

Special Issue Reprint

Differential Geometry, Geometric Analysis and Their Related Applications

Edited by
Mića S. Stanković

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Differential Geometry, Geometric Analysis and Their Related Applications

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Guest Editor

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About the Editor

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Article

On Curvature Pinching for Submanifolds with Parallel Normalized Mean Curvature Vector

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Abstract: In this note, we investigate the pinching problem for oriented compact submanifolds of dimension n with parallel normalized mean curvature vector in a space form $F^{n+p}(c)$. We first prove a codimension reduction theorem for submanifolds under lower Ricci curvature bounds. Moreover, if the submanifolds have constant normalized scalar curvature $R \geq c$, we obtain a classification theorem for submanifolds under lower Ricci curvature bounds. It should be emphasized that our Ricci pinching conditions are sharp for even n and $p = 2$.

Keywords: submanifold; rigidity theorems; Ricci curvature; scalar curvature

MSC: 53C24; 53C40; 53C42

1. Introduction

The geometric rigidity of compact submanifolds plays an important role in submanifold geometry. In 1968, Simons [1] first studied the rigidity for minimal submanifolds in spheres. Later, a series of striking rigidity theorems for minimal submanifolds were obtained by some geometers [2–6]. Let M^n be an n -dimensional oriented compact submanifold in the complete and simply connected space form $F^{n+p}(c)$ with constant curvature c . In 1979, Ejiri [7] proved a rigidity result for minimal submanifolds with pinched Ricci curvatures in spheres.

Theorem 1 ([7]). *Let $M^n (n \geq 4)$ be a simply connected, compact oriented minimal submanifold in S^{n+p} . If the Ricci curvature of M satisfies $\text{Ric}_M \geq n - 2$, then M is either the totally geodesic submanifold S^n , the Clifford torus $S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$ in S^{n+1} with $n = 2m$, or $\mathbb{CP}^2(\frac{4}{3})$ in S^7 . Here $\mathbb{CP}^2(\frac{4}{3})$ denotes the two-dimensional complex projective space minimally immersed into S^7 with constant holomorphic sectional curvature $\frac{4}{3}$.*

Later, Shen [8] and Li [9] obtained that if M^3 is a compact oriented minimal submanifold in S^{3+p} , and $\text{Ric}_M \geq 1$, then M is totally geodesic. Afterward, Xu and Tian [10] got a refined version of Theorem 1, where the condition “ M is simply connected” was removed. In 2013, Xu and Gu [11] generalized the Ejiri rigidity theorem to compact submanifolds with parallel mean curvature vector in space forms.

Theorem 2 ([11]). *Let $M^n (n \geq 3)$ be an oriented compact submanifold with parallel mean curvature vector in $F^{n+p}(c)$ with $c + H^2 > 0$. If*

$$\text{Ric}_M \geq (n - 2)(c + H^2),$$

then M is either the totally umbilic sphere $S^n(\frac{1}{\sqrt{c+H^2}})$, the Clifford hypersurface $S^m(\frac{1}{\sqrt{2(c+H^2)}}) \times S^m(\frac{1}{\sqrt{2(c+H^2)}})$ in the totally umbilic sphere $S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ with $n = 2m$, or $\mathbb{CP}^2(\frac{4}{3}(c + H^2))$

in $S^7(\frac{1}{\sqrt{c+H^2}})$. Here $\mathbb{C}P^2(\frac{4}{3}(c+H^2))$ denotes the two-dimensional complex projective space minimally immersed in $S^7(\frac{1}{\sqrt{c+H^2}})$ with constant holomorphic sectional curvature $\frac{4}{3}(c+H^2)$.

Further discussions for submanifolds with parallel mean curvature vector have been carried out by many authors (see [12–14], etc.).

On the other hand, it is important to study the rigidity problem for compact submanifolds with constant scalar curvature. In 1977, Cheng and Yau [15] constructed a self-adjoint second-order differential operator to study n -dimensional closed hypersurfaces with constant scalar curvature in the space form $F^{n+1}(c)$, and obtained a classification result.

Theorem 3 ([15]). *Let M^n be a compact hypersurface with constant normalized scalar curvature R in the space form $F^{n+1}(c)$ with constant curvature c . If $R - c \geq 0$, and the sectional curvature of M satisfies $K_M \geq 0$, then M is either a totally umbilical hypersurface, or a (Riemannian) product of two totally umbilical constantly curved submanifolds.*

In 1996, Li [16] studied Cheng-Yau's self-adjoint operator, and proved a rigidity theorem for submanifolds with pinched scalar curvature.

Theorem 4 ([16]). *Let $M^n (n \geq 3)$ be a compact hypersurface with constant normalized scalar curvature R in the unit sphere S^{n+1} . If $\bar{R} = R - 1 \geq 0$, and the norm square S of the second fundamental form of M satisfies*

$$n\bar{R} \leq S \leq \frac{n}{(n-2)(n\bar{R}+2)} [n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n],$$

then either $S \equiv n\bar{R}$, and M is a totally umbilical hypersurface, or

$$S = \frac{n}{(n-2)(n\bar{R}+2)} [n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n],$$

and $M = S^1(\sqrt{1-r^2}) \times S^{n-1}(r), r = \sqrt{\frac{n-2}{n(\bar{R}+1)}}$ and $\bar{R} = R - 1$.

After that, some rigidity theorems for submanifolds with constant scalar curvature were obtained [17,18]. In 2013, Guo and Li [19] generalized Theorem 4 to the case of $n(\geq 4)$ -dimensional submanifolds with parallel normalized mean curvature vector in spheres.

The pinching problem of submanifolds with parallel normalized mean curvature vector seems interesting. In this paper, we first study the compact submanifolds with parallel normalized mean curvature vector in space forms. Using Li-Li's inequality [20] and the DDVV inequality proved by Lu, Ge-Tang [21,22], we obtain a codimension reduction theorem.

Theorem 5. *Let $M^n (n \geq 3)$ be an oriented compact submanifold with parallel normalized mean curvature vector in the space form $F^{n+p}(c) (p > 1)$. If the Ricci curvature of M satisfies*

$$\text{Ric}_M > (n-2+\delta(n,p))(c+H^2) > 0,$$

then M lies in the totally geodesic space form $F^{n+1}(c)$. Here

$$\delta(n,p) = \begin{cases} 0, & \text{for } p = 2, \\ \frac{n-1}{3n-5}, & \text{for } p \geq 3. \end{cases}$$

If $p \geq 3, \frac{1}{3} < \delta(n,p) \leq \frac{1}{2}$. Moreover, we investigate the compact submanifolds with constant scalar curvature and parallel normalized mean curvature vector, and prove a rigidity result.

Theorem 6. Let $M^n (n \geq 3)$ be an oriented compact submanifold with constant normalized scalar curvature R and parallel normalized mean curvature vector in the space form $F^{n+p}(c)$. If $R \geq c$, and

$$Ric_M > (n - 2 + \delta(n, p))(c + H^2) > 0,$$

then M must be the totally umbilical sphere $S^n(\frac{1}{\sqrt{c+H^2}})$. Here $\delta(n, p)$ is defined as in Theorem 5.

2. Notation and Lemmas

Let M^n be an n -dimensional oriented compact submanifold in the $(n + p)$ -dimensional complete and simply connected space form $F^{n+p}(c)$ with constant curvature c . We make the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n + p, 1 \leq i, j, k, \dots \leq n, n + 1 \leq \alpha, \beta, \gamma \leq n + p.$$

For an arbitrary fixed point $x \in M \subset F^{n+p}(c)$, we choose an orthonormal local frame field $\{e_A\}$ in $F^{n+p}(c)$, where $\{e_i\}$ are tangent to M . Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection 1-forms of $F^{n+p}(c)$ respectively. Let h, ξ and Rm be the second fundamental form, mean curvature vector and the Riemannian curvature tensor of M , respectively, and $\bar{R}m$ the Riemannian curvature tensor of $F^{n+p}(c)$. Then

$$\begin{aligned} \omega_{\alpha i} &= \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \\ h &= \sum_{\alpha, i, j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n} \sum_{\alpha, i} h_{ii}^\alpha e_\alpha, \\ R_{ijkl} &= c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \end{aligned} \quad (1)$$

$$R_{\alpha\beta kl} = \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta). \quad (2)$$

The squared norm S of the second fundamental form of M are give by $S := \sum_{\alpha, i, j} (h_{ij}^\alpha)^2$. Define $H_\alpha = (h_{ij}^\alpha)_{n \times n}$ and choose e_{n+1} such that it is parallel to ξ . Hence, we have

$$tr H_{n+1} = nH, \quad tr H_\alpha = 0, \quad \text{for } \alpha \neq n + 1.$$

Here H is the mean curvature of M . Denote by $Ric(u)$ the Ricci curvature of M in the direction of $u \in UM$, where UM is the unit tangent bundle. From (1) we have

$$Ric(e_i) = (n - 1)c + \sum_{\alpha, j} [h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2], \quad (3)$$

and

$$n(n - 1)(R - c) = n^2 H^2 - S, \quad (4)$$

where R is the normalized scalar curvature, given by $R = \frac{1}{n(n-1)} \sum_{i, j} R_{ijij}$. Denoting the first and second covariant derivatives of h_{ij}^α by h_{ijk}^α and h_{ijkl}^α respectively. Then, by definition

$$\begin{aligned} \sum_k h_{ijk}^\alpha \omega_k &= dh_{ij}^\alpha - \sum_k h_{ik}^\alpha \omega_{kj} - \sum_k h_{kj}^\alpha \omega_{ki} - \sum_\beta h_{ij}^\beta \omega_{\beta\alpha}, \\ \sum_l h_{ijkl}^\alpha \omega_l &= dh_{ijk}^\alpha - \sum_l h_{ijl}^\alpha \omega_{lk} - \sum_l h_{ilk}^\alpha \omega_{lj} - \sum_l h_{ljk}^\alpha \omega_{li} - \sum_\beta h_{ijk}^\beta \omega_{\beta\alpha}. \end{aligned}$$

The Laplacian of h_{ij}^α is defined by $\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha$.

Now, we assume that M has parallel normalized mean curvature vector. Then, $\omega_{n+1, \alpha} = 0$ for any α . Therefore, $\sum_i \omega_{n+1, i} \wedge \omega_i \alpha = 0$, for any α . Thus,

$$H_{n+1} H_\alpha = H_\alpha H_{n+1},$$

and

$$R_{n+1,\alpha kl} = 0, \text{ for any } \alpha, k, l.$$

Following [5,6], we have

$$\Delta h_{ij}^{n+1} = \sum_k h_{kij}^{n+1} + \sum_{k,m} (h_{mk}^{n+1} R_{mijk} + h_{im}^{n+1} R_{mkjk}), \quad (5)$$

$$\Delta h_{ij}^\alpha = \sum_k h_{kij}^\alpha + \sum_{k,m} h_{km}^\alpha R_{mijk} + \sum_{k,m} h_{mi}^\alpha R_{mkjk} - \sum_{\beta \neq n+1, k} h_{ki}^\beta R_{\alpha\beta jk}, \text{ for } \alpha \neq n+1. \quad (6)$$

We define the gradient and Hessian of f by

$$df = \sum_i f_i \omega_i, \sum_j f_{ij} \omega_j = df_i + \sum_j f_j \omega_{ji},$$

where f is a function f defined on M . We make appeal to the differential operator due to Cheng and Yau [15], acting on any C^2 -function f by

$$\square f = \sum_{i,j} \left(nH\delta_{ij} - h_{ij}^{n+1} \right) f_{ij}. \quad (7)$$

It follows from [15] that the operator $\square$ is self-adjoint, that is,

$$\int_M \square f dv = 0, \text{ for any } f \in C^2(M). \quad (8)$$

The following lemma will be used to prove our main theorems, which is essentially due to Cheng-Yau [15], and see also [16,19].

Lemma 1. Assume the normalized scalar curvature $R = \text{constant}$ and $R - c \geq 0$, then

$$|\nabla h|^2 \geq n^2 |\nabla H|^2. \quad (9)$$

Denote by $N(A)$ the square of the norm of A , where $A = (a_{ij})$ is a matrix. Then, $N(A) = \text{tr}(A^t A) = \sum_{i,j} a_{ij}^2$, and we have the following lemmas.

Lemma 2 ([20]). Let $B_1, \dots, B_m$ be symmetric $(n \times n)$ -matrices. Set $S_{\alpha\beta} = \text{tr}(B_\alpha^t B_\beta)$, $S_\alpha = S_{\alpha\alpha}$, then

$$\sum_{\alpha,\beta}^m N(B_\alpha B_\beta - B_\beta B_\alpha) + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq \left(1 + \frac{1}{2} \text{sgn}(m-1)\right) \left(\sum_\alpha S_\alpha\right)^2,$$

and the equality holds if and only if one of the following conditions holds:

- (1) $B_1 = B_2 = \dots = B_m = 0$
- (2) only two of the matrices $B_1, B_2, \dots, B_m$ are different from zero. Moreover, assuming $B_1 \neq 0, B_1 \neq 0, B_3 = \dots = B_m = 0$, one has $S_1 = S_2$, and there exists an orthogonal $(n \times n)$ -matrix T such that

$$TB_1^t T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right),$$

$$TB_2^t T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right).$$

The following DDVV inequality is proved by Lu and Ge-Tang [21,22].

Lemma 3 (DDVV Inequality). *Let $B_1, \dots, B_m$ be symmetric $(n \times n)$ -matrices. Then,*

$$\sum_{\alpha, \beta} N(B_\alpha B_\beta - B_\beta B_\alpha) \leq \left(\sum_{\alpha} N(B_\alpha) \right)^2,$$

where the equality holds if and only if under some rotation all B_r 's are zero except two matrices which can be written as

$$\tilde{B}_r = P \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t, \quad \tilde{B}_s = P \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t,$$

for an orthogonal $(n \times n)$ -matrix P .

3. A Codimension Reduction Theorem

In this section, we assume M is a compact submanifold with parallel normalized mean curvature vector in $F^{n+p}(c)$ for $p > 1$. Since $\text{tr} H_\alpha = 0$, we have $\sum_k h_{kkij}^\alpha = 0$ for $\alpha \neq n+1$. Then, we obtain from (6) that

$$\begin{aligned} \sum_{i,j,\alpha \neq n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha &= \sum_{i,j,k,m,\alpha \neq n+1} (h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}) \\ &\quad - \sum_{i,j,k,\alpha,\beta \neq n+1} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk}. \end{aligned} \quad (10)$$

We obtain from (1) and (2) that

$$\begin{aligned} &\sum_{i,j,k,m,\alpha \neq n+1} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m,\alpha \neq n+1} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \\ &= ncS_I + \sum_{\alpha \neq n+1, \beta} \text{tr} H_\beta \cdot \text{tr}(H_\alpha^2 H_\beta) - \sum_{\alpha \neq n+1, \beta} [\text{tr}(H_\alpha H_\beta)]^2 \\ &\quad - \sum_{\beta, \alpha \neq n+1} [\text{tr}(H_\alpha^2 H_\beta^2) - \text{tr}(H_\alpha H_\beta)^2], \end{aligned}$$

and

$$\sum_{i,j,k,\alpha,\beta \neq n+1} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk} = \sum_{\alpha, \beta \neq n+1} [\text{tr}(H_\alpha^2 H_\beta^2) - \text{tr}(H_\alpha H_\beta)^2].$$

For submanifolds with parallel normalized mean curvature, $H_{n+1} H_\alpha = H_\alpha H_{n+1}$ for any α . $(\text{tr}(H_\alpha H_\beta))$ is a symmetric $(p-1) \times (p-1)$ -matrix for $\alpha, \beta \neq n+1$. Then, we choose the normal vector fields $\{e_\alpha\}_{\alpha \neq n+1}$ such that

$$\text{tr}(H_\alpha H_\beta) = \text{tr} H_\alpha^2 \cdot \delta_{\alpha\beta}.$$

This implies

$$\sum_{\alpha, \beta \neq n+1} [\text{tr}(H_\alpha H_\beta)]^2 = \sum_{\alpha \neq n+1} \text{tr}(H_\alpha^2)^2. \quad (11)$$

Then, we have

$$\begin{aligned} \sum_{i,j,\alpha \neq n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha &= ncS_I - 2 \sum_{\alpha, \beta \neq n+1} [\text{tr}(H_\alpha^2 H_\beta^2) - \text{tr}(H_\alpha H_\beta)^2] - \sum_{\alpha \neq n+1} (\text{tr} H_\alpha^2)^2 \\ &\quad + \sum_{\alpha \neq n+1} \text{tr}(H_\alpha^2 H_{n+1}) \cdot \text{tr} H_{n+1} - \sum_{\alpha \neq n+1} [\text{tr}(H_\alpha H_{n+1})]^2. \end{aligned} \quad (12)$$

Hence,

$$\begin{aligned}\frac{1}{2}\Delta S_I &= \sum_{i,j,k,\alpha \neq n+1} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha \neq n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha \\ &= X_1 + Y_1,\end{aligned}\quad (13)$$

where

$$\begin{aligned}X_1 &:= - \sum_{\alpha, \beta \neq n+1} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha \neq n+1} (tr H_\alpha^2)^2 \\ &\quad + \sum_{\alpha \neq n+1} tr(H_\alpha^2 H_{n+1}) \cdot tr H_{n+1} - \sum_{\alpha \neq n+1} [tr(H_\alpha H_{n+1})]^2 + ncS_I, \\ Y_1 &:= \sum_{i,j,k,\alpha \neq n+1} (h_{ijk}^\alpha)^2.\end{aligned}$$

Lemma 4. $X_1 \geq nS_I[Ric_{\min} - (n-2)(c+H^2)] - sgn(p-2)\frac{n-1}{2(n-2)}S_I^2$.

Proof. Since $H_{n+1}H_\alpha = H_\alpha H_{n+1}$, H_{n+1} and H_α can be simultaneously diagonalized for every fixed α :

(i) If $p = 2$, we choose $\{e_i\}$ such that H_{n+2} is a diagonal matrix, i.e., $h_{ij}^{n+2} = 0$ for $i \neq j$. Then, it can be seen from (3) that

$$(n-2)H(h_{ii}^{n+1} - H) \geq Ric_{\min} - (n-1)(c+H^2) + (h_{ii}^{n+1} - H)^2 + (h_{ii}^{n+2})^2.$$

Hence, we obtain

$$\begin{aligned}& tr(H_{n+2}^2 H_{n+1}) \cdot tr H_{n+1} - [tr(H_{n+2} H_{n+1})]^2 \\ &= -[tr(H_{n+1} - HI)H_{n+2}]^2 + nHtr[(H_{n+1} - HI)H_{n+2}^2] + nH^2S_I \\ &= nH \sum_i (h_{ii}^{n+1} - H)(h_{ii}^{n+2})^2 - \left[\sum_i (h_{ii}^{n+1} - H)(h_{ii}^{n+2}) \right]^2 + nH^2S_I \\ &\geq \frac{n}{n-2}[Ric_{\min} - (n-1)(c+H^2)]S_I + \frac{1}{n-2} \left[\left(\sum_i (h_{ii}^{n+1} - H)h_{ii}^{n+2} \right)^2 \right. \\ &\quad \left. + (tr H_{n+2}^2)^2 \right] - \left[\sum_i (h_{ii}^{n+1} - H)h_{ii}^{n+2} \right]^2 + nH^2S_I \\ &\geq \frac{n}{n-2}[Ric_{\min} - (n-1)(c+H^2)]S_I \\ &\quad + \frac{1}{n-2}(tr H_{n+2}^2)^2 - \frac{n-3}{n-2}(S_H - nH^2)S_I + nH^2S_I,\end{aligned}\quad (14)$$

where I is the unit $(n \times n)$ -matrix. Then, it follows from (4) and (14) that

$$\begin{aligned}X_1 &\geq \frac{n}{n-2}[Ric_{\min} - (n-1)(c+H^2)]S_I \\ &\quad - \frac{n-3}{n-2}(tr H_{n+2}^2)^2 - \frac{n-3}{n-2}(S_H - nH^2)S_I + nH^2S_I + ncS_I \\ &\geq \frac{n}{n-2}[Ric_{\min} - (n-1)(c+H^2)]S_I \\ &\quad - \frac{n-3}{n-2}(S - nH^2)S_I + nH^2S_I + ncS_I \\ &\geq nS_I[Ric_{\min} - (n-2)(c+H^2)].\end{aligned}$$

(ii) If $p \geq 3$, for a fixed α , let $\{e_i\}$ be a frame diagonalizing the matrix H_α such that $h_{ij}^\alpha = 0$ for $i \neq j$. So we observe that these terms can be written as follows:

$$\begin{aligned} & \sum_{\alpha \neq n+1} \text{tr}(H_\alpha^2 H_{n+1}) \cdot \text{tr} H_{n+1} - \sum_{\alpha \neq n+1} [\text{tr}(H_\alpha H_{n+1})]^2 \\ &= nH \sum_{\alpha \neq n+1} [\text{tr}(H_{n+1} - HI) H_\alpha^2] - \sum_{\alpha \neq n+1} [\text{tr}(H_{n+1} - HI) H_\alpha]^2 + nH^2 S_I \quad (15) \\ &= nH \sum_{\alpha \neq n+1} \left[\sum_i (h_{ii}^{n+1} - H)(h_{ii}^\alpha)^2 \right] - \sum_{\alpha \neq n+1} \left[\sum_i (h_{ii}^{n+1} - H) h_{ii}^\alpha \right]^2 + nH^2 S_I. \end{aligned}$$

We also obtain from (3) that

$$(n-1)(c+H^2) + (n-2)H(h_{ii}^{n+1} - H) - (h_{ii}^{n+1} - H)^2 - (h_{ii}^\alpha)^2 - \sum_{j, \beta \neq \alpha, n+1} (h_{ij}^\beta)^2 \geq Ric_{\min}.$$

This implies that

$$\begin{aligned} nH \sum_i (h_{ii}^{n+1} - H)(h_{ii}^\alpha)^2 &\geq \frac{n}{n-2} \sum_i (h_{ii}^{n+1} - H)^2 (h_{ii}^\alpha)^2 + \frac{n}{n-2} \sum_i (h_{ii}^\alpha)^4 \\ &\quad + \frac{n}{n-2} \sum_{\substack{\beta \neq \alpha, n+1 \\ i, j}} (h_{ij}^\beta)^2 (h_{ii}^\alpha)^2 \\ &\quad + \frac{n}{n-2} \left[Ric_{\min} - (n-1)(c+H^2) \right] \text{tr} H_\alpha^2 \quad (16) \\ &\geq \frac{1}{n-2} \left[\sum_i (h_{ii}^{n+1} - H)(h_{ii}^\alpha) \right]^2 + \frac{1}{n-2} (\text{tr} H_\alpha^2)^2 \\ &\quad + \frac{n}{n-2} \left[Ric_{\min} - (n-1)(c+H^2) \right] \text{tr} H_\alpha^2. \end{aligned}$$

From (15) and (16), we obtain

$$\begin{aligned} & - \sum_{\beta, \alpha \neq n+1} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha \neq n+1} (\text{tr} H_\alpha^2)^2 \\ & \quad + \sum_{\alpha \neq n+1} \text{tr}(H_\alpha^2 H_{n+1}) \cdot \text{tr} H_{n+1} - \sum_{\alpha \neq n+1} [\text{tr}(H_\alpha H_{n+1})]^2 \\ &\geq - \sum_{\beta, \alpha \neq n+1} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha \neq n+1} (\text{tr} H_\alpha^2)^2 + nH \sum_{\alpha \neq n+1} \sum_i (h_{ii}^{n+1} - H)(h_{ii}^\alpha)^2 \\ & \quad - \sum_{\alpha \neq n+1} \left[\sum_i (h_{ii}^{n+1} - H) h_{ii}^\alpha \right]^2 + nH^2 S_I \quad (17) \\ &\geq - \left(\frac{1}{n-2} + \frac{n-3}{n-2} \right) \sum_{\beta, \alpha \neq n+1} N(H_\alpha H_\beta - H_\beta H_\alpha) - \frac{n-3}{n-2} \sum_{\alpha \neq n+1} (\text{tr} H_\alpha^2)^2 + nH^2 S_I \\ & \quad - \frac{n-3}{n-2} \sum_{\alpha \neq n+1} \left[\sum_i (h_{ii}^{n+1} - H)(h_{ii}^\alpha) \right]^2 + \frac{n}{n-2} \left[Ric_{\min} - (n-1)(c+H^2) \right] S_I. \end{aligned}$$

It follows from Lemmas 2 and 3 that

$$\sum_{\beta, \alpha \neq n+1} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha \neq n+1} (\text{tr} H_\alpha^2)^2 \leq \frac{3}{2} S_I^2,$$

and

$$\sum_{\beta, \alpha \neq n+1} N(H_\alpha H_\beta - H_\beta H_\alpha) \leq S_I^2.$$

These together with (4) and (17) imply that

$$\begin{aligned} X_1 &\geq S_I \left\{ n(c + H^2) - \frac{n-3}{n-2} (S_H - nH^2) + \frac{n}{n-2} [Ric_{\min} - (n-1)(c + H^2)] \right\} \\ &\quad - \left[\frac{1}{n-2} + \frac{3(n-3)}{2(n-2)} \right] S_I^2 \\ &\geq nS_I [Ric_{\min} - (n-2)(c + H^2)] + \frac{n-1}{2(n-2)} S_I^2. \end{aligned} \quad (18)$$

This proves the lemma. $\square$

Theorem 7. If M^n ($n \geq 3$) is an oriented compact submanifold with parallel normalized mean curvature in the space forms $F^{n+p}(c)$ ($p \geq 2$), then

$$\int_M \left\{ nS_I [Ric_{\min} - (n-2)(c + H^2)] - \operatorname{sgn}(p-2) \frac{n-1}{2(n-2)} S_I^2 \right\} dM \leq 0.$$

Proof. Since M has parallel normalized mean curvature, $H_{n+1}H_\alpha = H_\alpha H_{n+1}$. This implies that $Y_1 = \sum_{i,j,k,\alpha \neq n+1} (h_{ijk}^\alpha)^2 \geq 0$. Then, it follows from Lemma 4 that

$$\frac{1}{2} \Delta S_I = X_1 + Y_1 \geq nS_I [Ric_{\min} - (n-2)(c + H^2)] - \operatorname{sgn}(p-2) \frac{n-1}{2(n-2)} S_I^2. \quad (19)$$

Hence,

$$\begin{aligned} 0 &= \frac{1}{2} \int_M \Delta S_I dM \\ &\geq \int_M \left\{ nS_I [Ric_{\min} - (n-2)(c + H^2)] - \operatorname{sgn}(p-2) \frac{n-1}{2(n-2)} S_I^2 \right\} dM. \end{aligned}$$

This proves Theorem 7. $\square$

Now, we are in the position to prove Theorem 5.

Proof of Theorem 5. It follows from (4) that

$$S_I \leq S - nH^2 \leq n[(n-1)(c + H^2) - Ric_{\min}]. \quad (20)$$

This together with (19) implies that

$$\begin{aligned} &\frac{1}{2} \Delta S_I \\ &\geq nS_I [Ric_{\min} - (n-2)(c + H^2) + \operatorname{sgn}(p-2) \frac{n-1}{2(n-2)} (Ric_{\min} - (n-1)(c + H^2))] \\ &\geq nS_I \left[1 + \operatorname{sgn}(p-2) \frac{n-1}{2(n-2)} \right] \times \left[Ric_{\min} - \left(n-2 + \operatorname{sgn}(p-2) \frac{n-1}{3n-5} \right) (c + H^2) \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &= \frac{1}{2} \int_M \Delta S_I dM \\ &\geq \int_M \left\{ nS_I \left[1 + \operatorname{sgn}(p-2) \frac{n-1}{2(n-2)} \right] \right. \\ &\quad \times \left. \left[Ric_{\min} - \left(n-2 + \operatorname{sgn}(p-2) \frac{n-1}{3n-5} \right) (c + H^2) \right] \right\} dM. \end{aligned}$$

If $Ric_{\min} > \left[n - 2 + \operatorname{sgn}(p - 2) \frac{n-1}{3n-5} \right] (c + H^2)$, then $S_I = 0$. It follows from a theorem due to Erbacher [23] that M lies in the totally geodesic submanifold $F^{n+1}(c)$ of $F^{n+p}(c)$. This proves Theorem 5. $\square$

4. Submanifolds with Constant Scalar Curvature

In this section, we assume M is an oriented compact submanifold with constant normalized scalar curvature R and parallel normalized mean curvature vector in the space form $F^{n+p}(c)$. Since $\operatorname{tr} H_{n+1} = nH$, we obtain from (5) that

$$\sum_{i,j} h_{ij}^{n+1} \Delta h_{ij}^{n+1} = \sum_{i,j} n H_{ij} h_{ij}^{n+1} + \sum_{i,j,k,m} (h_{ij}^{n+1} h_{mk}^{n+1} R_{mijk} + h_{ij}^{n+1} h_{im}^{n+1} R_{mkjk}). \quad (21)$$

Applying (1) and (2), we obtain

$$\begin{aligned} & \sum_{i,j,k,m} h_{ij}^{n+1} h_{km}^{n+1} R_{mijk} + \sum_{i,j,k,m} h_{ij}^{n+1} h_{mi}^{n+1} R_{mkjk} \\ &= nc(S_H - nH^2) + \sum_{\alpha} \operatorname{tr} H_{\alpha} \cdot \operatorname{tr}(H_{n+1}^2 H_{\alpha}) - \sum_{\alpha} [\operatorname{tr}(H_{n+1} H_{\alpha})]^2 \\ & \quad - \sum_{\alpha} [\operatorname{tr}(H_{n+1}^2 H_{\alpha}) - \operatorname{tr}(H_{n+1} H_{\alpha})^2]. \end{aligned}$$

Since $H_{n+1} H_{\alpha} = H_{\alpha} H_{n+1}$ for any α ,

$$\begin{aligned} \frac{1}{2} \Delta S_H &= \sum_{i,j,k} (h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1} \Delta h_{ij}^{n+1} \\ &= X_2 + Y_2, \end{aligned} \quad (22)$$

where

$$\begin{aligned} X_2 &:= nH \operatorname{tr} H_{n+1}^3 - (\operatorname{tr} H_{n+1}^2)^2 - \sum_{\alpha \neq n+1} [\operatorname{tr}(H_{n+1} H_{\alpha})]^2 + nc(S_H - nH^2), \\ Y_2 &:= \sum_{i,j,k} (h_{ijk}^{n+1})^2 + \sum_{i,j} n H_{ij} h_{ij}^{n+1}. \end{aligned}$$

Lemma 5. $X_2 \geq n(S_H - nH^2) [Ric_{\min} - (n-2)(c + H^2)]$.

Proof. We choose the orthonormal frame fields $\{e_i\}$ such that $h_{ij}^{n+1} = \lambda_i^{n+1} \delta_{ij}$. Let

$$\begin{aligned} f_k &= \sum_i (\lambda_i^{n+1})^k, \quad \mu_i^{n+1} = H - \lambda_i^{n+1}, \quad i = 1, 2, \dots, n, \\ B_k &= \sum_i (\mu_i^{n+1})^k, \end{aligned}$$

and we have

$$B_1 = 0, \quad B_2 = S_H - nH^2, \quad B_3 = 3HS_H - 2nH^3 - f_3.$$

Then

$$\begin{aligned} X_2 &= -S_H^2 + nHf_3 - \sum_{\alpha \neq n+1} \left(\sum_i \mu_i^{n+1} h_{ii}^{\alpha} \right)^2 + nc(S_H - nH^2) \\ &= -S_H^2 + nH(3HS_H - 2nH^3 - B_3) - \sum_{\alpha \neq n+1} \left(\sum_i \mu_i^{n+1} h_{ii}^{\alpha} \right)^2 \\ & \quad + nc(S_H - nH^2) \\ &= B_2[nc + 2nH^2 - S_H] - nHB_3 - \sum_{\alpha \neq n+1} \left(\sum_i \mu_i^{n+1} h_{ii}^{\alpha} \right)^2. \end{aligned} \quad (23)$$

According to Equation (3) in space form $F^{n+p}(c)$, we have

$$(n-1)c + nH\lambda_i^{n+1} - (\lambda_i^{n+1})^2 - \sum_{\alpha \neq n+1, j} (h_{ij}^\alpha)^2 = Ric(e_i) \geq Ric_{\min},$$

$$S - nH^2 \leq n[(n-1)(c + H^2) - Ric_{\min}], \quad (24)$$

and

$$(n-2)H(\lambda_i^{n+1} - H) - (\lambda_i^{n+1} - H)^2 + (n-1)(c + H^2) - \sum_{\alpha \neq n+1, j} (h_{ij}^\alpha)^2 - Ric_{\min} \geq 0,$$

from which it can be deduced that

$$H(\lambda_i^{n+1} - H) \geq \frac{(\lambda_i^{n+1} - H)^2}{n-2} + \frac{\sum_{\alpha \neq n+1, j} (h_{ij}^\alpha)^2}{n-2} + \frac{Ric_{\min}}{n-2} - \frac{n-1}{n-2}(c + H^2).$$

So,

$$\begin{aligned} -nHB_3 &\geq \frac{n}{n-2} \sum_i (\mu_i^{n+1})^4 + \frac{n}{n-2} \sum_{\alpha \neq n+1} \sum_{i,j} (h_{ij}^\alpha)^2 (\mu_i^{n+1})^2 \\ &\quad + \frac{n}{n-2} [Ric_{\min} - (n-1)(c + H^2)] B_2. \end{aligned}$$

This together with (23) and (24) implies that

$$\begin{aligned} X_2 &\geq B_2 \left\{ nc + 2nH^2 - S_H + \frac{n}{n-2} [Ric_{\min} - (n-1)(c + H^2)] \right\} \\ &\quad + \frac{n}{n-2} \sum_i (\mu_i^{n+1})^4 + \sum_{\alpha \neq n+1} \left[\frac{n}{n-2} \sum_i (h_{ii}^\alpha)^2 (\mu_i^{n+1})^2 - \left(\sum_i \mu_i^{n+1} h_{ii}^\alpha \right)^2 \right] \\ &\geq B_2 \left\{ nc + 2nH^2 - S_H + \frac{n}{n-2} [Ric_{\min} - (n-1)(c + H^2)] \right\} \\ &\quad - \frac{B_2^2}{n-2} - \frac{n-3}{n-2} \sum_{\alpha \neq n+1} \left(\sum_i \mu_i^{n+1} h_{ii}^\alpha \right)^2 \\ &\geq B_2 \left\{ nc + nH^2 - \frac{n-3}{n-2} (S - nH^2) \right. \\ &\quad \left. + \frac{n}{n-2} [Ric_{\min} - (n-1)(c + H^2)] \right\} \\ &\geq \frac{n}{n-2} B_2 \left\{ (n-2)(c + H^2) \right. \\ &\quad \left. - (n-3)[(n-1)(c + H^2) - Ric_{\min}] + [Ric_{\min} - (n-1)(c + H^2)] \right\} \\ &= n(S_H - nH^2) [Ric_{\min} - (n-2)(c + H^2)]. \end{aligned} \quad (25)$$

This completes the proof of the lemma. $\square$

Proof of Theorem 6. Since the normalized scalar curvature R is constant, we obtain from (4) that

$$n^2 \Delta H^2 = 2n^2 H \Delta H + 2n^2 |\nabla H|^2 = \Delta S. \quad (26)$$

On the other hand, we obtain from (13) and (22) that

$$\frac{1}{2} \Delta S = |\nabla h|^2 + \sum_{i,j} n H_{ij}^{n+1} h_{ij}^{n+1} + X_1 + X_2. \quad (27)$$

This together with (7) implies that

$$\begin{aligned}\square(nH) &= \sum_{i,j} (nH\delta_{ij} - h_{ij})(nH)_{ij} \\ &= \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_{i,j} nh_{ij}^{n+1}H_{ij} \\ &= (|\nabla h|^2 - n^2|\nabla H|^2) + X_1 + X_2.\end{aligned}\quad (28)$$

Therefore, we obtain from Lemmas 1, 4 and 5 that

$$\begin{aligned}\square(nH) &\geq n(S - nH^2) \left[Ric_{\min} - (n-2)(c + H^2) \right] - sgn(p-2) \frac{n-1}{2(n-2)} S_I^2 \\ &\geq n(S - nH^2) \left[Ric_{\min} - (n-2)(c + H^2) - \frac{sgn(p-2)(n-1)}{2n(n-2)} (S - nH^2) \right] \\ &\geq n(S - nH^2) \left\{ Ric_{\min} - (n-2)(c + H^2) \right. \\ &\quad \left. + sgn(p-2) \frac{n-1}{2(n-2)} \left[Ric_{\min} - (n-1)(c + H^2) \right] \right\} \\ &= n(S - nH^2) \left[1 + sgn(p-2) \frac{n-1}{2(n-2)} \right] \\ &\quad \times \left[Ric_{\min} - \left(n-2 + sgn(p-2) \frac{n-1}{3n-5} \right) (c + H^2) \right].\end{aligned}\quad (29)$$

Since the operator $\square$ is self-adjoint, we conclude

$$\begin{aligned}0 &\geq \int_M n(S - nH^2) \left[1 + sgn(p-2) \frac{n-1}{2(n-2)} \right] \\ &\quad \times \left[Ric_{\min} - \left(n-2 + sgn(p-2) \frac{n-1}{3n-5} \right) (c + H^2) \right].\end{aligned}$$

Thus, we obtain from the assumption $Ric_M > (n-2 + sgn(p-2) \frac{n-1}{3n-5})(c + H^2)$ that

$$S - nH^2 = 0.$$

This means M must be the totally umbilical sphere $S^n(\frac{1}{\sqrt{c+H^2}})$. This proves Theorem 6. $\square$

5. Discussion

The following example shows the pinching constant is the best possible in even dimensions and $p = 2$.

Example 1. Let $S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ be the totally umbilical sphere in $F^{n+p}(c)$ with $c + H^2 > 0$. Here the mean curvature H is a constant.

Let $M = S^m(\frac{1}{\sqrt{2(c+H^2)}}) \times S^m(\frac{1}{\sqrt{2(c+H^2)}})$ be a Clifford hypersurface in $S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ with $n = 2m$. Then, M is a compact submanifold in $F^{n+2}(c)$ with parallel normalized mean curvature vector, the Ricci curvature $Ric_M \equiv (n-2)(c + H^2)$, and the normalized scalar curvature $R = \frac{n-2}{n-1}(c + H^2)$.

More generally, M is also a submanifold in $F^{n+p}(c)$ with parallel normalized mean curvature vector for $p \geq 3$, and the Ricci curvature of M satisfies $Ric_M \equiv (n-2)(c + H^2)$.

For $n = 4$ and $p \geq 4$, we have the following example.

Example 2. Let $S^7(\frac{1}{\sqrt{c+H^2}})$ be the totally umbilic sphere in $F^{4+p}(c)$ with $c + H^2 > 0$. Here the mean curvature H is a constant. Let $\mathbb{CP}^2(\frac{4}{3}(c + H^2))$ be the two-dimensional complex projective space minimally immersed in $S^7(\frac{1}{\sqrt{c+H^2}})$ with constant holomorphic sectional curvature $\frac{4}{3}(c + H^2)$. Then, M is a compact submanifold in $F^{4+p}(c)$ with parallel normalized mean curvature vector for $p \geq 4$, the Ricci curvature $\text{Ric}_M \equiv 2(c + H^2)$, and the normalized scalar curvature $R = \frac{2}{3}(c + H^2)$.

Motivated by Theorem 6, Examples 1 and 2, we propose the following conjecture.

Conjecture 1. Let M be an $n(\geq 3)$ -dimensional oriented compact submanifold with constant normalized scalar curvature R and parallel normalized mean curvature vector in the space form $F^{n+p}(c)$. If $R \geq c$, and

$$\text{Ric}_M > (n - 2)(c + H^2) > 0,$$

then M must be the totally umbilical sphere $S^n(\frac{1}{\sqrt{c+H^2}})$.

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The Shape Operator of Real Hypersurfaces in $S^6(1)$

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Abstract: The aim of the paper is to present two results concerning real hypersurfaces in the six-dimensional sphere $S^6(1)$. More precisely, we prove that real hypersurfaces with the Lie-parallel shape operator A must be totally geodesic hyperspheres. Additionally, we classify real hypersurfaces in a nearly Kähler sphere $S^6(1)$ whose Lie derivative of the shape operator coincides with its covariant derivative.

Keywords: Hopf hypersurfaces; nearly Kähler manifolds; real hypersurfaces; shape operator; Lie derivative

MSC: 53B25; 53B35; 53B21

1. Introduction

An almost Hermitian manifold with an almost complex structure J and the Levi-Civita connection $\tilde{\nabla}$ is, respectively, Kähler or nearly Kähler if the tensor field $G(X, Y) = (\tilde{\nabla}_X J)Y$ is vanishing or skew-symmetric. In [1], it was shown that an arbitrary nearly Kähler manifold can be locally decomposed into manifolds of three particular types, with six-dimensional nearly Kähler manifolds being one of those types. We recall that it has been shown by Butruille, see [2], that only four homogeneous, six-dimensional, strictly nearly Kähler manifolds exist: the six-dimensional sphere $S^6(1)$, the manifold $S^3 \times S^3$, the projective space CP^3 , and the flag manifold $SU(3)/U(1) \times U(1)$. We note that, out of these four, only the sphere $S^6(1)$ is endowed with standard metrics.

If we denote by N the unit normal vector field on the hypersurface M of an almost Hermitian manifold, the tangent vector field $\xi = -JN$ is called the Reeb vector field or characteristic vector field. We denote by g the metric on $S^6(1)$ induced by the standard Euclidean metric $\langle \cdot, \cdot \rangle$ in the space $\mathbb{R}^7$. The shape operator of real hypersurfaces M of $S^6(1)$ is denoted by A and satisfies $g(AX, Y) = g(h(X, Y), N)$ for all X, Y tangent to M , where h is the second fundamental form of M .

We say that a hypersurface M is Hopf if the vector field ξ satisfies $A\xi = \alpha\xi$ for a certain differentiable function α , and then ξ is a principal vector field. We also note that the function α is locally constant, see [3]. It was shown in [3] that a connected Hopf hypersurface of a nearly Kähler $S^6(1)$ is an open part of either a geodesic hypersphere or a tube around an almost complex curve in $S^6(1)$. Therefore, at each point of a Hopf hypersurface in $S^6(1)$, there exist either one, two, or three different principal curvatures.

The parts of totally geodesic hyperspheres are, of course, umbilical and have only one principal curvature α . All other Hopf hypersurfaces in $S^6(1)$ are parts of tubes around almost complex curves in $S^6(1)$, and the principal curvature α has a multiplicity of 3. If the almost complex curve is totally geodesic, there is only one other principal curvature μ that has a multiplicity of 2. If the almost complex curve is not totally geodesic, then it is one of types (I), (II), or (III), and α has two other principal curvatures, μ and λ , that have a multiplicity of 1. For details, we refer the reader to [4].

In recent years, the study of Riemannian hypersurfaces in different ambient spaces endowed with an almost complex structure has been an active field of research. In particular, many studies deal with the question of the existence and classification of hypersurfaces that satisfy various conditions related to their parallelism, particularly related to its shape operator, such as η -parallelness or pseudo-parallelness; we refer the reader to [5,6]. The non-existence of real hypersurfaces in complex space forms with a parallel shape operator was observed in [7]. Additionally, Kimura and Maeda [8] classified real hypersurfaces in complex projective space whose shape operator is ξ -parallel, i.e., $\nabla_{\xi}A = 0$.

Recall that Lie derivative $\mathcal{L}$ of a vector field on M is given by $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$, while the Lie derivative of the shape operator is given by $(\mathcal{L}_X A)Y = \nabla_X (AY) + A\nabla_Y X - \nabla_{AY} X - A\nabla_X Y$ for the X, Y tangent of M . The Lie derivative, with respect to the vector field, has many applications in physics, in particular in mechanics, hydrodynamics, theory of relativity, and cosmology. Hence, hypersurfaces whose shape operator is invariant or has at least a certain regularity in its behavior with respect to the Lie derivative are of particular interest. Ki, Kim, and Lee, see [9], classified real hypersurfaces in complex space forms whose shape operator is Lie ξ -parallel, i.e., $\mathcal{L}_{\xi}A = 0$. Suh, in [10], provides a characterization of real hypersurfaces of type A in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, which are tubes over totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in terms of the vanishing Lie derivative of the shape operator A along the direction of the Reeb vector field ξ . In [11], the authors prove the nonexistence of real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$, whose shape operator satisfies the relation $\mathcal{L}_X A = \nabla_X A$, from the X orthogonal to ξ .

Motivated by these results, we consider a similar line of research regarding the nearly Kähler sphere S^6 and prove the following theorems.

Theorem 1. *Let M be a real hypersurface in $S^6(1)$. The shape operator A on M is Lie-parallel, i.e., $\mathcal{L}_X A = 0$, $X \in TM$, if and only if M is a totally geodesic hypersphere in $S^6(1)$.*

Theorem 2. *Let M be a real hypersurface in $S^6(1)$. The shape operator A on M satisfies $\mathcal{L}_{\xi}A = \nabla_{\xi}A$, if and only if M is a totally geodesic hypersphere in $S^6(1)$.*

Additionally, given that totally geodesic hyperspheres also satisfy the stronger condition $\mathcal{L}_X A = \nabla_X A$, for all $X \in TM$, we obtain the following:

Corollary 1. *Let M be a real hypersurface in $S^6(1)$. The shape operator A on M satisfies $\mathcal{L}_X A = \nabla_X A$, for all $X \in TM$, if and only if M is a totally geodesic hypersphere in $S^6(1)$.*

In particular, we note that most of the known results dealing with this type of problem regard hypersurfaces of Kähler manifolds. In this case, the almost complex structure is parallel, making it easier to conduct calculations. For hypersurfaces of the nearly Kähler sphere S^6 , the covariant derivative of the almost complex structure, tensor G , is skew-symmetric, which, in a technical sense, imposes the need to approach the problem in a different manner, see [12,13]. As a consequence, we need to use a suitable moving frame along the hypersurface, which is nicely suited to the given structure, in order to obtain its properties and analyze them.

2. Preliminaries

First, we provide a brief exposition of how the standard nearly Kähler structure J on $S^6(1)$ arises in a natural manner from the multiplication of Cayley numbers $\mathcal{O}$. A vector cross product $\times$ given as follows:

$$u \times v = \frac{1}{2}(uv - vu),$$

and is well defined on the space of purely imaginary Cayley numbers $\mathcal{O}$, which we may identify with $\mathbb{R}^7$.

Then, vectors $e_1, \dots, e_7$ of the standard orthonormal basis of the space $\mathbb{R}^7$ satisfy the relations provided in the following table of multiplication.

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

Any orthonormal basis or frame for which the relations of this table hold is called a G_2 basis or frame. Then, for an arbitrary point $p \in S^6(1)$ and $X \in T_p S^6(1)$, the (1,1)-tensor field J is defined by

$$J_p X = p \times X$$

and is an almost complex structure.

We will denote by $\langle, \rangle$ the standard metric in the space $\mathbb{R}^7$ and, by g , the induced metric on $S^6(1)$. Further, we denote D and $\bar{\nabla}$ by the corresponding Levi-Civita connections.

Let M be a Riemannian submanifold of the sphere $S^6(1)$ with a Hermitian structure (J, g) . The tensor field G of type $(2, 1)$, defined by $G(X, Y) = (\bar{\nabla}_X J)Y$, is skew symmetric, which makes the almost complex structure a nearly Kähler one. The tensor field G has the following properties:

$$G(X, JY) + JG(X, Y) = 0, \quad g(G(X, Y), Z) + g(G(X, Z), Y) = 0.$$

Additionally, for tangent vector fields X, Y , and Z , see [14], it holds that

$$(\bar{\nabla} G)(X, Y, Z) = g(X, Z)JY - g(X, Y)JZ - g(JY, Z)X. \quad (1)$$

Let ∇ and $\nabla^\perp$ be, respectively, the Levi-Civita connection of the submanifold M and the normal connection in the normal bundle $T^\perp M$ of M in $S^6(1)$. The formulae of Gauss and Weingarten for the hypersurface are as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X N = -AX + \nabla_X^\perp N,$$

where X and Y are tangent vector fields and N is a normal vector field on the hypersurface. Here, A and h denote the shape operator and the second fundamental form. The relationship between them is given by $g(h(X, Y), N) = g(AX, Y)$. The Gauss equation for the hypersurface is as follows:

$$R(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \quad X, Y, Z, W \in TM, \quad (2)$$

where we denote by R the Riemannian curvature tensor of M .

Now, let M be a hypersurface in $S^6(1)$. Using the almost complex structure J on $S^6(1)$, the normal vector field N on M , we define the corresponding Reeb vector field $\xi = -JN$ with dual 1-form $\eta(X) = g(X, \xi)$. Then, $\mathcal{D} = \text{Ker } \eta = \{X \in TM \mid \eta(X) = 0\}$ is a four-dimensional almost complex distribution on M .

3. The Moving Frame for Hypersurfaces in $S^6(1)$

Now, we will present the construction of one of the local moving frames that is, roughly speaking, compatible with the structure on the hypersurface, and hence, easier to work with. Also, we will present the relationship between the connection coefficients in this particular frame, for more details see [12,15].

For any unit vector field $E_1 \in \mathcal{D}$, let $E_2 = JE_1$, $E_3 = G(E_1, \xi)$, and $E_4 = JE_3$. Then, as shown in [15], the set $\{E_1, E_2, E_3, E_4, E_5 = \xi\}$ is a local orthonormal moving frame. One such frame is uniquely determined by the choice of the vector field $E_1 \in \mathcal{D}$.

We note that, additionally, the following relations are also valid for each frame.

Lemma 1 ([15]). *The orthonormal frame $\{E_1, E_2, E_3, E_4, E_5 = \xi\}$ satisfies the following relations:*

$$\begin{aligned} G(E_1, E_2) &= 0, & G(E_1, E_3) &= -\xi, & G(E_1, E_4) &= N, & G(E_1, \xi) &= E_3, \\ G(E_1, N) &= -E_4, & G(E_2, E_3) &= -N, & G(E_2, E_4) &= \xi, & G(E_2, \xi) &= -E_4, \\ G(E_2, N) &= -E_3, & G(E_3, E_4) &= 0, & G(E_3, \xi) &= -E_1, & G(E_3, N) &= E_2, \\ G(E_4, \xi) &= E_2, & G(E_4, N) &= E_1. \end{aligned} \quad (3)$$

We further denote the coefficients of the covariant derivatives in the given frame as follows:

$$g_{ij}^k = g(D_{E_i} E_j, E_k), \quad h_{ij} = g(D_{E_i} E_j, N), \quad 1 \leq i, j, k \leq 5. \quad (4)$$

Since D is a metric connection and the second fundamental form is symmetric, straightforwardly, we obtain $g_{ij}^k = -g_{ik}^j$, and $h_{ij} = h_{ji}$.

Using the definition of $G(X, Y) = (\bar{\nabla}_X J)Y$ and the expression for its covariant derivative, we obtain, the following two lemmas, see [12].

Lemma 2 ([12]). *For the previously defined coefficients, we obtain the following:*

$$\begin{aligned} g_{12}^3 &= -g_{11}^4, & g_{12}^4 &= g_{11}^3, & h_{11} &= -g_{12}^5, & h_{12} &= g_{11}^5, & g_{22}^3 &= -g_{21}^4, \\ g_{22}^4 &= g_{21}^3, & g_{22}^5 &= -g_{11}^5, & h_{22} &= g_{21}^5, & g_{32}^3 &= -g_{31}^4, & g_{32}^4 &= g_{31}^3, \\ h_{13} &= 1 - g_{32}^5, & h_{23} &= g_{31}^5, & g_{42}^3 &= -g_{41}^4, & g_{42}^4 &= g_{41}^3, & h_{14} &= -g_{42}^5, \\ h_{24} &= -1 + g_{41}^5, & g_{52}^3 &= -1 - g_{51}^4, & g_{52}^4 &= g_{51}^3, & h_{15} &= -g_{52}^5, & h_{25} &= g_{51}^5, \\ g_{32}^5 &= 2 + g_{14}^5, & g_{42}^5 &= -g_{13}^5, & g_{31}^5 &= -g_{24}^5, & g_{41}^5 &= 2 + g_{23}^5, & h_{33} &= -g_{43}^5, \\ h_{34} &= g_{33}^5, & g_{44}^5 &= -g_{33}^5, & h_{44} &= g_{43}^5, & h_{35} &= -g_{54}^5, & h_{45} &= g_{53}^5. \end{aligned}$$

Proof. By taking $X \in \{E_1, \dots, \xi\}$ and $Y \in \{E_1, \dots, \xi, N\}$ in the relation

$$\bar{\nabla}_X (JY) = G(X, Y) + J(\bar{\nabla}_X Y),$$

and using (3), we obtain the proof of the lemma.

For $X = E_1$ and $Y = E_1$, we obtain the following:

$$g_{12}^3 = -g_{11}^4, \quad g_{12}^4 = g_{11}^3, \quad h_{11} = -g_{12}^5, \quad h_{12} = g_{11}^5.$$

For $X = E_2$ and $Y = E_1$, we obtain the following:

$$g_{22}^3 = -g_{21}^4, \quad g_{22}^4 = g_{21}^3, \quad g_{22}^5 = -h_{12} = -g_{11}^5, \quad h_{22} = g_{21}^5.$$

In a similar manner, we obtain other relations. $\square$

Lemma 3 ([12]). *For the coefficient (4), the following holds:*

$$\begin{aligned} g_{52}^5 &= g_{11}^2 + g_{13}^4, & g_{51}^5 &= -g_{21}^2 - g_{23}^4, & g_{54}^5 &= g_{31}^2 + g_{33}^4, \\ g_{53}^5 &= -g_{41}^2 - g_{43}^4, & h_{55} &= -g_{51}^2 - g_{53}^4. \end{aligned} \quad (5)$$

Proof. By taking $X = E_1$, $Y = E_1$, and $Z = E_3$ in (1), we obtain the following: $g_{52}^5 = g_{11}^2 + g_{13}^4$. Similarly, by taking $X = E_2$, $Y = E_1$, and $Z = E_3$, we obtain $g_{51}^5 = -g_{21}^2 - g_{23}^4$; for $X = E_3$, $Y = E_1$, and $Z = E_3$, we obtain $g_{54}^5 = g_{31}^2 + g_{33}^4$. Finally, for $X = E_4$, $Y = E_1$, and $Z = E_3$ and $X = \zeta$, $Y = E_1$, and $Z = E_3$, respectively, we get $g_{53}^5 = -g_{41}^2 - g_{43}^4$ and $h_{55} = -g_{51}^2 - g_{53}^4$. Straightforward computation shows that (1) is then satisfied for an arbitrary choice of vector fields which completes the proof. $\square$

Recall that we still have the possibility of choosing $E_1 \in \mathcal{D}$. We can decompose the vector field $A\zeta$ into an orthogonal sum of two vector fields, parallel to $\mathcal{D}$ and ζ , respectively. Let E_1 be the unit vector field collinear with the first component. Then, we can write

$$A\zeta = \beta E_1 + \alpha \zeta, \quad (6)$$

for differentiable functions α and β . Moreover, by choosing the direction of E_1 , we may assume that $\beta \geq 0$. Of course, in the case of the Hopf hypersurface, when $\beta = 0$, the vector field E_1 is still not uniquely determined by this.

Since $A\zeta$ has no components in the direction of vector fields E_2, E_3, E_4 , causing it to vanish, we obtain:

$$g_{13}^4 = -g_{11}^2 - \beta, \quad g_{23}^4 = -g_{21}^2, \quad g_{33}^4 = -g_{31}^2, \quad g_{43}^4 = -g_{41}^2, \quad g_{53}^4 = -g_{51}^2 - \alpha.$$

From the Gauss equation we obtain the next two relations between the coefficients. By implementing $X = E_2$, $Y = E_3$, $Z = E_1$, and $W = E_2$ into the Gauss Equation (2) we obtain the following:

$$\begin{aligned} E_2(g_{31}^2) &= g_{21}^2 g_{21}^4 - 3g_{21}^5 - 2g_{14}^5 g_{21}^5 + 2g_{11}^5 g_{24}^5 + g_{11}^2 (-g_{21}^3 + g_{31}^2) - 2g_{21}^4 g_{31}^3 \\ &\quad + 2g_{21}^3 g_{31}^4 + g_{31}^2 g_{31}^4 - g_{21}^2 g_{41}^2 - g_{31}^3 g_{41}^2 - 2g_{51}^2 - g_{14}^5 g_{51}^2 + g_{23}^5 g_{51}^2 + E_3(g_{21}^2), \end{aligned}$$

and then by implementing $(X, Y, Z, W) = (E_2, E_3, E_3, E_4)$ in (2), we also obtain the following:

$$\begin{aligned} (3 + 2g_{14}^5)g_{21}^5 - 2g_{24}^5(g_{11}^5 + g_{33}^5) + (1 + 2g_{23}^5)g_{34}^5 + (-2 - g_{14}^5 + g_{23}^5)\alpha \\ + (-g_{21}^3 + g_{31}^2)\beta = 0. \end{aligned} \quad (7)$$

4. Proof of the Theorem 1

Let us denote $L_{ij}^k = g((\mathcal{L}_{E_i} A)E_j, E_k)$. The condition that the shape operator A is Lie-parallel is equivalent to $L_{ij}^k = 0$, $1 \leq i, j, k \leq 5$.

Lemma 4. *Let M be a hypersurface in $S^6(1)$ with Lie ζ -parallel shape operator A . Then, M is a Hopf hypersurface.*

Proof. Suppose that M is a non-Hopf hypersurface, i.e., $\beta \neq 0$. From $L_{55}^i = 0$, $1 \leq i \leq 4$, we have, respectively,

$$\zeta(\beta) = -g_{11}^5 \beta, \quad g_{51}^2 = -g_{12}^5, \quad g_{51}^3 = -g_{13}^5, \quad g_{51}^4 = -g_{14}^5.$$

Using this, from

$$\begin{aligned} 0 &= L_{51}^5 = -3g_{11}^5 \beta, \\ 0 &= L_{53}^5 = (-g_{13}^5 + 2g_{24}^5)\beta, \\ 0 &= L_{54}^5 = (-3 - g_{14}^5 - 2g_{23}^5)\beta, \end{aligned}$$

we get, respectively $g_{11}^5 = 0$, $g_{13}^5 = 2g_{24}^5$ and $g_{14}^5 = -3 - 2g_{23}^5$. From $0 = L_{51}^2 = 6(1 + g_{23}^5)$, we obtain $g_{12}^5 = -1$, and

$$0 = L_{52}^1 = (g_{12}^5 + g_{21}^5)^2 + (g_{24}^5)^2 + \beta^2$$

gives us a contradiction. $\square$

Therefore, now we assume M to be a Hopf hypersurface, implying that $\beta = 0$ and α are constant.

Recall that, for Hopf hypersurfaces, we still have the option of choosing $E_1 \in \mathcal{D}$, and we can assume that E_1 is an eigenvector field for the shape operator A . As

$$AE_1 = -g_{12}^5 E_1 + g_{11}^5 E_2 - (1 + g_{14}^5) E_3 + g_{13}^5 E_4,$$

we obtain $g_{11}^5 = 0$, $g_{14}^5 = -1$, $g_{13}^5 = 0$.

Now, from $0 = L_{22}^5 = 2g_{24}^5$, we obtain $g_{24}^5 = 0$, and then from $0 = L_{24}^5 + L_{13}^5 = -4g_{33}^5$, we obtain $g_{33}^5 = 0$. Now let us see what the shape operator looks like:

$$\begin{aligned} AE_1 &= -g_{12}^5 E_1, & AE_2 &= g_{21}^5 E_2 + (1 + g_{23}^5) E_4, & AE_3 &= -g_{34}^5 E_3, \\ AE_4 &= (1 + g_{23}^5) E_2 + g_{43}^5 E_4, & AE_5 &= \alpha E_5. \end{aligned} \quad (8)$$

Lemma 5. *Let M be a Hopf hypersurface in $S^6(1)$ with Lie-parallel shape operator A . Then, the moving frame can be chosen so that both E_1 and E_3 are the eigenvectors of the shape operator A for the eigenvalues α .*

Proof. From (8), we want to prove that $-g_{12}^5 = \alpha = -g_{34}^5$.

Let us first assume that $g_{34}^5 \neq -\alpha$. From

$$\begin{aligned} 0 &= L_{23}^5 = (1 - g_{23}^5)(g_{34}^5 + \alpha), \\ 0 &= L_{45}^3 = -(g_{34}^5 + \alpha)(g_{43}^5 + g_{51}^2 + \alpha), \\ 0 &= L_{43}^5 = (g_{34}^5 - g_{43}^5)(g_{34}^5 + \alpha), \end{aligned}$$

we obtain, respectively, $g_{23}^5 = 1$, $g_{51}^2 = -g_{43}^5 - \alpha$, and $g_{43}^5 = g_{34}^5$. Next, from $0 = L_{54}^3 = -2g_{51}^4$ we have $g_{51}^4 = 0$, and then, from $0 = L_{53}^4 = -4(1 + (g_{34}^5)^2)$, we get a contradiction. Hence, $g_{34}^5 = -\alpha$.

Now, suppose that $g_{12}^5 \neq -\alpha$. From

$$0 = L_{41}^5 = -(3 + g_{23}^5)(g_{12}^5 + \alpha), \quad 0 = L_{25}^1 = -(g_{21}^5 - g_{51}^2)(g_{12}^5 + \alpha),$$

we have $g_{23}^5 = -3$ and $g_{51}^2 = g_{21}^5$. Then, from $0 = L_{45}^1 = -(-1 - g_{51}^4)(g_{12}^5 + \alpha)$, we obtain $g_{51}^4 = -1$ and now, from $0 = L_{51}^2 = -4 - (g_{12}^5 + g_{21}^5)^2$, we obtain a contradiction, which completes the proof of this lemma. $\square$

Now, (7) becomes $g_{21}^5 - (2 + g_{23}^5)\alpha = 0$, so $g_{21}^5 = (2 + g_{23}^5)\alpha$, and we obtain the following:

$$0 = L_{12}^5 = -(1 + g_{23}^5)(3 + g_{23}^5)(1 + \alpha^2).$$

If we assume that $g_{23}^5 = -3$, then from $0 = L_{32}^5 = 2g_{43}^5 - 6\alpha$, we obtain $g_{43}^5 = 3\alpha$, and from $0 = L_{34}^5 = -8(1 + \alpha^2)$, we obtain a contradiction. Therefore, it must be $g_{23}^5 = -1$.

From (8), we now see that the shape operator A of the hypersurface M has a quadruple eigenvalue α and a single g_{43}^5 . Since M is a Hopf surface, it is possible that M is a totally geodesic hypersphere, i.e., $g_{34}^5 = \alpha = 0$.

Straightforward computation shows that all L_{ij}^k , $1 \leq i, j, k \leq 5$ are equal to zero, which completes the proof.

5. Proof of the Theorem 2

Let us denote $a_{ij} = g(A\nabla_{E_i}\xi - \nabla_{AE_i}\xi, E_j)$. The condition $\mathcal{L}_\xi A = \nabla_\xi A$ is equivalent to $a_{ij} = 0, 1 \leq i, j \leq 5$.

Lemma 6. *Let M be a hypersurface in $S^6(1)$. If the shape operator A on M satisfies $\mathcal{L}_\xi A = \nabla_\xi A$, then M is a Hopf hypersurface.*

Proof. Suppose that M is a non-Hopf hypersurface, i.e., $\beta \neq 0$. From $a_{i5} = 0, 1 \leq i \leq 4$, we obtain, respectively,

$$-g_{11}^5\beta = 0, \quad -g_{21}^5\beta = 0, \quad g_{24}^5\beta = 0, \quad -(2 + g_{23}^5)\beta = 0,$$

so $g_{11}^5 = g_{21}^5 = g_{24}^5 = 0$ and $g_{23}^5 = -2$. Now, from $0 = a_{54} = (-1 + g_{14}^5)\beta$, we obtain $g_{14}^5 = 1$, and then, $a_{21} = -4 \neq 0$, which is a contradiction. $\square$

Suppose now that M is a Hopf hypersurface, i.e., $\beta = 0$.

To determine the moving frame, we can choose E_1 to be an eigenvector field for the shape operator A . As

$$AE_1 = -g_{12}^5E_1 + g_{11}^5E_2 - (1 + g_{14}^5)E_3 + g_{13}^5E_4,$$

we obtain $g_{11}^5 = 0, g_{14}^5 = -1, g_{13}^5 = 0$.

Now, from $0 = a_{22} = -2g_{24}^5$, we obtain $g_{24}^5 = 0$; then, we obtain $0 = a_{12} = g_{33}^5$. From $a_{14} = 0$ and $a_{32} = 0$, we obtain, respectively,

$$-g_{12}^5g_{23}^5 + g_{43}^5 = 0, \quad -g_{21}^5 - (2 + g_{23}^5)g_{34}^5 = 0,$$

so $g_{43}^5 = g_{12}^5g_{23}^5$ and $g_{21}^5 = -(2 + g_{23}^5)g_{34}^5$. The only non-zero a_{ij} are now the following:

$$\begin{aligned} a_{12} &= 1 - (g_{12}^5)^2 + g_{23}^5 + g_{12}^5(2 + g_{23}^5)g_{34}^5, \\ a_{21} &= [2 + g_{23}^5][1 + g_{23}^5 - g_{12}^5g_{34}^5 + (2 + g_{23}^5)(g_{34}^5)^2], \\ a_{23} &= g_{23}^5(1 + g_{23}^5)(g_{12}^5 - g_{34}^5), \\ a_{34} &= -1 - g_{23}^5 - g_{34}^5(g_{12}^5g_{23}^5 + g_{34}^5), \\ a_{41} &= (1 + g_{23}^5)(2 + g_{23}^5)(g_{12}^5 - g_{34}^5), \\ a_{43} &= g_{23}^5(1 + g_{23}^5 + (g_{12}^5)^2g_{23}^5 + g_{12}^5g_{34}^5). \end{aligned}$$

We begin further discussion from $a_{23} = g_{23}^5(1 + g_{23}^5)(g_{12}^5 - g_{34}^5) = 0$.

If we assume that $g_{23}^5 = 0$, then $a_{34} = 0$ becomes $-1 - (g_{34}^5)^2 = 0$, which is a contradiction. If we assume that $g_{12}^5 = g_{34}^5$, from $0 = a_{12} = (1 + (g_{12}^5)^2)(1 + g_{23}^5)$, we obtain $1 + g_{23}^5 = 0$, so it is enough to consider only the case $g_{23}^5 = -1$.

Now, $0 = a_{12} = g_{12}^5(g_{34}^5 - g_{12}^5)$ gives us $g_{12}^5 = 0$ or $g_{34}^5 = g_{12}^5$.

Assuming $g_{12}^5 = 0, a_{21} = 0$ becomes $(g_{34}^5)^2 = 0$, so $g_{34}^5 = 0$ and now all a_{ij} are zero. In this case, the shape operator A has a quadruple eigenvalue 0 and one is α . As M is a Hopf hypersurface, this is possible if and only if $\alpha = 0$, i.e., M is a totally geodesic hypersphere.

Assuming $g_{34}^5 = g_{12}^5$, we assume that all a_{ij} are zero. In this case, the shape operator A has a quadruple eigenvalue $-g_{12}^5$ and one is α . As M is a Hopf hypersurface, this is possible if and only if $-g_{12}^5 = \alpha = 0$, i.e., M is a totally geodesic hypersphere.

Hence, if the shape operator A on M satisfies $\mathcal{L}_\xi A = \nabla_\xi A$, then M is a totally geodesic hypersphere.

If M is a totally geodesic hypersphere then the shape operator A on M obviously satisfies $\mathcal{L}_\xi A = \nabla_\xi A$, and that completes the proof.

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Article

Alternative View of Inextensible Flows of Curves and Ruled Surfaces via Alternative Frame

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Abstract: In this paper, we present the evolutions of ruled surfaces generated by the principal normal, the principal normal's derivative, and the Darboux vector fields along a space curve that are the elements of an alternative frame. The comprehension of an object's rotational behavior is crucial knowledge relevant to various realms, and this can be accomplished by analyzing the Darboux vector along the path of a point on the object as it moves through space. In that regard, examining the evolutions of the ruled surfaces based on the changes in their directrices, including the Darboux vector in the alternative frame along a space curve, is significant. As the first step of this study, we express the evolution of the alternative frame elements of a space curve. Subsequently, the conditions for the ruled surfaces generated by them to be minimal, developable, and inextensible are investigated. These findings can allow some physical phenomena to be well understood through surface evolutions satisfying these conditions. In the final step, we provide graphical representations of some examples of inextensible ruled surfaces and curve evolutions.

Keywords: ruled surface; alternative frame; evolution of curve

MSC: 53A05; 53A25; 58C27

1. Introduction

Differential geometry places significant emphasis on the study of curves and surfaces, particularly their time evolution within Euclidean 3-space E^3 . The transformation of curves and surfaces over time is described through their corresponding flows. When examining the time variation of a curve, maintaining constant arc length (zero arc-length variation) in its initial and final positions classifies it as an inelastic curve [1]. Similarly, for surfaces, preserving the main curvature at both initial and final positions designates them as inelastic surfaces [2]. Physically, inelastic curves and surfaces arise in situations where movements occur without tension energy, leading to the flow of surfaces. Examples include the oscillating motion of a fixed-length cable or the fluttering of a piece of paper in the wind, both of which are represented by inelastic curves and surface flows [1–4]. In a mathematical sense, the evolution of a surface $X(s, u, t)$ for all time points t is expressed as the isometric image of the original surface $X(s, u, t_0)$ defined at the initial time t_0 . Research in the field of differential geometry has enlightened inelastic curves, and surface flows in various domains, such as computer imaging [5,6], computer animation [7], and mechanical motion science [8]. The concepts of surface evolution, particularly the minimization of mean curvature flow and surface energy, provide a clear understanding of physical phenomena, such as the behavior of soap films and bubbles [9,10]. While there is a considerable body of research on curve flows, surface flows have not received as much attention in the existing literature. In this study, we obtained ruled surfaces with the help of a space evaluation

using the alternative frame. This alternative frame consists of the principal normal vector of the Frenet frame, the derivative of the principal normal, and the Darboux vectors. In 2016, Uzunoğlu et al. defined an alternative moving frame of a space curve [11]. This research has caught the attention of many researchers, and the new approach affects studies on curves [12–14] and surfaces [15–21] based on the alternative frame.

In this work, we intend to present the evolutions of the ruled surfaces constructed by the principal normal, the derivative in the direction of the principal normal, and the Darboux vector fields of the space curves that are the elements of the alternative frame. It is a fact that understanding the rotational behavior of objects is crucial in various fields, and this can be achieved by analyzing the Darboux vector along a curve to find out how the curve rotates as it moves through space. Here, the evolutions of the ruled surfaces depend on the evolutions of their directrices, including the Darboux vector in the alternative frame along a space curve. In that regard, some geometric properties of the ruled surface evolutions are investigated based on the benefit of knowing in which direction the curve flows are rotating by their Darboux vectors. Examples of the curve evolutions and the surfaces constructed by their alternative frame elements are given, and their graphics are drawn.

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \rightarrow E^3$ be a unit-speed curve parametrized by an arc-length s . The Frenet apparatus of α is $\{T(s), N(s), B(s), \kappa(s), \tau(s)\}$ where $T(s) = \alpha'(s)$, $N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$, and $B(s) = T(s) \times N(s)$ are the unit tangent, principal normal, and binormal vector fields of the curve α . The functions $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = \langle N'(s), B(s) \rangle$ are the curvature and torsion of it, respectively.

