open set in η containing j. Then, there exist neutrosophic δ-open sets T_1 and T_1 in η such that $j \subseteq T_1$, $\eta \cap N^c \subseteq T_2$ and $T_1 \subseteq \eta \cap (T_2)^c$. So $j_{m,n,p} \subseteq T_1 \subseteq \eta \cap (T_2)^c \subseteq N$. Thus $j_{m,n,p} \subseteq T_1 \subseteq \overline{T_1}^{\eta}_{\delta} \subseteq \overline{(\eta \cap (T_2)^c)}^{\eta}_{\delta} = \eta \cap (T_2)^c \subseteq N$.
(2) $\Rightarrow$ (3) Let C be a neutrosophic δ-closed subset in η and $j_{m,n,p}$ be a neutrosophic point in η such that $j_{m,n,p} \subseteq \eta \cap C^c$. Then, $\eta \cap C^c$ is a neutrosophic δ-open set in η with $j_{m,n,p} \subseteq \eta \cap C^c$. By (2), there is a neutrosophic δ-open set T in η such that $j_{m,n,p} \subseteq T \subseteq \overline{T}^{\eta}_{\delta} \subseteq \eta \cap C^c$. Since U is a neutrosophic δ-open set in η containing $j_{m,n,p}$, there is a neutrosophic δ-open set N in η such that $j_{m,n,p} \subseteq N \subseteq \overline{N}^{\eta}_{\delta} \subseteq T \subseteq \overline{T}^{\eta}_{\delta} \subseteq \eta \cap C^c$. Put $U_1 = N$ and $U_2 = \eta \cap (\overline{T}^{\eta}_{\delta})^c$. Then, T_1 and T_2 are neutrosophic δ-open sets in η with $j \subseteq T_1$,$K \subseteq T_2$. Furthermore, $\overline{T_2}^{\eta} \subseteq \overline{(\eta \cap (\eta \cap (\overline{T}^{\eta}_{\delta})^c)}^{\eta} \subseteq \overline{(\eta \cap T^c)}^{\eta} = \eta \cap T^c$. Since, $\overline{N}^{\eta} \subseteq \overline{N}^{\eta}_{\delta}$, $\overline{T_1}^{\eta} = \overline{N}^{\eta} \subseteq T \subseteq \eta \cap (\overline{T_2}^{\eta})^c$.
(3) $\Rightarrow$ (4) This is obvious.
(4) $\Rightarrow$ (1) Let C be a neutrosophic δ-closed subset in η and $j_{m,n,p}$ be a neutrosophic point in η such that $j_{m,n,p} \subseteq \eta \cap C^c$. By (4), there are neutrosophic open sets T and N in η such that $j_{m,n,p} \subseteq T$, $C \subseteq N$ and $\overline{T}^{\eta} \subseteq \eta \cap (\overline{N}^{\eta})^c$. Since $j_{m,n,p} \subseteq T \subseteq \overline{T}^{\eta}$, $j_{m,n,p} \subseteq T^{\eta\circ} = T \subseteq (\overline{U}^{\eta})^{\eta\circ}$. Put $T_1 = (\overline{T}^{\eta})^{\eta\circ}$, then T_1 is neutrosophic δ-open in η and $j \subseteq T_1$. Since $C \subseteq N \subseteq \overline{N}^{\eta}$, $C \subseteq N^{\eta\circ} = N \subseteq (\overline{N}^{\eta})^{\eta\circ}$. Put $T_2 = (\overline{N}^{\eta})^{\eta\circ}$, then T_2 is neutrosophic δ-open in η and $C \subseteq T_2$. Furthermore, since $(\overline{T}^{\eta})^{\eta\circ} \subseteq \overline{T}^{\eta} \subseteq \eta \cap (\overline{V}^{\eta})^c \subseteq \eta \cap ((\overline{V}^{\eta})^{\eta\circ})^c$, $U_1 \subseteq \eta \cap (U2)^c$. □

Definition 41. *A neutrosophic subspace (η, σ^{η}) is called neutrosophic δ-normal if for any pair of neutrosophic δ-closed subsets C, S in η with $C \subseteq S^c$, there are neutrosophic δ-open sets T_1, T_2 in η with $K \subseteq T_1$, $S \subseteq T_2$ and $T_1 \subseteq (T_2)^c$. A neutrosophic subspace (η, σ^{η}) is called neutrosophic $\delta - T_4$ if it is neutrosophic $\delta - T_1$ and neutrosophic δ-normal.*

Clearly every neutrosophic $\delta - T_4$ subspace is neutrosophic $\delta - T_3$.

Theorem 19. *For a neutrosophic subspace (η, σ^{η}), the following are equivalent:*

(1) *(η, σ^{η}) is neutrosophic δ-normal.*
(2) *For any neutrosophic δ-closed set C and any neutrosophic δ-open set T containing C in (η, σ^{η}), there exists a neutrosophic δ-open set N in η such that $C \subseteq N \subseteq \overline{N}^{\eta}_{\delta} \subseteq T$.*
(3) *For any neutrosophic δ-closed set C and any neutrosophic δ-open set T containing C in (η, σ^{η}), there exists a neutrosophic δ-open set N in η such that $C \subseteq N \subseteq \overline{N}^{\eta} \subseteq T$.*
(4) *For any neutrosophic δ-closed set C and any neutrosophic δ-open set T containing C in (η, σ^{η}), there exists a neutrosophic open set N in η such that $C \subseteq N \subseteq \overline{N}^{\eta} \subseteq T$.*
(5) *For any pair of neutrosophic δ-closed subsets C and S with $C \subseteq \eta \cap S^c$ in (η, σ^{η}), there are neutrosophic open sets T_1 and T_2 in η with $C \subseteq T_1$, $S \subseteq T_2$ and $\overline{T_1}^{\eta} \subseteq \eta \cap (\overline{T_2}^{\eta})^c$.*

Proof. (1) $\Rightarrow$ (2) Let C be a neutrosophic δ-closed set in η and T be a neutrosophic δ-open set in η containing C. Then, $\eta \cap T^c$ is a neutrosophic δ-closed set in η with $\eta \cap T^c \subseteq \eta \cap C^c$. Thus, there are neutrosophic δ-open sets N_1 and N_2 in η such that $C \subseteq N_1$, $\eta \cap T^c \subseteq N_2$ and $N_1 \subseteq \eta \cap (N_2)^c$. So, $K \subseteq N_1 \subseteq \eta \cap (N_2)^c \subseteq T$ and $N_1 \subseteq \overline{N_1}^{\eta}_{\delta} \subseteq \overline{(\eta \cap (N_2)^c)}^{\eta}_{\delta} = \eta \cap (N_2)^c$. Hence, $C \subseteq N_1 \subseteq \overline{(N_1)}^{\eta}_{\delta} \subseteq T$.
(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) This is obvious.
(4) $\Rightarrow$ (5) Let C and S be neutrosophic δ-closed subsets in η with $C \subseteq \eta \cap S^c$. Then, $\eta \cap S^c$ is a neutrosophic δ-open set in η containing C. By (3), there is a neutrosophic open set N in η such that $C \subseteq N \subseteq \overline{V}^{\eta}_{\delta} \subseteq \eta \cap S^c$. Since $\overline{N}^{\eta}_{\delta}$ is neutrosophic δ-closed in η and $\eta \cap S^c$ is a neutrosophic δ-open set in η containing $\overline{N}^{\eta}_{\delta}$, there is a neutrosophic open set T in η

such that $K \subseteq N \subseteq \overline{N}^{\eta}_{\delta} \subseteq T \subseteq \overline{T}^{\eta}_{\delta} \subseteq \eta \cap S^c$. Let $T_1 = N$ and $T_2 = \eta \cap (\overline{T}^{\eta}_{\delta})^c$, then T_1 and T_2 are neutrosophic open sets in η with $C \subseteq T_1$ and $S \subseteq T_2$. Also, $(\overline{T_2}^{\eta}) = \overline{(\eta \cap (\overline{T}^{\eta}_{\delta})^c)}^{\eta} \subseteq \overline{(\eta \cap T^c)}^{\eta} = \eta \cap T^c$. So, $(\overline{T_1}^{\eta}) = \overline{N}^{\eta} \subseteq T \subseteq \eta \cap ((\overline{T_2}^{\eta}))^c$.
(5) $\Rightarrow$ (1) Let C and S be neutrosophic δ-closed subsets in η with $C \subseteq \eta \cap S^c$. Then, by (4), there are neutrosophic open sets T and N in η such that $C \subseteq T$,$S \subseteq N$ and $\overline{T}^{\eta}_{\delta} \subseteq \eta \cap (\overline{N}^{\eta}_{\delta})^c$. Since $K \subseteq T \subseteq \overline{T}^{\eta}_{\delta}$, we have $C \subseteq T = T^{\eta\circ} \subseteq (\overline{T}^{\eta}_{\delta})^{\eta\circ}$. Put $T_1 = (\overline{T}^{\eta}_{\delta})^{\eta\circ}$, then T_1 is neutrosophic δ-open in η and $C \subseteq T_1$. Similarly put $T_2 = (\overline{N}^{\eta}_{\delta})^{\eta\circ}$, then T_2 is neutrosophic δ-open in η and $S \subseteq T_2$. Since $(\overline{T}^{\eta}_{\delta})^{\eta\circ} \subseteq \overline{T}^{\eta}_{\delta} \subseteq \eta \cap (\overline{N}^{\eta}_{\delta})^c \subseteq \eta \cap ((\overline{N}^{\eta}_{\delta})^{\eta\circ})^c$, we have $T_1 \subseteq \eta \cap (T_2)^c$. □

Theorem 20. *Let (U, σ) be a neutrosophic δ-regular space. Suppose that $Q \subseteq U$ and η is neutrosophic regular open in U, where $(\eta, \sigma^{\eta}) = \chi Q$. Then, χQ is neutrosophic δ-regular.*

Proof. Let T be a neutrosophic δ-open set in η and $j_{m,n,p}$ a neutrosophic point in η with $j_{m,n,p} \subseteq T$. Since η is neutrosophic regular open in U, T is also neutrosophic δ-open in U. Since X is neutrosophic δ-regular, there is a neutrosophic δ-open set G of U such that $j_{m,n,p} \subseteq G \subseteq \overline{G}_{\delta} \subseteq T$. Thus, $G \cap \eta$ is a neutrosophic δ-open set in η such that $j_{m,n,p} \subseteq G \cap \eta \subseteq \overline{G}_{\delta} \cap \eta = \overline{(G \cap \eta)}^{\eta}_{\delta} \subseteq T \cap \eta = T$. Hence, η is neutrosophic δ-regular. □

Lemma 4. *Let (U, σ) be a neutrosophic δ-normal space. Suppose that $Q \subseteq U$ and η is neutrosophic regular closed in U, where $(\eta, \sigma^{\eta}) = \chi Q$. If $C \in U^{\eta}$ is neutrosophic regular closed in η, then C is also neutrosophic regular closed in U.*

Proof. $C \subseteq \overline{C}_{\delta} = \overline{(C \cap \eta)}_{\delta} \subseteq \overline{C}_{\delta} \cap \overline{\eta}_{\delta} = \overline{C}_{\delta} \cap \eta = C$. □

Theorem 21. *Let (U, σ) be a neutrosophic δ-normal space. Suppose that $Q \subseteq U$ and η is neutrosophic regular open in U, where $(\eta, \sigma^{\eta}) = \chi Q$. Then, χQ is neutrosophic δ-normal.*

Proof. Let U be a neutrosophic δ-normal space and η be a neutrosophic δ-open subspace of U. Let C, S be neutrosophic δ-closed subsets in η with $C \subseteq \eta \cap S^c$. Since η is neutrosophic δ-closed in U, C and S are also neutrosophic δ-closed in U with $C \subseteq S^c$. Since U is neutrosophic δ-normal, there exist neutrosophic δ-open sets T_1 and T_2 in U with $C \subseteq T_1$, $S \subseteq T_2$ and $T_1 \subseteq (T_2)^c$. So, there exist neutrosophic δ-open sets $\eta \cap T_1$ and $\eta \cap T_2$ in η with $C \subseteq \eta \cap T_1$, $S \subseteq \eta \cap T_2$ and $\eta \cap T_2 \subseteq \eta \cap (\eta \cap T_2)^c$ in the subspace η. □

7. Conclusions

In this study, the concept of δ-separation axioms, which have been defined in different ways in general topological spaces and various types of topological spaces, was extended to neutrosophic topological spaces. The relationships between these newly introduced separation axioms, which are defined for the first time in this paper, were analyzed and clarified with the aid of a diagram. Additionally, the properties of these novel separation axioms in neutrosophic subspaces were investigated. This work is anticipated to lay the groundwork for further exploration in the field of mathematics and contribute to human life.

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A Novel IVBPRT-ELECTRE III Algorithm Based on Bidirectional Projection and Its Application

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Abstract: Fuzzy semantics have a wide range of applications in life, and especially when expressing people's evaluation information, it is more specific. As people increasingly prefer to express their personal opinions through media platforms, the opinions of the general public have become an indispensable reference. However, information asymmetry can have a significant impact on the rationality of decision-making. Based on the above considerations, this paper extends bidirectional projection to probabilistic linguistic term sets to preserve the completeness of information as much as possible. The large-scale group decision-making problem under the probabilistic linguistic environment is extended to limited interval values, and a new group decision-making method named IVBPRT-ELECTRE III algorithm (ELECTRE III based on bidirectional projection and regret theory under limited interval-valued probabilistic linguistic term set) is proposed. The method is an extended ELECTRE III method based on limited interval-valued probabilistic linguistic term set (l-IVPLTS) bidirectional projection by regret theory approach. Firstly, this involves mining the online text comment information on social media about an emergency and considering the effect of the number of fans, determining the attributes and their initial weights for judging the strengths and weaknesses of the emergency management alternative using the TF-IDF and the Word2vec technology, and using the entropy value to adjust the initial weight of attributes, not only considering the real opinions of the public, but also combining with the views of experts, making the decision-making alternative selection more scientific and reasonable. Secondly, this paper fills the gap of bidirectional projection under l-IVPLTS environment; then, combining l-IVPLTS bidirectional projection and regret theory to determine the objective weights of experts, combines the differences in individual expertise of experts to obtain the comprehensive weights of experts, and uses the extended ELECTRE III method to rank the alternatives. Finally, the feasibility and validity of the provided method is verified through the Yanjiao explosion incident as a case.

Keywords: limited interval-valued probabilistic linguistic term set; bidirectional projection; regret theory; ELECTRE; fuzzy set

1. Introduction

In recent years, the frequency of unconventional sudden events has been increasing, especially fire and combustion events, and the emergency response capability of unconventional emergencies is more directly related to the on-site emergency response situation, such as casualties and property damage. Therefore, emergency decision-making research has become one of the current global hot topics. For a decision-making problem, it is crucial for decision makers to evaluate the information truly and effectively. Considering the different expression habits of decision makers, linguistic term sets (LTS) are commonly used as a reference quantitative scale for evaluating all alternatives. However, due to the complexity of decision-making problems, these sets may not accurately capture the cognitive information of decision makers. In order to overcome this limitation, Rodriguez et al. [1] proposed hesitant fuzzy linguistic term set (HFLTS) based on hesitant fuzzy set (HFS) [2]. Subsequently, Pang et al. [3] extended the hesitant fuzzy linguistic term set by adding probability values and gave the first definition of the probabilistic linguistic term set (PLTS). The emergence of probabilistic linguistic term set (PLTS) has enabled some decision-making problems to be solved. Based on the uncertain linguistic term set (ULTS) [4], scholars proposed the probabilistic uncertain linguistic term set (PULTS) [5], while considering the shortcomings of PLTS. In actual decision-making cases, the decision makers may not be able to give complete probabilistic information, which requires the probabilities to be complemented. Traditional normalization methods distribute unknown probabilities uniformly over known linguistic terms, which may lead to inaccuracies. Based on this, Jin et al. [6] introduced the concept of uncertain probabilistic linguistic term set (UPLTS) as a method to complete probabilities. Based on this, Li et al. [7] proposed limited interval-valued probabilistic linguistic term sets (l-IVPLTS) by introducing the membership degree of unknown probability to solve the problem of assigning unknown probabilities.

The ELECTRE method is one of the most classical multi-attribute group decision-making (MAGDM) methods. It was first proposed by Roy [8], and through intensive research, various versions of the ELECTRE method have emerged [9–12]. Among these versions, the ELECTRE III method stands out due to its three thresholds and has been applied in a variety of MAGDM scenarios. Wu et al. [13] proposed the PH2TL-ELECTRE III method by combining ELECTRE III and the proportional hesitant 2-tuple linguistic term set (PH2TLTS). Fei et al. [14] introduced the evidential linguistic term set (ELTS) and combined it with the ELECTRE method. Chen et al. [15] proposed a multi-attribute decision support framework based on the concept of basic uncertain linguistic information by combining the Bayesian best-worst method with the ELECTRE III method. Pang et al. [16] used the ELECTRE III method to establish the weak advantage relationship among indicators, and Fan et al. [17] proposed the CRITIC-ELECTRE III method, which comprehensively considers conflicting attributes using the CRITIC method.

As one of the most influential behavioral decision theories, regret theory [18] has attracted the attention of related scholars and is widely applied to MAGDM problems. Liang et al. [19] combined regret theory with the ELECTRE III method to solve decision problems. Song et al. [20] combined regret theory with DEA to calculate the total utility value. Song et al. [21] extended the regret theory to a probabilistic hesitant fuzzy environment and introduced related concepts, such as the utility function, the reject-rejoice function, and the perceived utility value of the probabilistic hesitant fuzzy element (P-HFE). Bisht et al. [22] considered the psychological behavior of decision makers and combined the advantages of prospect theory and regret theory to address the decision-making problem. Yan et al. [23] introduced a three-way decision-making method that integrates regret theory with the TOPSIS method.

Projection measure is a very suitable method to deal with decision-making problems because it can consider not only the distance, but also the angle between the evaluated objects [24,25]. However, the classical projection method suffers from similarity misclassification; Tsao et al. [26] noted that at certain times, the projection value fails, so that the projection method results in decision-making errors. Therefore, in order to overcome the limitations of the existing projection methods, Ye [27] proposed the bidirectional projection (BP) method and extended it to single-valued neutral set environments, which is more comprehensive and reasonable compared with the projection method. In addition, Liang [19] extended the BP method to the probabilistic interval-valued hesitant fuzzy set (PIVHFS), and Liu et al. [28] applied bidirectional projection to probabilistic linguistic information (PLI). However, the use of bidirectional projection in the context of limited interval-valued probabilistic linguistic term sets is still relatively rare in current research.

However, existing MAGDM research still has shortcomings: the distance under the l-IVPLTS environment only considers the differences between the evaluated objects and does not take into account the angles between the evaluated objects. The existing methods for obtaining event attributes and their weights based on social platforms only consider the opinions of the public and ignore expert opinions, which will result in a lack of scientific determination of attributes and their weights. The method of determining expert weights has the problem of neglecting the expertise of experts themselves.

Based on the above analysis, this paper proposes a new IVBPRT-ELECTRE III algorithm. The following are the innovations and advantages of this paper: (1) this paper proposes a new bidirectional projection measurement method based on l-IVPLTS, which fills the gap of bidirectional projection in the l-IVPLTS language environment; (2) based on social media comment data, the comment data is divided into public and official levels according to the number of fans, taking into account the impact of fan base on attribute weights; (3) after obtaining the initial weights of attributes based on the relevant information of comment data, the entropy value of attributes is calculated based on expert evaluation information, and the initial information of attributes is judged and adjusted to make the attribute weights more reasonable; and (4) this paper combines the l-IVPLTS bidirectional projection with regret theory, calculates the objective weights of decision-makers based on this, and considers their own factors to obtain individual weights. The combination of the two obtains the comprehensive weights of decision-makers, making their weights more reasonable and scientific.

The structure of this paper is arranged as follows. In Section 2, we introduce some basic concepts and propose a new bidirectional projection measurement method based on l-IVPLTS. In Section 3, we present a method for obtaining attributes and their weights, a method for calculating the probability missing value compensation parameter x, a method for obtaining expert weights, and propose an extended ELECTRE III algorithm based on l-IVPLTS bidirectional projection and regret theory. In Section 4, we validate the effectiveness and feasibility of our method through the "3.13" Yanjiao explosion accident and compare it with other methods. Meanwhile, sensitivity analysis is also proposed. In Section 5, we present our conclusions and future work.

2. Preliminaries

This section introduces the basic theory of l-IVPLTS, proposes a new bidirectional projection measurement method based on l-IVPLTS, and then introduces the relevant basic knowledge of LDA, TF-IDF, and regret theory. This will help to better understand the methods presented in this article in the future.

2.1. Limited Interval-Valued Probabilistic Linguistic Term Set

Definition 1 ([3]). *Let $S = \{s_\alpha | \alpha = -\tau, -\tau+1, \ldots, \tau-1, \tau, \tau \in N^+\}$ be a LTS, then a PLTS can be defined as follows:*

$$L(p) = \left\{ L^{(k)}\left(p^{(k)}\right) \middle| L^{(k)} \in S, p^{(k)} \geq 0, \sum_{k=1}^{\#L(p)} p^{(k)} \leq 1, k = 1, 2, \ldots, \#L(p) \right\} \quad (1)$$

where $L^{(k)}\left(p^{(k)}\right)$ denotes the associated probability of the set of linguistic terms $L^{(k)}$ with $p^{(k)}$; $\#L(p)$ denotes the number of linguistic terms in the set of probabilistic linguistic terms.

Definition 2 ([7]). *Let $S=\{s_\alpha|\alpha=-\tau,-\tau+1,\ldots,\tau-1,\tau,\tau\in N^+\}$ be a LTS, then a limited interval-valued probabilistic linguistic term set (l-IVPLTS) can be defined as follows:*

$$_{l-}L(p)=\left\{{}_{l-}L^{(k)}\left[p^{(k)l},p^{(k)u}\right]\middle|L^{(k)}\in S,0\le p^{(k)l}\le p^{(k)u}\le 1,k=1,2,\ldots,\#_{l-}L(p)\right\} \quad (2)$$

where $_{l-}L^{(k)}\left[p^{(k)l},p^{(k)u}\right]$ (called limited interval-valued probabilistic linguistic term element (l-IVPLTE)) denotes the kth linguistic term $_{l-}L^{(k)}$ with the limited interval-valued probability $\left[p^{(k)l},p^{(k)u}\right]$, $\#_{l-}L(p)$ denotes the number of linguistic term in $_{l-}L(p)$. $p^{(k)l}$ and $p^{(k)u}$ are determined respectively as:

$$p^{(k)l}=p^{(k)} \quad (3)$$

$$p^{(k)u}=p^{(k)l}+\left(1-\sum_{k=1}^{\#L(p)}p^{(k)l}\right)x^{(k)} \quad (4)$$

where $k=1,2,\ldots,\#_{l-}L(p)$, $0\le x^{(k)}\le 1$, $p^{(k)}$ is the original probability in the corresponding PLTS with $\sum_{k=1}^{\#L(p)}p^{(k)}\le 1$, $\left(1-\sum_{k=1}^{\#L(p)}p^{(k)l}\right)$ represents the unknown probability information, and $x^{(k)}$ denotes the probabilistic missing value compensation parameter belonging to the kth linguistic term.

Definition 3 ([7]). *If $p^{(k)l}=p^{(k)u}$ is satisfied for any $k\in\{1,2,\ldots,\#_{l-}L(p)\}$, then $_{l-}L^{(k)}\left[p^{(k)l},p^{(k)u}\right]$ is reduced to $L^{(k)}\left(p^{(k)}\right)$, $_{l-}L(p)$ is reduced to $L(p)$.*

Definition 4 ([7]). *Given two l-IVPLTSs $_{l-}L_1(p)$ and $_{l-}L_2(p)$, $\#_{l-}L_1(p)$ and $\#_{l-}L_2(p)$ represent the number of linguistic terms in $_{l-}L_1(p)$ and $_{l-}L_2(p)$, respectively. If $\#_{l-}L_1(p)\succ\#_{l-}L_2(p)$, then we add $\#_{l-}L_1(p)-\#_{l-}L_2(p)$ linguistic terms to $_{l-}L_2(p)$. The added linguistic term is the linguistic term with the smallest subscript in $_{l-}L_2(p)$ with probability zero.*

Definition 5 ([7]). *For a l-IVPLTS $_{l-}L(p)$, its score is defined as follows:*

$$E\left({}_{l-}L(p)\right)=[l,u] \quad (5)$$

where $l=\sum_{k=1}^{\#_{l-}L(p)}\gamma^{(k)}p^{(k)l}/\sum_{k=1}^{\#_{l-}L(p)}\left(p^{(k)l}+p^{(k)u}\right)$, $u=\sum_{k=1}^{\#_{l-}L(p)}\gamma^{(k)}p^{(k)u}/\sum_{k=1}^{\#_{l-}L(p)}\left(p^{(k)l}+p^{(k)u}\right)$, and $\gamma^{(k)}$ denotes the subscript of kth linguistic term L^k.

Definition 6 ([7]). *For a l-IVPLTS $_{l-}L(p)$, its deviation degree is defined as follows:*

$$\sigma\left({}_{l-}L(p)\right)=[\sigma_l,\sigma_u] \quad (6)$$

where $\sigma_l=\sqrt{\sum_{k=1}^{\#_{l-}L(p)}\left(p^{(k)l}\left(\gamma^{(k)}-l\right)\right)^2}/\sum_{k=1}^{\#_{l-}L(p)}\left(p^{(k)l}+p^{(k)u}\right)$, $\sigma_u=\sqrt{\sum_{k=1}^{\#_{l-}L(p)}\left(p^{(k)u}\left(\gamma^{(k)}-u\right)\right)^2}/\sum_{k=1}^{\#_{l-}L(p)}\left(p^{(k)l}+p^{(k)u}\right)$, and $\gamma^{(k)}$ denotes the subscript of kth linguistic term L^k.

Definition 7 ([7]). *The comparison rules corresponding to two limited interval-valued probabilistic linguistic term sets $_{l-}L_1(p)$ and $_{l-}L_2(p)$ are as follows:*

(1) *If $E\left({}_{l-}L_1(p)\right)\succ E\left({}_{l-}L_2(p)\right)$, then $_{l-}L_1(p)\succ{}_{l-}L_2(p)$;*
(2) *If $E\left({}_{l-}L_1(p)\right)\prec E\left({}_{l-}L_2(p)\right)$, then $_{l-}L_1(p)\prec{}_{l-}L_2(p)$;*

(3) *If* $E\left({}_{l-}L_1(p)\right) = E\left({}_{l-}L_2(p)\right)$*, while* $\sigma\left({}_{l-}L_1(p)\right) \succ \sigma\left({}_{l-}L_2(p)\right)$*, then* ${}_{l-}L_1(p) \prec {}_{l-}L_2(p)$*; while* $\sigma\left({}_{l-}L_1(p)\right) \prec \sigma\left({}_{l-}L_2(p)\right)$*, then* ${}_{l-}L_1(p) \succ {}_{l-}L_2(p)$*; while* $\sigma\left({}_{l-}L_1(p)\right) = \sigma\left({}_{l-}L_2(p)\right)$*, then* ${}_{l-}L_1(p) = {}_{l-}L_2(p)$*.*

Definition 8 ([29])**.** *The operation laws associated with linguistic terms in l-IVPLTS are defined as follows:*

(1) $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$;
(2) $\lambda s_\alpha = s_{\lambda\alpha}$;
(3) $neg(s_\alpha) = s_{-\alpha}$*, especially,* $neg(s_0) = s_0$*.*

where, $neg(s_\alpha)$ *denotes the negation operator of* s_α*.*

In addition, we define some operators and PLWA operators applicable to l-IVPLTS:

$$ {}_{l-}L(p)_1 \oplus {}_{l-}L(p)_2 = \left\{ {}_{l-}L_3^{l_3}\left[p_3^{(l)l_3}, p_3^{(u)l_3}\right] \middle| l_3 = 1,2,\ldots \#_{l-}L(p)_3 \right\} \tag{7} $$

$$ \lambda_{l-}L(p)_1 = \left\{ \lambda_{l-}L_1^{l_1}(p) \middle| l_1 = 1,2,\ldots \#_{l-}L(p)_1 \right\} \tag{8} $$

where ${}_{l-}L_3^{l_3} = {}_{l-}L_1^{l_1} \oplus {}_{l-}L_2^{l_2}$, $\left[p_3^{(l)l_3}, p_3^{(u)l_3}\right] = \left[p_1^{(l)l_1} p_2^{(l)l_2}, p_1^{(u)l_1} p_2^{(u)l_2}\right]$ $(l_1 = 1,2,\ldots \#_{l-}L(p)_1;$ $l_2 = 1,2,\ldots \#_{l-}L(p)_2)$.

$$ \begin{aligned} &PLWA\left({}_{l-}L(p)_1, \ldots, {}_{l-}L(p)_n\right) = \underset{1\le i\le n}{\oplus} w_{i\,l-}L(p)_i = w_{1\,l-}L(p)_1 \oplus \cdots \oplus w_{n\,l-}L(p)_n = \\ &\cup_{{}_{l-}L(p)_1^{l_1} \in {}_{l-}L(p)_1}\left\{ w_{1\,l-}L(p)_1^{l_1} \right\} \oplus \cdots \oplus \cup_{{}_{l-}L(p)_n^{l_n} \in {}_{l-}L(p)_n}\left\{ w_{n\,l-}L(p)_n^{l_n} \right\} \end{aligned} \tag{9} $$

where $w_{i\,l-}L(p)_i^{l_i} = w_i L_i^{l_i}(p) = w_{i\,l-}s_\alpha^{l_i}(p)$, $s_\alpha^{l_i}$, are the linguistic terms.