On the other hand, the alternative moving frame along the curve α is given by $\{N(s), C(s), W(s)\}$ where $N(s)$, $C(s) = \frac{N'(s)}{\|N'(s)\|}$, and $W(s) = \frac{\tau(s)T(s) + \kappa(s)B(s)}{\sqrt{\kappa(s)^2 + \tau(s)^2}}$ are the unit principal normal, the derivative of the principal normal, and the Darboux vector fields, respectively [11]. In addition, the alternative moving frame derivatives are given in matrix form

$$\begin{bmatrix} N \\ C \\ W \end{bmatrix}_s = \begin{bmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & \mu \\ 0 & -\mu & 0 \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix}. \quad (1)$$

Here, for $\eta(s) = \frac{\kappa(s)^2}{(\kappa(s)^2 + \tau(s)^2)^{3/2}} \left(\frac{\tau(s)}{\kappa(s)} \right)'$, the functions

$$\lambda(s) = \sqrt{\kappa(s)^2 + \tau(s)^2} \text{ and } \mu(s) = \eta(s)\lambda(s)$$

are the first and second curvature of α with respect to alternative moving frame, respectively [11].

Therefore, there is the following relationship between the new alternative frame $\{N, C, W\}$ and Frenet frame $\{T, N, B\}$:

$$T(s) = \frac{-\kappa(s)C(s) + \tau(s)W(s)}{\sqrt{\kappa(s)^2 + \tau(s)^2}} \text{ and } B(s) = \frac{\tau(s)C(s) + \kappa(s)W(s)}{\sqrt{\kappa(s)^2 + \tau(s)^2}},$$

and the principal normal vector $N(s)$ is also common in each frame for all $s \in I$.

Lemma 1. Let α be a unit speed curve with the alternative apparatus $\{N, C, W, \lambda, \mu\}$ in Euclidean 3-space, then the following expressions are satisfied:

- i. If the second curvature function $\mu(s)$ vanishes for $\kappa(s) \neq 0$, then the curve α is a helix.
- ii. If the function $\eta(s)$ is a constant, then the curve α is a slant helix [11].

Let $X(s, u, t)$ be a ruled surface evolution in E^3 , then some quantities associated with the ruled surface evolution at any value of t are given below

i. The unit normal vector field is

$$U(s, u, t) = \frac{X_s \times X_u}{\|X_s \times X_u\|}, \quad (2)$$

where the tangent vectors of $X(s, u, t)$ are $X_s = \frac{\partial X}{\partial s}$ and $X_u = \frac{\partial X}{\partial u}$.
ii. The first fundamental form is $I(s, u, t) = Eds^2 + 2Fdsdu + Gdu^2$, where coefficients of I are

$$E(s, u, t) = \langle X_s, X_s \rangle, \quad F(s, u, t) = \langle X_s, X_u \rangle, \quad G(s, u, t) = \langle X_u, X_u \rangle. \quad (3)$$

iii. The second fundamental form is $II(s, u, t) = eds^2 + 2fdsdu + gdu^2$, where coefficients of II are

$$e(s, u, t) = \langle X_{ss}, U \rangle, \quad f(s, u, t) = \langle X_{su}, U \rangle, \quad g(s, u, t) = \langle X_{uu}, U \rangle. \quad (4)$$

iv. The Gaussian and mean curvatures are

$$K(s, u, t) = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H(s, u, t) = \frac{Eg - 2Ef + Ge}{2(EG - F^2)}, \quad (5)$$

respectively.

Also, it is well-known that any surface with vanishing Gaussian curvature is called developable, and with vanishing mean curvature is called minimal [15].

3. Inextensible Flows of Curves with Alternative Frame

In this section, first, we recall the basic concepts about the evolution of a moving curve in Euclidean 3-space E^3 . Then, we give the intrinsic equations expressing its curvatures corresponding with a moving alternative frame $\{N, C, W\}$.

Let α be a moving space curve described in parametric form by a position vector $\alpha(s, t)$ for a time parameter t . Thus, the partial derivatives of the elements of the moving alternative frame $\{N, C, W\}$ of α with respect to t are given in the matrix form as follows

$$\begin{bmatrix} N \\ C \\ W \end{bmatrix}_t = \begin{bmatrix} 0 & f_1 & f_2 \\ -f_1 & 0 & f_3 \\ -f_2 & -f_3 & 0 \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix},$$

where f_1, f_2 , and f_3 are smooth function of s and t . If the curve flow $\alpha(s, t)$ is inextensible, then there exist $N_{st} = N_{ts}$, $C_{st} = C_{ts}$, and $W_{st} = W_{ts}$. Thus, considering the matrices above and (1) gives us the following relationships

$$\begin{cases} \lambda_t C + \lambda C_t = f_{1s} C + f_1 C_s + f_{2s} W + f_2 W_s, \\ \mu_t W + \mu W_t - \lambda_t N - \lambda N_t = f_{3s} W + f_3 W_s - f_{1s} N - f_1 N_s, \\ \mu_t C + \mu C_t = f_{2s} N + f_2 N_s + f_{3s} C + f_3 C_s, \end{cases}$$

such that

$$\begin{cases} f_1 = \frac{\lambda f_3 - f_{2s}}{\mu}, \\ f_2 = \frac{f_{1s} - \lambda_t}{\mu}, \\ f_3 = \frac{\mu f_1 + f_{2s}}{\lambda}. \end{cases}$$

Moreover, the curvatures of the evolving space curve satisfy the relations

$$\begin{cases} \lambda_t = f_{1s} - \mu f_2, \\ \mu_t = \lambda f_2 - f_{3s}. \end{cases}$$

So, we can say that these equations represent the motion of the space curve with the alternative frame.

4. Inextensible Flows of Ruled Surfaces

In this section, we investigate the ruled surface constructed by the evolution of a space curve with the alternative frame. The parametric equation of a ruled surface is presented by

$$X(s, u) = \alpha(s) + ur(s), \quad (6)$$

where $\alpha(s)$ is called the base curve and $r(s)$ is the director curve of the ruled surface. By moving with time t curves $\alpha(s)$ and $r(s)$, the evolution equation of this ruled surface can be written as

$$X(s, u, t) = \alpha(s, t) + ur(s, t).$$

Now, let's give the definition of inextensibility.

Definition 1. Let $X(s, u, t)$ be a surface evolution of the surface $X(s, u)$ given by Equation (6) in E^3 . Then the flow of the surface evolution $X(s, u, t)$ is inextensible if the coefficients E, F , and G of the first fundamental form of $X(s, u, t)$ satisfy

$$\frac{\partial E}{\partial t} = \frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} = 0,$$

for all time t [1].

4.1. Inextensible Flows of N -Ruled Surface

Let X_N be an N -ruled surface which is generated by the motion of the vector N in the alternative frame of the curve α , then the evolution equation of this surface is represented by

$$X_N(s, u, t) = \alpha(s, t) + uN(s, t). \quad (7)$$

So, the subscripts s and u represent partial derivatives of the equation of the surface X_N are easily calculated in the following form

$$\begin{aligned} \frac{\partial X_N(s, u, t)}{\partial s} &= (p + u\lambda)C(s, t) + qW(s, t), \\ \frac{\partial X_N(s, u, t)}{\partial u} &= N(s, t), \end{aligned} \quad (8)$$

where the functions $p = \frac{-\kappa}{\sqrt{\kappa^2 + \tau^2}}$ and $q = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}$ depend on the parameters s and t . By substituting Equation (8) into Equation (2), one can get the normal vector field of the ruled surface X_N given by Equation (7) as

$$U_N(s, u, t) = \frac{qC(s, t) - (p + u\lambda)W(s, t)}{\sqrt{q^2 + (p + u\lambda)^2}}$$

for $q^2 + (p + u\lambda)^2 \neq 0$. On the other hand, by considering Equations (3) and (8), the coefficients E_N, F_N , and G_N of the first fundamental form of the ruled surface X_N are obtained as follows

$$E_N(s, u, t) = q^2 + (p + u\lambda)^2, \quad F_N(s, u, t) = 0, \quad G_N(s, u, t) = 1. \quad (9)$$

Theorem 1. Let $X_N(s, u, t)$ be an N -ruled surface evolution of X_N with the alternative frame $\{N, C, W\}$. The ruled surface evolution $X_N(s, u, t)$ is inextensible provided that

$$\frac{\partial \lambda}{\partial t} = 0, \quad \frac{\partial p}{\partial t} = 0, \quad \text{and} \quad \frac{\partial q}{\partial t} = 0,$$

for all time t .

Proof. Let $X_N(s, u, t)$ be an N -ruled surface evolution of X_N with the alternative frame. Taking the partial derivatives of the coefficients of the first fundamental form given by Equation (9) with respect to t , we get

$$\frac{\partial E_N}{\partial t} = 2q \frac{\partial q}{\partial t} + 2(p + u\lambda) \left(\frac{\partial p}{\partial t} + u \frac{\partial \lambda}{\partial t} \right), \quad \frac{\partial F_N}{\partial t} = 0, \quad \frac{\partial G_N}{\partial t} = 0.$$

If there exists $\frac{\partial \lambda}{\partial t} = \frac{\partial p}{\partial t} = \frac{\partial q}{\partial t} = 0$ (in other words, if the curve α is a circular helix), the conditions given in Definition 1 are satisfied. This completes the proof. $\square$

Theorem 2. Let $X_N(s, u, t)$ be an N -ruled surface evolution of X_N with the alternative frame, then the Gaussian curvature K_N and the mean curvature H_N of X_N are

$$K_N(s, u, t) = -\frac{q^2 \lambda^2}{\left(q^2 + (p + u\lambda)^2 \right)^2}$$

and

$$H_N(s, u, t) = \frac{q \left(-q\mu + \frac{\partial p}{\partial s} + u \frac{\partial \lambda}{\partial s} \right) - (p + u\lambda) \left((p + u\lambda)\mu + \frac{\partial q}{\partial s} \right)}{2 \left(q^2 + (p + u\lambda)^2 \right)^{3/2}},$$

respectively.

Proof. Let $X_N(s, u, t)$ be a N -ruled surface evolution of X_N with the alternative frame. By differentiating Equation (8), one can get

$$\begin{aligned} \frac{\partial^2 X_N(s, u, t)}{\partial s^2} &= -\left(p\lambda + u\lambda^2 \right) N(s, t) + \left(\frac{\partial p}{\partial s} - q\mu + u \frac{\partial \lambda}{\partial s} \right) C(s, t) + \left(\frac{\partial q}{\partial s} + p\mu + u\lambda\mu \right) W(s, t), \\ \frac{\partial^2 X_N(s, u, t)}{\partial s \partial u} &= \lambda C(s, t), \\ \frac{\partial^2 X_N(s, u, t)}{\partial u^2} &= 0. \end{aligned}$$

Considering these last equations together with the principal normal vector field of X_N , the coefficients of the second fundamental form of X_N are given by

$$\begin{aligned} e_N(s, u, t) &= \frac{q \left(-q\mu + \frac{\partial p}{\partial s} + u \frac{\partial \lambda}{\partial s} \right) - (p + u\lambda) \left((p + u\lambda)\mu + \frac{\partial q}{\partial s} \right)}{\sqrt{q^2 + (p + u\lambda)^2}}, \\ f_N(s, u, t) &= \frac{q\lambda}{\sqrt{q^2 + (p + u\lambda)^2}}, \quad g_N(s, u, t) = 0. \end{aligned} \tag{10}$$

By substituting Equations (9) and (10) into Equation (5), one can obtain the Gaussian curvature and the mean curvature of the surface X_N as above. $\square$

Corollary 1. Let $X_N(s, u, t)$ be an N -ruled surface evolution of X_N with the alternative frame, then the surface X_N is

- developable if q vanishes,
- minimal if q vanishes, or $\mu = \lambda_s = 0$ and p, q are constants.

4.2. Inextensible Flows of C-Ruled Surface

Let X_C be a C-ruled surface which is generated by the motion of the vector C of the curve α with the alternative frame, then the evolution equation of this surface is represented by

$$X_C(s, u, t) = \alpha(s, t) + uC(s, t). \quad (11)$$

So, the partial derivatives of the ruled surface X_C with respect to s and u are easily calculated in the following form

$$\begin{aligned} \frac{\partial X_C(s, u, t)}{\partial s} &= -\lambda u N(s, t) + pC(s, t) + (q + u\mu)W(s, t), \\ \frac{\partial X_C(s, u, t)}{\partial u} &= C(s, t), \end{aligned} \quad (12)$$

where the functions p and q depend on the parameters s and t . By considering the partial derivatives of the surface X_C given by Equations (2) and (11), one can get the normal vector field of the ruled surface X_C as

$$U_C(s, u, t) = -\frac{(q + u\mu)N(s, t) + u\lambda W(s, t)}{\sqrt{u^2\lambda^2 + (q + u\mu)^2}}$$

for $u^2\lambda^2 + (q + u\mu)^2 \neq 0$. On the other hand, by considering Equations (3) and (12), the coefficients E_C , F_C , and G_C of the first fundamental form of the ruled surface X_C are obtained as follows

$$E_C(s, u, t) = p^2 + u^2\lambda^2 + (q + u\mu)^2, \quad F_C(s, u, t) = 0, \quad G_C(s, u, t) = 1. \quad (13)$$

Theorem 3. Let $X_C(s, u, t)$ be a C-ruled surface evolution of X_C with the alternative frame $\{N, C, W\}$. The ruled surface evolution $X_C(s, u, t)$ is inextensible if and only if

$$\frac{\partial \lambda}{\partial t} = \frac{\partial \mu}{\partial t} = \frac{\partial p}{\partial t} = \frac{\partial q}{\partial t} = 0 \text{ (or the curve } \alpha \text{ is a circular helix).}$$

Proof. Let $X_C(s, u, t)$ be a C-ruled surface evolution with the alternative frame. Taking the partial derivatives of the coefficients of the first fundamental form given by Equation (13) with respect to t , we have

$$\begin{aligned} \frac{\partial E_C}{\partial t} &= 2p \frac{\partial p}{\partial t} + 2u^2\lambda \frac{\partial \lambda}{\partial t} + 2(q + u\mu) \left(\frac{\partial q}{\partial t} + u \frac{\partial \mu}{\partial t} \right), \\ \frac{\partial F_C}{\partial t} &= \frac{\partial p}{\partial t}, \quad \frac{\partial G_C}{\partial t} = 0. \end{aligned}$$

If there exists $\frac{\partial \lambda}{\partial t} = \frac{\partial \mu}{\partial t} = \frac{\partial p}{\partial t} = \frac{\partial q}{\partial t} = 0$, or if the curve α is a circular helix, the conditions given in Definition 1 are satisfied. Thus, the proof is completed. $\square$

Theorem 4. Let $X_C(s, u, t)$ be a C-ruled surface evolution of X_C with the alternative frame, then the Gaussian curvature K_C and the mean curvature H_C of the surface X_C are

$$K_C(s, u, t) = -\frac{q^2\lambda^2}{(u^2\lambda^2 + (q + u\mu)^2)^2}$$

and

$$H_C(s, u, t) = \frac{u \left((q + u\mu) \frac{\partial \lambda}{\partial s} - \lambda \left(\frac{\partial q}{\partial s} + u \frac{\partial \mu}{\partial s} \right) \right) - pq\lambda}{2(u^2\lambda^2 + (q + u\mu)^2)^{3/2}},$$

respectively.

Proof. Let $X_C(s, u, t)$ be a C-ruled surface evolution of X_C with the alternative frame. By differentiating Equation (12), one can get

$$\begin{aligned} \frac{\partial^2 X_C(s, u, t)}{\partial s^2} &= \left(-p\lambda - u \frac{\partial \lambda}{\partial s} \right) N(s, t) + \left(\frac{\partial p}{\partial s} - u\lambda^2 - q\mu - u\mu^2 \right) C(s, t) \\ &\quad + \left(p\mu + \frac{\partial q}{\partial s} + u \frac{\partial \mu}{\partial s} \right) W(s, t), \\ \frac{\partial^2 X_C(s, u, t)}{\partial s \partial u} &= -\lambda N(s, t) + \mu W(s, t), \\ \frac{\partial^2 X_C(s, u, t)}{\partial u^2} &= 0. \end{aligned}$$

Considering these last equations together with the normal vector field of X_C , the coefficients of the second fundamental form of X_C are calculated as

$$\begin{aligned} e_C(s, u, t) &= \frac{pq\lambda + u \left((q + u\mu) \frac{\partial \lambda}{\partial s} - \lambda \left(\frac{\partial q}{\partial s} + u \frac{\partial \mu}{\partial s} \right) \right)}{\sqrt{u^2\lambda^2 + (q + u\mu)^2}}, \\ f_C(s, u, t) &= \frac{q\lambda}{\sqrt{u^2\lambda^2 + (q + u\mu)^2}}, \quad g_C(s, u, t) = 0. \end{aligned} \quad (14)$$

By substituting Equations (13) and (14) into Equation (4), one can obtain the Gaussian curvature and the mean curvature of the C-ruled surface as above. $\square$

Corollary 2. Let X_C be a C-ruled surface with the alternative frame. The surface X_C is developable and minimal, provided that q vanishes.

4.3. Inextensible Flows of W-Ruled Surface

Let X_W be a W-ruled surface that is generated by the motion of the Darboux vector W of the curve α with the alternative frame, then the evolution equation of this surface is represented by

$$X_W(s, u, t) = \alpha(s, t) + uW(s, t). \quad (15)$$

So, the partial derivatives of the equation of W-ruled surface X_W with respect to s and u are easily calculated in the following form

$$\begin{aligned} \frac{\partial X_W(s, u, t)}{\partial s} &= (p - u\mu)C(s, t) + qW(s, t), \\ \frac{\partial X_W(s, u, t)}{\partial u} &= W(s, t), \end{aligned} \quad (16)$$

where the functions p and q depend on the parameters s and t . By considering the partial derivatives of the surface X_W and Equation (2), one can get the normal vector field of the W-ruled surface as

$$U_W(s, u, t) = N(s, t),$$

where $p > u\mu$. On the other hand, by considering Equation (16), the coefficients E_W, F_W , and G_W of the first fundamental form of the W -ruled surface given by Equation (15) are obtained as follows

$$E_W(s, u, t) = q^2 + (p - u\mu)^2, F_W(s, u, t) = q, G_W(s, u, t) = 1. \quad (17)$$

Theorem 5. Let $X_W(s, u, t)$ be a W -ruled surface evolution of X_W with the alternative frame $\{N, C, W\}$. The W -ruled surface evolution $X_W(s, u, t)$ is inextensible if and only if

$$\frac{\partial p}{\partial t} = \frac{\partial q}{\partial t} = \frac{\partial \mu}{\partial t} = 0, \text{ or the curve } \alpha \text{ is a circular helix.}$$

Proof. Let $X_W(s, u, t)$ be a W -ruled surface evolution of X_W with the alternative frame. Taking the partial derivatives of the coefficients of the first fundamental form given by Equation (17) with respect to t , we have

$$\begin{aligned} \frac{\partial E_W(s, u, t)}{\partial t} &= 2q \frac{\partial q}{\partial t} + 2(p - u\mu) \left(\frac{\partial p}{\partial t} - u \frac{\partial \mu}{\partial t} \right), \\ \frac{\partial F_W(s, u, t)}{\partial t} &= \frac{\partial q}{\partial t}, \quad \frac{\partial G_W(s, u, t)}{\partial t} = 0. \end{aligned}$$

If there exists $\frac{\partial p}{\partial t} = \frac{\partial q}{\partial t} = \frac{\partial \mu}{\partial t} = 0$, or the curve α is a circular helix, the conditions given in Definition 1 are satisfied. Thus, the proof is completed. $\square$

Theorem 6. Let $X_W(s, u, t)$ be a W -ruled surface evolution of X_W with the alternative frame, then the Gaussian curvature K_W and the mean curvature H_W of the W -ruled surface are

$$K_W(s, u, t) = 0$$

and

$$H_W(s, u, t) = \frac{\lambda}{2(u\mu - p)},$$

respectively.

Proof. Let $X_W(s, u, t)$ be a W -ruled surface evolution of X_W with the alternative frame. By differentiating Equation (16), we find that

$$\begin{aligned} \frac{\partial^2 X_W(s, u, t)}{\partial s^2} &= (u\lambda\mu - p\lambda)N(s, t) + \left(p\mu - u\mu^2 + \frac{\partial q}{\partial t} \right)W(s, t) + \left(\frac{\partial p}{\partial t} - u \frac{\partial \mu}{\partial t} - q\mu \right)C(s, t), \\ \frac{\partial^2 X_W(s, u, t)}{\partial s \partial u} &= -\mu C(s, t), \\ \frac{\partial^2 X_W(s, u, t)}{\partial u^2} &= 0. \end{aligned}$$

Considering these last equations together with the normal vector field of X_W , the coefficients of the second fundamental form of X_W are calculated by

$$e_W(s, u, t) = \lambda(u\mu - p), f_W(s, u, t) = 0, g_W(s, u, t) = 0. \quad (18)$$

By substituting Equations (17) and (18) into Equation (5), one can obtain the Gaussian curvature and the mean curvature of the W -ruled surface as above. $\square$

Corollary 3. Let X_W be a W -ruled surface with the alternative frame, then the surface X_W is

- developable,
- not minimal.

Example 1. Let's take a curve defined in parametric form

$$\alpha(s) = (s - 2 \tanh s, -2 \operatorname{sech} s, -2 \operatorname{sech} s), \quad (19)$$

see Figure 1. Let its evolution be given in the form

$$\alpha(s, t) = (s - 2 \tanh s, -2 \operatorname{sech} s \cos t, -2 \operatorname{sech} s \sin t). \quad (20)$$

see Figure 2. The Frenet vectors and the curvatures of the curve evolution $\alpha(s, t)$ at any time t are

$$\begin{aligned} T(s, t) &= (1 - 2 \operatorname{sech}^2 s, 2 \cos t \operatorname{sech} s \tanh s, 2 \operatorname{sech} s \sin t \tanh s), \\ N(s, t) &= (2 \operatorname{sech} s \tanh s, \cos t (2 - \cosh^2 s) \operatorname{sech} s^2, (2 - \cosh^2 s) \operatorname{sech} s^2 \sin t), \\ B(s, t) &= (0, \sin t, -\cos t), \quad \kappa(s, t) = 2 \operatorname{sech} s, \quad \tau(s, t) = 0. \end{aligned}$$

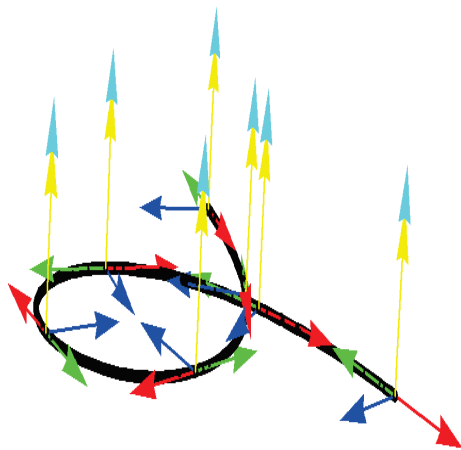


Figure 1. The tangent vectors (green), principal normal vectors (blue), binormal vectors (cyan), derivative of the principal normal vectors (red), and Darboux vectors (yellow) of $\alpha(s)$ given by Equation (19) where $s \in (-3, 3)$.

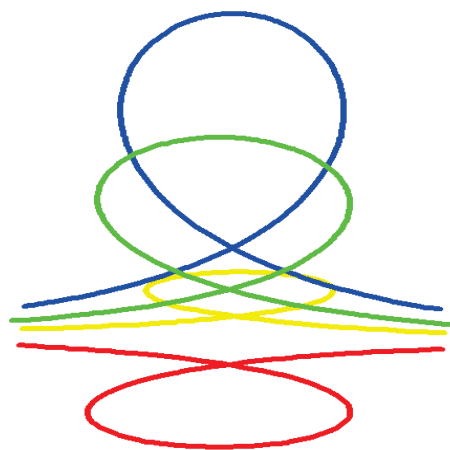


Figure 2. The curve evolution $\alpha(s, t)$ given by Equation (20) for $t = 0$ (red), $t = \pi/4$ (green), $t = \pi/2$ (blue), and $t = \pi$ (yellow) where $s \in (-3, 3)$.

The alternative frame vectors and the alternative curvatures of $\alpha(s, t)$ are

$$\begin{aligned} N(s, t) &= \left(2\operatorname{sech} s \tanh s, \operatorname{sech} s^2 (2 - \cosh s^2) \cos t, \operatorname{sech} s^2 (2 - \cosh s^2) \sin t \right), \\ C(s, t) &= \left(\frac{1}{2} (3 - \cosh 2s) \operatorname{sech} s^2, -2 \operatorname{sech} s \tanh s \cos t, -2 \operatorname{sech} s \tanh s \sin t \right), \\ W(s, t) &= (0, \sin t, -\cos t), \quad \lambda(s, t) = 2\operatorname{sech} s, \quad \mu(s, t) = 0. \end{aligned}$$

Thus, the evolution equations of the ruled surfaces, which are generated by the motion of the vector N, C , and W along the curve α , are represented by

$$\begin{cases} X_N(s, u, t) = \begin{pmatrix} s - 2(1 - u \operatorname{sech} s) \tanh s, \cos t (u - 2 \operatorname{sech} s (1 - u \operatorname{sech} s)), \\ (u - 2 \operatorname{sech} s (1 - u \operatorname{sech} s)) \sin t \end{pmatrix}, \\ X_C(s, u, t) = \begin{pmatrix} s - u + 2u \operatorname{sech} s^2 - 2 \tanh s, -2 \cos t \operatorname{sech} s (1 + u \tanh s), \\ -2 \operatorname{sech} s \sin t (1 + u \tanh s) \end{pmatrix}, \\ X_W(s, u, t) = (s - 2 \tanh s, -2 \cos t \operatorname{sech} s + u \sin t, -u \cos t - 2 \operatorname{sech} s \sin t), \end{cases} \quad (21)$$

see Figure 3, Figure 4, and Figure 5, respectively.

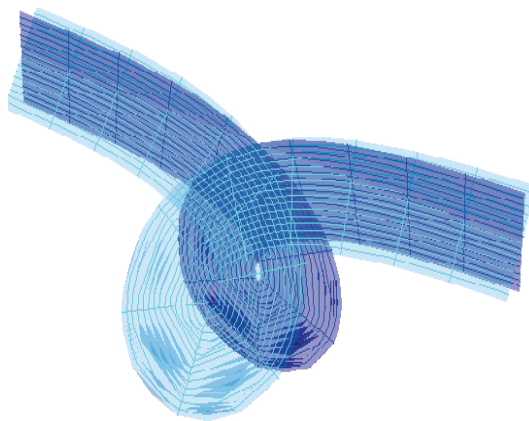


Figure 3. The surface evolution $X_N(s, u, t)$ given in Equations (21) for $t = \pi/4$ (cyan) and $t = \pi/2$ (blue) where $s \in (-5, 5)$ and $u \in (-0.5, 0.5)$.



Figure 4. The surface evolution $X_C(s, u, t)$ given in Equations (21) for $t = \pi/4$ (red) and $t = \pi/2$ (yellow) where $s \in (-5, 5)$ and $u \in (-2, 2)$.

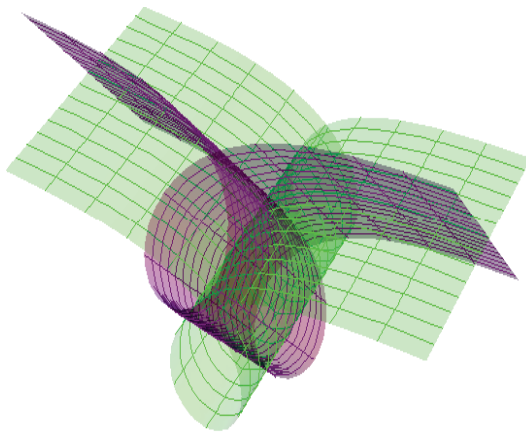


Figure 5. The surface evolution $X_W(s, u, t)$ given in Equations (21) for $t = \pi/4$ (purple) and $t = \pi/2$ (green) where $s \in (-5, 5)$ and $u \in (-2, 2)$.

Thus, one can get the normal vector fields of the ruled surfaces as:

$$\begin{cases} U_N(s, u, t) = \left(0, \frac{(\cosh s - 2u)\operatorname{sech} s \sin t}{\sqrt{(1 - 2u\operatorname{sech} s)^2}}, \frac{\cos t(2u\operatorname{sech} s - 1)}{\sqrt{(1 - 2u\operatorname{sech} s)^2}} \right), & u\operatorname{sech} s \neq \frac{1}{2}, \\ U_C(s, u, t) = (0, -\sin t, \cos t), \\ U_W(s, u, t) = \left(-2\operatorname{sech} s \tanh s, \frac{\cos t(\cosh 2s - 3)\operatorname{sech} s^2}{2}, \frac{(\cosh 2s - 3)\operatorname{sech} s^2 \sin t}{2} \right). \end{cases}$$

The coefficients of the first fundamental form of the ruled surfaces are obtained as follows

$$\begin{cases} E_N(s, u, t) = (1 - 2u\operatorname{sech} s)^2, & F_N(s, u, t) = 0, & G_N(s, u, t) = 1, \\ E_C(s, u, t) = 1 + 4u^2\operatorname{sech} s^2, & F_C(s, u, t) = -1, & G_C(s, u, t) = 1, \\ E_W(s, u, t) = 1, & F_W(s, u, t) = 0, & G_W(s, u, t) = 1. \end{cases}$$

Taking the partial derivatives of the coefficients of the first fundamental form of X_N , X_C , and X_W with respect to t , we get

$$\begin{cases} \frac{\partial E_N}{\partial t} = \frac{\partial F_N}{\partial t} = \frac{\partial G_N}{\partial t} = 0, \\ \frac{\partial E_C}{\partial t} = \frac{\partial F_C}{\partial t} = \frac{\partial G_C}{\partial t} = 0, \\ \frac{\partial E_W}{\partial t} = \frac{\partial F_W}{\partial t} = \frac{\partial G_W}{\partial t} = 0. \end{cases}$$

Therefore, it is said that the ruled surface evolutions are inextensible. The Gaussian curvatures and the mean curvatures of X_N , X_C and X_W are

$$\begin{cases} K_N(s, u, t) = 0, & H_N(s, u, t) = 0, \\ K_C(s, u, t) = 0, & H_C(s, u, t) = 0, \\ K_W(s, u, t) = 0, & H_W(s, u, t) = -\operatorname{sech} s. \end{cases}$$

Consequently, the surface evolutions

- X_N , X_C , and X_W are developable,
- X_N and X_C are minimal but X_W is not minimal.

Example 2. Let us consider a curve defined in parametric form,

$$\begin{aligned}\beta : (-\pi, \pi) \subset \mathbb{R} &\rightarrow E^3 \\ s &\rightarrow \beta(s) = \frac{1}{\sqrt{2}}(\sin s, \cos s, s),\end{aligned}\quad (22)$$

see Figure 6, and its evolution is defined in the form

$$\begin{aligned}\beta : (-\pi, \pi) \times (0, w) &\rightarrow E^3 \\ (s, t) &\rightarrow \beta(s, t) = \frac{1}{\sqrt{2}}\left(\frac{\sin s(t+1)}{t+1}, \frac{\cos s(t+1)}{t+1}, s\right),\end{aligned}\quad (23)$$

see Figure 7. The Frenet vectors and the curvatures of the curve evolution $\beta(s, t)$ at any time t are found as

$$\begin{aligned}T(s, t) &= \left(\frac{\cos s(1+t)}{\sqrt{2}}, -\frac{\sin s(1+t)}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \\ N(s, t) &= (-\sin s(1+t), -\cos s(1+t), 0), \\ B(s, t) &= \left(\frac{\cos s(1+t)}{\sqrt{2}}, -\frac{\sin s(1+t)}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \\ \kappa(s, t) &= \frac{1+t}{\sqrt{2}}, \quad \tau(s, t) = -\frac{1+t}{\sqrt{2}},\end{aligned}$$

and the alternative frame vectors and the alternative curvatures of $\beta(s, t)$ are

$$\begin{aligned}N(s, t) &= (-\sin s(1+t), -\cos s(1+t), 0), \\ C(s, t) &= (-\cos s(1+t), \sin s(1+t), 0), \\ W(s, t) &= (0, 0, -1), \\ \lambda(s, t) &= 1+t, \quad \mu(s, t) = 0.\end{aligned}$$

Thus, the evolution equations of the ruled surfaces, which are generated by the motion of the vectors N , C , and W in the alternative frame of the curve β , are represented by

$$\begin{cases} Y_N(s, u, t) = \left(\frac{(\sqrt{2}-2(1+t)u)\sin s(1+t)}{2(1+t)}, \frac{(\sqrt{2}-2(1+t)u)\cos s(1+t)}{2(1+t)}, \frac{s}{\sqrt{2}}\right), \\ Y_C(s, u, t) = \left(\frac{\sin s(1+t)}{\sqrt{2}(1+t)} - u\cos s(1+t), \frac{\cos s(1+t)}{\sqrt{2}(1+t)} + u\sin s(1+t), \frac{s}{\sqrt{2}}\right), \\ Y_W(s, u, t) = \left(\frac{\sin s(1+t)}{\sqrt{2}(1+t)}, \frac{\cos s(1+t)}{\sqrt{2}(1+t)}, \frac{s}{\sqrt{2}} - u\right), \end{cases}\quad (24)$$

see Figure 8, Figure 9, and Figure 10, respectively.

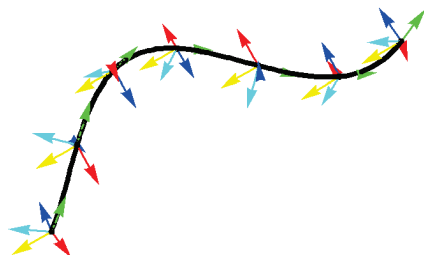


Figure 6. The tangent vectors (green), principal normal vectors (blue), binormal vectors (cyan), derivative of the principal normal vectors (red) and Darboux vectors (yellow) of $\beta(s)$ given by Equation (22) where $s \in (-\pi, \pi)$.

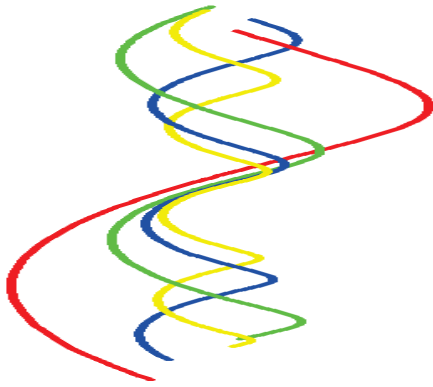


Figure 7. The curve evolution $\beta(s, t)$ given by Equation (23) for $t = 0$ (red), $t = 1$ (green), $t = 2$ (blue), $t = 3$ (yellow) where $s \in (-\pi, \pi)$.

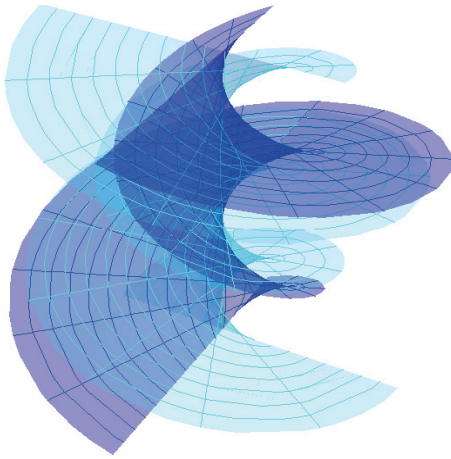


Figure 8. The surface evolution $Y_N(s, u, t)$ given by Equations (24) for $t = \pi/4$ (blue) and $t = \pi/2$ (cyan) where $s \in (-\pi/2, \pi/2)$ and $u \in (-1, 1)$.

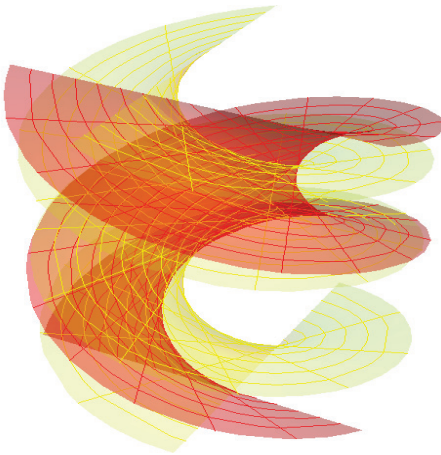


Figure 9. The surface evolution $Y_C(s, u, t)$ given by Equations (24) for $t = \pi/4$ (red) and $t = \pi/2$ (yellow) where $s \in (-\pi/2, \pi/2)$ and $u \in (-1, 1)$.

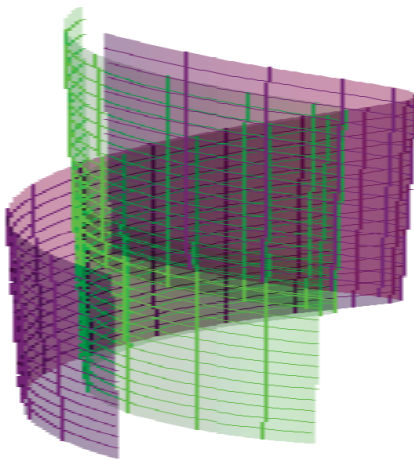


Figure 10. The surface evolution $T_W(s, u, t)$ given by Equations (24) for $t = \pi/4$ (purple) and $t = \pi/2$ (green) with $s \in (-\pi/2, \pi/2)$ and $u \in (-1, 1)$.

The equations of the normal vector fields of the ruled surfaces are found

$$\left\{ \begin{array}{l} U_N(s, u, t) = \left(\frac{\sqrt{2} \cos s(1+t)}{2\sqrt{1+(1+t)((1+t)u - \sqrt{2})u}}, \frac{-\sqrt{2} \sin s(1+t)}{2\sqrt{1+(1+t)((1+t)u - \sqrt{2})u}}, \right. \\ \left. \frac{\sqrt{2}u(1+t) - 1}{2\sqrt{1+(1+t)((1+t)u - \sqrt{2})u}} \right), \\ U_C(s, u, t) = \left(-\frac{\sin s(1+t)}{\sqrt{1+2(1+t)^2u^2}}, -\frac{\cos s(1+t)}{\sqrt{1+2(1+t)^2u^2}}, \frac{\sqrt{2}(1+t)u}{\sqrt{1+2(1+t)^2u^2}} \right), \\ U_W(s, u, t) = (\sin s(1+t), \cos s(1+t), 0), \\ 1 + (1+t)((1+t)u - \sqrt{2})u \neq 0. \end{array} \right.$$

The coefficients of the first fundamental form of the ruled surfaces are obtained as follows

$$\left\{ \begin{array}{l} E_N(s, u, t) = 1 + (1+t)((1+t)u - \sqrt{2})u, \quad F_N(s, u, t) = 0, \quad G_N(s, u, t) = 1, \\ E_C(s, u, t) = 1 + (1+t)^2u^2, \quad F_C(s, u, t) = -\frac{1}{\sqrt{2}}, \quad G_C(s, u, t) = 1, \\ E_W(s, u, t) = 1, \quad F_W(s, u, t) = -\frac{1}{\sqrt{2}}, \quad G_W(s, u, t) = 1. \end{array} \right.$$

Taking the partial derivatives of the coefficients of the first fundamental form of Y_N , Y_C , and Y_W with respect to t , we get

$$\left\{ \begin{array}{l} \frac{\partial E_N}{\partial t} = (2(1+t)u - \sqrt{2})u, \quad \frac{\partial F_N}{\partial t} = \frac{\partial G_N}{\partial t} = 0, \\ \frac{\partial E_C}{\partial t} = 2(1+t)u^2, \quad \frac{\partial F_C}{\partial t} = \frac{\partial G_C}{\partial t} = 0, \\ \frac{\partial E_W}{\partial t} = \frac{\partial F_W}{\partial t} = \frac{\partial G_W}{\partial t} = 0. \end{array} \right.$$

So, it is said that the ruled surface Y_W is inextensible, but the ruled surface Y_N and Y_C are not inextensible.

The Gaussian curvatures and the mean curvatures of Y_N , Y_C , and Y_W are

$$\begin{cases} K_N(s, u, t) = -\frac{(1+t)^2}{2\left(1 + (1+t)\left(-\sqrt{2} + (1+t)u\right)u\right)^2}, & H_N(s, u, t) = 0, \\ K_C(s, u, t) = -\frac{2(1+t)^2}{\left(1 + 2(1+t)^2u^2\right)^2}, & H_C(s, u, t) = -\frac{1+t}{\sqrt{2}\left(1 + 2(1+t)^2u^2\right)^{3/2}}, \\ K_W(s, u, t) = 0, & H_W(s, u, t) = -\frac{1+t}{\sqrt{2}}. \end{cases}$$

As a result, the surface

- Y_N is not developable but minimal,
- Y_C is not both developable and minimal,
- Y_W is developable but not minimal.

5. Conclusions

This paper presents the evolution of ruled surfaces formed by the vector fields in the alternative frame, such as the principal normal vector, its derivative, and the Darboux vector. Since analyzing the Darboux vector along a curve allows us to determine how a curve is rotating as it moves through space, the obtained results may have a crucial role in various fields like robotics, biomechanics, and mechanical engineering, where understanding the rotational behavior of objects is essential. Therefore, as the first step of this study, the evolution of a space curve is expressed with an alternative frame. Then, the conditions for these ruled surfaces to be minimal, developable, and inextensible are investigated. The evolution of these surfaces has not been investigated so far. It is thought that this subject may have important applications as well as in the aforementioned areas and in the field of physics.

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Article

Exploring Conformal Soliton Structures in Tangent Bundles with Ricci-Quarter Symmetric Metric Connections

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Abstract: In this study, we investigate the tangent bundle TM of an n -dimensional (pseudo-)Riemannian manifold M equipped with a Ricci-quarter symmetric metric connection $\hat{\nabla}$. Our primary goal is to establish the necessary and sufficient conditions for TM to exhibit characteristics of various solitons, specifically conformal Yamabe solitons, gradient conformal Yamabe solitons, conformal Ricci solitons, and gradient conformal Ricci solitons. We determine that for TM to be a conformal Yamabe soliton, the potential vector field must satisfy certain conditions when lifted vertically, horizontally, or completely from M to TM , alongside specific constraints on the conformal factor λ and the geometric properties of M . For gradient conformal Yamabe solitons, the conditions involve λ and the Hessian of the potential function. Similarly, for TM to be a conformal Ricci soliton, we identify conditions involving the lift of the potential vector field, the value of λ , and the curvature properties of M . For gradient conformal Ricci solitons, the criteria include the Hessian of the potential function and the Ricci curvature of M . These results enhance the understanding of the geometric properties of tangent bundles under Ricci-quarter symmetric metric connections and provide insights into their transition into various soliton states, contributing significantly to the field of differential geometry.

Keywords: complete lift metric; Ricci-quarter symmetric metric connection; tangent bundle; conformal Yamabe soliton; gradient conformal Yamabe soliton; conformal Ricci soliton; gradient conformal Ricci soliton

MSC: 53C07; 53A45

1. Introduction

Golab [1] introduced the notion of a quarter-symmetric connection, defined by a linear connection ∇ on a differentiable manifold (M, g) of dimension n . The torsion tensor T^∇ associated with this connection must satisfy a certain criterion:

$$T^\nabla(\xi_1, \xi_2) = \eta(\xi_2)\phi\xi_1 - \eta(\xi_1)\phi\xi_2. \quad (1)$$

In this context, η represents a non-zero 1-form, ϕ denotes a $(1, 1)$ -tensor, and ξ_i ($i = 1, 2$) represent vector fields. The most comprehensive form of quarter-symmetric metric connections on Riemannian, Hermitian and Kaehlerian manifolds was introduced by Yano and Imai [2]. In particular, when the $(1, 1)$ -tensor ϕ coincides with the identity tensor ($\phi = id$), the quarter-symmetric connection simplifies to a semi-symmetric connection. Friedmann and Schouten [3] were the pioneers in introducing the notion of a semi-symmetric linear connection on a differentiable manifold. The form of the semi-symmetric metric connection that is currently known was first proposed by Yano [4], who employed Hayden's method [5]. Consequently, one can perceive a quarter-symmetric connection as an expansion of the notion of a semi-symmetric connection. Fundamentally, a quarter-symmetric

metric connection aligns with a Hayden connection, distinguished by its torsion tensor properties (1).

When the tensor ϕ is specified as a $(1, 1)$ type Ricci tensor, expressed as

$$g(\phi\tilde{\xi}_1, \tilde{\xi}_2) = R(\tilde{\xi}_1, \tilde{\xi}_2),$$

the resulting quarter-symmetric connection is termed a Ricci quarter-symmetric connection. Such a connection, denoted as ∇ , on a Riemannian manifold is classified as a Ricci quarter-symmetric metric connection (abbreviated as RQSMC) if it adheres to the condition:

$$(\nabla_{\tilde{\xi}_1}g)(\tilde{\xi}_2, \tilde{\xi}_3) = 0$$

for all vector fields $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3$ on M . Kamilya and De introduced the concept of a RQSMC on a Riemannian manifold and determined the required conditions for the symmetry of the Ricci tensor of a RQSMC [6].

The Ricci flow transforms the metric g (i.e., the shape of the manifold M) into a metric of appropriate constant curvature (i.e., spherical) proportional to the Ricci tensor. In differential geometry, self-similar solutions of the Ricci flow are known as Ricci soliton. On a Riemannian manifold (M, g) , a smooth vector field V is considered to define a Ricci soliton if it satisfies the equation:

$$\frac{1}{2}L_Vg + Ric = \lambda g,$$

where L_Vg denotes the Lie derivative of the Riemannian metric g according to V and λ is a constant. A Ricci soliton is represented by the triple (g, V, λ) , with its classification as either shrinking, steady, or expanding whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

The Yamabe flow, introduced by Hamilton [7] shortly after the Ricci flow addresses the Yamabe problem—a quest to discover a metric on a given compact Riemannian manifold (M, g) of dimension $n > 3$ that preserves the original metric g while upholding a constant scalar curvature. An intriguing aspect of this flow is the concept of a Yamabe soliton, elucidated by Barbosa and Ribeiro [8], which represents a solution showcasing self-similar behavior on the manifold (M, g) .

A Yamabe soliton, defined on a Riemannian or pseudo-Riemannian manifold (M, g) , is characterized by the equation

$$\frac{1}{2}L_Vg = (r - \lambda)g,$$

where r signifies the scalar curvature of (M, g) . The soliton is categorized as shrinking, steady, or expanding based on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

In 2005, Fischer [9] introduced a novel concept termed conformal Ricci flow, inspired by Hamilton's pioneering work on Ricci and Yamabe flows [7]. This approach, a modification of the traditional Ricci flow equation, replaces the unit volume restriction with a scalar curvature constraint. The resulting equations, known as the conformal Ricci flow equations, emerge from a blend of conformal flow and Ricci flow equations in terms of vector fields.

These equations, expressed as

$$\frac{\partial g}{\partial t} + 2\left(Ric(g) + \frac{1}{n}g\right) = -\rho g \quad \text{with } r(g) = 1,$$

where Ric denotes the Ricci tensor of M , r denotes the scalar curvature of the metric g and ρ is a scalar non-dynamical field, exhibit a notable resemblance to the Navier–Stokes equations from fluid mechanics:

$$\frac{\partial v}{\partial t} + \nabla_v v + v\Delta v = -grad\rho \quad divv = 0$$

is analogous to the Navier–Stokes equations in fluid mechanics. In this context, the time-dependent scalar field ρ , termed conformal pressure, plays a pivotal role, akin to real physical pressure, in ensuring the incompressibility of the fluid. Similarly, the conformal pressure serves as a Lagrange multiplier, conformally deforming the metric flow in order to maintain the scalar curvature constraint.

In their initial research contribution, Gezer and Karakas [10] introduced the innovative concept of an RQSMC $\tilde{\nabla}$ on the tangent bundle TM , in conjunction with the complete lift metric ${}^Cg$ defined on the Riemannian manifold M . This seminal work delved into an in-depth exploration of various curvature tensors and their associated properties under the framework of $\tilde{\nabla}$, accompanied by the introduction of the mean connection to enhance the understanding of the geometric structures involved. The central focus of their research was the clarification of the fundamental conditions that govern the transformation of the tangent bundle TM into a Ricci soliton, gradient Ricci soliton, and the manifestation of local conformal flatness. This elucidated the manifold's dynamic behavior in response to the RQSMC $\tilde{\nabla}$.

Subsequently, in a follow-up research endeavor, Gezer and Karakas [11] undertook a classification of distinct special vector fields on TM based on the RQSMC $\tilde{\nabla}$. This encompassed a diverse range of vector fields, including incompressible vector fields, harmonic vector fields, concurrent vector fields, conformal vector fields, projective vector fields and $\tilde{\varphi}(\text{Ric})$ vector fields on TM . This comprehensive categorization provided the foundation for the delineation of the intricate interactions and behaviors of these vector fields within the tangent bundle structure. Furthermore, their investigation extended to defining the criteria under which the tangent bundle TM can function as a Riemannian soliton and a generalized Ricci–Yamabe soliton under the influence of the connection $\tilde{\nabla}$. This unlocked further insights into the manifold's soliton dynamics.

The overarching objective of their collective body of research work is to establish the fundamental conditions that dictate the transition of the tangent bundle TM into a diverse array of solitons, including the manifestation of a conformal Yamabe soliton, gradient conformal Yamabe soliton, conformal Ricci soliton, and gradient conformal Ricci soliton relative to the RQSMC $\tilde{\nabla}$. This comprehensive exploration not only expands the existing knowledge base concerning the intricate geometrical properties of the tangent bundle TM but also provides insight into the transition of the tangent bundle into a diverse array of solitons, including the manifestation of a conformal Yamabe soliton, gradient conformal Yamabe soliton, conformal Ricci soliton, and gradient conformal Ricci soliton relative to the RQSMC $\tilde{\nabla}$. While both this paper and the paper [11] deal with soliton theory and vector fields on TM under a Ricci quarter-symmetric metric connection $\tilde{\nabla}$, they differ in their primary focus: This paper focuses on conditions for specific types of solitons (conformal Yamabe solitons, gradient conformal Yamabe solitons, conformal Ricci solitons, and gradient conformal Ricci solitons) under $\tilde{\nabla}$, while the paper [11] focuses primarily on specific types of vector fields (incompressible, harmonic, etc.) and establish conditions for solitons such as Riemannian solitons and generalized Ricci–Yamabe solitons within this framework. The motivation behind considering the article is to contribute new insights into soliton theory under Ricci quarter-symmetric metric connections, providing a rigorous framework to explore and understand the emergence and properties of solitons on tangent bundles TM of (pseudo-)Riemannian manifolds M . By addressing these gaps, the article aimed to advance theoretical knowledge and potentially open avenues for future research and applications in mathematical and physical sciences.

2. Preliminaries

2.1. The Tangent Bundle

Consider an n -dimensional differentiable manifold M . The tangent bundle TM of the manifold M is defined as:

$$TM = \bigcup_{p \in M} T_p M$$

where $T_P M$ denotes the tangent space of M at P . Let us choose a local coordinate system $\{U, x^h\}$ within M , and use Cartesian coordinates (y^h) in each tangent space $T_P M$ at a point $P \in M$. These Cartesian coordinates are established using the natural basis $\left\{\frac{\partial}{\partial x^h} \Big|_P\right\}$. With this setup, we can define a local coordinate system in TM denoted as $\{\pi^{-1}(U), x^h, y^h\}$. Here, π represents the natural projection function defined as: $\pi : TM \mapsto M$, and P stands for an arbitrary point belonging to U . Moreover, the coordinate system (x^h, y^h) is referred to as the induced coordinates on $\pi^{-1}(U)$, which comes from the original coordinate system $\{U, x^h\}$ within M .

Now, consider a vector field ξ defined within the open subset U of M . This can be locally expressed as $\xi = \xi^h \frac{\partial}{\partial x^h}$. Given that ∇ is a torsion-free linear connection on M , we are able to provide the following geometric objects without delay.

1. The vertical lift ${}^V \xi$, the horizontal lift ${}^H \xi$ and the complete lift ${}^C \xi$ are, respectively, as follows:

$${}^V \xi = \xi^h \partial_{\bar{h}}, \quad {}^H \xi = \xi^h \partial_h - y^s \Gamma_{sk}^h \xi^k \partial_{\bar{h}} \quad \text{and} \quad {}^C \xi = \xi^h \partial_h + y^s \partial_s \xi^h \partial_{\bar{h}}.$$

2. The adapted frame $\{E_\beta\} = \{E_j, E_{\bar{j}}\}$ on the tangent bundle is given by:

$$E_j = \partial_j - y^s \Gamma_{sj}^h \partial_{\bar{h}}, \quad E_{\bar{j}} = \partial_{\bar{j}}.$$

The expressions of the lifts of a vector field ξ with regards to the adapted frame immediately follow:

$${}^V \xi = \xi^j E_{\bar{j}}, \quad {}^H \xi = \xi^j E_j, \quad {}^C \xi = \xi^j E_j + y^s \nabla_s \xi^j E_{\bar{j}}. \quad (2)$$

3. The following is the complete lift metric ${}^C g$ with respect to the adapted frame:

$${}^C g_{\alpha\beta} = \begin{pmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{pmatrix}.$$

For the fundamental concepts and details, we refer to [12,13].

2.2. The RQSMC on $(TM, {}^C g)$

The RQSMC $\tilde{\nabla}$ functions within the tangent bundle TM of a (pseudo-)Riemannian manifold (M, g) , where it is defined with respect to the complete lift metric ${}^C g$. This metric, derived from the underlying metric g on M , serves as the basis for establishing geometric relationships and structures within the tangent space. In our previous investigation detailed in [10], we delve into the intricacies of the RQSMC $\tilde{\nabla}$ and the associated curvature tensors. By representing the components of these entities as Γ_{ij}^k and R_{sij}^k , respectively, we elucidate the behavior of the RQSMC $\tilde{\nabla}$ and its associated curvature tensors in a structured manner.

1. The RQSMC $\tilde{\nabla}$ on TM :

$$\begin{cases} \tilde{\nabla}_{E_i} E_j = \Gamma_{ij}^k E_k + \{y^s R_{sij}^k + y_j R_i^k - y^k R_{ij}^s\} E_{\bar{k}}, \\ \tilde{\nabla}_{E_i} E_{\bar{j}} = \Gamma_{ij}^k E_{\bar{k}}, \\ \tilde{\nabla}_{E_{\bar{i}}} E_j = 0, \quad \tilde{\nabla}_{E_{\bar{i}}} E_{\bar{j}} = 0 \end{cases}$$

2. The curvature tensor $\tilde{R}$ of the RQSMC $\tilde{\nabla}$:

$$\begin{aligned}
 \tilde{R}(E_i, E_j)E_k &= R_{ijk}^l E_l + \{y^s \nabla_s R_{ijk}^l\} E_{\bar{l}}, \\
 \tilde{R}(E_i, E_j)E_{\bar{k}} &= R_{ijk}^l E_{\bar{l}}, \\
 \tilde{R}(E_i, E_{\bar{j}})E_k &= \{R_{ijk}^l + R_{ik}\delta_j^l - g_{jk}R_i^l\} E_{\bar{l}}, \\
 \tilde{R}(E_{\bar{i}}, E_j)E_k &= \{R_{ijk}^l + g_{ik}R_j^l - R_{jk}\delta_i^l\} E_{\bar{l}}, \\
 \tilde{R}(E_{\bar{i}}, E_{\bar{j}})E_k &= 0, \quad \tilde{R}(E_{\bar{i}}, E_j)E_{\bar{k}} = 0, \\
 \tilde{R}(E_i, E_{\bar{j}})E_{\bar{k}} &= 0, \quad \tilde{R}(E_{\bar{i}}, E_{\bar{j}})E_{\bar{k}} = 0
 \end{aligned}$$

with respect to the adapted frame $\{E_\beta\}$. The Ricci tensor $\tilde{K}$ of the RQSMC $\tilde{\nabla}$:

$$\begin{aligned}
 \tilde{K}_{jk} &= (3-n)R_{jk}, \quad \tilde{K}_{\bar{j}\bar{k}} = 0, \\
 \tilde{K}_{\bar{j}k} &= 0, \quad \tilde{K}_{j\bar{k}} = 0.
 \end{aligned} \tag{3}$$

The scalar curvature $\tilde{r}$ of the RQSMC $\tilde{\nabla}$ in relation to ${}^Cg$ is established as zero.

A vector field $\tilde{V} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ on TM within the adapted frame $\{E_\beta\}$ is identified as a fiber-preserving vector field if the components v^h solely depend on the variables (x^h) . The Lie derivative of the fiber-preserving vector field $\tilde{V}$ with respect to the RQSMC $\tilde{\nabla}$ is expressed to be used in forthcoming proofs [11]:

$$\left\{ \begin{array}{l} (i) \quad L_{\tilde{V}} \tilde{g}_{ij} = \left(\nabla_i v^h \right) g_{hj} + \left(E_{\bar{j}} v^{\bar{h}} \right) g_{hi} \\ (ii) \quad L_{\tilde{V}} \tilde{g}_{ij} = \left(E_{\bar{i}} v^{\bar{h}} \right) g_{hj} + \left(\nabla_j v^h \right) g_{hi} \\ (iii) \quad L_{\tilde{V}} \tilde{g}_{ij} = \left[E_i v^{\bar{h}} + \left(y^s R_{sia}^h + y_a R_i^h - y^h R_{ia} \right) v^a + \Gamma_{ia}^h v^{\bar{a}} \right] g_{hj} \\ \quad + \left[E_j v^{\bar{h}} + \left(y^s R_{sja}^h + y_a R_j^h - y^h R_{ja} \right) v^a + \Gamma_{ja}^h v^{\bar{a}} \right] g_{hi}. \end{array} \right. \tag{4}$$

3. Conformal Yamabe Soliton Structure on $(TM, {}^Cg)$ in Relation to RQSMC $\tilde{\nabla}$

In recent years, geometric flows and associated solitons have attracted the attention of many geometers. In 2021, Roy, Dey and Bhattacharyya [14] developed the idea of Conformal Yamabe solitons, defined on an n -dimensional Riemannian manifold as follows:

$$L_V g + \left[2\lambda - 2r - \left(\rho + \frac{2}{n} \right) \right] g = 0. \tag{5}$$

The classification of a conformal Yamabe soliton into shrinking, steady, or expanding categories is determined by the value of the constant λ . Specifically, a soliton is considered shrinking if $\lambda > 0$, steady if $\lambda = 0$, and expanding if $\lambda < 0$. In this context, the scalar curvature r , a scalar non-dynamical field ρ (which is time-dependent), and the Lie derivative $L_V g$ of the metric g along the vector field V are all key components.

Theorem 1. When TM is regarded as the tangent bundle of an n -dimensional (pseudo-)Riemannian manifold M equipped with the RQSMC $\tilde{\nabla}$, for the triplet $({}^Cg, \tilde{V}, \lambda)$ to be a conformal Yamabe soliton, it is necessary and sufficient that the specified conditions need to be satisfied:

$$\begin{aligned}
 (i) \quad {}^V V &= (v^a, v^{\bar{a}}) = (0, v^{\bar{a}}), \\
 (ii) \quad \lambda &= \frac{1}{2} \left(\rho + \frac{1}{n} \right), \\
 (iii) \quad V &\text{ is a Killing vector field on } M.
 \end{aligned}$$

In here, the potential vector field is constructed as the vertical lift ${}^V V$ of a vector field V defined on M to the tangent bundle TM .

Proof. Starting from the relation (5) we write

$$L_{\tilde{V}}\tilde{g}_{\varepsilon\beta} + [2\lambda - 2\tilde{r} - (\rho + \frac{2}{2n})]\tilde{g}_{\varepsilon\beta} = 0. \quad (6)$$

Consider $(\varepsilon, \beta) = (i, \bar{j})$ from the previous equation:

$$L_{\tilde{V}}\tilde{g}_{i\bar{j}} + [2\lambda - (\rho + \frac{2}{2n})]\tilde{g}_{i\bar{j}} = 0.$$

Based on the formulation of $L_{\tilde{V}}\tilde{g}$ in (4), we have

$$(\nabla_i v^h)g_{hj} + (E_{\bar{j}}v^{\bar{h}})g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Consider the potential vector field $\tilde{V}$ as the vertical lift ${}^V V = (v^a, v^{\bar{a}}) = (0, v^a)$ of a vector field V on M , then

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}).$$

When we consider $(\varepsilon, \beta) = (i, j)$ in (6), we obtain

$$L_{\tilde{V}}\tilde{g}_{ij} + [2\lambda - 2\tilde{r} - (\rho + \frac{2}{2n})]\tilde{g}_{ij} = 0.$$

From (4) we have

$$\begin{aligned} & \left[E_i v^{\bar{h}} + (y^s R_{sia}^h + y_a R_i^h - y^h R_{ia})v^a + \Gamma_{ia}^h v^{\bar{a}} \right] g_{hj} \\ & + \left[E_j v^{\bar{h}} + (y^s R_{sja}^h + y_a R_j^h - y^h R_{ja})v^a + \Gamma_{ja}^h v^{\bar{a}} \right] g_{hi} = 0 \end{aligned}$$

and if we choose the potential vector field $\tilde{V}$ as the vertical lift ${}^V V = (v^a, v^{\bar{a}}) = (0, v^a)$ of a vector field V on M , then

$$(\nabla_i v^h)g_{hj} + (\nabla_j v^h)g_{hi} = 0.$$

It follows

$$L_V g_{ij} = 0.$$

This shows V is a Killing vector field on M with respect to the Levi-Civita connection. $\square$

A (pseudo-)Riemannian manifold (M, g) is called an Einstein manifold if the relation

$$K_{jk} = \lambda g_{jk}$$

holds with a scalar function λ , where K is the Ricci tensor of (M, g) .

Theorem 2. Consider TM as the tangent bundle of an n -dimensional (pseudo-)Riemannian manifold M equipped with the RQSMC $\tilde{\nabla}$. For the triplet $({}^C g, \tilde{V}, \lambda)$ to be a conformal Yamabe soliton, it is necessary and sufficient that the specified conditions need to be satisfied:

$$\begin{aligned} (i) \quad & {}^H V = (v^a, v^{\bar{a}}) = (v^a, 0), \\ (ii) \quad & \lambda = \frac{1}{2}(\rho + \frac{1}{n}) - \frac{\nabla_h v^h}{2n}, \\ (iii) \quad & R_{sa} = \frac{r}{2}g_{sa}, \text{ i.e., } M \text{ is an Einstein manifold,} \end{aligned}$$

where the potential vector field is formed as the horizontal lift ${}^H V$ of a vector field V defined on M to the tangent bundle TM .

Proof. Referring to the relation (5), the following equations are derived for a manifold M in connection with a potential vector field V .

The equation

$$L_{\tilde{V}}\tilde{g}_{i\bar{j}} + [2\lambda - 2\tilde{r} - (\rho + \frac{2}{2n})]\tilde{g}_{i\bar{j}} = 0$$

is obtained by considering specific values for $(\varepsilon, \beta) = (i, \bar{j})$. Using the expression of $L_{\tilde{V}}\tilde{g}$ in (4), the equation

$$(\nabla_i v^h)g_{hj} + (E_{\bar{j}}v^h)g_{hi} + [2\lambda - 2\tilde{r} - (\rho + \frac{1}{n})]g_{ij} = 0$$

is obtained.

For the potential vector field $\tilde{V}$ as the horizontal lift $^H V = (v^a, v^{\bar{a}}) = (v^a, 0)$ with $\tilde{r} = 0$, the resulting equation is

$$(\nabla_i v^h)g_{hj} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Contracting both sides of the previous equation with g^{ij} yields

$$\nabla_h v^h + [2\lambda - (\rho + \frac{1}{n})]n = 0$$

and

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}) - \frac{\nabla_h v^h}{2n}.$$

By considering the indices $(\varepsilon, \beta) = (i, j)$, the equation

$$L_{\tilde{V}}\tilde{g}_{ij} + [2\lambda - 2\tilde{r} - (\rho + \frac{2}{2n})]\tilde{g}_{ij} = 0$$

is obtained. Referring to (4), we have

$$\begin{aligned} & [E_i v^{\bar{h}} + (y^s R_{sia}^h + y_a R_i^h - y^h R_{ia})v^a + \Gamma_{ia}^h v^{\bar{a}}]g_{hj} \\ & + [E_j v^{\bar{h}} + (y^s R_{sja}^h + y_a R_j^h - y^h R_{ja})v^a + \Gamma_{ja}^h v^{\bar{a}}]g_{hi} = 0. \end{aligned}$$

If we consider the potential vector field $\tilde{V}$ as the horizontal lift $^H V = (v^a, v^{\bar{a}}) = (v^a, 0)$, then

$$\begin{aligned} & \left[(y^s R_{sia}^h + y_a R_i^h - y^h R_{ia})v^a \right]g_{hj} \\ & + \left[(y^s R_{sja}^h + y_a R_j^h - y^h R_{ja})v^a \right]g_{hi} = 0. \end{aligned}$$

By contracting both sides of the last equation with g^{ij} , we have

$$R_{sa} = \frac{r}{2}g_{sa}.$$

This shows M is an Einstein manifold. Here, r is the scalar curvature of M . $\square$

Theorem 3. When TM is regarded as the tangent bundle of an n -dimensional (pseudo-)Riemannian manifold M equipped with the RQSMC $\tilde{\nabla}$, for the triplet $({}^C g, \tilde{V}, \lambda)$ to be a conformal Yamabe soliton, it is necessary and sufficient that the specified conditions need to be satisfied:

$$\begin{aligned} (i) \quad & {}^C V = (v^a, v^{\bar{a}}) = (v^a, y^s \nabla_s v^a), \\ (ii) \quad & \lambda = \frac{1}{2}(\rho + \frac{1}{n}) - \frac{\nabla_h v^h}{n}, \\ (iii) \quad & \nabla_h \nabla_s v^h - (2R_{sa} - g_{sa}r)v^a = 0, \end{aligned}$$

where the potential vector field is generated as the complete lift ${}^C V$ of a vector field V defined on M to the tangent bundle TM .

Proof. Utilizing the concept from relation (5), we represent the equation as follows:

$$L_{\tilde{V}}\tilde{g}_{\varepsilon\beta} + [2\lambda - 2\tilde{r} - (\rho + \frac{2}{2n})]\tilde{g}_{\varepsilon\beta} = 0. \quad (7)$$

When considering $(\varepsilon, \beta) = (i, \bar{j})$, this equation simplifies to:

$$L_{\tilde{V}}\tilde{g}_{i\bar{j}} + [2\lambda - 2\tilde{r} - (\rho + \frac{2}{2n})]\tilde{g}_{i\bar{j}} = 0.$$

Expanding using the expression of $L_{\tilde{V}}\tilde{g}$ in (4), we obtain

$$(\nabla_i v^h)g_{hj} + (E_{\bar{j}}v^{\bar{h}})g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

By considering the potential vector field $\tilde{V}$ as the complete lift ${}^C V = (v^a, v^{\bar{a}}) = (v^a, y^s \nabla_s v^{\bar{a}})$ of a vector field V on M , the equation simplifies to:

$$(\nabla_i v^h)g_{hj} + [\partial_{\bar{j}}(y^s \nabla_s v^h)]g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Further simplification yields:

$$(\nabla_i v^h)g_{hj} + (\nabla_j v^h)g_{hi} + \left[2\lambda - (\rho + \frac{1}{n})\right]g_{ij} = 0.$$

Contracting both sides of the last equation with g^{ij} , we obtain:

$$2\nabla_h v^h + \left[2\lambda - (\rho + \frac{1}{n})\right]n = 0,$$

leading to the relationship:

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}) - \frac{\nabla_h v^h}{n}.$$

Upon analyzing $(\varepsilon, \beta) = (i, j)$ in (7), we obtain

$$L_{\tilde{V}}\tilde{g}_{ij} + [2\lambda - 2\tilde{r} - (\rho + \frac{2}{2n})]\tilde{g}_{ij} = 0.$$

Referring back to (4), we express the equation as:

$$\begin{aligned} & \left[E_i v^{\bar{h}} + (y^s R_{sia}^h + y_a R_i^h - y^h R_{ia})v^a + \Gamma_{ia}^h v^{\bar{a}} \right] g_{hj} \\ & + \left[E_j v^{\bar{h}} + (y^s R_{sja}^h + y_a R_j^h - y^h R_{ja})v^a + \Gamma_{ja}^h v^{\bar{a}} \right] g_{hi} = 0. \end{aligned}$$

Opting for the potential vector field $\tilde{V}$ as the complete lift ${}^C V = (v^a, v^{\bar{a}}) = (v^a, y^s \nabla_s v^{\bar{a}})$, then

$$\begin{aligned} & [(\partial_i - y^s \Gamma_{si}^m \partial_{\bar{m}})(y^t \nabla_t v^a) + (y^s R_{sia}^h + y_a R_i^h - y^h R_{ia})v^a + \Gamma_{ia}^h (y^s \nabla_s v^{\bar{a}})]g_{hj} \\ & + [(\partial_j - y^s \Gamma_{sj}^m \partial_{\bar{m}})(y^t \nabla_t v^a) + (y^s R_{sja}^h + y_a R_j^h - y^h R_{ja})v^a + \Gamma_{ja}^h (y^s \nabla_s v^{\bar{a}})]g_{hi} \\ & = 0. \end{aligned}$$

This simplifies to:

$$\begin{aligned} & \left[y^s \nabla_i \nabla_s v^h + (y^s R_{sia}^h + y_a R_i^h - y^h R_{ia})v^a \right] g_{hj} \\ & + \left[y^s \nabla_j \nabla_s v^h + (y^s R_{sja}^h + y_a R_j^h - y^h R_{ja})v^a \right] g_{hi} = 0. \end{aligned}$$

Contracting both sides with g^{ij} gives

$$2y^s \left[\nabla_h \nabla_s v^h - (2R_{sa} - g_{sa}r)v^a \right] = 0,$$

from which we conclude:

$$\nabla_h \nabla_s v^h - (2R_{sa} - g_{sa}r)v^a = 0.$$

Here, r stands for the scalar curvature of M . This completes the proof. $\square$

4. Gradient Conformal Yamabe Soliton Structure on $(TM, {}^C g)$ with Respect to RQSMC $\tilde{\nabla}$

A Conformal Yamabe soliton (g, V, λ) is labeled as a gradient Conformal Yamabe soliton when V is equal to the gradient of a smooth function f , denoted as ∇f . In this context, the smooth function f is denoted as the potential function, and the Equation (5) is expressed as:

$$\text{Hess}f + \left[2\lambda - 2r - \left(\rho + \frac{2}{n} \right) \right] g = 0, \quad (8)$$

where ∇f represents the gradient of f and Hess represents the Hessian. Typically, the Hessian of any function f on M , in relation to the connection ∇ , is typically expressed as:

$$(\text{Hess}_{\nabla} f)(X, Y) = XYf - (\nabla_X Y)f,$$

for any vector fields X and Y on M .