Definition 9 *Let* ${}_{l-}L_{i1}(p), {}_{l-}L_{i2}(p)$ *be the evaluation information of expert* e_1 *and expert* e_2 *on alternative* A_i*:*

$$ {}_{l-}L_{i1}(p) = \left\{ {}_{l-}L_{i1}^{(k)}\left[p_{i1}^{(k)l}, p_{i1}^{(k)u}\right] | k = 1,2,\ldots,\#_{l-}L_{i1}(p) \right\} \tag{10} $$

$$ {}_{l-}L_{i2}(p) = \left\{ {}_{l-}L_{i2}^{(k)}\left[p_{i2}^{(k)l}, p_{i2}^{(k)u}\right] | k = 1,2,\ldots,\#_{l-}L_{i2}(p) \right\} \tag{11} $$

The modules of ${}_{l-}L_{i1}(p), {}_{l-}L_{i2}(p)$, are defined as:

$$ \left|{}_{l-}L_{i1}(p)\right| = \sqrt{\sum_{k=1}^{\#_{l-}L_{i1}(p)} \left[\left(g\left({}_{l-}L_{i1}^k\right) p_{i1}^{ku} \right)^2 + \left(g\left({}_{l-}L_{i1}^k\right) p_{i1}^{kl} \right)^2 \right]} \tag{12} $$

$$ \left|{}_{l-}L_{i2}(p)\right| = \sqrt{\sum_{k=1}^{\#_{l-}L_{i2}(p)} \left[\left(g\left({}_{l-}L_{i2}^k\right) p_{i2}^{ku} \right)^2 + \left(g\left({}_{l-}L_{i2}^k\right) p_{i2}^{kl} \right)^2 \right]} \tag{13} $$

The inner product of ${}_{l-}L_{i1}(p), {}_{l-}L_{i2}(p)$, are given by:

$$ {}_{l-}L_{i1}(p) \cdot {}_{l-}L_{i2}(p) = \sum_{k=1}^{\#_{l-}L_{i1}(p)} \left[\left(g\left({}_{l-}L_{i1}^k\right) p_{i1}^{ku} \right)\left(g\left({}_{l-}L_{i2}^k\right) p_{i2}^{ku} \right) + \left(g\left({}_{l-}L_{i1}^k\right) p_{i1}^{kl} \right)\left(g\left({}_{l-}L_{i2}^k\right) p_{i2}^{kl} \right) \right] \tag{14} $$

where $g\left({}_{l-}L_{ij}^k\right) = \frac{r_{ij}^k+\tau}{2\tau}$, r_{ij}^k is the subscript of ${}_{l-}L_{ij}^k$.

Then the l-IVPLTS bidirectional projection of ${}_{l-}L_{i1}(p)$ and ${}_{l-}L_{i2}(p)$ is defined as:

$$Bproj\left({}_{l-}L_{i1}(p), {}_{l-}L_{i2}(p)\right) = \frac{\left|{}_{l-}L_{i1}(p)\right|\left|{}_{l-}L_{i2}(p)\right|}{\left|{}_{l-}L_{i1}(p)\right|\left|{}_{l-}L_{i2}(p)\right| + \left|\left|{}_{l-}L_{i1}(p)\right| - \left|{}_{l-}L_{i2}(p)\right|\right|{}_{l-}L_{i1}(p) \cdot {}_{l-}L_{i2}(p)} \tag{15}$$

Lemma 1. *The l-$IVPLTS$ bidirectional projection has the following properties:*

(1) $Bproj\left({}_{l-}L_{i1}(p), {}_{l-}L_{i2}(p)\right) \in [0,1]$;

(2) *If* ${}_{l-}L_{i1}(p) = {}_{l-}L_{i2}(p)$, $Bproj\left({}_{l-}L_{i1}(p), {}_{l-}L_{i2}(p)\right) = 1$.

Proof of Lemma 1. (1) It is obvious that $\left|{}_{l-}L_{i1}(p)\right| \geq 0$, $\left|{}_{l-}L_{i2}(p)\right| \geq 0$, ${}_{l-}L_{i1}(p) \cdot {}_{l-}L_{i2}(p) \geq 0$, then $Bproj\left({}_{l-}L_{i1}(p), {}_{l-}L_{i2}(p)\right) \geq 0$, since $\left|{}_{l-}L_{i1}(p)\right|\left|{}_{l-}L_{i2}(p)\right| \leq \left|{}_{l-}L_{i1}(p)\right|\left|{}_{l-}L_{i2}(p)\right| + \left|\left|{}_{l-}L_{i1}(p)\right| - \left|{}_{l-}L_{i2}(p)\right|\right|$, then $Bproj\left({}_{l-}L_{i1}(p), {}_{l-}L_{i2}(p)\right) \leq 1$; (2) When ${}_{l-}L_{i1}(p) = {}_{l-}L_{i2}(p)$, we have $\left|{}_{l-}L_{i1}(p)\right| = \left|{}_{l-}L_{i2}(p)\right|$, then $Bproj\left({}_{l-}L_{i1}(p), {}_{l-}L_{i2}(p)\right) = 1$.

The proof is completed. □

When an event occurs, the public shares their thoughts and opinions in the form of text on social media, reflecting the aspects they are concerned about regarding the event. Accordingly, data mining techniques can be employed to crawl the text data posted by the social public and extract relevant attributes of the event. The data mining algorithms used in this paper primarily include LDA theme models and TF-IDF technology, which will be briefly introduced below.

2.2. LDA

Latent Dirichlet allocation (LDA) theme model was proposed in 2003 by Blei et al. [30]. It is an extension of potential semantic analysis and probabilistic potential semantic analysis. It consists of a three-layer Bayesian probabilistic structure of document layer, topic layer, and word layer. It generates a "document-word item" of a document by extracting a polynomial distribution of implicit topics from the Dirichlet prior distribution of each document, and then iteratively sampling the resulting topics to map the document to a topic vector space. It is widely used, notably within the domain of text mining, it can analyze a large amount of text data and analyze its topic structure, and in multi-attribute group decision-making, it is mostly used to extract attributes based on text data. The LDA model diagram is shown in Figure 1.

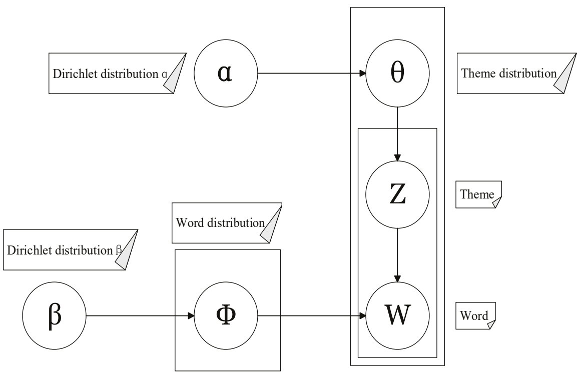

Figure 1. LDA model diagram.

2.3. TF-IDF

The TF-IDF algorithm represents a common weighting mechanism employed in the realms of information retrieval and text mining, and is widely used to determine keyword weights in group decision making. The equation for calculating the weight of keywords is presented below:

$$TF\text{-}IDF(T) = td_d(T) \cdot log_d\left(\frac{N}{df(T)}\right) \tag{16}$$

where $td_d(T)$ denotes the frequency of the word in the text, N is the number of all texts, and $df(T)$ denotes the number of texts in the text collection that contain the word T.

2.4. Regret Theory

Bell (1982) [18], and Loomes and Sugden, introduced regret theory, which, as a highly influential behavioral decision-making theory, has garnered the interest of scholars and is extensively applied to MAGDM issues. Regret theory uses utility function, utility value, and the regret-rejoice function to model the psychology of decision makers. According to regret theory, decision makers compare the outcomes of different choices to assess their regret and satisfaction, and choose the best choice that they will not regret.

Definition 10 ([19])**.** *Let Δu as the utility difference between two alternatives, the regret-rejoice function $R(\Delta u)$ is given by:*

$$R(\Delta u) = 1 - exp(-\alpha \Delta u) \tag{17}$$

where α denotes the regret aversion coefficient of the decision maker and $\alpha \in (0, +\infty)$. The larger the value of α, the higher the regret aversion of the decision maker. If $R(\Delta u) \succ 0$, then $R(\Delta u)$ is called the rejoice value; if $R(\Delta u) \leq 0$, then $R(\Delta u)$ is called the regret value.

3. Emergency Decision-Making Method Based on Bidirectional Projection—Regret Theory

This section firstly introduces how to obtain event attributes and their weights through comment data on social platforms, then introduces the method of obtaining probability missing value compensation parameters by constructing a model, followed by the method of obtaining expert comprehensive weights through personal weights and objective weights, and finally proposes an extended ELECTRE III algorithm based on l-IVPLTS bidirectional projection and regret theory.

3.1. Obtaining Attributes and Initial Weights

Crawling data under event-related words using Python3.7.4 code, including text data, the number of likes, comments, retweets, and fans data, and performing data preprocessing, such as removing duplicate and useless data, and segmenting text data, and determining the number of attributes and attributes using the LDA model.

Due to the significant difference in the number of user followers in the comment data, in order to prevent the influence of user followers on the clustering effect, the text data is divided into mass level and official level based on the number of user followers in the social platform text data. Then, the text data is clustered according to the likes, comments, shares, and followers of the mass level and official level text data, respectively. When a certain viewpoint on social media is recognized by more people, it will have more likes, comments, shares, and followers. It is reasonable to calculate the attention coefficient of each dataset based on these specific attributes. The calculation formulas for the attention coefficient of each dataset at the public and official levels are as follows:

$$\gamma_i^g = \frac{\partial_a^g \cdot DS_i^g(AN) + \partial_c^g \cdot DS_i^g(CN) + \partial_r^g \cdot DS_i^g(RN) + \partial_f^g \cdot DS_i^g(FSN)}{\sum_{i=1}^t \partial_a^g \cdot DS_i^g(AN) + \partial_c^g \cdot DS_i^g(CN) + \partial_r^g \cdot DS_i^g(RN) + \partial_f^g \cdot DS_i^g(FSN)} \tag{18}$$

$$\gamma_i^q = \frac{\partial_a^q \cdot DS_i^q(AN) + \partial_c^q \cdot DS_i^q(CN) + \partial_r^q \cdot DS_i^q(RN) + \partial_f^q \cdot DS_i^q(FSN)}{\sum_{i=1}^t \partial_a^q \cdot DS_i^q(AN) + \partial_c^q \cdot DS_i^q(CN) + \partial_r^q \cdot DS_i^q(RN) + \partial_f^q \cdot DS_i^q(FSN)} \tag{19}$$

where

$${}^g_i(AN) = \frac{\sum_{d_{ij}^g \in DS_i^g} AN_{ij}^g}{n_i^g}, DS_i^g(CN) = \frac{\sum_{d_{ij}^g \in DS_i^g} CN_{ij}^g}{n_i^g}, DS_i^g(RN) = \frac{\sum_{d_{ij}^g \in DS_i^g} RN_{ij}^g}{n_i^g}, DS_i^g(FSN) = \frac{\sum_{d_{ij}^g \in DS_i^g} FSN_{ij}^g}{n_i^g} \tag{20}$$

$$AN) = \frac{\sum_{d_{ij}^q \in DS_i^q} AN_{ij}^q}{n_i^q}, DS_i^q(CN) = \frac{\sum_{d_{ij}^q \in DS_i^q} CN_{ij}^q}{n_i^q}, DS_i^q(RN) = \frac{\sum_{d_{ij}^q \in DS_i^q} RN_{ij}^q}{n_i^q}, DS_i^q(FSN) = \frac{\sum_{d_{ij}^q \in DS_i^q} FSN_{ij}^q}{n_i^q} \tag{21}$$

where $DS_i^g(AN)$, $DS_i^g(CN)$, $DS_i^g(RN)$ and $DS_i^g(FSN)$ represent the mean likes, mean comments, mean retweets, and mean followers in the official ith data set, respectively, and $DS_i^q(AN)$, $DS_i^q(CN)$, $DS_i^q(RN)$ and $DS_i^q(FSN)$ represent the mean likes, mean comments, mean retweets, and mean followers in the mass ith data set, respectively. The denominators n_i^g, n_i^q denote the amount of data in the official, mass dataset respectively.

We established a linear programming model to obtain the maximum influence as the goal, Z signifies the overall impact of the data, while $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ correspond to the quantities of likes, comments, retweets, and followers, respectively. The $\partial_a, \partial_c, \partial_r, \partial_f$ denote the respective weights attributed to each metric. Assuming an equal influence exerted by each factor, we find the maximum influence of the data and $\partial_a, \partial_c, \partial_r, \partial_f$.

$$\begin{aligned} &MaxZ = A \cdot \partial_a + B \cdot \partial_c + C \cdot \partial_r + D \cdot \partial_f \\ &s.t. \begin{cases} A \cdot \partial_a - B \cdot \partial_c = 0 \\ A \cdot \partial_a - C \cdot \partial_r = 0 \\ A \cdot \partial_a - D \cdot \partial_f = 0 \\ \ldots \\ C \cdot \partial_r - D \cdot \partial_f = 0 \\ \partial_a + \partial_c + \partial_r + \partial_f = 1 \\ \partial_a, \partial_c, \partial_r, \partial_f \geq 0 \end{cases} \end{aligned} \tag{22}$$

We ensure that the sum of the indicator weights equals to 1 by solving the linear programming model to obtain the weight of each indicator; the use of the model to determine the weight of the indicator is more objective, and can effectively avoid the decision-making risk brought about by the subjective determination of the indicator weight by experts, and is conducive to the improvement of the applicability of the method and the scientific nature of the method.

Combining the attention coefficients of the datasets, the TF-IDF algorithm is used to obtain the TF-IDF values under each dataset under the official and mass, respectively, and then the initial weights of the attributes are obtained based on the TF-IDF values:

$$w_j^g = \frac{\sum_{i=1}^t \gamma_i \cdot w\left(c_{ij}^g\right)}{\sum_{j=1}^n \sum_{i=1}^t \gamma_i \cdot w\left(c_{ij}^g\right)} \tag{23}$$

$$w_j^q = \frac{\sum_{i=1}^{t} \gamma_i \cdot w\left(c_{ij}^q\right)}{\sum_{j=1}^{n} \sum_{i=1}^{t} \gamma_i \cdot w\left(c_{ij}^q\right)} \tag{24}$$

where $w\left(c_{ij}^g\right), w\left(c_{ij}^q\right)$ are the weight of the attribute c_j under the ith dataset at the official and mass level, w_j^g, w_j^q are the weight of the attribute c_j at the official and mass level.

The initial weights of the attributes are obtained by combining the official and mass level attribute weights:

$$w_j = \beta w_j^g + (1-\beta) w_j^q \tag{25}$$

where $0 \le \beta \le 1$.

3.2. Solving for Probabilistic Missing Value Compensation Parameter x

Using Equation (15) to denote the similarity between the evaluation information provided by expert e_1 and expert e_2 for alternative A_i, where $p_{i1}^{ku} = p_{i1}^{kl} + \left(1 - \sum_{k=1}^{\#_{l-}L_{i1}(p)} p_{i1}^{kl}\right) x_{i1}^k, p_{i2}^{ku} = p_{i2}^{kl} + \left(1 - \sum_{k=1}^{\#_{l-}L_{i2}(p)} p_{i2}^{kl}\right) x_{i2}^k, 0 \le x_{i1}^k \le 1, 0 \le x_{i2}^k \le 1$.

Based on $l\text{-}IVPLTS$ bidirectional projection, the similarity between experts under alternative A_i can be calculated, and the similarity matrix $S_i = \left[BProj\left({}_{l-}L_{il}(p), {}_{l-}L_{iq}(p)\right)\right]$ is derived as follows:

$$S_i = \begin{bmatrix} BProj\left({}_{l-}L_{i1}(p), {}_{l-}L_{i1}(p)\right) & BProj\left({}_{l-}L_{i1}(p), {}_{l-}L_{i2}(p)\right) & \cdots & BProj\left({}_{l-}L_{i1}(p), {}_{l-}L_{it}(p)\right) \\ BProj\left({}_{l-}L_{i2}(p), {}_{l-}L_{i1}(p)\right) & BProj\left({}_{l-}L_{i2}(p), {}_{l-}L_{i2}(p)\right) & \cdots & BProj\left({}_{l-}L_{i2}(p), {}_{l-}L_{it}(p)\right) \\ \vdots & \vdots & \ddots & \vdots \\ BProj\left({}_{l-}L_{it}(p), {}_{l-}L_{i1}(p)\right) & BProj\left({}_{l-}L_{it}(p), {}_{l-}L_{i2}(p)\right) & \cdots & BProj\left({}_{l-}L_{it}(p), {}_{l-}L_{it}(p)\right) \end{bmatrix}_{t \times t} \tag{26}$$

where $BProj\left({}_{l-}L_{il}(p), {}_{l-}L_{iq}(p)\right) = BProj\left({}_{l-}L_{iq}(p), {}_{l-}L_{il}(p)\right), BProj\left({}_{l-}L_{il}(p), {}_{l-}L_{il}(p)\right) = 1, l, q = 1, 2, \ldots, t$.

Then, based on the similarity matrix S_i, the total similarity between experts e_r and other experts under the alternative A_i can be obtained:

$$s_{ir} = \sum_{q=1}^{t} BProj\left({}_{l-}L_{il}(p), {}_{l-}L_{iq}(p)\right), i = 1, 2, \ldots, n; r = 1, 2, \ldots, t. \tag{27}$$

Then, based on s_{ir}, the total similarity of the alternatives A_i can be obtained as follows:

$$s_i = \sum_{r=1}^{t} s_{ir}. \tag{28}$$

For a multi-attribute group decision-making problem, alternatives are usually assigned to experts in the relevant fields. There are usually differences in the decision-making information provided by different experts. In this paper, bidirectional projection is utilized to measure the similarity between the decision information provided by the experts. To obtain effective decision-making results, the differences should be minimized, while similarities should be maximized. Based on this idea, the following model 1 is constructed to solve the missing value compensation parameter:

Model 1

$$\begin{cases} maxZ = s_i \\ \sum\limits_{k=1}^{\#} x_{ir}^k = 1, i = 1, 2, \ldots, n, r = 1, 2, \ldots, t \\ x_{ir}^k \ge 0 \end{cases} \tag{29}$$

The missing value compensation parameters are obtained by solving Model 1 to get the complete evaluation information.

3.3. Expert Weight Determination Method

As one of the most influential behavioral decision-making theories, regret theory [18] has attracted the attention of relevant scholars and is widely used in the MAGDM problem. Regret theory uses utility function, utility value, and regret-rejoice function to model the psychology of decision maker. The key step is to apply the utility function to obtain the utility value. In most cases, the utility function is determined by the probability density function. However, in uncertain environments, it is difficult to obtain the specific distribution of the variables.

Fang et al. [31] introduced regret theory to reduce the degree of regret that experts feel regarding the final decision. This theory is used to calculate the maximum regret value for each expert, and consequently, the expected weights of the experts are derived. While Fang et al. only uses the difference of simple transformation function as the utility difference, which only takes into account the difference between the evaluation value. In this paper, the distance and angle between the evaluation value and the ideal solution are considered. The utility difference is calculated using a bidirectional projection to determine the maximum regret value of the expert, and consequently, the objective weights of the experts are derived.

On this basis, this paper takes into account the authority of the expert from three aspects: age, education background, and professional status, obtains the expert's personal score based on the expert's personal situation and scoring attribute [32], and obtains the expert's individual weight based on the expert's personal score. The specific attributes for individual expert scores are shown in Table 1.

Table 1. Individual expert scoring attribute.

Factors	Standard	Score
Age	>50	4
	41–50	3
	30–40	2
	<30	1
Educational background	PhD	4
	Master's Degree	3
	Undergraduate	2
	Specialist	1
Professional status	Full Senior Engineer/Professor	4
	Senior Engineer/Associate Professor	3
	Engineer/Lecturer	2
	Assistant Engineer/Assistant Professor	1

The specific steps are as follows:

Firstly, calculate the regret value of each expert's linguistic evaluation:

$$R_{ij}^{r} = 1 - exp(-\alpha(\Delta BProj)) \tag{30}$$

where $\Delta BProj = BProj\left({}_{l-}L_{ij}^{t}(p), {}_{l-}L^{+}(p)\right) - BProj\left({}_{l-}L^{+}(p), {}_{l-}L^{+}(p)\right) = BProj\left({}_{l-}L_{ij}(p), {}_{l-}L^{+}(p)\right) - 1$, ${}_{l-}L^{+}(p) = \left\{S_{\tau}\left[\frac{1}{k}, \frac{1}{k}\right], S_{\tau}\left[\frac{1}{k}, \frac{1}{k}\right], \ldots, S_{\tau}\left[\frac{1}{k}, \frac{1}{k}\right]\right\}$. $\Delta BProj$ is the difference between the expert e_r's evaluation value ${}_{l-}L_{ij}^{t}(p)$ of the alternative A_i under the attribute c_j and the positive ideal solution ${}_{l-}L^{+}(p)$.

Secondly, calculate the objective weights of the experts based on the regret value:

$$w_r^1 = \frac{\left(\frac{1}{R^r}\right)}{\sum_{r=1}^{t}\left(\frac{1}{R^r}\right)} \tag{31}$$

where $R^r = max\left\{\left|R_{ij}^r\right|\right\}$.

And then, calculate the expert's personal score based on the expert's personal situation and scoring attribute, and obtain the expert's individual weight:

$$w_r^2 = \frac{S^r}{\sum_{r=1}^{t} S^r} \tag{32}$$

where S^r is the total value of the expert e_r's scores in the three aspects of age, education background and professional status.

Finally, the comprehensive expert weight is obtained by combining the expert personal weight and objective weight:

$$W_r = \gamma w_r^1 + (1-\gamma) w_r^2 \tag{33}$$

where $0 \le \gamma \le 1$.

3.4. Adjustment of Attribute Weights Based on Information Entropy

Based on the expert weights obtained from Section 3.3, the PLWA operator is used to aggregate the expert evaluation matrix, thereby obtaining the comprehensive decision matrix ${}_{l-}L'_{ij}$. Next, the entropy value for each attribute is calculated based on the comprehensive decision matrix [33], and the initial attribute weights are adjusted according to the entropy value. The equation for computing the entropy value is presented as follows:

$$Entropy(c_j) = -\frac{1}{lnm}\sum_{i=1}^{m} f_i^j ln f_i^j \tag{34}$$

where $f_i^j = \frac{\sum_{k=1}^{\#_{l-}L(p)}\left(p'^{kl}_{ij}\cdot g\left({}_{l-}L'^{k}_{ij}\right)+p'^{ku}_{ij}\cdot g\left({}_{l-}L'^{k}_{ij}\right)\right)}{\sum_{i=1}^{m}\sum_{k=1}^{\#_{l-}L(p)}\left(p'^{kl}_{ij}\cdot g\left({}_{l-}L'^{k}_{ij}\right)+p'^{ku}_{ij}\cdot g\left({}_{l-}L'^{k}_{ij}\right)\right)}, j = 1,2,\ldots,n$, $g\left({}_{l-}L'^{k}_{ij}\right) = \frac{r'^{k}_{ij}+\tau}{2\tau}$, $m = \#\{A_i\}$.

Entropy [34] is a commonly used term in information theory that describes the uncertainty associated with the occurrence of an event, and is also employed to determine attribute weights. If the entropy value of an attribute is larger, the information uncertainty of the attribute is higher, the difference in the ratings of the attribute by the alternatives is smaller, which means that the attribute provides less information, and the weight of the attribute should be smaller.

Zhu et al. [35] proposed a method for dynamic updating of attribute weights based on decision quality. This method determines the attribute weights of the next stage based on the attribute weights of the current stage and the decision quality, which ensures that the decision problem is more closely related to real-world conditions. Wang et al. [36] used text comments to determine the attributes of the decision attribute and the attribute weights, but since most of the text comments are posted by the public, they lack professionalism.

On this basis, this paper calculates the attribute entropy value based on the expert evaluation, and proposes the following attribute weight adjustment rules based on the attribute initial weight and attribute entropy value, which integrates the perspectives of both experts and the general public:

- If $w_j > \bar{w}$ and $E(c_j) > \bar{E}$, the weight w_j need to be adjusted downward;

- If $w_j \leq \bar{w}$ and $E(c_j) \leq \bar{E}$, the weight w_j need to be adjusted upward;
- if $w_j > \bar{w}$ and $E(c_j) \leq \bar{E}$, the weight w_j is reasonable and no adjustment is needed;
- if $w_j \leq \bar{w}$ and $E(c_j) > \bar{E}$, the weight w_j is reasonable and no adjustment is needed;

where $\bar{w} = \frac{1}{n}, \bar{E} = \frac{\sum E(c_j)}{n}$.

Adjust and update the initial attribute weights based on the entropy values and obtain the initial attribute weights:

$$w_j^* = \begin{cases} w_j \cdot \left(1 - \left|\overline{E} - E(c_j)\right|\right), & w_j > \overline{w} \text{ and } E(c_j) > \overline{E} \\ w_j \cdot \left(1 + \left|\overline{E} - E(c_j)\right|\right), & w_j \leq \overline{w} \text{ and } E(c_j) \leq \overline{E}. \\ w_j & else \end{cases} \tag{35}$$

Finally, the adjusted attribute weights can be obtained from the following equation:

$$W_j = \frac{w_j^*}{\sum_{j=1}^{n} w_j^*}. \tag{36}$$

3.5. ELECTRE III Method Based on Bidirectional Projection—Regret Theory

The ELECTRE III method involves three thresholds and is able to deal with imprecise, ambiguous or uncertain data, and its main idea is to use the concordance index and the discordance index to eliminate the shortcomings of weighting and large-value compensation. Therefore, the ELECTRE III method is applied to various MAGDM problems. This section presents an extended ELECTRE III method that considers the psychology of expert regret.

In Section 3.3, the regret value evaluation matrix of each expert is obtained, and the regret value matrix of each expert is linearly weighted using expert weights to obtain the comprehensive evaluation regret value matrix:

$$R_{ij} = \sum_{r=1}^{t} W_r \cdot R_{ij}^r. \tag{37}$$

Based on the comprehensive regret value evaluation matrix, calculate the concordance index and the discordance index of alternative A_i over alternative A_s under attribute c_j:

$$CI_j(A_i, A_s) = \begin{cases} 1, & R_{sj} - R_{ij} \leq q_j \\ 0, & R_{sj} - R_{ij} \geq p_j \\ \frac{R_{ij} - R_{sj} + p_j}{p_j - q_j}, & otherwise \end{cases} \tag{38}$$

$$DI_j(A_i, A_s) = \begin{cases} 1, & R_{sj} - R_{ij} \geq v_j \\ 0, & R_{sj} - R_{ij} \leq p_j \\ \frac{R_{sj} - R_{ij} - p_j}{v_j - p_j}, & otherwise \end{cases} \tag{39}$$

where p_j, q_j, v_j are the preference threshold, indifferent threshold and veto threshold, respectively, and R_{ij} is the evaluation regret value of alternative A_i under attribute c_j.

The comprehensive concordance index is determined using the concordance index:

$$CI(A_i, A_s) = \sum_{j=1}^{n} W_j CI_j(A_i, A_s) \tag{40}$$

where W_j is the weight of the attribute c_j.

Calculate the credibility index of alternative A_i over alternative A_s based on $CI(A_i, A_s)$ and $DI_j(A_i, A_s)$:

$$\varphi(A_i, A_s) = \begin{cases} CI(A_i, A_s), & \forall j, DI_j(A_i, A_s) \leq CI(A_i, A_s) \\ CI(A_i, A_s) \times \prod\limits_{j \in J} \dfrac{1 - DI_j(A_i, A_s)}{1 - CI(A_i, A_s)}, & otherwise \end{cases} \tag{41}$$

where $J = \{j | DI_j(A_i, A_s) > CI(A_i, A_s)\}$.

Calculate concordance credibility degrees, discordance credibility degrees and net credibility degree.

Concordance credibility degrees:

$$\Phi^+(A_i) = \sum\nolimits_{A_s \in A} \varphi(A_i, A_s), \forall X_i \in X. \tag{42}$$

Discordance credibility degrees:

$$\Phi^-(A_i) = \sum\nolimits_{A_s \in A} \varphi(A_s, A_i), \forall A_i \in A. \tag{43}$$

Net credibility degree:

$$\Phi(A_i) = \Phi^+(A_i) - \Phi^-(A_i). \tag{44}$$

Finally rank the alternatives based on the size of the net credibility degree.

3.6. Decision Making Steps of IVBPRT-ELECTRE III Algorithm

For a multi-attribute group decision problem, let $A = \{A_i | A_1, A, \ldots, A_m\}$ denote a set of alternatives, $C = \{c_j | c_1, c_2, \ldots, c_n\}$ is a set of attributes, $E = \{e_r | e_1, e_2, \ldots, e_t\}$ is a set of experts. Based on the above analysis, the specific process is shown in Figure 2 and the specific steps of the extend ELECTRE III method based on regret theory—bidirectional projection—are as follows:

Step 1. According to Section 3.1, crawl the event-related text comments, combined with the big data network behavior mining event keywords, to get the event evaluation attribute, and use the TF-IDF technology to get the initial weight of the event attributes;

Step 2. Based on Section 3.2, the probabilistic missing value compensation parameter x is solved by using Model 1 to convert the expert evaluation into l-IVPLTS form;

Step 3. Based on Section 3.3 combined with Equations (30)–(32) to calculate the expert personal weights and expert objective weights, using Equation (33) to calculate the expert comprehensive weights;

Step 4. Calculate the entropy value of each attribute using Equation (34), and adjust the attribute weights in combination with the adjustment rules in Section 3.4;

Step 5. Calculated using Equation (37) to obtain a comprehensive regret value matrix;

Step 6. Based on the comprehensive regret value matrix, combine Equations (38)–(40) to calculate the comprehensive concordance index;

Step 7. Combine the Equations (41)–(44) to calculate the net credibility degree and rank the alternatives.

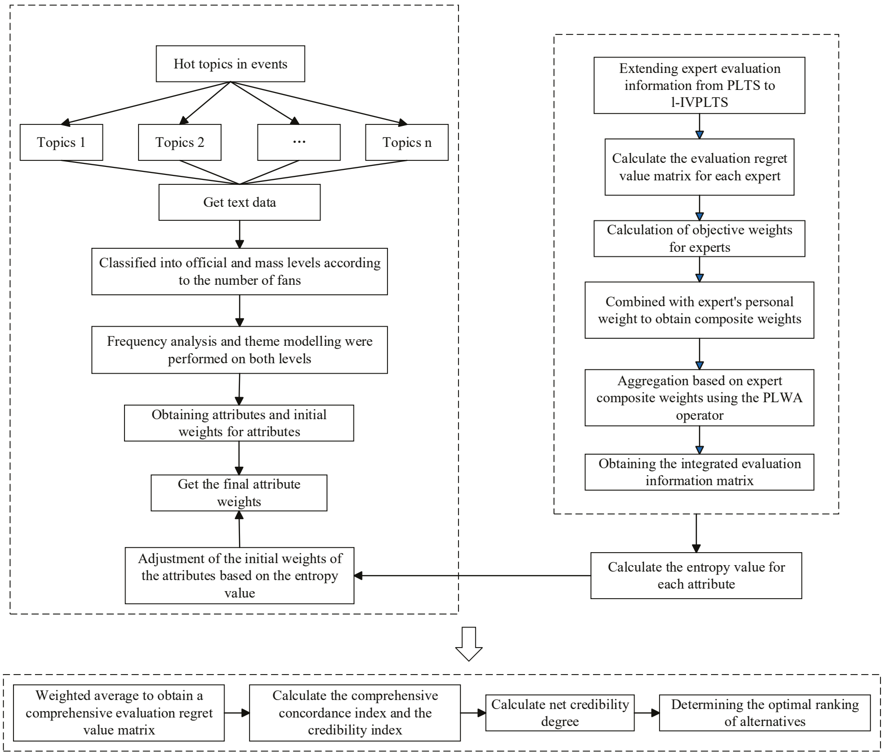

Figure 2. Emergency decision-making flow chart.