Lemma 1. *The Hessian of the vertical lift of a smooth function f on a Riemannian manifold (M, g) can be described in terms of the RQSMC $\tilde{\nabla}$ on the tangent bundle $(TM, {}^C g)$ as follows:*

$$\begin{aligned} \text{Hess}_{\tilde{\nabla}} {}^V f({}^H X, {}^H Y) &= {}^H X {}^H Y {}^V f - (\tilde{\nabla}_{{}^H X} {}^H Y) {}^V f \\ \text{Hess}_{\tilde{\nabla}} {}^V f(E_i, E_j) &= E_i E_j {}^V f - (\tilde{\nabla}_{E_i} E_j) {}^V f \\ &= (\partial_i - y^s \Gamma_{si}^h \partial_{\bar{h}})(\partial_j - y^m \Gamma_{mj}^l \partial_{\bar{l}})f \\ &\quad - [\Gamma_{ij}^h E_h + (y^s R_{sij}^h + y_j R_i^h - y^h R_{ij})E_{\bar{h}}] {}^V f \\ &= (\partial_i - y^s \Gamma_{si}^h \partial_{\bar{h}})(\partial_j f) - \Gamma_{ij}^h (\partial_h - y^s \Gamma_{sh}^m \partial_{\bar{m}})f \\ &= \partial_i \partial_j f - \Gamma_{ij}^h \partial_h f \\ &= \nabla_i \nabla_j f, \end{aligned} \quad (9)$$

$$\begin{aligned} \text{Hess}_{\tilde{\nabla}} {}^V f({}^V X, {}^V Y) &= {}^V X {}^V Y {}^V f - (\tilde{\nabla}_{{}^V X} {}^V Y) {}^V f \\ \text{Hess}_{\tilde{\nabla}} {}^V f(E_{\bar{i}}, E_{\bar{j}}) &= E_{\bar{i}} E_{\bar{j}} {}^V f - (\tilde{\nabla}_{E_{\bar{i}}} E_{\bar{j}}) {}^V f \\ &= 0, \end{aligned} \quad (10)$$

$$\begin{aligned} \text{Hess}_{\tilde{\nabla}} {}^V f({}^H X, {}^V Y) &= {}^H X {}^V Y {}^V f - (\tilde{\nabla}_{{}^H X} {}^V Y) {}^V f \\ \text{Hess}_{\tilde{\nabla}} {}^V f(E_i, E_{\bar{j}}) &= E_i E_{\bar{j}} {}^V f - (\tilde{\nabla}_{E_i} E_{\bar{j}}) {}^V f \\ &= -(\Gamma_{ij}^h E_h + \Gamma_{ij}^{\bar{h}} E_{\bar{h}}) {}^V f \\ &= 0, \end{aligned} \quad (11)$$

$$\begin{aligned}
 Hess_{\tilde{\nabla}}^V f(VX, {}^H Y) &= {}^V X {}^H Y^V f - (\tilde{\nabla}_V X {}^H Y)^V f \\
 Hess_{\tilde{\nabla}}^V f(E_{\tilde{i}}, E_j) &= E_{\tilde{i}} E_j^V f - (\tilde{\nabla}_{E_{\tilde{i}}} E_j)^V f \\
 &= \partial_{\tilde{i}}(\partial_j - y^s \Gamma_{sj}^h \partial_{\tilde{h}})^V f \\
 &= \partial_{\tilde{i}} \partial_j f \\
 &= 0.
 \end{aligned} \tag{12}$$

Now, we focus on the gradient Conformal Yamabe soliton.

Theorem 4. Consider TM as the tangent bundle of an n -dimensional (pseudo-)Riemannian manifold M equipped with the RQSMC $\tilde{\nabla}$. The triplet $({}^C g, \tilde{V}, \lambda)$ forms a conformal gradient Yamabe soliton if and only if the following conditions are met:

$$\begin{aligned}
 (i) \quad \lambda &= \frac{1}{2}(\rho + \frac{1}{n}), \\
 (ii) \quad \nabla_i \tilde{\nabla}_j f &= 0.
 \end{aligned}$$

Proof. With help of the relation (8), we can represent the equation as:

$$(Hess f)_{\varepsilon\beta} + \left[2\lambda - 2\tilde{r} - (\rho + \frac{2}{2n}) \right] \tilde{g}_{\varepsilon\beta} = 0. \tag{13}$$

Setting $(\varepsilon, \beta) = (i, \tilde{j})$, we derive

$$(Hess f)_{i\tilde{j}} + [2\lambda - 2\tilde{r} - (\rho + \frac{1}{n})] \tilde{g}_{i\tilde{j}} = 0$$

from the previous equation. Referring to the expression for $Hess f$ in (11), we establish:

$$[2\lambda - (\rho + \frac{1}{n})] g_{ij} = 0.$$

Contracting both sides with g^{ij} yields the relationship

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}).$$

Considering $(\varepsilon, \beta) = (i, j)$ in (13), we write

$$(Hess f)_{ij} + [2\lambda - 2\tilde{r} - (\rho + \frac{1}{n})] \tilde{g}_{ij} = 0.$$

Finally, the conclusion from (9) is that:

$$\nabla_i \nabla_j f = 0.$$

□

5. Conformal Ricci Soliton Structure on $(TM, {}^C g)$ in Relation to RQSMC $\tilde{\nabla}$

In the context of an n -dimensional Riemannian manifold M , a conformal Ricci soliton is characterized by the equation:

$$L_V g + 2Ric = \left[2\lambda - (\rho + \frac{2}{n}) \right] g = 0, \tag{14}$$

where Ric denotes the Ricci tensor, λ represents a constant, and ρ is a scalar non-dynamical field (time-dependent scalar field) [15].

Theorem 5. Consider TM as the tangent bundle of an n -dimensional (pseudo-)Riemannian manifold M equipped with the RQSMC $\tilde{\nabla}$. The triplet $({}^Cg, \tilde{V}, \lambda)$ constitutes a conformal Ricci soliton if and only if the following conditions are met:

$$\begin{aligned} (i) \quad {}^V V &= (v^a, v^{\bar{a}}) = (0, v^a), \\ (ii) \quad \lambda &= \frac{1}{2}(\rho + \frac{1}{n}), \\ (iii) \quad \text{For } n \neq 3, \quad &\frac{1}{2(3-n)}L_V g_{ij} + R_{ij} = 0. \end{aligned}$$

Here, the potential vector field is obtained by vertically lifting V , a vector field defined on M , to the tangent bundle TM as ${}^V V$.

Proof. We commence by considering relation (14), which is represented as:

$$L_{\tilde{V}} \tilde{g}_{\varepsilon\beta} + 2\tilde{K}_{\varepsilon\beta} + [2\lambda - (\rho + \frac{2}{2n})]\tilde{g}_{\varepsilon\beta} = 0. \quad (15)$$

Substituting $(\varepsilon, \beta) = (i, \bar{j})$ into this equation, we derive:

$$L_{\tilde{V}} \tilde{g}_{i\bar{j}} + 2\tilde{K}_{i\bar{j}} + [2\lambda - (\rho + \frac{2}{2n})]\tilde{g}_{i\bar{j}} = 0.$$

Using the expression for $L_{\tilde{V}} \tilde{g}$ in (4), we have:

$$(\nabla_i v^h)g_{hj} + (E_{\bar{j}} v^{\bar{h}})g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Choosing the potential vector field $\tilde{V}$ as the vertical lift ${}^V V = (v^a, v^{\bar{a}}) = (0, v^a)$ of a vector field V on M , we simplify to:

$$[2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Contracting with g^{ij} on both sides of the last equation we have

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}).$$

Next, we examine $(\varepsilon, \beta) = (i, j)$ in (15), which yields:

$$L_{\tilde{V}} \tilde{g}_{ij} + 2\tilde{K}_{ij} + [2\lambda - (\rho + \frac{2}{2n})]\tilde{g}_{ij} = 0$$

Utilizing (4), this can be expressed as:

$$\begin{aligned} &[E_i v^{\bar{h}} + (y^s R_{sia}^h + y_a R_i^h - y^h R_{ia})v^a + \Gamma_{ia}^h v^{\bar{a}}]g_{hj} \\ &+ [E_j v^{\bar{h}} + (y^s R_{sja}^h + y_a R_j^h - y^h R_{ja})v^a + \Gamma_{ja}^h v^{\bar{a}}]g_{hi} \\ &+ 2(3-n)R_{ij} = 0 \end{aligned}$$

and also step by step with further simplifications:

$$[E_i v^h + \Gamma_{ia}^h v^a]g_{hj} + [E_j v^h + \Gamma_{ja}^h v^a]g_{hi} + 2(3-n)R_{ij} = 0$$

$$\begin{aligned} &[(\partial_i - y^s \Gamma_{si}^m \partial_{\bar{m}})v^h + \Gamma_{ia}^h v^a]g_{hj} + [(\partial_j - y^s \Gamma_{sj}^m \partial_{\bar{m}})v^h + \Gamma_{ja}^h v^a]g_{hi} \\ &+ 2(3-n)R_{ij} = 0 \end{aligned}$$

$$\begin{aligned} &[(\partial_i v^h + \Gamma_{ia}^h v^a]g_{hj} + [(\partial_j v^h + \Gamma_{ja}^h v^a]g_{hi} \\ &+ 2(3-n)R_{ij} = 0 \end{aligned}$$

$$(\nabla_i v^h)g_{hj} + (\nabla_j v^h)g_{hi} + 2(3-n)R_{ij} = 0$$

and

$$L_v g_{ij} + 2(3-n)R_{ij} = 0.$$

This expression leads to:

$$\frac{1}{2(3-n)}L_v g_{ij} + R_{ij} = 0.$$

□

Theorem 6. Consider TM as the tangent bundle of an n -dimensional (pseudo-)Riemannian manifold M equipped with the RQSMC $\tilde{\nabla}$. The triplet $({}^Cg, \tilde{V}, \lambda)$ constitutes a conformal Ricci soliton if and only if the following conditions hold:

$$\begin{aligned} (i) \quad {}^H V &= (v^a, v^{\bar{a}}) = (v^a, 0), \\ (ii) \quad \lambda &= \frac{1}{2}\left(\rho + \frac{1}{n}\right) - \frac{\nabla_h v^h}{2n}, \\ (iii) \quad r &= 0 \text{ and } M \text{ is Ricci flat,} \end{aligned}$$

where the potential vector field is determined by horizontally lifting the vector field V from the base manifold M to the tangent bundle TM .

Proof. As defined in (14), we begin with the equation:

$$L_{\tilde{V}} \tilde{g}_{\varepsilon\beta} + 2\tilde{K}_{\varepsilon\beta} + [2\lambda - (\rho + \frac{2}{2n})]\tilde{g}_{\varepsilon\beta} = 0. \quad (16)$$

When applied to $(\varepsilon, \beta) = (i, \bar{j})$, the equation simplifies to:

$$L_{\tilde{V}} \tilde{g}_{i\bar{j}} + 2\tilde{K}_{i\bar{j}} + [2\lambda - (\rho + \frac{2}{2n})]\tilde{g}_{i\bar{j}} = 0.$$

Using the expression of $L_{\tilde{V}} \tilde{g}$ in (4), we derive:

$$(\nabla_i v^h)g_{hj} + (E_{\bar{j}} v^{\bar{h}})g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Assuming $\tilde{V}$ as the horizontal lift vector field ${}^H V = (v^a, v^{\bar{a}}) = (v^a, 0)$, we can express:

$$(\nabla_i v^h)g_{hj} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Upon contracting with g^{ij} , we determine:

$$\nabla_h v^h + [2\lambda - (\rho + \frac{1}{n})]n = 0$$

leading to:

$$\lambda = \frac{1}{2}\left(\rho + \frac{1}{n}\right) - \frac{\nabla_h v^h}{2n}.$$

For $(\varepsilon, \beta) = (i, j)$ in (16), the equation simplifies to:

$$L_{\tilde{V}} \tilde{g}_{ij} + 2\tilde{K}_{ij} + [2\lambda - (\rho + \frac{2}{2n})]\tilde{g}_{ij} = 0.$$

From (4), the expression transforms into:

$$\begin{aligned} & \left[E_i v^{\bar{h}} + \left(y^s R_{sia}^h + y_a R_i^h - y^h R_{ia} \right) v^a + \Gamma_{ia}^h v^{\bar{a}} \right] g_{hj} \\ & + \left[E_j v^{\bar{h}} + \left(y^s R_{sja}^h + y_a R_j^h - y^h R_{ja} \right) v^a + \Gamma_{ja}^h v^{\bar{a}} \right] g_{hi} \\ & + 2(3-n)R_{ij} = 0 \\ & \left[\left(y^s R_{sia}^h + y_a R_i^h - y^h R_{ia} \right) v^a \right] g_{hj} + \left[\left(y^s R_{sja}^h + y_a R_j^h - y^h R_{ja} \right) v^a \right] g_{hi} \\ & + 2(3-n)R_{ij} = 0. \end{aligned}$$

Contracting with g^{ij} , we obtain:

$$[-2y^s(2R_{sa} - g_{sa}r)v^a] + 2(3-n)r = 0.$$

This leads to the conclusion:

$$R_{sa} = \frac{r}{2}g_{sa} \text{ and } (3-n)r = 0,$$

implying that $r = 0$, signifying the Ricci flatness of M with r denoting the scalar curvature of M . $\square$

Theorem 7. *If we consider TM as the tangent bundle of an n -dimensional (pseudo-)Riemannian manifold M equipped with the RQSMC $\tilde{\nabla}$, the triplet $({}^Cg, \tilde{V}, \lambda)$ constitutes a conformal Ricci soliton if and only if the following conditions are met:*

$$\begin{aligned} (i) \quad & {}^C V = (v^a, v^{\bar{a}}) = (v^a, y^s \nabla_s v^a), \\ (ii) \quad & \lambda = \frac{1}{2}(\rho + \frac{1}{n}) - \frac{\nabla_h v^h}{n}, \\ (iii) \quad & r = 0, \\ (iv) \quad & \nabla_h \nabla_s v^h - 2R_{sa}v^a = 0, \end{aligned}$$

where the potential vector field is formed by completely lifting V , a vector field defined on M , to the tangent bundle TM as ${}^C V$.

Proof. Based on the relation (14), we write

$$L_{\tilde{V}}\tilde{g}_{\varepsilon\beta} + 2\tilde{K}_{\varepsilon\beta} + [2\lambda - (\rho + \frac{2}{2n})]\tilde{g}_{\varepsilon\beta} = 0. \quad (17)$$

If we consider $(\varepsilon, \beta) = (i, \bar{j})$, from the previous equation,

$$L_{\tilde{V}}\tilde{g}_{i\bar{j}} + 2\tilde{K}_{i\bar{j}} + [2\lambda - (\rho + \frac{2}{2n})]\tilde{g}_{i\bar{j}} = 0$$

and from the expression of $L_{\tilde{V}}\tilde{g}$ in (4) we have

$$(\nabla_i v^h)g_{hj} + (E_{\bar{j}} v^{\bar{h}})g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Taking ${}^C V = (v^a, v^{\bar{a}}) = (v^a, y^s \nabla_s v^a)$, we obtain

$$(\nabla_i v^h)g_{hj} + [\partial_{\bar{j}}(y^s \nabla_s v^h)]g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0$$

and

$$(\nabla_i v^h)g_{hj} + (\nabla_j v^h)g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Contracting with g^{ij} on both sides of the last equation we have

$$2\nabla_h v^h + [2\lambda - (\rho + \frac{1}{n})]n = 0$$

and it follows

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}) - \frac{\nabla_h v^h}{n}.$$

Now we consider $(\varepsilon, \beta) = (i, j)$ in (17), we obtain

$$L_{\tilde{V}}\tilde{g}_{ij} + 2\tilde{K}_{ij} + [2\lambda - (\rho + \frac{2}{2n})]\tilde{g}_{ij} = 0$$

and from (4) we have

$$\begin{aligned} & \left[E_i v^h + \left(y^s R_{sia}^h + y_a R_i^h - y^h R_{ia} \right) v^a + \Gamma_{ia}^h v^{\bar{a}} \right] g_{hj} \\ & + \left[E_j v^h + \left(y^s R_{sja}^h + y_a R_j^h - y^h R_{ja} \right) v^a + \Gamma_{ja}^h v^{\bar{a}} \right] g_{hi} \\ & + 2(3-n)R_{ij} = 0 \end{aligned}$$

and it follows

$$\begin{aligned} & \left[y^s \nabla_i \nabla_s v^h + \left(y^s R_{sia}^h + y_a R_i^h - y^h R_{ia} \right) v^a \right] g_{hj} \\ & + \left[y^s \nabla_j \nabla_s v^h + \left(y^s R_{sja}^h + y_a R_j^h - y^h R_{ja} \right) v^a \right] g_{hi} \\ & + 2(3-n)R_{ij} = 0. \end{aligned}$$

Contracting with g^{ij} on both sides of the last equation we have

$$2y^s \left[\nabla_h \nabla_s v^h - (2R_{sa} - g_{sa}r)v^a \right] + 2(3-n)r = 0$$

and it follows

$$\nabla_h \nabla_s v^h - (2R_{sa} - g_{sa}r)v^a = 0 \text{ and } (3-n)r = 0.$$

This shows $r = 0$. Therefore, we obtain $\nabla_h \nabla_s v^h - 2R_{sa}v^a = 0$. Here, r is the scalar curvature of M . $\square$

6. Gradient Conformal Ricci Soliton Structure on $(TM, {}^Cg)$ in Relation to RQSMC $\tilde{\nabla}$

A gradient conformal Ricci soliton, defined by $(g, \nabla f, \lambda)$ with $V = \nabla f$, involves a smooth function f known as the potential function. The Equation (14) takes the form:

$$Hessf + 2Ric = \left[2\lambda - (\rho + \frac{2}{n}) \right] g = 0. \quad (18)$$

In this context, ∇f represents the gradient of f , and $Hess$ denotes the Hessian. The Hessian of any function f on M , with respect to the connection ∇ , is typically denoted as:

$$(Hess_{\nabla}f)(X, Y) = XYf - (\nabla_X Y)f,$$

for any vector fields X and Y on M .

Theorem 8. *If we consider TM as the tangent bundle of an n -dimensional (pseudo-)Riemannian manifold M equipped with the RQSMC $\tilde{\nabla}$. The conformal gradient Ricci soliton can be identified by the triplet $({}^Cg, \tilde{V}, \lambda)$ if and only if the following conditions are satisfied:*

- (i) $\lambda = \frac{1}{2}(\rho + \frac{1}{n})$,
- (ii) $\nabla_i \nabla_j f = 2(n-3)R_{ij}$.

Proof. In line with relation (18), we derive the equation:

$$(Hessf)_{\varepsilon\beta} + 2\bar{K}_{\varepsilon\beta} + [2\lambda - (\rho + \frac{2}{2n})]\tilde{g}_{\varepsilon\beta} = 0. \quad (19)$$

By setting $(\varepsilon, \beta) = (i, \bar{j})$, we find:

$$(Hessf)_{i\bar{j}} + 2\bar{K}_{i\bar{j}} + [2\lambda - (\rho + \frac{1}{n})]\tilde{g}_{i\bar{j}} = 0.$$

Using the expression for $Hessf$ from (11) and (3), we conclude:

$$[2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Contracting both sides with g^{ij} leads us to:

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}).$$

When we substitute $(\varepsilon, \beta) = (i, j)$ in (19), we can rewrite the equation as:

$$(Hessf)_{ij} + 2\bar{K}_{ij} + [2\lambda - (\rho + \frac{1}{n})]\tilde{g}_{ij} = 0.$$

Finally, (9) and (3) lead us to the outcome:

$$\nabla_i \nabla_j f = 2(n - 3)R_{ij}.$$

□

7. Conclusions

This paper presents a comprehensive investigation into the geometric attributes of the tangent bundle TM of a (pseudo-)Riemannian manifold M , equipped with a RQSMC $\tilde{\nabla}$. The primary objective is to identify the essential conditions for TM to exhibit different types of solitons and soliton-like structures under this connection. Through rigorous mathematical analysis, we have provided explicit conditions characterizing each type of soliton within the framework of the RQSMC $\tilde{\nabla}$. By defining and scrutinizing the properties of potential vector fields lifted from M to TM , we have elucidated the geometric and algebraic structures underlying these soliton formations. These solitons represent critical solutions of geometric flow equations and play pivotal roles in understanding the global and local geometry of (pseudo-)Riemannian manifolds.

The results presented here provide a comprehensive understanding of the interplay between geometric structures and connection properties within the tangent bundle TM . By delineating the precise conditions under which each type of soliton emerges, we contribute to the advancement of differential geometry and its applications in physics. These findings have significant implications for various fields, including general relativity, where soliton-like structures often arise as solutions to Einstein's field equations.

The contributions of this paper extend beyond pure mathematics, as the results obtained have implications in both differential geometry and physics. From a differential geometry perspective, our work enriches the understanding of soliton-like structures and their emergence in tangent bundles endowed with specific metric connections. Furthermore, our findings provide valuable insights into the geometric flows and curvature properties of tangent bundles, contributing to ongoing research in geometric analysis and mathematical physics.

In the context of physics, the elucidation of soliton formations in tangent bundles opens avenues for exploring the geometric aspects of physical theories. Solitons play a significant role in various physical phenomena, including nonlinear wave equations and field theories.

Therefore, our results pave the way for investigating the geometric underpinnings of soliton dynamics in the context of tangent bundles, offering new perspectives on the interplay between geometry and physics.

In conclusion, this paper represents a significant contribution to the fields of differential geometry and theoretical physics, providing valuable insights into the geometric properties of tangent bundles and their relevance to soliton theory. We anticipate that our findings will inspire further research and foster interdisciplinary collaborations aimed at unraveling the intricate geometrical structures of (pseudo-)Riemannian manifolds.

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Article

Right Conoids Demonstrating a Time-like Axis within Minkowski Four-Dimensional Space

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Abstract: In the four-dimensional Minkowski space, hypersurfaces classified as right conoids with a time-like axis are introduced and studied. The computation of matrices associated with the fundamental form, the Gauss map, and the shape operator specific to these hypersurfaces is included in our analysis. The intrinsic curvatures of these hypersurfaces are determined to provide a deeper understanding of their geometric properties. Additionally, the conditions required for these hypersurfaces to be minimal are established, and detailed calculations of the Laplace–Beltrami operator are performed. Illustrative examples are provided to enhance our comprehension of these concepts. Finally, the umbilical condition is examined to determine when these hypersurfaces become umbilic, and also the Willmore functional is explored.

Keywords: Minkowski four-dimensional space; right conoid; hypersurface; time-like axis; Gauss map

MSC: 53A10; 53A35; 53B30; 53C35; 53C42

1. Introduction

The parametric equation

$$\begin{aligned} \mathbf{r}(u, v) &= \alpha(v) + u\beta(v) \\ &= (0, 0, g(v)) + u(\cos(f(v)), \sin(f(v)), 0) \end{aligned}$$

determines a ruled surface in three-dimensional Minkowski space $\mathbb{L}^3$. This specific surface, distinguished by its unique geometric properties, is accurately identified as a right conoid. It features a time-like axis defined by the vector $(0, 0, 1)$ and adheres to the metric signatures $(+, +, -)$ inherent to Minkowski space.

To unravel the intricacies inherent in this parametric representation, let us break down its components. The curve $\alpha(v)$ acts as a foundational axis along the z -direction. Concurrently, the vector function $\beta(v)$ defines the generating vector shaping the ruled surface. The parameter u moves dynamically along this generating vector, while v parametrizes the curve $\alpha(v)$, influencing the overall structure of the ruled surface.

The essential geometric features of the right conoid are encapsulated by this parametric formulation, which also serves as a mathematical framework for understanding its behavior within $\mathbb{L}^3$. A foundation is laid by the intricate relationship among the reference curve, generating vector, and parametric variables, allowing for the exploration of the geometric intricacies and properties of this ruled surface within the context of Minkowski space.

This classification gains significance as the surface is generated by moving a straight line intersecting a predetermined linear reference, ensuring that perpendicular alignment between these lines is maintained throughout the generation process. By defining the xy -plane as the orthogonal plane and the z -axis as the baseline, we derive the parametric equation that defines the right conoid.

Moreover, identifying this ruled surface as a right conoid with a time-like axis in $\mathbb{L}^3$ emphasizes its geometric finesse and structural integrity. This classification not only emphasizes its importance within the context of Minkowski space but also enables a deeper understanding of its curvature characteristics and spatial configuration.

This study aims to thoroughly investigate the core attributes of hypersurfaces categorized as right conoids with a time-like axis (RCH-T) within the expansive realm of four-dimensional Minkowski space $\mathbb{L}^4$. Our primary objective is to compute differential geometric objects crucial to these hypersurfaces. Leveraging the robust framework provided by the Cayley–Hamilton theorem, our aim is to scrutinize and quantify the intrinsic curvatures inherent in these particular hypersurfaces.

In addition to exploring curvature properties, a core objective of our research is to establish criteria that govern minimization within this particular geometric framework. This entails a thorough examination of factors influencing the reduction in specific geometric attributes, thereby enriching our understanding of the dynamics of RCH-T in four-dimensional Minkowski space.

An essential focus of our investigation is the intricate relationship between RCH-T and the Laplace–Beltrami ($\mathcal{L}$ – $\mathcal{B}$) operator in the expansive domain of $\mathbb{L}^4$. By exploring this connection, our aim is to gain insights into the intrinsic geometric properties of these hypersurfaces, thereby deepening our understanding of their behavior within the Minkowski space framework.

Furthermore, our analysis extends to include the elucidation of the umbilical condition, which reveals specific geometric characteristics that RCH-T may manifest. To provide a thorough overview of these geometric properties, we introduce the Willmore functional for RCH-T, enabling a quantitative assessment of their shape attributes.

This multifaceted approach not only advances our theoretical understanding but also lays the foundation for practical applications and further advancements in the study of hypersurfaces within Minkowski space.

In Section 2, we delve into the foundational principles and concepts of four-dimensional Minkowski geometry. Section 3 focuses on detailing curvature formulas applicable to hypersurfaces in $\mathbb{L}^4$. Moving forward to Section 4, we provide a comprehensive definition of right conoid hypersurfaces with a time-like axis, highlighting their distinct properties and characteristics. Section 5 shifts the discussion to the $\mathcal{L}$ – $\mathcal{B}$ operator for a smooth function in $\mathbb{L}^4$, utilizing the hypersurfaces analyzed previously. Section 6 addresses umbilical right conoid hypersurfaces with a time-like axis in $\mathbb{L}^4$. The Willmore functional of these hypersurfaces is introduced in Section 7. Finally, the present study concludes in the last section.

2. Preliminaries

In Minkowski $(n + 1)$ -space $\mathbb{L}^{n+1}$, the symbol s_j is expressed as σ_j , which denotes the j -th elementary symmetric function defined by

$$\sigma_j(a_1, a_2, \dots, a_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq n} a_{i_1} a_{i_2} \dots a_{i_j},$$

and we use the notation

$$r_i^j = \sigma_j(k_1, k_2, \dots, k_{i-1}, k_{i+1}, k_{i+2}, \dots, k_n).$$

An oriented hypersurface M is considered within the intricate domain of Minkowski four-dimensional space $\mathbb{L}^4$. The shape operator of this hypersurface, denoted by $\mathcal{S} = (s_{ij})_{3 \times 3}$, is employed in the exploration of various geometric measures.

Let $r_i^j = \sigma_j(k_1, k_2, k_3)$ with the defined condition, leading to $r_i^0 = 1$. The function s_k is recognized as the k -th average curvature of the oriented surface M . Specifically, the average curvature H is given by $H = \frac{1}{3}s_1$. Moreover, the Gauss–Kronecker ($\mathcal{G}$ – $\mathcal{K}$) curvature of M is denoted as $K = s_3$.

These curvature terms play crucial roles in differential geometry, particularly in understanding the geometric properties of hypersurfaces. The mean curvature H represents the average of the principal curvatures and is fundamental in various geometric and physical contexts. For M , H is directly related to the first mean curvature s_1 .

The Gauss–Kronecker curvature K , also known as the Gaussian curvature in dimension three, characterizes the intrinsic curvature of M independent of its embedding in higher-dimensional space. It is given by the third mean curvature s_3 .

Together, these curvatures provide comprehensive insights into the geometry of M , facilitating analyses ranging from surface classification to applications in fields such as physics and materials science.

The characteristic polynomial, denoted as $\sum_{k=0}^3 (-1)^k s_k \lambda^{3-k} = 0$, of $\mathcal{S}$ is indicated by the equation

$$\det(\mathcal{S} - \lambda \mathcal{I}_3) = 0. \quad (1)$$

Here, $\mathcal{I}_3$ denotes the identity matrix. The curvature relations are revealed as $\binom{3}{i} \mathbb{K}_i = s_i$, highlighting the intrinsic geometric properties of the hypersurface.

In this investigation, a vector is deemed identical to its transpose. We analyze the embedding of $\mathfrak{x} = \mathfrak{x}(u, v, w)$ from $M^3 \subset \mathbb{E}^3$ into $\mathbb{L}^4$.

Definition 1. An inner product for two vectors $\mathbf{t}^1 = (t_1^1, t_2^1, t_3^1, t_4^1)$, $\mathbf{t}^2 = (t_1^2, t_2^2, t_3^2, t_4^2)$ of $\mathbb{L}^4$ is achieved through the expression

$$\langle \mathbf{t}^1, \mathbf{t}^2 \rangle = t_1^1 t_1^2 + t_2^1 t_2^2 + t_3^1 t_3^2 - t_4^1 t_4^2.$$

Definition 2. A triple vector product for three vectors $\mathbf{t}^1 = (t_1^1, t_2^1, t_3^1, t_4^1)$, $\mathbf{t}^2 = (t_1^2, t_2^2, t_3^2, t_4^2)$, $\mathbf{t}^3 = (t_1^3, t_2^3, t_3^3, t_4^3)$ in $\mathbb{L}^4$ is given by the determinant

$$\mathbf{t}^1 \times \mathbf{t}^2 \times \mathbf{t}^3 = \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \mathfrak{e}_3 & -\mathfrak{e}_4 \\ t_1^1 & t_2^1 & t_3^1 & t_4^1 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^3 & t_2^3 & t_3^3 & t_4^3 \end{vmatrix}.$$

Here, the base components in $\mathbb{L}^4$ are determined by $\mathfrak{e}_k$.

Definition 3. The shape operator matrix $\mathcal{S}$ of the hypersurface $\mathfrak{x}$ in Minkowski four-dimensional space $\mathbb{L}^4$ is indicated by $(\mathfrak{g}_{ij})^{-1} \cdot (\mathfrak{h}_{ij})$, where $(\mathfrak{g}_{ij})_{3 \times 3}$ and $(\mathfrak{h}_{ij})_{3 \times 3}$ represent the first and the second fundamental form matrices, respectively. The components of the matrices are defined as $\mathfrak{g}_{ij} = \langle \mathfrak{x}_i, \mathfrak{x}_j \rangle$, $\mathfrak{h}_{ij} = \langle \mathfrak{x}_{ij}, \mathbb{G} \rangle$, for $i, j = 1, 2, 3$, and the Gauss map of $\mathfrak{x}$ is obtained through the expression

$$\mathbb{G} = \frac{\mathfrak{x}_u \times \mathfrak{x}_v \times \mathfrak{x}_w}{\|\mathfrak{x}_u \times \mathfrak{x}_v \times \mathfrak{x}_w\|}. \quad (2)$$

Refer to Chen et al. [1], Li, Güler, and Toda [2], and O'Neill [3] for detailed explanations.

3. Curvatures in $\mathbb{L}^4$

In this section, we explore the curvature expressions associated with a hypersurface characterized by $\phi = \phi(u, v, w)$ within the context of $\mathbb{L}^4$. We delve into the geometric properties of the hypersurface, considering its embedding into the space $\mathbb{L}^4$.

Proposition 1. The following curvature formulas are associated with a hypersurface ϕ in $\mathbb{L}^4$: $\mathbb{K}_0 = 1$ by definition,

$$3\mathbb{K}_1 = -\frac{\partial_2}{\partial_3}, \quad 3\mathbb{K}_2 = \frac{\partial_1}{\partial_3}, \quad \mathbb{K}_3 = -\frac{\partial_0}{\partial_3}, \quad (3)$$

where $\mathfrak{d}_3\lambda^3 + \mathfrak{d}_2\lambda^2 + \mathfrak{d}_1\lambda + \mathfrak{d}_0 = 0$ describes the characteristic polynomial Eq. $P_S(\lambda) = 0$ of the shape operator matrix $\mathcal{S}$, $\mathfrak{d}_3 = \det(\mathfrak{g}_{ij})$, $\mathfrak{d}_0 = \det(\mathfrak{h}_{ij})$, and $(\mathfrak{g}_{ij}), (\mathfrak{h}_{ij})$ denote the first and the second fundamental form matrices, respectively.

Proof. The curvatures are indicated by the characteristic polynomial equation of $\mathcal{S}$ in $\mathbb{L}^4$:

$$\begin{aligned}\mathbb{K}_0 &= 1, \\ 3\mathbb{K}_1 &= k_1 + k_2 + k_3 = -\frac{\mathfrak{d}_2}{\mathfrak{d}_3}, \\ 3\mathbb{K}_2 &= k_1k_2 + k_1k_3 + k_2k_3 = \frac{\mathfrak{d}_1}{\mathfrak{d}_3}, \\ \mathbb{K}_3 &= k_1k_2k_3 = -\frac{\mathfrak{d}_0}{\mathfrak{d}_3}.\end{aligned}$$

□

Definition 4. A hypersurface ϕ in $\mathbb{L}^4$ is termed j -minimal when $\mathbb{K}_{j=1,2,3} = 0$.

Proposition 2. The following relation between the curvatures and the fundamental forms holds for a hypersurface $\phi = \phi(u, v, w)$ in $\mathbb{L}^4$:

$$\mathbb{K}_0(\mathfrak{f}_{ij}) - 3\mathbb{K}_1(\mathfrak{t}_{ij}) + 3\mathbb{K}_2(\mathfrak{h}_{ij}) - \mathbb{K}_3(\mathfrak{g}_{ij}) = \mathcal{O}_3,$$

where $(\mathfrak{g}_{ij}), (\mathfrak{h}_{ij}), (\mathfrak{t}_{ij}), (\mathfrak{f}_{ij})$ give the first, second, third, and fourth fundamental form matrices, and $\mathcal{O}_3$ determines the zero matrix.

4. RCH-T in $\mathbb{L}^4$

In this section, the right conoid hypersurface with a time-like axis is described, and its geometric qualities are examined in Minkowski four-dimensional space $\mathbb{L}^4$.

In $\mathbb{L}^4$, attention is given to a ruled hypersurface determined by

$$\begin{aligned}\phi(u, v, w) &= \alpha(v, w) + u\beta(v, w) \\ &= (0, 0, 0, \Psi) + u(\cos \wp \cos \rho, \sin \wp \cos \rho, \sin \rho, 0).\end{aligned}$$

Here, α, β denote the surfaces, $u \in \mathbb{R} - \{0\}$, $\langle \beta, \beta \rangle = 1$, $\wp = \wp(v)$, $\rho = \rho(w)$, $\Psi = \Psi(v, w)$ describe differentiable functions, and $0 \leq \wp, \rho < 2\pi$.

The following definition supplies the hypersurface $\phi(u, v, w) = \mathbf{R} \cdot \eta^T$, where the profile hypersurface $\eta = (u, 0, 0, \Psi)$ rotates around the time-like axis $\ell = (0, 0, 0, 1)$ through

$$\mathbf{R}(v, w) = \begin{pmatrix} \cos \wp \cos \rho & -\sin \wp & -\cos \wp \sin \rho & 0 \\ \sin \wp \cos \rho & \cos \wp & -\sin \wp \sin \rho & 0 \\ \sin \rho & 0 & \cos \rho & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\mathbf{R} \in SO(4)$, $\mathbf{R} \cdot \ell^T = \ell^T$, $\mathbf{R}^T \cdot \varepsilon \cdot \mathbf{R} = \mathbf{R} \cdot \varepsilon \cdot \mathbf{R}^T = \varepsilon$, $\varepsilon = \text{diag}(1, 1, 1, -1)$.

Definition 5. A right conoid hypersurface with a time-like axis is immersed by $\phi : M^3 \subset \mathbb{E}^3 \rightarrow \mathbb{L}^4$, parametrized by

$$\phi(u, v, w) = \begin{pmatrix} \phi_1(u, v, w) \\ \phi_2(u, v, w) \\ \phi_3(u, v, w) \\ \phi_4(u, v, w) \end{pmatrix} = \begin{pmatrix} u \cos \wp \cos \rho \\ u \sin \wp \cos \rho \\ u \sin \rho \\ \Psi \end{pmatrix}. \quad (4)$$

Here, $u \in \mathbb{R} - \{0\}$, $\wp = \wp(v)$, $\rho = \rho(w)$, $0 \leq \wp, \rho < 2\pi$, and $\Psi = \Psi(v, w)$.

For brevity, the following notations are chosen: $c = \cos \varphi$, $s = \sin \varphi$, $C = \cos \rho$, and $S = \sin \rho$.

The computation of (g_{ij}) is determined by the first derivatives of the RCH-T, as given in Equation (4) with respect to u, v , and w , resulting in the following matrix:

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u^2 C^2 \wp_v^2 - \Psi_v^2 & -\Psi_v \Psi_w \\ 0 & -\Psi_v \Psi_w & u^2 \rho_w^2 - \Psi_w^2 \end{pmatrix}, \quad (5)$$

and $\wp_v = \frac{\partial \varphi}{\partial v}$, $\wp_v^2 = \frac{\partial^2 \varphi}{\partial v^2}$, etc. Therefore, $\det(g_{ij}) = u^2 \mathbb{W}$, where $\mathbb{W} = \wp_v^2 (u^2 \rho_w^2 - \Psi_w^2) C^2 - \rho_w^2 \Psi_v^2$. The characterization of the RCH-T described by Equation (4) as a space-like (resp., time-like, light-like) hypersurface is contingent upon whether $\mathbb{W} > 0$ (resp., $\mathbb{W} < 0$, $\mathbb{W} = 0$).

Applying the Gauss map formula (Formula (2)), the Gauss map of the RCH-T, as determined by Equation (4), is derived as follows:

$$\mathbb{G} = \frac{1}{|\mathbb{W}|^{1/2}} \begin{pmatrix} -s \rho_w \Psi_v - c S C \wp_v \Psi_w \\ c \rho_w \Psi_v - s S C \wp_v \Psi_w \\ C^2 \wp_v \Psi_w \\ C u \wp_v \rho_w \end{pmatrix}. \quad (6)$$

By taking the second derivatives with respect to u, v , and w of the RCH-T defined by Equation (4) and employing the Gauss map as given by Equation (6), the subsequent matrix is acquired:

$$(h_{ij}) = \begin{pmatrix} 0 & \frac{C \wp_v \rho_w \Psi_v}{|\mathbb{W}|^{1/2}} & \frac{C \wp_v \rho_w \Psi_w}{|\mathbb{W}|^{1/2}} \\ \frac{C \wp_v \rho_w \Psi_v}{|\mathbb{W}|^{1/2}} & \frac{C u (C S \wp_v^3 \Psi_w + \rho_w (\wp_{vv} \Psi_v - \wp_v \Psi_{vv}))}{|\mathbb{W}|^{1/2}} & -\frac{u \wp_v \rho_w (C \Psi_{vw} + \rho_w S \Psi_v)}{|\mathbb{W}|^{1/2}} \\ \frac{C \wp_v \rho_w \Psi_w}{|\mathbb{W}|^{1/2}} & -\frac{u \wp_v \rho_w (\rho_w S \Psi_v + C \Psi_{vw})}{|\mathbb{W}|^{1/2}} & \frac{C u \wp_v (\rho_{ww} \Psi_w - \rho_w \Psi_{ww})}{|\mathbb{W}|^{1/2}} \end{pmatrix}, \quad (7)$$

where $\wp_{uu} = \frac{\partial^2 \varphi}{\partial u^2}$, $\wp_{uv} = \frac{\partial^2 \varphi}{\partial u \partial v}$, etc. By employing (5) and (7), the computation of the shape operator matrix $\mathcal{S} = (s_{ij})_{3 \times 3}$ for (4) is carried out. Therefore, utilizing (3) in conjunction with (5) and (7), the curvatures of the RCH-T are determined.

Theorem 1. Let ϕ be an RCH-T defined by the equation in (4) within $\mathbb{L}^4$. The curvatures associated with ϕ are outlined, with $\mathbb{K}_0 = 1$ by default, as follows:

$$\begin{aligned} \mathbb{K}_1 &= \frac{1}{3u(C^2 \wp_v^2 (u^2 \rho_w^2 - \Psi_w^2) - \rho_w^2 \Psi_v^2)^{3/2}} \left[C \wp_v \rho_w (C^2 u^2 \wp_v^2 - \Psi_v^2) \Psi_{ww} + 2C \wp_v \rho_w \Psi_v \Psi_w \Psi_{vw} \right. \\ &\quad + C \wp_v \rho_w (u^2 \rho_w^2 - \Psi_w^2) \Psi_{vv} + \wp_v (C \rho_{ww} + 2S \rho_w^2) \Psi_v^2 \Psi_w + C (\wp_{vv} \rho_w \Psi_v + C S \wp_v^3 \Psi_w) \Psi_w^2 \\ &\quad \left. - C u^2 \wp_{vv} \rho_w^3 \Psi_v - C^2 u^2 \wp_v^3 (C \rho_{ww} + S \rho_w^2) \Psi_w \right], \\ \mathbb{K}_2 &= \frac{\wp_v}{3(C^2 \wp_v^2 (u^2 \rho_w^2 - \Psi_w^2) - \rho_w^2 \Psi_v^2)^2} \left[-C^2 \rho_w (\rho_w \wp_{vv} \Psi_v - \rho_w \wp_v \Psi_{vv} + C S \wp_v^3 \Psi_w) \Psi_{ww} \right. \\ &\quad - C \wp_v \rho_w^2 (2S \rho_w \Psi_v + C \Psi_{vw}) \Psi_{vw} - C^2 \wp_v \rho_w \rho_{ww} \Psi_w \Psi_{vv} - \wp_v \rho_w^4 \Psi_v^2 \\ &\quad \left. + C^3 \wp_v^3 (S \rho_{ww} - C \rho_w^2) \Psi_w^2 + C^2 \wp_{vv} \rho_w \rho_{ww} \Psi_v \Psi_w \right], \\ \mathbb{K}_3 &= \frac{C^2 \wp_v^2 \rho_w^2}{u(C^2 \wp_v^2 (u^2 \rho_w^2 - \Psi_w^2) - \rho_w^2 \Psi_v^2)^{5/2}} \left[-C \rho_w \wp_v (\Psi_w^2 \Psi_{vv} - 2\Psi_v \Psi_w \Psi_{vw} + \Psi_v^2 \Psi_{ww}) \right. \\ &\quad \left. + (\wp_v (C \rho_{ww} + 2S \rho_w^2) \Psi_v^2 + C (\rho_w \wp_{vv} \Psi_v + C S \wp_v^3 \Psi_w) \Psi_w) \Psi_w \right], \end{aligned}$$

Proof. By applying the theorem of Cayley–Hamilton, the curvatures $\mathbb{K}_i$ associated with ϕ are determined through the characteristic polynomial, mentioned by Equation (1), of the RCH-T defined by Equation (4):

$$\mathbb{K}_0\lambda^3 - 3\mathbb{K}_1\lambda^2 + 3\mathbb{K}_2\lambda - \mathbb{K}_3 = 0.$$

□

Corollary 1. Let ϕ be an RCH-T mentioned by Equation (4) in $\mathbb{L}^4$. ϕ is deemed 1-minimal if the ensuing partial differential equation emerges:

$$\begin{aligned} & C_{\wp v}\rho_w[(C^2u^2\wp_v^2 - \Psi_v^2)\Psi_{ww} + 2\Psi_v\Psi_w\Psi_{vw} + (u^2\rho_w^2 - \Psi_w^2)\Psi_{vv}] \\ & + \wp_v(C\rho_{ww} + 2S\rho_w^2)\Psi_v^2\Psi_w + C(\wp_{vv}\rho_w\Psi_v + CS\wp_v^3\Psi_w)\Psi_w^2 \\ & - Cu^2\wp_{vv}\rho_w^3\Psi_v - C^2u^2\wp_v^3(C\rho_{ww} + S\rho_w^2)\Psi_w = 0, \end{aligned}$$

where $u \neq 0$, $C_{\wp v}(u^2\rho_w^2 - \Psi_w^2)^{1/2} \neq \pm\rho_w\Psi_v$.

Corollary 2. Consider ϕ as an RCH-T indicated by Equation (4) within $\mathbb{L}^4$. ϕ is regarded as 2-minimal if the subsequent partial differential equation is satisfied:

$$\begin{aligned} & -C^2\rho_w(\wp_{vv}\rho_w\Psi_v - \wp_v\rho_w\Psi_{vv} + CS\wp_v^3\Psi_w)\Psi_{ww} \\ & - C_{\wp v}\rho_w^2(2S\rho_w\Psi_v + C\Psi_{vw})\Psi_{vw} - C^2\wp_v\rho_w\rho_{ww}\Psi_w\Psi_{vv} \\ & - \wp_v\rho_w^4\Psi_v^2 + C^3\wp_v^3(S\rho_{ww} - C\rho_w^2)\Psi_w^2 + C^2\wp_{vv}\rho_w\rho_{ww}\Psi_v\Psi_w = 0, \end{aligned}$$

where $C_{\wp v}(u^2\rho_w^2 - \Psi_w^2)^{1/2} \neq \pm\rho_w\Psi_v$.

Corollary 3. Let ϕ be an RCH-T given by Equation (4) in $\mathbb{L}^4$. ϕ is deemed 3-minimal if the following partial differential equation is provided:

$$\begin{aligned} & -C\rho_w\wp_v(\Psi_v^2\Psi_{ww} - 2\Psi_v\Psi_w\Psi_{vw} + \Psi_w^2\Psi_{vv}) \\ & + (\wp_v(C\rho_{ww} + 2S\rho_w^2)\Psi_v^2 + C(\rho_w\wp_{vv}\Psi_v + CS\wp_v^3\Psi_w)\Psi_w)\Psi_w = 0, \end{aligned}$$

where $u \neq 0$, $C_{\wp v}(u^2\rho_w^2 - \Psi_w^2)^{1/2} \neq \pm\rho_w\Psi_v$.

It should be noted that the resolutions for Ψ in the corollaries are challenges yet to be addressed.

5. The Laplace–Beltrami Operator of the RCH-T in $\mathbb{L}^4$

In this section, the focus shifts to the application of the $\mathcal{L}$ – $\mathcal{B}$ operator on a smooth function in $\mathbb{L}^4$. The subsequent steps involve calculating this operator using the RCH-T defined by the equation in (4).

Definition 6. The Laplace–Beltrami operator is defined for a smooth function $\varsigma = \varsigma(x^1, x^2, x^3) \mid_{\mathcal{D}}$ ($\mathcal{D} \subset \mathbb{R}^3$) of class C^3 and depends on the first fundamental form (g_{ij}) :

$$\Delta\varsigma = \frac{1}{g^{1/2}} \sum_{i,j=1}^3 \frac{\partial}{\partial x^i} \left(g^{1/2} g^{ij} \frac{\partial \varsigma}{\partial x^j} \right), \quad (8)$$

where $(g^{ij}) = (g_{kl})^{-1}$ and $g = \det(g_{ij})$.

For comprehensive insights into the Laplace–Beltrami operator, refer to the works of Chen et al. [1] and Lawson [4]. Hence, the following is provided.

Theorem 2. The Laplace–Beltrami operator of the RCH-T ϕ expressed by Equation (4) is formulated as $\Delta\phi = 3\mathbb{K}_1\mathbb{G}$, where $\mathbb{K}_1$ represents the mean curvature and $\mathbb{G}$ depicts the Gauss map of ϕ .

Proof. The $\mathcal{L}$ – $\mathcal{B}$ operator applied to the RCH-T as described by Equation (4) is represented as follows:

$$\Delta\phi = \frac{1}{\mathbf{g}^{1/2}} \left[\frac{\partial}{\partial u} \left(\mathbf{g}^{1/2} \mathbf{g}^{11} \frac{\partial\phi}{\partial u} \right) + \frac{\partial}{\partial v} \left(\mathbf{g}^{1/2} \mathbf{g}^{22} \frac{\partial\phi}{\partial v} \right) + \frac{\partial}{\partial w} \left(\mathbf{g}^{1/2} \mathbf{g}^{23} \frac{\partial\phi}{\partial w} \right) + \frac{\partial}{\partial w} \left(\mathbf{g}^{1/2} \mathbf{g}^{32} \frac{\partial\phi}{\partial v} \right) + \frac{\partial}{\partial w} \left(\mathbf{g}^{1/2} \mathbf{g}^{33} \frac{\partial\phi}{\partial w} \right) \right], \quad (9)$$

where

$$(\mathbf{g}^{ij}) = \frac{1}{u^2\mathbb{W}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & u^2\rho_w^2 - \Psi_w^2 & \Psi_v\Psi_w \\ 0 & \Psi_v\Psi_w & C^2u^2\wp_v^2 - \Psi_v^2 \end{pmatrix}, \quad (10)$$

and $\mathbb{W} = C^2\wp_v^2(u^2\rho_w^2 - \Psi_w^2) - \rho_w^2\Psi_v^2$. Through the substitution of the derivatives of the components, as determined by (10), into the formula presented by (9), the formation of $\Delta\phi = (\Delta\phi_1, \Delta\phi_2, \Delta\phi_3, \Delta\phi_4)$ along with its individual components is accomplished.

$$\begin{aligned} \Delta\phi_1 = & \frac{1}{u\mathbb{W}^2} (-C\rho_w\wp_v(s\Psi_v\rho_w + CSc\Psi_w\wp_v))((u^2\rho_w^2 - \Psi_w^2)\Psi_{vv} + 2\Psi_v\Psi_w\Psi_{vw} + (C^2u^2\wp_v^2 - \Psi_v^2)\Psi_{ww}) \\ & - cC^3S^2\wp_v^4\Psi_w^4 - C(s\wp_{vv}\rho_w^2 + 2cS^2\wp_v^2\rho_w^2 + cCS\wp_v^2\rho_{ww})\Psi_v^2\Psi_w^2 + cC^3Su^2\wp_v^4(C\rho_{ww} + S\rho_w^2)\Psi_w^2 \\ & + C^2u^2\wp_v\rho_w(sS\wp_v^2\rho_w^2 + cS\wp_{vv}\rho_w^2 + cS\wp_v^2\rho_{ww})\Psi_v\Psi_w - s\wp_v\rho_w(C\rho_{ww} + 2S\rho_w^2)\Psi_v^3\Psi_w \\ & - C^2S\wp_v\rho_w(c\wp_{vv} + s\wp_v^2)\Psi_v\Psi_w^3 + sCu^2\wp_{vv}\rho_w^4\Psi_v^2, \end{aligned}$$

$$\begin{aligned} \Delta\phi_2 = & \frac{1}{u\mathbb{W}^2} (C\rho_w\wp_v(c\Psi_v\rho_w - CSc\Psi_w\wp_v))((u^2\rho_w^2 - \Psi_w^2)\Psi_{vv} + 2\Psi_v\Psi_w\Psi_{vw} + (C^2u^2\wp_v^2 - \Psi_v^2)\Psi_{ww}) \\ & - sC^3S^2\wp_v^4\Psi_w^4 - C(-c\wp_{vv}\rho_w^2 + 2sS^2\wp_v^2\rho_w^2 + sCS\rho_{ww}\wp_v^2)\Psi_v^2\Psi_w^2 + sC^3Su^2\wp_v^4(C\rho_{ww} + S\rho_w^2)\Psi_w^2 \\ & - C^2u^2\wp_v\rho_w(cS\wp_v^2\rho_w^2 + cC\wp_v^2\rho_{ww} - sS\wp_{vv}\rho_w^2)\Psi_v\Psi_w + c\wp_v\rho_w(C\rho_{ww} + 2S\rho_w^2)\Psi_v^3\Psi_w \\ & - C^2S\wp_v\rho_w(s\wp_{vv} - c\wp_v^2)\Psi_v\Psi_w^3 - cCu^2\wp_{vv}\rho_w^4\Psi_v^2, \end{aligned}$$

$$\begin{aligned} \Delta\phi_3 = & \frac{1}{u\mathbb{W}^2} (C^3\wp_v^2\rho_w(u^2\rho_w^2 - \Psi_w^2)\Psi_w\Psi_{vv} + 2C^3\wp_v^2\rho_w\Psi_v\Psi_w^2\Psi_{vw} \\ & + C^3\wp_v^2\rho_w(-\Psi_v^2 + sCSu^2\wp_v^2)\Psi_w\Psi_{ww} - sC^4Su^2\wp_v^4\rho_{ww}\Psi_w^2 + C^4S\wp_v^4(\Psi_w^2 - u^2\rho_w^2)\Psi_w^2 \\ & - C^2\wp_v(-C\wp_v\rho_{ww}\Psi_v\Psi_w - 2S\wp_v\rho_w^2\Psi_v\Psi_w - C\wp_{vv}\rho_w\Psi_w^2 + Cu^2\rho_w^3\wp_{vv})\Psi_v\Psi_w), \end{aligned}$$

$$\begin{aligned} \Delta\phi_4 = & \frac{1}{u^2\mathbb{W}^2} (C^2\wp_v^2\rho_w^2(u^2\rho_w^2 - \Psi_w^2)\Psi_{vv} + 2C^2\wp_v^2\rho_w^2\Psi_v\Psi_w\Psi_{vw} + C^2\wp_v^2\rho_w^2(C^2u^2\wp_v^2 - \Psi_v^2)\Psi_{ww} \\ & + C\rho_w\wp_v(2S\wp_v\rho_w^2\Psi_v^2 + C\wp_v\rho_{ww}\Psi_v^2 + C\rho_w\wp_{vv}\Psi_v\Psi_w + C^2S\wp_v^3\Psi_w^2 - C^2u^2\wp_v^3(C\rho_{ww} + S\rho_w^2))\Psi_w \\ & - C^2u^2\wp_v\wp_{vv}\rho_w^4\Psi_v). \end{aligned}$$

□

Definition 7. The hypersurface ϕ is termed harmonic if each component of $\Delta\phi$ is null.

Example 1. Upon substituting $\wp(v) = v$, $\rho(w) = w$, and $\Psi(v, w) = w$ into an RCH-T defined by Equation (4), the expression is simplified to

$$\phi(u, v, w) = (u \cos v \cos w, u \sin v \cos w, u \sin w, w).$$

Its Gauss map and the shape operator matrix are characterized by

$$\mathbb{G} = \frac{1}{(1 - u^2)^{1/2}} (-\cos v \sin w, -\sin v \sin w, \cos w, u),$$

$$S = \begin{pmatrix} 0 & 0 & \frac{1}{(1-u^2)^{1/2}} \\ 0 & \frac{\tan w}{u(1-u^2)^{1/2}} & 0 \\ -\frac{1}{(1-u^2)^{3/2}} & 0 & 0 \end{pmatrix}.$$

Subsequently, the principal curvatures are $k_1 = \frac{i}{u^2-1} = -k_2$ and $k_3 = \frac{\tan w}{u(1-u^2)^{1/2}}$, and the curvatures are determined through

$$\begin{aligned} \mathbb{K}_1 &= \frac{\tan w}{3u(1-u^2)^{1/2}}, \\ \mathbb{K}_2 &= \frac{1}{3(1-u^2)^2}, \\ \mathbb{K}_3 &= \frac{\tan w}{u(1-u^2)^{5/2}}. \end{aligned}$$

Hence,

$$\Delta\phi = \frac{\tan w}{u(1-u^2)}(-\cos v \sin w, -\sin v \sin w, \cos w, u).$$

In conclusion, the hypersurface is neither minimal nor harmonic.

Example 2. Upon choosing $\wp(v) = v$, $\rho(w) = w$, and $\Psi(v, w) = v$ for an RCH-T defined by Equation (4), the RCH-T is parametrized by

$$\phi(u, v, w) = (u \cos v \cos w, u \sin v \cos w, u \sin w, v).$$

ϕ possesses the Gauss map

$$\mathbb{G} = \frac{1}{(1-u^2 \cos^2 w)^{1/2}}(-\sin v, \cos v, 0, u \cos w).$$

Therefore, the shape operator matrix is determined by

$$S = \begin{pmatrix} 0 & \frac{\cos w}{(1-u^2 \cos^2 w)^{1/2}} & 0 \\ -\frac{\cos w}{(1-u^2 \cos^2 w)^{3/2}} & 0 & \frac{u \sin w}{(1-u^2 \cos^2 w)^{3/2}} \\ 0 & -\frac{\sin w}{u(1-u^2 \cos^2 w)^{1/2}} & 0 \end{pmatrix}.$$

The curvatures are given by

$$\begin{aligned} \mathbb{K}_1 &= 0, \\ \mathbb{K}_2 &= \frac{1}{3(1-u^2 \cos^2 w)^2}, \\ \mathbb{K}_3 &= 0. \end{aligned}$$

Finally, $\Delta\phi = (0, 0, 0, 0)$. This implies that the hypersurface is 1-minimal, 3-minimal, and harmonic.

A comprehensive understanding of the diverse expressions of the geometric structures of the RCH-T indicated by Equation (4) is provided by exploring the hypersurfaces classified as minimal, non-minimal, harmonic, and non-harmonic.

6. The Umbilical Condition of the RCH-T in $\mathbb{L}^4$

In this section, the umbilical condition for the RCH-T defined by Equation (4) is disclosed.

Definition 8. A point $\mathbf{p}$ on a hypersurface is termed umbilical if and only if $k_1 = k_2 = k_3$. This implies $\mathbb{K}_1^3 = \mathbb{K}_3$.

As a result, we offer the following, laying down an essential foundation for deciphering the fundamental principles behind our findings.

Theorem 3. A point $\mathbf{p}$ on the hypersurface of the right conoid, $\phi : M^3 \subset \mathbb{E}^3 \rightarrow \mathbb{L}^4$, that features a time-like axis, is considered umbilic if and only if it satisfies the partial differential equation $\mathbb{K}_1^3 - \mathbb{K}_3 = 0$. Here, $\mathbb{K}_1$ and $\mathbb{K}_3$ indicate the mean and Gauss–Kronecker curvatures, respectively.

Proof. By using the curvatures $\mathbb{K}_1$ and $\mathbb{K}_3$ of the right conoid hypersurface with the time-like axis ϕ , the following partial differential equation arises:

$$\Phi^3 - 27C^2u^2\mathbb{W}^2\wp_v^2\rho_w^2\Theta = 0, \quad (11)$$

where

$$\begin{aligned} \Phi &= C\wp_v\rho_w(C^2u^2\wp_v^2 - \Psi_v^2)\Psi_{ww} + 2C\wp_v\rho_w\Psi_v\Psi_w\Psi_{vw} \\ &\quad + C\wp_v\rho_w(u^2\rho_w^2 - \Psi_w^2)\Psi_{vv} + \wp_v(C\rho_{ww} + 2S\rho_w^2)\Psi_v^2\Psi_w \\ &\quad + C(\wp_{vv}\rho_w\Psi_v + CS\wp_v^3\Psi_w)\Psi_w^2 - Cu^2\wp_{vv}\rho_w^3\Psi_v \\ &\quad - C^2u^2\wp_v^3(C\rho_{ww} + S\rho_w^2)\Psi_w, \\ \Theta &= -C\rho_w\wp_v(\Psi_w^2\Psi_{vv} - 2\Psi_v\Psi_w\Psi_{vw} + \Psi_v^2\Psi_{ww}) \\ &\quad + (\wp_v(C\rho_{ww} + 2S\rho_w^2)\Psi_v^2 + C(\rho_w\wp_{vv}\Psi_v + CS\wp_v^3\Psi_w)\Psi_w)\Psi_w, \\ \mathbb{W} &= C^2\wp_v^2(u^2\rho_w^2 - \Psi_w^2) - \rho_w^2\Psi_v^2. \end{aligned}$$

□

This theorem highlights a significant geometric property of the points on the hypersurface. It states that a point on the right conoid hypersurface with a time-like axis is umbilic exactly when it satisfies the given partial differential equation. The primary challenge is to find solutions Ψ that fulfil the partial differential equations described in Equation (11).

7. The Willmore Functional of the RCH-T in $\mathbb{L}^4$

In this section, we introduce the Willmore property of the RCH-T.

Definition 9. Let $s : M \subset \mathbb{E}^2 \rightarrow \mathbb{R}^n$ be a smooth immersion such that $W(s) < \infty$. The function s is deemed a critical point for W if

$$\forall n \in C^\infty(M, \mathbb{R}^n), \quad \frac{d}{dq}W(s + qn)|_{q=0} = 0.$$

An immersion satisfying this condition is referred to as Willmore.

Refer to the studies conducted by Li and Yau [5], Toda [6], and the research carried out by Willmore [7,8] for a thorough exploration of Euclidean properties. Hence, the Willmore functional is employed for the RCH-T given by Equation (4) in Minkowski four-dimensional space.

Theorem 4. An immersion $\phi : M^3 \subset \mathbb{E}^3 \rightarrow \mathbb{L}^4$ is considered Willmore if and only if it satisfies the partial differential equation

$$\Delta\mathbb{K}_1 + 3\mathbb{K}_1(\mathbb{K}_1^3 - \mathbb{K}_3) = 0,$$

where Δ denotes the Laplace–Beltrami operator, and $\mathbb{K}_1$ and $\mathbb{K}_3$ represent the mean and Gauss–Kronecker curvatures, respectively.

Proof. By using the curvatures determined by Theorem 1, and by utilizing the expression $\Delta\mathbb{K}_1 = (\mathbb{K}_1)_{uu} + (\mathbb{K}_1)_{vv} + (\mathbb{K}_1)_{ww}$ and incorporating the umbilical condition $\mathbb{K}_1^3 - \mathbb{K}_3 = 0$, the following partial differential equation is unveiled:

$$(\mathbb{K}_1)_{uu} + (\mathbb{K}_1)_{vv} + (\mathbb{K}_1)_{ww} + \Phi(\Phi^3 - 27C^2u^2\mathbb{W}^2\wp_v^2\rho_w^2\Theta) = 0, \quad (12)$$

where

$$\begin{aligned} \Phi &= C\wp_v\rho_w(C^2u^2\wp_v^2 - \Psi_v^2)\Psi_{ww} + 2C\wp_v\rho_w\Psi_v\Psi_w\Psi_{vw} \\ &\quad + C\wp_v\rho_w(u^2\rho_w^2 - \Psi_w^2)\Psi_{vv} + \wp_v(C\rho_{ww} + 2S\rho_w^2)\Psi_v^2\Psi_w \\ &\quad + C(\wp_{vv}\rho_w\Psi_v + CS\wp_v^3\Psi_w)\Psi_w^2 - Cu^2\wp_{vv}\rho_w^3\Psi_v \\ &\quad - C^2u^2\wp_v^3(C\rho_{ww} + S\rho_w^2)\Psi_w, \\ \Theta &= -C\rho_w\wp_v(\Psi_w^2\Psi_{vv} - 2\Psi_v\Psi_w\Psi_{vw} + \Psi_v^2\Psi_{ww}) \\ &\quad + (\wp_v(C\rho_{ww} + 2S\rho_w^2)\Psi_v^2 + C(\rho_w\wp_{vv}\Psi_v + CS\wp_v^3\Psi_w)\Psi_w)\Psi_w, \\ \mathbb{W} &= C^2\wp_v^2(u^2\rho_w^2 - \Psi_w^2) - \rho_w^2\Psi_v^2. \end{aligned}$$

□

A profound connection between the immersion of the hypersurface and its Willmore property is established by this theorem. It is asserted that the immersion ϕ from $\mathbb{E}^3$ into $\mathbb{L}^4$ achieves the status of being Willmore if and only if it aligns with the specified equation. The identification of solutions Ψ for the partial differential equation referred to as (12) remains an unresolved matter.

8. Conclusions

In conclusion, this study has offered a detailed exploration of hypersurfaces identified as right conoids with a time-like axis (RCH-Ts) in four-dimensional Minkowski space $\mathbb{L}^4$. Through meticulous analysis and computation of essential matrices—specifically, the fundamental form, Gauss map, and shape operator—specific curvatures intrinsic to these geometrical entities have been uncovered. The application of the Cayley–Hamilton theorem has been instrumental in shedding light on the complex geometric properties of RCH-T hypersurfaces.

The foundational investigation into four-dimensional Minkowski geometry laid a solid groundwork, elucidating crucial curvature formulas applicable to hypersurfaces within this space. The comprehensive definition of RCH-Ts has highlighted their unique characteristics and has distinguished them within the realm of hypersurfaces.

Moreover, practical implications of the Laplace–Beltrami operator on these hypersurfaces have been explored, demonstrating its computational utility and furthering our understanding of their geometric behavior. The introduction of umbilical RCH-T hypersurfaces has expanded our exploration, revealing additional geometric insights.

Lastly, the presentation and analysis of the Willmore functional has provided a quantitative measure to evaluate the shape characteristics of RCH-T hypersurfaces, contributing to both theoretical advancements and potential practical applications.

In summary, this comprehensive investigation not only deepens our theoretical understanding of hypersurfaces in Minkowski space but also sets the stage for future research and applications in geometry and mathematical physics.

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Article

Disaffinity Vectors on a Riemannian Manifold and Their Applications

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Abstract: A disaffinity vector on a Riemannian manifold (M, g) is a vector field whose affinity tensor vanishes. In this paper, we observe that nontrivial disaffinity functions offer obstruction to the topology of M and show that the existence of a nontrivial disaffinity function on M does not allow M to be compact. A characterization of the Euclidean space is also obtained by using nontrivial disaffinity functions. Further, we study properties of disaffinity vectors on M and prove that every Killing vector field is a disaffinity vector. Then, we prove that if the potential field ζ of a Ricci soliton (M, g, ζ, λ) is a disaffinity vector, then the scalar curvature is constant. As an application, we obtain conditions under which a Ricci soliton (M, g, ζ, λ) is trivial. Finally, we prove that a Yamabe soliton (M, g, ζ, λ) with a disaffinity potential field ζ is trivial.

Keywords: affinity tensor; disaffinity function; disaffinity vector; Ricci soliton; Yamabe soliton; isometric to Euclidean space; Killing vector field

MSC: 53C21; 53C15

1. Introduction

Throughout this paper, we assume that M^n is a connected Riemannian n -manifold with a Riemannian metric g . Let $\mathfrak{X}(M^n)$ denote the space of smooth vector fields on M^n .

The affinity tensor, $\mathcal{L}_\zeta \nabla$, of a vector field ζ on a Riemannian manifold M is defined by (cf. [1], p. 109):

$$(\mathcal{L}_\zeta \nabla)(X, Y) = \mathcal{L}_\zeta \nabla_X Y - \nabla_{\mathcal{L}_\zeta X} Y - \nabla_X \mathcal{L}_\zeta Y, \quad X, Y \in \mathfrak{X}(M), \quad (1)$$

where ∇ is the Riemannian connection and $\mathcal{L}$ represents the Lie derivative. An affinity tensor was used in [2–4], where the authors used it to obtain a characterization of a trivial Ricci soliton and used differential equations to obtain different characterizations of a sphere, respectively. We were informed by one of the reviewers that W. A. Poor was not the first to use the affinity tensor of a vector field, but rather, slightly earlier, S. Kobayashi and K. Nomizu (cf. [5], Chapter IV, pp. 225–236) used these vector fields under the name “affine infinitesimal transformations” and extensively studied them on manifolds equipped with an affine connection, which is a more general context than that which we are considering here.

A vector field ζ on M is called a *disaffinity vector* if its affinity tensor vanishes, that is $\mathcal{L}_\zeta \nabla = 0$. From (1), we obtain

$$\begin{aligned} (\mathcal{L}_\zeta \nabla)(X, Y) &= \mathcal{L}_\zeta \nabla_X Y - \nabla_{\mathcal{L}_\zeta X} Y - \nabla_X \mathcal{L}_\zeta Y \\ &= [\zeta, \nabla_X Y] - \nabla_{[\zeta, X]} Y - \nabla_X [\zeta, Y] \end{aligned}$$

$$\begin{aligned}
&= \nabla_{\xi} \nabla_X Y - \nabla_{\nabla_X Y} \xi - \nabla_{[\xi, X]} Y - \nabla_X \nabla_{\xi} Y + \nabla_X \nabla_Y \xi \\
&= R(\xi, X)Y + \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi.
\end{aligned}$$

Hence, we obtain

$$R(X, \xi)Y = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi \quad (2)$$

for a disaffinity vector ξ on M .

Now, we discuss a few examples of disaffinity vectors on some well-known Riemannian manifolds:

(i) Considering the Euclidean n -space $\mathbb{E}^n = (R^n, \langle \cdot, \cdot \rangle)$ and the position vector field

$$\xi = \sum_{j=1}^n u^j \frac{\partial}{\partial u^j} \quad (3)$$

on $\mathbb{E}^n$, where $u^1, \dots, u^n$ are Euclidean coordinates, then it follows that ξ satisfies Equation (2) and, consequently, ξ is a disaffinity vector on $\mathbb{E}^n$. Also, all parallel vector fields on the Euclidean n -space are disaffinity vectors.

(ii) It is well known that on the standard unit sphere S^{2n-1} in the complex n -space $\mathbb{C}^n$, the Reeb vector field $\xi = -JN$, where J is the complex structure on $\mathbb{C}^n$ and N is the unit normal to S^{2n-1} , defines a Sasakian structure on S^{2n-1} (cf. [6]), and it satisfies Equation (2); therefore, ξ is a disaffinity vector on S^{2n-1} .

Indeed, disaffinity vectors are in abundance and, as will be explored herein, we observe that they play an important role in shaping the geometry of the Riemannian manifold on which they live. Given a (smooth) function f on a Riemannian manifold M^n , we say f is a *disaffinity function* if its gradient ∇f is a disaffinity vector. Observe that if we consider the function $f = \frac{1}{2} \|\xi\|^2$ on $\mathbb{E}^n$, where ξ is the position vector field appearing in Equation (3), then it follows that $\nabla f = \xi$. And, as we have seen, because ξ is a disaffinity vector, it follows that f is a non-constant disaffinity function on $\mathbb{E}^n$. Since each constant function is a disaffinity function, we call a non-constant disaffinity function a *nontrivial disaffinity function*.

In this article, we are interested in the impact of the existence of a nontrivial disaffinity function on a Riemannian manifold. Indeed, we prove that if a Riemannian manifold M^n admits a nontrivial disaffinity function, then it is non-compact (cf. Theorem 1), showing that the existence of a nontrivial disaffinity function provides an obstruction to the topology of M^n . This result may appear as a counter argument to example (ii); however, it is actually not so, as the Reeb vector field ξ on compact S^{2n-1} is not the gradient of a function. Next, we prove that each Killing vector is a disaffinity vector (cf. Proposition 1), supporting the statement that disaffinity vectors are in abundance. To show the importance of disaffinity functions on a complete Riemannian manifold M^n , we provide a characterization of $\mathbb{E}^n$ using a non-harmonic disaffinity function on M^n (see Theorem 2). The next important question to consider involves finding conditions under which a Ricci soliton (M^n, g, ξ, λ) is trivial (cf. Theorem 3). In addition, we prove that if the potential field ξ of a Yamabe soliton (M^n, g, ξ, λ) (cf. [7]) is a disaffinity vector, then (M^n, g, ξ, λ) is trivial (see Theorem 4).