4. Case Study

4.1. Case Background

Fire safety is an important part of the public safety of society and the national emergency rescue system. According to the National Fire and Rescue Bureau statistics, in the first half of 2023, a total of 550,000 fires were reported nationwide, with 959 deaths, 1311 injuries, and 3.94 billion yuan in direct property losses [https://www.119.gov.cn/qmxfgk/sjtj/2023/38420.shtml (accessed on 29 September 2024)]; in the first half of 2024, the Guangdong Provincial Fire and Rescue Corps reported and disposed of 84,411 various types of alarms, with 212,806 vehicle trips, 1,109,000 trips of firefighting and rescue personnel, and rescuing 25,363 trapped people [https://www.119.gov.cn/include/inc/detail.html?id=1815231765137403905 (29 September 2024)].

At 7:55 p.m. on 13 March 2024, an explosion occurred at the Culture Building intersection in Yanjiao Township, Sanhe City, with a fireball shooting straight up into the sky and damaging a number of cars. In order to ensure the scientificity and authority of the decision-making results, five experts from the scientific research field, fire-fighting field, and chemical industry field were invited to constitute the expert group, and each expert had a bachelor's degree and more than 5 years of related work or scientific research experience. This paper adopts a 5-grain language evaluation environment:

$$S = \{S_{-2} : very\ low, S_{-1} : low, S_0 : fair, S_1 : high, S_2 : very\ high\}.$$

Attributes by crawling the microblog comment text data, using the LDA topic model to determine the number of attributes and obtain the attributes, the expert needs to evaluate each alternative under different attributes and give the relevant probability of that evaluation, with initial expert evaluation in the form of PLTS. For the evaluation of an alternative under one attribute, the expert can give one or more evaluation, but the evaluation correlation probability needs to be less than or equal to 1. The five alternatives were evaluated in terms of their attributes by five experts in emergency decision-making from the fire-fighting, chemical and other relevant sectors; the five alternatives are listed in the Supplementary Materials.

4.2. Data Analysis

Step 1. Obtain attributes and their initial weights

By crawling the data related to the Yanjiao explosion, a total of 1223 data points were extracted after processing, each of which included microblog content, the count of likes, comments, retweets, and followers. Following the data cleansing and filtration processes, a word cloud map, as shown in Figure 3, is formed.

Figure 3. Word cloud of '3.13' Yanjiao deflagration accident.

Use the LDA model to determine the number of attributes and attributes:

As shown in Figure 4, when the topic count is four, the coherence achieves its peak value, and the number of attributes is determined to be 4. According to the keywords, the attributes are determined to be $C = \{c_j | c_1, c_2, c_3, c_4\}$, where c_1 is "emergency response", c_2 is "fire suppression and derivative disaster control", c_3 is "site and surrounding environment monitoring", c_4 is "casualties and rescue".

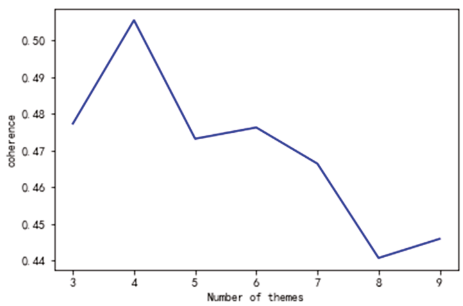

Figure 4. Topic consistency.

The threshold of the number of followers of the official and the mass is set as 500,000, accounts with more than 500,000 followers are considered official, and accounts with fewer than 500,000 followers are considered the mass. Based on the public attention of the event, the data at the official level and the mass level are classified using $k-$ means clustering to classify the data information and calculate the attention coefficient of each cluster. The results of the number of clusters are shown in Figures 5 and 6.

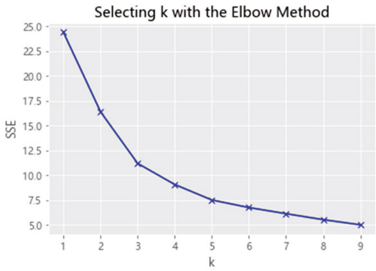

Figure 5. SSE value of clustering for mass data.

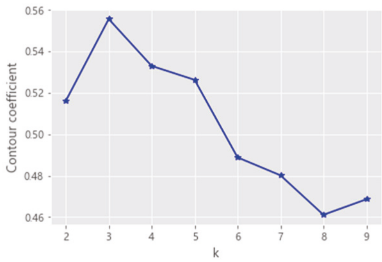

Figure 6. Silhouette coefficient of clustering for mass data.

(1) Mass level:

The determination of the optimal cluster count, $k = 3$, is based on the SSE and silhouette coefficient method, and the data is visualized through Python code as shown in Figure 7:

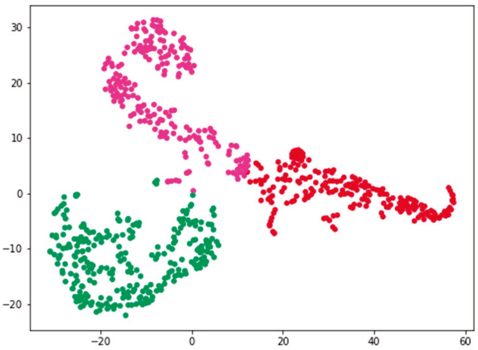

Figure 7. Visualization of clustering results for mass data.

Calculate the weight of each indicator using the linear programming model, and obtain the coefficient of attention of the mass level dataset according to Equation (19), and the weights of the indicators and the coefficient of attention of the mass level dataset are shown in Tables 2 and 3.

Table 2. Information of each indicator at the mass level.

	Retweets	Comments	Likes	Fans (Ten Thousand)
Total number of indicators	9086	16,139	88,666	9256.5626
Indicator weights	0.3778	0.2127	0.0387	0.3708

Table 3. Attention coefficient of each dataset at the mass level.

Dataset	Average Retweets	Average Comments	Average Likes	Average Fans (Ten Thousand)	Attention Coefficient
DS0	6.1896	7.9100	27.1280	26.4919	0.2237
DS1	37.0850	61.8400	386.8750	16.1031	0.7226
DS2	1.3444	7.7852	20.6185	1.6524	0.0537

(2) Official level:

The optimal number of clusters $k = 4$ is determined based on the SSE and silhouette coefficient method, the results of the number of clusters are shown in Figures 8 and 9, and the data are visualized through Python code as shown in Figure 10:

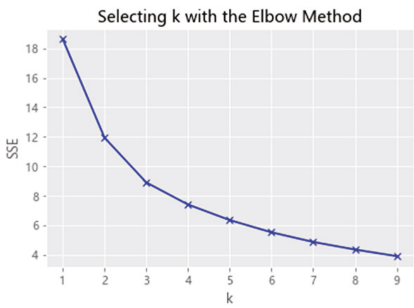

Figure 8. SSE value of clustering for official data.

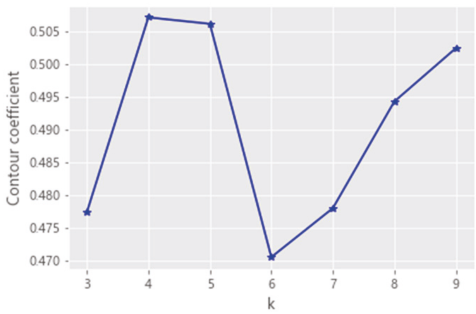

Figure 9. Silhouette coefficient of clustering for official data.

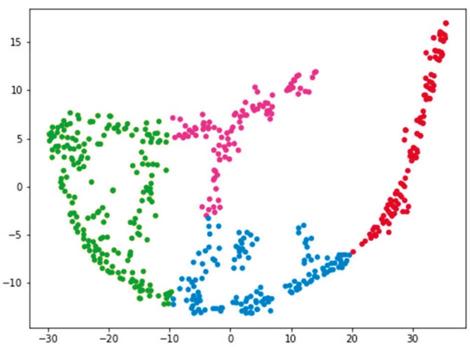

Figure 10. Visualization of clustering results for official data.

Calculate the weight of each indicator using the linear programming model and obtain the official level dataset attention coefficient according to Equation (18); the results are shown in Tables 4 and 5.

Table 4. Weight information of each indicator at the official level.

	Retweets	Comments	Likes	Fans (Ten Thousand)
Total number of indicators	19,280	33,913	204,324	737,759.1
Indicator weights	0.5920	0.3366	0.0559	0.0155

Table 5. Attention coefficient of each dataset at the official level.

Dataset	Average Retweets	Average Comments	Average Likes	Average Fans (Ten Thousand)	Attention Coefficient
DS3	9.1762	12.8342	39.6684	100.9782	0.0312
DS4	31.6389	49.6389	252.6574	5546.9935	0.3120
DS5	127.7368	244.0737	1667.1789	551.5832	0.5973
DS6	13.4041	19.7808	75.3356	457.4973	0.0595

Use the TF-IDF algorithm to obtain the initial attribute weights under each dataset at the official and mass levels, respectively, and obtain the initial attribute weights at the official and mass levels by using Equations (23) and (24); the results are shown in Tables 6 and 7.

Table 6. Initial attributes' weights of mass level.

Attribute	DS0	DS1	DS2
c_1	0.1763	0.1989	0.1655
c_2	0.2594	0.2970	0.3415
c_3	0.3061	0.2275	0.2491
c_4	0.2582	0.2766	0.2439

Table 7. Initial attributes' weights of official level.

Attribute	DS3	DS4	DS5	DS6
c_1	0.1640	0.1943	0.1794	0.2283
c_2	0.3551	0.3160	0.2924	0.3467
c_3	0.2006	0.2057	0.2407	0.1370
c_4	0.2803	0.2840	0.2875	0.2880

Initial attribute weights are obtained by combining official and mass level attribute weights through Equation (25); the results are shown in Table 8.

Table 8. Initial weights of attributes.

Attribute	c_1	c_2	c_3	c_4
Official level weights	0.1865	0.3050	0.2224	0.2862
Mass level weights	0.1921	0.2910	0.2462	0.2707
Initial weights	0.1893	0.2980	0.2343	0.2784

where $\beta = 0.5$.

Step 2. Solve the probabilistic missing value compensation parameter x

In this paper, five experts are invited to evaluate five emergency alternatives from four attributes: emergency response, fire suppression and derivative disaster control, site and surrounding environment monitoring and casualties, and rescue. The original evaluation matrix is as Table 9. For the incomplete probabilistic evaluation information, the probabilistic missing value compensation parameters are calculated using Model 1 in Section 3.2 to complete it. For space reasons, only the calculation results of probabilistic missing value compensation parameters under attribute c_1 are given in Table 10, and the rest of the results are shown as Tables S2–S4 in the Supplementary Materials.

Table 9. Original evaluation matrix of decision-making experts.

		c_1	c_2	c_3	c_4
e_1	A_1	$\{S_1(0.7), S_2(0.1)\}$	$\{S_{-1}(0.5), S_0(0.2)\}$	$\{S_0(0.7), S_1(0.1)\}$	$\{S_{-1}(0.1), S_0(0.5)\}$
	A_2	$\{S_1(0.1), S_2(0.8)\}$	$\{S_1(0.6), S_2(0.2)\}$	$\{S_0(0.2), S_1(0.6)\}$	$\{S_1(0.2), S_2(0.6)\}$
	A_3	$\{S_1(0.7), S_2(0.2)\}$	$\{S_1(0.7), S_2(0.2)\}$	$\{S_1(0.6), S_2(0.1)\}$	$\{S_1(0.9)\}$
	A_4	$\{S_1(0.7), S_2(0.1)\}$	$\{S_0(0.6), S_1(0.1)\}$	$\{S_{-2}(0.6), S_{-1}(0.2)\}$	$\{S_{-2}(0.8), S_{-1}(0.1)\}$
	A_5	$\{S_0(0.2), S_1(0.6)\}$	$\{S_1(0.7), S_2(0.1)\}$	$\{S_1(0.3), S_2(0.6)\}$	$\{S_{-1}(0.6), S_0(0.1)\}$
e_2	A_1	$\{S_{-2}(0.2), S_{-1}(0.5)\}$	$\{S_{-2}(0.5), S_{-1}(0.3)\}$	$\{S_{-1}(0.2), S_0(0.5)\}$	$\{S_{-2}(0.6), S_{-1}(0.1)\}$
	A_2	$\{S_1(0.1), S_2(0.5)\}$	$\{S_1(0.5), S_2(0.4)\}$	$\{S_1(0.4), S_2(0.4)\}$	$\{S_1(0.5), S_2(0.3)\}$
	A_3	$\{S_1(0.3), S_2(0.4)\}$	$\{S_1(0.2), S_2(0.1)\}$	$\{S_1(0.4), S_2(0.5)\}$	$\{S_0(0.2), S_1(0.4)\}$
	A_4	$\{S_1(0.3), S_2(0.1)\}$	$\{S_1(0.8), S_2(0.1)\}$	$\{S_1(0.7), S_2(0.2)\}$	$\{S_1(0.7), S_2(0.1)\}$
	A_5	$\{S_1(0.7), S_2(0.1)\}$	$\{S_1(0.6), S_2(0.2)\}$	$\{S_1(0.6), S_2(0.3)\}$	$\{S_1(0.7), S_2(0.3)\}$
e_3	A_1	$\{S_{-2}(0.3), S_{-1}(0.4)\}$	$\{S_{-2}(0.6), S_{-1}(0.3)\}$	$\{S_{-1}(0.3), S_0(0.5)\}$	$\{S_{-2}(0.8), S_{-1}(0.1)\}$
	A_2	$\{S_1(0.6), S_2(0.1)\}$	$\{S_1(0.8)\}$	$\{S_1(0.2), S_2(0.4)\}$	$\{S_1(0.6), S_2(0.2)\}$
	A_3	$\{S_1(0.5), S_2(0.4)\}$	$\{S_1(0.2), S_2(0.4)\}$	$\{S_1(0.4), S_2(0.2)\}$	$\{S_0(0.5), S_1(0.1)\}$
	A_4	$\{S_1(0.3), S_2(0.2)\}$	$\{S_1(0.6), S_2(0.1)\}$	$\{S_1(0.6), S_2(0.3)\}$	$\{S_1(0.8), S_2(0.1)\}$
	A_5	$\{S_1(0.8), S_2(0.1)\}$	$\{S_1(0.7), S_2(0.2)\}$	$\{S_1(0.8), S_2(0.2)\}$	$\{S_1(0.8)\}$
e_4	A_1	$\{S_0(0.2), S_1(0.6)\}$	$\{S_1(0.6), S_2(0.1)\}$	$\{S_{-2}(0.8), S_{-1}(0.1)\}$	$\{S_{-2}(0.6), S_{-1}(0.2)\}$
	A_2	$\{S_1(0.8), S_2(0.1)\}$	$\{S_0(0.2), S_1(0.6)\}$	$\{S_1(0.5), S_2(0.2)\}$	$\{S_1(0.7), S_2(0.1)\}$
	A_3	$\{S_{-1}(0.2), S_0(0.7)\}$	$\{S_1(0.7), S_2(0.1)\}$	$\{S_{-1}(0.2), S_0(0.6)\}$	$\{S_1(0.8), S_2(0.2)\}$
	A_4	$\{S_0(0.7)\}$	$\{S_0(0.2), S_1(0.7)\}$	$\{S_{-1}(0.1), S_0(0.8)\}$	$\{S_{-1}(0.6), S_0(0.2)\}$
	A_5	$\{S_{-2}(0.8)\}$	$\{S_1(0.8), S_2(0.1)\}$	$\{S_{-2}(0.6), S_{-1}(0.2)\}$	$\{S_1(0.2), S_2(0.7)\}$
e_5	A_1	$\{S_2(0.8)\}$	$\{S_{-1}(0.6), S_0(0.2)\}$	$\{S_{-2}(0.8), S_{-1}(0.1)\}$	$\{S_{-2}(0.7), S_{-1}(0.2)\}$
	A_2	$\{S_2(0.9)\}$	$\{S_1(0.1), S_2(0.8)\}$	$\{S_1(0.1), S_2(0.7)\}$	$\{S_1(0.6)\}$
	A_3	$\{S_2(0.8)\}$	$\{S_1(0.8), S_2(0.1)\}$	$\{S_1(0.6), S_2(0.1)\}$	$\{S_0(0.6), S_1(0.1)\}$
	A_4	$\{S_0(0.8)\}$	$\{S_{-1}(0.6), S_0(0.1)\}$	$\{S_0(0.9)\}$	$\{S_1(0.6), S_2(0.1)\}$
	A_5	$\{S_1(0.8), S_2(0.1)\}$	$\{S_0(0.6), S_1(0.2)\}$	$\{S_0(0.5), S_1(0.2)\}$	$\{S_1(0.5), S_2(0.1)\}$

Table 10. Missing value compensation parameters under attribute c_1.

c_1	e_1	e_2	e_3	e_4	e_5
A_1	(1, 0)	(0.57, 0.43)	(0.01, 0.99)	(0.05, 0.95)	1
A_2	(1, 0)	(0, 1)	(1, 0)	(1, 0)	1
A_3	(0.38, 0.62)	(0.21, 0.79)	(0.01, 0.99)	(0.01, 0.99)	1
A_4	(0.01, 0.99)	(0, 1)	(0.06, 0.94)	1	1
A_5	(0, 1)	(1, 0)	(0, 1)	1	(0, 1)

Complete evaluation information for each expert is provided in Table 11.

Table 11. Evaluation matrix of decision experts for complementary probabilities.

		c_1	c_2	c_3	c_4
e_1	A_1	$\{S_1[0.7,0.899], S_2[0.1,0.101]\}$	$\{S_{-1}[0.5,0.8], S_0[0.2,0.2]\}$	$\{S_0[0.7,0.9], S_1[0.1,0.1]\}$	$\{S_{-1}[0.1,0.5], S_0[0.5,0.5]\}$
	A_2	$\{S_1[0.1,0.2], S_2[0.8,0.8]\}$	$\{S_1[0.6,0.8], S_2[0.2,0.2]\}$	$\{S_0[0.2,0.202], S_1[0.6,0.798]\}$	$\{S_1[0.2,0.4], S_2[0.6,0.6]\}$
	A_3	$\{S_1[0.7,0.738], S_2[0.2,0.262]\}$	$\{S_1[0.7,0.795], S_2[0.2,0.205]\}$	$\{S_1[0.6,0.894], S_2[0.1,0.106]\}$	$\{S_1[0.9,1]\}$
	A_4	$\{S_1[0.7,0.702], S_2[0.1,0.298]\}$	$\{S_0[0.6,0.9], S_1[0.1,0.1]\}$	$\{S_{-2}[0.6,0.6], S_{-1}[0.2,0.4]\}$	$\{S_{-2}[0.8,0.8], S_{-1}[0.1,0.2]\}$
	A_5	$\{S_0[0.2,0.2], S_1[0.6,0.8]\}$	$\{S_1[0.7,0.792], S_2[0.1,0.208]\}$	$\{S_1[0.3,0.4], S_2[0.6,0.6]\}$	$\{S_{-1}[0.6,0.6], S_0[0.1,0.4]\}$
e_2	A_1	$\{S_{-2}[0.2,0.372], S_{-1}[0.5,0.628]\}$	$\{S_{-2}[0.5,0.5], S_{-1}[0.3,0.5]\}$	$\{S_{-1}[0.2,0.5], S_0[0.5,0.5]\}$	$\{S_{-2}[0.6,0.726], S_{-1}[0.1,0.274]\}$
	A_2	$\{S_1[0.1,0.1], S_2[0.5,0.9]\}$	$\{S_1[0.5,0.6], S_2[0.4,0.4]\}$	$\{S_1[0.4,0.596], S_2[0.4,0.404]\}$	$\{S_1[0.5,0.5], S_2[0.3,0.5]\}$
	A_3	$\{S_1[0.3,0.363], S_2[0.4,0.637]\}$	$\{S_1[0.2,0.2], S_2[0.1,0.8]\}$	$\{S_1[0.4,0.498], S_2[0.5,0.502]\}$	$\{S_0[0.2,0.2], S_1[0.4,0.8]\}$
	A_4	$\{S_1[0.3,0.3], S_2[0.1,0.7]\}$	$\{S_1[0.8,0.8], S_2[0.1,0.2]\}$	$\{S_1[0.7,0.8], S_2[0.2,0.2]\}$	$\{S_1[0.7,0.9], S_2[0.1,0.1]\}$
	A_5	$\{S_1[0.7,0.9], S_2[0.1,0.1]\}$	$\{S_1[0.6,0.798], S_2[0.2,0.202]\}$	$\{S_1[0.6,0.7], S_2[0.3,0.3]\}$	$\{S_1[0.7,0.7], S_2[0.3,0.3]\}$
e_3	A_1	$\{S_{-2}[0.3,0.304], S_{-1}[0.4,0.696]\}$	$\{S_{-2}[0.6,0.6], S_{-1}[0.3,0.4]\}$	$\{S_{-1}[0.3,0.5], S_0[0.5,0.5]\}$	$\{S_{-2}[0.8,0.801], S_{-1}[0.1,0.199]\}$
	A_2	$\{S_1[0.6,0.9], S_2[0.1,0.1]\}$	$\{S_1[0.8,1]\}$	$\{S_1[0.2,0.48], S_2[0.4,0.52]\}$	$\{S_1[0.6,0.8], S_2[0.2,0.2]\}$
	A_3	$\{S_1[0.5,0.501], S_2[0.4,0.499]\}$	$\{S_1[0.2,0.308], S_2[0.4,0.692]\}$	$\{S_1[0.4,0.8], S_2[0.2,0.2]\}$	$\{S_0[0.5,0.5], S_1[0.1,0.5]\}$
	A_4	$\{S_1[0.3,0.33], S_2[0.2,0.67]\}$	$\{S_1[0.6,0.9], S_2[0.1,0.1]\}$	$\{S_1[0.6,0.7], S_2[0.3,0.3]\}$	$\{S_1[0.8,0.8], S_2[0.1,0.2]\}$
	A_5	$\{S_1[0.8,0.8], S_2[0.1,0.2]\}$	$\{S_1[0.7,0.705], S_2[0.2,0.295]\}$	$\{S_1[0.8,0.8], S_2[0.2,0.2]\}$	$\{S_1[0.8,1]\}$
e_4	A_1	$\{S_0[0.2,0.21], S_1[0.6,0.79]\}$	$\{S_1[0.6,0.9], S_2[0.1,0.1]\}$	$\{S_{-2}[0.8,0.8], S_{-1}[0.1,0.2]\}$	$\{S_{-2}[0.6,0.786], S_{-1}[0.2,0.214]\}$
	A_2	$\{S_1[0.8,0.9], S_2[0.1,0.1]\}$	$\{S_0[0.2,0.2], S_1[0.6,0.8]\}$	$\{S_1[0.5,0.8], S_2[0.2,0.2]\}$	$\{S_1[0.7,0.7], S_2[0.1,0.3]\}$
	A_3	$\{S_{-1}[0.2,0.201], S_0[0.7,0.799]\}$	$\{S_1[0.7,0.848], S_2[0.1,0.152]\}$	$\{S_{-1}[0.2,0.202], S_0[0.6,0.798]\}$	$\{S_1[0.8,0.8], S_2[0.2,0.2]\}$
	A_4	$\{S_0[0.7,1]\}$	$\{S_0[0.2,0.2], S_1[0.7,0.8]\}$	$\{S_{-1}[0.1,0.1], S_0[0.8,0.9]\}$	$\{S_{-1}[0.6,0.6], S_0[0.2,0.4]\}$
	A_5	$\{S_{-2}[0.8,1]\}$	$\{S_1[0.8,0.801], S_2[0.1,0.199]\}$	$\{S_{-2}[0.6,0.6], S_{-1}[0.2,0.4]\}$	$\{S_1[0.2,0.3], S_2[0.7,0.7]\}$
e_5	A_1	$\{S_2[0.8,1]\}$	$\{S_{-1}[0.6,0.8], S_0[0.2,0.2]\}$	$\{S_{-2}[0.8,0.8], S_{-1}[0.1,0.2]\}$	$\{S_{-2}[0.7,0.787], S_{-1}[0.2,0.213]\}$
	A_2	$\{S_2[0.9,1]\}$	$\{S_1[0.1,0.2], S_2[0.8,0.8]\}$	$\{S_1[0.1,0.3], S_2[0.7,0.7]\}$	$\{S_1[0.6,0.1]\}$
	A_3	$\{S_2[0.8,1]\}$	$\{S_1[0.8,0.801], S_2[0.1,0.199]\}$	$\{S_1[0.6,0.894], S_2[0.1,0.106]\}$	$\{S_0[0.6,0.9], S_1[0.1,0.1]\}$
	A_4	$\{S_0[0.8,1]\}$	$\{S_{-1}[0.6,0.6], S_0[0.1,0.4]\}$	$\{S_0[0.9,1]\}$	$\{S_1[0.6,0.9], S_2[0.1,0.1]\}$
	A_5	$\{S_1[0.8,0.8], S_2[0.1,0.2]\}$	$\{S_0[0.6,0.602], S_1[0.2,0.398]\}$	$\{S_0[0.5,0.8], S_1[0.2,0.2]\}$	$\{S_1[0.5,0.9], S_2[0.1,0.1]\}$

Step 3. Obtain expert weights

Through the Equation (30) to calculate the evaluation information regret value of each expert in different attributes for each alternative, due to space reasons, only the evaluation information regret value of decision-making experts e_1 is given in Table 12, and the rest of the calculation results are shown in the Supplementary Materials Tables S5–S8, where $\alpha = 0.3, q_j = 0.03, p_j = 0.06, v_j = 0.1$

Table 12. Regret value of evaluation information for decision-making expert e_1.

R_{ij}^1	c_1	c_2	c_3	c_4
A_1	−0.0297	−0.1305	−0.0798	−0.1109
A_2	−0.0313	−0.0467	−0.0502	−0.0223
A_3	−0.0415	−0.0377	−0.0387	−0.0019
A_4	−0.0453	−0.0838	−0.1185	−0.1234
A_5	−0.0499	−0.0401	−0.0191	−0.1260

It can be obtained that the maximum absolute regret value of each expert is 0.1305, 0.1183, 0.1234, 0.1283, and 0.1294, in order. The objective weights of expects can be calculated by Equation (31):

$$w_r^1 = (0.1928, 0.2127, 0.2039, 0.1961, 0.1945).$$

The individual scores of each expert are as follows and the individual weights of the experts are calculated by Equation (32), the specific results are shown in Table 13.

Table 13. Scores of experts and individual weights of experts.

	Age Score	Education Score	Professional Status Score	Total Score	Individual Weight
e_1	4	2	4	10	0.1961
e_2	4	2	4	10	0.1961
e_3	4	3	3	10	0.1961
e_4	3	3	3	9	0.1765
e_5	4	4	4	12	0.2352

Finally, the comprehensive weights of experts are calculated by Equation (33), the specific results are shown in Table 14.

Table 14. Comprehensive weights of experts.

	Objective Weights	Individual Weight	Comprehensive Weight
e_1	0.1928	0.1961	0.1945
e_2	0.2127	0.1961	0.2044
e_3	0.2039	0.1961	0.2000
e_4	0.1961	0.1765	0.1863
e_5	0.1945	0.2352	0.2148

where $\gamma = 0.5$.

Step 4. Adjustment of attribute weights based on information entropy.

Aggregate each evaluation matrix based on the expert weights calculated in Step 3 to obtain a comprehensive decision matrix, and calculate the entropy value of each attribute based on the entropy value Equation (34):

$$E_j = (0.9874, 0.9674, 0.9719, 0.9283).$$

According to the attribute weight adjustment criterion in Section 3.4, the weight of attribute c_2 needs to be adjusted downward, and the adjusted final attribute weight is calculated according to Equations (35) and (36):

$$W_j = (0.1895, 0.2972, 0.2346, 0.2787).$$

Step 5. Calculate the comprehensive evaluation regret value matrix

Using the comprehensive weights of the experts, the evaluation regret value matrix of the five experts is linearly weighted according to Equation (37) to obtain the comprehensive evaluation regret value matrix; the results are shown in Table 15.

Table 15. Regret values for comprehensive evaluations.

R_{ij}	c_1	c_2	c_3	c_4
A_1	−0.0714	−0.1084	−0.1110	−0.1210
A_2	−0.0319	−0.0365	−0.0434	−0.0381
A_3	−0.0519	−0.0340	−0.0521	−0.0551
A_4	−0.0567	−0.0645	−0.0658	−0.0679
A_5	−0.0277	−0.0522	−0.0615	−0.0447

Step 6. Calculate the consistency index of the alternative under attributes

Based on the comprehensive evaluation regret value matrix and Equations (38)–(40), calculate the concordance index, discordance index, and the comprehensive concordance index. We list $CI_1\left(A_i,\ A_q\right)$, $DI_1\left(A_i,\ A_q\right)$ and $CI\left(A_i,\ A_q\right)$ as Tables 16 and 17, and the rest of the calculation results are shown as Tables S9–S14 in the Supplementary Materials.

Table 16. Concordance index and discordance index under attribute c_1.

	CI_1	A_1	A_2	A_3	A_4	A_5
c_1	A_1	1.0000	0.6833	1.0000	1.0000	0.5433
	A_2	1.0000	1.0000	1.0000	1.0000	1.0000
	A_3	1.0000	1.0000	1.0000	1.0000	1.0000
	A_4	1.0000	1.0000	1.0000	1.0000	1.0000
	A_5	1.0000	1.0000	1.0000	1.0000	1.0000
	DI_1	A_1	A_2	A_3	A_4	A_5
c_1	A_1	1.0000	0.0000	0.0000	0.0000	0.0000
	A_2	0.0000	1.0000	0.0000	0.0000	0.0000
	A_3	0.0000	0.0000	1.0000	0.0000	0.0000
	A_4	0.0000	0.0000	0.0000	1.0000	0.0000
	A_5	0.0000	0.0000	0.0000	0.0000	1.0000

Table 17. Comprehensive concordance index.

CI	A_1	A_2	A_3	A_4	A_5
A_1	1.0000	0.1295	0.1981	0.5298	0.2227
A_2	1.0000	1.0000	1.0000	1.0000	1.0000
A_3	1.0000	1.0000	1.0000	1.0000	1.0000
A_4	1.0000	1.0000	0.9941	1.0000	1.0000
A_5	1.0000	1.0000	1.0000	1.0000	1.0000

Step 7. Calculate the net credibility degree and rank the alternatives

Based on the Equation (41), the credibility index of alternatives is calculated as Table 18:

Table 18. Credibility index.

$\varphi(A_i, A_j)$	A_1	A_2	A_3	A_4	A_5
A_1	1.0000	0.0479	0.1581	0.5298	0.1698
A_2	1.0000	1.0000	1.0000	1.0000	1.0000
A_3	1.0000	1.0000	1.0000	1.0000	1.0000
A_4	1.0000	1.0000	0.9941	1.0000	1.0000
A_5	1.0000	1.0000	1.0000	1.0000	1.0000

Calculate the concordance credibility degree, discordance credibility degree and net credibility of each alternative by Equations (42)–(44); the results are shown in Table 19.