2. Preliminaries

The curvature tensor R of a Riemannian manifold M^n is defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, \quad X, Y, Z \in \mathfrak{X}(M) \quad (4)$$

and the Ricci tensor Ric is given by

$$Ric(X, Y) = \sum_l g(R(e_l, X)Y, e_l), \quad (5)$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M^n . Associated to the Ricci tensor is the Ricci operator Q , defined by

$$\text{Ric}(X, Y) = g(QX, Y). \quad (6)$$

The scalar curvature S of M^n is

$$S = \sum_l \text{Ric}(e_l, e_l). \quad (7)$$

The following formula for the gradient ∇S of S is well known (cf. [7–12]):

$$\frac{1}{2} \nabla S = \sum_l (\nabla_{e_l} Q)(e_l), \quad (8)$$

where the covariant derivative $\nabla_X Q$ is defined by

$$(\nabla_X Q)(Y) = \nabla_X QY - Q(\nabla_X Y).$$

Given a (smooth) function $f : M^n \rightarrow \mathbb{R}$, the Laplacian Δ acting on f is given by

$$\Delta f = \text{div}(\nabla f), \quad (9)$$

where ∇f is the gradient of f and $\text{div} X = \sum_l g(\nabla_{e_l} X, e_l)$. If M^n is compact, then the Stokes's Theorem implies that

$$\int_{M^n} (\text{div} X) dV_g = 0, \quad (10)$$

where dV_g is the volume element of M^n .

For a function $f : M^n \rightarrow \mathbb{R}$, the Hessian operator H_f is defined by

$$H_f X = \nabla_X \nabla f, \quad X \in \mathfrak{X}(M) \quad (11)$$

and the Hessian of f , $\text{Hess}(f)$, is given by

$$\text{Hess}(f)(X, Y) = g(H_f X, Y). \quad (12)$$

A function f on M^n is called a disaffinity function if its gradient ∇f is a disaffinity vector. We see from the introduction that there is a non-constant disaffinity function on $\mathbb{E}^n$.

Recall that a vector field ξ on a Riemannian manifold M^n is said to be a Killing vector field if it satisfies (cf. [13]):

$$\mathcal{L}_\xi g = 0. \quad (13)$$

It is known that the Reeb vector field ξ on the unit sphere S^{2n-1} of example (ii) in the introduction is a Killing vector field, which is also a disaffinity vector. Indeed, we prove in the next result that every Killing vector field is a disaffinity vector, broadening the area of influence of disaffinity vectors.

Proposition 1. *Every Killing vector field on a Riemannian manifold is a disaffinity vector.*

Proof. Let ξ be a Killing vector field on a Riemannian manifold M^n and let η be a 1-form dual to ξ , that is, $\eta(X) = g(X, \xi)$. Define $G : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ by

$$\frac{1}{2} d\eta(X, Y) = g(GX, Y), \quad X, Y \in \mathfrak{X}(M), \quad (14)$$

which shows that G is a skew symmetric operator. Now, using (13) and (14), we have

$$\begin{aligned} 2g(\nabla_X \xi, Y) &= g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) + g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) \\ &= 2g(GX, Y). \end{aligned}$$

Thus, we have

$$\nabla_X \xi = GX. \quad (15)$$

The above equation implies that

$$\nabla_X \nabla_Y \xi = \nabla_X GY = (\nabla_X G)(Y) + G(\nabla_X Y),$$

which gives

$$R(X, Y)\xi = (\nabla_X G)(Y) - (\nabla_Y G)(X). \quad (16)$$

Note that as the 2-form $d\eta$ is closed, we can apply Equation (14) to confirm

$$g((\nabla_X G)(Y), Z) + g((\nabla_Y G)(Z), X) + g((\nabla_Z G)(X), Y) = 0.$$

The above equation, on using skew symmetry of G , implies

$$g((\nabla_X G)(Y), Z) - g((\nabla_Y G)(X), Z) + g((\nabla_Z G)(X), Y) = 0,$$

which, in view of Equation (16), confirms

$$g(R(X, Y)\xi, Z) + g((\nabla_Z G)(X), Y) = 0.$$

Thus, we have

$$(\nabla_Z G)(X) = R(Z, \xi)X. \quad (17)$$

Now, Equation (2) for a Killing vector field ξ and Equation (15) implies that

$$R(X, \xi)Y = \nabla_X GY - G(\nabla_X Y) = (\nabla_X G)(Y),$$

which is Equation (17). Therefore, ξ is a disaffinity vector on M^n . $\square$

3. Disaffinity Functions on a Riemannian Manifolds

Recall that a function f on a Riemannian manifold M^n is a disaffinity function if ∇f is a disaffinity vector. In this section, we study the influence of the existence of a nontrivial disaffinity function on M^n .

Theorem 1. *If a Riemannian n -manifold M^n admits a nontrivial disaffinity function, then it is non-compact.*

Proof. Suppose that M^n is a Riemannian manifold and f is a nontrivial disaffinity function on M^n . Then, ∇f satisfies Equation (2), which gives

$$R(X, \nabla f)Y = (\nabla_X H_f)(Y), \quad X, Y \in \mathfrak{X}(M). \quad (18)$$

Taking a local orthonormal frame $\{e_1, \dots, e_n\}$ on M^n and using the symmetry of the Hessian operator, Equation (18) implies

$$\text{Ric}(\nabla f, Y) = \sum_j g(Y, (\nabla_{e_j} H_f)(e_j)). \quad (19)$$

Note that using Equations (4) and (11), we find

$$R(X, Y)\nabla f = (\nabla_X H_f)(Y) - (\nabla_Y H_f)(X). \quad (20)$$

Also, using $\Delta f = \sum_j g(H_f e_j, e_j)$, we obtain

$$X(\Delta f) = \sum_j (g((\nabla_X H_f)(e_j) + H_f(\nabla_X e_j), e_j) + g(H_f e_j, \nabla_X e_j))$$

$$= \sum_j g((\nabla_X H_f)(e_j), e_j) + 2 \sum_j g(H_f e_j, \nabla_X e_j). \quad (21)$$

Note that $H_f e_j = \sum_l \text{Hess}(f)(e_j, e_l) e_l$, where $\text{Hess}(f)(e_j, e_l)$ is symmetric, and

$$\nabla_X e_j = \sum_k \omega_j^k(X) e_k,$$

where the connection forms ω_j^k are skew symmetric. Consequently, we have

$$\sum_j g(H_f e_j, \nabla_X e_j) = 0.$$

Thus, by using Equation (20) and the above equation in Equation (21), we find that

$$X(\Delta f) = \sum_j g(R(X, e_j) \nabla f, e_j) + \sum_j g((\nabla_{e_j} H_f) X, e_j),$$

which, in view of Equation (5) and the symmetry of the Hessian operator H_f , yields

$$X(\Delta f) = -\text{Ric}(X, \nabla f) + \sum_j g(X, (\nabla_{e_j} H_f)(e_j)).$$

Using Equation (19) in the above equation, we obtain

$$X(\Delta f) = 0, \quad X \in \mathfrak{X}(M), \quad (22)$$

and it confirms that $\Delta f = c$, where c is a constant. Thus, after integrating $\Delta f = c$, we obtain $c = 0$ and $\Delta f = 0$ on a compact M^n . Therefore, f is a constant, which is a contradiction as f is a nontrivial disaffinity function. $\square$

Now, we prove the following characterization of a Euclidean space.

Theorem 2. A complete Riemannian n -manifold M^n admits a non-harmonic disaffinity function f such that $h = \frac{1}{2} \|\nabla f\|^2$ satisfies the inequality

$$\text{Ric}(\nabla f, \nabla f) \geq \Delta h - \frac{1}{n} (\Delta f)^2$$

if and only if M^n is isometric to the Euclidean n -space.

Proof. Let M^n be a complete Riemannian n -manifold and $f : M^n \rightarrow \mathbb{R}$ be a non-harmonic disaffinity function. Suppose that the function $h = \frac{1}{2} \|\nabla f\|^2$ satisfies the inequality

$$\text{Ric}(\nabla f, \nabla f) \geq \Delta h - \frac{1}{n} (\Delta f)^2. \quad (23)$$

Note that for $X \in \mathfrak{X}(M)$, we have $X(h) = g(\mathcal{H}_f X, \nabla f)$, which gives

$$\nabla h = \mathcal{H}_f(\nabla f).$$

Differentiating the above equation with respect to $X \in \mathfrak{X}(M)$ gives

$$\mathcal{H}_h(X) = (\nabla_X \mathcal{H}_f)(\nabla f) + \mathcal{H}_f^2(\nabla f). \quad (24)$$

Now, since f is a disaffinity function, Equation (18) implies

$$R(X, \nabla f) \nabla f = (\nabla_X H_f)(\nabla f)$$

and inserting this equation in Equation (24) gives

$$\mathcal{H}_h(X) = R(X, \nabla f) \nabla f + \mathcal{H}_f^2(\nabla f).$$

Contracting the above equation yields

$$\Delta h = Ric(\nabla f, \nabla f) + \|\mathcal{H}_f\|^2.$$

We then rearrange the above equation into the form

$$\|\mathcal{H}_f\|^2 - \frac{1}{n}(\Delta f)^2 = \Delta h - \frac{1}{n}(\Delta f)^2 - Ric(\nabla f, \nabla f).$$

Now, using the inequality (23) in the above equation, we find

$$\|\mathcal{H}_f\|^2 - \frac{1}{n}(\Delta f)^2 \leq 0,$$

which, by virtue of Cauchy–Schwartz’s inequality $\|\mathcal{H}_f\|^2 \geq \frac{1}{n}(\Delta f)^2$, gives

$$\|\mathcal{H}_f\|^2 = \frac{1}{n}(\Delta f)^2.$$

The above equality holds if and only if

$$\mathcal{H}_f = \frac{\Delta f}{n} I. \quad (25)$$

Since f is a disaffinity function by (23), we see that $\Delta f =$ is a constant, say, nc . Moreover, since f is non-harmonic function, we obtain $c \neq 0$. Thus, Equation (25) becomes

$$Hess(f) = cg$$

and it guarantees that M^n is isometric to the Euclidean n -space (cf. [12]).

Conversely, suppose that M^n is isometric to the Euclidean space $\mathbb{E}^n$. Then, we know that the position vector field ξ given by Equation (3) is a disaffinity vector. If we consider

$$f = \frac{1}{2} \|\xi\|^2,$$

then we obtain $\nabla f = \xi$. This shows that f is a disaffinity function on $\mathbb{E}^n$. Also, $\Delta f = \operatorname{div} \xi = n$ implies that f is a non-harmonic disaffinity function. Further, here,

$$h = \frac{1}{2} \|\nabla f\|^2 = \frac{1}{2} \|\xi\|^2 = f.$$

Thus, $\Delta h = \Delta f = n$ and $Ric(\nabla f, \nabla f) = 0$ for $\mathbb{E}^n$ as well as

$$\Delta h - \frac{1}{n}(\Delta f)^2 = 0.$$

Consequently, the converse holds. $\square$

4. Disaffinity Vectors and Triviality of Ricci Solitons

For a Ricci soliton (M^n, g, ζ, λ) , the potential field ζ and constant λ satisfy (cf. [7]),

$$\frac{1}{2} \mathcal{L}_\zeta g + Ric = \lambda g. \quad (26)$$

A Ricci soliton (M^n, g, ζ, λ) is called trivial if either $\zeta = 0$ or $\mathcal{L}_\zeta g = 0$.

In this section, we are interested in studying the impart of the condition that the potential field ζ of the Ricci soliton (M^n, g, ζ, λ) is a disaffinity vector. It is worth noting that if (M^n, g, ζ, λ) is nontrivial and the potential field ζ is a disaffinity vector, then the Ricci soliton is non-compact. For a compact Ricci soliton (M^n, g, ζ, λ) to be a gradient Ricci soliton, that is $\zeta = \nabla \varphi$ for a function φ (cf. [7]), this implies that φ is a nontrivial disaffinity function. Thus, Theorem 1 implies that M^n is non-compact.

On a Ricci soliton (M^n, g, ζ, λ) , let β denote the 1-form dual to the potential field ζ , that is, $\beta(X) = g(X, \zeta)$. Define a skew symmetric operator $G : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ by

$$\frac{1}{2}d\beta(X, Y) = g(GX, Y), \quad X, Y \in \mathfrak{X}(M). \quad (27)$$

Then, using

$$2g(\nabla_X \zeta, Y) = (\mathcal{L}_\zeta g)(X, Y) + d\beta(X, Y)$$

and Equations (6), (26), and (27), we arrive at

$$\nabla_X \zeta = \lambda X - QX + GX. \quad (28)$$

Using the above equation, we obtain

$$R(X, Y)\zeta = -(\nabla_X Q)(Y) + (\nabla_Y Q)(X) + (\nabla_X G)(Y) - (\nabla_Y G)(X)$$

and, using a local orthonormal frame $\{e_1, \dots, e_n\}$, the symmetry of Q , skew symmetry of G , and Equation (8) in the above equation, we obtain

$$Ric(Y, \zeta) = -\frac{1}{2}Y(S) + Y(S) - \sum_j g(Y, (\nabla_{e_j} G)(e_j)).$$

Thus, we have

$$Q(\zeta) = \frac{1}{2}\nabla S - \sum_j (\nabla_{e_j} G)(e_j). \quad (29)$$

Proposition 2. *If (M^n, g, ζ, λ) is a Ricci soliton with the potential field ζ as a disaffinity vector, then the scalar curvature S of M^n is constant.*

Proof. Suppose that the potential field ζ of the Ricci soliton (M^n, g, ζ, λ) is a disaffinity vector. Then, using Equations (2) and (28), we obtain

$$\begin{aligned} R(X, \zeta)Y &= \nabla_X(\lambda Y - QY + GY) - (\lambda \nabla_X Y - Q(\nabla_X Y) + G(\nabla_X Y)) \\ &= -(\nabla_X Q)(Y) + (\nabla_X G)(Y). \end{aligned}$$

Using a local orthonormal frame $\{e_1, \dots, e_n\}$, the symmetry of Q , skew symmetry of G , and Equation (8) in the above equation, we obtain

$$Ric(Y, \zeta) = -\frac{1}{2}Y(S) - \sum_j g(Y, (\nabla_{e_j} G)(e_j)),$$

that is,

$$Q(\zeta) = -\frac{1}{2}\nabla S - \sum_j (\nabla_{e_j} G)(e_j). \quad (30)$$

From Equations (29) and (30), we arrive at $\nabla S = 0$, and hence, S is a constant. $\square$

Note that one of the interesting questions in the geometry of Ricci solitons is to find conditions under which a Ricci soliton (M^n, g, ζ, λ) is trivial. This question is relatively

easier if M^n is compact, and hence, it is important to find conditions under which a non-compact Ricci soliton (M^n, g, ζ, λ) is trivial.

Theorem 3. A Ricci soliton (M^n, g, ζ, λ) with ζ as a disaffinity vector and the Ricci operator Q satisfying $Q(\zeta) = \lambda\zeta$ is trivial.

Proof. Suppose that (M^n, g, ζ, λ) is a Ricci soliton with potential field ζ as a disaffinity vector satisfying

$$Q(\zeta) = \lambda\zeta. \quad (31)$$

Differentiating the above equation and using Equation (28), we arrive at

$$(\nabla_X Q)(\zeta) + Q(\lambda X - QX + GX) = \lambda(\lambda X - QX + GX).$$

Let us choose a local orthonormal frame $\{e_1, \dots, e_n\}$ with $X = e_j$. Then, after taking the inner product with e_j and summing up the resulting equations, we find

$$\sum_j g(\zeta, (\nabla_{e_j} Q)(e_j)) + \lambda S - \|Q\|^2 = n\lambda^2 - \lambda S, \quad (32)$$

where we have applied the skew symmetry of G and the symmetry of Q to confirm

$$\sum_j g(QGe_j, e_j) = 0 \text{ and } \sum_j g(Ge_j, e_j) = 0.$$

Using Proposition 2, we see that S is constant. Thus, on using Equation (8) in Equation (32), we find that

$$-\|Q\|^2 = n\lambda^2 - 2\lambda S,$$

which could be rearranged as

$$\|Q\|^2 - \frac{1}{n}S^2 = -\frac{1}{n}(n\lambda - S)^2. \quad (33)$$

Now, applying Cauchy–Schwartz’s inequality

$$\|Q\|^2 \geq \frac{1}{n}S^2$$

in (33), we obtain

$$\|Q\|^2 = \frac{1}{n}S^2 \quad \text{and} \quad S = n\lambda. \quad (34)$$

The first equation in (34) is the equality case in Cauchy–Schwartz’s inequality, which holds if and only if $Q = \frac{S}{n}I$. Thus, by virtue of the second equation in (34) gives $Q = \lambda I$. This gives $\mathcal{L}_\zeta g = 0$. Consequently, the Ricci soliton (M^n, g, ζ, λ) is trivial. $\square$

5. Disaffinity Vectors and Yamabe Solitons

In this section, we study Yamabe solitons (M^n, g, ζ, λ) , $n \geq 2$, whose potential field ζ satisfies (cf. [7,10])

$$\frac{1}{2}\mathcal{L}_\zeta g = (S - \lambda)g, \quad (35)$$

for a constant λ , where S is the scalar curvature of M^n .

A Yamabe soliton (M^n, g, ζ, λ) is called trivial if its scalar curvature S is a constant. If the potential field ζ of the Yamabe soliton (M^n, g, ζ, λ) is the gradient of a function φ , that is, $\zeta = \nabla\varphi$, then $(M^n, g, \nabla\varphi, \lambda)$ is called a gradient Yamabe soliton. It is known that a compact gradient Yamabe soliton is always trivial (cf. [7]). In this section, we are interested in the impact of the potential field ζ being a disaffinity vector on the geometry of Yamabe soliton (M^n, g, ζ, λ) .

As before, we denote by β the dual of the potential field ξ , and define the skew symmetric operator $F : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ by

$$\frac{1}{2}d\beta(X, Y) = g(FX, Y). \quad (36)$$

Then, using (35) and (36), we have the following expression:

$$\nabla_X \xi = (S - \lambda)X + FX. \quad (37)$$

Theorem 4. *A Yamabe soliton (M^n, g, ξ, λ) , $n \geq 2$, with potential field ξ as a disaffinity vector is trivial.*

Proof. Suppose that (M^n, g, ξ, λ) is a Yamabe soliton, $n \geq 2$ whose potential field ξ satisfies Equation (2). Then, using Equations (2) and (37), we have

$$\begin{aligned} R(X, \xi)Y &= \nabla_X((S - \lambda)Y + FY) - (S - \lambda)\nabla_X Y - F(\nabla_X Y) \\ &= X(S)Y + (\nabla_X F)(Y). \end{aligned}$$

Using the skew symmetry of F and a local orthonormal frame $\{e_1, \dots, e_n\}$ of M^n , we may take the trace from the above equation to obtain

$$\text{Ric}(\xi, Y) = Y(S) - \sum_j g(Y, (\nabla_{e_j} F)(e_j)),$$

which is equivalent to

$$Q(\xi) = \nabla S - \sum_j (\nabla_{e_j} F)(e_j). \quad (38)$$

Next, we use Equations (4) and (37) to obtain

$$R(X, Y)\xi = X(S)Y - Y(S)X + (\nabla_X F)(Y) - (\nabla_Y F)(X),$$

which, on using a local orthonormal frame $\{e_1, \dots, e_n\}$ and skew symmetry of the operator F while taking the trace from the above equation, we obtain

$$\text{Ric}(Y, \xi) = -(n - 1)Y(S) - \sum_j g(Y, (\nabla_{e_j} F)(e_j)). \quad (39)$$

Equation (39) implies

$$Q(\xi) = -(n - 1)\nabla S - \sum_j (\nabla_{e_j} F)(e_j). \quad (40)$$

Combining Equations (38) and (40) gives $\nabla S = 0$. Therefore, the Yamabe soliton (M^n, g, ξ, λ) is trivial. $\square$

6. Epilogue

In this article, we have seen the impact of a nontrivial disaffinity function as well as disaffinity vector on the geometry of the Riemannian manifold M^n . In particular, we use the non-harmonic disaffinity function to derive a characterization of the Euclidean space. We also use the constraint on the potential field of a Ricci or Yamabe soliton to be a disaffinity vector to obtain the triviality of the solitons. We shall elaborate two situations for future study, one for nontrivial disaffinity functions and the other for the disaffinity vectors on a Riemannian manifold.

Recall that the notion of Eikonal equation $\|\nabla f\| = 1$ on a Riemannian manifold M^n comes from physics, specially medical imaging (cf. [9,11]). Note that the existence of the Eikonal equation $\|\nabla f\| = 1$ on M^n forces M^n to be non-compact. A similar role is played

by the nontrivial disaffinity function as well. Thus, it would be interesting to analyze the geometry of the Riemannian manifold that admits a nontrivial disaffinity function f , which satisfies the Eikonal equation $\|\nabla f\| = 1$.

Recall that a vector field ξ on M^n is called torse-forming (cf. [14]) if it satisfies

$$\nabla_X \xi = \sigma X + \omega(X)\xi, \quad X \in \mathfrak{X}(M^n),$$

where σ is a function called the conformal scalar and ω is a 1-form called the generating form. For another situation, we may consider a compact Riemannian manifold M^n that admits a torse-forming vector field ξ . It would be interesting to study a torse-forming vector field ξ on a Riemannian manifold M^n that is also the disaffinity vector, and to investigate its impact on the geometry of M^n .

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Article

First Eigenvalues of Some Operators Under the Backward Ricci Flow on Bianchi Classes

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Abstract: Let $\lambda(t)$ be the first eigenvalue of the operator $-\Delta + aR^b$ on locally three-dimensional homogeneous manifolds along the backward Ricci flow, where a, b are real constants and R is the scalar curvature. In this paper, we study the properties of $\lambda(t)$ on Bianchi classes. We begin by deriving an evolution equation for the quantity $\lambda(t)$ on three-dimensional homogeneous manifolds in the context of the backward Ricci flow. Utilizing this equation, we subsequently establish a monotonic quantity that is contingent upon $\lambda(t)$. Additionally, we present both upper and lower bounds for $\lambda(t)$ within the framework of Bianchi classes.

Keywords: Ricci flow; p -Laplacian operator; eigenvalue

MSC: 53E20; 53C21; 53C40

1. Introduction

Over the last few years, the study of the eigenvalues of geometric operators has become more important in mathematics and physics. Understanding the behavior and nature of the eigenvalues of the operators reveals useful information about the topology and geometry of the underlying Riemannian manifolds.

Suppose that (M, g) is a Riemannian manifold, $-\Delta$ is the Laplace–Beltrami operator, and R is the scalar curvature corresponding to the metric g . Richard Hamilton first defined and introduced the concept of Ricci flow (see [1]). Perelman was able to prove Poincaré’s conjecture by using Ricci flow. In [2], Perelman showed that the first non-zero eigenvalue of the operator $-4\Delta + R$ was non-decreasing under Ricci flow. Perelman’s results were built on by many authors. Under the Ricci flow, Cao [3] extended Perelman’s results by considering evolving the first eigenvalue of $-\Delta + \frac{1}{2}R$ on Riemannian manifolds with $R \geq 0$, and in [4], he investigated the monotonicity of the first eigenvalue of $-\Delta + cR$, $c \geq \frac{1}{4}$ on closed Riemannian manifolds. Li [5] removed Cao’s curvature assumption for the first eigenvalue of $-\Delta + \frac{1}{2}R$ and came to a similar conclusion. In [6], Fang and Yang studied the first eigenvalues of $-\Delta_\phi + \frac{R}{2}$ under the Yamabe flow, where $-\Delta_\phi$ is the Witten–Laplacian. In [7], the first author investigated the first eigenvalue of $-\Delta_\phi + cS$ for $c > \frac{1}{4}$ under the rescaled List extended Ricci flow, where $S = R - \alpha|\nabla\phi|^2$, $\alpha > 0$ is a constant, and $\phi(t) : M \rightarrow \mathbb{R}$ is a smooth function such that the couple $(g(t), \phi(t))$ evolves under rescaled List extended Ricci flow as $\frac{\partial g}{\partial t} = -2Ric + 2\alpha\nabla \otimes \nabla\phi + \frac{2r}{n}g$ and $\frac{\partial \phi}{\partial t} = \Delta\phi$ and $r(t)$ is a smooth function only dependent on the t . Yang and Zhang [8] studied the first eigenvalue of $-\Delta + cR^a$ along the Ricci flow, where a, c are real constants. It is important to recognize that the Laplacian expressed as $\nabla^*\nabla + \frac{1}{4}R$ corresponds to the familiar Dirac Laplacian, where the equation $D = \nabla^*\nabla + \frac{1}{4}R$ defines the Dirac operator D where ∇^* is an adjoint of ∇ . For further details, please refer to sources [9] (p. 352), [10], [11] (p. 55), and [12].

Recently, the authors studied the eigenvalue of $-\Delta_\phi + cS^a$ along the rescaled List extended Ricci flow [13]. Hou, in [14], studied the first eigenvalue of $-\Delta$ under the backward Ricci flow. Then, Korouki and Razavi [15] obtained similar results for $(-\Delta - R)$ along the Ricci flow. In [16], Hou and Yang investigated the eigenvalues of $-\Delta + aR$, $a > 0$ on locally homogeneous three-dimensional manifolds along the backward Ricci flow.

In this paper, we consider the first eigenvalues of the operators $-\Delta + aR^b$ on locally three-dimensional homogeneous manifolds along the backward Ricci flow. Then, on Bianchi classes, we obtain some monotonic quantities and bounds for the first eigenvalues of the operators $-\Delta + aR^b$ along the backward Ricci flow. If $a = 0$, the findings presented in paper [14] will be derived. In the case where $a = -1$ and $b = 1$, the outcomes from article [15] will be achieved. Furthermore, when $b = 1$, the results from article [16] will also be obtained. Consequently, our findings represent a novel contribution and serve as a generalization of the results from the aforementioned studies.

2. Preliminaries

There are nine classes of three-dimensional locally homogeneous closed manifolds, which are divided into two families [17]. The first family includes of $H(3)$, $SO(3) \times \mathbb{R}^1$, and $H(2) \times \mathbb{R}^1$; the second family consists $\mathbb{R}^3$, $SL(2, \mathbb{R})$, $SU(2)$, $E(1, 1)$, $E(2)$, and *Heisenberg*. The second family is called Bianchi classes. Milnor [18] obtain a basis $\{X_i\}_{i=1}^3$ on Bianchi classes such that the initial metric and the Ricci tensors can be diagonalized with respect to this basis, and this property is maintained by the Ricci flow. Suppose that $\{\theta\}_{i=1}^3$ is the frame dual to $\{X_i\}_{i=1}^3$. Hence, we consider the following metric:

$$g(t) = \mathcal{J}(t)(\theta_1)^2 + \mathcal{K}(t)(\theta_2)^2 + \mathcal{L}(t)(\theta_3)^2.$$

Consider M to be a closed n -dimensional manifold with the Riemannian metric g_0 . The Ricci flow [19,20] is defined as follows:

$$\frac{d}{dt}g(t) + 2Ric(g(t)) = 0, \quad g(0) = g_0. \quad (1)$$

Here, Ric indicates the Ricci curvature tensor of $g(t)$. Let $g(t)$ be a solution of (1) and $d\mu$ be the volume form of $g(t)$. For $r = \frac{\int_M R d\mu}{\int_M d\mu}$, we assume

$$\gamma(t) = \frac{1}{1 - \frac{2}{n} \int_0^t r(\tau) d\tau}$$

and $\bar{t} = \int_0^t \gamma(\tau) d\tau$; then, $\bar{g}(\bar{t}) - \gamma(t)g(t) = 0$ solves the normalized Ricci flow

$$\frac{\partial \bar{g}_{ij}}{\partial t} = -2(\bar{R}_{ij} - \frac{r}{n} \bar{g}_{ij}). \quad (2)$$

Under the normalized Ricci flow, the volume of $(M^n, g(t))$ is a constant. Suppose that $g(t), t \in (-T_b, T_c)$ is the maximal solution of (2). We consider the backward Ricci flow equation

$$\frac{d}{dt}g(t) + 2Ric(g(t)) + \frac{2r}{n}g = 0, \quad g(0) = g_0, \quad (3)$$

which on locally homogeneous closed three-dimensional manifolds becomes

$$\frac{d}{dt}g(t) + 2Ric(g(t)) + \frac{2R}{3}g = 0, \quad g(0) = g_0. \quad (4)$$

See [21–24] for more detail of the backward Ricci flow.

Suppose that $\mathcal{I} \in [0, +\infty]$ is the maximal existence time for (4).

For real constants a and b , we say that λ is an eigenvalue of the operator $-\Delta + aR^b$ if there is a function u defined on M that satisfies the equation

$$-\Delta u + aR^b u = \lambda u. \quad (5)$$

In this scenario, u is called the eigenfunction corresponding to the eigenvalue λ . Now, we give an evolution formula for $\lambda(t)$ on three-homogeneous manifold along the backward Ricci flow.

Proposition 1. Suppose that the backward Ricci flow (4) on a locally homogeneous three-dimensional manifold has the solution $(M, g(t))$ and $\lambda(t)$ is the first eigenvalue of $-\Delta + aR^b$. Then,

$$\begin{aligned} \frac{d}{dt}\lambda(t) &= -\frac{2R}{3}\lambda(t) - 2 \int_M R_{ij}u_{ij}u \, d\mu + 2abR^{b-1}|Ric|^2 \\ &\quad + \frac{2}{3}a(1-b)R^{b+1}. \end{aligned} \quad (6)$$

Proof. Along the backward Ricci flow, we obtain

$$\frac{\partial}{\partial t}\Delta u = \Delta u_t + 2R_{ij}u_{ij} - \frac{2}{3}R\Delta u.$$

Therefore, we find

$$\begin{aligned} \frac{d}{dt}\lambda(t) &= \frac{d}{dt} \left(\int_M (-\Delta u + aR^b u) u \, d\mu \right) \\ &= \int_M \left(-2R_{ij}u_{ij} + \frac{2}{3}R\Delta u + abR^{b-1} \frac{\partial R}{\partial t} u \right) u \, d\mu \\ &\quad + \int_M (-\Delta u_t + aR^b u_t) u \, d\mu + \int_M (-\Delta u + aR^b u) \frac{d}{dt}(u \, d\mu). \end{aligned} \quad (7)$$

Integration by parts implies that

$$\int_M (-\Delta u_t + aR^b u_t) u \, d\mu = \int_M (-\Delta u + aR^b u) u_t \, d\mu.$$

From the normalization condition $\int_M u^2 d\mu = 1$, we conclude that

$$0 = \frac{d}{dt} \left(\int_M u^2 d\mu \right) = \int_M u(u_t d\mu + (u d\mu)_t).$$

Hence, we infer

$$\begin{aligned} &\int_M (-\Delta u_t + aR^b u_t) u \, d\mu + \int_M (-\Delta u + aR^b u) \frac{d}{dt}(u \, d\mu) \\ &= \int_M (-\Delta u + aR^b u) (u_t d\mu + (u d\mu)_t) = \lambda(t_0) \int_M u(u_t d\mu + (u d\mu)_t) = 0. \end{aligned} \quad (8)$$

Plugging (8) into (7), we have

$$\frac{d}{dt}\lambda(t) = \int_M \left(-2R_{ij}u_{ij} + \frac{2}{3}R\Delta u + abR^{b-1} \frac{\partial R}{\partial t} u \right) u \, d\mu. \quad (9)$$

Along the backward Ricci flow, we arrive at

$$\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2 - \frac{2}{3}R^2. \quad (10)$$

Applying (10) in the evolution formula (9), we deduce

$$\begin{aligned} \frac{d}{dt}\lambda(t) &= \int_M \left[-2R_{ij}u_{ij}u + \frac{2R}{3}u\Delta u \right] d\mu \\ &\quad + ab \int_M R^{b-1} \left[\Delta R + 2|Ric|^2 - \frac{2}{3}R^2 \right] u^2 d\mu. \end{aligned} \quad (11)$$

On locally homogeneous three-dimensional manifolds, the scalar curvature R is a constant; then, $\Delta R = 0$. Also, $|Ric|^2$ is a constant; hence, we obtain (6). $\square$

3. Bounds for $\lambda(t)$ on Bianchi Classes

Now, we are going to estimate the eigenvalue in Bianchi classes.

Case 1: $\mathbb{R}^3$

On $\mathbb{R}^3$, we have $g(t) = g_0$; thus, $\lambda(t)$ becomes a constant.

Case 2: Heisenberg H^3

For a given the metric $g_0 = \mathcal{J}_0(\theta_1)^2 + \mathcal{K}_0(\theta_2)^2 + \mathcal{L}_0(\theta_3)^2$ on Heisenberg, we have the frame $\{X_i\}_{i=1}^3$ as follows:

$$[X_2, X_3] = 2X_1, \quad [X_3, X_1] = 0, \quad [X_1, X_2] = 0,$$

also under the normalization $\mathcal{J}_0\mathcal{K}_0\mathcal{L}_0 = 4$, we have

$$\begin{aligned} R_{11} &= \frac{1}{2}\mathcal{J}^3, \quad R_{22} = -\frac{1}{2}\mathcal{J}^2\mathcal{K}, \quad R_{33} = -\frac{1}{2}\mathcal{J}^2\mathcal{L}, \\ R &= -\frac{1}{2}\mathcal{J}^2, \quad |Ric|^2 = \frac{3}{4}\mathcal{J}^4. \end{aligned}$$

Therefore, the backward Ricci flow equations become

$$\begin{cases} \frac{d\mathcal{J}}{dt} = \frac{4}{3}\mathcal{J}^3, & \mathcal{J}(0) = \mathcal{J}_0, \\ \frac{d\mathcal{K}}{dt} = -\frac{2}{3}\mathcal{J}^2\mathcal{K}, & \mathcal{K}(0) = \mathcal{K}_0, \\ \frac{d\mathcal{L}}{dt} = -\frac{2}{3}\mathcal{J}^2\mathcal{L}, & \mathcal{L}(0) = \mathcal{L}_0, \end{cases}$$

and this system has a solution as follows:

$$\mathcal{J} = \mathcal{J}_0 \left(1 - \frac{8}{3}\mathcal{J}_0^2 t \right)^{-\frac{1}{2}}, \quad \mathcal{K} = \mathcal{K}_0 \left(1 - \frac{8}{3}\mathcal{J}_0^2 t \right)^{\frac{1}{4}}, \quad \mathcal{L} = \mathcal{L}_0 \left(1 - \frac{8}{3}\mathcal{J}_0^2 t \right)^{\frac{1}{4}}.$$

Theorem 1. Suppose that $\lambda(t)$ is the first eigenvalue of $-\Delta + aR^b$, $a \geq 0$ on $(M = H^3, g_0)$. Let $\mathcal{I}$ be the maximal existence time for (4). Then, for any $b \geq 0$ such that $(-1)^b$ is a real constant and $b \neq \frac{1}{2}$, the quantity

$$\lambda(t) \left(1 - \frac{8}{3}\mathcal{J}_0^2 t \right)^{-\frac{1}{4}} - a \left(-\frac{1}{2} \right)^b \left(\frac{5}{3} + \frac{5}{6}b \right) \frac{3\mathcal{J}_0^{2b}}{2(4b+1)} \left(\left(1 - \frac{8}{3}\mathcal{J}_0^2 t \right)^{-b-\frac{1}{4}} - 1 \right)$$

is non-increasing along flow (4) and

$$\lambda(t) \leq \left(1 - \frac{8}{3}\mathcal{J}_0^2 t \right)^{\frac{1}{4}} \left[\lambda(0) - a \left(-\frac{1}{2} \right)^b \left(\frac{5}{3} + \frac{5}{6}b \right) \frac{3\mathcal{J}_0^{2b}}{2(4b+1)} \left(\left(1 - \frac{8}{3}\mathcal{J}_0^2 t \right)^{-b-\frac{1}{4}} - 1 \right) \right]. \quad (12)$$

Also, the quantity

$$\lambda(t) \left(1 - \frac{8}{3}\mathcal{J}_0^2 t \right)^{\frac{1}{2}} - a \left(-\frac{1}{2} \right)^b \left(-\frac{1}{3} + \frac{5}{6}b \right) \frac{3\mathcal{J}_0^{2b}}{4(-2b+1)} \left(\left(1 - \frac{8}{3}\mathcal{J}_0^2 t \right)^{-b+\frac{1}{2}} - 1 \right)$$

is non-decreasing along flow (4) and

$$\lambda(t) \geq \left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{-\frac{1}{2}} \left[\lambda(0) - a\left(-\frac{1}{2}\right)^b \left(-\frac{1}{3} + \frac{5}{6}b\right) \frac{3\mathcal{J}_0^{2b}}{4(-2b+1)} \left(\left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{-b+\frac{1}{2}} - 1 \right) \right] \quad (13)$$

on $[0, \mathcal{I})$.

Proof. Proposition 1 implies that

$$\begin{aligned} \frac{d}{dt}\lambda(t) &\leq \frac{1}{3}\mathcal{J}^2\lambda + \mathcal{J}^2 \int_M u \Delta u d\mu + \frac{3}{2}ab\left(-\frac{1}{2}\mathcal{J}^2\right)^{b-1}\mathcal{J}^4 + \frac{2}{3}\left(-\frac{1}{2}\mathcal{J}^2\right)^{b+1}a(1-b) \\ &= -\frac{2}{3}\mathcal{J}^2\lambda + a\left(-\frac{1}{2}\mathcal{J}^2\right)^b\mathcal{J}^2 + \frac{3}{2}ab\left(-\frac{1}{2}\mathcal{J}^2\right)^{b-1}\mathcal{J}^4 + \frac{2}{3}\left(-\frac{1}{2}\mathcal{J}^2\right)^{b+1}a(1-b) \\ &= -\frac{2}{3}\mathcal{J}^2\lambda + a\left(-\frac{1}{2}\right)^b\mathcal{J}^{2b+2}\left(\frac{5}{3} + \frac{5}{6}b\right). \end{aligned} \quad (14)$$

Then,

$$\frac{d}{dt} \left(\lambda(t) e^{\frac{2}{3} \int_0^t \mathcal{J}^2(s) ds} - \int_0^t a\left(-\frac{1}{2}\right)^b \mathcal{J}^{2b+2}(s) \left(\frac{5}{3} + \frac{5}{6}b\right) e^{\frac{2}{3} \int_0^s \mathcal{J}^2(\tau) d\tau} ds \right) \leq 0.$$

Since $\mathcal{J} = \mathcal{J}_0(1 - \frac{8}{3}\mathcal{J}_0^2 t)^{-\frac{1}{2}}$, then we have

$$\frac{2}{3} \int_0^t \mathcal{J}^2(s) ds = \ln(1 - \frac{8}{3}\mathcal{J}_0^2 t)^{-\frac{1}{4}}$$

and

$$\begin{aligned} \int_0^t \mathcal{J}^{2b+2}(s) e^{\frac{2}{3} \int_0^s \mathcal{J}^2(\tau) d\tau} ds &= \mathcal{J}_0^{2b+2} \int_0^s (1 - \frac{8}{3}\mathcal{J}_0^2 s)^{-b-\frac{5}{4}} ds \\ &= \frac{3\mathcal{J}_0^{2b}}{2(4b+1)} \left((1 - \frac{8}{3}\mathcal{J}_0^2 t)^{-b-\frac{1}{4}} - 1 \right). \end{aligned}$$

Therefore, the quantity

$$\lambda(t) \left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{-\frac{1}{4}} - a\left(-\frac{1}{2}\right)^b \left(\frac{5}{3} + \frac{5}{6}b\right) \frac{3\mathcal{J}_0^{2b}}{2(4b+1)} \left((1 - \frac{8}{3}\mathcal{J}_0^2 t)^{-b-\frac{1}{4}} - 1 \right)$$

is non-increasing along flow (4), and it follows from the integration that

$$\lambda(t) \leq \left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{\frac{1}{4}} \left[\lambda(0) - a\left(-\frac{1}{2}\right)^b \left(\frac{5}{3} + \frac{5}{6}b\right) \frac{3\mathcal{J}_0^{2b}}{2(4b+1)} \left((1 - \frac{8}{3}\mathcal{J}_0^2 t)^{-b-\frac{1}{4}} - 1 \right) \right]. \quad (15)$$

In a similar way, we also have

$$\begin{aligned} \frac{d}{dt}\lambda(t) &\geq \frac{1}{3}\mathcal{J}^2\lambda - \mathcal{J}^2 \int_M u \Delta u d\mu + \frac{3}{2}ab\left(-\frac{1}{2}\mathcal{J}^2\right)^{b-1}\mathcal{J}^4 + \frac{2}{3}\left(-\frac{1}{2}\mathcal{J}^2\right)^{b+1}a(1-b) \\ &= \frac{4}{3}\mathcal{J}^2\lambda + a\left(-\frac{1}{2}\right)^b\mathcal{J}^{2b+2}\left(-\frac{1}{3} + \frac{5}{6}b\right). \end{aligned} \quad (16)$$

Hence,

$$\frac{d}{dt} \left(\lambda(t) e^{-\frac{4}{3} \int_0^t \mathcal{J}^2(s) ds} - \int_0^t a\left(-\frac{1}{2}\right)^b \mathcal{J}^{2b+2}(s) \left(-\frac{1}{3} + \frac{5}{6}b\right) e^{-\frac{4}{3} \int_0^s \mathcal{J}^2(\tau) d\tau} ds \right) \geq 0.$$

Therefore, the quantity

$$\lambda(t) \left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{\frac{1}{2}} - a\left(-\frac{1}{2}\right)^b \left(-\frac{1}{3} + \frac{5}{6}b\right) \frac{3\mathcal{J}_0^{2b}}{4(-2b+1)} \left((1 - \frac{8}{3}\mathcal{J}_0^2 t)^{-b+\frac{1}{2}} - 1 \right)$$

is non-decreasing along flow (4), and from the integration, we obtain

$$\lambda(t) \geq \left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{-\frac{1}{2}} \left[\lambda(0) - a\left(-\frac{1}{2}\right)^b \left(-\frac{1}{3} + \frac{5}{6}b\right) \frac{3\mathcal{J}_0^{2b}}{4(-2b+1)} \left(\left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{-b+\frac{1}{2}} - 1 \right) \right]. \quad (17)$$

This completes the proof of our theorem. $\square$

Case 3: E(2)

E(2) represents the collection of isometries within the Euclidean plane.

The manifold E(2) is an Einstein manifold. For a given metric g_0 , there is a Milnor frame $\{X_i\}_{i=0}^3$ such that

$$[X_1, X_2] = 0, \quad [X_3, X_1] = 2X_2, \quad [X_2, X_3] = 2X_1.$$

In this case, under the normalization $\mathcal{J}_0\mathcal{K}_0\mathcal{L}_0 = 4$, we have

$$\begin{aligned} R_{11} &= \frac{1}{2}\mathcal{J}(\mathcal{J}^2 - \mathcal{K}^2), \quad R_{22} = \frac{1}{2}\mathcal{K}(\mathcal{K}^2 - \mathcal{J}^2), \quad R_{33} = -\frac{1}{2}\mathcal{L}(\mathcal{J} - \mathcal{K})^2, \\ R &= -\frac{1}{2}(\mathcal{J} - \mathcal{K})^2, \quad |\text{Ric}|^2 = \frac{1}{4}[2(\mathcal{J}^2 - \mathcal{K}^2)^2 + (\mathcal{J} - \mathcal{K})^4]. \end{aligned}$$

Cao and Saloff-Coste [22] proved the following:

- if $\mathcal{J}_0 = \mathcal{K}_0$, then there holds $\mathcal{J} = \mathcal{K}$, $\mathcal{I} = \infty$, and $g(t) = g_0$;
- If $\mathcal{J}_0 > \mathcal{K}_0$, then $\mathcal{J} > \mathcal{K}$, $\mathcal{I} < \infty$.

$$\mathcal{J}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I} - t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \eta_1(\mathcal{I} - t)^{\frac{1}{4}}, \quad \mathcal{L}(t) \sim \eta_2(\mathcal{I} - t)^{\frac{1}{4}}$$

as $t \rightarrow \mathcal{I}$, where η_1 and η_2 are two positive constants.

Theorem 2. Suppose that $\lambda(t)$ is a first eigenvalue of $-\Delta + aR^b$, $a \geq 0$ on a three-dimensional homogeneous Riemannian manifold ($M = E(2), g_0$). Let $1 \leq b$ be an odd number. Then, the following is true:

- If $\mathcal{J}_0 = \mathcal{K}_0$ then $\lambda(t) = \lambda(0)$;
- If $\mathcal{J}_0 > \mathcal{K}_0$ and $\mathcal{L}_0 > \mathcal{K}_0$, then the quantity

$$\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}^2(s)-\mathcal{K}^2(s))ds} - 3ab\left(\frac{1}{2}\right)^b \int_0^t (\mathcal{J}^2(s) - \mathcal{K}^2(s))^{b+1} e^{\frac{1}{3}\int_0^s(\mathcal{J}^2(r)-\mathcal{K}^2(r))dr} ds$$

is non-increasing along flow (4) and the quantity

$$\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}(s)-\mathcal{K}(s))^2ds} + \frac{1}{3}a(2+b)\left(\frac{1}{2}\right)^b \int_0^t (\mathcal{J}(s) - \mathcal{K}(s))^{2b+2} e^{\frac{1}{3}\int_0^s(\mathcal{J}(r)-\mathcal{K}(r))^2dr} ds$$

is non-decreasing under (4). Also,

$$\begin{aligned} \lambda(\tau) &\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} - \frac{c_2}{b+c_1}(\mathcal{I}-t)^{c_1} \left((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\right) \\ &\leq \lambda(t) \leq \lambda(\tau) \left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} + \frac{c_3}{b+c_1}(\mathcal{I}-t)^{c_1} \left((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\right) \end{aligned}$$

for $t \geq \tau$ and some positive constants c_1, c_2 and c_3 .

Proof. For the case $\mathcal{J}_0 = \mathcal{K}_0$, we have $g(t) = g_0$, and consequently $\lambda(t) = \lambda(0)$. For $\mathcal{J}_0 > \mathcal{K}_0$, $1 \leq b$ is an odd number, and $a > 0$; then, from [21], we obtain $\mathcal{J} > \mathcal{K}$ and from Proposition 1, so we have

$$\begin{aligned} \frac{d}{dt}\lambda(t) &\leq \frac{2}{3}(\mathcal{J} - \mathcal{K})^2\lambda(t) + (\mathcal{J}^2 - \mathcal{K}^2) \int_M u \Delta u d\mu + \frac{2}{3}a(1-b)\left(-\frac{1}{2}\right)^{b-1}(\mathcal{J} - \mathcal{K})^{2b+2} \\ &\quad + \frac{1}{2}ab\left(-\frac{1}{2}\right)^{b-1}(\mathcal{J} - \mathcal{K})^{2b-2}[2(\mathcal{J}^2 - \mathcal{K}^2)^2 + (\mathcal{J} - \mathcal{K})^4] \\ &= \left(\frac{2}{3}(\mathcal{J} - \mathcal{K})^2 - (\mathcal{J}^2 - \mathcal{K}^2)\right)\lambda(t) + a(\mathcal{J}^2 - \mathcal{K}^2)\left(-\frac{1}{2}\right)^b(\mathcal{J} - \mathcal{K})^{2b} \\ &\quad + \frac{2}{3}a(1-b)\left(-\frac{1}{2}\right)^{b-1}(\mathcal{J} - \mathcal{K})^{2b+2} \\ &\quad + \frac{1}{2}ab\left(-\frac{1}{2}\right)^{b-1}(\mathcal{J} - \mathcal{K})^{2b-2}[2(\mathcal{J}^2 - \mathcal{K}^2)^2 + (\mathcal{J} - \mathcal{K})^4]. \end{aligned}$$

Therefore, we obtain

$$\frac{d}{dt}\lambda(t) \leq -\frac{1}{3}(\mathcal{J}^2 - \mathcal{K}^2)\lambda(t) + 3ab\left(\frac{1}{2}\right)^b(\mathcal{J}^2 - \mathcal{K}^2)^{b+1}.$$

Then,

$$\frac{d}{dt}\left(\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}^2(s)-\mathcal{K}^2(s))ds} - 3ab\left(\frac{1}{2}\right)^b \int_0^t(\mathcal{J}^2(s) - \mathcal{K}^2(s))^{b+1}e^{\frac{1}{3}\int_0^s(\mathcal{J}^2(r)-\mathcal{K}^2(r))dr}ds\right) \leq 0$$

and it yields the quantity

$$\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}^2(s)-\mathcal{K}^2(s))ds} - 3ab\left(\frac{1}{2}\right)^b \int_0^t(\mathcal{J}^2(s) - \mathcal{K}^2(s))^{b+1}e^{\frac{1}{3}\int_0^s(\mathcal{J}^2(r)-\mathcal{K}^2(r))dr}ds$$

which is non-increasing along flow (4). Also, we conclude

$$\frac{d}{dt}\left(\lambda(t)e^{c_1\int_\tau^t(\mathcal{I}-s)^{-1}ds} - c_3 \int_\tau^t(\mathcal{I} - s)^{-b-1}e^{c_1\int_\tau^s(\mathcal{I}-r)^{-1}dr}ds\right) \leq 0.$$

Then,

$$\lambda(t) \leq \lambda(\tau)\left(\frac{\mathcal{I} - t}{\mathcal{I} - \tau}\right)^{c_1} + \frac{c_3}{b + c_1}(\mathcal{I} - t)^{c_1}\left((\mathcal{I} - t)^{-b-c_1} - (\mathcal{I} - \tau)^{-b-c_1}\right).$$

In a similar way, we obtain

$$\frac{d}{dt}\lambda(t) \geq -\frac{1}{3}(\mathcal{J} - \mathcal{K})^2\lambda(t) - a\left(\frac{1}{2}\right)^b\left(\frac{2}{3} + \frac{b}{3}\right)(\mathcal{J} - \mathcal{K})^{2b+2},$$

which implies that

$$\frac{d}{dt}\left(\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}(s)-\mathcal{K}(s))^2ds} + \frac{1}{3}a(2+b)\left(\frac{1}{2}\right)^b \int_0^t(\mathcal{J}(s) - \mathcal{K}(s))^{2b+2}e^{\frac{1}{3}\int_0^s(\mathcal{J}(r)-\mathcal{K}(r))^2dr}ds\right) \geq 0$$

and it yields the quantity

$$\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}(s)-\mathcal{K}(s))^2ds} + \frac{1}{3}a(2+b)\left(\frac{1}{2}\right)^b \int_0^t(\mathcal{J}(s) - \mathcal{K}(s))^{2b+2}e^{\frac{1}{3}\int_0^s(\mathcal{J}(r)-\mathcal{K}(r))^2dr}ds$$

which is non-decreasing along (4). We also infer

$$\frac{d}{dt}\left(\lambda(t)e^{c_1\int_\tau^t(\mathcal{I}-s)^{-1}ds} + c_2 \int_\tau^t(\mathcal{I} - s)^{-b-1}e^{c_1\int_\tau^s(\mathcal{I}-r)^{-1}dr}ds\right) \geq 0.$$

Then,

$$\lambda(t) \geq \lambda(\tau) \left(\frac{\mathcal{I} - t}{\mathcal{I} - \tau} \right)^{c_1} - \frac{c_2}{b + c_1} (\mathcal{I} - t)^{c_1} \left((\mathcal{I} - t)^{-b-c_1} - (\mathcal{I} - \tau)^{-b-c_1} \right).$$

□

Case 4: E(1,1)

$E(1,1)$ represents the collection of isometries within a two-dimensional plane characterized by a flat Lorentzian metric. In this context, there is an absence of an Einstein metric. Instead, all elements within this group exhibit asymptotic behavior akin to cigar degeneracies. Given a metric g_0 on $E(1,1)$, we choose a Milnor frame $\{X_i\}_{i=0}^3$ such that

$$[X_3, X_1] = 0, \quad [X_2, X_3] = 2X_1, \quad [X_1, X_2] = -2X_3. \quad .$$

Also, under the normalization $\mathcal{J}_0 \mathcal{K}_0 \mathcal{L}_0 = 4$, we have

$$R_{11} = \frac{1}{2} \mathcal{J} (\mathcal{J}^2 - \mathcal{L}^2), \quad R_{22} = -\frac{1}{2} \mathcal{K} (\mathcal{J} + \mathcal{L})^2, \quad R_{33} = \frac{1}{2} \mathcal{L} (\mathcal{L}^2 - \mathcal{J}^2), \\ R = -\frac{1}{2} (\mathcal{J} + \mathcal{L})^2, \quad |Ric|^2 = \frac{1}{4} [(2(\mathcal{J}^2 - \mathcal{L}^2)^2 + (\mathcal{J} + \mathcal{L})^2].$$

Under the condition $\mathcal{J}_0 \geq \mathcal{L}_0$, Cao and Saloff-Coste [22] proved the following:

- (i) if $\mathcal{J}_0 = \mathcal{L}_0$, then we have $\mathcal{J}(t) = \mathcal{L}(t)$, $\mathcal{I} = \frac{3}{32} \mathcal{K}_0$, and

$$\mathcal{J}(t) \sim \frac{\sqrt{6}}{4} (\mathcal{I} - t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \frac{32}{3} (\mathcal{I} - t), \quad t \in [0, \mathcal{I}).$$

- (ii) If $\mathcal{J}_0 > \mathcal{L}_0$, then $\mathcal{J} > \mathcal{L}$, $\mathcal{I} < \infty$, and

$$\mathcal{J}(t) \sim \frac{\sqrt{6}}{4} (\mathcal{I} - t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \eta_1 (\mathcal{I} - t)^{\frac{1}{4}}, \quad \mathcal{L}(t) \sim \eta_2 (\mathcal{I} - t)^{\frac{1}{4}}$$

as t goes to $\mathcal{I}$, where η_1 and η_2 are two positive constants and $\mathcal{I}$ is the maximal existence time.

Theorem 3. Suppose that $\lambda(t)$ is the first eigenvalue of $-\Delta + aR^b$, $a \geq 0$ on $(M = E(1,1), g_0)$. Let $1 \leq b$ be a odd number. Then, the quantity

$$\lambda(t) e^{\frac{2}{3} \int_0^t (\mathcal{J} + \mathcal{L})^2(s) ds} - \frac{1}{3} a \left(\frac{1}{2} \right)^b (2 + b) \int_0^s (\mathcal{J} + \mathcal{L})^{2b+2}(s) e^{\frac{2}{3} \int_0^t (\mathcal{J} + \mathcal{L})^2(r) dr} ds$$

is non-increasing along flow (4), and the quantity

$$\lambda(t) e^{\frac{2}{3} \int_0^t (\mathcal{J}^2(s) - \mathcal{L}^2(s)) ds} - \frac{1}{3} a \left(\frac{1}{2} \right)^b (2 + b) \int_0^s (\mathcal{J} + \mathcal{L})^{2b+2}(s) e^{\frac{2}{3} \int_0^t (\mathcal{J}^2(r) - \mathcal{L}^2(r)) dr} ds$$

is non-decreasing along flow (4). If $\mathcal{J}_0 \geq \mathcal{L}_0$, then

$$\lambda(\tau) \left(\frac{\mathcal{I} - t}{\mathcal{I} - \tau} \right)^{c_3} + \frac{c_2}{b + c_3} (\mathcal{I} - t)^{c_3} \left((\mathcal{I} - t)^{-b-c_3} - (\mathcal{I} - \tau)^{-b-c_3} \right) \\ \leq \lambda(t) \leq \lambda(\tau) \left(\frac{\mathcal{I} - t}{\mathcal{I} - \tau} \right)^{c_1} + \frac{c_2}{b + c_1} (\mathcal{I} - t)^{c_1} \left((\mathcal{I} - t)^{-b-c_1} - (\mathcal{I} - \tau)^{-b-c_1} \right) \quad (18)$$

for $t \geq \tau$ and some positive constants c_1 and c_3 .

Proof.

$$\begin{aligned} \frac{d}{dt} \lambda(t) &\leq \frac{1}{3}(\mathcal{J} + \mathcal{L})^2 \lambda(t) + (\mathcal{J} + \mathcal{L})^2 \int_M u \Delta u \, d\mu \\ &\quad + \frac{1}{2}ab \left(-\frac{1}{2}(\mathcal{J} + \mathcal{L})^2 \right)^{b-1} \left[(2(\mathcal{J}^2 - \mathcal{L}^2)^2 + (\mathcal{J} + \mathcal{L})^2) \right] \\ &\quad + \frac{2}{3}a(1-b) \left(-\frac{1}{2}(\mathcal{J} + \mathcal{L})^2 \right)^{b+1} \\ &\leq -\frac{2}{3}(\mathcal{J} + \mathcal{L})^2 \lambda(t) + \frac{1}{3}a \left(\frac{1}{2} \right)^b (2+b) (\mathcal{J} + \mathcal{L})^{2b+2}. \end{aligned}$$

Thus,

$$\frac{d}{dt} \left(\lambda(t) e^{\frac{2}{3} \int_0^t (\mathcal{J} + \mathcal{L})^2(s) ds} - \frac{1}{3} a \left(\frac{1}{2} \right)^b (2+b) \int_0^s (\mathcal{J} + \mathcal{L})^{2b+2}(s) e^{\frac{2}{3} \int_0^s (\mathcal{J} + \mathcal{L})^2(r) dr} ds \right) \leq 0,$$

which yields the quantity

$$\lambda(t) e^{\frac{2}{3} \int_0^t (\mathcal{J} + \mathcal{L})^2(s) ds} - \frac{1}{3} a \left(\frac{1}{2} \right)^b (2+b) \int_0^s (\mathcal{J} + \mathcal{L})^{2b+2}(s) e^{\frac{2}{3} \int_0^s (\mathcal{J} + \mathcal{L})^2(r) dr} ds$$

which is non-increasing along flow (4). Similarly, we have

$$\begin{aligned} \frac{d}{dt} \lambda(t) &\geq -\frac{2}{3}(\mathcal{J}^2 - \mathcal{L}^2) \lambda(t) + a(\mathcal{J}^2 - \mathcal{L}^2) \left(-\frac{1}{2}(\mathcal{J} + \mathcal{L})^2 \right)^b + \frac{2}{3}a(1-b) \left(-\frac{1}{2}(\mathcal{J} + \mathcal{L})^2 \right)^{b+1} \\ &\geq -\frac{2}{3}(\mathcal{J}^2 - \mathcal{L}^2) \lambda(t) \lambda(t) - \frac{1}{3}a \left(\frac{1}{2} \right)^b (2+b) (\mathcal{J} + \mathcal{L})^{2b+2}. \end{aligned}$$

Thus,

$$\frac{d}{dt} \left(\lambda(t) e^{\frac{2}{3} \int_0^t (\mathcal{J}^2(s) - \mathcal{L}^2(s)) ds} - \frac{1}{3} a \left(\frac{1}{2} \right)^b (2+b) \int_0^s (\mathcal{J} + \mathcal{L})^{2b+2}(s) e^{\frac{2}{3} \int_0^s (\mathcal{J}^2(r) - \mathcal{L}^2(r)) dr} ds, \right) \geq 0$$

which yields the quantity

$$\lambda(t) e^{\frac{2}{3} \int_0^t (\mathcal{J}^2(s) - \mathcal{L}^2(s)) ds} - \frac{1}{3} a \left(\frac{1}{2} \right)^b (2+b) \int_0^s (\mathcal{J} + \mathcal{L})^{2b+2}(s) e^{\frac{2}{3} \int_0^s (\mathcal{J}^2(r) - \mathcal{L}^2(r)) dr} ds$$

which is non-decreasing along flow (4). If $\mathcal{J}_0 \leq \mathcal{L}_0$, then $\frac{2}{3}(\mathcal{J} + \mathcal{L})^2 \sim c_1(\mathcal{I} - t)^{-1}$ and $\frac{2}{3}(\mathcal{J}^2 - \mathcal{L}^2) \sim c_3(\mathcal{I} - t)^{-1}$ for some positive constant c_1 . Hence,

$$\lambda(t) \leq \lambda(\tau) \left(\frac{\mathcal{I} - t}{\mathcal{I} - \tau} \right)^{c_1} + \frac{c_2}{b + c_1} (\mathcal{I} - t)^{c_1} \left((\mathcal{I} - t)^{-b-c_1} - (\mathcal{I} - \tau)^{-b-c_1} \right)$$

and

$$\lambda(t) \geq \lambda(\tau) \left(\frac{\mathcal{I} - t}{\mathcal{I} - \tau} \right)^{c_3} + \frac{c_2}{b + c_3} (\mathcal{I} - t)^{c_3} \left((\mathcal{I} - t)^{-b-c_3} - (\mathcal{I} - \tau)^{-b-c_3} \right)$$

which implies that (18) holds. $\square$

Case 5: SU(2)

A Manifold $SU(2)$ is an Einstein manifold. Given a metric g_0 on $SU(2)$, we choose a Milnor frame $\{X_i\}_{i=0}^3$ such that

$$[X_1, X_2] = 2X_3, \quad [X_3, X_1] = 2X_2, \quad [X_2, X_3] = 2X_1.$$

In this case, under the normalization $\mathcal{J}_0\mathcal{K}_0\mathcal{L}_0 = 4$, we have

$$\begin{aligned} R_{11} &= \frac{1}{2}\mathcal{J}[\mathcal{J}^2 - (\mathcal{K} - \mathcal{L})^2], & R_{22} &= \frac{1}{2}\mathcal{K}[\mathcal{K}^2 - (\mathcal{J} - \mathcal{L})^2], \\ R_{33} &= \frac{1}{2}\mathcal{L}[\mathcal{L}^2 - (\mathcal{J} - \mathcal{K})^2]. \end{aligned}$$

Then,

$$R = \frac{1}{2}[\mathcal{J}^2 - (\mathcal{K} - \mathcal{L})^2] + \frac{1}{2}[\mathcal{K}^2 - (\mathcal{J} - \mathcal{L})^2] + \frac{1}{2}[\mathcal{L}^2 - (\mathcal{J} - \mathcal{K})^2],$$

and

$$|\text{Ric}|^2 = \frac{1}{4}([\mathcal{J}^2 - (\mathcal{K} - \mathcal{L})^2]^2 + [\mathcal{K}^2 - (\mathcal{J} - \mathcal{L})^2]^2 + [\mathcal{L}^2 - (\mathcal{J} - \mathcal{K})^2]^2).$$

From [22], we have:

Theorem 4. Suppose that $g(t)$, $t \in [0, \mathcal{I})$ is the solution of (4) and $\mathcal{L}_0 \leq \mathcal{K}_0 \leq \mathcal{J}_0$. Then, the following is true:

- (i) If $\mathcal{L}_0 = \mathcal{K}_0 = \mathcal{J}_0$, then $\mathcal{I} = \infty$ and $g(t) = g_0$, $t \in [0, \infty)$;
- (ii) If $\mathcal{L}_0 < \mathcal{J}_0 = \mathcal{K}_0$, then $\mathcal{I} = \infty$, $\mathcal{J} = \mathcal{K} > \mathcal{L}$ and $\mathcal{J} \sim \frac{8}{3}t$, $\mathcal{L} \sim \frac{9}{16}t^{-2}$ as $t \in \infty$;
- (iii) If $\mathcal{L}_0 \leq \mathcal{K}_0 < \mathcal{J}_0$, then $\mathcal{I} < \infty$, $\mathcal{J} > \mathcal{K} \geq \mathcal{L}$ and there are two positive constants η_1 , η_2 , such that

$$\mathcal{J}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I} - t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \eta_1(\mathcal{I} - t)^{\frac{1}{4}}, \quad \mathcal{L}(t) \sim \eta_2(\mathcal{I} - t)^{\frac{1}{4}}$$

as t tends to $\mathcal{I}$.

Theorem 5. Suppose that $\lambda(t)$ is a first eigenvalue of $-\Delta + aR^b$, $a \geq 0$ on $(M = SU(2), g_0)$. Let $1 \leq b$ be an odd number. The following is then true:

- (i) If $\mathcal{J}_0 = \mathcal{K}_0 = \mathcal{L}_0$, then $\lambda(t) = \lambda(0)$.
- (ii) If $\mathcal{J}_0 = \mathcal{K}_0 > \mathcal{L}_0$, then the quantity

$$\lambda(t) - \frac{9}{4}ab2^b \int_0^t \mathcal{J}^{2b+2}(s)ds$$

is non-increasing along (4) and the quantity

$$\lambda(t)e^{-\frac{2}{3} \int_0^t \mathcal{J}^2(s)ds} + \frac{2}{3}a(-2 + 4b)2^b \int_0^t \mathcal{J}^{2b+2}(s)e^{-\frac{2}{3} \int_0^s \mathcal{J}^2(r)dr}ds$$

is non-decreasing under (4). Also,

$$\lambda(t) \leq \lambda(\tau) + \frac{6ab2^b}{2b+3} (t^{2b+3} - \tau^{2b+3})$$

for $\tau \leq t$, so τ is a fixed time.

- (iii) If $\mathcal{J}_0 > \mathcal{K}_0 \geq \mathcal{L}_0$, then the quantity

$$\lambda(t)e^{\frac{2}{3} \int_0^t \mathcal{J}^2(s)ds} - a2^b(1 + \frac{3}{4}b) \int_0^t \mathcal{J}^{2b+2}(s)e^{\frac{2}{3} \int_0^s \mathcal{J}^2(r)dr}ds$$

is non-increasing along (4) and the quantity

$$\lambda(t)e^{-\frac{2}{3} \int_0^t \mathcal{J}^2(s)ds} - \frac{2}{3}a(1-b)(\frac{9}{2})^{b+1} \int_0^t \mathcal{J}^{2b+2}(s)e^{-\frac{2}{3} \int_0^s \mathcal{J}^2(r)dr}ds$$

is non-decreasing under (4). Also,

$$\begin{aligned} \lambda(\tau) \left(\frac{\mathcal{I} - t}{\mathcal{I} - \tau} \right)^{c_1} + \frac{c_2}{b + c_1} (\mathcal{I} - t)^{c_1} \left((\mathcal{I} - t)^{-b-c_1} - (\mathcal{I} - \tau)^{-b-c_1} \right) \\ \leq \lambda(t) \leq \lambda(\tau) \left(\frac{\mathcal{I} - t}{\mathcal{I} - \tau} \right)^{c_1} + \frac{c_3}{b + c_1} (\mathcal{I} - t)^{c_1} \left((\mathcal{I} - t)^{-b-c_1} - (\mathcal{I} - \tau)^{-b-c_1} \right). \end{aligned}$$

Proof. (i) If $\mathcal{J}_0 = \mathcal{K}_0 = \mathcal{L}_0$, then we obtain $g(t) = g_0$, $t \in [0, \infty)$ and it follows that $\lambda(t) = \lambda(0)$.

(ii) If $\mathcal{J}_0 = \mathcal{K}_0 > \mathcal{L}_0$, then we have $\mathcal{J} = \mathcal{K} > \mathcal{L}$,

$$\begin{aligned} -2R_{11} = -2R_{22} = \mathcal{J} \left[\mathcal{L}^2 - 2\mathcal{J}\mathcal{L} \right] &\leq -\mathcal{J}\mathcal{L}^2, \\ -R_{33} &= -\mathcal{L}^3, \\ -\frac{2}{3}R &= \frac{1}{3}(\mathcal{L}^2 - 4\mathcal{J}\mathcal{L}) \leq -\mathcal{L}^2, \\ 2abR^{b-1}|\text{Ric}|^2 &= \frac{ab}{2} \left(-\frac{1}{2}\mathcal{L}^2 + 2\mathcal{J}\mathcal{L} \right)^{b-1} \left[2(-\mathcal{L}^2 + 2\mathcal{J}\mathcal{L})^2 + \mathcal{L}^4 \right] \\ &\leq \frac{9}{4}ab2^b \mathcal{J}^{2b+2}, \end{aligned}$$

and

$$\frac{2}{3}a(1-b)R^{b+1} = \frac{2}{3}a(1-b) \left(-\frac{1}{2}\mathcal{L}^2 + 2\mathcal{J}\mathcal{L} \right)^{b+1} \leq 0.$$

Therefore, based on Proposition 1, we can write

$$\begin{aligned} \frac{d}{dt}\lambda(t) &\leq -\mathcal{L}^2\lambda(t) - \mathcal{L}^2 \int_M u \Delta u d\mu + \frac{9}{4}ab2^b \mathcal{J}^{2b+2} \\ &= -a\mathcal{L}^2 \left(-\frac{1}{2}\mathcal{L}^2 + 2\mathcal{J}\mathcal{L} \right)^b + \frac{9}{4}ab2^b \mathcal{J}^{2b+2} \\ &\leq \frac{9}{4}ab2^b \mathcal{J}^{2b+2} \end{aligned}$$

which implies that the quantity

$$\lambda(t) - \frac{9}{4}ab2^b \int_0^t \mathcal{J}^{2b+2}(s) ds$$

is non-increasing along (4). Also, we infer

$$\lambda(t) \leq \lambda(\tau) + \frac{6ab2^b}{2b+3} \left(t^{2b+3} - \tau^{2b+3} \right)$$

for $t \geq \tau$. In a similar way for $b \geq 1$, we conclude

$$\begin{aligned} -2R_{11} = -2R_{22} = \mathcal{J} \left[\mathcal{L}^2 - 2\mathcal{J}\mathcal{L} \right] &\geq -2\mathcal{J}^3, \\ -R_{33} = -\mathcal{L}^3 &\geq -2\mathcal{J}^2\mathcal{L}, \\ -\frac{2}{3}R &= \frac{1}{3}(\mathcal{L}^2 - 4\mathcal{J}\mathcal{L}) \geq -\frac{4}{3}\mathcal{J}^2, \\ 2abR^{b-1}|\text{Ric}|^2 &\geq 0, \\ \frac{2}{3}a(1-b)R^{b+1} &\geq \frac{2}{3}a(1-b) \left(2\mathcal{J}^2 \right)^{b+1}. \end{aligned}$$

It follows from the above inequalities and Proposition 1 that

$$\begin{aligned}\frac{d}{dt}\lambda(t) &\geq -\frac{4}{3}\mathcal{J}^2\lambda(t) - 2\mathcal{J}^2 \int_M u\Delta u d\mu + \frac{4}{3}a(1-b)2^b \mathcal{J}^{2b+2} \\ &= \frac{2}{3}\mathcal{J}^2\lambda(t) - 2a\mathcal{J}^2\left(-\frac{1}{2}\mathcal{L}^2 + 2\mathcal{J}\mathcal{L}\right)^b + \frac{4}{3}a(1-b)2^b \mathcal{J}^{2b+2} \\ &\geq \frac{2}{3}\mathcal{J}^2\lambda(t) - 2a\mathcal{J}^2(2\mathcal{J}^2)^b + \frac{4}{3}a(1-b)2^b \mathcal{J}^{2b+2} \\ &= \frac{2}{3}\mathcal{J}^2\lambda(t) - \frac{1}{3}a2^b(-2+4b)\mathcal{J}^{2b+2}.\end{aligned}$$

It is easy to check that the quantity

$$\lambda(t)e^{-\frac{2}{3}\int_0^t \mathcal{J}^2(s)ds} + \frac{2}{3}a(-2+4b)2^b \int_0^t \mathcal{J}^{2b+2}(s)e^{-\frac{2}{3}\int_0^s \mathcal{J}^2(r)dr}ds$$

is non-decreasing along flow (4).