Table 19. The net credibility of each alternative.

	Concordance Credibility Degree	Discordance Credibility Degree	Net Credibility
A_1	1.9056	5.0000	−3.0944
A_2	5.0000	4.0479	0.9521
A_3	5.0000	4.1522	0.8478
A_4	4.9941	4.5298	0.4643
A_5	5.0000	4.1698	0.8302

The five alternatives are ranked based on net credibility and the final ranking is obtained as follows:

$$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1.$$

Therefore, option A_2 is selected.

4.3. Comparative Analysis

4.3.1. Solving the Case by the PLTS-Based Method

In this subsection, we solve this problem with linguistic information in the form of $PLTS$, still using the attributes and their initial weights from the paper.

Step 1. Normalize the incomplete probability of the decision expert evaluation information to obtain the evaluation information with complete probability (the specific results of the calculations are shown as Tables S15–S19 in Supplementary Materials).

Step 2. Obtaining expert weights (the specific results of the calculations are shown as Tables S20–S26 in Supplementary Materials).

$$Wr = 0.1979, 0.1987, 0.1979, 0.1881, 0.2174.$$

Step 3. Adjust the attribute weights based on information entropy. The adjusted attribute weights are as follows:

$$Wj = 0.1893, 0.2978, 0.2344, 0.2785.$$

Step 4. Calculate the net credibility of each alternative, and the result is shown as Table 20 (the specific results of the calculations are shown as Tables S27–S38 in Supplementary Materials).

The five alternatives are ordered according to their net credibility and the final ranking is obtained:

$$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1.$$

Therefore, option A_2 is selected.

Table 20. The net credibility of each alternative based on PLTS.

	Concordance Credibility Degree	Discordance Credibility Degree	Net Credibility
A_1	2.5178	5.0000	−2.4822
A_2	5.0000	4.2065	0.7935
A_3	5.0000	4.2848	0.7152
A_4	5.0000	4.6989	0.3011
A_5	5.0000	4.3276	0.6724

4.3.2. Solving the Case by the UPLTS-Based Method

The expert evaluation information is changed from PLTS to the form of UPLTS, and the rest of the calculation steps are consistent with the paper.

Step 1. Change the evaluation information into the form of UPLTS (the specific results of the calculations are shown in Supplementary Materials Tables S39–S43).

Step 2. Obtain expert weights (the specific results of the calculations are shown as Tables S44–S50 in Supplementary Materials).

$$W_r = (0.1974, 0.2027, 0.1974, 0.1866, 0.2159).$$

Step 3. Adjust the attribute weights based on information entropy, and the adjusted attribute weights are as follows:

$$W_j = (0.1886, 0.2948, 0.2391, 0.2775).$$

Step 4. Calculate the credibility index of each alternative (the specific results of the calculations are shown as Tables S51–S62 in Supplementary Materials), and the results are shown in Table 21.

Table 21. The credibility index of each alternative based on UPLTS.

	Concordance Credibility Degree	Discordance Credibility Degree	Net Credibility
A_1	1.7603	5.0000	−3.2397
A_2	5.0000	3.9611	1.0389
A_3	5.0000	3.9891	1.0109
A_4	4.8258	4.5121	0.3137
A_5	5.0000	4.1238	0.8762

The five alternatives are ordered according to their net credibility and the final ranking is obtained:

$$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$$

Therefore, option A_2 is selected.

4.3.3. Solving the Case with the Extended TOPSIS Method

Step 1. Attribute weights and expert weights are calculated in the same way as Sections 3.1, 3.3 and 3.4.

Step 2. Aggregate the expert evaluation information using PLWA operators (the specific result of the calculation is shown as Table S1 in Supplementary Materials).

Step 3. Obtaining positive ideal solutions PIS ($L(p)^+$) and negative ideal solutions NIS ($L(p)^-$) (the specific results of the calculations are shown as Tables S63–S64 in Supplementary Materials).

Step 4. Calculate the distance between each alternative and PIS and the distance between each alternative and NIS.

$$d_1^+ = 0.0491,\ d_2^+ = 0.0051,\ d_3^+ = 0.0122,\ d_4^+ = 0.0245,\ d_5^+ = 0.0158;$$
$$d_1^- = 0.0001,\ d_2^- = 0.0441,\ d_3^- = 0.0370,\ d_4^- = 0.0247,\ d_5^- = 0.0334.$$

Step 5. Determine the minimum and maximum distances.

$$d_{min}\left(A_i, L(p)^+\right) = 0.0051,\ d_{max}\left(A_i, L(p)^-\right) = 0.0441$$

Step 6. Calculate the closeness coefficient.

$$CI(A_1) = -9.6253, CI(A_2) = 0,\ CI(A_3) = -1.5532,$$
$$CI(A_4) = -4.2438, CI(A_5) = -2.3407.$$

Step 7. Obtain the ranking of the alternatives by the closeness coefficient as follows:

$$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1.$$

4.3.4. Solving the Case with the Extended PROMETHEE Method

Step 1. Attribute weights and expert weights are calculated in the same way as Sections 3.1, 3.3 and 3.4.

Step 2. Aggregate the expert evaluation information using PLWA operators (the specific result of the calculation is shown as Table S1 in Supplementary Materials).

Step 3. Calculate the score matrix and deviation matrix.

$$E = \begin{bmatrix} [-0.1143, -0.0034] & [-1.0233, -0.2823] & [-0.9231, -0.4303] & [-0.3160, -0.1290] & [-0.5132, -0.2672] \\ [0.4395, 0.9111] & [0.3883, 1.1322] & [0.2655, 0.9432] & [0.3013, 0.7768] & [0.4644, 1.1459] \\ [0.3857, 0.7518] & [0.1268, 1.2758] & [0.1836, 1.0112] & [0.2076, 0.3753] & [0.2652, 0.7559] \\ [-0.4286, -0.1958] & [0.2403, 1.0061] & [0.2711, 0.9994] & [-0.0012, 0.0324] & [0.0166, 0.0948] \\ [0.2051, 0.4944] & [0.4604, 0.7823] & [0.4459, 0.7115] & [0.0124, 0.0500] & [0.1572, 0.5368] \end{bmatrix}$$

$$\sigma = \begin{bmatrix} [0.0142, 0.1026] & [0.0273, 0.2370] & [0.0563, 0.2138] & [0.0327, 0.1422] & [0.0774, 0.2328] \\ [0.1358, 0.1416] & [0.0900, 0.1048] & [0.1014, 0.1046] & [0.0996, 0.0998] & [0.2410, 0.2768] \\ [0.1177, 0.1278] & [0.0334, 0.0958] & [0.0506, 0.0767] & [0.0633, 0.0700] & [0.0976, 0.1178] \\ [0.0596, 0.1582] & [0.0826, 0.0847] & [0.0793, 0.0825] & [0.0215, 0.0583] & [0.0136, 0.0684] \\ [0.0593, 0.0672] & [0.0965, 0.0991] & [0.1198, 0.1413] & [0.0212, 0.0488] & [0.0440, 0.0562] \end{bmatrix}$$

Step 4. Calculating the comprehensive possibility degree matrix for all experts.

$$R = [r_{ie}]_{5\times5} = \begin{bmatrix} 0.5 & 0 & 0 & 0.1945 & 0 \\ 1 & 0.5 & 0.6577 & 0.8120 & 0.8012 \\ 1 & 0.3423 & 0.5 & 0.8013 & 0.7130 \\ 0.8055 & 0.1880 & 0.1987 & 0.5 & 0.2663 \\ 1 & 0.1988 & 0.2870 & 0.7337 & 0.5 \end{bmatrix}$$

Step 5. Calculates the out-degree, in-degree, and the net flow, and ranks the alternatives based on the net flow, the result is shown as Table 22 (the specific results of the calculations are shown as Tables S65 in Supplementary Materials).

Table 22. The out-degree, in-degree, and net flow of each alternative the Extended PROMETHEE.

Alternative	Out-Degree	In-Degree	Net Flow
A_1	0.1389	0.8611	−0.7222
A_2	0.75418	0.24582	0.50836
A_3	0.67132	0.32868	0.34264
A_4	0.3917	0.6083	−0.2166
A_5	0.5439	0.4561	0.0878

Subsequently, the final ranking of the alternatives is delineated as follows:

$$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1.$$

4.3.5. Solve This Problem with the Method from Reference [17]

Step 1. Aggregate the expert evaluation information using PLWA operators (the specific result of the calculation is shown as Table S1 in Supplementary Materials).

Step 2. Calculate the score matrix (taking the middle of the interval), and the result is shown as Table 23.

Table 23. Score value matrix.

E_{ij}	c_1	c_2	c_3	c_4
A_1	0.1238	−0.3749	−0.4473	−0.7603
A_2	0.7982	0.6318	0.6757	0.6555
A_3	0.6308	0.7001	0.4849	0.3437
A_4	0.4596	0.2459	0.0843	0.1131
A_5	0.2467	0.5141	0.2825	0.4485

Step 3. Using attribute weights from the paper:

$$W_j = (0.1895, 0.2972, 0.2346, 0.2787).$$

Step 4. Calculate the net credibility based on the score values and rank the alternatives based on the net credibility; the results are shown in Table 24.

Table 24. The credibility index of each alternative based on Reference [17].

	Concordance Credibility Degree	Discordance Credibility Degree	Net Credibility
A_1	1.0000	5.0000	−4
A_2	4.8912	1.0000	3.8912
A_3	3.7213	1.8912	1.8301
A_4	2.0000	3.8105	−1.8105
A_5	2.8105	2.7213	0.0892

Step 5. Sort the five alternatives according to the net credibility and get the results of the sorting as follows (the specific results of the calculations are shown as Tables S66–S77 in Supplementary Materials):

$$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1.$$

Therefore, option A_2 is selected.

4.3.6. Solving the Problem Using the Method from Reference [21]

Step 1. Aggregate the expert evaluation information using PLWA operators (the specific result of the calculation is shown as Table S1 in Supplementary Materials).

Step 2. Calculate the regret-rejoice value of the alternatives; the results are shown in Table 25 (the specific results of the calculations are shown as Tables S78–S80 in Supplementary Materials).

Table 25. Regret-rejoice values of alternatives under each attribute.

R_{ij}	c_1	c_2	c_3	c_4
A_1	−0.0515	−0.0862	−0.0908	−0.1195
A_2	0.0000	−0.0050	0.0000	0.0000
A_3	−0.0123	0.0000	−0.0142	−0.0234
A_4	−0.0252	−0.0344	−0.0454	−0.0415
A_5	−0.0417	−0.0138	−0.0297	−0.0154

Step 3. Calculate the perceived utility values of the alternatives; the results are shown in Table 26 (the specific results of the calculations are shown as Table S81 in Supplementary Materials).

Table 26. Perceived utility values of alternatives.

u_{ij}	c_1	c_2	c_3	c_4
A_1	0.2757	0.1091	0.0844	−0.0355
A_2	0.4948	0.4493	0.4650	0.4601
A_3	0.4418	0.4710	0.4039	0.3594
A_4	0.3866	0.3239	0.2717	0.2829
A_5	0.3168	0.4115	0.3378	0.3936

Step 4. Determine the PIS and NIS.

$$u(L_j(p))^+ = \left\{ \underset{1\le i\le m}{max}\ u_{ij} \right\} = [0.4948, 0.4710, 0.4650, 0.4601];$$
$$u(L_j(p))^- = \left\{ \underset{1\le i\le m}{min}\ u_{ij} \right\} = [0.2757, 0.1091, 0.0844, -0.0355].$$

Step 5. Using attribute weights from the paper:

$$W_j = (0.1895, 0.2972, 0.2346, 0.2787)$$

Step 6. Calculate the distance between the perceived utility value of the alternative and PIS and the distance between the perceived utility value of the alternative and NIS, respectively, and determine the maximum and minimum distances as follows:

$$d_i^+ = [0.3879, 0.0118, 0.0651, 0.1617, 0.1099];$$
$$d_i^- = [0.0000, 0.3820, 0.3340, 0.2292, 0.3064].$$

Then, the maximum and minimum distances are as follows:

$$d_{min}(A_i, A^+) = \underset{1\le i\le m}{min}\ d_i^+ = 0.0118,\ d_{max}(A_i, A^-) = \underset{1\le i\le m}{max}\ d_i^- = 0.3820$$

Step 7. Obtains the closeness coefficients of the alternatives and determines the ranking result of the alternatives based on the closeness coefficients as follows:

$$ICI(A_i) = [-32.8729, 0, -4.6426, -13.1034, -8.5115]$$

The final sorting result is as follows:

$$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$$

4.3.7. Summary of Comparative Analysis

This section compares six methods with the method proposed in this paper. Among them, PLTS and UPLTS are classic language environments which are different from the l-IVPLTS in this paper. The extended TOPSIS algorithm and extended PROMETHEE algorithm are classic ranking methods which are different from the ELETRE III algorithm in this paper. The methods in references [17,21] have similarities with the method proposed in this paper. The former and the ranking method proposed in this paper are both ELETRE III algorithms, while the latter and this paper both use regret theory. These methods have a certain correlation with the research method of this paper, so the comparative analysis of these methods can prove the rationality and effectiveness of the research method in this paper. The results are shown as Figure 11 and Table 27.

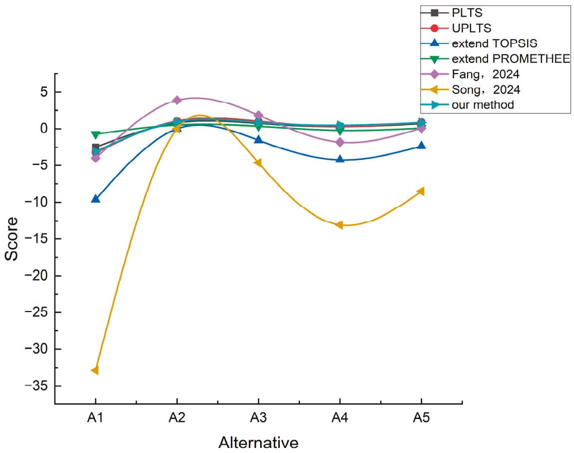

Figure 11. Comparison of the results of the six methods [21,31].

Table 27. Experimental results compared with other methods.

Method	A_1	A_2	A_3	A_4	A_5	Rank
Our Method	−3.0944	0.9521	0.8478	0.4643	0.8302	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
PLTS	−2.4822	0.7935	0.7152	0.3011	0.6724	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
UPLTS	−3.2397	1.0389	1.0109	0.3137	0.8762	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
extended TOPSIS	−9.6253	0.0000	−1.5532	−4.2438	−2.3407	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
extended PROME THEE	−0.7222	0.5084	0.3426	−0.2166	0.0878	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
reference [17]	−4.0000	3.8912	1.8301	−1.8105	0.0892	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
reference [21]	−32.8729	0.0000	−4.6426	−13.1034	−8.5115	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$

Note: The specific steps of the comparison experiment are shown in the Supplementary Materials.

Observing Table 27, it can be seen that the difference between the optimal alternative A_2 and the other alternatives is 4.0465, 0.1043, 0.4878, 0.1219, respectively, while in the PLTS method, the differences between the optimal alternative A_2 and the other alternatives are 3.2757, 0.0783, 0.4924, and 0.1211, respectively, which shows that in this paper's method, the difference between the optimal alternative and the other alternatives is basically greater than the method; in the UPLTS method, the differences between the optimal alternative A_2 and the other alternatives are 4.2786, 0.028, 0.7252, 0.1627, respectively, compared with the method of this paper. In comparing A_2 and A_3, the difference in this study is more tangible, which indicates that the method of this paper has better differentiation.

Observing Table 27, it can be seen that the sorting results of this paper's method are consistent with the sorting results of other methods, both of which are optimal for alternative A_2 and worse for alternative A_1, which indicates that the sorting method of this paper has a certain degree of rationality and credibility, and also proves the usability of the method presented in this paper. The reason for the consistency of the sorting results may be since the evaluation information of the experts for the five alternatives is too differentiated, which also proves that the method of this paper fully retains the original evaluation information.

4.4. Sensitivity Analysis

4.4.1. Sensitivity Analysis of Regret Aversion Coefficient α

As mentioned before in this paper, regret theory is one of the most widely used behavioral decision theories, and the final decision result is closely related to the degree of regret avoidance of experts. Therefore, we introduce a sensitivity analysis to study the effect of the regret aversion coefficient α.

In this paper, the value of α is varied from 0.1 to 0.9 in increments of 0.1. The final decision result is shown in Table 28 and Figure 12, which varies with the change of α value. The main reason for this phenomenon is that the change in the value of α affects the distinction between different regret values, which leads to the failure of the predefined thresholds. For example, when α is 0.1, the proposed method fails to give reasonable results, and when α is from 0.5 to 0.7, the ordering of A_3 and A_5 changes, while the ordering of the rest of the alternatives remains unchanged.

Table 28. Results of α sensitivity analysis.

α	A_1	A_2	A_3	A_4	A_5	Rank
0.1	0	0	0	0	0	$A_1 = A_2 = A_3 = A_4 = A_5$
0.2	−1.1664	0.4953	0.3645	0.0427	0.2639	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.3	−3.0944	0.9521	0.8478	0.4643	0.8302	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.4	−3.8398	1.2062	1.1005	0.4459	1.0872	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.5	−3.9819	1.5786	1.1509	−0.0070	1.2594	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
0.6	−4.0000	2.0827	1.1205	−0.5139	1.3107	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
0.7	−4.0000	2.4606	1.0938	−0.8369	1.2825	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
0.8	−4.0000	2.7154	1.4365	−1.392	1.2401	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.9	−4.0000	2.9134	1.6610	−1.9098	1.3354	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$

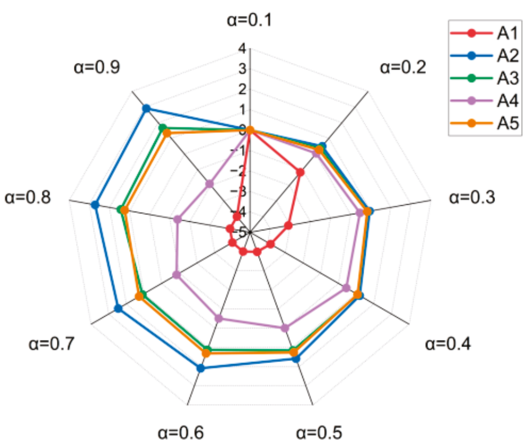

Figure 12. Visualization of the results of the α sensitivity analysis.

4.4.2. Sensitivity Analysis of β

In this paper, β appears in the Equation (19) for calculating the initial weights of the attributes. In a multi-attribute group decision-making problem, attribute weights have been a greater concern for researchers, and changes in attribute weights may have an impact on the decision results. Therefore, we introduce a sensitivity analysis to investigate the effect of the coefficient β.

In this paper, the values β are varied from 0 to 1 in increments of 0.1. The final decision result is shown in Table 29 and Figure 13. As coefficient β varied, the decision outcome remained consistent, and the main reason for this phenomenon is that the difference between the attribute weights of the official and mass levels obtained based on the text comments is not large, and the proportion of the initial weight adjustment of the attributes based on the entropy value is also small. Therefore, regardless of the value of β, the change in the final attribute weights remains small, which is why the decision results remain unchanged.

Table 29. Results of β sensitivity analysis.

β	A_1	A_2	A_3	A_4	A_5	Rank
0	−3.0824	0.9511	0.8445	0.4613	0.8255	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.1	−3.0848	0.9513	0.8451	0.4619	0.8265	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.2	−3.0872	0.9515	0.8459	0.4624	0.8274	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.3	−3.0897	0.9517	0.8465	0.4631	0.8284	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.4	−3.0921	0.9519	0.8472	0.4637	0.8293	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.5	−3.0944	0.9521	0.8478	0.4643	0.8302	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.6	−3.0969	0.9523	0.8486	0.4648	0.8312	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.7	−3.0993	0.9525	0.8492	0.4655	0.8321	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.8	−3.1018	0.9527	0.8499	0.4661	0.8331	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.9	−3.1042	0.9529	0.8506	0.4666	0.8341	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
1	−3.1065	0.9531	0.8513	0.4672	0.8349	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$

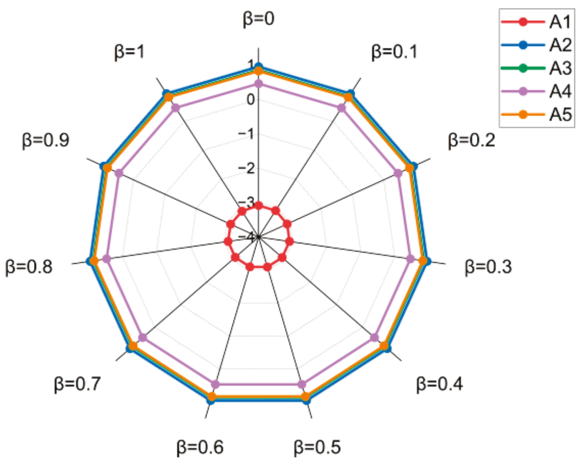

Figure 13. Visualization of the results of the β sensitivity analysis.

4.4.3. Sensitivity Analysis of γ

In this paper, γ appears in the Equation (33) for calculating the combined expert weights. In a MAGDM problem, the professional evaluation information provided by the experts is an indispensable part of the decision-making problem in that changes in the weights of the experts may have an impact on the decision outcome. Therefore, we introduce a sensitivity analysis to study the effect of the coefficient γ.

In this paper, the value of γ is varied from 0 to 1 in increments of 0.1. The final decision result is shown in Table 30 and Figure 14, which varies with the value of γ. The comprehensive expert weight is calculated by integrating the expert's objective weight with their personal weight, when γ is taken as 0, the combined weight of the experts only takes the part of the personal weight of the experts, and when γ is taken as 1, the combined weight of the experts only takes the part of the objective weight of the experts. When γ is taken from 0 to 0.7, the decision result is consistent with the decision result of the paper. When γ is taken from 0.7 to 1, the ordering of A_3 and A_5 changes, while the ordering of the rest of the alternatives remains unchanged, which also shows the reasonableness of the expert comprehensive weights proposed in this paper, which considers both the professionalism of the experts themselves and the objectivity of the expert evaluation.

Table 30. Results of γ sensitivity analysis.

γ	A_1	A_2	A_3	A_4	A_5	Rank
0	−3.0873	0.9532	0.8789	0.4404	0.8148	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.1	−3.0889	0.9520	0.8726	0.4462	0.8181	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.2	−3.0906	0.9520	0.8664	0.4509	0.8213	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.3	−3.092	0.9520	0.8600	0.4556	0.8244	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.4	−3.0933	0.9520	0.8536	0.4602	0.8275	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.5	−3.0944	0.9521	0.8478	0.4643	0.8302	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.6	−3.0968	0.9521	0.8409	0.4696	0.8342	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.7	−3.0975	0.9522	0.8369	0.4718	0.8366	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
0.8	−3.099	0.9523	0.8345	0.4725	0.8397	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
0.9	−3.1002	0.9524	0.8321	0.4732	0.8425	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
1	−3.1016	0.9526	0.8296	0.4739	0.8455	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$

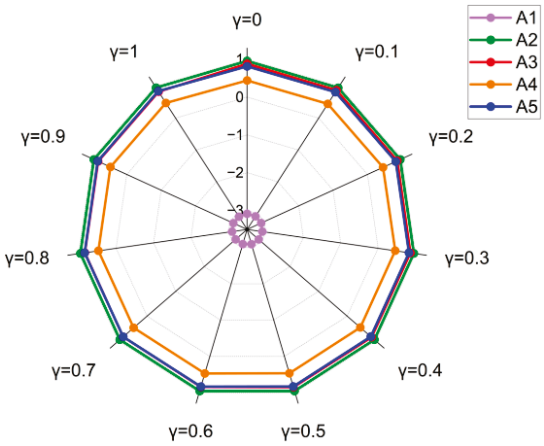

Figure 14. Visualization of the results of the γ sensitivity analysis.

4.4.4. Summary of Sensitivity Analysis

From the sensitivity analysis of α, β and γ above, it can be seen that with the change of variable α, the final decision result changes. From the regret theory, it can be seen that the smaller the value of α, the lower the degree of regret avoidance by experts, resulting in a smaller difference in regret values, which leads to different results. As the variable β changes, the final decision result remains unchanged, which indirectly verifies the stability of the model studied in this paper. As the variable γ changes, the final decision result also changes. As γ appears in the process of calculating the comprehensive weights of experts, this indicates that the value of expert weights will have an impact on the decision model.

5. Conclusions and Future Work

In this paper, we studied the emergency decision-making events in the social media environment in the big data era, and proposed a decision-making method based on the l-IVPLTS bidirectional projection-regret theory, which is characterized as follows:

1. After crawling the microblog text comment data, the text data is divided into the official level and the mass level based on the number of fans, the influence of the number of fans on the attributes and their initial weights taken into account, and the entropy value is calculated through the expert evaluation information and the initial weights of the attributes are adjusted; it also ensures the scientificity of the attribute weights;
2. The application of bidirectional projection to the limited interval-valued probabilistic linguistic term sets presents the l-IVPLTS bidirectional projection, which bridges the gap of bidirectional projection in the limited interval-valued probabilistic linguistic term sets;
3. In the calculation of expert weights, the l-IVPLTS bidirectional projection is combined with the regret theory, from which the expert's regret-rejoice value is calculated, according to which the expert's objective weights are obtained, in addition to which the expert's own professionalism and authority are also taken into account, and the expert's personal weights are added, which makes the expert's comprehensive weights more scientific and ensures the credibility of the evaluation information.

This study proposes a new IVBPRT-ELECTRE III algorithm, which provides a new research approach for multi-attribute emergency decision-making, especially in the case of incomplete expert evaluation information, providing a new probability completion method. This method is not only applicable to a limited interval-valued probabilistic linguistic term set, but also to situations with incomplete probabilities and probabilities with interval values, and has strong applicability. Meanwhile, the research method presented in this paper can not only be applied to emergency decision-making, but also to other decision making, such as supplier selection, factory site selection, medical program selection, and so on.

In addition, based on social platform comment data and the research results of this paper, the government should quickly deploy rescue plans when responding to sudden explosion accidents, focus on fire extinguishing and rescue matters, constantly pay attention to the occurrence of derivative disasters caused by explosion fires, and reduce the health and property losses of the people. It provides optional response measures for the government to deal with sudden explosion accidents and has a certain applicability.

In addition, this paper only determines the event-related attributes based on the comment information of a single social platform, without considering the comment information of other social platforms. At the same time, the determination of attributes does not take into account the opinions of relevant experts, which will make the attributes lack rationality and scientificity. In future research, we will crawl comment data from multiple platforms and consider the professional opinions of experts to make the attributes more scientific and reasonable. Meanwhile, in future research, we will try to find a scientific method to divide evaluation information into public and official levels, making the study more scientific.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/sym17010026/s1, Table S1: Results of operator aggregation; Table S2: Missing value compensation parameters under attribute c_2; Table S3: Missing value compensation parameters under attribute c_3; Table S4: Missing value compensation parameters under attribute c_4; Table S5: Regret value of evaluation information for decision-making expert e_2; Table S6: Regret value of evaluation information for decision-making expert e_3; Table S7: Regret value of evaluation information for decision-making expert e_4; Table S8: Regret value of evaluation information for decision-making expert e_5; Table S9: Concordance index under attribute c_2; Table S10: Discordance index under attribute c_2; Table S11: Concordance index under attribute c_3; Table S12: Discordance index under attribute c_3; Table S13: Concordance index under attribute c_4; Table S14: Discordance index under attribute c_4; Table S15: Evaluation matrix for decision expert e_2 after probability normalization;

Table S16: Evaluation matrix for decision expert e_2 after probability normalization; Table S17: Evaluation matrix for decision expert e_3 after probability normalization; Table S18: Evaluation matrix for decision expert e_4 after probability normalization; Table S19: Evaluation matrix for decision expert e_5 after probability normalization; Table S20: Regret value of evaluation information for decision-making expert e_1; Table S21: Regret value of evaluation information for decision-making expert e_2; Table S22: Regret value of evaluation information for decision-making expert e_3; Table S23: Regret value of evaluation information for decision-making expert e_4; Table S24: Regret value of evaluation information for decision-making expert e_5; Table S25: Scores of experts and individual weights of experts; Table S26: Comprehensive weights of experts; Table S27: Regret values for comprehensive evaluations; Table S28: Concordance index under attribute c_1; Table S29: Discordance index under attribute c_1; Table S30: Concordance index under attribute c_2; Table S31: Discordance index under attribute c_2; Table S32: Concordance index under attribute c_3; Table S33: Discordance index under attribute c_3; Table S34: Concordance index under attribute c_4; Table S35: Discordance index under attribute c_4; Table S36: Comprehensive concordance index; Table S37: Credibility index; Table S38: The net credibility of each alternative; Table S39: Evaluation matrix in the form of UPLTS for decision expert e_1; Table S40: Evaluation matrix in the form of UPLTS for decision expert e_2; Table S41: Evaluation matrix in the form of UPLTS for decision expert e_3; Table S42: Evaluation matrix in the form of UPLTS for decision expert e_4; Table S43: Evaluation matrix in the form of UPLTS for decision expert e_5; Table S44: Regret value of evaluation information for decision-making expert e_1; Table S45: Regret value of evaluation information for decision-making expert e_2; Table S46: Regret value of evaluation information for decision-making expert e_3; Table S47: Regret value of evaluation information for decision-making expert e_4; Table S48: Regret value of evaluation information for decision-making expert e_5; Table S49: Scores of experts and individual weights of experts; Table S50: Comprehensive weights of experts; Table S51: Regret values for comprehensive evaluations; Table S52: Concordance index under attribute c_1; Table S53: Discordance index under attribute c_1; Table S54: Concordance index under attribute c_2; Table S55: Discordance index under attribute c_2; Table S56: Concordance index under attribute c_3; Table S57: Discordance index under attribute c_3; Table S58: Concordance index under attribute c_4; Table S59: Discordance index under attribute c_4; Table S60: Comprehensive concordance index; Table S61: Credibility index; Table S62: The net credibility of each alternative; Table S63: Positive ideal solutions; Table S64: Negative ideal solutions; Table S65: The out degree, in degree and net flow of each alternative; Table S66: Score value matrix; Table S67: Concordance index under attribute c_1; Table S68: Discordance index under attribute c_1; Table S69: Concordance index under attribute c_2; Table S70: Discordance index under attribute c_2; Table S71: Concordance index under attribute c_3; Table S72: Discordance index under attribute c_3; Table S73: Concordance index under attribute c_4; Table S74: Discordance index under attribute c_4; Table S75: Comprehensive concordance index; Table S76: Credibility index; Table S77: The net credibility of each alternative; Table S78: Score matrix; Table S79: The utility value of each alternative; Table S80: Regret-rejoice values of alternatives under each attribute; Table S81: Perceived utility values of alternatives.

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