(iii) If $\mathcal{J}_0 > \mathcal{K}_0 \geq \mathcal{L}_0$ then $\mathcal{J} > \mathcal{K} \geq \mathcal{L}$,

$$\begin{aligned}-2R_{11} &\leq \mathcal{J}^3, \quad -2R_{22} \leq \mathcal{J}^2\mathcal{K}, \quad -2R_{33} \leq \mathcal{J}^2\mathcal{L}, \\ -\frac{2}{3}R &= \frac{1}{3}\left(\mathcal{J}^2 + \mathcal{K}^2 + \mathcal{L}^2 - 2\mathcal{J}\mathcal{K} - 2\mathcal{A}\mathcal{L} - 2\mathcal{K}\mathcal{L}\right) \leq \frac{1}{3}\mathcal{J}^2, \\ 2abR^{b-1}|Ric|^2 &= ab\left(\frac{1}{2}\right)^b\left(-\mathcal{J}^2 - \mathcal{K}^2 - \mathcal{L}^2 + 2\mathcal{J}\mathcal{K} + 2\mathcal{J}\mathcal{L} + 2\mathcal{K}\mathcal{L}\right) \\ &\quad \times \left([\mathcal{J}^2 - (\mathcal{K} - \mathcal{L})^2]^2 + [\mathcal{K}^2 - (\mathcal{J} - \mathcal{L})^2]^2 + [\mathcal{L}^2 - (\mathcal{J} - \mathcal{K})^2]^2\right) \\ &\leq ab\left(\frac{1}{2}\right)^b(4\mathcal{J}^2)^{b-1}(3\mathcal{J}^4) = \frac{3}{4}ab2^b \mathcal{J}^{2b+2}, \\ \frac{2}{3}a(1-b)R^{b+1} &\leq 0.\end{aligned}$$

According to Proposition 1 and the above inequalities, we find

$$\begin{aligned}\frac{d}{dt}\lambda(t) &\leq \frac{1}{3}\mathcal{J}^2\lambda(t) + \mathcal{J}^2 \int_M u\Delta u d\mu + \frac{3}{4}ab2^b \mathcal{J}^{2b+2} \\ &= -\frac{2}{3}\mathcal{J}^2\lambda(t) + a2^b \mathcal{J}^{2b+2} + \frac{3}{4}ab2^b \mathcal{J}^{2b+2} \\ &= -\frac{2}{3}\mathcal{J}^2\lambda(t) + a2^b\left(1 + \frac{3}{4}b\right)\mathcal{J}^{2b+2},\end{aligned}$$

which leads to the quantity

$$\lambda(t)e^{\frac{2}{3}\int_0^t \mathcal{J}^2(s)ds} - a2^b\left(1 + \frac{3}{4}b\right) \int_0^t \mathcal{J}^{2b+2}(s)e^{\frac{2}{3}\int_0^s \mathcal{J}^2(r)dr}ds$$

which is non-increasing along flow (4) and

$$\lambda(t) \leq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} + \frac{c_3}{b+c_1}(\mathcal{I}-t)^{c_1}\left((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\right).$$

Similarly, we have

$$\begin{aligned} -2R_{11} &= -\mathcal{J}[\mathcal{J}^2 - (\mathcal{K} - \mathcal{L})^2] \geq \mathcal{J}[-\mathcal{J}^2 + \mathcal{L}^2 - \mathcal{K}^2] \geq -2\mathcal{J}^3, \\ -2R_{22} &\leq -2\mathcal{J}^2\mathcal{K}, \quad -2R_{33} \leq -2\mathcal{J}^2\mathcal{L}, \quad -\frac{2}{3}R \geq -\frac{4}{3}\mathcal{J}^2, \\ 2abR^{b-1}|Ric|^2 &\geq 0, \\ \frac{2}{3}a(1-b)R^{b+1} &\geq \frac{2}{3}a(1-b)\left(\frac{9}{2}\right)^{b+1}\mathcal{J}^{2b+2}, \end{aligned}$$

then

$$\begin{aligned} \frac{d}{dt}\lambda(t) &\geq -\frac{4}{3}\mathcal{J}^2\lambda(t) - 2\mathcal{J}^2 \int_M u \Delta u d\mu + \frac{2}{3}a(1-b)\left(\frac{9}{2}\right)^{b+1}\mathcal{J}^{2b+2} \\ &\geq \frac{2}{3}\mathcal{J}^2\lambda(t) + \frac{2}{3}a(1-b)\left(\frac{9}{2}\right)^{b+1}\mathcal{J}^{2b+2}. \end{aligned}$$

We deduce that

$$\lambda(t)e^{-\frac{2}{3}\int_0^t \mathcal{J}^2(s)ds} - \frac{2}{3}a(1-b)\left(\frac{9}{2}\right)^{b+1} \int_0^t \mathcal{J}^{2b+2}(s)e^{-\frac{2}{3}\int_0^s \mathcal{J}^2(r)dr} ds$$

is non-decreasing under (4) and

$$\lambda(t) \geq \lambda(\tau) \left(\frac{\mathcal{I} - t}{\mathcal{I} - \tau} \right)^{c_1} + \frac{c_2}{b + c_1} (\mathcal{I} - t)^{c_1} \left((\mathcal{I} - t)^{-b-c_1} - (\mathcal{I} - \tau)^{-b-c_1} \right).$$

This completes the proof of our theorem. $\square$

Case 6: $SL(2, \mathbb{R})$

Using the Milnor frame $\{X_i\}_{i=0}^3$ on $SL(2, \mathbb{R})$, we obtain

$$[X_1, X_2] = 2X_3, \quad [X_3, X_1] = 2X_2, \quad [X_2, X_3] = -2X_1,$$

and in this case, under the normalization $\mathcal{J}_0\mathcal{K}_0\mathcal{L}_0 = 4$, we have

$$\begin{aligned} R_{11} &= \frac{1}{2}\mathcal{J}[\mathcal{J}^2 - (\mathcal{K} - \mathcal{L})^2], \quad R_{22} = \frac{1}{2}\mathcal{K}[\mathcal{K}^2 - (\mathcal{J} + \mathcal{L})^2] \\ R_{33} &= \frac{1}{2}\mathcal{L}[\mathcal{L}^2 - (\mathcal{J} + \mathcal{K})^2]. \end{aligned}$$

Then,

$$R = \frac{1}{2}[\mathcal{J}^2 - (\mathcal{K} - \mathcal{L})^2] + \frac{1}{2}[\mathcal{K}^2 - (\mathcal{J} + \mathcal{L})^2] + \frac{1}{2}[\mathcal{L}^2 - (\mathcal{J} + \mathcal{K})^2],$$

and

$$|Ric|^2 = \frac{1}{4}([\mathcal{J}^2 - (\mathcal{K} - \mathcal{L})^2]^2 + [\mathcal{K}^2 - (\mathcal{J} + \mathcal{L})^2]^2 + [\mathcal{L}^2 - (\mathcal{J} + \mathcal{K})^2]^2).$$

Under the condition $\mathcal{K}_0 \geq \mathcal{L}_0$, from [22], we have

(i) if $\mathcal{J}_0 \geq \mathcal{K}_0 \geq \mathcal{L}_0$, then $\mathcal{J} \geq \mathcal{K} \geq \mathcal{L}$ and

$$\mathcal{J}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I} - t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \eta_1(\mathcal{I} - t)^{\frac{1}{4}}, \quad \mathcal{L}(t) \sim \eta_2(\mathcal{I} - t)^{\frac{1}{4}}$$

as t goes to $\mathcal{I}$, where η_1 and η_2 are two positive constants.

(ii) If $\mathcal{J}_0 \leq \mathcal{K}_0 - \mathcal{L}_0$, then $\mathcal{J} \leq \mathcal{K} - \mathcal{L}$ and

$$\mathcal{J}(t) \sim \eta_1(\mathcal{I} - t)^{\frac{1}{4}}, \quad \mathcal{K}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I} - t)^{-\frac{1}{2}}, \quad \mathcal{L}(t) \sim \eta_2(\mathcal{I} - t)^{\frac{1}{4}}$$

as t goes to $\mathcal{I}$, where η_1 and η_2 are two positive constants.

(iii) If $\mathcal{J} < \mathcal{K} < \mathcal{J} + \mathcal{L}$ for all time $t \in [0, \mathcal{I})$, then we obtain

$$\mathcal{J}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I} - t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I} - t)^{-\frac{1}{2}}, \quad \mathcal{L}(t) \sim \eta_2(\mathcal{I} - t)$$

as t goes to $\mathcal{I}$, where η_1 is a positive constant.

Theorem 6. Suppose that $\lambda(t)$ is the first eigenvalue of $-\Delta + aR^b$, $a \geq 0$ on $(M = SL(2, \mathbb{R}), g_0)$. Let $1 \geq b$ be an even number.

If $\mathcal{J}_0 \geq \mathcal{K}_0 \geq \mathcal{L}_0$, then the quantity

$$\lambda(t)e^{\frac{11}{3} \int_0^t \mathcal{J}^2(s)ds} - \frac{1}{3}a\left(\frac{1}{2}\right)^b(11+b) \int_0^t \mathcal{J}^{2b+2}(s)e^{\frac{11}{3} \int_0^t \mathcal{J}^2(r)dr} ds$$

is non-increasing along flow (4) and the quantity

$$\lambda(t)e^{-\int_0^t \mathcal{J}^2(s)ds} - a\left(\frac{1}{2}\right)^b(1+9b) \int_0^t \mathcal{J}^{2b+2}(s)e^{-\int_0^t \mathcal{J}^2(r)dr} ds$$

is non-decreasing along (4). Also,

$$\begin{aligned} & \lambda(\tau) \left(\frac{\mathcal{I} - \tau}{\mathcal{I} - t} \right)^{\frac{3}{8}} + a\left(\frac{1}{2}\right)^b(1+9b) \frac{1}{b+\frac{3}{8}} (\mathcal{I} - t)^{-\frac{3}{8}} \left[(\mathcal{I} - t)^{-b+\frac{3}{8}} - (\mathcal{I} - \tau)^{-b+\frac{3}{8}} \right] \\ & \leq \lambda(t) \leq \lambda(\tau) \left(\frac{\mathcal{I} - t}{\mathcal{I} - \tau} \right)^{c_1} + \frac{c_2}{b+c_1} (\mathcal{I} - t)^{c_1} \left((\mathcal{I} - t)^{-b-c_1} - (\mathcal{I} - \tau)^{-b-c_1} \right), \end{aligned}$$

for $t \geq \tau$ and some positive constants c_1 and c_2 , so τ is a fixed time.

Proof. If $\mathcal{J}_0 \geq \mathcal{K}_0 \geq \mathcal{L}_0$, then $\mathcal{J} \geq \mathcal{K} \geq \mathcal{L}$, $R_{11} > 0$, $R_{22} < 0$, $R_{33} < 0$,

$$\begin{aligned} & -2R_{11} < 0, \\ & -2R_{22} = -\mathcal{K} \left[\mathcal{K}^2 - (\mathcal{J} + \mathcal{L})^2 \right] \leq 4\mathcal{K}\mathcal{J}^2, \\ & -2R_{33} = -\mathcal{L} \left[\mathcal{L}^2 - (\mathcal{J} + \mathcal{K})^2 \right] \leq 4\mathcal{L}\mathcal{J}^2, \\ & -\frac{2}{3}R = \frac{1}{3}(\mathcal{J} - \mathcal{K} - \mathcal{L})^2 \leq \frac{1}{3}\mathcal{J}^2, \\ & 2abR^{b-1}|\text{Ric}|^2 = -2ab\left(\frac{1}{2}\right)^{b-1}(\mathcal{J} - \mathcal{K} - \mathcal{L})^{2b-2}|\text{Ric}|^2 \leq 0, \end{aligned}$$

and

$$\frac{2}{3}a(1-b)R^{b+1} = \frac{2}{3}a(b-1)\left(\frac{1}{2}\right)^{b+1}(\mathcal{J} - \mathcal{K} - \mathcal{L})^{2b+2} \leq \frac{2}{3}a(b-1)\left(\frac{1}{2}\right)^{b+1}\mathcal{J}^{2b+2}.$$

Hence, Proposition 1 yields

$$\begin{aligned}\frac{d}{dt}\lambda(t) &\leq \frac{1}{3}\mathcal{J}^2\lambda(t) + 4\mathcal{J}^2 \int_M u\Delta u d\mu + \frac{2}{3}a(b-1)\left(\frac{1}{2}\right)^{b+1}\mathcal{J}^{2b+2} \\ &= -\frac{11}{3}\mathcal{J}^2\lambda(t) + 4a\mathcal{J}^2\left(\frac{1}{2}\right)^b(\mathcal{J} - \mathcal{K} - \mathcal{L})^b + \frac{2}{3}a(b-1)\left(\frac{1}{2}\right)^{b+1}\mathcal{J}^{2b+2} \\ &\leq -\frac{11}{3}\mathcal{J}^2\lambda(t) + \frac{1}{3}a\left(\frac{1}{2}\right)^b(11+b)\mathcal{J}^{2b+2}.\end{aligned}$$

Then, the quantity

$$\lambda(t)e^{\frac{11}{3}\int_0^t \mathcal{J}^2(s)ds} - \frac{1}{3}a\left(\frac{1}{2}\right)^b(11+b)\int_0^t \mathcal{J}^{2b+2}(s)e^{\frac{11}{3}\int_0^t \mathcal{J}^2(r)dr}ds$$

is non-increasing along flow (4) and

$$\lambda(t) \leq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} + \frac{c_2}{b+c_1}(\mathcal{I}-t)^{c_1}\left((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\right).$$

Also,

$$\begin{aligned}-2R_{11} &= \mathcal{J}\left[-\mathcal{J}^2 + (\mathcal{K} - \mathcal{L})^2\right] \geq -\mathcal{J}^3, \\ -2R_{22} &\geq -\mathcal{K}\mathcal{J}^2, \\ -2R_{33} &\geq -\mathcal{L}\mathcal{J}^2, \\ -\frac{2}{3}R &= \frac{1}{3}(\mathcal{J} - \mathcal{K} - \mathcal{L})^2 \geq 0, \\ 2abR^{b-1}|Ric|^2 &= -2ab\left(\frac{1}{2}\right)^{b-1}(\mathcal{J} - \mathcal{K} - \mathcal{L})^{2b-2}|Ric|^2 \geq -\frac{9ab}{2}\left(\frac{1}{2}\right)^{b-1}\mathcal{J}^{2b+2}, \\ \frac{2}{3}a(1-b)R^{b+1} &\geq 0.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{d}{dt}\lambda(t) &\geq -\mathcal{J}^2 \int_M u\Delta u d\mu - \frac{9ab}{2}\left(\frac{1}{2}\right)^{b-1}\mathcal{J}^{2b+2} \\ &\geq \mathcal{J}^2\lambda(t) - a\left(\frac{1}{2}\right)^b(1+9b)\mathcal{J}^{2b+2}.\end{aligned}$$

This yields the quantity

$$\lambda(t)e^{-\int_0^t \mathcal{J}^2(s)ds} - a\left(\frac{1}{2}\right)^b(1+9b)\int_0^t \mathcal{J}^{2b+2}(s)e^{-\int_0^t \mathcal{J}^2(r)dr}ds$$

is non-decreasing under (4) and

$$\lambda(t) \geq \lambda(\tau)\left(\frac{\mathcal{I}-\tau}{\mathcal{I}-t}\right)^{\frac{3}{8}} + a\left(\frac{1}{2}\right)^b(1+9b)\frac{1}{b+\frac{3}{8}}(\mathcal{I}-t)^{-\frac{3}{8}}\left[(\mathcal{I}-t)^{-b+\frac{3}{8}} - (\mathcal{I}-\tau)^{-b+\frac{3}{8}}\right].$$

□

4. Conclusions

In this paper, we study the first eigenvalues $\lambda(t)$ of the geometric perators $-\Delta + aR^b$ on locally three-dimensional homogeneous manifolds along the backward Ricci flow, where a, b are constants. First, we obtain an evolution formula for $\lambda(t)$ on three-homogeneous manifold along the backward Ricci flow. Then, on Bianchi classes, we obtain some monotonic quantities and bounds for $\lambda(t)$, which is constant on $\mathbb{R}^3$.

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Article

Modified $F(R, T)$ -Gravity Model Coupled with Magnetized Strange Quark Matter Fluid

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Abstract: This research note presents the properties of the $F(R, T)$ -gravity model in combination with magnetized strange quark matter. We obtain the equation of state for the magnetized strange quark matter in the $F(R, T)$ -gravity model endowed with the Lagrangian through of Ricci curvature. We also examine the Ricci solitons supported by a time-like conformal vector field in $F(R, T)$ -gravity, attached with magnetized strange quark matter fluid. Within this ongoing research, we give an estimate of the total quark pressure and total density in the phantom barrier and the radiation epochs of the Universe. Finally, using Ricci solitons, we study the various energy conditions, some black holes criteria, and Penrose's singularity theorem for magnetized strange quark matter fluid spacetime coupled with the $F(R, T)$ -gravity model.

Keywords: $F(R, T)$ -gravity; Ricci soliton; magnetized strange quark matter; conformal vector field; black holes

MSC: 53C44; 53B30; 53C50; 53C80

1. Introduction

Einstein's gravitational field equations present a standard method for examining known cosmic dynamics [1,2]. The most reliable description of observed data is given by Einstein's field equation. The $F(R, T)$ -gravity theory was formulated to tackle issues like the late-time acceleration of the Universe and the problems of dark matter and inflation in cosmological models.

In response to this scenario, several mathematicians and physicists have crafted advanced gravitational theories like $F(R)$ -gravity [3–5], $F(G)$ -gravity [6], and $F(R, T)$ -gravity theory [7], among others. These theories originated from the Einstein–Hilbert action. Diverging from Einstein's conventional gravity theory, these models hold promise as potential accurate approximations to quantum gravity [8]. Utilizing the Einstein–Hilbert Lagrangian density, General Relativity (GR) can be expanded into $F(R)$ gravity, where $F(R)$ is a function of the Ricci scalar R . Neutron stars of considerable size within $F(R)$ gravity are characterized by higher-order curvature effects; refer to works such as [9–11] for more details. However, concerns have been raised about the applicability of $F(R)$ gravity due to its limitations in achieving equilibrium within the solar system and its inability to support various astronomical models, including stable star structures (refer to [12,13] for further insights).

Harko et al. [14] introduced a more inclusive gravity model known as $F(R, T)$ -gravity theory, where they consider the Lagrangian to be an arbitrary function of T and R , with T denoting the trace of the energy–momentum tensor. This framework proved effective in describing the rapid expansion of the Universe during late cosmic times.

The notion of a quark star or a compact star, supported by the degenerate pressure of quark matter, has been proposed for stars smaller than neutron stars. Many researchers have delved into the characteristics of such quark stars. Alcock et al. [15] and Haensel et al. [16] suggest that a few neutron stars could potentially be exotic stars composed entirely of strange materials. Cheng et al. [17] investigated the properties of strange quark stars.

In this model, quarks are perceived as a degenerate Fermi gas restricted to the spatial area occupied by vacuum energy density. The quark content within this model comprises electrons, massless quarks u , massive quarks s , and quarks d .

According to the bag model, in scenarios where quarks are massless and non-interacting, quark pressure is anticipated.

$$\rho_q = 3p_q, \quad (1)$$

where the energy density of quarks is denoted by ρ_q . Furthermore, the total pressure p_M and total energy density ρ_M are expressed as

$$\rho_M = \rho_q + \mathcal{B}_c, \quad p_M = p_q - \mathcal{B}_c. \quad (2)$$

Ultimately, the equation of the state is employed to generate quark matter ($\mathcal{E}oS$):

$$p_M = \frac{\rho_M - 4\mathcal{B}_c}{3}, \quad (3)$$

Here, $\mathcal{B}_c$ is referred to as the bag constant. This concept is in line with the phenomenological bag model of quark matter, where the quark confinement is described by an energy term that is proportional to the volume.

Substantial alterations in the properties of strange matter occur when the electromagnetic scale coincides with nuclear scales. In a study by [18], a phenomenological bag model has been employed to examine quark matter in a strong magnetic field, revealing that the stability of strange quark matter is enhanced when the magnetic field surpasses a specific critical strength. Literature references [18,19] highlight that the presence of a magnetic field induces pressure anisotropy, with the bag model being considered the most suitable approach for investigating magnetized strange quark matter (MSQM) as discussed in [20–22].

Moreover, the extension of the quasiparticle model has been applied to analyze MSQM in [23], unveiling a bag function dependent on density and magnetic field that achieves the maximum saturation density at the quantum chromodynamic (QCD) scale parameter [24]. General relativity (GR) and cosmology are both employed to represent a time-constrained, 4-dimensional connected Lorentzian manifold [25,26].

When the Ricci tensor exhibits a specific form, these spacetimes are referred to as perfect fluid spacetimes within quasi-Einstein Lorentzian manifolds [26,27].

$$Ric = A_1 d + A_2 \eta \otimes \eta \quad (4)$$

In this scenario, scalars A_1 and A_2 are included, d is the Lorentzian metric, while the 1-form η is metrically equivalent to a unit time-like vector field. Furthermore, a Lorentzian spacetime manifold is one that accommodates a vector field resembling time [28].

The concept of a generalized quasi-Einstein manifold (GQE) is elaborated in the manner outlined in [29,30].

Definition 1. A non-flat Riemannian manifold (M^n, d) ($n > 2$) is termed a generalized quasi-Einstein Lorentzian manifold (GQE) when its Ricci tensor S of type $(0, 2)$ is non-zero and meets the condition:

$$Ric = A_1 d + A_2 \eta \otimes \eta + A_3 \theta \otimes \theta \quad (5)$$

wherein A_1, A_2 , and A_3 are smooth scalar functions of which $A_2 \neq 0$, $A_3 \neq 0$ and η, θ are 1-form such that

$$d(p, \gamma) = \eta(p), \quad d(p, \zeta) = \theta(p)$$

for any vector field $p \in \chi(M^n, d)$.

The unit vectors γ and ζ corresponding to 1-form η and θ are orthogonal to each other. In addition, γ and ζ are the generators of the manifold. If $A_3 = 0$, then (M^n, d) reduces to a perfect fluid spacetime.

Spacetime geometry, especially in the context of the General Theory of Relativity (GR), is intricately linked to the symmetry of physical matter. The solutions to field equations are often streamlined by the presence of symmetries in the metric. One notable form of symmetry is represented by solitons, which are connected to the geometric evolution of spacetime.

In 1988, Hamilton [31] introduced the concept of Ricci flow. The Ricci soliton emerges as the ultimate limit of solutions to the Ricci flow. The behavior of self-similar solutions or Ricci solitons often arises as limits of dilations of singularities in Ricci flow. They can be viewed as fixed points of Ricci flow, as a dynamical system, and the spaces of Riemannian metrics.

The Ricci flow equation is detailed in [31].

$$\frac{\partial}{\partial t} d = -2Ric, \quad (6)$$

A Riemannian manifold (M, d) that allows for a smooth vector field $\mathcal{F}$ is termed a Ricci soliton as per [32].

$$\frac{1}{2} \mathcal{L}_{\mathcal{F}} d + \lambda d + Ric = 0, \quad (7)$$

where the Lie derivative the Ricci tensor, and a real number are denoted by $\mathcal{L}_{\mathcal{F}}$, Ric , and λ , respectively. A Ricci soliton (RS) is said to be expanding, stable, or declining, with reference to (7), depending on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

Researchers in [33] delved into spacetime through the lens of the Ricci soliton. Additionally, Venkatesha and Aruna discussed Ricci solitons within the context of ideal fluid spacetime in [34]. Siddiqi et al. (refer to [35–37]) scrutinized spacetime utilizing solitons across a range of methodologies.

Currently, the investigation of quark matter fluid represents a captivating research area. Within General Relativity (GR), the exploration of quark matter fluid is conducted under various assumptions. Mak and Harko [38] have employed conformal motion to analyze quark matter within spherically symmetric spacetime. Agarwal and Pawar [39] delved into a cosmological model incorporating quark matter within the context of $F(R, T)$ -gravity theory. In a recent study in 2022, Siddiqi et al. [40,41] delved into the characteristics of $F(R, T)$ -gravity featuring perfect fluid matter and accommodating Ricci solitons, Yamabe solitons, Einstein solitons, gradient Ricci solitons, and gradient Yamabe solitons. Building upon prior investigations, we explore an $F(R, T)$ -gravity model within a magnetized strange quark matter fluid spacetime that admits a Ricci soliton in this paper.

2. Gravitational Field Equation of $F(R, T) = F_1(R) + F_2(T)$ -Gravity Model Attached with Magnetized Strange Quark Matter Fluid

In this section, we delve into the $F(R, T) = F_1(R) + F_2(T)$ -gravity model coupled with quark matter fluid. Given that this model hinges on the physical characteristics of the magnetized strange quark matter fluid, we can formulate a range of theoretical models across different values of R and T , as outlined in [14]. For illustrative purposes, we will consider the following model.

$$F(R, T) = F_1(R) + F_2(T), \quad (8)$$

where the functions of R and T are represented by $F_1(R)$ and $F_2(T)$, respectively.

The adjusted Einstein–Hilbert action term is considered as:

$$\Pi_E = \frac{1}{16\pi} \int [\mathbf{L}_m + F(R, T)] \sqrt{(-d)} d^4x, \quad (9)$$

where the Lagrangian is symbolized as $\mathbf{L}_m$. The energy tensor of the matter is given by

$$T_{ab} = \frac{-2\delta(\sqrt{-d})\mathbf{L}_m}{\sqrt{-d}\delta^{ab}}. \quad (10)$$

Assume that $\mathbf{L}_m$ is not contingent on its derivatives but is solely reliant on d_{ab} . The variation in action (9) concerning d_{ab} suggests

$$\begin{aligned} F'_1(R)Ric_{ab} - \frac{1}{2}F_1(R)d_{ab} + (d_{ab}\nabla_c\nabla^c - \nabla_a\nabla_b)F'_1(R) \\ = 8\pi T_{ab} - F'_2(T)T_{ab} - F'_2(T)\Xi_{ab} + \frac{1}{2}F_2(T)d_{ab}. \end{aligned} \quad (11)$$

where $F'_1(R) = \frac{\partial F(R, T)}{\partial R}$ and $F'_2(T) = \frac{\partial F(R, T)}{\partial T}$.

Standard notation is employed here; ∇_a and $\square \equiv \nabla_c\nabla^c$ represent the d'Alembert operator and covariant derivative, respectively. Additionally, we have

$$\Xi_{ab} = -2T_{ab} + d_{ab}\mathbf{L}_m - 2d^{lk}\frac{\partial^2\mathbf{L}_m}{\partial d^{ab}\partial d^{lk}}. \quad (12)$$

The field equation of the standard $F(R)$ -gravity model can be obtained anew by setting $F_2(T)$ to zero.

Consider a quark matter fluid characterized by the total pressure p_M , total energy density ρ_M , and velocity vector η^α . Given our flexibility in selecting $\mathbf{L}_m$, we set $\mathbf{L}_m = -p_M$.

The energy–momentum tensor for magnetized strange quark (MSQ) matter fluid is defined as per reference [23,42].

$$T_{ab}^{(MSQ)} = (p_M + \rho_M)\eta_a\eta_b + \left(\frac{h^2}{2} - p_M\right)d_{ab} - \theta_a\theta_b, \quad (13)$$

where

$$\eta^a\nabla_b\eta_a = 0, \quad \eta_a \cdot \eta^a = 1. \quad (14)$$

Moreover, where $\eta^a = (0, 0, 0, 1)$ serves as the four-velocity vector, the magnetic flux h^2 aligns in the x -direction due to the condition $\eta_a\theta_a = 0$. In this context, p_M stands for the proper pressure, and ρ_M denotes the energy density.

By utilizing Equations (12) and (13), we derive the variation of the stress energy–momentum tensor for the magnetized strange quark fluid as follows.

$$\Xi_{ab} = -2T_{ab} - p_M d_{ab}. \quad (15)$$

After adopting (8) and (11), we obtain

$$F'_1(R)Ric_{ab} = \frac{1}{2}F_1(R)d_{ab} + 8\pi T_{ab} - F'_2(T)T_{ab} - F'_2(T)\Xi_{ab} + \frac{1}{2}F_2(T)d_{ab}. \quad (16)$$

In view of (13)–(15), the gravitational field equation for magnetized strange quark matter fluid in $F(R, T) = F_1(R) + F_2(T)$ -gravity model (16) becomes

$$\begin{aligned} Ric_{ab} = & \frac{1}{F'_1(R)} \left\{ \frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F'_2(T)) \left(\frac{h^2}{2} - p_M \right) \right\} d_{ab} \\ & + \frac{1}{F'_1(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F'_2(T)) \right\} \eta_a \eta_b + \frac{1}{F'_1(R)} \left\{ (8\pi + F'_2(T)) \right\} \theta_a \theta_b, \end{aligned} \quad (17)$$

Upon contraction, Equation (17) simplifies to

$$R = \frac{2[F_1(R) + F_2(T)]}{F'_1(R)} + \frac{(32\pi + 4F'_2(T))}{F'_1(R)} [(\rho_M - 3p_M) + h^2]. \quad (18)$$

Therefore, for a spacetime (M^4, d) containing magnetized strange quark matter fluid within $F(R, T) = F_1(R) + F_2(T)$ -gravity, the Ricci tensor takes on the following structure:

$$Ric_{ab} = \alpha d_{ab} + \beta \eta_a \eta_b + \gamma \theta_a \theta_b, \quad (19)$$

where

$$\alpha = \frac{1}{F'_1(R)} \left\{ \frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F'_2(T)) \left(\frac{h^2}{2} - p_M \right) \right\}, \quad (20)$$

$$\beta = \frac{1}{F'_1(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F'_2(T)) \right\}, \quad \gamma = \frac{1}{F'_1(R)} \left\{ (8\pi + F'_2(T)) \right\}. \quad (21)$$

We assume that a and b are non-simultaneously zero throughout the document. A similar approach was employed in [40] to establish the formulation of the Ricci tensor. For the sake of coherence, we present the proof as well. As a result, we obtain

Theorem 1. *The Ricci tensor for the spacetime of magnetized strange quark matter fluid in the $F(R, T) = F_1(R) + F_2(T)$ -gravity model is given by*

$$\begin{aligned} Ric_{ab} = & \frac{1}{F'_1(R)} \left\{ \frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F'_2(T)) \left(\frac{h^2}{2} - p_M \right) \right\} d_{ab} \\ & + \frac{1}{F'_1(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F'_2(T)) \right\} \eta_a \eta_b + \frac{1}{F'_1(R)} \left\{ (8\pi + F'_2(T)) \right\} \theta_a \theta_b. \end{aligned}$$

Corollary 1. *The scalar curvature for the $F(R, T) = F_1(R) + F_2(T)$ -gravity model featuring magnetized strange quark matter fluid is defined as:*

$$R = \frac{2[F_1(R) + F_2(T)]}{F_1'(R)} + \frac{(32\pi + 4F_2'(T))}{F_1'(R)}[(\rho_M - 3p_M) + h^2].$$

In a notation devoid of indices, Equation (17) can now be represented as

$$Ric = \alpha d + \beta \eta \otimes \eta + \gamma \theta \otimes \theta. \quad (22)$$

By referencing Equations (1) and (17), we arrive at the subsequent outcome.

Theorem 2. A spacetime (M^4, d) in $F(R, T) = F_1(R) + F_2(T)$ -gravity featuring magnetized strange quark matter fluid represents a generalized quasi-Einstein spacetime.

Now, employing Equation (18), we derive

$$p_M = -\frac{\rho_M}{3} - \frac{1}{3} \left\{ h^2 - \frac{R}{F_1'}(R) - 2(F_1(R) + F_2(T))(32\pi + 4F_2'(T)) \right\}. \quad (23)$$

Referring to [43], for the fact that $p_M = \rho_M + F(r)$ is the equation of state for dark energy, with t being the cosmic time and $F(r)$ being a function of the scale factor " r ". The author also demonstrated that whereas $\omega < -1$ and $\omega > -1$ indicate a shift from phantom to non-phantom, $\omega = \frac{p_M}{\rho_M} = -1$ yields a phantom barrier.

Theorem 3. If the matter of $F(R, T) = F_1(R) + F_2(T)$ -gravity model is magnetized strange quark matter fluid, then $\mathcal{EoS}$ is given by (23).

Assume that the strange quark matter's $\mathcal{EoS}$ is $p_M = \frac{(\rho_M - 4\mathcal{B}_c)}{3}$. This finding, when used with Equation (23), yields

$$p_M = \left\{ \frac{RF_1'(R) - 2(F_1(R) + F_2(T))}{6(32\pi + 4F_2'(T))} - h^2 \right\} - \frac{2}{3}\mathcal{B}_c. \quad (24)$$

$$\rho_M = \left\{ \frac{RF_1'(R) - 2(F_1(R) + F_2(T))}{2(32\pi + 4F_2'(T))} - h^2 \right\} + 2\mathcal{B}_c. \quad (25)$$

Thus, we turn up the following outcomes.

Corollary 2. If the magnetized strange quark matter fluid in the $F(R, T) = F_1(R) + F_2(T)$ -gravity model obeys the $\mathcal{EoS}$ (23) for strange quark matter. Then (24) and (25) determines the total pressure p_M and the total energy density ρ_M , respectively.

In the case of phantom barrier, $\rho_M = -p_M = \left\{ h^2 - \frac{RF_1'(R) - 2(F_1(R) + F_2(T))}{2(32\pi + 4F_2'(T))} \right\} + \frac{2}{3}\mathcal{B}_c$. As a result, we may infer

Corollary 3. If a phantom barrier type source of matter is used in a $F(R, T) = F_1(R) + F_2(T)$ -gravity model filled with magnetized strange quark matter fluid, the total pressure and total energy density are calculated as

$$\rho_M = -p_M = \left\{ h^2 - \frac{RF_1'(R) - 2(F_1(R) + F_2(T))}{2(32\pi + 4F_2'(T))} \right\} + \frac{2}{3}\mathcal{B}_c. \quad (26)$$

3. Ricci Soliton on $F(R, T) = F_1(R) + F_2(T)$ -Gravity Model Attached with Magnetized Strange Quark Matter

In this section, we analyze the Ricci soliton (RS) within the $F(R, T)$ -gravity model, coupled with magnetized strange quark matter. The conformal vector field ζ in this scenario acts as the timelike velocity vector field.

As defined by Kuhnel and Rademacher [44], a conformal vector field $\mathcal{F}$ (CVF) on a Lorentzian spacetime manifold (M^4, d) is given by:

$$\mathcal{L}_{\mathcal{F}}d = 2\Omega d \quad (27)$$

wherein Ω is a smooth function on M^4 . $\mathcal{F}$ is homothetic when Ω is constant, and is Killing if $\Omega = 0$. Next, adopting $\mathcal{F} = \zeta$, Equation (7) becomes

$$(\mathcal{L}_{\zeta}d)(a, b) + 2Ric(a, b) + 2\lambda d(a, b) = 0. \quad (28)$$

In light of (27), we gain

$$Ric(a, b) + (\lambda + \Omega)d(a, b) = 0. \quad (29)$$

Substituting (22) into the preceding equation, we obtain

$$(\alpha + \lambda + \Omega)d(a, b) + \beta\eta(a)\eta(b) = 0. \quad (30)$$

By substituting $a = b = \zeta$ into (30), we deduce

$$\begin{aligned} \lambda = \frac{1}{F_1(R)} \left\{ \frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T)) \left(\frac{h^2}{2} - p_M \right) \right\} \\ - \left(\frac{1}{F_1'(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F_2'(T)) \right\} + \Omega \right). \end{aligned} \quad (31)$$

As such, we obtain the subsequent outcome.

Theorem 4. *If a spacetime (M^4, g) in the $F(R, T) = F_1(R) + F_2(T)$ -gravity model attached with magnetized strange quark matter fluid admits a RS (d, ζ, λ) with a CVF ζ , then RS is growing, stable, or decreasing, referring as*

1. $\frac{1}{F_1(R)} \left\{ \frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T)) \left(\frac{h^2}{2} - p_M \right) \right\} > \frac{1}{F_1'(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F_2'(T)) \right\} + \Omega,$
2. $\frac{1}{F_1(R)} \left\{ \frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T)) \left(\frac{h^2}{2} - p_M \right) \right\} = \frac{1}{F_1'(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F_2'(T)) \right\} + \Omega, \text{ and}$
3. $\frac{1}{F_1(R)} \left\{ \frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T)) \left(\frac{h^2}{2} - p_M \right) \right\} < \frac{1}{F_1'(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F_2'(T)) \right\} + \Omega, \text{ respectively, provided } \frac{1}{F_1'(R)} \neq 0.$

4. Energy Conditions in $F(R, T) = F_1(R) + F_2(T)$ -Gravity Model Attached with Magnetized Strange Quark Matter Fluid Admits Ricci Soliton

Referring to [45], we determine whether the criterion is met by the Ricci tensor Ric in the spacetime described by Equation (32), where

$$\text{Ric}(\zeta, \zeta) > 0, \quad (32)$$

holds for all timelike vector fields $\zeta \in \chi(M^4)$. In such cases, Equation (32) is termed the time-like convergence condition (TCC).

From Equation (17), we have

$$\text{Ric}(\zeta, \zeta) = \alpha + \beta.$$

The spacetime in question is valid if it satisfies the time-like convergence condition (TCC), meaning $\text{Ric}(\zeta, \zeta) > 0$.

$$\begin{aligned} & \frac{1}{F_1'(R)} \left\{ \frac{1}{2} (F_1(R) + F_2(T)) + (8\pi + F_2'(T)) \left(\frac{h^2}{2} - p_M \right) \right\} \\ & > \frac{1}{F_1'(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F_2'(T)) \right\} + \Omega. \end{aligned} \quad (33)$$

The spacetime adheres to the cosmological strong energy condition (SEC) [46]. With the provided details and considering Equation (33), we can conclude that

Theorem 5. *If a spacetime (M^4, d) in the $F(R, T) = F_1(R) + F_2(T)$ -gravity model attached with magnetized strange quark matter fluid admits a RS (d, ζ, λ) with a CVF ζ and satisfies TCC, then RS is growing.*

Remark 1. *In 1973, Hawking and Ellis demonstrated that [47]:*

- (i) *The time-like convergence condition (TCC) implies the cosmological strong energy condition (SEC) and the null convergence condition (NCC).*
- (ii) *The strong energy condition (SEC) implies the null energy condition (NEC).*

Consequently, it follows that TCC implies NCC as well.

Using Theorem 5 and Remark 1 together, we turn up the following outcomes:

Theorem 6. *If a spacetime (M^4, d) in the $F(R, T) = F_1(R) + F_2(T)$ -gravity model attached with magnetized strange quark matter fluid admits a growing RS (d, ζ, λ) with a CVF ζ , if (33) holds, then the quark matter fluid spacetime (M^4, d) in the $f(R, T)$ satisfies SEC.*

Corollary 4. *In the framework of the $F(R, T) = F_1(R) + F_2(T)$ -gravity model coupled with magnetized strange quark matter fluid, if a spacetime (M^4, d) allows for an expanding RS geometry (d, ζ, Λ) characterized by a conformal vector field ζ , and if condition (33) is met, then the spacetime of the quark matter fluid in the $f(R, T)$ model satisfies the null convergence condition (NCC).*

Corollary 5. *In the context of the $F(R, T) = F_1(R) + F_2(T)$ -gravity model coupled with magnetized strange quark matter fluid, if a spacetime (M^4, d) allows for an RS geometry (d, ζ, λ) characterized by a conformal vector field (CVF) ζ and satisfies the strong energy condition (SEC), then the Ricci tensor Ric in the expanding RS geometry belongs to the second Segre type [47].*

The Universe's evolution and the formation of galaxies were significantly influenced by dark matter, which is implied by gravitational effects that general relativity cannot account for unless there is more matter than can be seen [48].

Various changes to the standard rules of general relativity are advocated by dark matter. These consist of entropic gravity, tensor–vector–scalar gravity, and modified Newtonian

dynamics. None of the modified gravity theories that have been put forth thus far are able to account for all of the observational data simultaneously, indicating that dark matter of some kind will still be necessary even if gravity must be changed [48].

Remark 2. Strong energy conditions are mathematically imposed boundary restrictions that aim to reflect the idea that energy should be positive rather than physical constraints. However, negative pressure is a characteristic of the dark energy component. Dark energy slows the motion of large-scale structures while speeding up the Universe's expansions because it defies gravity [49]. The SEC violation is correlated with this negative.

Moreover, a necessary condition for the late accelerated expansion of the Universe is that the parameter of the dark energy equation (EoS) $\frac{p_M}{\rho_M} = \omega$ must be negative, i.e., $\omega < 0$. Therefore, if $Ric(\zeta, \zeta) \leq 0$, then the violation of SEC is necessary. Therefore, (33) holds only for non-expanding solitons such that

$$\begin{aligned} & \frac{1}{F_1'(R)} \left\{ \frac{1}{2} (F_1(R) + F_2(T)) + (8\pi + F_2'(T)) \left(\frac{h^2}{2} - p_M \right) \right\} \\ & \leq \frac{1}{F_1'(R)} \left\{ \left(\omega + \frac{h^2}{\rho_M} \right) (8\pi + F_2'(T)) \right\} + \Omega. \end{aligned} \quad (34)$$

The right-hand side of (34) disappears in the case of dark energy, $\omega = -\frac{h^2}{\rho_M} = -1$. Thus, we can articulate the following outcome.

Theorem 7. Let the source of matter be dark energy in the $F(R, T) = F_1(R) + F_2(T)$ -gravity model attached with magnetized strange quark matter fluid, which admits a non-growing RS ($d, \zeta, \lambda \leq 0$) with a CVF ζ , then SEC violates.

Moreover, we gain an interesting result with the violation of strong energy conditions.

Theorem 8. If the source of matter is dark energy in the $F(R, T) = F_1(R) + F_2(T)$ -gravity model attached with magnetized strange quark matter fluid admits a non-growing RS ($d, \zeta, \lambda \leq 0$) with a CVF ζ , then the magnetic flux h is equal to the total density ρ_M .

5. Application of Singularity Theorem in $F(R, T) = F_1(R) + F_2(T)$ -Gravity Model Attached with Magnetized Strange Quark Matter Fluid Admits a Ricci Soliton

Remark 3. Based on Penrose's singularity theorem, Vilenkin and Wall ([50]) showed that the spacetime M satisfies the null convergence condition (NCC), indicating the presence of black holes and a trapped surface outside these black holes within M .

In view of Theorem 5, Remarks 1 and 2, and Corollary 4, we can articulate the upcoming result.

Theorem 9. If a spacetime (M^4, d) in the $F(R, T) = F_1(R) + F_2(T)$ -gravity model coupled with magnetized strange quark matter fluid accommodates a growing RS geometry (d, ζ, Λ) characterized by a conformal vector field (CVF) ζ , and if the spacetime (M^4, g) satisfies the null convergence condition (NCC), then the spacetime of the magnetized strange quark matter fluid in (M^4, d) includes black holes with a trapped surface outside these black holes within the $F(R, T)$ -gravity model.

Corollary 6. *If a spacetime (M^4, d) in the $F(R, T) = F_1(R) + F_2(T)$ -gravity model coupled with magnetized strange quark matter fluid features a growing RS geometry (d, ζ, Λ) with a Killing vector field ζ , and if the spacetime (M^4, d) satisfies the null convergence condition (NCC), then the spacetime of the magnetized strange quark matter fluid in (M^4, d) includes black holes with a trapped surface outside these black holes within the $F(R, T)$ -gravity model.*

6. Solitonic Solutions for Magnetic Dilaton

The inclusion of a hypothetical particle known as the magnetic dilaton in the standard model. The magnetic dilaton is a scalar field that appears in gravity theories with additional dimensions [51] and in the field theory when conformal symmetry is spontaneously violated [52].

Certain cosmological issues are clarified by the magnetic dilaton field. Magnetic dilaton field theory provides an explanation for inflation and the existence of the cosmological constant [53]. Additionally, using a magnetic dilaton to study black holes could lead to new insights on black hole thermodynamics [54]. Additionally, composite Higgs models contain magnetic dilaton. In this case, a magnetic dilaton might act as a link between dark matter particles and the standard model.

Multidimensional theories of gravity, such as the superstring theory [55] and the 5-dimensional Kaluza–Klein theory [51], which combines gravitational and electromagnetic interactions, also predict magnetic dilaton. The diagonal elements of a multidimensional metric tensor d_{ab} , which correspond to the coordinates of additional spacetime dimensions, are where a magnetic dilaton is regarded in these theories as a scalar field. While elements d_{00} , d_{11} , d_{22} , and d_{33} reflect ordinary 4-dimensional spacetime, the magnetic dilaton field in a 5-dimensional scenario might be found in metric tensor element d_{44} [51].

Maxwell-dilaton theory [56], which specifically derives from the multidimensional theories of gravity [55], takes the magnetic dilaton into account. Electromagnetic fields can produce magnetic dilation, which can be found in investigations looking for particles that resemble axion [57].

In some astrophysical objects, magnetic dilatons [58], produced by different arrangements of electromagnetic fields may be investigated. Investigating revolving neutron stars (pulsars and magnetars) as dilaton sources is particularly noteworthy. These objects are known to generate coherent electromagnetic radiation and have strong magnetic fields between 10^8 and 10^{15} Gauss [59]. A detailed discussion of how revolving neutron stars produce dilatons may be found in [60].

The creation of magnetic dilaton during magnetic dipole radiation transmission in a galactic magnetic field. For relatively long distances L_{coh} , where L_{coh} is the magnetic field's coherence length, the galactic magnetic field can be regarded as uniform and constant [61]. $L_{coh} = \text{approx. } 100 \text{ pc}$ for the galactic magnetic field [62]. This is why galactic magnetic fields can contribute significantly to dilaton formation even though they are only about 10^{-5} Gauss [61].

The following is an expression for the density of the Lagrange function for the magnetic dilaton interacting with an electromagnetic field [60]:

$$\mathcal{L} = C_0(\partial\Psi) + C_1 e^{-2K\Psi} \mathcal{F}_{ab} \mathcal{F}^{ab}, \quad (35)$$

where the Maxwell tensor is $\mathcal{F}_{ab}$ and the gauge constants are C_0 , C_1 , and K . A massless scalar field Ψ is said to be a magnetic dilaton.

Superstring theory predicts the value of the constant $K = 1$. $K = \sqrt{3}$ is the value obtained from the 5-dimensional Kaluza–Klein theory result [51].

The density of the Lagrange function (35) yields the field equations, which take the following form:

$$\partial_k \partial^k \Psi = \frac{-C_1 K}{C_0} e^{-2K\Psi} \mathcal{F}_{ab} \mathcal{F}^{ab}, \quad (36)$$

$$\partial_a [e^{-2K\Psi} \mathcal{F}^{ab}] = 0. \quad (37)$$

The self-interacting magnetic dilaton field Ψ is represented by Equation (36). An electromagnetic field equation is shown in Equation (37). It also explains how the electromagnetic is affected by the magnetic dilaton field.

In the spacetime, the field Equations (36) and (37), will look like this with this approximation:

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi = \frac{C_1 K}{C_0} \mathcal{F}_{ab} \mathcal{F}^{ab}, \quad (38)$$

$$\partial_a \mathcal{F}^{ab} = 0. \quad (39)$$

Maxwellian electrodynamics without charges or currents is described by Equation (39).

The following is a representation of the invariant $\mathcal{F}_{ab} \mathcal{F}^{ab}$ in terms of the electric field E and magnetic field B :

$$\mathcal{F}_{ab} \mathcal{F}^{ab} = 2[B^2 - E^2]. \quad (40)$$

The Equation (38) will be written as follows in this case:

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi = \frac{C_1 K}{C_0} 2[B^2 - E^2]. \quad (41)$$

The source of the magnetic dilaton field can only be electromagnetic fields for which the invariant (39) is not equal to zero, per Equations (38) and (41) in the approximation under consideration. As a result, the magnetic dilaton field can be produced during the propagation of electromagnetic waves in an external electromagnetic field or in the vicinity of an electromagnetic source.

However, Kong and Liu studied the hyperbolic Ricci flow in 2010 [63]. This flow is composed of a system of second-order nonlinear evolution of partial differential equations. The wave characteristics of metrics and manifold curvatures are described by hyperbolic Ricci flow. Furthermore, gravity has the ability to create waves. Gravity waves generate spacetime rippling that spreads over the cosmos. A gravitational wave is essentially an oscillation of spacetime curvature that is traveling away from Earth.

For example, the hyperbolic Ricci flow [63], which is thus inspired by the Ricci flow, is characterized by the resulting evolution equation

$$\frac{1}{2} \frac{\partial^2}{\partial t^2} d(t) = -Ric(t) d(t), \quad d_0 = d(0), \quad \frac{\partial}{\partial t} d_{ij} = \Delta \Psi, \quad (42)$$

Therefore, a hyperbolic Ricci soliton is a self-similar solution of hyperbolic Ricci flow that is characterized as:

Definition 2 ([64]). *A hyperbolic Ricci soliton is a semi-Riemannian manifold (M^n, g) if, and only if, there is a vector field ζ on M and real scalars μ and λ such that*

$$\frac{1}{2} \mathcal{L}_\zeta \mathcal{L}_\zeta g + \lambda \mathcal{L}_\zeta g + Ric = \mu d, \quad (43)$$

where Ric is the Ricci curvature of M .

Now, in light of Equations (41)–(43), we can express the geometric flow for magnetic dilaton such that

$$\frac{\partial^2}{\partial t^2} \Psi = -c^2 \Delta \Psi + 2c^2 \frac{C_1 K}{C_0} [B^2 - E^2]. \quad (44)$$

Thus, the soliton is a self-similar solution of geometric flow (44) that is expressed as:

$$\frac{1}{2} \mathcal{L}_\zeta \mathcal{L}_\zeta \Psi + c^2 \Delta \Psi - 2c^2 \frac{C_1 K}{C_0} [B^2 - E^2] + \lambda \mathcal{L}_\zeta \Psi = \mu \Psi. \quad (45)$$

where metric d_Ψ induced by Ψ and d_Ψ depends nonlinearly on Ψ .

In (45), λ and μ show the types of solitons for magnetic dilaton and the rate of the underlying type, respectively. Moreover, μ represents the rate of change in the solutions and has geometric meaning as well. Depending on the constant μ , the soliton rate of change can be either shrinking, expanding, or approximately stable, regardless of whether $\mu < 0$, $\mu > 0$, or $\mu = 0$. Still, it is an open problem to discuss the rate of change of μ of hyperbolic Ricci solitons for Theorems 4–7 (see the chart in Figure 1).

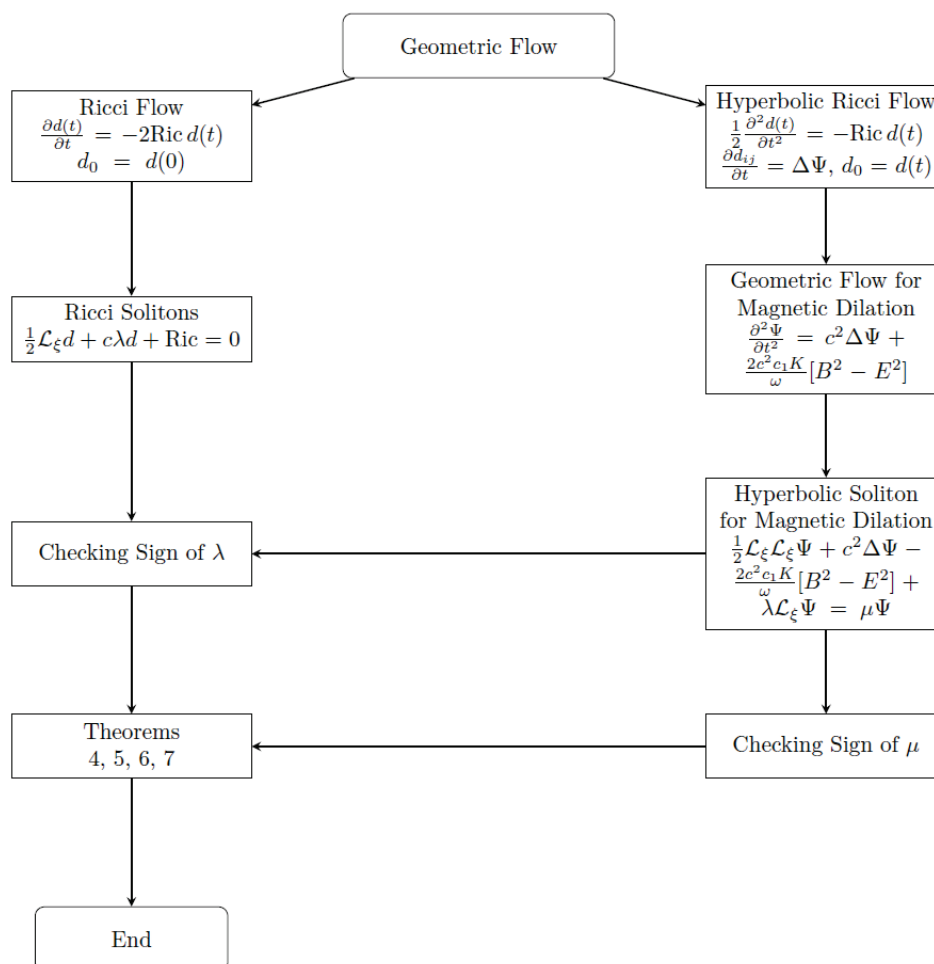


Figure 1. Analysis chart for the Ricci solitons and the solitons for magnetic dilaton.

7. Conclusions

$F(R, T)$ -gravity theory is effective in representing the fast expansion of the Universe throughout late cosmic eras.

We have revealed an extensive variety of phenomena from our study of the properties of magnetized strange quark matter in $F(R, T)$ -gravity, especially its interesting coupling

to Ricci solitons with a conformal vector field. We derive the gravitational field equation of the $F(R, T) = F_1(R) + F_2(T)$ -gravity model attached with magnetized strange quark matter fluid and derive the expression for the Ricci scalar for the spacetime of magnetized strange quark matter fluid in the specific $F(R, T)$ -gravity model.

Moreover, we gain the equation of state for magnetized strange quark matter fluid attached with the $F(R, T)$ -gravity framework and determine the total quark density and total pressure during the radiation and phantom barrier epochs of the Universe. Furthermore, we delve into the characteristics of magnetized strange quark matter in $F(R, T)$ -gravity, highlighting its association with Ricci solitons endowed with a time-like conformal vector field. After that, we provide certain criteria for the magnetized strange quark matter combined with the given $F(R, T)$ -gravity model, taking into account the black holes and different energy circumstances in terms of Ricci solitons.

Finally, we obtain a singularity theorem, which is based on the application of Penrose's singularity theorem for a spacetime that satisfies the null convergence condition indicating the presence of black holes, and explore a solitonic method on magnetic dilaton.

Some aspects of $F(R, T)$ -gravity, and important other field theory aspects:

Researchers have investigated the singularity in the research of black holes using contour integrals and assessed it in the energy component using Fourier transforms; some theories, such as transistors and perturbation theory, are even highly helpful in this regard. In addition to being commonly utilized to give dark matter abundances a geometric origin, the $F(R, T)$ -gravity model has garnered more interest recently in the context of astrophysical applications. Furthermore, it has been suggested as an explanation for cosmic inflation.

The $F(R, T)$ -gravity theory, which generalizes the EH Lagrangian to be an arbitrary function of the Ricci scalar R and energy–momentum tensor T , is one of the most extensively researched variations of Einstein's classical explanation of gravitational waves. The technical nature of these theories restricts analytical knowledge to systems with high symmetry or perturbations of such systems, despite the fact that significant progress has been made in grasping the consequences of such theories. Numerical relativity has made significant contributions to our understanding of space–time dynamics in classical general relativity in recent years, and it offers some optimism that concerns in $F(R, T)$ gravity will be similarly applicable.

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Article

Differential Geometry and Matrix-Based Generalizations of the Pythagorean Theorem in Space Forms

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Abstract: In this work, we consider Pythagorean triples and quadruples using fundamental form matrices of hypersurfaces in three- and four-dimensional space forms and illustrate various figures. Moreover, we generalize that an immersed hypersphere M^n with radius r in an $(n + 1)$ -dimensional Riemannian space form $\mathbb{M}^{n+1}(\mathbf{c})$, where the constant sectional curvature is $\mathbf{c} \in \{-1, 0, 1\}$, satisfies the $(n + 1)$ -tuple Pythagorean formula $\mathcal{P}_{n+1}$. Remarkably, as the dimension $n \rightarrow \infty$ and the fundamental form $N \rightarrow \infty$, we reveal that the radius of the hypersphere converges to $r \rightarrow \frac{1}{\sqrt{2}}$. Finally, we propose that the determinant of the $\mathcal{P}_{n+1}$ formula characterizes an umbilical round hypersphere satisfying $k_1 = k_2 = \dots = k_n$, i.e., $H^n = K_c$ in $\mathbb{M}^{n+1}(\mathbf{c})$.

Keywords: space forms; Pythagorean triples; Pythagorean quadruples; Pythagorean $(n + 1)$ -tuples; hypersurface; hypersphere; radius; fundamental form matrices

MSC: 53A07; 53C42; 15A09; 15A24

1. Introduction

The Pythagorean theorem is one of the most fundamental and enduring results in mathematics. This article explores its geometric significance through differential geometry, extending its classical formulation to higher-dimensional space forms.

The Pythagorean (or Pythagoras' of Samos (570–495 BC)) Theorem: *Geometrically: The sum of (the areas of) the two small squares equals (the area of) the big one. Algebraically: $a^2 + b^2 = c^2$, where a and b are the legs of the triangle and c is the hypotenuse.*

For centuries, people have been cited and used this theorem. Mordell [1] and Nagell [2] gave the Pythagorean triples and quadruples, algebraically, in their books.

Arnold and Eydelzon [3] studied matrix Pythagorean triples. Crasmareanu [4] introduced a new method to find Pythagorean triples with matrices. Aydın and Mihai [5] worked on surfaces with Pythagorean fundamental forms in 3-space forms. Palmer, Ahuja, and Tikoo [6] studied the Pythagorean triples preserving matrices. Some authors, such as [7–9], gave the Pythagorean theorem in their books, historically.

Remarkably, the Babylonians empirically knew the hypotenuse length of a square, i.e., $\sqrt{2}$. See Figure 1 (Left and Middle: written in the original Babylonian sexagesimal system using base 60; Right: converted to our decimal system using base 10) for the best-

known ancient Babylonian tablet, known as YBC 7289, which is approximately 4000 years old and related to the Pythagorean theorem [7,9].

See [10,11] for more details on the Babylonian clay tablet YBC 7289.

Next, we give the fundamental notions of the differential geometry of hypersurfaces. See [12] for details.

Let $\mathbb{E}^m$ denote the Euclidean m -space, where $(x_1, x_2, \dots, x_m)$ is a rectangular coordinate system, with the canonical Euclidean metric tensor defined by $\tilde{g} = \langle \cdot, \cdot \rangle = \sum_{i=1}^m dx_i^2$.

We denote the Levi-Civita connections [13] of the manifold $\tilde{M}$, and its submanifold M of $\mathbb{E}^m$ by $\tilde{\nabla}$, ∇ , respectively. We use the letters X, Y, Z, W (resp., ξ, η) to denote the vectors fields tangent (resp., normal) to M . The Gauss and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (1)$$

$$\tilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \quad (2)$$

where h , D , and A are the second fundamental form, the metric connection, and the shape operator of M , respectively. The shape operator A_ξ is a symmetric endomorphism of the tangent space $T_p M$ at $p \in M$ for each $\xi \in T_p^\perp M$. The shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The Gauss and Codazzi equations are, respectively, defined by

$$\langle R(X, Y)Z, W \rangle = \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle, \quad (3)$$

$$(\tilde{\nabla}_X h)(Y, Z) = (\tilde{\nabla}_Y h)(X, Z), \quad (4)$$

where R, R^D are the curvature tensors related with connections ∇ and D , respectively, and then $\tilde{\nabla}h$ is given by the Levi-Civita connection

$$(\tilde{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

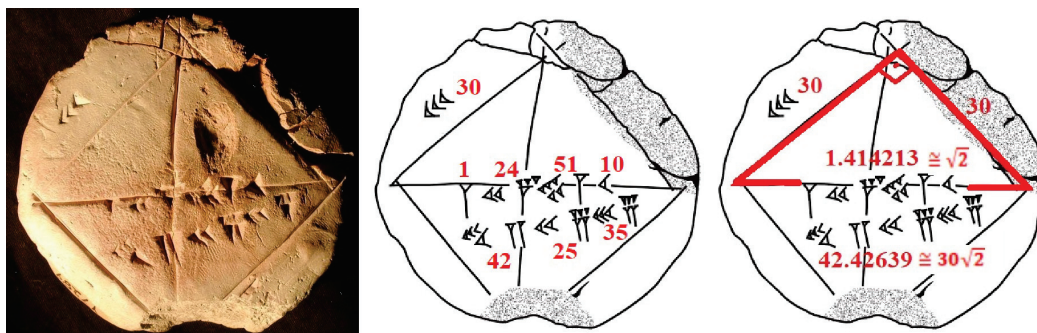


Figure 1. The best known old Babylonian tablet from Mesopotamia: YBC 7289. (Left) Original tablet. (Middle) Babylonian numbers and their decoded values. (Right) Decimal number transformation.

Let M be an oriented hypersurface in $\mathbb{E}^{n+1}$, S its shape operator, and x its position vector. Considering a local orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ consisting of principal directions of M corresponding from the principal curvature k_i for $i = 1, 2, \dots, n$, we assume

the dual basis of this frame field be $\{\theta_1, \theta_2, \dots, \theta_n\}$. Hence, the first structural equation of Cartan is given by

$$d\theta_i = \sum_{j=1}^n \theta_j \wedge \omega_{ij}, \quad i, j = 1, 2, \dots, n, \quad (5)$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field. Denoting the Levi-Civita connections of M in $\mathbb{E}^{n+1}$ by ∇ and $\tilde{\nabla}$, respectively, from the Codazzi Equation (3), we obtain

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j), \quad (6)$$

$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l) \quad (7)$$

for distinct $i, j, l = 1, 2, \dots, n$.

Taking $s_j = \sigma_j(k_1, k_2, \dots, k_n)$, where σ_j is the j -th symmetric function defined by

$$\sigma_j(a_1, a_2, \dots, a_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq n} a_{i_1} a_{i_2} \dots a_{i_j},$$

we use the following notation:

$$r_i^j = \sigma_j(k_1, k_2, \dots, k_{i-1}, k_{i+1}, k_{i+2}, \dots, k_n).$$

By the definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \dots = 0$. Calling the function s_k as the k -th mean curvature of M , the functions $H = \frac{1}{n}s_1$ and $K = s_n$ are named the mean curvature and the Gauss–Kronecker curvature of M , respectively. M is called the j -minimal if $s_j \equiv 0$ on M .

In $\mathbb{E}^{n+1}$, finding the i -th curvature formulas $\mathfrak{C}_i$, where $i = 0, \dots, n$, we use the following characteristic polynomial of $\mathbf{S}$:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda \mathcal{I}_n) = \sum_{k=0}^n (-1)^k s_k \lambda^{n-k}, \quad (8)$$

and $\mathcal{I}_n$ denotes the identity matrix of order n . Then, we obtain curvature formulas $\binom{n}{i} \mathfrak{C}_i = s_i$. Here, $\binom{n}{0} \mathfrak{C}_0 = s_0 = 1$ (by definition), $\binom{n}{1} \mathfrak{C}_1 = s_1, \dots, \binom{n}{n} \mathfrak{C}_n = s_n = K$.

k -th fundamental form of M is defined by

$$(\mathbf{S}^{k-1}(X), Y) = \langle \mathbf{S}^{k-1}(X), Y \rangle.$$

Then, one obtains

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \mathfrak{C}_i I(\mathbf{S}^{n-i}(X), Y) = 0. \quad (9)$$

Recently, Güler [14] introduced curvature formulas for hypersurfaces in four-dimensional space. For further details on the nature of hypersurfaces, see also [15–18].

In this work, we present several results related to the Pythagorean $\mathcal{P}_{n+1}$ formula. In Section 2, we introduce the Pythagorean triples $\mathcal{P}_3$ using fundamental form matrices of surfaces in three-dimensional space forms.

In Section 3, we examine Pythagorean quadruples $\mathcal{P}_4$ using fundamental form matrices of hypersurfaces in four-dimensional space forms $\mathbb{M}^4(\mathbf{c})$, where $\mathbf{c} \in \{-1, 0, 1\}$.

In Section 4, we generalize the Pythagorean formula for hypersurfaces immersed in $\mathbb{M}^{n+1}(\mathbf{c})$ using matrices corresponding to the fundamental forms $I, II, III, \dots, N, N+1$ of

the hypersurface. Additionally, we show that an immersed hypersphere M^n with radius r in $\mathbb{M}^{n+1}(\mathbf{c})$ satisfies the Pythagorean $(n+1)$ -tuple equation $\mathcal{P}_{n+1}$. Remarkably, as $n \rightarrow \infty$ and $N \rightarrow \infty$, we reveal that the radius converges to $r \rightarrow \frac{1}{\sqrt{2}}$ for the hypersphere M^n . Finally, we show that the determinant of the $\mathcal{P}_{n+1}$ formula characterizes an umbilical round hypersphere satisfying $k_1 = k_2 = \dots = k_n$, i.e., $H^n = K_e$ in $\mathbb{M}^{n+1}(\mathbf{c})$.

We also provide a conclusion in Section 5.

2. The Pythagorean Triples $\mathcal{P}_3$

Using the fundamental form matrices, we consider the Pythagorean triples Equation as follows

$$I^2 + II^2 = III^2. \quad (10)$$

For any surface, taking $n = 2$ in Equation (9), we have

$$\mathfrak{C}_0 III - 2\mathfrak{C}_1 II + \mathfrak{C}_2 I = 0, \quad (11)$$

where $\mathfrak{C}_0 = 1$ (by definition), $\mathfrak{C}_1 = H$ (mean curvature), $\mathfrak{C}_2 = K = K_e$ (extrinsic Gaussian curvature, where $K = K_e + \mathbf{c}$).

Next, we obtain the solutions of the Pythagorean triples formula $\mathcal{P}_3$ with the fundamental form matrices, using two different ways, excluding the way of Aydın and Mihai [5].

2.1. First Solution of the Pythagorean Triples Formula $\mathcal{P}_3$

Substituting III of Equation (11) into the right side of Equation (10), we obtain the Pythagorean triples formula $\mathcal{P}_3$, as follows

$$I^2 + II^2 = (2H II - K_e I)^2,$$

considering that $\mathcal{A}\mathcal{B} \neq \mathcal{B}\mathcal{A}$ for matrices in general, and taking $II = I\mathbf{S}$, we have the following equation

$$\left[(1 - 4H^2)(\mathbf{S}) + 2HK_e(\mathcal{I}_2) \right] (I\mathbf{S}) - \left[(K_e^2 - 1)(\mathcal{I}_2) - 2HK_e(\mathbf{S}) \right] (I) = 0. \quad (12)$$

Substituting the following matrices: shape operator $\mathbf{S} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$, the first fundamental form $I = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, and identity $\mathcal{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ into the Equation (12), we obtain

$$\begin{pmatrix} a_{11}\mathcal{X}_{11} & a_{12}\mathcal{X}_{12} \\ a_{12}\mathcal{X}_{12} & a_{22}\mathcal{X}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (13)$$

where

$$\mathcal{X}_{11} = (1 - 4H^2)k_1^2 + 4HK_ek_1 + 1 - K_e^2, \quad (14)$$

$$\mathcal{X}_{22} = (1 - 4H^2)k_2^2 + 4HK_ek_2 + 1 - K_e^2, \quad (15)$$

$$\mathcal{X}_{12} = (1 - 4H^2)k_1k_2 + 2HK_e(k_1 + k_2) + 1 - K_e^2. \quad (16)$$

Here, $a_{11} \neq 0$, $a_{12} \neq 0$, $a_{22} \neq 0$, since $I \neq 0$. Hence, we have $\mathcal{X}_{ij} = 0$, as follows

$$(1 - 4H^2)k_1^2 + 4HK_e k_1 + 1 - K_e^2 = 0, \quad (17)$$

$$(1 - 4H^2)k_2^2 + 4HK_e k_2 + 1 - K_e^2 = 0, \quad (18)$$

$$(1 - 4H^2)k_1 k_2 + 2HK_e(k_1 + k_2) + 1 - K_e^2 = 0. \quad (19)$$

Adding Equations (17) and (18), and extending Equation (19) by 2, respectively, we obtain

$$(1 - 4H^2)(k_1^2 + k_2^2) + 4HK_e(k_1 + k_2) + 2(1 - K_e^2) = 0, \quad (20)$$

$$2(1 - 4H^2)k_1 k_2 + 4HK_e(k_1 + k_2) + 2(1 - K_e^2) = 0. \quad (21)$$

Replacing $(k_1 + k_2)^2 = k_1^2 + k_2^2 + 2k_1 k_2$, $k_1 + k_2 = 2H$, $k_1 k_2 = K_e$, $k_1^2 + k_2^2 = 4H^2 - 2K_e$, into Equation (20) and Equation (21), respectively, we obtain the following

$$K_e^2 + K_e - 1 = 2H^2(1 + 4(K_e - H^2)), \quad (22)$$

$$K_e^2 - K_e - 1 = 0. \quad (23)$$

K_e solutions of Equation (23) are the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$, and its conjugate $\varphi^* = \frac{1-\sqrt{5}}{2}$ as in [5]. Together with complex solutions, all $\{H, K_e\}$ solutions of Equations (22) and (23) are the following

$$\left\{-\frac{1}{2}, \varphi^*\right\}, \left\{-\frac{1}{2}, \varphi\right\}, \left\{\frac{1}{2}, \varphi^*\right\}, \left\{\frac{1}{2}, \varphi\right\}, \\ \left\{-i\sqrt{-\varphi^*}, \varphi^*\right\}, \left\{i\sqrt{-\varphi^*}, \varphi^*\right\}, \left\{-\sqrt{\varphi}, \varphi\right\}, \left\{\sqrt{\varphi}, \varphi\right\}.$$

Corollary 1. Whenever $H^2 = K_e$ in Equation (22), we obtain Equation (23). Therefore, $H^2 = K_e$ is the common solution, and then the real $\{H, K_e\}$ solutions of $\mathcal{P}_3$ are $\{-\sqrt{\varphi}, \varphi\}$, $\{\sqrt{\varphi}, \varphi\}$.

Definition 1. The surface is called totally umbilical if all its points are umbilical, i.e., $k_1 = k_2$ or equivalently $H^2 = K_e$.

For simplicity, we will use the term “umbilical surface” instead of “totally umbilical surface” for the rest of the paper.

The only totally umbilical surfaces are (pieces of) planes and spheres. See [19] for details.

2.2. Second Solution of the Pythagorean Triples Formula $\mathcal{P}_3$

Taking the determinant of both sides of the following

$$\begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{12} & \mathcal{X}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (24)$$

where $\mathcal{X}_{ij}$ is given by Equations (14), (15), and (16), respectively, we obtain

$$(-4H^2 - K_e^2 + 1)(k_1^2 + k_2^2) + 2(4H^2 + K_e^2 - 1)k_1 k_2 = 0.$$

Substituting $k_1^2 + k_2^2 = 4H^2 - 2K_e$, $k_1 k_2 = K_e$, we obtain an implicit algebraic curve $\Phi_1(H, K_e) = 0$ (See Figure 2, Left):

$$4(K_e - H^2)(4H^2 + K_e^2 - 1) = 0. \quad (25)$$

Corollary 2 (Geometric Conclusion 1). *The determinant described by Equation (25) of the $\mathcal{P}_3$ matrix given by Equation (24) draws a parabola $K_e = H^2$ or an ellipse $(2H)^2 + K_e^2 = 1$.*

Taking $H = \frac{k_1 + k_2}{2}$ and $K_e = k_1 k_2$ in Equation (25), we obtain the implicit algebraic curve $\Phi_2(k_1, k_2) = 0$, which depends on the principal curvatures, curvature k_1 , and torsion k_2 , as follows (see Figure 2, right):

$$-k_1^4 k_2^2 - k_1^2 k_2^4 + 2k_1^3 k_2^3 - k_1^4 - k_2^4 + 2k_1^2 k_2^2 + k_1^2 + k_2^2 - 2k_1 k_2 = 0. \quad (26)$$

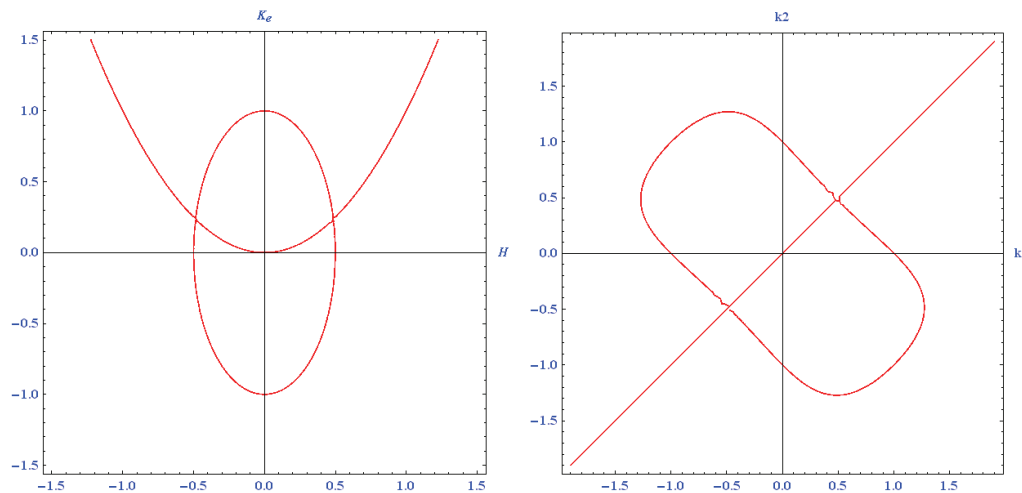


Figure 2. Algebraic curves. (Left) $\Phi_1(H, K_e) = 0$, (Right) $\Phi_2(k_1, k_2) = 0$.

Corollary 3. *The determinant of the $\mathcal{P}_3$ matrix defined in Equation (24) and given by Equation (25), corresponds to an umbilical round sphere where the principal curvatures satisfy $k_1 = k_2$, i.e., $H^2 = K_e$, in $\mathbb{M}^3(\mathbf{c})$ with $\mathbf{c} \in \{-1, 0, 1\}$.*

3. The Pythagorean Quadruples $\mathcal{P}_4$

Let $a, b, c, d \in \mathbb{N} - \{0\}$, and let $\{a, b, c, d\}$ be a quadruple with $a^2 + b^2 + c^2 = d^2$, called a *Pythagorean quadruple* (we called it $\mathcal{P}_4$).

In addition, if $\{a, b, c, d\}$ is a $\mathcal{P}_4$, so is $\{ta, tb, tc, td\}$ for any $t \in \mathbb{N} - \{0\}$. If $\gcd(a, b, c, d) = 1$, the quadruple $\{a, b, c, d\}$ is named a *primitive Pythagorean quadruple*. Here are some of the quadruples $\{1, 2, 2, 3\}$, $\{2, 10, 11, 15\}$, $\{3, 4, 12, 13\}$, $\{7, 24, 312, 313\}$, $\{17, 144, 408, 433\}$, and $\{1089, 4840, 6480, 8161\}$.

See [1,2,20,21] for the algebraic cases, and for the geometric cases [3–6] of the Pythagorean theorem. Considering the algebraic findings of it, we continue our computations with the geometric ways. Nelsen [22] gave the proof of the Pythagorean quadruples, virtually.

The set of the primitive Pythagorean quadruples for which a is odd can be obtained by

$$\left\{ \underbrace{m^2 + n^2 - p^2 - q^2}_a, \underbrace{2(mp + nq)}_b, \underbrace{2(nq - mp)}_c, \underbrace{m^2 + n^2 + p^2 + q^2}_d \right\}, \quad (27)$$

where m, n, p, q are the non-negative integers, $\gcd(m, n, p, q) = 1$, and $m + n + p + q$ is odd. Here, $a^2 + b^2 + c^2 = d^2$ is also known as Lebesgue's identity. See [2] for Lebesgue's identity.

Better understanding the Pythagorean quadruples $\mathcal{P}_4$, we consider the Hopf fibration map defined by Hopf: $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{E}^3$,

$$\text{Hopf}(z, w) = (|z|^2 - |w|^2, 2\text{Re}(z\bar{w}), 2\text{Im}(z\bar{w})),$$

or briefly, it is defined by Hopf: $\mathbb{S}^3 \rightarrow \mathbb{S}^2$. Adding on the last term d of Equation (27) on the image $\text{Hopf}(z, w)$, we can define $\mathcal{P}_4 : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{E}^4$, as follows:

$$(z, w) \mapsto \mathcal{P}_4(z, w) = \left(\underbrace{|z|^2 - |w|^2}_a, \underbrace{2\text{Re}(z\bar{w})}_b, \underbrace{2\text{Im}(z\bar{w})}_c, \underbrace{|z|^2 + |w|^2}_d \right),$$

where $z = m + ni$, $w = p + qi$, m, n, p, q are the non-negative integers; $\gcd(m, n, p, q) = 1$; and $m + n + p + q$ is odd. Then, we obtain that $\mathcal{P}_4(z, w)$ transforms into a Pythagorean quadruple (27).

Let $\mathbb{M}^4(\mathbf{c})$ be a 4-dimensional Riemannian space form, which has constant sectional curvature $\mathbf{c} \in \{-1, 0, 1\}$, with metric $\langle \cdot, \cdot \rangle_{\mathbb{M}}$. While $\mathbf{c} \in \{-1, 0, 1\}$, $\mathbb{M}^4(\mathbf{c})$ represents a hyperbolic space $\mathbb{H}^4$, Euclidean space $\mathbb{E}^4$, 4-sphere $\mathbb{S}^4$, respectively. The hypersphere with radius r in $\mathbb{E}^5$ is defined by $\mathbb{S}^4(r) = \{\mathbf{x} \in \mathbb{E}^5 \mid \langle \mathbf{x}, \mathbf{x} \rangle_{\mathbb{E}} = r^2\}$, and the hyperquadric in Lorentz–Minkowski space $\mathbb{L}^5 = \mathbb{E}_1^5$ is defined by $\mathbb{H}^4(r) = \{\mathbf{x} \in \mathbb{L}^5 \mid \langle \mathbf{x}, \mathbf{x} \rangle_{\mathbb{L}} = -r^2\}$. We remark that the open hemi-hypersphere, which has all points $\mathbf{x} = (x_1, \dots, x_5)$ of $\mathbb{S}^4$, is $\mathbb{S}_{i,+}^4$, where $x_i > 0$.

We assume M^3 to be an orientable hypersurface immersed into $\mathbb{M}^4(\mathbf{c})$. For any hypersurface of $\mathbb{M}^4(\mathbf{c})$, taking $n = 3$ in Equation (9), the fundamental forms and the curvatures are related by

$$\mathfrak{C}_0 IV - 3\mathfrak{C}_1 III + 3\mathfrak{C}_2 II - \mathfrak{C}_3 I = 0. \quad (28)$$

Here, $\mathfrak{C}_0 = 1$, $\mathfrak{C}_1 = H$, $\mathfrak{C}_3 = K_e$. See [14] for details.

Next, we consider a hypersurface M^3 immersed in a space form $\mathbb{M}^4(\mathbf{c})$, $\mathbf{c} \in \{-1, 0, 1\}$, satisfying Pythagorean quadruples Equation $\mathcal{P}_4$:

$$I^2 + II^2 + III^2 = IV^2, \quad (29)$$

geometrically. In the next theorem, we only use the following 3-sphere $\mathbb{S}^3(r)$ with spherical representing $\mathbf{x} : \mathbb{S}^3 \rightarrow \mathbb{S}^3(r) \subset \mathbb{E}^4$:

$$(r \cos \theta_1 \cos \theta_2 \cos \theta_3, r \sin \theta_1 \cos \theta_2 \cos \theta_3, r \sin \theta_2 \cos \theta_3, r \sin \theta_3), \quad (30)$$

where $r > 0$ and $0 \leq \theta_i \leq 2\pi$. It can be given for $\mathbb{H}^3$ and $\mathbb{E}^4$, similarly.

Theorem 1. *The Pythagorean quadruple given Equation (29) of the hypersphere determined by Equation (30) can be denoted by*

$$(\mathcal{I}_3)^2 + (\mathcal{S})^2 + (\mathcal{S}^2)^2 = (\mathcal{S}^3)^2.$$

$\mathbb{S}^3(r)$ has the Pythagorean quadruple described by Equation (29) if and only if the following algebraic equation holds:

$$x^3 + x^2 + x - 1 = 0, \quad (31)$$

where $x = r^2$.

Proof. Let $\mathbb{M}^4(\mathbf{c})$ with $\mathbf{c} = 0$ be a Euclidean 4-space $\mathbb{E}^4$, and let $\mathbb{S}^3(r)$ be a hypersphere with radius r given by Equation (30). Then, we compute the fundamental forms of the hypersphere as follows

$$\begin{aligned} I &= \text{diag}\left(r^2 \cos^2 \theta_2 \cos^2 \theta_3, r^2 \cos^2 \theta_3, r^2\right), \\ II &= \text{diag}\left(-r \cos^2 \theta_2 \cos^2 \theta_3, -r \cos^2 \theta_3, -r\right), \\ III &= \text{diag}\left(\cos^2 \theta_2 \cos^2 \theta_3, \cos^2 \theta_3, 1\right), \\ IV &= \text{diag}\left(-\frac{1}{r} \cos^2 \theta_2 \cos^2 \theta_3, -\frac{1}{r} \cos^2 \theta_3, -\frac{1}{r}\right). \end{aligned}$$

Using Equation (29), and with the help of the fundamantel form matrices of the $\mathbb{S}^3(r)$, we obtain $r^6 + r^4 + r^2 - 1 = 0$, and $r \neq 0$. The shape operator matrix of the hypersphere described by Equation (30) is given by $\mathbf{S} = -\frac{1}{r}\mathcal{I}_3$, where $\mathcal{I}_3$ is the identity matrix. See [14] for details.

Numerical solutions of the Equation (31) are $x_1 \approx 0.54369$, $x_2, x_3 \in \mathbb{C}$. Since $x = r^2$, we have $r_1 \approx 0.73735$. $\square$

Next, we indicate the solutions of the Pythagorean quadruples formula $\mathcal{P}_4$, using two different ways with the fundamental form matrices.

3.1. First Solution of the Pythagorean Quadruples Formula $\mathcal{P}_4$

Substituting IV of Equation (28) into the right side of Equation (29), we have the Pythagorean quadruple formula $\mathcal{P}_4$, as follows

$$I^2 + II^2 + III^2 = (3H III - 3\mathfrak{C}_2 II + K_e I)^2. \quad (32)$$

Taking into account $\mathcal{A}\mathcal{B} \neq \mathcal{B}\mathcal{A}$ for matrices in general, and $\mathbf{S} = I^{-1}.II$, $\mathbf{S}^{-1} = II^{-1}.I$, $III = II.\mathbf{S} = I.\mathbf{S}^2$, after some computations, we obtain the following equation

$$\begin{aligned} 0 &= \left[-K_e M(\mathcal{I}_3) + MT(\mathbf{S}) + (1 - M^2)(\mathbf{S}^2) \right] (I.\mathbf{S}^2) \\ &+ \left[TK_e(\mathcal{I}_3) + (1 - T^2)(\mathbf{S}) + MT(\mathbf{S}^2) \right] (I.\mathbf{S}) \\ &+ \left[(1 - K_e^2)(\mathcal{I}_3) + TK_e(\mathbf{S}) - MK_e(\mathbf{S}^2) \right] (I). \end{aligned} \quad (33)$$

To shorten the computations, we use

$$\begin{aligned} k_1 + k_2 + k_3 &= 3H = M, \\ k_1 k_2 + k_1 k_3 + k_2 k_3 &= 3\mathfrak{C}_2 = T, \\ k_1 k_2 k_3 &= K_e. \end{aligned}$$

Substituting the shape operator $\mathbf{S} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}$, the first fundamental form $I = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{pmatrix}$, and the identity $\mathcal{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ matrices into the Equation (33), we obtain

$$\begin{pmatrix} a_1 \mathcal{Y}_{11} & a_2 \mathcal{Y}_{12} & a_3 \mathcal{Y}_{13} \\ a_2 \mathcal{Y}_{12} & a_4 \mathcal{Y}_{22} & a_5 \mathcal{Y}_{23} \\ a_3 \mathcal{Y}_{13} & a_5 \mathcal{Y}_{23} & a_6 \mathcal{Y}_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\mathcal{Y}_{11} = (1 - M^2)k_1^4 + 2TMk_1^3 + (1 - T^2 - 2MK_e)k_1^2 + 2TK_ek_1 + 1 - K_e^2, \quad (34)$$

$$\mathcal{Y}_{22} = (1 - M^2)k_2^4 + 2TMk_2^3 + (1 - T^2 - 2MK_e)k_2^2 + 2TK_ek_2 + 1 - K_e^2, \quad (35)$$

$$\mathcal{Y}_{33} = (1 - M^2)k_3^4 + 2TMk_3^3 + (1 - T^2 - 2MK_e)k_3^2 + 2TK_ek_3 + 1 - K_e^2, \quad (36)$$

$$\mathcal{Y}_{12} = -MK_e(k_1^2 + k_2^2) + TK_e(k_1 + k_2) + (1 - M^2)k_1^2k_2^2 + TM(k_1^2k_2 + k_1k_2^2) + (1 - T^2)k_1k_2 + 1 - K_e^2, \quad (37)$$

$$\mathcal{Y}_{13} = -MK_e(k_1^2 + k_3^2) + TK_e(k_1 + k_3) + (1 - M^2)k_1^2k_3^2 + TM(k_1^2k_3 + k_1k_3^2) + (1 - T^2)k_1k_3 + 1 - K_e^2, \quad (38)$$

$$\mathcal{Y}_{23} = -MK_e(k_2^2 + k_3^2) + TK_e(k_2 + k_3) + (1 - M^2)k_2^2k_3^2 + TM(k_2^2k_3 + k_2k_3^2) + (1 - T^2)k_2k_3 + 1 - K_e^2. \quad (39)$$

Since $I \neq 0$, $a_i \neq 0$, where $i = 1, 2, \dots, 6$. Hence, $\mathcal{Y}_{11} = 0$, $\mathcal{Y}_{12} = 0$, $\mathcal{Y}_{13} = 0$, $\mathcal{Y}_{22} = 0$, $\mathcal{Y}_{23} = 0$, and $\mathcal{Y}_{33} = 0$. Adding diagonal elements $\sum \mathcal{Y}_{ii}$ and other elements $\sum \mathcal{Y}_{jk}$ ($j \neq k$), we find

$$\begin{aligned} -3K_e^2 - 2M(3M^2 - 6T - 2)K_e + M^2T(6M^2 - 9T - 4) \\ - M^6 + M^4 + M^2 + 1 + 2(T^3 + T^2 - T + 1) \end{aligned} = 0, \quad (40)$$

$$-3K_e^2 + M(3T - 2)K_e + 2 - (T^3 - T^2 - T - 1) = 0, \quad (41)$$

respectively. We use

$$\begin{aligned} k_1^4 + k_2^4 + k_3^4 &= (M^2 - 2T)^2 - 2(T^2 - 2MK_e) \\ &= (9H^2 - 6\mathfrak{C}_2)^2 - 2(9\mathfrak{C}_2^2 - 6HK_e), \\ k_1^3 + k_2^3 + k_3^3 &= M^3 - 3MT + 3K_e = 27H^3 - 27H\mathfrak{C}_2 + 3K_e, \\ k_1^2 + k_2^2 + k_3^2 &= M^2 - 2T = 9H^2 - 6\mathfrak{C}_2, \\ k_1^2k_2^2 + k_1^2k_3^2 + k_2^2k_3^2 &= T^2 - 2MK_e = 9\mathfrak{C}_2^2 - 6HK_e, \end{aligned}$$

and

$$k_1k_2^2 + k_1^2k_2 + k_1k_3^2 + k_1^2k_3 + k_2k_3^2 + k_2^2k_3 = MT - 3K_e = 9H\mathfrak{C}_2 - 3K_e$$

to obtain Equations (40) and (41). Therefore, by replacing T, M with $3\mathfrak{C}_2, 3H$, respectively, Equations (40) and (41) reduce to $\Omega_1(H, \mathfrak{C}_2, K_e) = 0$ and $\Omega_2(H, \mathfrak{C}_2, K_e) = 0$, as follows

$$\Omega_1(H, \mathfrak{C}_2, K_e) = -3(K_e^2 - 2H(18\mathfrak{C}_2 - 27H^2 + 2))K_e \quad (42)$$

$$\begin{aligned} &+ 9\mathfrak{C}_2H^2(27\mathfrak{C}_2 - 54H^2 + 4) \\ &+ 243H^6 - 27H^4 - 3H^2 - 18\mathfrak{C}_2^3 - 6\mathfrak{C}_2^2 + 2\mathfrak{C}_2 - 1), \\ \Omega_2(H, \mathfrak{C}_2, K_e) &= -3(K_e^2 - H(9\mathfrak{C}_2 - 2)K_e + 9\mathfrak{C}_2^3 - 3\mathfrak{C}_2^2 - \mathfrak{C}_2 - 1), \end{aligned} \quad (43)$$

respectively. Equations (42) and (43) are the implicit algebraic surfaces. See Figure 3 and Figure 4, respectively.

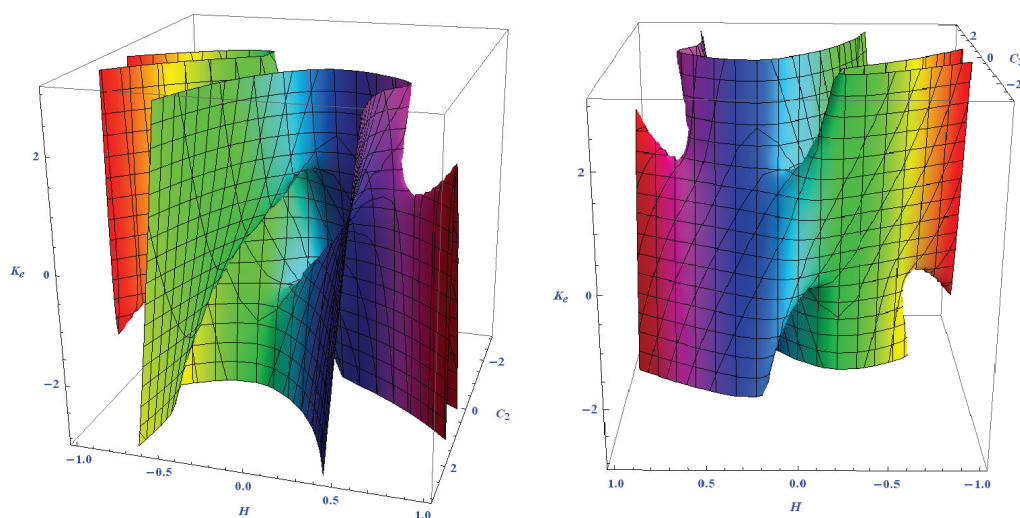


Figure 3. Algebraic surface $\Omega_1(H, \mathfrak{C}_2, K_e) = 0$. (Left) Front view, (Right) back view.

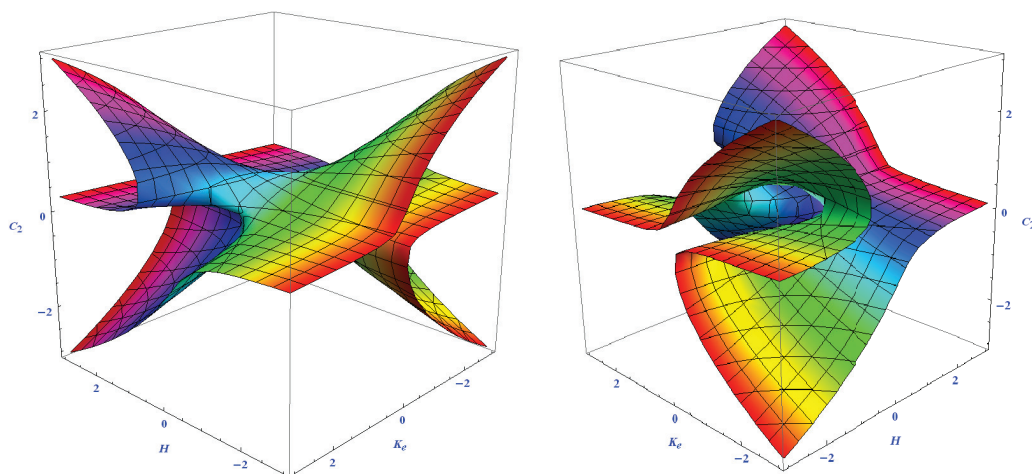


Figure 4. Algebraic surface $\Omega_2(H, \mathfrak{C}_2, K_e) = 0$ in (43). (Left) Front side view, (Right) back side view.

K_e solutions of Equations (42) and (43) are as follows

$$\begin{aligned} K_e &= \left(18\mathfrak{C}_2 - 27H^2 + 2\right)H \pm (18\mathfrak{C}_2^3 + 6\mathfrak{C}_2^2 - 2\mathfrak{C}_2 + 1 + 81\mathfrak{C}_2^2H^2 \\ &\quad - 486\mathfrak{C}_2H^4 + 36\mathfrak{C}_2H^2 + 486H^6 - 81H^4 + 7H^2)^{1/2}, \\ K_e &= \frac{1}{2} \left((9\mathfrak{C}_2 - 2)H \pm \left((9\mathfrak{C}_2 - 2)^2H^2 - 4(9\mathfrak{C}_2^3 - 3\mathfrak{C}_2^2 - \mathfrak{C}_2 - 1) \right)^{1/2} \right), \end{aligned}$$

respectively.

We also obtain that the implicit algebraic surfaces depend on the principal curvatures k_1, k_2, k_3 as $\Omega_1(k_1, k_2, k_3) = 0$ and $\Omega_2(k_1, k_2, k_3) = 0$, replacing $H, \mathfrak{C}_2, K_e$ with $\frac{k_1+k_2+k_3}{3}, \frac{k_1k_2+k_1k_3+k_2k_3}{3}, k_1k_2k_3$, respectively, in Equation (42) and Equation (43). See Figure 5 and Figure 6 for the algebraic surfaces Ω_1, Ω_2 , respectively.

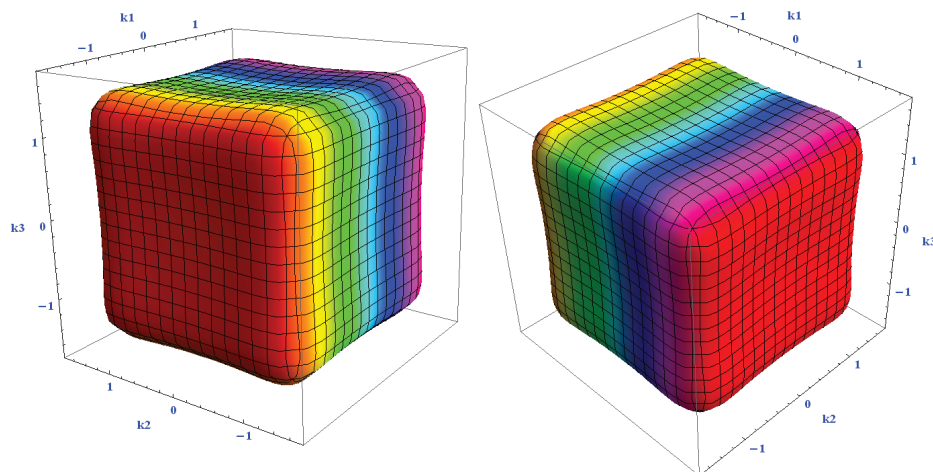


Figure 5. Algebraic surface $\Omega_1(k_1, k_2, k_3) = 0$. (Left) Front side view, (Right) back side view.

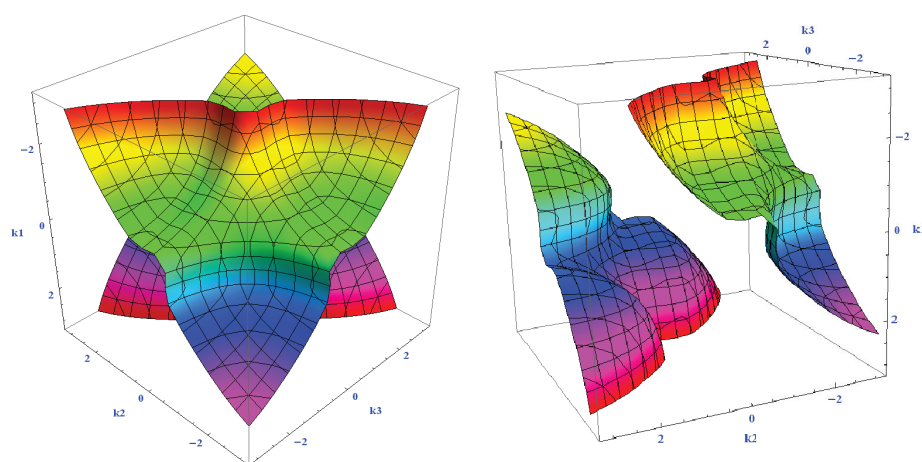


Figure 6. Algebraic surface $\Omega_2(k_1, k_2, k_3) = 0$. (Left) Front side view, (Right) back side view.

Eliminating $\mathfrak{C}_2, H, K_e$, respectively, in Equation (42) and Equation (43), we have the following implicit algebraic equations. $\Omega_3(H, K_e) = 0$ (see Figure 7, Left), $\Omega_3(\mathfrak{C}_2, K_e) = 0$ (See Figure 7, Middle), and $\Omega_3(H, \mathfrak{C}_2) = 0$ (see Figure 7, Right), respectively.

$$\begin{aligned} \Omega_3(H, K_e) = & -387420489H^{18} + 387420489H^{16} + 516560652H^{15}K_e \\ & -33480783H^{14} - 420901272H^{13}K_e - 200884698H^{12}K_e^2 \\ & +60584274H^{12} + 74401740H^{11}K_e + 136580337H^{10}K_e^2 \\ & +18068994H^9K_e^3 - 49483062H^{10} - 49837356H^9K_e \\ & -30114990H^8K_e^2 - 12045996H^7K_e^3 + 177147H^6K_e^4 \\ & +2827791H^8 + 15348366H^7K_e + 8815797H^6K_e^2 \\ & +2033910H^5K_e^3 + 347733H^4K_e^4 - 39366H^3K_e^5 \\ & -1081107H^6 - 783918H^5K_e - 1579014H^4K_e^2 \\ & -265842H^3K_e^3 - 23328H^2K_e^4 - 7290HK_e^5 - 729K_e^6 \\ & -43335H^4 + 278208H^3K_e + 41292H^2K_e^2 + 17100HK_e^3 \\ & +2871K_e^4 - 10332H^2 - 9810HK_e - 3699K_e^2 + 1557, \end{aligned}$$

$$\begin{aligned}\Omega_3(\mathfrak{C}_2, K_e) = & -387420489\mathfrak{C}_2^{18} + 774840978\mathfrak{C}_2^{17} + 516560652\mathfrak{C}_2^{15}K_e^2 \\ & -387420489\mathfrak{C}_2^{16} - 903981141\mathfrak{C}_2^{14}K_e^2 - 200884698\mathfrak{C}_2^{12}K_e^4 \\ & + 114791256\mathfrak{C}_2^{15} + 449599086\mathfrak{C}_2^{13}K_e^2 + 306110016\mathfrak{C}_2^{11}K_e^4 \\ & + 18068994\mathfrak{C}_2^9K_e^6 - 286978140\mathfrak{C}_2^{14} - 157306536\mathfrak{C}_2^{12}K_e^2 \\ & - 150929244\mathfrak{C}_2^{10}K_e^4 - 24446286\mathfrak{C}_2^8K_e^6 + 177147\mathfrak{C}_2^6K_e^8 \\ & + 153055008\mathfrak{C}_2^{13} + 199113228\mathfrak{C}_2^{11}K_e^2 + 39090438\mathfrak{C}_2^9K_e^4 \\ & + 10628820\mathfrak{C}_2^7K_e^6 - 39366\mathfrak{C}_2^3K_e^{10} + 2125764\mathfrak{C}_2^{12} \\ & - 99615663\mathfrak{C}_2^{10}K_e^2 - 24229773\mathfrak{C}_2^8K_e^4 + 2342277\mathfrak{C}_2^6K_e^6 \\ & - 91854\mathfrak{C}_2^4K_e^8 + 45927\mathfrak{C}_2^2K_e^{10} - 729K_e^{12} + 46766808\mathfrak{C}_2^{11} \\ & + 17990262\mathfrak{C}_2^9K_e^2 + 14464818\mathfrak{C}_2^7K_e^4 - 4531464\mathfrak{C}_2^5K_e^6 \\ & + 77274\mathfrak{C}_2^3K_e^8 - 10206\mathfrak{C}_2K_e^{10} - 25863462\mathfrak{C}_2^{10} \\ & - 13161366\mathfrak{C}_2^8K_e^2 - 2138886\mathfrak{C}_2^6K_e^4 + 1972188\mathfrak{C}_2^4K_e^6 \\ & - 125145\mathfrak{C}_2^2K_e^8 + 5994K_e^{10} - 5117580\mathfrak{C}_2^9 + 7619508\mathfrak{C}_2^7K_e^2 \\ & - 57834\mathfrak{C}_2^5K_e^4 - 259686\mathfrak{C}_2^3K_e^6 + 19818\mathfrak{C}_2K_e^8 - 5078214\mathfrak{C}_2^8 \\ & - 1570995\mathfrak{C}_2^6K_e^2 - 141426\mathfrak{C}_2^4K_e^4 + 106146\mathfrak{C}_2^2K_e^6 - 16551K_e^8 \\ & + 2152008\mathfrak{C}_2^7 - 303750\mathfrak{C}_2^5K_e^2 - 2970\mathfrak{C}_2^3K_e^4 + 18396\mathfrak{C}_2K_e^6 \\ & + 819396\mathfrak{C}_2^6 - 326268\mathfrak{C}_2^4K_e^2 - 76158\mathfrak{C}_2^2K_e^4 + 20700K_e^6 \\ & + 454896\mathfrak{C}_2^5 + 97524\mathfrak{C}_2^3K_e^2 - 49716\mathfrak{C}_2K_e^4 - 43740\mathfrak{C}_2^4 \\ & + 80919\mathfrak{C}_2^2K_e^2 - 13383K_e^4 - 40824\mathfrak{C}_2^3 + 26082\mathfrak{C}_2K_e^2 \\ & - 24057\mathfrak{C}_2^2 + 4698K_e^2 - 4374\mathfrak{C}_2 - 729,\end{aligned}$$

$$\begin{aligned}\Omega_3(H, \mathfrak{C}_2) = & 4782969H^{12} - 9565938\mathfrak{C}_2H^{10} + 4782969\mathfrak{C}_2^2H^8 \\ & - 3188646H^{10} + 1062882\mathfrak{C}_2^3H^6 + 6731586\mathfrak{C}_2H^8 \\ & - 1062882\mathfrak{C}_2^4H^4 - 4901067\mathfrak{C}_2^2H^6 + 413343H^8 \\ & + 1417176\mathfrak{C}_2^3H^4 - 944784\mathfrak{C}_2H^6 + 59049\mathfrak{C}_2^6 \\ & - 157464\mathfrak{C}_2^4H^2 + 597051\mathfrak{C}_2^2H^4 - 223074H^6 \\ & + 13122\mathfrak{C}_2^5 - 83106\mathfrak{C}_2^3H^2 + 266814\mathfrak{C}_2H^4 \\ & - 12393\mathfrak{C}_2^4 - 78003\mathfrak{C}_2^2H^2 + 50301H^4 - 1458\mathfrak{C}_2^3 \\ & - 27702\mathfrak{C}_2H^2 + 729\mathfrak{C}_2^2 - 2916H^2.\end{aligned}$$

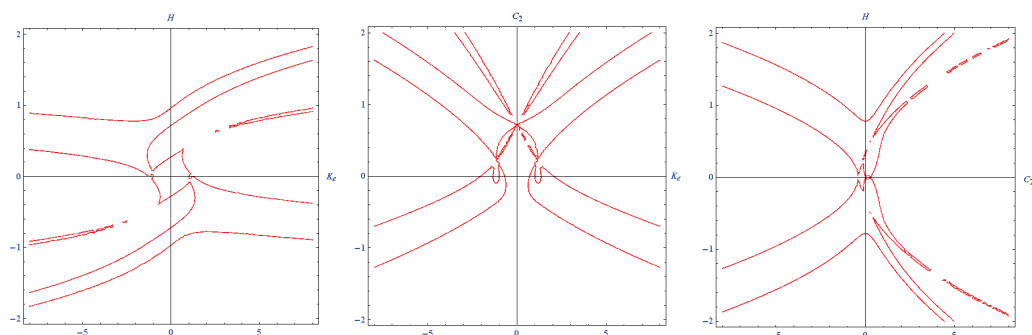


Figure 7. Algebraic curves. (Left) $\Omega_3(H, K_e) = 0$, (Middle) $\Omega_3(\mathfrak{C}_2, K_e) = 0$, and (Right) $\Omega_3(H, \mathfrak{C}_2) = 0$.

We obtain the implicit algebraic surface, which depends on principal curvatures k_1, k_2, k_3 as $\Omega_3(k_1, k_2, k_3) = 0$, replacing H, K_e with $\frac{k_1+k_2+k_3}{3}, k_1k_2k_3$, respectively, in $\Omega_3(H, K_e) = 0$. See Figure 8 for the algebraic surface Ω_3 .

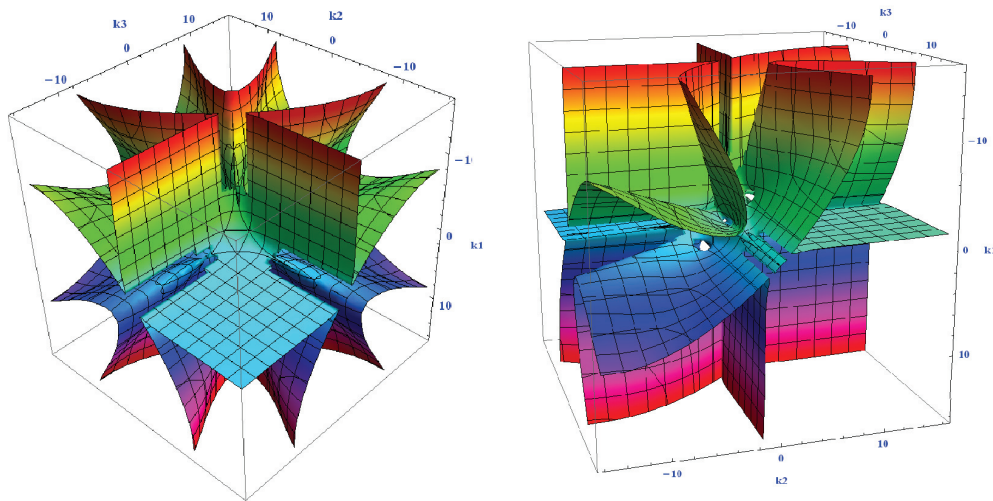


Figure 8. Algebraic surface $\Omega_3(k_1, k_2, k_3) = 0$. (Left) Front side view, (Right) back side view.

3.2. Second Solution of the Pythagorean Quadruple Formula $\mathcal{P}_4$

Before defining the umbilical hypersurface in 4-space, we make some remarks.

Remark 1. The following are equivalent

$$\begin{aligned}
 k_1 = k_2 = k_3 &\Leftrightarrow k_1 - k_2 = 0 \wedge k_1 - k_3 = 0 \wedge k_2 - k_3 = 0 \\
 &\Leftrightarrow (k_1 - k_2)^2 = 0 \wedge (k_1 - k_3)^2 = 0 \wedge (k_2 - k_3)^2 = 0 \\
 &\Leftrightarrow \begin{cases} k_1^2 - 2k_1k_2 + k_2^2 = 0 \\ k_1^2 - 2k_1k_3 + k_3^2 = 0 \\ k_2^2 - 2k_2k_3 + k_3^2 = 0 \end{cases} \\
 &\Leftrightarrow k_1^2 + k_2^2 + k_3^2 = k_1k_2 + k_1k_3 + k_2k_3 \text{ (Adding above eqs.)} \\
 &\Leftrightarrow H^2 = \mathfrak{C}_2.
 \end{aligned}$$

Remark 2. The following are equivalent

$$\begin{aligned}
 k_1 = k_2 = k_3 &\Leftrightarrow k_1 - k_2 = 0 \wedge k_1 - k_3 = 0 \wedge k_2 - k_3 = 0 \\
 &\Leftrightarrow (k_1 - k_2)^2 = 0 \wedge (k_1 - k_3)^2 = 0 \wedge (k_2 - k_3)^2 = 0 \\
 &\Leftrightarrow \begin{cases} k_1^2 + 2k_1k_2 + k_2^2 = 4k_1k_2 \\ k_1^2 + 2k_1k_3 + k_3^2 = 4k_1k_3 \\ k_2^2 + 2k_2k_3 + k_3^2 = 4k_2k_3 \end{cases} \\
 &\Leftrightarrow (k_1 + k_2)^2 = 4k_1k_2 \wedge (k_1 + k_3)^2 = 4k_1k_3 \wedge (k_2 + k_3)^2 = 4k_2k_3 \\
 &\Leftrightarrow (k_1 + k_2)(k_1 + k_3)(k_2 + k_3) = 8k_1k_2k_3 \\
 &\Leftrightarrow H\mathfrak{C}_2 = K_e.
 \end{aligned}$$

Remark 3. Using the results of the Remarks 1 and 2 together, we have

$$k_1 = k_2 = k_3 \Leftrightarrow H^3 = K_e.$$

Next, we describe the umbilical hypersurface of four dimensional space.

Definition 2. The hypersurface M^3 immersed into a $\mathbb{M}^4(\mathbf{c})$, $\mathbf{c} \in \{-1, 0, 1\}$ is called umbilical if all its points are umbilical, i.e., $k_1 = k_2 = k_3$ or, equivalently, $H^3 - K_e = 0$, with $H\mathfrak{C}_2 = K_e$, $H^2 = \mathfrak{C}_2$.

The only umbilical hypersurfaces are (open to) hyperplanes and hyperspheres in $\mathbb{M}^4(\mathbf{c})$.

Next, taking determinants of both sides of

$$\begin{pmatrix} \mathcal{Y}_{11} & \mathcal{Y}_{12} & \mathcal{Y}_{13} \\ \mathcal{Y}_{12} & \mathcal{Y}_{22} & \mathcal{Y}_{23} \\ \mathcal{Y}_{13} & \mathcal{Y}_{23} & \mathcal{Y}_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (44)$$

we obtain

$$\begin{aligned} 0 = & -[(k_1^2 k_2^4 + k_1^4 k_2^2 + k_1^2 k_3^4 + k_1^4 k_3^2 + k_2^2 k_3^4 + k_2^4 k_3^2) \\ & + 6k_1^2 k_2^2 k_3^2 + 2(k_1^3 k_2^3 + k_1^3 k_3^3 + k_2^3 k_3^3) + 2k_1 k_2 k_3 (k_1^3 + k_2^3 + k_3^3) \\ & - 2k_1 k_2 k_3 (k_1 k_2^2 + k_1^2 k_2 + k_1 k_3^2 + k_1^2 k_3 + k_2 k_3^2 + k_2^2 k_3)] (K_e^2 + M^2 + T^2 - 1). \end{aligned}$$

Here, $\mathcal{Y}_{ij}$ are given by Equations (34), (35), (36), (37), (38), and (39), respectively. Hence, the above Equation reduces to

$$-(k_1 - k_2)^2 (k_1 - k_3)^2 (k_2 - k_3)^2 (K_e^2 + M^2 + T^2 - 1) = 0. \quad (45)$$

Then, $-(k_1 - k_2)^2 (k_1 - k_3)^2 (k_2 - k_3)^2$ transforms to the following

$$-(k_2 - k_3)^2 (k_1 - k_3)^2 (k_1 - k_2)^2 = 27(4H^3 K_e - 3\mathfrak{C}_2^2 H^2 + 4\mathfrak{C}_2^3 + K_e^2 - 6\mathfrak{C}_2 H K_e),$$

by using

$$\begin{aligned} k_1^2 k_2^4 + k_1^4 k_2^2 + k_1^2 k_3^4 + k_1^4 k_3^2 + k_2^2 k_3^4 + k_2^4 k_3^2 &= (T^2 - 2M K_e)(M^2 - 2T) - 3K_e^2 \\ &= (9\mathfrak{C}_2^2 - 6H K_e)(9H^2 - 6\mathfrak{C}_2) - 3K_e^2. \end{aligned}$$

Therefore, we obtain the following.

Corollary 4. Since $H^2 = \mathfrak{C}_2$, $H\mathfrak{C}_2 = K_e$, $H^3 = K_e$, the determinant of the $\mathcal{P}_4$ matrix (44) satisfies

$$4H^3 K_e - 3\mathfrak{C}_2^2 H^2 + 4\mathfrak{C}_2^3 + K_e^2 - 6\mathfrak{C}_2 H K_e = 0,$$

by using

$$\begin{aligned} k_1^3 k_2^3 + k_1^3 k_3^3 + k_2^3 k_3^3 &= T^3 - 3T M K_e + 3K_e^2 \\ &= 27\mathfrak{C}_2^3 - 27H\mathfrak{C}_2 K_e + 3K_e^2. \end{aligned}$$

We also have the following.

Corollary 5 (Geometric Conclusion 2). The determinant given by Equation (45) of the $\mathcal{P}_4$ matrix described by Equation (44) draws the surface $K_e = H^3$ with $H^2 = \mathfrak{C}_2$, $H\mathfrak{C}_2 = K_e$ and the ellipsoid surface $(3H)^2 + (3\mathfrak{C}_2)^2 + K_e^2 = 1$.

The determinant of the $\mathcal{P}_4$ is also given by as follows

$$27(4H^3 K_e - 3\mathfrak{C}_2^2 H^2 + 4\mathfrak{C}_2^3 + K_e^2 - 6\mathfrak{C}_2 H K_e)((3H)^2 + (3\mathfrak{C}_2)^2 + K_e^2 - 1) = 0.$$

Here, $4H^3K_e - 3\mathfrak{C}_2^2H^2 + 4\mathfrak{C}_2^3 + K_e^2 - 6\mathfrak{C}_2HK_e = 0$ is the implicit surface as $\Theta_1(H, \mathfrak{C}_2, K_e) = 0$ (See Figure 9, Left), satisfying $H^3 - K_e = 0$, with $H\mathfrak{C}_2 = K_e$, $H^2 = \mathfrak{C}_2$, and the implicit ellipsoid, or ellipsoidal surface $(3H)^2 + (3\mathfrak{C}_2)^2 + K_e^2 = 1$ as $\Theta_2(H, \mathfrak{C}_2, K_e) = 0$ (See Figure 9, Right).

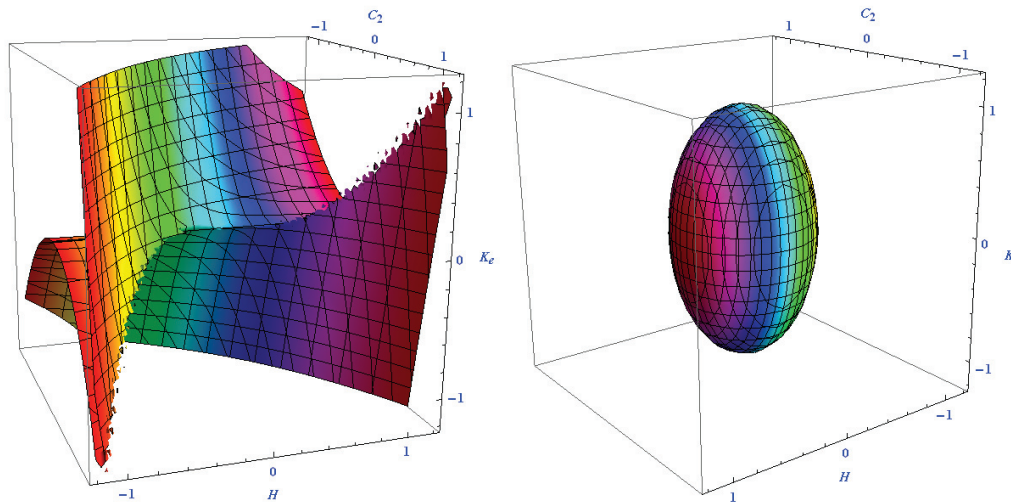


Figure 9. Algebraic surfaces. (Left) $\Theta_1(H, \mathfrak{C}_2, K_e) = 0$, (Right) $\Theta_2(H, \mathfrak{C}_2, K_e) = 0$.

Hence, extending the determinant of the $\mathcal{P}_4$, we obtain the implicit algebraic surface $\Psi(H, \mathfrak{C}_2, K_e) = 0$, as follows (see Figure 10):

$$\begin{aligned} \Psi(H, \mathfrak{C}_2, K_e) = & -108H^3K_e + 972H^5K_e + 81\mathfrak{C}_2^2H^2 + 972\mathfrak{C}_2^3H^2 - 729\mathfrak{C}_2^2H^4 \\ & - 729\mathfrak{C}_2^4H^2 + 243\mathfrak{C}_2^2K_e^2 + 108\mathfrak{C}_2^3K_e^2 + 243H^2K_e^2 + 108H^3K_e^3 \\ & - 108\mathfrak{C}_2^3 + 972\mathfrak{C}_2^5 - 27K_e^2 + 27K_e^4 - 81\mathfrak{C}_2^2H^2K_e^2 - 162\mathfrak{C}_2HK_e^3 \\ & - 1458\mathfrak{C}_2H^3K_e - 1458\mathfrak{C}_2^3HK_e + 972\mathfrak{C}_2^2H^3K_e + 162\mathfrak{C}_2HK_e. \end{aligned}$$

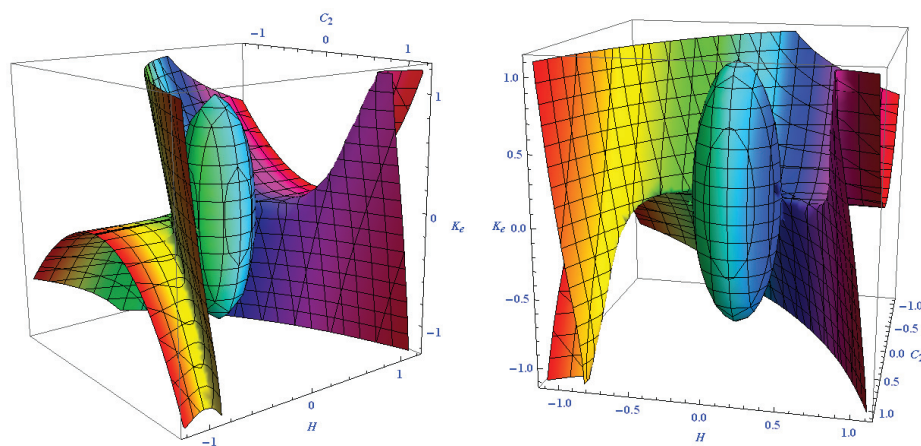


Figure 10. Algebraic surface $\Psi(H, \mathfrak{C}_2, K_e) = 0$. (Left) Front side view, (Right) back side view.

Corollary 6. The determinant given by Equation (45) of the $\mathcal{P}_4$ matrix described by Equation (44) corresponds to an umbilical round hypersphere where the principal curvatures satisfy $k_1 = k_2 = k_3$. In other words, it satisfies $H^3 = K_e$ with $H\mathfrak{C}_2 = K_e$ and $H^2 = \mathfrak{C}_2$ in $\mathbb{M}^4(\mathfrak{c})$, where $\mathfrak{c} \in \{-1, 0, 1\}$.

We remark that this work relies on the Pythagorean formula but does not depend on the distance between points in the running space form.

On the other hand, we use the upper hypersphere $\mathbb{S}^3(r)$ with the Cartesian map $\mathbf{x}: \mathbb{E}^3 \rightarrow \mathbb{S}^3(r) \subset \mathbb{E}^4$ given by

$$(u, v, w) \mapsto \left(u, v, w, \sqrt{r^2 - u^2 - v^2 - w^2}\right), \quad (46)$$

where $r > 0$. Next, we compute the fundamental forms of the upper hypersphere given by Equation (46). The first fundamental form matrix of it as follows

$$I = \begin{pmatrix} \frac{r^2 - v^2 - w^2}{r^2 - u^2 - v^2 - w^2} & \frac{uv}{r^2 - u^2 - v^2 - w^2} & \frac{uw}{r^2 - u^2 - v^2 - w^2} \\ \frac{uv}{r^2 - u^2 - v^2 - w^2} & \frac{r^2 - u^2 - w^2}{r^2 - u^2 - v^2 - w^2} & \frac{vw}{r^2 - u^2 - v^2 - w^2} \\ \frac{uw}{r^2 - u^2 - v^2 - w^2} & \frac{vw}{r^2 - u^2 - v^2 - w^2} & \frac{r^2 - u^2 - v^2}{r^2 - u^2 - v^2 - w^2} \end{pmatrix},$$

Then, the shape operator matrix is given by $\mathbf{S} = \frac{1}{r}\mathcal{I}_3$. We obtain $II = I\mathbf{S}$, $III = II\mathbf{S}$, and $IV = III\mathbf{S}$, easily.

Taking into account the Pythagorean quadruple formula $\underbrace{I^2 + II^2 + III^2}_{\Omega} = \underbrace{IV^2}_{\Phi}$, of the hypersphere described by Equation (46), and considering $\det(\Omega) = \det(\Phi)$, we have the following.

Corollary 7. *The hypersphere given by Equation (46) has the Pythagorean quadruple formula mentioned by (29) if and only if the following holds*

$$r^6(r^{12} + 3r^{10} + 6r^8 + 7r^6 + 6r^4 + 3r^2 + 1) - 1 = 0.$$

The roots of above equation are $r_1 = -0.73735 \in \mathbb{R}^-$, $r_2 = 0.73735 \in \mathbb{R}^+$, $r_{3,4,\dots,18} \in \mathbb{C}$. Here, the positive and real r_2 solution is also given in the second row of the Table 1.

Let M^3 be an immersed hypersurface into $\mathbb{H}^4$, $\mathbb{E}^4$, or $\mathbb{S}^4$. M^3 is named totally geodesic while $II = 0$ and totally umbilical while $II = \mu I$, where $\mu \neq 0$ is a constant. When $II = 0$, then $III = 0$ and $IV = 0$. This is not possible for the Pythagorean $\mathcal{P}_4$ formula. On the other hand, while II is degenerate (i.e., $\det(II) = 0$), using Equations (28) and (29), and considering $III = II\mathbf{S} = I\mathbf{S}^2$, $K_e = \det \mathbf{S}$, we obtain $I = 0$, which contradicts $\det(II) = 0$. So, $\det(II) \neq 0$.

When M^3 is minimal, and $K_e \neq 0$, using Equations (28) and (29) again, we find the following:

$$(K_e^2 - 1)I^2 = (1 - (3\mathfrak{C}_2)^2)II^2.$$

Taking both sides of the determinant, we have

$$K_e^6 - 3K_e^4 + ((3\mathfrak{C}_2)^2 + 2)K_e^2 - 1 = 0. \quad (47)$$

Here, K_e has 4 complex roots, 1 negative real root ($K_e = -1.3936$, $\mathfrak{C}_2 = -0.251538$), and the following positive real root ($K_e = 0.317103$, $\mathfrak{C}_2 = 0.956641$)

$$K_e = \left(0.381571c - \frac{7.86222(\mathfrak{C}_2)^2}{c} + \frac{0.87358}{c} + 1\right)^{1/2},$$

where

$$c = \left(-81(\mathfrak{C}_2)^2 + 1.73205(2916(\mathfrak{C}_2)^6 + 1215(\mathfrak{C}_2)^4 - 378(\mathfrak{C}_2)^2 + 23)^{1/2} + 9\right)^{1/3}.$$

Then, K_c is a constant. When the space is $\mathbb{H}^4$ or $\mathbb{E}^4$, then M^3 should be totally geodesic (see [23,24] for M^2 case.). Then, it gives a contradiction.

Therefore, the case of $K_e = 1$ and M^3 is totally geodesic and does not take place. The case of $K_e = 0$, M^3 is an open piece of the Clifford torus. Then, $K_e = -1$, which is incompatible with Equation (47). So, the immersed hypersurface into $\mathbb{H}^4$, $\mathbb{E}^4$, or $\mathbb{S}_+^4$ supplying the $\mathcal{P}_4$ formula mentioned by Equation (29) cannot be totally geodesic, may not be minimal, and does not have a degenerate second fundamental form.

On the other side, we want to see the real solutions of $\sum_{k=1}^n x^k - 1 = 0$ for some integers. See Tables 1–4 for some solutions to it. To see the real solutions of $\sum_{k=1}^{99999} x^k - 1 = 0$ in $\mathbb{E}^{100000}$ on graphics depending on x and r , respectively, see Figure 11.

Table 1. x and r solutions of Pythagorean 3–26-tuples.

Dimension	x (11 Digits)	r (11 Digits)
3	0.6180349887	0.7861513777
4	0.5436890126	0.7373527058
5	0.5187900636	0.7202708266
6	0.5086603916	0.7132043127
7	0.5041382583	0.7100269420
8	0.5020170551	0.7085316191
9	0.5009941779	0.7078094220
10	0.5004931182	0.7074553826
11	0.5002454622	0.7072803279
12	0.5001224294	0.7071933466
13	0.5000611322	0.7071500069
14	0.5000305436	0.7071283785
15	0.5000152657	0.7071175757
16	0.5000076312	0.7071121773
17	0.5000038151	0.7071094789
18	0.5000019074	0.7071081300
19	0.5000009537	0.7071074556
20	0.5000004768	0.7071071183
21	0.5000002384	0.7071069498
22	0.5000001192	0.7071068655
23	0.5000000596	0.7071068233
24	0.5000000298	0.7071068023
25	0.5000000149	0.7071067917
26	0.5000000074	0.7071067864

Table 2. x and r solutions of Pythagorean 30, 40, 50-tuples.

Dimension	x (17 Digits)	r (17 Digits)
30	0.5000000004656613	0.7071067815158197
40	0.5000000000004547	0.7071067811868691
50	0.5000000000000004	0.7071067811865478

Table 3. x and r solutions of Pythagorean 75-tuples of $S^{74}(r)$ in $\mathbb{E}^{75}$.

[illegible]

Theorem 2. Let a hypersurface M^n immersed into a $(n + 1)$ -dimensional Riemannian space form $\mathbb{M}^{n+1}(\mathbf{c})$, $\mathbf{c} \in \{-1, 0, 1\}$ satisfy the following Pythagorean $(n + 1)$ -tuples $\mathcal{P}_{n+1}$ equation

$$I^2 + II^2 + III^2 + \dots + N^2 = (N + 1)^2, \quad (48)$$

if and only if the following algebraic equation holds:

$$\sum_{k=1}^n x^k - 1 = 0,$$

where $x = r^2$. Here, $I, II, III, \dots, N, N + 1$ are the fundamental form matrices of the hypersurface M^n .

Proof. Let M^n be an immersed hypersurface into $\mathbb{H}^{n+1}$, $\mathbb{E}^{n+1}$, or $\mathbb{S}^{n+1}$. The shape operator matrix is given by $\mathbf{S} = \frac{1}{r}\mathcal{I}_n$, where $\mathcal{I}_n$ is the identity matrix. The $\mathcal{P}_{n+1}$ given by Equation (48) of the hypersurface can be denoted by

$$(\mathcal{I}_n)^2 + (\mathbf{S})^2 + (\mathbf{S}^2)^2 + \dots + (\mathbf{S}^{n-1})^2 = (\mathbf{S}^n)^2.$$

Considering the $\mathcal{P}_{n+1}$ determined by Equation (48) with the fundamental form matrices of the hypersurface, we obtain $r^{2n} + r^{2n-2} + \dots + r^2 - 1 = 0$, and $r \neq 0$. The hypersurface M^n has the Pythagorean $(n + 1)$ -tuples $\mathcal{P}_{n+1}$ described by Equation (48) if and only if algebraic equation $x^n + x^{n-1} + \dots + x - 1 = 0$ holds, where $x = r^2$.

Additionally, the geometric series is defined by $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, where $|x| < 1$. Therefore, we have the following

$$\begin{aligned} \sum_{k=1}^{\infty} x^k - 1 = 0 &\Leftrightarrow \sum_{k=0}^{\infty} x^k - 2 = 0 \\ &\Leftrightarrow \frac{-1+2x}{1-x} = 0, \quad x \neq 1 \\ &\Leftrightarrow x = \frac{1}{2} \\ &\Leftrightarrow r = \frac{1}{\sqrt{2}}, \quad (x = r^2, r > 0). \end{aligned} \quad (49)$$

□

Corollary 8. Let M^n be an immersed hypersphere into $\mathbb{M}^{n+1}(\mathbf{c})$, $\mathbf{c} \in \{-1, 0, 1\}$ with radius r satisfying the Pythagorean $(n + 1)$ -tuples $\mathcal{P}_{n+1}$ mentioned by Equation (48). When $n \rightarrow \infty$, then $N \rightarrow \infty$, and $r \rightarrow \frac{1}{\sqrt{2}}$.

Next, we define the umbilic hypersurface, then give a generalization of the determinant of the Pythagorean $(n + 1)$ -tuples $\mathcal{P}_{n+1}$ formula.

Definition 3. The hypersurface M^n immersed into a $(n + 1)$ -dimensional Riemannian space form $\mathbb{M}^{n+1}(\mathbf{c})$, $\mathbf{c} \in \{-1, 0, 1\}$ is called umbilical if all its points are umbilical, i.e., $k_1 = k_2 = \dots = k_n$, or, equivalently, $H^n - K_e = 0$.

Proposition 1. Let a hypersurface M^n immersed into a $\mathbb{M}^{n+1}(\mathbf{c})$, $\mathbf{c} \in \{-1, 0, 1\}$ satisfy the Pythagorean $(n+1)$ -tuples $\mathcal{P}_{n+1}$ formula

$$I^2 + II^2 + III^2 + \dots + N^2 = \left(\sum_{i=1}^n (-1)^{i+1} \binom{n}{i} \mathfrak{C}_i I(\mathbf{S}^{n-i}(X), Y) \right)^2. \quad (50)$$

Then, the determinant of Equation (50) is given by

$$n^n (\mathfrak{C}_n - \mathfrak{C}_1^n) \left((n\mathfrak{C}_1)^2 + (n\mathfrak{C}_2)^2 + \dots + (n\mathfrak{C}_{n-1})^2 + \mathfrak{C}_n^2 - 1 \right) = 0, \quad (51)$$

or equivalently by

$$- \prod_{1 \leq i < j \leq n} (k_i - k_j)^2 \left(\sum_{\ell=1}^{n-1} (n\mathfrak{C}_\ell)^2 + \mathfrak{C}_n^2 - 1 \right) = 0, \quad (52)$$

where $\mathfrak{C}_1 = H$, $\mathfrak{C}_n = K_e$.

Finally, we have the following.

Conjecture 1. The determinant of the $\mathcal{P}_{n+1}$ matrix, as defined in Equation (50), generates a hypersurface $H^n = K_e$ and a hyperellipsoid $(nH)^2 + (n\mathfrak{C}_2)^2 + (n\mathfrak{C}_3)^2 + \dots + (n\mathfrak{C}_{n-1})^2 + K_e^2 = 1$ in $\mathbb{M}^{n+1}(\mathbf{c})$, where $\mathbf{c} \in \{-1, 0, 1\}$.

Then, we present the following generalization for the determinant of $\mathcal{P}_{n+1}$.

Conjecture 2. The determinant given by Equation (51) (or equivalently by Equation (52)) of the $\mathcal{P}_{n+1}$ formula mentioned by Equation (50) corresponds to an umbilical round hypersphere satisfying $k_1 = k_2 = \dots = k_n$, i.e., $H^n = K_e$, in $\mathbb{M}^{n+1}(\mathbf{c})$, where $\mathbf{c} \in \{-1, 0, 1\}$.

5. Conclusions

This article provides a thorough investigation of the Pythagorean theorem from the perspective of differential geometry, offering insightful generalizations in higher-dimensional space forms. It establishes a connection between classical geometric identities and the intrinsic properties of hypersurfaces, filling a notable gap in the field.

Specifically, the paper generalizes that an immersed hypersphere M^n with radius r in $\mathbb{M}^{n+1}(\mathbf{c})$, where $\mathbf{c} \in \{-1, 0, 1\}$, satisfies the $(n+1)$ -tuple Pythagorean formula. Furthermore, as the dimension $n \rightarrow \infty$ and the fundamental form $N \rightarrow \infty$, it is shown that $r \rightarrow \frac{1}{\sqrt{2}}$.

Finally, the paper proposes that the determinant of the Pythagorean formula characterizes an umbilical round hypersphere with equal principal curvatures, satisfying $H^n = K_e$ in $\mathbb{M}^{n+1}(\mathbf{c})$. These findings contribute to a broader understanding of curvature relations and their role in the geometry of hypersurfaces.

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Article

Geometry of LP-Sasakian Manifolds Admitting a General Connection

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Abstract: This paper concerns certain properties of projective curvature tensor, conharmonic curvature tensor, quasi-conharmonic curvature tensor, and Ricci semi-symmetric conditions with respect to the general connection in an LP-Sasakian manifold. We also provide the applications of LP-Sasakian manifolds admitting general connections in the context of the general theory of relativity.

Keywords: LP-Sasakian manifold; general connection; projective curvature tensor; conharmonic curvature tensor; quasi-conharmonic curvature tensor; Ricci semi-symmetric manifold

MSC: 53C15; 53C25; 53C40

1. Introduction

In the realm of differential geometry, the study of manifolds with specialized geometric forms is basic for grasping complex spaces that appear in physics, especially in general relativity and cosmology. One such form is the Para-Sasakian manifold, which like Sasakian manifolds, provides an interesting framework for geometries with some symmetry properties. When these forms are combined with a Lorentzian metric, they bring to light Lorentzian Para-Sasakian manifolds. These manifolds have a fascinating relation between the geometric properties of both Riemannian and Lorentzian spaces. Lorentzian Para-Sasakian manifolds are remarkable in the study of spacetime symmetries and cosmological models. Their structure allows for the discovery of physical systems where time-like and space-like directions reveal a specific association, which is crucial in several areas of theoretical physics. These manifolds appeared in 1989 and were introduced by Matsumoto. In 1992, Mihai and Rosca independently introduced the idea of Lorentzian para-Sasakian manifolds in classical analysis [1,2].

In 2019, Biswas and Baishya [3,4] introduced the notion of a new connection, called general connection in the context of Sasakian geometry and discussed several cases. They defined general connection D^G in the following manner:

$$D_{\mathcal{X}_1}^G \mathcal{X}_2 = D_{\mathcal{X}_1} \mathcal{X}_2 + a[(D_{\mathcal{X}_1} \eta)(\mathcal{X}_2)\zeta - \eta(\mathcal{X}_2)D_{\mathcal{X}_1} \zeta] + b\eta(\mathcal{X}_1)\phi \mathcal{X}_2 \quad (1)$$

for any $\varkappa_1, \varkappa_2 \in \chi(M)$, where a, b are real constants, ϕ is a $(1, 1)$ -type tensor field, ζ is a vector field, η is a 1-form, $\chi(M)$ is the set of all vector fields on M and D is the Levi-Civita connection.

From (1), we have the following particular cases:

- (i) if $a = 0, b = -1$, then (1) reduces to the quarter-symmetric metric connection [5,6];
- (ii) if $a = 1, b = 0$, then (1) reduces to the Schouten–Van Kampen connection [7];
- (iii) if $a = 1, b = -1$, then (1) reduces to the Tanaka Webster connection [8];
- (iv) if $a = 1, b = 1$, then (1) reduces to the Zamkovoy connection [9].

Recently, Bhatt and Chanyal [10] studied η -Ricci solitons on Sasakian manifolds admitting general connection, ref. [11] investigated conformal Ricci soliton in Sasakian manifolds admitting general connection. Moreover, ref. [12] studied Ricci solitons on concircularly flat and W_i -flat Sasakian manifolds admitting a general connection.

In a Lorentzian para-Sasakian manifold (briefly LP-Sasakian manifold) M of dimension n ($n > 2$), the projective curvature tensor $\bar{P}$ and the conharmonic curvature tensor $\bar{C}$ in accordance with general connection D^G are given by

$$\bar{P}(\varkappa_1, \varkappa_2)\varkappa_3 = \bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 - \frac{1}{n-1}[\bar{S}(\varkappa_2, \varkappa_3)\varkappa_1 - \bar{S}(\varkappa_1, \varkappa_3)\varkappa_2], \quad (2)$$

$$\begin{aligned} \bar{C}(\varkappa_1, \varkappa_2)\varkappa_3 = & \bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 - \frac{1}{n-2}[\bar{S}(\varkappa_2, \varkappa_3)\varkappa_1 - \bar{S}(\varkappa_1, \varkappa_3)\varkappa_2 \\ & + g(\varkappa_2, \varkappa_3)\bar{Q}\varkappa_1 - g(\varkappa_1, \varkappa_3)\bar{Q}\varkappa_2] \end{aligned} \quad (3)$$

for any $\varkappa_1, \varkappa_2, \varkappa_3 \in \chi(M)$, where $\bar{R}, \bar{S}, \bar{Q}$ are the Riemannian curvature tensor, the Ricci tensor and the Ricci operator in accordance with general connection D^G , respectively [13–16].

Definition 1. An n -dimensional LP-Sasakian manifold M is said to be a generalized η -Einstein manifold if the Ricci tensor of type $(0, 2)$ satisfies

$$S(\varkappa_2, \varkappa_3) = \alpha_1 g(\varkappa_2, \varkappa_3) + \alpha_2 \eta(\varkappa_2)\eta(\varkappa_3) + \alpha_3 \gamma(\varkappa_2, \varkappa_3) \quad (4)$$

for any $\varkappa_2, \varkappa_3 \in \chi(M)$, with $\chi(M)$ being the set of all vector fields of the manifold M , where α_1, α_2 and α_3 are scalars and γ is a 2-form.

The paper is designed as follows: After the introduction, some postulates of the LP-Sasakian manifold are given in Section 2. In Section 3, we have discussed the LP-Sasakian manifold admitting the general connection and obtaining certain curvature tensors in an LP-Sasakian manifold. Section 4 contains projectively flat, ζ -projectively flat and ϕ -projectively flat LP-Sasakian manifolds in accordance with the general connection. Section 5 discusses the conharmonically flat, ζ -conharmonically flat and ϕ -conharmonically flat LP-Sasakian manifolds in accordance with the general connection. Quasi-conharmonically flat LP-Sasakian manifolds in accordance with the general connection are discussed in Section 6. Section 7 deals with Ricci semi-symmetric LP-Sasakian manifold, reviewing the general connection. Section 8 deals with the applications of the general connection to the general theory of relativity by considering Einstein’s field equation without cosmological constant. Finally, with the help of Koszul’s formula, an example is given for the validation of the work.

2. Preliminaries

An n -dimensional differentiable manifold M is classified as an LP-Sasakian manifold if it admits a $(1, 1)$ -type tensor field ϕ , a vector field ζ , a 1-form η and a Lorentzian metric g that satisfy the following conditions:

$$\phi^2 \varkappa_1 = \varkappa_1 + \eta(\varkappa_1)\zeta, \quad \eta(\zeta) = -1, \quad \eta(\phi \varkappa_1) = 0, \quad \phi \zeta = 0, \quad (5)$$

$$g(\phi \varkappa_1, \phi \varkappa_2) = g(\varkappa_1, \varkappa_2) + \eta(\varkappa_1)\eta(\varkappa_2), \quad (6)$$

$$\eta(\varkappa_1) = g(\varkappa_1, \zeta), \quad g(\varkappa_1, \phi \varkappa_2) = g(\phi \varkappa_1, \varkappa_2), \quad (7)$$

$$D_{\varkappa_1} \zeta = \phi \varkappa_1, \quad (8)$$

$$(D_{\varkappa_1} \phi)(\varkappa_2) = \eta(\varkappa_2)\varkappa_1 + g(\varkappa_1, \varkappa_2)\zeta + 2\eta(\varkappa_1)\eta(\varkappa_2), \quad (9)$$

where D denotes the operator of covariant differentiation in accordance with the Lorentzian metric g .

Let γ be a symmetric $(0, 2)$ -type tensor field such that $\gamma(\varkappa_1, \varkappa_2) = g(\varkappa_1, \phi \varkappa_2)$. Moreover, in an LP-Sasakian manifold M , the vector field η is closed, then we have

$$(D_{\varkappa_1} \eta)(\varkappa_2) = g(\varkappa_1, \phi \varkappa_2) = \gamma(\varkappa_1, \varkappa_2), \quad \gamma(\varkappa_1, \zeta) = 0 \quad (10)$$

for all $\varkappa_1, \varkappa_2$ on M .

Furthermore, in an LP-Sasakian manifold M , the following condition holds:

$$\eta(R(\varkappa_1, \varkappa_2)\varkappa_3) = g(\varkappa_2, \varkappa_3)\eta(\varkappa_1) - g(\varkappa_1, \varkappa_3)\eta(\varkappa_2), \quad (11)$$

$$R(\varkappa_1, \varkappa_2)\zeta = \eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2, \quad (12)$$

$$R(\zeta, \varkappa_2)\varkappa_3 = g(\varkappa_2, \varkappa_3)\zeta - \eta(\varkappa_3)\varkappa_2, \quad (13)$$

$$R(\zeta, \varkappa_2)\zeta = \eta(\varkappa_2)\zeta + \varkappa_2, \quad (14)$$

$$S(\varkappa_1, \zeta) = (n - 1)\eta(\varkappa_1), \quad (15)$$

$$S(\phi \varkappa_1, \phi \varkappa_2) = S(\varkappa_1, \varkappa_2) + (n - 1)\eta(\varkappa_1)\eta(\varkappa_2), \quad (16)$$

$$Q\zeta = (n - 1)\zeta, \quad Q\phi = \phi Q, \quad S(\varkappa_1, \varkappa_2) = g(Q\varkappa_1, \varkappa_2), \quad S^2(\varkappa_1, \varkappa_2) = S(Q\varkappa_1, \varkappa_2) \quad (17)$$

for all $\varkappa_1, \varkappa_2 \in \chi(M)$, where R, S and Q are the curvature tensor, the Ricci tensor and the Ricci operator with respect to the Levi-Civita connection D , respectively.

3. Curvature Tensors in Accordance with General Connection in an LP-Sasakian Manifolds

With the help of (8) and (10) in (1), we obtain

$$D_{\varkappa_1}^G \varkappa_2 = D_{\varkappa_1} \varkappa_2 + a[g(\varkappa_1, \phi \varkappa_2)\zeta - \eta(\varkappa_2)\phi \varkappa_1] + b\eta(\varkappa_1)\phi \varkappa_2, \quad (18)$$

where a and b are real constants. The torsion tensor is given by

$$\bar{T}(\varkappa_1, \varkappa_2) = (a + b)[\eta(\varkappa_1)\phi \varkappa_2 - \eta(\varkappa_2)\phi \varkappa_1]. \quad (19)$$

From (1) and (10), we obtain

$$(D_{\varkappa_1}^G g)(\varkappa_2, \varkappa_3) = -2b\eta(\varkappa_1)g(\varkappa_2, \phi \varkappa_3). \quad (20)$$

Replacing $\varkappa_2$ by ζ in (18), we obtain

$$D_{\varkappa_1}^G \zeta = (a + 1)\phi \varkappa_1. \quad (21)$$

Again, from (7)–(9) and (18), we have

$$D_{\varkappa_1}^G \phi \varkappa_2 = (a+1)g(\varkappa_1, \varkappa_2)\zeta + (a+b+2)\eta(\varkappa_1)\eta(\varkappa_2)\zeta \\ + \phi(D_{\varkappa_1} \varkappa_2) + \eta(\varkappa_2)\varkappa_1 + b\eta(\varkappa_1)\varkappa_2, \quad (22)$$

$$D_{\varkappa_1}^G g(\varkappa_2, \varkappa_3) = g(D_{\varkappa_1} \varkappa_2, \phi \varkappa_3) + g(\varkappa_2, D_{\varkappa_1} \varkappa_3), \quad (23)$$

$$D_{\varkappa_1}^G g(\varkappa_2, \phi \varkappa_3) = g(D_{\varkappa_1} \varkappa_2, \phi \varkappa_3) + g(\varkappa_2, \phi D_{\varkappa_1} \varkappa_3) + g(\varkappa_1, \varkappa_3)\eta(\varkappa_2) \\ + g(\varkappa_1, \varkappa_2)\eta(\varkappa_3) + 2\eta(\varkappa_1)\eta(\varkappa_2)\eta(\varkappa_3). \quad (24)$$

Let $\bar{R}$ be the Riemannian curvature tensor in accordance with general connection D^G defined by

$$\bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 = D_{\varkappa_1}^G D_{\varkappa_2}^G \varkappa_3 - D_{\varkappa_2}^G D_{\varkappa_1}^G \varkappa_3 - D_{[\varkappa_1, \varkappa_2]}^G \varkappa_3. \quad (25)$$

Using (18), (20) and (22)–(24) in (25), we obtain

$$\bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 = R(\varkappa_1, \varkappa_2)\varkappa_3 + (a+ab+b)[g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\zeta - g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\zeta] \\ + a(a+2)[g(\varkappa_2, \phi \varkappa_3)\phi \varkappa_1 - g(\varkappa_1, \phi \varkappa_3)\phi \varkappa_2] \\ + (ab+b-a)[\eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1 - \eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2]. \quad (26)$$

Contracting (26), we have

$$\bar{S}(\varkappa_2, \varkappa_3) = S(\varkappa_2, \varkappa_3) + (ab+b-a^2-a)g(\varkappa_2, \varkappa_3) + a(a+2)\lambda g(\varkappa_2, \phi \varkappa_3) \\ + [ab+b-a^2-a+(n-1)(ab+b-a)]\eta(\varkappa_2)\eta(\varkappa_3), \quad (27)$$

where $\lambda = \text{trace}\phi$.

By direct calculation, we are able to obtain the following relations:

$$\bar{S}(\zeta, \varkappa_3) = (a+1)(1-b)(n-1)\eta(\varkappa_3), \quad (28)$$

$$\bar{Q}\varkappa_2 = Q\varkappa_2 + [ab+b-a^2-a]\varkappa_2 + a(a+2)\lambda\phi\varkappa_2 \\ + [ab+b-a^2-a+(n-1)(ab+b-a)]\eta(\varkappa_2)\zeta, \quad (29)$$

$$\bar{Q}\zeta = (a+1)(1-b)(n-1)\zeta, \quad (30)$$

$$\bar{r} = r - a^2(n-1) + a(a+2)\lambda^2, \quad (31)$$

$$\bar{R}(\varkappa_1, \varkappa_2)\zeta = (a-ab-b+1)[\eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2], \quad (32)$$

$$\bar{R}(\zeta, \varkappa_2)\varkappa_3 = (a+ab+b+1)g(\varkappa_2, \varkappa_3)\zeta + 2(a+1)b\eta(\varkappa_2)\eta(\varkappa_3)\zeta \\ + (ab+b-a-1)\eta(\varkappa_3)\varkappa_2 \quad (33)$$

$$\bar{R}(\varkappa_1, \zeta)\varkappa_3 = -(a+ab+b+1)g(\varkappa_1, \varkappa_3)\zeta - 2(a+1)b\eta(\varkappa_1)\eta(\varkappa_3)\zeta \\ + (a-ab-b+1)\eta(\varkappa_3)\varkappa_1 \quad (34)$$

for any $\varkappa_1, \varkappa_2, \varkappa_3 \in \chi(M)$, where $\lambda = \text{trace}\phi$. Therefore, we can state the following:

Theorem 1. An n -dimensional LP-Sasakian manifold M equipped with general connection D^G satisfies the following:

- (i) $\bar{R}$ of D^G is specified by (26);
- (ii) $\bar{S}$ of D^G is specified by (27);
- (iii) $\bar{r}$ of D^G is specified by (31);
- (iv) $\bar{S}$ of D^G is symmetric;

$$(v) \quad \bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 + \bar{R}(\varkappa_2, \varkappa_3)\varkappa_1 + \bar{R}(\varkappa_3, \varkappa_1)\varkappa_2 = 0.$$

Remark 1. We can see that in an n -dimensional LP-Sasakian manifold M admitting general connection D^G , the following properties are obtained from (1), (26), (27) and (31) under the following conditions:

Case 1	Results
$a = 0, b = -1$	Quarter-symmetric metric connection
$\bar{R}$	$R(\varkappa_1, \varkappa_2)\varkappa_3 + g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\zeta - g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\zeta + \eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2$ $- \eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1$
$\bar{S}$	$S(\varkappa_2, \varkappa_3) - g(\varkappa_2, \varkappa_3) - n\eta(\varkappa_2)\eta(\varkappa_3)$
$\bar{r}$	r
Case 2	Results
$a = 1, b = 0$	Schouten–Van Kampen connection
$\bar{R}$	$R(\varkappa_1, \varkappa_2)\varkappa_3 - g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\zeta + g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\zeta + 3g(\varkappa_2, \phi\varkappa_3)\phi\varkappa_1$ $- 3g(\varkappa_1, \phi\varkappa_3)\phi\varkappa_2 + \eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 - \eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1$
$\bar{S}$	$S(\varkappa_2, \varkappa_3) - 2g(\varkappa_2, \varkappa_3) - (n+1)\eta(\varkappa_2)\eta(\varkappa_3) + 3\lambda g(\varkappa_2, \phi\varkappa_3)$
$\bar{r}$	$r - n + 1 + 3\lambda^2$
Case 3	Results
$a = 1, b = -1$	Tanaka Webster connection
$\bar{R}$	$R(\varkappa_1, \varkappa_2)\varkappa_3 + g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\zeta - g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\zeta + 3g(\varkappa_2, \phi\varkappa_3)\phi\varkappa_1$ $- 3g(\varkappa_1, \phi\varkappa_3)\phi\varkappa_2 + 3\eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 - 3\eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1$
$\bar{S}$	$S(\varkappa_2, \varkappa_3) - 4g(\varkappa_2, \varkappa_3) - (3n+1)\eta(\varkappa_2)\eta(\varkappa_3) + 3\lambda g(\varkappa_2, \phi\varkappa_3)$
$\bar{r}$	$r - n + 1 + 3\lambda^2$
Case 4	Results
$a = 1, b = 1$	Zamkovoy connection
$\bar{R}$	$R(\varkappa_1, \varkappa_2)\varkappa_3 - 3g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\zeta + 3g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\zeta + 3g(\varkappa_2, \phi\varkappa_3)\phi\varkappa_1$ $- 3g(\varkappa_1, \phi\varkappa_3)\phi\varkappa_2 - \eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 + \eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1$
$\bar{S}$	$S(\varkappa_2, \varkappa_3) + (n-1)\eta(\varkappa_2)\eta(\varkappa_3) + 3\lambda g(\varkappa_2, \phi\varkappa_3)$
$\bar{r}$	$r - n + 1 + 3\lambda^2$

4. Projectively Flat LP-Sasakian Manifolds in Accordance with General Connection

From (2), (26) and (27), we obtain

$$\begin{aligned}
 \bar{P}(\varkappa_1, \varkappa_2)\varkappa_3 = & R(\varkappa_1, \varkappa_2)\varkappa_3 + (a + ab + b)[g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\zeta - g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\zeta] \\
 & + a(a + 2)[g(\varkappa_2, \phi\varkappa_3)\phi\varkappa_1 - g(\varkappa_1, \phi\varkappa_3)\phi\varkappa_2] \\
 & + (ab + b - a)[\eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1 - \eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2] \\
 & - \frac{1}{n-1}[S(\varkappa_2, \varkappa_3)\varkappa_1 + (ab + b - a^2 - a)g(\varkappa_2, \varkappa_3)\varkappa_1 \\
 & + \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1 \\
 & + a(a + 2)\lambda g(\varkappa_2, \phi\varkappa_3)\varkappa_1 - S(\varkappa_1, \varkappa_3)\varkappa_2 \\
 & - (ab + b - a^2 - a)g(\varkappa_1, \varkappa_3)\varkappa_2 \\
 & - \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 \\
 & - a(a + 2)\lambda g(\varkappa_1, \phi\varkappa_3)\varkappa_2].
 \end{aligned} \tag{35}$$

Let M be projectively flat with respect to D^G , i.e., $\bar{P} = 0$. Then, taking the inner product with $\varkappa_4$ in (35), we obtain

$$\begin{aligned} R(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) = & (a + ab + b)[g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\eta(\varkappa_4) - g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\eta(\varkappa_4)] \\ & - a(a + 2)[g(\varkappa_2, \phi\varkappa_3)g(\phi\varkappa_1, \varkappa_4) - g(\varkappa_1, \phi\varkappa_3)g(\phi\varkappa_2, \varkappa_4)] \\ & - (ab + b - a)[\eta(\varkappa_2)\eta(\varkappa_3)g(\varkappa_1, \varkappa_4) - \eta(\varkappa_1)\eta(\varkappa_3)g(\varkappa_2, \varkappa_4)] \\ & + \frac{1}{n-1}[S(\varkappa_2, \varkappa_3)g(\varkappa_1, \varkappa_4) \\ & + (ab + b - a^2 - a)g(\varkappa_2, \varkappa_3)g(\varkappa_1, \varkappa_4) \\ & + \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_2)\eta(\varkappa_3)g(\varkappa_1, \varkappa_4) \\ & + a(a + 2)\lambda g(\varkappa_2, \phi\varkappa_3)g(\varkappa_1, \varkappa_4) - S(\varkappa_1, \varkappa_3)g(\varkappa_2, \varkappa_4) \\ & - (ab + b - a^2 - a)g(\varkappa_1, \varkappa_3)g(\varkappa_2, \varkappa_4) \\ & - \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_1)\eta(\varkappa_3)g(\varkappa_2, \varkappa_4) \\ & - a(a + 2)\lambda g(\varkappa_1, \phi\varkappa_3)g(\varkappa_2, \varkappa_4)]. \end{aligned} \quad (36)$$

Setting $\varkappa_1 = \varkappa_4 = \zeta$ in (36) with (5) and (14), we get

$$S(\varkappa_2, \varkappa_3) = d_1g(\varkappa_2, \varkappa_3) + d_2\eta(\varkappa_2)\eta(\varkappa_3) + d_3\lambda\gamma(\varkappa_2, \varkappa_3), \quad (37)$$

where $d_1 = -2ab - 2b + a^2 + bn + abn + n - 1$, $d_2 = -2ab - 2b + a^2 + bn + abn + bn$, $d_3 = -a(a + 2)$, $\gamma(\varkappa_2, \varkappa_3) = g(\phi\varkappa_2, \varkappa_3) = g(\varkappa_2, \phi\varkappa_3)$ and $\lambda = \text{trace}\phi$. Therefore, the following theorem can be stated as:

Theorem 2. *If an n -dimensional LP-Sasakian manifold M is projectively flat in accordance with general connection D^G , then it is found to be a generalized η -Einstein manifold.*

Again, setting $\varkappa_3 = \zeta$ in (35), we obtain

$$\begin{aligned} \bar{P}(\varkappa_1, \varkappa_2)\zeta = & R(\varkappa_1, \varkappa_2)\zeta + (a + ab + b)[g(\varkappa_1, \zeta)\eta(\varkappa_2)\zeta - g(\varkappa_2, \zeta)\eta(\varkappa_1)\zeta] \\ & + a(a + 2)[g(\varkappa_2, \phi\zeta)\phi\varkappa_1 - g(\varkappa_1, \phi\zeta)\phi\varkappa_2] \\ & + (ab + b - a)[\eta(\varkappa_2)\eta(\zeta)\varkappa_1 - \eta(\varkappa_1)\eta(\zeta)\varkappa_2] \\ & - \frac{1}{n-1}[S(\varkappa_2, \zeta)\varkappa_1 + (ab + b - a^2 - a)g(\varkappa_2, \zeta)\varkappa_1 \\ & + \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_2)\eta(\zeta)\varkappa_1 \\ & + a(a + 2)\lambda g(\varkappa_2, \phi\zeta)\varkappa_1 - S(\varkappa_1, \zeta)\varkappa_2 \\ & - (ab + b - a^2 - a)g(\varkappa_1, \zeta)\varkappa_2 \\ & - \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_1)\eta(\zeta)\varkappa_2 \\ & - a(a + 2)\lambda g(\varkappa_1, \phi\zeta)\varkappa_2]. \end{aligned}$$

Inserting (2), (5), (7), (12) and (15) in the above equation, we obtain

$$\bar{P}(\varkappa_1, \varkappa_2)\zeta = P(\varkappa_1, \varkappa_2)\zeta.$$

Therefore, the theorem states the following:

Theorem 3. *An n -dimensional LP-Sasakian manifold M is ζ -projectively flat with respect to the general connection D^G if and only if it is ζ -projectively flat with respect to the Levi-Civita connection D .*

Now, let us consider the ϕ -projectively flat LP-Sasakian manifold M in accordance with general connection D^G .

Definition 2. An n -dimensional LP-Sasakian manifold M is said to be ϕ -projectively flat in accordance with the general connection D^G if

$$g(\bar{P}(\phi\kappa_1, \phi\kappa_2)\phi\kappa_3, \phi\kappa_4) = 0 \quad (38)$$

for all the vector fields $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \chi(M)$.

Theorem 4. If an n -dimensional LP-Sasakian manifold M ($n > 2$) is ϕ -projectively flat in accordance with the general connection D^G , then M is a generalized η -Einstein manifold.

Proof. Taking the inner product with κ_4 in (4), we get

$$g(\bar{R}(\phi\kappa_1, \phi\kappa_2)\phi\kappa_3, \phi\kappa_4) = \frac{1}{n-1} [\bar{S}(\phi\kappa_2, \phi\kappa_3)g(\phi\kappa_1, \phi\kappa_4) - \bar{S}(\phi\kappa_1, \phi\kappa_3)g(\phi\kappa_2, \phi\kappa_4)]. \quad (39)$$

Let $\{v_1, v_2, v_3, \dots, v_{n-1}, \zeta\}$ be a local orthonormal basis of vector fields in M by using the fact that $\{\phi v_1, \phi v_2, \phi v_3, \dots, \phi v_{n-1}, \zeta\}$ is also a local orthonormal basis of vector fields in M .

Setting $\kappa_2 = \kappa_3 = v_i$ in (39) and summing over i ($1 \leq i \leq n-1$), we obtain

$$\sum_{i=1}^{n-1} g(\bar{R}(\phi\kappa_1, \phi v_i)\phi v_i, \phi\kappa_4) = \frac{1}{n-1} \sum_{i=1}^{n-1} [\bar{S}(\phi v_i, \phi v_i)g(\phi\kappa_1, \phi\kappa_4) - \bar{S}(\phi\kappa_1, \phi v_i)g(\phi v_i, \phi\kappa_4)]. \quad (40)$$

It can be easily seen that

$$\sum_{i=1}^{n-1} g(\phi v_i, \phi v_i) = n-1, \quad (41)$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi\kappa_1, \phi v_i)g(\phi v_i, \phi\kappa_4) = \bar{S}(\phi\kappa_1, \phi\kappa_4), \quad (42)$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi v_i, \phi v_i) = \bar{r}. \quad (43)$$

Now, using (41)–(43) in (40), we have

$$\bar{S}(\phi\kappa_1, \phi\kappa_4) = \frac{\bar{r}}{n}g(\phi\kappa_1, \phi\kappa_4). \quad (44)$$

From (6), (16), (31) and (44), we get

$$S(\kappa_1, \kappa_4) = f_1g(\kappa_1, \kappa_4) + f_2\eta(\kappa_1)\eta(\kappa_4) + f_3\lambda\gamma(\kappa_1, \kappa_4), \quad (45)$$

where

$$\begin{aligned} f_1 &= \frac{r - a^2(n-1) + a(a+2)\lambda^2}{n} - (ab + b - a^2 - a), \\ f_2 &= \frac{r - a^2(n-1) + a(a+2)\lambda^2}{n} - (ab + b - a^2 - a + n - 1), \\ f_3 &= -a(a+2), \end{aligned}$$

$\lambda = \text{trace}\phi$ and $\gamma(\mathcal{X}_1, \mathcal{X}_4) = g(\phi\mathcal{X}_1, \mathcal{X}_4) = g(\mathcal{X}_1, \phi\mathcal{X}_4)$.

Thus, M is a generalized η -Einstein manifold. $\square$

5. Conharmonically Flat LP-Sasakian Manifolds in Accordance with General Connection

In view of (3), (26), (27) and (29), we obtain

$$\begin{aligned} \bar{C}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 = & C(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 + (a + ab + b)[g(\mathcal{X}_1, \mathcal{X}_3)\eta(\mathcal{X}_2)\zeta - g(\mathcal{X}_2, \mathcal{X}_3)\eta(\mathcal{X}_1)\zeta] \\ & + a(a + 2)[g(\mathcal{X}_2, \phi\mathcal{X}_3)\phi\mathcal{X}_1 - g(\mathcal{X}_1, \phi\mathcal{X}_3)\phi\mathcal{X}_2] \\ & + (ab + b - a)[\eta(\mathcal{X}_2)\eta(\mathcal{X}_3)\mathcal{X}_1 - \eta(\mathcal{X}_1)\eta(\mathcal{X}_3)\mathcal{X}_2] \\ & - \frac{1}{n-2}[2(ab + b - a^2 - a)\{g(\mathcal{X}_2, \mathcal{X}_3)\mathcal{X}_1 - g(\mathcal{X}_1, \mathcal{X}_3)\mathcal{X}_2\} \\ & + \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\{\eta(\mathcal{X}_2)\mathcal{X}_1 \\ & - \eta(\mathcal{X}_1)\mathcal{X}_2\}\eta(\mathcal{X}_3) \\ & + \{ab + b - a^2 - a + (n-1)(ab + b - a)\}g(\mathcal{X}_2, \mathcal{X}_3)\eta(\mathcal{X}_1)\zeta \\ & - \{ab + b - a^2 - a + (n-1)(ab + b - a)\}g(\mathcal{X}_1, \mathcal{X}_3)\eta(\mathcal{X}_2)\zeta \\ & + a(a + 2)\lambda\{g(\mathcal{X}_2, \phi\mathcal{X}_3)\mathcal{X}_1 - g(\mathcal{X}_1, \phi\mathcal{X}_3)\mathcal{X}_2\} \\ & - a(a + 2)\lambda g(\mathcal{X}_1, \mathcal{X}_3)\phi\mathcal{X}_2 - \lambda g(\mathcal{X}_2, \mathcal{X}_3)\phi\mathcal{X}_1]. \end{aligned} \quad (46)$$

If M is a conharmonically flat (i.e., $\bar{C} = 0$) LP-Sasakian manifold in accordance with general connection, then from (3), we obtain

$$\bar{R}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 = \frac{1}{n-2}[\bar{S}(\mathcal{X}_2, \mathcal{X}_3)\mathcal{X}_1 - \bar{S}(\mathcal{X}_1, \mathcal{X}_3)\mathcal{X}_2 + g(\mathcal{X}_2, \mathcal{X}_3)\bar{Q}\mathcal{X}_1 - g(\mathcal{X}_1, \mathcal{X}_3)\bar{Q}\mathcal{X}_2]. \quad (47)$$

Taking the inner product of (47) with vector field $\mathcal{X}_4$, we obtain

$$\begin{aligned} g(\bar{R}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3, \mathcal{X}_4) = & \frac{1}{n-2}[\bar{S}(\mathcal{X}_2, \mathcal{X}_3)g(\mathcal{X}_1, \mathcal{X}_4) - \bar{S}(\mathcal{X}_1, \mathcal{X}_3)g(\mathcal{X}_2, \mathcal{X}_4) \\ & + g(\mathcal{X}_2, \mathcal{X}_3)\bar{S}(\mathcal{X}_1, \mathcal{X}_4) - g(\mathcal{X}_1, \mathcal{X}_3)\bar{S}(\mathcal{X}_2, \mathcal{X}_4)], \end{aligned} \quad (48)$$

where $g(\bar{Q}\mathcal{X}_1, \mathcal{X}_4) = \bar{S}(\mathcal{X}_1, \mathcal{X}_4)$.

Performing an orthonormal frame field of M and contracting (48) over $\mathcal{X}_1$ and $\mathcal{X}_4$, gives

$$\bar{r} = 0. \quad (49)$$

Using (31) in (49), we have

$$r = a^2n - a^2 - a^2\lambda^2 - 2a\lambda^2. \quad (50)$$

Therefore, the following theorem arises:

Theorem 5. *If an n -dimensional LP-Sasakian manifold M ($n > 2$) is conharmonically flat in accordance with general connection D^G , then the scalar curvature is given by $\bar{r} = 0$, meaning that, $r = a^2n - a^2 - a^2\lambda^2 - 2a\lambda^2$.*

Theorem 6. *An n -dimensional LP-Sasakian manifold M is ζ -conharmonically flat in accordance with general connection D^G if and only if it is so in accordance with Levi-Civita connection D , assuming that the vector fields are horizontal vector fields.*

Proof. Now, we will show for the ζ -conharmonically flat LP-Sasakian manifold M in accordance with general connection D^G .

Replacing $\varkappa_3$ by ζ in (46) and with the help of (3) and (5), we obtain

$$\begin{aligned}\bar{C}(\varkappa_1, \varkappa_2)\zeta &= C(\varkappa_1, \varkappa_2)\zeta + \frac{a}{n-2} [\eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2] \\ &\quad + \frac{a(a+2)\lambda}{n-2} [\eta(\varkappa_1)\phi\varkappa_2 - \eta(\varkappa_2)\phi\varkappa_1] \\ &= C(\varkappa_1, \varkappa_2)\zeta,\end{aligned}\quad (51)$$

if $\varkappa_1, \varkappa_2$ are horizontal vector fields on M , and a is a real constant.

Hence, it is proven. $\square$

Now, $\varkappa_3$ is replaced with ζ in (3). Then, using (28) and (32), we have

$$\bar{C}(\varkappa_1, \varkappa_2)\zeta = \frac{(a+1)(b-1)}{n-2} [\eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2] + \frac{1}{n-2} [\eta(\varkappa_1)\bar{Q}\varkappa_2 - \eta(\varkappa_2)\bar{Q}\varkappa_1]. \quad (52)$$

If M is ζ -conharmonically flat in accordance with general connection D^G , then from (52), we obtain

$$(a+1)(b-1) [\eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2] = \eta(\varkappa_2)\bar{Q}\varkappa_1 - \eta(\varkappa_1)\bar{Q}\varkappa_2. \quad (53)$$

After taking the inner product with $\varkappa_4$ in the above equation, the new equation becomes

$$\begin{aligned}(a+1)(b-1) [\eta(\varkappa_2)g(\varkappa_1, \varkappa_4) - \eta(\varkappa_1)g(\varkappa_2, \varkappa_4)] &= \eta(\varkappa_2)\bar{S}(\varkappa_1, \varkappa_4) \\ &\quad - \eta(\varkappa_1)\bar{S}(\varkappa_2, \varkappa_4).\end{aligned}\quad (54)$$

Setting $\varkappa_2 = \zeta$ in (54), we obtain

$$\bar{S}(\varkappa_1, \varkappa_4) = (a+1)(1-b) [g(\varkappa_1, \varkappa_4) + n\eta(\varkappa_1)\eta(\varkappa_4)]. \quad (55)$$

Contracting (55) over $\varkappa_1$ and $\varkappa_4$, we obtain

$$\bar{r} = 0. \quad (56)$$

Theorem 7. An n -dimensional LP-Sasakian manifold M is ζ -conharmonically flat in accordance with general connection D^G , then the scalar curvature in accordance with general connection D^G vanishes.

Moreover, let us consider the ϕ -conharmonically flat LP-Sasakian manifold M in accordance with general connection D^G .

Definition 3. An n -dimensional LP-Sasakian manifold M is said to be ϕ -conharmonically flat in accordance with general connection D^G if

$$g(\bar{C}(\phi\varkappa_1, \phi\varkappa_2)\phi\varkappa_3, \phi\varkappa_4) = 0 \quad (57)$$

for all the vector fields $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4 \in \chi(M)$.

Theorem 8. If an n -dimensional LP-Sasakian manifold M ($n > 2$) is ϕ -conharmonically flat in accordance with general connection D^G , then M is generalized η -Einstein manifold.

Proof. Taking the inner product with $\varkappa_4$ in (3), we obtain

$$\begin{aligned}
 g(\bar{R}(\phi\kappa_1, \phi\kappa_2)\phi\kappa_3, \phi\kappa_4) &= \frac{1}{n-2} [\bar{S}(\phi\kappa_2, \phi\kappa_3)g(\phi\kappa_1, \phi\kappa_4) \\
 &\quad - \bar{S}(\phi\kappa_1, \phi\kappa_3)g(\phi\kappa_2, \phi\kappa_4) \\
 &\quad + g(\phi\kappa_2, \phi\kappa_3)\bar{S}(\phi\kappa_1, \phi\kappa_4) \\
 &\quad - g(\phi\kappa_1, \phi\kappa_3)\bar{S}(\phi\kappa_2, \phi\kappa_4)].
 \end{aligned} \tag{58}$$

Let $\{v_1, v_2, v_3, \dots, v_{n-1}, \zeta\}$ be a local orthonormal basis of vector fields in M . Utilizing the fact that $\{\phi v_1, \phi v_2, \phi v_3, \dots, \phi v_{n-1}, \zeta\}$ also forms a local orthonormal basis of vector fields in M , we proceed by setting $\kappa_2 = \kappa_3 = v_i$ in Equation (58) and summing over i for $1 \leq i \leq n-1$, we obtain

$$\begin{aligned}
 \sum_{i=1}^{n-1} g(\bar{R}(\phi\kappa_1, \phi v_i)\phi v_i, \phi\kappa_4) &= \frac{1}{n-2} \sum_{i=1}^{n-1} [\bar{S}(\phi v_i, \phi v_i)g(\phi\kappa_1, \phi\kappa_4) \\
 &\quad - \bar{S}(\phi\kappa_1, \phi v_i)g(\phi v_i, \phi\kappa_4) + g(\phi v_i, \phi v_i)\bar{S}(\phi\kappa_1, \phi\kappa_4) \\
 &\quad - g(\phi\kappa_1, \phi v_i)\bar{S}(\phi v_i, \phi\kappa_4)].
 \end{aligned} \tag{59}$$

It is evident that

$$\sum_{i=1}^{n-1} g(\phi v_i, \phi v_i) = n-1, \tag{60}$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi\kappa_1, \phi v_i)g(\phi v_i, \phi\kappa_4) = \bar{S}(\phi\kappa_1, \phi\kappa_4), \tag{61}$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi v_i, \phi v_i) = \bar{r}. \tag{62}$$

Using (60)–(62) in (59), we obtain

$$\bar{S}(\phi\kappa_1, \phi\kappa_4) = \bar{r}g(\phi\kappa_1, \phi\kappa_4). \tag{63}$$

From (6), (16), (31) and (63), we obtain

$$S(\kappa_1, \kappa_4) = k_1g(\kappa_1, \kappa_4) + k_2\eta(\kappa_1)\eta(\kappa_4) + k_3\lambda\gamma(\kappa_1, \kappa_4), \tag{64}$$

where

$$\begin{aligned}
 k_1 &= r - a^2(n-1) + a(a+2)\lambda^2 - (ab + b - a^2 - a), \\
 k_2 &= r - a^2(n-1) + a(a+2)\lambda^2 - (ab + b - a^2 - a + n-1), \\
 k_3 &= -a(a+2),
 \end{aligned}$$

$\lambda = \text{trace}\phi$ and $\gamma(\kappa_1, \kappa_4) = g(\phi\kappa_1, \kappa_4) = g(\kappa_1, \phi\kappa_4)$. This verifies that M is generalized η -Einstein manifold. $\square$

6. Quasi-Conharmonically Flat LP-Sasakian Manifolds in Accordance with General Connection

Definition 4. An n -dimensional LP-Sasakian manifold M is said to be quasi-conharmonically flat in accordance with general connection D^G if

$$g(\bar{C}(\phi\kappa_1, \kappa_2)\kappa_3, \phi\kappa_4) = 0 \tag{65}$$

for all vector fields $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \chi(M)$.

Theorem 9. *If an n -dimensional LP-Sasakian manifold M ($n > 2$) is quasi-conharmonically flat in accordance with general connection D^G , then its scalar curvature in accordance with general connection D^G vanishes.*

Proof. Taking the inner product with $\varkappa_4$ in (3), we obtain

$$g(\bar{R}(\phi \varkappa_1, \varkappa_2) \varkappa_3, \phi \varkappa_4) = \frac{1}{n-2} [\bar{S}(\varkappa_2, \varkappa_3) g(\phi \varkappa_1, \phi \varkappa_4) - \bar{S}(\phi \varkappa_1, \varkappa_3) g(\varkappa_2, \phi \varkappa_4) + g(\varkappa_2, \varkappa_3) \bar{S}(\phi \varkappa_1, \phi \varkappa_4) - g(\phi \varkappa_1, \varkappa_3) \bar{S}(\varkappa_2, \phi \varkappa_4)]. \quad (66)$$

Let $\{v_1, v_2, v_3, \dots, v_{n-1}, \zeta\}$ be a local orthonormal basis of vector fields in M . Using the fact that $\{\phi v_1, \phi v_2, \phi v_3, \dots, \phi v_{n-1}, \zeta\}$ also form a local orthonormal basis of vector fields in M , we proceed as follows. Setting $\varkappa_2 = \varkappa_3 = v_i$ in (66) and summing over i ($1 \leq i \leq n-1$), we obtain

$$\sum_{i=1}^{n-1} g(\bar{R}(\phi \varkappa_1, v_i) v_i, \phi \varkappa_4) = \frac{1}{n-2} \sum_{i=1}^{n-1} [\bar{S}(v_i, v_i) g(\phi \varkappa_1, \phi \varkappa_4) - \bar{S}(\phi \varkappa_1, v_i) g(v_i, \phi \varkappa_4) + g(v_i, v_i) \bar{S}(\phi \varkappa_1, \phi \varkappa_4) - g(\phi \varkappa_1, v_i) \bar{S}(v_i, \phi \varkappa_4)]. \quad (67)$$

It is readily apparent that

$$\sum_{i=1}^{n-1} g(\phi v_i, \phi v_i) = n, \quad (68)$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi \varkappa_1, \phi v_i) g(\phi v_i, \phi \varkappa_4) = \bar{S}(\phi \varkappa_1, \phi \varkappa_4), \quad (69)$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi v_i, \phi v_i) = \bar{r}. \quad (70)$$

Using (68)–(70) in (67), we obtain

$$\frac{\bar{r}}{n-2} g(\phi \varkappa_1, \phi \varkappa_4) = 0. \quad (71)$$

Since $g(\phi \varkappa_1, \phi \varkappa_4) \neq 0$. Then, the above equation reduces to

$$\bar{r} = 0. \quad (72)$$

Thus, the proof is completed. $\square$

7. Ricci Semi-Symmetric LP-Sasakian Manifolds in Accordance with General Connection

Numerous geometers have explored the characteristics of semi-symmetric and Ricci semi-symmetric manifolds, as documented in [17,18]. In this section, we will examine Ricci semi-symmetric LP-Sasakian manifolds in the context of a general connection D^G .

Definition 5. *An n -dimensional LP-Sasakian manifold M is said to be Ricci semi-symmetric in accordance with general connection D^G if the curvature tensor $\bar{R}$ in accordance with D^G satisfies*

$$(\bar{R}(\varkappa_1, \varkappa_2) \cdot \bar{S})(\varkappa_3, \varkappa_4) = 0, \quad (73)$$

where $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4$ are horizontal vector fields on M .

Theorem 10. *A Ricci semi-symmetric LP-Sasakian manifold M of dimension n ($n > 2$) equipped with general connection D^G is a generalized η -Einstein manifold.*

Proof. Equation (73) can be written as

$$\bar{S}(\bar{R}(\kappa_1, \kappa_2)\kappa_3, \kappa_4) + \bar{S}(\kappa_3, \bar{R}(\kappa_1, \kappa_2)\kappa_4) = 0. \quad (74)$$

Setting $\kappa_1 = \kappa_4 = \zeta$ in (74), we obtain

$$\bar{S}(\bar{R}(\zeta, \kappa_2)\kappa_3, \zeta) + \bar{S}(\kappa_3, \bar{R}(\zeta, \kappa_2)\zeta) = 0. \quad (75)$$

Using (28) and (33) in (75), we have

$$(a - ab - b + 1)\bar{S}(\kappa_2, \kappa_3) = (a + ab + b + 1)(n - 1)(a + 1)(1 - b)[g(\kappa_2, \kappa_3) + 2(a + 1)b\eta(\kappa_2)\eta(\kappa_3)]. \quad (76)$$

Now, inserting (27) in (76), we obtain

$$S(\kappa_2, \kappa_3) = l_1g(\kappa_2, \kappa_3) + l_2\eta(\kappa_2)\eta(\kappa_3) + l_3\lambda\gamma(\kappa_2, \kappa_3), \quad (77)$$

where

$$\begin{aligned} l_1 &= \frac{1}{a - ab - b + 1} [(a - ab - b + 1)(ab + b - a^2 - a) \\ &\quad - (a + ab + b + 1)(n - 1)(a + 1)(1 - b)], \\ l_2 &= \frac{1}{a - ab - b + 1} [(a - ab - b + 1)(ab + b - a^2 - a + abn + bn \\ &\quad - an - ab - b + a) - 2(a + ab + b + 1)(n - 1)(a + 1)(1 - b)], \\ l_3 &= a(a + 2), \end{aligned}$$

$\lambda = \text{trace}\phi$ and $\gamma(\kappa_2, \kappa_3) = g(\kappa_1, \phi\kappa_3)$. Therefore, we can say that M is a generalized η -Einstein manifold. $\square$

8. Applications to the General Theory of Relativity

The objective of this section is to explore the applications of the general connection D^G on an n -dimensional LP-Sasakian manifold M within the framework of the general theory of relativity. We assume that an LP-Sasakian manifold M of dimension n represents a fluid space-time and its Ricci tensor $\bar{S}$ defined with respect to D^G satisfies $\bar{S} = 0$.

From (27) and (31), we obtain

$$\begin{aligned} S(\kappa_1, \kappa_2) &= (a - ab - b + a^2)g(\kappa_1, \kappa_2) - a(a + 2)\lambda g(\kappa_1, \phi\kappa_2) \\ &\quad + [a - ab - b + a^2 - (n - 1)(ab + b - a)]\eta(\kappa_1)\eta(\kappa_2), \end{aligned} \quad (78)$$

$$r = a^2(n - 1) - a(a + 2)\lambda^2. \quad (79)$$

The Einstein field equation in the absence of a cosmological constant is expressed by

$$S(\kappa_1, \kappa_2) - \frac{r}{2}g(\kappa_1, \kappa_2) = \kappa T(\kappa_1, \kappa_2) \quad (80)$$

for all $\kappa_1, \kappa_2 \in \chi(M)$, where T is the energy momentum tensor and κ is the Einstein's gravitational constant. If possible, we consider that the fluid space-time is a perfect fluid space-time, then the energy-momentum tensor T assumes the form

$$T(\kappa_1, \kappa_2) = pg(\kappa_1, \kappa_2) + (p + \mu)\eta(\kappa_1)\eta(\kappa_2) \quad (81)$$

for $\varkappa_1, \varkappa_2 \in \chi(M)$, where p is the isotropic pressure, μ is the energy density, and ζ is the fluid flow velocity, i.e., $g(\zeta, \zeta) = -1$ of the perfect fluid space-time.

Now, from (78), (79) and (81), Equation (80) becomes

$$\begin{aligned} & \frac{a(a+2)\lambda^2 - a^2(n-1) + 2(a-ab-b+a^2)}{2} g(\varkappa_1, \varkappa_2) \\ & + \{a-ab-b+a^2 - (n-1)(ab+b-a)\} \eta(\varkappa_1) \eta(\varkappa_2) \\ & - a(a+2)\lambda g(\varkappa_1, \phi \varkappa_2) \\ & = \kappa \{pg(\varkappa_1, \varkappa_2) + (p+\mu)\eta(\varkappa_1)\eta(\varkappa_2)\}. \end{aligned} \quad (82)$$

Setting $\varkappa_1 = \varkappa_2 = \zeta$ and using (5) and (7) in (82), we find

$$(n-1)(a^2 + 2a - 2ab - 2b) - a(a+2)\lambda^2 = 2\kappa\mu. \quad (83)$$

By selecting a frame field and subsequently contracting Equation (80), we obtain

$$a(n-2)(a-an+a\lambda^2+2\lambda^2) = 2\kappa[p(n-1)-\mu]. \quad (84)$$

From (83) and (84), we obtain

$$\frac{p}{\mu} = \frac{\Delta}{(n-1)\{(n-1)(a^2+2a-2ab-2b)-a(a+2)\lambda^2\}}, \quad (85)$$

where $\Delta = a(n-2)(a-an+a\lambda^2+2\lambda^2) + (n-1)(a^2+2a-2ab-2b) - a(a+2)\lambda^2$. Thus, (85) represents the equation of state for the perfect fluid space-time. As a special case, if we assume $\lambda = \text{trace } \phi = 0$ and $n = 4$, the equation of state for the perfect fluid space-time reduces to

$$\frac{p}{\mu} = \frac{1}{3} \left[\frac{a(2-a) - 2b(a+1)}{a(a-2) - 2b(a+1)} \right]. \quad (86)$$

If the real constants $a = 0, b = -1$ in (86), then Equation (86) reduces to the quarter-symmetric metric connection, which can be seen from (1). Therefore, we obtain

$$3p = \mu. \quad (87)$$

Thus, this indicates that the space-time is isotropic and homogeneous [19]. De et al. [20] also investigated the properties of isotropic and homogeneous space-times. Numerous researchers, including [20–23] have explored the characteristics of such space-times. Based on these findings, we can formulate the following theorem:

Theorem 11. Consider a perfect fluid space-time equipped with the general connection D^G , where the Ricci tensor associated with D^G vanishes. If this perfect fluid space-time satisfies Einstein field equations without a cosmological constant, then the equation of state for the space-time is given by

$$\frac{p}{\mu} = \frac{a(n-2)(a-an+a\lambda^2+2\lambda^2) + (n-1)(a^2+2a-2ab-2b) - a(a+2)\lambda^2}{(n-1)\{(n-1)(a^2+2a-2ab-2b)-a(a+2)\lambda^2\}}.$$

In particular, for $n = 4, \lambda = \text{trace } \phi = 0$, and the real constants $a = 0, b = -1$, the space-time is filled with an isotropic and homogeneous perfect fluid.

9. Example

Let $M = (x, y, z, u) \in \mathbf{R}^4 : u \neq 0$ be a 4-dimensional differentiable manifold, where $\mathbf{R}^4$ denotes the Euclidean space of dimension 4.

Let us assume that

$$v_1 = e^{x+u} \frac{\partial}{\partial x}, \quad v_2 = e^{y+u} \frac{\partial}{\partial y}, \quad v_3 = e^{z+u} \frac{\partial}{\partial z}, \quad v_4 = \frac{\partial}{\partial u}.$$

are the linearly independent vector fields, and therefore, they form a basis of M . The non-vanishing components of the Lie bracket are given by

$$[v_1, v_4] = -v_1, \quad [v_2, v_4] = -v_2, \quad [v_3, v_4] = -v_3.$$

Let the semi-Riemannian metric g of M be defined by

$$g(v_i, v_j) = 1 \quad \text{for } i, j = 1, 2, 3, \quad g(v_i, v_j) = 0 \quad \text{for } i \neq j$$

and

$$g(v_i, v_j) = -1 \quad \text{for } i = j = 4, \quad \text{where } i, j = 1, 2, 3, 4.$$

We assume that the associated 1-form corresponding to the unit timelike vector field $\zeta = v_4$ on M is related by the condition $g(\varkappa_1, v_4) = \eta(\varkappa_1)$. Let the $(1, 1)$ -tensor field ϕ be defined by

$$\phi(v_1) = -v_1, \quad \phi(v_2) = -v_2, \quad \phi(v_3) = -v_3, \quad \phi(v_4) = 0.$$

Then, it is clear that

$$\phi^2 \varkappa_1 = \varkappa_1 + \eta(\varkappa_1) v_4, \quad \eta(v_4) = -1$$

hold for all vector fields $\varkappa_1$. The manifold M , equipped with the compatible metric g , forms a 4-dimensional Lorentzian para-contact metric manifold. Using Koszul's formula, we obtain

$$2g(D_{\varkappa_1} \varkappa_2, \varkappa_3) = \varkappa_1 g(\varkappa_2, \varkappa_3) + \varkappa_2 g(\varkappa_3, \varkappa_1) - \varkappa_3 g(\varkappa_1, \varkappa_2) - g(\varkappa_1, [\varkappa_2, \varkappa_3]) \\ - g(\varkappa_2, [\varkappa_1, \varkappa_3]) + g(\varkappa_3, [\varkappa_1, \varkappa_2]).$$

It can easily verify that

$$D_{v_i} v_4 = -v_i \quad \text{for } i = 1, 2, 3,$$

$$D_{v_i} v_i = -2v_4 \quad \text{for } i = 1, 2, 3,$$

and

$$D_{v_i} v_j = 0 \quad \text{for } i \neq j, \quad \text{where } i = j = 4.$$

From the above analysis, we establish that $D_{\varkappa_1} v_4 = \phi \varkappa_1$ for all $\varkappa_1 \in \chi(M)$. This allows us to assert that (M, g) is a 4-dimensional LP-Sasakian manifold. The non-zero components of the curvature tensor R corresponding to the Levi-Civita connection are

$$R(v_1, v_2)v_1 = -v_2, \quad R(v_1, v_3)v_1 = -v_3, \quad R(v_1, v_4)v_1 = -v_4, \quad R(v_1, v_2)v_2 = v_1,$$

$$R(v_2, v_3)v_2 = -v_3, \quad R(v_2, v_4)v_2 = -v_4, \quad R(v_1, v_3)v_3 = v_1, \quad R(v_2, v_3)v_3 = v_2,$$

$$R(v_3, v_4)v_3 = -v_4, \quad R(v_1, v_4)v_4 = -v_1, \quad R(v_2, v_4)v_4 = -v_2, \quad R(v_3, v_4)v_4 = -v_2.$$

Additionally, the non-zero components of the Ricci tensor S are expressed as

$$S(v_i, v_i) = 3 \quad \text{for } i = 1, 2, 3 \quad \text{and} \quad S(v_j, v_j) = -3 \quad \text{for } j = 4.$$

Proceeding further, based on the preceding discussions, the general connection D^G on M defined by Equation (1) takes the following form

$$D_{v_i}^G v_i = -(a+2)v_4 \quad \text{for } i = 1, 2, 3, \quad D_{v_i}^G v_4 = -(a+1)v_i \quad \text{for } i = 1, 2, 3$$

and

$$D_{v_i}^G v_j = 0 \quad \text{for } i \neq j, \quad \text{where } i = j = 4.$$

The non-zero components of the curvature tensor R in accordance with the general connection D^G are given as

$$\bar{R}(v_i, v_4)v_i = -(a+ab+b+1)v_4 \quad \text{for } i = 1, 2, 3.$$

Additionally, we compute the Ricci tensor $\bar{S}$ with respect to the connection D^G as follows

$$\bar{S}(v_i, v_i) = 3+ab+b-a^2-a-a^2\lambda-2a\lambda \quad \text{for } i = 1, 2, 3$$

and

$$\bar{S}(v_j, v_j) = -3(a-b-ab+1) \quad \text{for } j = 4,$$

where a and b are real constants. From the above findings, the scalar curvatures regarding the Levi-Civita connection D and the general connection D^G are $r = 12$ and $\bar{r} = r - 3a^2$, respectively, knowing that $\lambda = \text{trace}\phi = 0$. Hence, these results verify Equation (31) and Theorem 5.

10. Conclusions

From all of the above results, we can conclude that an n -dimensional LP-Sasakian manifold admitting general connection D^G satisfies the following:

Conditions	Results
$\bar{P} = 0$	generalized η -Einstein manifold
ϕ -projectively flat	generalized η -Einstein manifold
$\bar{C} = 0$	$\bar{r} = 0$
ξ -conharmonically flat	$\bar{r} = 0$
ϕ -conharmonically flat	generalized η -Einstein manifold
quasi-conharmonically flat	$\bar{r} = 0$
$\bar{R} \cdot \bar{S} = 0$	generalized η -Einstein manifold

From Section 8, we established that a perfect fluid space-time equipped with the general connection D^G , whose Ricci tensor associated with D^G vanishes and satisfies the Einstein field equation without cosmological constant. Under these conditions, the space-time is filled with an isotropic and homogeneous perfect fluid, particularly for $n = 4$, $\lambda = \text{trace}\phi = 0$, and the real constants $a = 0, b = -1$. Further, connections are fundamental in both general relativity and differential geometry for understanding the geometry of curved spaces. In general relativity, they are used to describe space-time curvature, the movement of particles along geodesics, parallel transport, and covariant derivatives, which help explain how matter and energy shape space-time. In differential geometry, connections are employed to define curvature, facilitate parallel transport, and allow differentiation of tensor fields on curved manifolds, extending the concept of differentiation to curved spaces. In both fields, connections play a key role in describing how geometric objects change as they move through curved spaces, enhancing our understanding of curvature, structure, and geometry.

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Article

Ruled Surfaces and Their Geometric Invariants via the Orthogonal Modified Frame in Minkowski 3-Space

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Abstract: Ruled surfaces in Minkowski 3-space play a crucial role in differential geometry and have significant applications in physics and engineering. This study explores the fundamental properties of ruled surfaces via orthogonal modified frame in Minkowski space E_1^3 , focusing on their minimality, developability, and curvature characteristics. We examine the necessary and sufficient conditions for a ruled surface to be minimal, considering the mean curvature and its implications. Furthermore, we analyze the developability of such surfaces, determining the conditions under which they can be locally unfolded onto a plane without distortion. The Gaussian and mean curvatures of ruled surfaces in Minkowski space are computed and discussed, providing insights into their geometric behavior. Special attention is given to spacelike, timelike, and lightlike rulings, highlighting their unique characteristics. This research contributes to the broader understanding of the geometric properties of ruled surfaces within the framework of Minkowski geometry.

Keywords: ruled surfaces; Minkowski 3-space; orthogonal modified frame

MSC: 53A05; 53A25; 58C27

1. Introduction

The requirement that there is at least one straight line, known as a ruling, that lies wholly on the surface through each point on the surface defines ruled surfaces, a fundamental class of surfaces in differential geometry. These surfaces' straightforward geometric structure and simplicity of creation make them useful in physics, engineering, and architecture. Compared to their Euclidean counterparts, governed surfaces in Minkowski 3-space E_1^3 have special geometric features, especially when categorized according to spacelike, timelike, or lightlike rulings.

In the parametric representation of a ruled surface, denoted as $\Lambda(\sigma, v) = \vartheta(\sigma) + v U(\sigma)$, the base curve $\vartheta(\sigma)$ describes the shape of the surface, while the director vector $U(\sigma)$ determines the direction of the rulings. We can convey the points on the surface using two parameters σ and v owing to this representation; see [1,2].

The study of null geodesics, which characterize the motions of light beams in curved space–time, naturally involves ruled surfaces. For instance, governed surfaces made up of families of null geodesics can be thought of as light cones, which determine the causal structure of space–time (see [3]).

Ruled surfaces are used to explain wavefronts moving through gravitational fields in general relativity and optics. Due to the curvature of space–time, these wavefronts

frequently create cusps and caustics, which are crucial for gravitational lensing effects (see [4]). Since the paths of charged particles frequently form helical or conical ruled surfaces in space–time diagrams, ruled surfaces may be used to examine the motion of charged particles in electromagnetic fields (see [5]). Also, since null geodesics establish the border between parts of space–time that can communicate with infinity and those that cannot, the event horizon of a revolving black hole (such as a Kerr black hole) is composed of ruled surfaces (see [6]).

Observing the behaviors of the ruled surfaces under Lorentz transformations, the orthogonal modified frame (OMF) provides a sophisticated geometric and algebraic framework, which modifies the Lorentz group for ruled surfaces in E_1^3 . Because traditional Frenet–Serret frames frequently falter in degenerate situations, such as lightlike decisions, the OMF is a more reliable and flexible instrument. We may create frame-adaptive Lorentz transformations using an OMF that maintain the core geometric characteristics of ruled surfaces, such as their causal nature (spacelike, timelike, or lightlike). Specifically, the ruling structure can be preserved while applying modified Lorentz boosts and rotations to spacelike and timelike governed surfaces. It is necessary to perform certain null rotations (parabolic Lorentz transformations) for lightlike ruled surfaces in order to preserve the causal nature of the ruling. Since ruled surfaces are commonly seen in shock wavefronts, string theory, and relativistic kinematics, this method has obvious uses in relativity, differential geometry, and wave propagation in spacetime. This version of the Lorentz group preserves the basic structure of the governed surface while expanding our knowledge of symmetry transformations in E_1^3 .

Analyzing ruled surfaces' curvature characteristics, minimality requirements, and developability is a crucial part of the study process. In [7,8], Struik and Eisenhart have performed classical studies on ruled surfaces, establishing basic conclusions in Euclidean and Lorentzian geometry. These investigations have been expanded to Minkowski space due to developments in contemporary differential geometry, where the behavior of ruled surfaces under various frame structures has emerged as a key field of study. In recent times, there has been a particular focus on the orthogonal modified frame with $\kappa \neq 0$ and $\tau \neq 0$ of a space curve in Minkowski 3-space. This framework involves studying curves in a three-dimensional space with specific curvature and torsion properties. Researchers have made notable contributions to this area, as referenced in works such as [9–17]. The purpose of this work is to use an orthogonal modified frame to investigate the geometric characteristics of ruled surfaces in Minkowski 3-space. We advance our knowledge of these surfaces and their uses in Lorentzian geometry by investigating the requirements for minimality, developability, and curvature properties. The results of this study give a wider view of the function of modified frame structures in differential geometry by comparing them with previous findings.

2. Preliminaries

The definition of the Lorentzian product in E_1^3 is

$$\mathcal{L} = d\sigma_1^2 + d\sigma_2^2 - d\sigma_3^2,$$

where $(\sigma_1, \sigma_2, \sigma_3)$ is the coordinate system of E_1^3 . The characteristics of an arbitrary $v \in E_1^3$ vector are as follows: Null if $\mathcal{L}(v, v) = 0$ and $v \neq 0$, spacelike if $\mathcal{L}(v, v) > 0$ or $v = 0$, and timelike if $\mathcal{L}(v, v) < 0$. Likewise, if a curve $\varphi = \varphi(q)$ has a $\varphi'(q)$ that is spacelike, timelike, or null, then it can also be spacelike, timelike, or null [18,19].

Let $\vartheta = \vartheta(\sigma)$ be a spacelike curve in E_1^3 . Let $\{t, n, b\}$ denote the moving Frenet frame of ϑ , then $\{t, n, b\}$ satisfying [2,18,19]:

$$\begin{aligned} \dot{t}(\sigma) &= \kappa(\sigma) n(\sigma), \\ \dot{n}(\sigma) &= -\varepsilon \kappa(\sigma) t(\sigma) + \tau(\sigma) b(\sigma), \\ \dot{b}(\sigma) &= \tau(\sigma) n(\sigma), \end{aligned} \quad (1)$$

where $\left(\cdot = \frac{d}{d\sigma}\right)$, $\mathfrak{L}(t, t) = 1$, $\mathfrak{L}(n, n) = \varepsilon$, $\mathfrak{L}(b, b) = -\varepsilon$, and $\mathfrak{L}(t, n) = \mathfrak{L}(t, b) = \mathfrak{L}(n, b) = 0$. When $\varepsilon = 1$, $\vartheta(\sigma)$ is first kind spacelike and when $\varepsilon = -1$, $\vartheta(\sigma)$ is second kind spacelike.

Assume that $\vartheta(\sigma)$ is an analytic curve with discrete zero points in E_1^3 . The orthogonal modified frames are provided under the assumption that $\kappa(\sigma)$ is not identically zero as [9–11]:

$$\begin{aligned} \dot{T}(\sigma) &= N(\sigma), \\ \dot{N}(\sigma) &= -\varepsilon \kappa^2(\sigma) T(\sigma) + \left(\frac{\dot{\kappa}(\sigma)}{\kappa(\sigma)}\right) N(\sigma) + \tau(\sigma) B(\sigma), \\ \dot{B}(\sigma) &= \tau(\sigma) N(\sigma) + \left(\frac{\dot{\kappa}(\sigma)}{\kappa(\sigma)}\right) B(\sigma), \end{aligned} \quad (2)$$

where $\mathfrak{L}(T, T) = 1$, $\mathfrak{L}(N, N) = \varepsilon \kappa^2$, $\mathfrak{L}(B, B) = -\varepsilon \kappa^2$, $\mathfrak{L}(T, B) = 0$, $\mathfrak{L}(N, B) = 0$, and $\mathfrak{L}(T, N) = 0$.

A ruled surface Λ in E_1^3 can be represented as

$$\Lambda(\sigma, v) = \vartheta(\sigma) + v \ell(\sigma), \quad (3)$$

where $\vartheta(\sigma)$ is base curve and $\ell(\sigma)$ is director. The unit normal vector field $\mathcal{N}$ on Λ can be defined by [19]

$$N_\Lambda = \frac{\Lambda_\sigma \times \Lambda_v}{\|\Lambda_\sigma \times \Lambda_v\|},$$

where $\Lambda_\sigma = \frac{\partial \Lambda}{\partial \sigma}$ and $\Lambda_v = \frac{\partial \Lambda}{\partial v}$. These are the first and second fundamental forms of Λ , respectively,

$$I : E = \|\Lambda_\sigma\|^2, F = \mathfrak{L}(\Lambda_\sigma, \Lambda_v), G = \|\Lambda_v\|^2,$$

$$II : e = \mathfrak{L}(\Lambda_{\sigma\sigma}, N_\Lambda), f = \mathfrak{L}(\Lambda_{\sigma v}, N_\Lambda), g = \mathfrak{L}(\Lambda_{vv}, N_\Lambda).$$

The Gaussian and mean curvatures of Λ , respectively, are given by [19]

$$\begin{aligned} K_\Lambda &= \frac{eg - f^2}{EG - F^2}, \\ H_\Lambda &= \frac{Eg + Ge - 2Ff}{2(EG - F^2)}. \end{aligned}$$

Proposition 1 ([20,21]). *A ruled surface Λ is developable if and only if $K_\Lambda = 0$, and minimal if and only if $H_\Lambda = 0$.*

3. Characteristics of Ruled Surfaces via Orthogonal Modified Frame

We derive descriptions of the minimal and developable ruled surfaces from the orthogonal modified frame in E_1^3 .

The representation of a ruled surface for a regular spacelike curve $\beta = \beta(\sigma)$ via orthogonal modified frame in E_1^3 is handed as

$$\Gamma(\sigma, v) = \beta(\sigma) + v \ell(\sigma), \quad (4)$$

for $\beta(\sigma)$ the base curve and $\ell(\sigma)$ the director curve such that $\|\ell(\sigma)\| = 1$. Then, we can write (4) in the form

$$\Gamma(\sigma, v) = \beta(\sigma) + v [r_1 T(\sigma) + r_2 N(\sigma) + r_3 B(\sigma)], r_1^2 + \varepsilon \kappa^2 (r_2^2 - r_3^2) = 1. \quad (5)$$

Based on (5), the equations of I 's components and the unit normal vector field basic form of Γ are as follows:

$$\begin{aligned} E_\Gamma &= (1 - \varepsilon v r_2 \kappa^2)^2 + \varepsilon v^2 \kappa^2 \left[\left(r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau \right)^2 - \left(r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right) \right)^2 \right], \\ F_\Gamma &= r_1 (1 - \varepsilon v r_2 \kappa^2) + \varepsilon v^2 \kappa^2 \left[r_2 \left(r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau \right) - r_3 \left(r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right) \right) \right], \\ G_\Gamma &= 1, \end{aligned} \quad (6)$$

$$N_\Gamma = \frac{\alpha_1 T(\sigma) + \alpha_2 N(\sigma) + \alpha_3 B(\sigma)}{\sqrt{\alpha_1^2 + \varepsilon \kappa^2 (\alpha_2^2 - \alpha_3^2)}}, \quad (7)$$

where

$$\begin{aligned} \alpha_1 &= v r_3 \left[r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau \right] - v r_2 \left[r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right) \right], \\ \alpha_2 &= v r_1 \left[r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right) \right] - r_3 (1 - \varepsilon v r_2 \kappa^2), \\ \alpha_3 &= v r_1 \left[r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau \right] - r_2 (1 - \varepsilon v r_2 \kappa^2). \end{aligned} \quad (8)$$

The features of Γ 's second fundamental form are as follows:

$$\begin{aligned} e_\Gamma &= \frac{\alpha_1 \delta_1 + \varepsilon \kappa^2 (\alpha_2 \delta_2 - \alpha_3 \delta_3)}{\sqrt{\alpha_1^2 + \varepsilon \kappa^2 (\alpha_2^2 - \alpha_3^2)}}, \\ f_\Gamma &= \frac{-\varepsilon \alpha_1 r_2 \kappa^2 + \varepsilon \kappa^2 [\alpha_2 (r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau) - \alpha_3 (r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right))]}{\sqrt{\alpha_1^2 + \varepsilon \kappa^2 (\alpha_2^2 - \alpha_3^2)}}, \\ g_\Gamma &= 0, \end{aligned} \quad (9)$$

where

$$\begin{aligned} \delta_1 &= -\varepsilon v \left\{ 2 r_2 \dot{\kappa} + \kappa \left[r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau \right] \right\}, \\ \delta_2 &= 1 - \varepsilon v r_2 \kappa^2 + v \left\{ r_3 \dot{\tau} + r_2 \frac{d}{d\sigma} \left(\frac{\dot{\kappa}}{\kappa} \right) + \left(\frac{\dot{\kappa}}{\kappa} \right) \left[r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau \right] + \tau \left[r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right) \right] \right\}, \\ \delta_3 &= v \left\{ r_2 \dot{\tau} + r_3 \frac{d}{d\sigma} \left(\frac{\dot{\kappa}}{\kappa} \right) + \tau \left[r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau \right] + \left(\frac{\dot{\kappa}}{\kappa} \right) \left[r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right) \right] \right\}. \end{aligned} \quad (10)$$

Through using above-mentioned information, the Gaussian and mean curvatures are calculated based on the following:

$$K_\Gamma = - \frac{\left\{ -\varepsilon \alpha_1 r_2 \kappa^2 + \varepsilon \kappa^2 [\alpha_2 (r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau) - \alpha_3 (r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right))] \right\}^2}{\Delta [\alpha_1^2 + \varepsilon \kappa^2 (\alpha_2^2 - \alpha_3^2)]}, \quad (11)$$

$$H_{\Gamma} = \frac{1}{2\Delta\sqrt{\alpha_1^2 + \varepsilon\kappa^2(\alpha_2^2 - \alpha_3^2)}} \left\{ \alpha_1\delta_1 + \varepsilon\kappa^2(\alpha_2\delta_2 - \alpha_3\delta_3) - 2\left[r_1(1 - \varepsilon v r_2 \kappa^2) + \varepsilon v^2 \kappa^2 \left[r_2 \left(r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau \right) - r_3 \left(r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right) \right) \right] \right] \left[-\varepsilon \alpha_1 r_2 \kappa^2 + \varepsilon \kappa^2 \left[\alpha_2 \left(r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau \right) - \alpha_3 \left(r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right) \right) \right] \right] \right\}, \quad (12)$$

where

$$\Delta = (1 - \varepsilon v r_2 \kappa^2)^2 + \varepsilon v^2 \kappa^2 \left[\left(r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau \right)^2 - \left(r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right) \right)^2 \right] - \left\{ r_1(1 - \varepsilon v r_2 \kappa^2) + \varepsilon v^2 \kappa^2 \left[r_2 \left(r_1 + r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau \right) - r_3 \left(r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right) \right) \right] \right\}^2.$$

As a result of the preceding findings, we can deduce the the next theorems.

Theorem 1. *At the point $(\sigma, 0)$, the ruled surface (5) via orthogonal modified frame fulfills the following requirements:*

1. *The surface is developable if and only if $\tau = \frac{r_1 r_3}{r_2^2 - r_3^2}$.*
2. *The surface is minimal if and only if $\tau = \frac{r_3(1 - 2r_1^2)}{2r_1(r_2^2 - r_3^2)}$.*

Case 1. *At $r_1 = 0$, the ruled surface (5) has to have the specific traits:*

$$K_{\Gamma} = -\frac{\varepsilon\kappa^2 \{ r_3 [r_2 \left(\frac{\dot{\kappa}}{\kappa} \right) + r_3 \tau] - r_2 [r_2 \tau + r_3 \left(\frac{\dot{\kappa}}{\kappa} \right)] \}}{r_2^2 - r_3^2},$$

$$H_{\Gamma} = -\frac{\varepsilon r_3 \kappa}{2\sqrt{\varepsilon(r_2^2 - r_3^2)}}. \quad (13)$$

Corollary 1. *At the point $(\sigma, 0)$, the ruled surface (5) via orthogonal modified frame with $r_1 = 0$ is developable if $\tau = 0$.*

Case 2. *For $\varepsilon = 1$ at $r_2 = 0$, the ruled surface (5) has to have the specific traits:*

$$K_{\Gamma} = -\frac{\kappa^2(r_1 + r_3 \tau)}{r_3(r_1^2 - 1)},$$

$$H_{\Gamma} = -\frac{\kappa[1 - 2r_1(r_1 + r_3 \tau)]}{2\sqrt{r_1^2 - 1}}. \quad (14)$$

Corollary 2. *For $\varepsilon = 1$ and at the point $(\sigma, 0)$, the ruled surface (5) via orthogonal modified frame with $r_2 = 0$ is as follows:*

1. *Developable if only and only if $\tau = -\left(\frac{r_1}{r_3}\right)$.*
2. *Minimal if only and only if $\tau = \frac{1 - 2r_1^2}{2r_1 r_3}$.*

Case 3. *For $\varepsilon = -1$ at $r_3 = 0$, the ruled surface (5) has to have the specific traits:*

$$K_{\Gamma} = \frac{r_2^2 \tau \kappa^2}{1 - r_1^2}, H_{\Gamma} = \frac{r_1 r_2 \tau \kappa}{1 - r_1^2}. \quad (15)$$

Corollary 3. For $\varepsilon = -1$ and at the point $(\sigma, 0)$, the ruled surface (5) via orthogonal modified frame with $r_2 = 0$ is an (H_{Γ}, K_{Γ}) -linear Weingarten surface if only and only if $\beta(\sigma)$ is a W-curve.

Case 4. For $\varepsilon = 1$ at $r_1 = r_2 = 0$, the ruled surface (5) has to have the specific traits:

$$K_{\Gamma} = -\tau \kappa^2, H_{\Gamma} = -r_3 \kappa. \quad (16)$$

Corollary 4. For $\varepsilon = 1$ and at the point $(\sigma, 0)$, the ruled surface (5) via orthogonal modified frame with $r_1 = r_2 = 0$ is as follows:

1. Developable if only and only if $\tau = 0$.
2. An (H_{Γ}, K_{Γ}) -linear Weingarten surface if only and only if $\beta(\sigma)$ is a W-curve.

Case 5. For $\varepsilon = -1$ at $r_1 = r_3 = 0$, the ruled surface (5) has to have the specific traits:

$$K_{\Gamma} = \tau \kappa^2, H_{\Gamma} = 0. \quad (17)$$

Corollary 5. For $\varepsilon = -1$ and during the point $(\sigma, 0)$, the ruled surface (5) via orthogonal modified frame with $r_1 = r_3 = 0$ is as follows:

1. A minimal surface.
2. Developable if only and only if $\tau = 0$.
3. An (H_{Γ}, K_{Γ}) -linear Weingarten surface if only and only if $\beta(\sigma)$ is a W-curve.

Example 1. For $\varepsilon = -1$, let $\beta(\sigma) = \left(\frac{\sqrt{3}\sigma}{2}, \sinh\left(\frac{\sigma}{2}\right), \cosh\left(\frac{\sigma}{2}\right) \right)$ be a spacelike curve in E_1^3 . It is simple to demonstrate the Frenet invariants as follows:

$$\begin{aligned} t(\sigma) &= \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \cosh\left(\frac{\sigma}{2}\right), \frac{1}{2} \sinh\left(\frac{\sigma}{2}\right) \right), \\ n(\sigma) &= \left(0, \sinh\left(\frac{\sigma}{2}\right), \cosh\left(\frac{\sigma}{2}\right) \right), \\ b(\sigma) &= \left(\frac{1}{2}, -\frac{\sqrt{3}}{2} \cosh\left(\frac{\sigma}{2}\right), -\frac{\sqrt{3}}{2} \sinh\left(\frac{\sigma}{2}\right) \right), \\ \kappa &= \frac{1}{4}, \tau = -\frac{\sqrt{3}}{4}. \end{aligned}$$

Then, the orthogonal modified frames are

$$\begin{aligned} T(\sigma) &= \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \cosh\left(\frac{\sigma}{2}\right), \frac{1}{2} \sinh\left(\frac{\sigma}{2}\right) \right), \\ N(\sigma) &= \left(0, \frac{1}{4} \sinh\left(\frac{\sigma}{2}\right), \frac{1}{4} \cosh\left(\frac{\sigma}{2}\right) \right), \\ B(\sigma) &= \left(\frac{1}{8}, -\frac{\sqrt{3}}{8} \cosh\left(\frac{\sigma}{2}\right), -\frac{\sqrt{3}}{8} \sinh\left(\frac{\sigma}{2}\right) \right). \end{aligned}$$

Taking $r_1 = \frac{3}{2\sqrt{2}}$, $r_2 = \sqrt{3}$, and $r_3 = 1$, the ruled surfaces via Frenet frame and orthogonal modified frame are then computed as follows (see Figures 1 and 2):

$$\begin{aligned}\Gamma_1(\sigma, v) &= \left(3\sigma + (2\sqrt{3} + \sqrt{2})v, (v+1) \sinh\left(\frac{\sigma}{2}\right) + \left(\frac{(2-\sqrt{6})v}{2\sqrt{3}}\right) \cosh\left(\frac{\sigma}{2}\right) \right. \\ &\quad \left. , \left(\frac{(2-\sqrt{6})v}{2\sqrt{3}}\right) \sinh\left(\frac{\sigma}{2}\right) + (v+1) \cosh\left(\frac{\sigma}{2}\right) \right), \\ \Gamma_2(\sigma, v) &= \left(\frac{1}{8\sqrt{2}}[4\sqrt{6}\sigma + (\sqrt{2} + 6\sqrt{3})v], \left(\frac{(4+\sqrt{3}v)}{4}\right) \sinh\left(\frac{\sigma}{2}\right) + \left(\frac{(6-\sqrt{6})v}{8\sqrt{2}}\right) \cosh\left(\frac{\sigma}{2}\right) \right. \\ &\quad \left. , \left(\frac{(6-\sqrt{6})v}{8\sqrt{2}}\right) \sinh\left(\frac{\sigma}{2}\right) + \left(\frac{(4+\sqrt{3}v)}{4}\right) \cosh\left(\frac{\sigma}{2}\right) \right)\end{aligned}$$

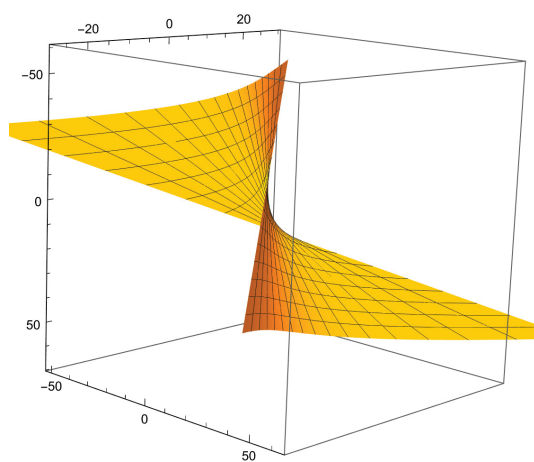


Figure 1. $\Gamma_1(\sigma, v) = \beta(\sigma) + v\ell(\sigma)$.

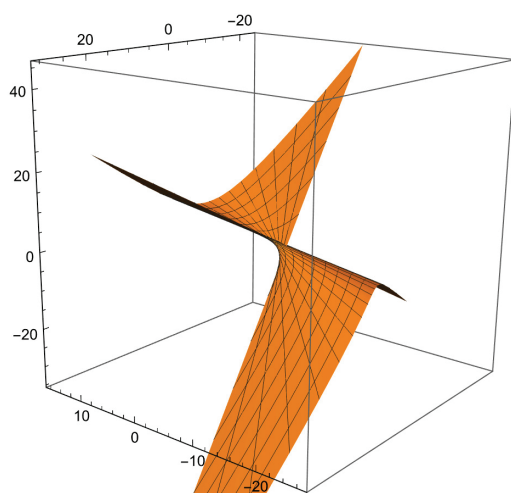


Figure 2. $\Gamma_2(\sigma, v) = \beta(\sigma) + v(r_1 T(\sigma) + r_2 N(\sigma) + r_3 B(\sigma))$.

4. Evolutionary Characteristics of Spectacular Ruled Surfaces via Orthogonal Modified Frame

Within this section, we investigate the T -, N -, and B -ruled surfaces along an analytic space curve using orthogonal modified frames. If the $\beta(\sigma)$ and $\ell(\sigma)$ curves do, in fact, transit in time t , the ruled surface (4) looks like this:

$$\Gamma(\sigma, v, t) = \beta(\sigma, t) + v\ell(\sigma, t). \quad (18)$$

4.1. T-Ruled Surface via Orthogonal Modified Frame

For $\varepsilon = -1$, the tangent-ruled surface is generated by the motion of the tangent vector $T(\sigma)$ of $\beta(\sigma)$, and its equation is represented by

$$\Psi(\sigma, v, t) = \beta(\sigma, t) + v T(\sigma, t). \quad (19)$$

The Ψ 's partial differentiations are

$$\begin{aligned} \Psi_\sigma &= T(\sigma) + v N(\sigma), \\ \Psi_v &= T(\sigma). \end{aligned} \quad (20)$$

Using (20), in reality, we may obtain the I_Ψ' quantities and the Ψ unit normal vector field as follows:

$$\begin{aligned} E_\Psi &= 1 - v^2 \kappa^2, \\ F_\Psi &= 1, \\ G_\Psi &= 1, \end{aligned} \quad (21)$$

$$N_\Psi = \frac{B(\sigma)}{\kappa}. \quad (22)$$

Differentiating (20) in relation to σ and v , respectively, and using (22), the II_Ψ' quantities of Ψ are

$$\begin{aligned} e_\Psi &= v \tau \kappa, \\ f_\Psi &= 0, \\ g_\Psi &= 0. \end{aligned} \quad (23)$$

Then, K_Ψ and H_Ψ of a T -ruled surface Ψ are

$$K_\Psi = 0, H_\Psi = -\frac{\tau}{2v\kappa}. \quad (24)$$

Corollary 6. For $\varepsilon = -1$, the T -ruled surface (19) via an orthogonal modified frame is as follows:

1. A developable surface.
2. Minimal if only and only if $\tau = 0$.
3. An (H_Ψ, K_Ψ) -linear Weingarten surface if only and only if $\beta(\sigma)$ is a helix.

4.2. N-Ruled Surface via Orthogonal Modified Frame

The formula for the normal-ruled surface produced by the movement of the normal vector $N(\sigma)$ of $\beta(\sigma)$ is shown by

$$\Omega(\sigma, v, t) = \beta(\sigma, t) + v N(\sigma, t). \quad (25)$$

The Ω 's velocity vectors are

$$\begin{aligned} \Omega_\sigma &= [1 - \varepsilon v \kappa^2] T(\sigma) + v \left(\frac{\dot{\kappa}}{\kappa} \right) N(\sigma) + v \tau B(\sigma), \\ \Omega_v &= N(\sigma). \end{aligned} \quad (26)$$

Using (26), we could really obtain the Ω 's quantities of I_Ω and N_Ω as follows:

$$\begin{aligned} E_\Omega &= (1 - \varepsilon v \kappa^2)^2 + \varepsilon v^2 (\dot{\kappa}^2 - \tau^2 \kappa^2), \\ F_\Omega &= \varepsilon v \kappa \dot{\kappa}, \\ G_\Omega &= \varepsilon \kappa^2, \end{aligned} \quad (27)$$

$$N_\Omega = \frac{-v \tau T(\sigma) + (\varepsilon v \kappa^2 - 1) B(\sigma)}{\sqrt{\tau^2 \kappa^2 - \varepsilon \kappa^2 (\varepsilon v \kappa^2 - 1)^2}}. \quad (28)$$

Differentiating (26) in relation to σ and v , respectively, and using (28), the Ω 's quantities of II_Ω are

$$\begin{aligned} e_\Omega &= \frac{\varepsilon v \kappa [3v \tau \dot{\kappa} - (\varepsilon v \kappa^2 - 1)(\dot{\tau} \kappa + 2\tau \dot{\kappa})]}{\sqrt{\tau^2 \kappa^2 - \varepsilon \kappa^2 (\varepsilon v \kappa^2 - 1)^2}}, \\ f_\Omega &= \frac{\tau \kappa^2 [\varepsilon(v + 1) - v \kappa^2]}{\sqrt{\tau^2 \kappa^2 - \varepsilon \kappa^2 (\varepsilon v \kappa^2 - 1)^2}}, \\ g_\Omega &= 0. \end{aligned} \quad (29)$$

Then, K_Ω and H_Ω of an N -ruled surface Ω are

$$\begin{aligned} K_\Omega &= \frac{\tau^2 \kappa^2 [\varepsilon(v + 1) - v \kappa^2]^2}{[\varepsilon(1 - \varepsilon v \kappa^2)^2 - v^2 \tau^2 \kappa^2] [\tau^2 \kappa^2 - \varepsilon \kappa^2 (\varepsilon v \kappa^2 - 1)^2]}, \\ H_\Omega &= \frac{v \kappa [3v \tau \dot{\kappa} - (\varepsilon v \kappa^2 - 1)(\dot{\tau} \kappa + 2\tau \dot{\kappa}) - 2\varepsilon \tau \dot{\kappa} [\varepsilon(v + 1) - v \kappa^2]]}{2[\varepsilon(1 - \varepsilon v \kappa^2)^2 - v^2 \tau^2 \kappa^2] \sqrt{\tau^2 \kappa^2 - \varepsilon \kappa^2 (\varepsilon v \kappa^2 - 1)^2}}. \end{aligned} \quad (30)$$

Corollary 7. The N -ruled surface (25) via an orthogonal modified frame is as follows:

1. A developable surface iff $\tau = 0$.
2. Minimal if only and only if

$$3v \tau \dot{\kappa} - (\varepsilon v \kappa^2 - 1)(\dot{\tau} \kappa + 2\tau \dot{\kappa}) - 2\varepsilon \tau \dot{\kappa} [\varepsilon(v + 1) - v \kappa^2] = 0.$$

4.3. B-Ruled Surface via Orthogonal Modified Frame

The binormal-ruled surface is generated by the motion of the binormal vector $B(\sigma)$ of $\beta(\sigma)$, and its equation is represented by

$$\Phi(\sigma, v, t) = \beta(\sigma, t) + v B(\sigma, t). \quad (31)$$

The Φ 's velocity vectors are

$$\begin{aligned} \Phi_\sigma &= T(\sigma) + (v \tau) N(\sigma) + v \left(\frac{\dot{\kappa}}{\kappa} \right) B(\sigma), \\ \Phi_v &= B(\sigma). \end{aligned} \quad (32)$$

Using (32), we could really obtain the Φ 's quantities of I_Φ and N_Φ as follows:

$$\begin{aligned} E_\Phi &= 1 + \varepsilon v^2 [\tau^2 \kappa^2 - \dot{\kappa}^2], \\ F_\Phi &= -\varepsilon v \kappa \dot{\kappa}, \\ G_\Phi &= -\varepsilon \kappa^2, \end{aligned} \quad (33)$$

$$N_{\Phi} = \frac{v\tau T(\sigma) - N(\sigma)}{\sqrt{\varepsilon\kappa^2 + v^2\tau^2}}. \quad (34)$$

Differentiating (32) in relation to σ and v , respectively, and using (34), the Φ 's quantities of II_{Φ} are

$$\begin{aligned} e_{\Phi} &= -\frac{\varepsilon\kappa[v^2\tau^2\kappa + 2v\tau\dot{\kappa} + \kappa(1 + v\dot{\tau})]}{\sqrt{\varepsilon\kappa^2 + v^2\tau^2}}, \\ f_{\Phi} &= -\frac{\varepsilon\tau\kappa^2}{\sqrt{\varepsilon\kappa^2 + v^2\tau^2}}, \\ g_{\Phi} &= 0. \end{aligned} \quad (35)$$

Then, K_{Φ} and H_{Φ} of a B -ruled surface Φ are

$$\begin{aligned} K_{\Phi} &= \frac{\tau^2\kappa^2}{[\varepsilon + v^2\tau^2\kappa^2][\varepsilon\kappa^2 + v^2\tau^2]}, \\ H_{\Phi} &= -\frac{\kappa^2[v\dot{\tau} + v^2\tau^2 + 1]}{2[\varepsilon + v^2\tau^2\kappa^2]\sqrt{\varepsilon\kappa^2 + v^2\tau^2}}. \end{aligned} \quad (36)$$

Corollary 8. The B -ruled surface (31) via orthogonal modified frame is as follows:

1. A developable surface iff $\tau = 0$.
2. Minimal if only and only if $v\dot{\tau} + v^2\tau^2 + 1 = 0$.

Example 2. For $\varepsilon = 1$, let $\beta(\sigma) = \left(\sqrt{2}\sigma, \int_0^{\sigma} \sinh\left(\frac{\pi t^2}{2}\right) dt, \int_0^{\sigma} \cosh\left(\frac{\pi t^2}{2}\right) dt\right)$ be a space-like curve in E_1^3 . It is simple to demonstrate the Frenet invariants as follows:

$$\begin{aligned} t(\sigma) &= \left(\sqrt{2}, \sinh\left(\frac{\pi\sigma^2}{2}\right), \cosh\left(\frac{\pi\sigma^2}{2}\right)\right), \\ n(\sigma) &= \left(0, \left(\frac{\sigma}{|\sigma|}\right) \cosh\left(\frac{\pi\sigma^2}{2}\right), \left(\frac{\sigma}{|\sigma|}\right) \sinh\left(\frac{\pi\sigma^2}{2}\right)\right), \\ b(\sigma) &= \left(-\frac{\sigma}{|\sigma|}, -\left(\frac{\sqrt{2}\sigma}{|\sigma|}\right) \sinh\left(\frac{\pi\sigma^2}{2}\right), -\left(\frac{\sqrt{2}\sigma}{|\sigma|}\right) \cosh\left(\frac{\pi\sigma^2}{2}\right)\right), \\ \kappa &= \pi|\sigma|, \tau = \sqrt{2}\pi\sigma. \end{aligned}$$

Then, the orthogonal modified frames are

$$\begin{aligned} T(\sigma) &= \left(\sqrt{2}, \sinh\left(\frac{\pi\sigma^2}{2}\right), \cosh\left(\frac{\pi\sigma^2}{2}\right)\right), \\ N(\sigma) &= \left(0, \pi\sigma \cosh\left(\frac{\pi\sigma^2}{2}\right), \pi\sigma \sinh\left(\frac{\pi\sigma^2}{2}\right)\right), \\ B(\sigma) &= \left(-\pi\sigma, -\sqrt{2}\pi\sigma \sinh\left(\frac{\pi\sigma^2}{2}\right), -\sqrt{2}\pi\sigma \cosh\left(\frac{\pi\sigma^2}{2}\right)\right). \end{aligned}$$

The T -ruled surface via an orthogonal modified frame is computed as follows (see Figure 3):

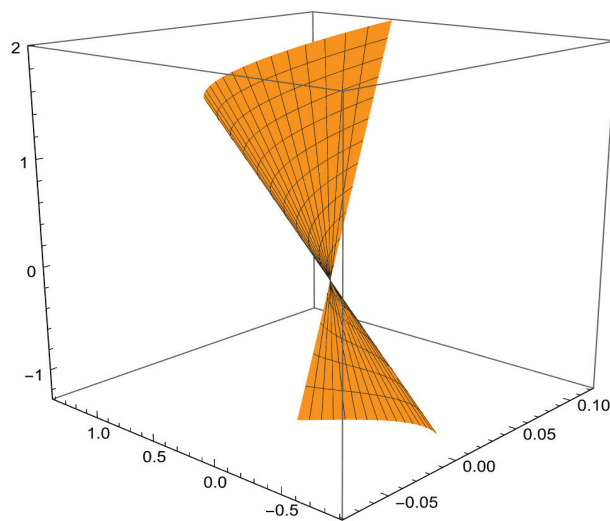


Figure 3. T-ruled surface.

$$\Psi(\sigma, v, t) = \left(\sqrt{2}(\sigma + v), \int_0^\sigma \sinh\left(\frac{\pi t^2}{2}\right) dt + v \sinh\left(\frac{\pi \sigma^2}{2}\right), \int_0^\sigma \cosh\left(\frac{\pi t^2}{2}\right) dt + v \cosh\left(\frac{\pi \sigma^2}{2}\right) \right).$$

The N-ruled surfaces via Frenet frame and orthogonal modified frame are then computed as (see Figures 4 and 5).

$$\begin{aligned} \Omega_1(\sigma, v, t) &= \left(\sqrt{2}\sigma, \int_0^\sigma \sinh\left(\frac{\pi t^2}{2}\right) dt + \left(\frac{v\sigma}{|\sigma|}\right) \cosh\left(\frac{\pi \sigma^2}{2}\right), \int_0^\sigma \cosh\left(\frac{\pi t^2}{2}\right) dt + \left(\frac{v\sigma}{|\sigma|}\right) \sinh\left(\frac{\pi \sigma^2}{2}\right) \right), \\ \Omega_2(\sigma, v, t) &= \left(\sqrt{2}\sigma, \int_0^\sigma \sinh\left(\frac{\pi t^2}{2}\right) dt + \pi v \sigma \cosh\left(\frac{\pi \sigma^2}{2}\right), \int_0^\sigma \cosh\left(\frac{\pi t^2}{2}\right) dt + \pi v \sigma \sinh\left(\frac{\pi \sigma^2}{2}\right) \right). \end{aligned}$$

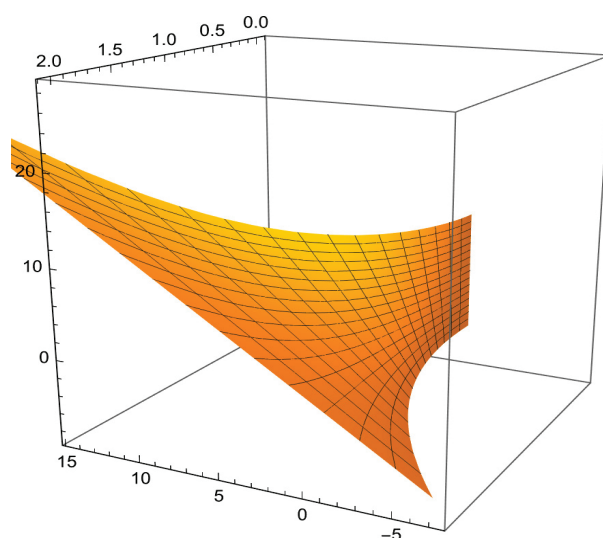


Figure 4. N-ruled surface via Frenet frame.

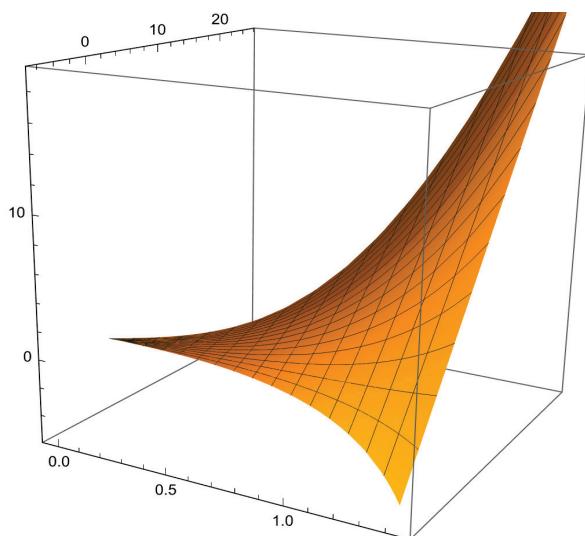


Figure 5. *N*-ruled surface via orthogonal modified frame.

The *B*-ruled surfaces via Frenet frame and orthogonal modified frame are then computed as follows (see Figures 6 and 7):

$$\begin{aligned}\Phi_1(\sigma, v, t) &= \left(\sqrt{2}\sigma - \frac{v\sigma}{|\sigma|}, \int_0^\sigma \sinh\left(\frac{\pi t^2}{2}\right) dt - \left(\frac{\sqrt{2}v\sigma}{|\sigma|} \right) \sinh\left(\frac{\pi\sigma^2}{2}\right), \int_0^\sigma \cosh\left(\frac{\pi t^2}{2}\right) dt \right. \\ &\quad \left. - \left(\frac{\sqrt{2}v\sigma}{|\sigma|} \right) \cosh\left(\frac{\pi\sigma^2}{2}\right) \right), \\ \Phi_2(\sigma, v, t) &= \left(\sqrt{2}\sigma - v\sigma, \int_0^\sigma \sinh\left(\frac{\pi t^2}{2}\right) dt - \sqrt{2}v\sigma \sinh\left(\frac{\pi\sigma^2}{2}\right), \int_0^\sigma \cosh\left(\frac{\pi t^2}{2}\right) dt \right. \\ &\quad \left. - \sqrt{2}v\sigma \cosh\left(\frac{\pi\sigma^2}{2}\right) \right).\end{aligned}$$

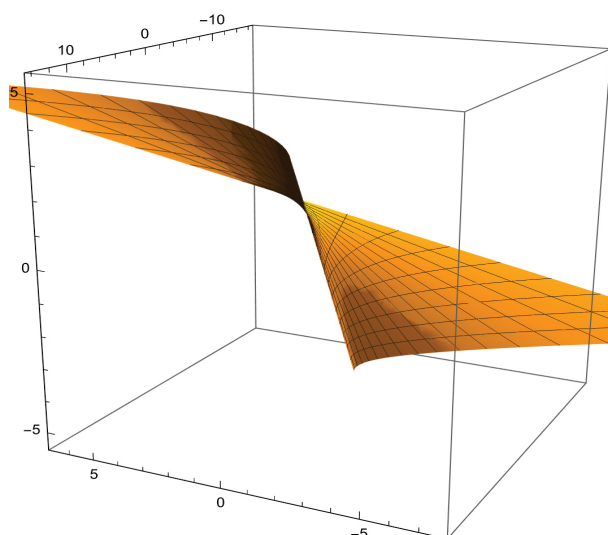


Figure 6. *B*-ruled surface via Frenet frame.

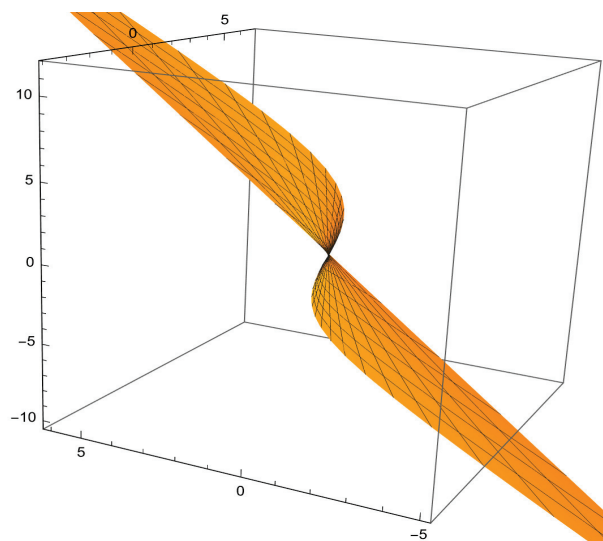


Figure 7. B-ruled surface via orthogonal modified frame.

5. Conclusions

In this work, we used an orthogonal modified frame to investigate the geometric features of ruled surfaces in Minkowski 3-space E_1^3 . The research emphasized the influence of spacelike rulings on the behavior of these surfaces by concentrating on important features like minimality, developability, and curvature attributes. According to our research, ruled surfaces may be studied more methodically and effectively by using an orthogonal modified frame, which also offers fresh perspectives on the curvature requirements and categorization of these surfaces. In order to better comprehend minimum ruled surfaces in Lorentzian geometry, the requirements for these surfaces were investigated, and the connections between Gaussian and mean curvature were developed. We illustrate the benefits of using modified frames to analyze the geometric features of ruled surfaces by contrasting our findings with those of previous research. These results may have use in differential geometry, mathematical physics, and relativity, where Minkowski space is essential. By examining higher-dimensional governed surfaces or their dynamic behaviors under other geometric flows, further study might be built on this work. This work lays the groundwork for future investigations into ruled surfaces in the Minkowski framework, improving our theoretical knowledge of them and their possible uses in a range of scientific domains.

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Article

Ricci Semi-Symmetric Robertson–Walker Spacetime in $f(\mathcal{R})$ -Gravity

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Abstract: We investigated the properties of Ricci semi-symmetric Robertson–Walker spacetimes within the framework of $f(\mathcal{R})$ -gravity theory. Initially, we established that Ricci semi-symmetric Robertson–Walker spacetimes are locally isometric to either Minkowski or de Sitter spacetimes. We then focused on the 4-dimensional formulation of these spacetimes in $f(\mathcal{R})$ -gravity, deriving expressions for the isotropic pressure p and energy density σ . To further develop our understanding, we explored various energy conditions to constrain the functional form of $f(\mathcal{R})$. We analyzed several models, namely $f(\mathcal{R}) = \mathcal{R} - \alpha(1 - e^{-\frac{\mathcal{R}}{\alpha}})$, $f(\mathcal{R}) = \mathcal{R} - \beta \tanh \mathcal{R}$, and $f(\mathcal{R}) = \mathcal{R} - \log(m\mathcal{R})$, where α , β , and m are constants. Our findings suggest that the equations of state parameters for these models are compatible with the universe’s accelerating expansion, indicating an equation of state parameter $\omega = -1$. Moreover, while these models satisfy the null, weak, and dominant energy conditions reflective of the observed accelerated expansion, our analysis reveals that they violate the strong energy condition.

Keywords: Ricci semi-symmetric spacetime; Robertson–Walker spacetime; $f(\mathcal{R})$ -gravity theory; energy conditions in modified gravity

MSC: 53C21; 53C50; 53Z05

1. Introduction

The geometry of generalized Robertson–Walker (GRW) spacetimes has emerged as a vital and dynamic area of research, bridging the fields of mathematics and theoretical physics. A GRW spacetime is characterized as a warped product manifold, where its foundational structure comprises a one-dimensional base manifold. In cosmological physics, the Friedmann–Lemaître–Robertson–Walker (FLRW) metrics serve as elementary models for describing the expansion of the universe, acknowledging the pioneering contributions of several scientists in the evolution of modern cosmological theories. Recent advances in the study of Lorentzian manifolds have unveiled numerous intriguing decomposition theorems that deepen our understanding of spacetime geometry. Notably, in [1], a groundbreaking decomposition of Lorentzian manifolds was introduced, demonstrating a novel connection between their geometric properties and the foundational principles of General Relativity.

This work significantly enhances our understanding of how Lorentzian manifolds relate to the more specialized FLRW spacetimes, revealing the potential for these structures to inform our understanding of cosmic evolution. A Lorentzian manifold can be modeled as a GRW spacetime if it satisfies a specific criterion: the presence of a time-like concircular vector field. However, this condition is relaxed when an additional condition [2] is taken into account, thus permitting a wider range of spacetime geometries to be classified as GRW. Specifically, a Lorentzian manifold M attains the GRW spacetime if it possesses a unit time-like torse-forming vector field $u^{\sharp b}$, which serves as an eigenvector of the Ricci tensor Ric_{ab} . In the presence of a unit time-like torse-forming vector field, it suggests the existence of a scalar function φ on M such that

$$\nabla_a u^{\sharp}_b = \varphi(g_{ab} + u^{\sharp}_a u^{\sharp}_b), \quad u^{\sharp a} u^{\sharp}_a = -1. \quad (1)$$

In [3], it has been shown that the Ricci tensor of a GRW spacetime is characterized by

$$\text{Ric}_{ab} = \frac{\mathcal{R} - \psi}{n-1} g_{ab} + \frac{\mathcal{R} - n\psi}{n-1} u^{\sharp}_a u^{\sharp}_b + (n-2) u^{\sharp k} u^{\sharp l} \mathcal{C}_{kabl},$$

where $\mathcal{C}_{kabl}$ is the Weyl conformal curvature tensor and $\mathcal{R}$ is the scalar curvature. A conformally flat GRW spacetime is known as classical Robertson–Walker (RW) spacetime, allowing the Ricci curvature form to change as follows:

$$\text{Ric}_{ab} = \frac{\mathcal{R} - \psi}{n-1} g_{ab} + \frac{\mathcal{R} - n\psi}{n-1} u^{\sharp}_a u^{\sharp}_b. \quad (2)$$

At this point, we recall that the Ricci tensor for homogeneous and isotropic spacetime has the form

$$\text{Ric}_{ab} = \alpha g_{ab} + \beta u^{\sharp}_a u^{\sharp}_b. \quad (3)$$

By virtue of (2) and (3), one can conclude that an RW-spacetime is a perfect fluid spacetime, where

$$\alpha = \frac{\mathcal{R} - \psi}{n-1}, \quad \beta = \frac{\mathcal{R} - n\psi}{n-1}.$$

For more information regarding perfect fluid spacetimes and the characterization of GRW- and RW-spacetimes, readers are encouraged to see the references [2,4–6]. An algebraic curvature condition for a spacetime to be classified as a perfect fluid spacetime is described in [7]. In differential geometry, manifolds that meet this algebraic curvature criterion are known as quasi-Einstein manifolds [8].

The quest to understand the dynamics of our universe, particularly during its late-time inflationary phase, has led some researchers to modify the action principle of Einstein’s field equations (EFEs), recognizing that the standard theory of gravity may not offer a complete explanation. A significant contribution to this field is found in Buchdahl’s proposal [9], which introduces a modified gravity theory by replacing the Ricci scalar $\mathcal{R}$ in the Einstein–Hilbert action with an arbitrary function $f(\mathcal{R})$. This innovative modification yields a new action principle that challenges conventional views of astrophysical phenomena. While the choice of functions $f(\mathcal{R})$ is indeed constrained by a range of observational data and scalar–tensor theoretical results, researchers can adopt a phenomenological approach that assumes specific forms for $f(\mathcal{R})$, allowing for empirical testing against existing criteria. A diverse array of models and functional forms has emerged in the literature, with

notable contributions from studies such as [10–15]. These explorations have demonstrated that certain astrophysical structures, including massive neutron stars, resist explanations grounded in GR but can be effectively described through the higher-order curvilinear features of $f(\mathcal{R})$ -gravity. Supporting evidence for these claims can be found in references [16–20], which highlight the notable advantages of investigating modified gravity theories. The equations of motion derived from $f(\mathcal{R})$ -gravity reflect increased complexity, offering solutions that can diverge significantly from those predicted by traditional GR.

Energy conditions (ECs) play a crucial role in establishing the positivity of the energy–momentum tensor, serving as fundamental criteria for various theories of gravity. However, the introduction of $f(\mathcal{R})$ -gravity models has posed significant challenges in effectively constraining their model parameters. By applying different energy conditions, researchers can derive potential constraints on the parameters governing the $f(\mathcal{R})$ framework [21]. Originally articulated within the confines of GR [22], these conditions have since been adapted for $f(\mathcal{R})$ -gravity by defining effective pressure and energy density, facilitating a more nuanced understanding of these modified theories. As evident from the preceding literature, it is imperative to dedicate more attention to $f(\mathcal{R})$ -gravity, which remains largely unexplored. Motivated by these gaps, the present study focuses on the Ricci semi-symmetric Robertson–Walker (RW) spacetime in the context of $f(\mathcal{R})$ -gravity. Our objective is to shed light on the unanswered questions that arise within this realm, specifically through an examination of the null, weak, strong, and dominant energy conditions applicable to various $f(\mathcal{R})$ -gravity models.

2. RW-Spacetime

The scalar curvature of RW spacetime, the eigenvalue of the Ricci tensor corresponding to $u^\sharp$, and the divergence of the one-form $u^\sharp$ can be easily deduced as follows:

$$\mathcal{R} = n\alpha - \beta, \quad \psi = \alpha - \beta, \quad (4)$$

$$\nabla^b u_b^\sharp = (n - 1)\varphi, \text{ respectively.} \quad (5)$$

An RW-spacetime is a conformally flat GRW-spacetime which holds significant interest in the standard theory of gravity [23] governed by Einstein’s field equations (EFEs):

$$\text{Ric}_{ab} - \frac{\mathcal{R}}{2}g_{ab} = k^2 T_{ab},$$

where T_{ab} denotes the energy–momentum tensor and k is a gravitational constant. Thus,

$$\left(\alpha - \frac{\mathcal{R}}{2}\right)g_{ab} + \beta u_a^\sharp u_b^\sharp = k^2 T_{ab}. \quad (6)$$

The energy momentum tensor for homogeneous and isotropic spacetime is defined by the symmetric tensor:

$$T_{ab} = p g_{ab} + (p + \sigma) u_a^\sharp u_b^\sharp, \quad (7)$$

where p and σ indicate isotropic pressure and energy density. Banerjee et al. [24] employed isotropic pressure to provide a detailed analysis of the characteristics of wormholes and energy conditions in the context of $f(\mathcal{R}, \mathcal{T})$ gravity. The authors of reference [25] explored bouncing cosmology with higher-order curvature in modified gravity by using isotropic pressure. Gadbail et al. [26] studied the corrections to the Lagrangian for bouncing cosmology in $f(\mathcal{Q})$ gravity, using the energy–momentum tensor for homogeneous and

isotropic spacetime. Extensive studies of isotropic pressure have been conducted in various cosmological and astrophysical phenomena [27–35]. Upon comparing (6) and (7), one can easily find that

$$\begin{aligned} k^2 p &= \alpha - \frac{\mathcal{R}}{2} = \frac{\mathcal{R} - \psi}{n-1} - \frac{\mathcal{R}}{2} = \frac{(3-n)\mathcal{R} - 2\psi}{2(n-1)}, \\ k^2(p + \sigma) &= \beta = \frac{\mathcal{R} - n\psi}{n-1}, \\ k^2 \sigma &= \frac{\mathcal{R} - n\psi}{n-1} - \frac{(3-n)\mathcal{R} - 2\psi}{2(n-1)} = \frac{\mathcal{R} - 2\psi}{2}. \end{aligned}$$

3. Characterization of Ricci Semi-Symmetric RW-Spacetime

In this section, we investigate some geometric behaviors of RW-spacetime within the background of curvature condition on the Ricci tensor. A spacetime is said to be Ricci semi-symmetric [23] if it satisfies

$$\nabla_l \nabla_k \text{Ric}_{ab} - \nabla_k \nabla_l \text{Ric}_{ab} = 0.$$

Taking the covariant derivative of (2) twice gives

$$\begin{aligned} \nabla_l \nabla_k \text{Ric}_{ab} - \nabla_k \nabla_l \text{Ric}_{ab} &= \nabla_l \nabla_k (\alpha g_{ab} + \beta u_a^\# u_b^\#) - \nabla_k \nabla_l (\alpha g_{ab} + \beta u_a^\# u_b^\#), \\ &= \beta (u_a^\# (\nabla_l \nabla_k - \nabla_k \nabla_l) u_b^\# + u_b^\# (\nabla_l \nabla_k - \nabla_k \nabla_l) u_a^\#). \end{aligned}$$

We can easily conclude from the previous equation that an RW spacetime is Ricci semi-symmetric if and only if either $\beta = 0$ or $\nabla_l \nabla_k u_b^\# = \nabla_k \nabla_l u_b^\#$. Remarkably, if $\beta = 0$ in (3), then this implies that an RW spacetime is Einstein, that is, $\text{Ric}_{ab} = \frac{\mathcal{R}}{n} g_{ab}$. Conversely, if the spacetime is Einstein, then

$$\frac{\mathcal{R}}{n} = (\alpha - n\alpha + \mathcal{R}).$$

The aforementioned equation yields $\alpha = \frac{\mathcal{R}}{n}$, which, when considered in (4), gives $\beta = 0$. The second condition is equivalent to $u_h^\# \text{Ric}_{abk}^h = 0$. Hence, we conclude the following.

Theorem 1. *An RW-spacetime M is Ricci semi-symmetric if and only if M is Einstein or $u_h^\# \text{Ric}_{abk}^h = 0$.*

The Ricci semi-symmetric RW-spacetime is Einstein, provided that $\beta = 0$. In this case, the Ricci tensor and energy–momentum tensor are, respectively, transformed into

$$\text{Ric}_{ab} = \frac{\mathcal{R}}{n} g_{ab}, \quad (8)$$

$$T_{ab} = -\frac{1}{k^2} \left(\frac{n-2}{2n} \right) \mathcal{R} g_{ab}. \quad (9)$$

In this situation, one can find

$$p = -\frac{1}{k^2} \left(\frac{n-2}{2n} \right) \mathcal{R}, \quad (10)$$

$$\sigma = \frac{1}{k^2} \left(\frac{n-2}{2n} \right) \mathcal{R}. \quad (11)$$

By virtue of the previous equation, one can obtain $p + \sigma = 0$, which indicates that a perfect fluid is referred to as dark energy matter.

Remark 1. Let M be a Ricci semi-symmetric RW-spacetime. Then, perfect fluid spacetime is referred to as the dark matter era.

Since the energy density cannot be negative, it follows from (11) that $\mathcal{R} \geq 0$, that is, $\mathcal{R} = 0$ or $\mathcal{R} > 0$:

- When $\mathcal{R} = 0$, then (8), which results in $\text{Ric}_{ab} = 0$, signifies that the spacetime is flat, and thus locally isometric to Minkowski spacetime [36].
- When $\mathcal{R} > 0$, this condition leads to a spacetime characterized by a constant positive curvature, specifically classifying it as a de Sitter spacetime [36].

Therefore, we state the following.

Theorem 2. A Ricci semi-symmetric RW-spacetime is either locally isometric to Minkowski spacetime or a de Sitter spacetime.

4. Ricci Semi-Symmetric RW-Spacetime Obeying $f(\mathcal{R})$ -Gravity

In this study, we delve into the implications of $f(\mathcal{R})$ -gravity within the context of Ricci semi-symmetric RW-spacetime, which is characterized by a specific geometric structure that allows for cosmological models consistent with modern observations. This examination is prompted by the recognition that $f(\mathcal{R})$ theories can extend our understanding of gravitational phenomena beyond that provided by GR.

To initiate our analysis, we consider a modified form of the Einstein–Hilbert action represented by the following integral:

$$H = \frac{1}{\kappa^2} \int \{f(\mathcal{R}) + \mathcal{L}_m\} \sqrt{-g} d^4x, \quad (12)$$

where $\mathcal{L}_m$ corresponds to the Lagrangian matter density of scalar field and κ^2 is a coupling constant that relates the gravitational phenomenon to the energy–momentum tensor. The stress–energy tensor of the matter is calculated using the following expression:

$$T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{ab}}. \quad (13)$$

By varying the modified action H with respect to the metric, we arrive at the field equations for $f(\mathcal{R})$ -gravity as

$$f_{\mathcal{R}}(\mathcal{R}) \text{Ric}_{ab} - \frac{1}{2} f(\mathcal{R}) g_{ab} + (g_{ab} \square - \nabla_a \nabla_b) f_{\mathcal{R}}(\mathcal{R}) = \kappa^2 T_{ab}, \quad (14)$$

where $\square$ denotes the d’Alembert operator. An interesting facet of this framework is the ability to simplify the field equations by substituting $f(\mathcal{R})$ with $\mathcal{R}$. We know that $\mathcal{R}$ is constant on Ricci semi-symmetric RW-spacetime, so the equation can be restructured to express the relationship more clearly, as follows:

$$\text{Ric}_{ab} - \frac{\mathcal{R}}{2} g_{ab} = \frac{\kappa^2}{f_{\mathcal{R}}(\mathcal{R})} T_{ab}^{eff}, \quad (15)$$

with $T_{ab}^{eff} = T_{ab} + \frac{f(\mathcal{R}) - \mathcal{R}f_{\mathcal{R}}(\mathcal{R})}{2\kappa^2} g_{ab}$. Due to Theorem 1, we obtain $\text{Ric}_{ab} = \frac{\mathcal{R}}{4} g_{ab}$. For a constant $\mathcal{R}$, the above field equation becomes

$$-\frac{\mathcal{R}}{2} g_{ab} = \left[\frac{\kappa^2}{f_{\mathcal{R}}(\mathcal{R})} p + \frac{f(\mathcal{R}) - \mathcal{R}f_{\mathcal{R}}(\mathcal{R})}{2f_{\mathcal{R}}(\mathcal{R})} \right] g_{ab} + \frac{\kappa^2}{f_{\mathcal{R}}(\mathcal{R})} (p + \sigma) u_a^\sharp u_b^\sharp. \quad (16)$$

Combining a and b in the previous equation, one can obtain

$$\mathcal{R} = \frac{3\kappa^2 p}{f_{\mathcal{R}}(\mathcal{R})} - \frac{\kappa^2 \sigma}{f_{\mathcal{R}}(\mathcal{R})} + \frac{2f(\mathcal{R})}{f_{\mathcal{R}}(\mathcal{R})}. \quad (17)$$

Once again, transvecting (16) by $(u^\sharp)^b$ and after some computation, we obtain

$$\mathcal{R} = \frac{2f(\mathcal{R})}{f_{\mathcal{R}}(\mathcal{R})} - \frac{4\kappa^2 \sigma}{f_{\mathcal{R}}(\mathcal{R})}. \quad (18)$$

Combining (17) together with (18) gives

$$p = \frac{\mathcal{R}f_{\mathcal{R}}(\mathcal{R})}{4\kappa^2} - \frac{f(\mathcal{R})}{2\kappa^2}, \quad \text{and} \quad \sigma = \frac{f(\mathcal{R})}{2\kappa^2} - \frac{\mathcal{R}f_{\mathcal{R}}(\mathcal{R})}{4\kappa^2}. \quad (19)$$

Hence, we state the following.

Theorem 3. In Ricci semi-symmetric RW-spacetime, isotropic pressure p and energy density σ are given by the expression (19).

5. Energy Conditions

In the context of GR, ECs are widely regarded as crucial tools for exploring the properties of compact objects in various modified-gravity theories, as highlighted in references [37–39]. ECs serve as invaluable tools for assessing the self-stability of cosmological models, typically derived from the widely recognized Raychaudhuri equation [40,41]. Additionally, it helps us to elucidate the geometrical characteristics of spacetime curvature, such as space-like, time-like, and light-like behaviors, while also shedding light on the nature of dreadful singularities [42,43]. The expressions of the Raychaudhuri equations are

$$\frac{d\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - \text{Ric}_{ab}u^a u^b, \quad (20)$$

$$\frac{d\Theta}{d\tau} = -\frac{1}{2}\Theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - \text{Ric}_{ab}n^a n^b. \quad (21)$$

Here, Θ refers to the expansion rate, n^a is the null vector, and σ^{ab} and ω^{ab} are the shear and rotation associated with the vector field u^a , respectively. For gravitational attraction, the conditions given by (20) and (21) are satisfied as follows:

$$\text{Ric}_{ab}u^a u^b \geq 0, \quad \text{Ric}_{ab}n^a n^b \geq 0.$$

We use (19) to deduce some relevant energy conditions for the present scenario:

- Null energy condition (NEC): if $\sigma + p \geq 0$.
- Strong energy condition (SEC): if $\sigma + 3p \geq 0$ with $\mathcal{R}f_{\mathcal{R}}(\mathcal{R}) - f_{\mathcal{R}}(\mathcal{R}) \geq 0$.

- Weak energy condition (WEC): if $\sigma \geq 0$, $\sigma + p \geq 0$ with $-\mathcal{R}f_{\mathcal{R}}(\mathcal{R}) + f_{\mathcal{R}}(\mathcal{R}) \geq 0$.
- Dominant energy condition (DEC): if $\sigma \geq 0$, $\|p\| \leq \sigma$.

We will now explore the ECs for particular functions of the Ricci scalar $\mathcal{R}$ within this framework in the subsequent section.

6. Features of $f(\mathcal{R})$ -Gravity Models

In this section, we explore a specific class of $f(\mathcal{R})$ -modified-gravity models by detailing the functional form of f . Generally, the field equations are influenced by the tensor T_{ab} , which incorporates the physical properties of the matter field. Consequently, in $f(\mathcal{R})$ -gravity, each choice of the function f can lead to a variety of theoretical models, depending on the specific nature of the matter source considered. Some of the early $f(\mathcal{R})$ models, including the Starobinsky [44] and Carroll [11] models, accounted for cosmic inflation without the need for scalar fields and explained late-time acceleration purely as a gravitational phenomenon, without invoking dark energy. Although these models have their limitations, they played a significant role in popularizing $f(\mathcal{R})$ models overall. They also highlighted the complexity involved in identifying a viable and stable $f(\mathcal{R})$ model. For an in-depth review of $f(\mathcal{R})$ -gravity theory, refer to [45]. Sotiriou et al. [37] described the $f(\mathcal{R})$ theory as a form of toy theory and provided a detailed explanation of this toy model in their research. Additionally, numerous researchers have examined various toy models in their studies [46–48]. We will analyze three different toy models to investigate energy conditions in the following subsections.

6.1. Model-1: $f(\mathcal{R}) = \mathcal{R} - \alpha(1 - e^{-\frac{\mathcal{R}}{\alpha}})$

In this context, we consider $f(\mathcal{R})$ to be an exponential function characterized by the free parameter α . Consequently, the expressions for energy density and pressure can be formulated as

$$\sigma = \frac{1}{4\kappa^2} \left(-2\alpha + e^{-\frac{\mathcal{R}}{\alpha}} (2\alpha + \mathcal{R}) + \mathcal{R} \right), \quad (22)$$

$$p = \frac{1}{4\kappa^2} \left(2\alpha - e^{-\frac{\mathcal{R}}{\alpha}} (2\alpha + \mathcal{R}) - \mathcal{R} \right). \quad (23)$$

Utilizing (22) and (23), one can examine the energy conditions within this framework. Figure 1 illustrates the profiles of various energy conditions within the context of cosmological evolution. The analysis reveals that the energy density remains strictly positive throughout the universe's evolution for parameters, where $\mathcal{R} > 1$ and $\alpha > 0$. This figure further demonstrates that elevated values of $\mathcal{R}$ correspond to higher energy density levels, implying that the universe possessed substantial energy density during its early stages of development. It is noteworthy that the NEC is a subset of the WEC, thereby confirming the satisfaction of both conditions. The profiles for the DEC are also depicted in Figure 1, indicating positivity throughout. However, a violation of the SEC is observed, which suggests that the universe is experiencing a phase of late-time acceleration [49,50]. Throughout this analysis, the EoS parameter $\omega = -1$. The EoS parameter is viewed as a viable metric for assessing our models in relation to the Λ CDM cosmological model. Remarkably, this nature is consistent with the Λ CDM theoretical framework for dark energy and is further validated by recent experimental findings [51]. In [13], the author elaborated on a significant observation regarding this result.

Moreover, the present analysis demonstrates that the condition $\omega = -1$ characterizes late-time cosmic acceleration. Utilizing this parameter in the context of the SEC, it is observed that $\rho + 3\omega\rho$ remains negative. In order to explain cosmic acceleration in a homo-

geneous and isotropic universe, it is necessary for the late universe to display substantial negative pressure. Consequently, these findings confirm that our model indicates that the universe is undergoing an accelerated expansion phase [49,52–54].

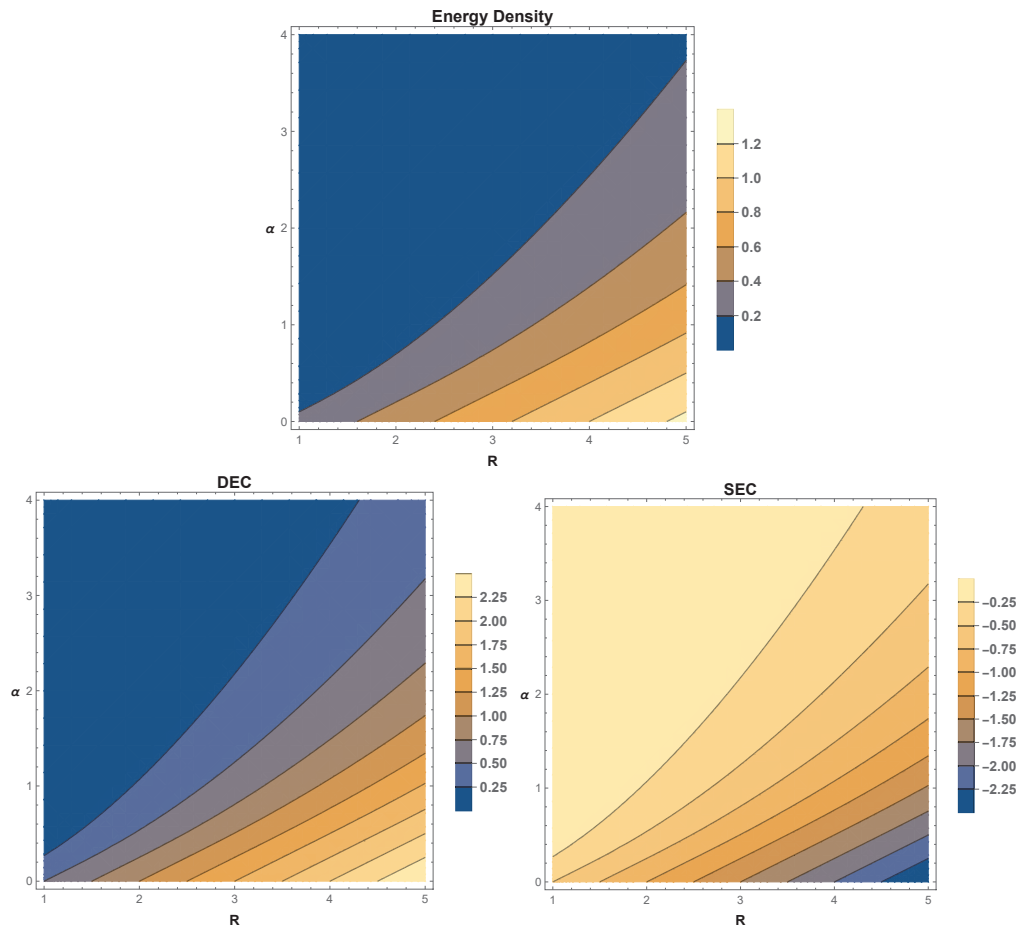


Figure 1. Contour plots showing the variations in energy density, DEC, and SEC as functions of R and α . In the energy density and DEC plots, the color gradient ranges from dark blue (lower values) to light yellow (higher values), indicating that both energy density and DEC increase with increasing R and α . In the SEC plot, the color gradient ranges from light yellow (higher values) to dark blue (lower values), showing that SEC decreases (becomes more negative) with increasing R and α .

6.2. Model-2: $f(\mathcal{R}) = \mathcal{R} - \beta \tanh \mathcal{R}$

In this part, we consider a model $f(\mathcal{R}) = \mathcal{R} - \beta \tanh \mathcal{R}$. Tsujikawa [55], Appleby, and Battye [56] described this type of $f(\mathcal{R})$ -gravity model. We can express the pressure and energy density in terms of this model as

$$p = \frac{1}{2\kappa^2} \beta \tanh(\mathcal{R}) - \frac{1}{4\kappa^2} \mathcal{R} (\beta \operatorname{sech}^2(\mathcal{R}) + 1), \quad (24)$$

$$\sigma = \frac{1}{4\kappa^2} (-2\beta \tanh(\mathcal{R}) + \beta \mathcal{R} \operatorname{sech}^2(\mathcal{R}) + \mathcal{R}). \quad (25)$$

By analyzing the behavior of the pressure and energy density as expressed in (24) and (25), we can deduce the EoS parameter ω in this scenario. Upon performing the necessary calculations, we find that $\omega = -1$ for any β . This result implies that the universe is primarily driven by a cosmological constant, which is consistent with the widely accepted Λ CDM theory [51]. In Figure 2, we present the profiles of energy density and energy conditions to visually illustrate their characteristics. The graphs demonstrate that while

both the energy density and DEC criteria are satisfied, the SEC is not met. Additionally, as $\sigma + p$ approaches zero in this analysis, it becomes clear that both the WEC and NEC are fulfilled.

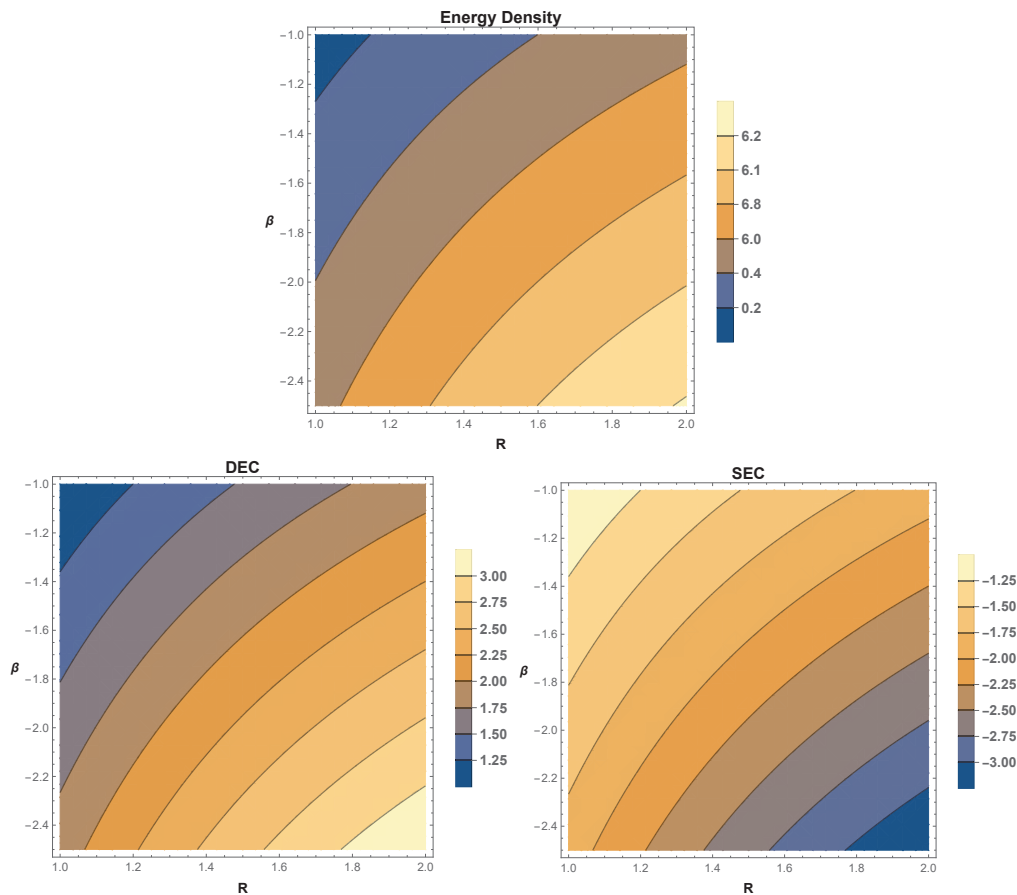


Figure 2. Contour plots showing the variations in energy density, DEC, and SEC as functions of R and β . In the energy density and DEC plots, the color gradient ranges from dark blue (lower values) to light yellow (higher values), indicating that both energy density and DEC increase with increasing R and β . In the SEC plot, the color gradient ranges from light yellow (higher values) to dark blue (lower values), showing that SEC decreases (becomes more negative) with increasing R and β .

6.3. Model-3: $f(\mathcal{R}) = \mathcal{R} - \log(m\mathcal{R})$

We consider $f(\mathcal{R})$ as a logarithmic function of the Ricci scalar with the free parameter $m > 0$. This model is characterized by the following expressions for energy density and pressure:

$$\sigma = \frac{1}{4}(\mathcal{R} + 1 - 2\log(m\mathcal{R})), \quad (26)$$

$$p = \frac{1}{4}(2\log(m\mathcal{R}) - \mathcal{R} - 1) \quad (27)$$

In this scenario, the EoS is equal to 1 for all values of m , suggesting that the universe is primarily governed by a cosmological constant, aligning with the widely accepted Λ CDM model. Since WEC combines NEC and positive density, and NEC is always zero in this context, we can observe behaviors related to SEC, DEC, and the density parameter. Figure 3 presents the profiles of energy density, as well as the DEC and SEC. An analysis of these graphs indicates that while both the energy density and DEC criteria are satisfied,

the SEC criterion is not. Furthermore, as $p + \sigma$ approaches zero in this construction, it is evident that both the WEC and NEC are also satisfied.

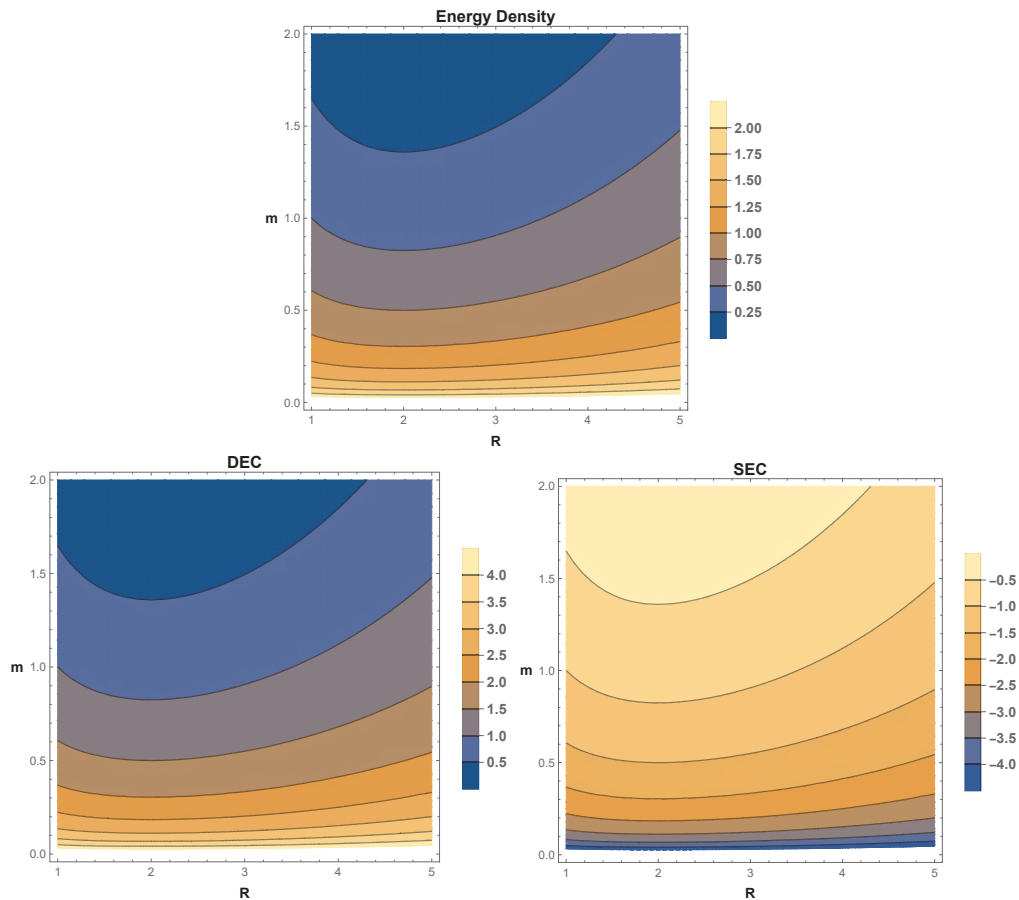


Figure 3. Contour plots showing the variations in energy density, DEC, and SEC as functions of R and m . In the energy density and DEC plots, the color gradient ranges from dark blue (lower values) to light yellow (higher values), indicating that both energy density and DEC increase with increasing R and m . In the SEC plot, the color gradient ranges from light yellow (higher values) to dark blue (lower values), showing that SEC decreases (becomes more negative) with increasing R and m .

7. Discussion

This work has provided a comprehensive exploration of Ricci semi-symmetric Robertson–Walker spacetime within the framework of $f(\mathcal{R})$ -gravity. By employing both analytical techniques and graphical analyses, we have effectively demonstrated the stability of several cosmological toy models, which were framed by carefully chosen functional forms of $f(\mathcal{R})$. Specifically, we investigated models defined by $f(\mathcal{R}) = \mathcal{R} - \alpha(1 - e^{-\frac{\mathcal{R}}{\alpha}})$, $f(\mathcal{R}) = \mathcal{R} - \beta \tanh \mathcal{R}$ and $f(\mathcal{R}) = \mathcal{R} - \log(m\mathcal{R})$, all of which allowed for a rich understanding of the underlying cosmological dynamics. In [13,57], the author employed these models to investigate the energy conditions, demonstrating its relevance and effectiveness in their analysis. Motivated by their findings, we adopted a similar model for our research, seeking to advance the understanding of this area.

Our study not only examined the expression and implications of energy conditions within these models but also highlighted their stability properties. In particular, for model-1, we found that the energy density remains positive across a range of parameters ($\mathcal{R} \geq 1$ and $\alpha > 0$) in Figure 1 and that the dominant, null, and weak energy conditions are satisfied, while the SEC is violated. This behavior aligns well with the observed accelerated expansion of the universe, supporting the notion that such models can represent a dark energy era,

characterized by an equation of state parameter ω that approaches -1 , consistent with predictions from the Λ CDM model. Further, recent data from the Planck collaboration and the Λ CDM model suggest that the equation of the state parameter is roughly $\omega \simeq -1$ [51]. This behavior indicates that the universe is experiencing a negative pressure regime, which plays a key role in its ongoing accelerated expansion. Therefore, the ω parameter serves as an effective basis for comparing our models with the Λ CDM framework. Moreover, we have confirmed that similar results hold for model-2 and model-3, further validating our findings. These insights reinforce the potential of $f(R)$ -gravity theories to provide viable alternatives to standard cosmological models. Comparable results have been documented in the literature [13,57–59], reinforcing the conclusions of the present study.

To relate the proposed $f(R)$ mathematical models to cosmological observational data, we will first determine the scale factor $a(t)$ and subsequently calculate the Hubble parameter $H(t) = a'/a$ based on the modified Friedmann equations. Next, we will utilize the relation $H = -z'/z + 1$ to express cosmic time t as a function of the redshift parameter z . This will enable us to derive the Hubble parameter $H(z)$ in terms of redshift, forming our model equation. We plan to fit this $H(z)$ using observational data from cosmic chronometers (CCs), baryon acoustic oscillations (BAOs), and Type Ia supernovae (SNeIa) in the near future. Through this method, we aim to test the viability of the proposed $f(R)$ model in explaining the universe's expansion history and ensuring its consistency with various cosmological observations.

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Article

Characterization of Finsler Space with Rander's-Type Exponential-Form Metric

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Abstract: This study explores a unique Finsler space with a Rander's-type exponential metric, $\mathcal{G}(\alpha, \beta) = (\alpha + \beta)e^{\frac{\beta}{\alpha + \beta}}$, where α is a Riemannian metric and β is a 1-form. We analyze the conditions under which its hypersurfaces behave like hyperplanes of the first, second, and third kinds. Additionally, we examine the reducibility of the Cartan tensor $\mathcal{C}$ for these hypersurfaces, providing insights into their geometric structure.

Keywords: Finslerian hypersurface; exponential (α, β) -metric; Cartan connection; hyperplane of first, second, and third kind

MSC: 53B40; 53C60

1. Introduction

The study of Finslerian hypersurfaces and their classification represents a significant advancement in the field of Finsler geometry, a branch of differential geometry that generalizes Riemannian geometry by allowing the metric to depend not only on position but also on direction. The notion of Finslerian hypersurfaces was first introduced by the eminent mathematician Matsumoto, who provided a systematic classification of these hypersurfaces into three distinct types: hyperplanes of the first kind, second kind, and third kind. This classification was based on the geometric and algebraic properties of the hypersurfaces and their relationship to the underlying Finsler metric. Matsumoto's work laid the groundwork for subsequent research, inspiring numerous mathematicians to explore the properties of these hypersurfaces under various modifications and generalizations of the Finsler metric. These investigations, as documented in references [1–12], have uncovered a rich array of geometric properties, deepening our understanding of the intrinsic structure of Finslerian hypersurfaces and their applications in geometry and physics.

In addition to his contributions to the theory of Finslerian hypersurfaces, Matsumoto also introduced the concept of an (α, β) -metric [13], which has since become a central topic of research in Finsler geometry. An (α, β) -metric is a generalization of the classical Riemannian metric, where α represents the Riemannian part of the metric and β is a 1-form that introduces a directional dependence. The exponential metric, expressed as $L = \alpha e^{\frac{\beta}{\alpha}}$, is a unique form of the (α, β) -metric that has attracted significant attention due to its elegant

structure and its connections to theoretical physics and cosmology. The exponential metric has been extensively examined by various authors [14–20], who have explored its geometric properties under different conditions. One notable feature of the exponential metric is its relationship to Rander’s metric. Specifically, under certain transformations, the exponential metric can be reduced to Rander’s metric, which has significant applications in theoretical physics, particularly in the study of spacetime geometries and cosmological models. This connection highlights the importance of (α, β) -metrics not only in pure mathematics but also in applied fields.

A hypersurface is a generalization of the concept of hyperplane. It is defined as follows.

Definition 1. A sub-manifold of dimension $(n - 1)$ is termed a hypersurface of an enveloping manifold $\mathcal{M}$ of dimension n , and the co-dimension of the hypersurface is one.

If $m > n$, then the space V^n is termed a subspace of V^m , and V^m serves as an enveloping space for V^n . Particularly, if $m = n + 1$, then V^n is referred to as a hypersurface of the enveloping space V^{n+1} .

Hence, a hypersurface $\mathcal{M}^{n-1}$ of the manifold $\mathcal{M}^n$ can be parametrically described by the equation

$$x^i = x^i(u^\alpha), \{\alpha = 1, 2, \dots, (n - 1)\},$$

where u represent the Gaussian coordinates on the hypersurface $\mathcal{M}^{n-1}$.

If y^i represents the supporting line element at a point (u^α) on the hypersurface $\mathcal{M}^{n-1}$, tangential to $\mathcal{M}^{n-1}$, then we have

$$y^i = \mathcal{B}_\alpha^i(u) v^\alpha$$

Thus, v^α is regarded as the supporting element of $\mathcal{M}^{n-1}$ at a point (u^α) . Considering the function $\underline{\mathcal{G}}(u, v) = \mathcal{G}(x(u), y(u, v))$, which generates a Finsler metric on $\mathcal{M}^{n-1}$, we obtain an $(n - 1)$ -dimensional Finsler space $\mathcal{U}^{(n-1)} = (\mathcal{M}^{n-1}, \underline{\mathcal{G}}(u, v))$.

In Finsler geometry, a hyperplane can be classified into different types based on its geometric properties. Below are some examples of such hypersurfaces:

Example 1. A sphere can be visualized as a three-dimensional structure and is an example of a two-dimensional manifold ($n = 2$) embedded in three-dimensional space. Thus, a hypersurface on this sphere would be a one-dimensional curve.

Example 2. A torus can be visualized as a three-dimensional structure and is an example of a two-dimensional manifold ($n = 2$) embedded in three-dimensional space, and a spiral curve would represent the hypersurface.

In Figure 1, the first image shows the sphere as the three-dimensional space, with the curve (red lines) representing a hypersurface within it. The second image features a torus as the three-dimensional space, where the spiral curve (red line) represents a hypersurface within that structure.

Example 3 (Hyperplane of first kind). A hyperplane in a three-dimensional Finsler space is classified as a “hyperplane of the first kind” if it intersects with a given curve or surface within that space.

Example 4 (Hyperplane of second kind). A “hyperplane of the second kind” in Finsler geometry is a hyperplane (represented by a surface) that does not intersect with a given curve or surface in the three-dimensional Finsler space.

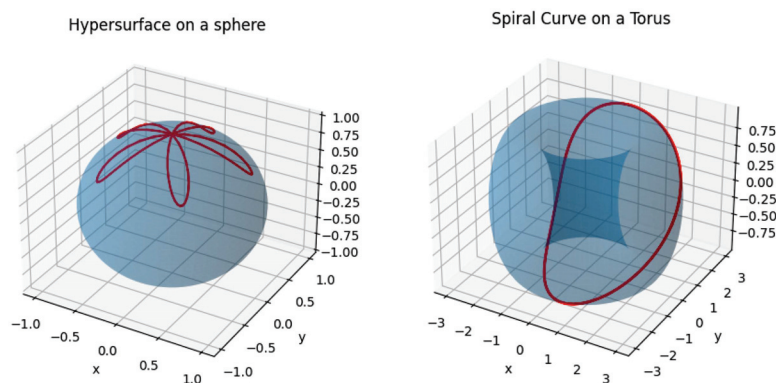


Figure 1. Examples of hypersurface in sphere and torus.

In Figure 2, the first image shows a hyperplane intersecting with a curve, and this intersection is classified as a “hyperplane of the first kind”. In the second image, the surface is shifted upward by adding 5 to the z-values, preventing any intersection with the curve. This is referred to as the “hyperplane of the second kind”.

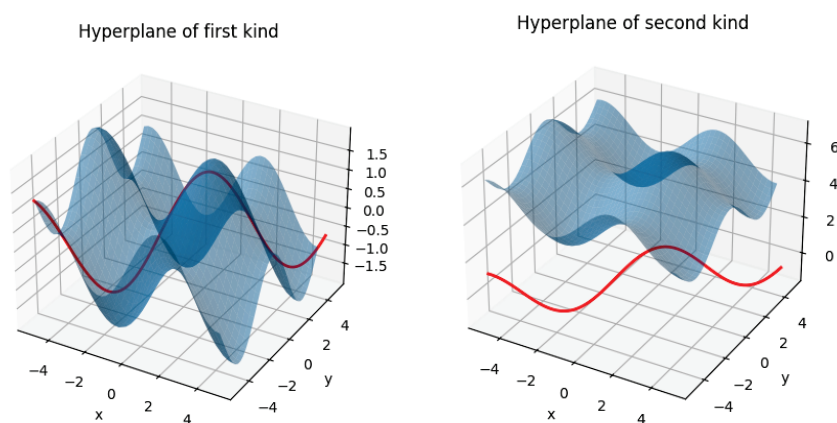


Figure 2. Examples of first- and second-kind hyperplane in 3D.

Example 5 (Hyperplane of third kind). We consider a flat plane embedded in 3D space. This plane has zero curvature, and its normal vector remains constant. This serves as a simplified analogy for a hyperplane of the third kind in Finsler geometry, where the conditions of vanishing curvature and tensor create a similar “flatness” within the Finsler space (Figure 3).

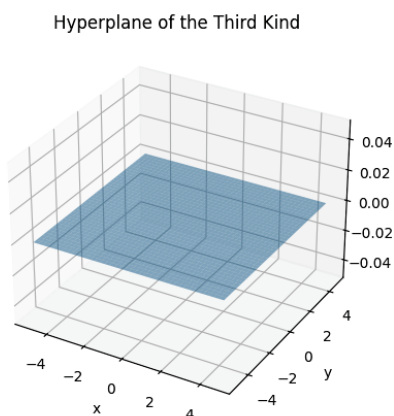


Figure 3. Example of hyperplane of third kind.

Thus “hyperplane of the third kind” is essentially “flat” with respect to the ambient Finsler space.

In this paper, we introduce and analyze a novel (α, β) -metric defined as $\mathcal{G}(\alpha, \beta) = (\alpha + \beta)e^{\frac{\beta}{\alpha + \beta}}$. We refer to this metric as the Rander’s-type exponential (α, β) -metric due to its structural similarity to Rander’s metric combined with an exponential factor. This metric represents a natural extension of the classical Rander’s metric and the exponential metric, combining their features in a way that opens up new avenues for geometric exploration. Our primary focus is on investigating the intrinsic properties of this Finsler space, particularly the conditions under which its hypersurfaces exhibit characteristics of hyperplanes of the first, second, and third kind. In Theorem 3, we derive the necessary and sufficient conditions for the hypersurfaces of the Rander’s-type exponential (α, β) -metric to be classified as hyperplanes of the first kind. These conditions are expressed in terms of the geometric invariants of the Finsler space and provide a clear characterization of the hypersurfaces in this category. Similarly, in Theorem 4, we establish the conditions under which the hypersurfaces exhibit properties of hyperplanes of the second kind. These conditions involve a deeper analysis of the interplay between the Riemannian part α and the 1-form β in the metric. Finally, in Theorem 5, we address the case of hyperplanes of the third kind, identifying the specific geometric constraints that must be satisfied for the hypersurfaces to fall into this classification.

In addition to the classification of hypersurfaces, we also investigate the reducibility of the Cartan tensor $\mathcal{C}$ for these hypersurfaces. The Cartan tensor is a fundamental object in Finsler geometry, encoding information about the anisotropy of the Finsler metric. Its reducibility, or the extent to which it can be decomposed into simpler components, provides valuable insights into the geometric structure of the Finsler space. In Propositions 1–3, we examine the reducibility of the Cartan tensor in various forms, focusing on the hypersurfaces associated with the Rander’s-type exponential (α, β) -metric. These propositions reveal the conditions under which the Cartan tensor can be reduced to simpler forms, shedding light on the geometric behavior of the hypersurfaces and their relationship to the underlying metric.

Through this detailed analysis, we aim to contribute to the broader understanding of Finslerian hypersurfaces and their geometric properties, particularly in the context of the newly introduced Rander’s-type exponential (α, β) -metric. Our results not only extend the existing theory of Finslerian hypersurfaces but also provide new tools for exploring the geometric and physical implications of (α, β) -metrics. By uncovering the intrinsic properties of this metric and its hypersurfaces, we hope to inspire further research into the rich and diverse world of Finsler geometry.

2. Preliminaries

In this study, we investigate an n -dimensional Finsler space represented as $\mathcal{U}^n = \{\mathcal{M}^n, \mathcal{G}(\alpha, \beta)\}$. Here, $\mathcal{U}^n$ consists of an n -dimensional differentiable manifold $\mathcal{M}^n$ coupled with a fundamental function $\mathcal{G}$ that takes on a Rander’s-type exponential form within a unique Finsler space metric, expressed as

$$\mathcal{G}(\alpha, \beta) = (\alpha + \beta)e^{\frac{\beta}{\alpha + \beta}} \quad (1)$$

Taking partial derivatives of Equation (1) with respect to α and β yields

$$\begin{aligned} \mathcal{G}_\alpha &= \frac{\alpha}{(\alpha + \beta)}e^{\frac{\beta}{\alpha + \beta}}, & \mathcal{G}_\beta &= \frac{(2\alpha + \beta)}{(\alpha + \beta)}e^{\frac{\beta}{\alpha + \beta}} \\ \mathcal{G}_{\alpha\alpha} &= \frac{\beta^2}{(\alpha + \beta)^3}e^{\frac{\beta}{\alpha + \beta}} & \mathcal{G}_{\beta\beta} &= \frac{\alpha^2}{(\alpha + \beta)^3}e^{\frac{\beta}{\alpha + \beta}} \end{aligned}$$

$$\mathcal{G}_{\alpha\beta} = -\frac{\alpha\beta}{(\alpha+\beta)^3} e^{\frac{\beta}{(\alpha+\beta)}}$$

$$\text{where } \mathcal{G}_\alpha = \frac{\partial \mathcal{G}}{\partial \alpha}, \quad \mathcal{G}_\beta = \frac{\partial \mathcal{G}}{\partial \beta}, \quad \mathcal{G}_{\alpha\alpha} = \frac{\partial \mathcal{G}_\alpha}{\partial \alpha}, \quad \mathcal{G}_{\beta\beta} = \frac{\partial \mathcal{G}_\beta}{\partial \beta}, \quad \mathcal{G}_{\alpha\beta} = \frac{\partial \mathcal{G}_\alpha}{\partial \beta}.$$

In the Finsler space $\mathcal{U}^n = \{\mathcal{M}^n, \mathcal{G}(\alpha, \beta)\}$, the normalized support element $l_i = \partial_i \mathcal{G}$ and the angular metric tensor h_{ij} are defined as, per reference [21],

$$l_i = \alpha^{-1} \mathcal{G}_\alpha Y_i + \mathcal{G}_\beta b_i$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1} (b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j$$

where $Y_i = a_{ij} y^j$. For the fundamental metric Function (1) above, the constants are

$$\begin{aligned} p &= e^{\frac{2\beta}{(\alpha+\beta)}}, \\ q_0 &= \frac{\alpha^2}{(\alpha+\beta)^2} e^{\frac{2\beta}{(\alpha+\beta)}}, \\ q_{-1} &= -\frac{\beta}{(\alpha+\beta)^2} e^{\frac{2\beta}{(\alpha+\beta)}}, \\ q_{-2} &= -\frac{(\alpha+2\beta)}{\alpha(\alpha+\beta)^2} e^{\frac{2\beta}{(\alpha+\beta)}} \end{aligned} \quad (2)$$

The fundamental metric tensor $g_{ij} = \frac{1}{2} \partial_i \partial_j \mathcal{G}^2$ and its corresponding reciprocal tensor g^{ij} for $\mathcal{G} = \mathcal{G}(\alpha, \beta)$ can be found in reference [21].

$$g_{ij} = p a_{ij} + p_0 b_i b_j + p_{-1} (b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j \quad (3)$$

where

$$\begin{aligned} p_0 &= \frac{\{\alpha^2 + (2\alpha + \beta)^2\}}{(\alpha + \beta)^2} e^{\frac{2\beta}{(\alpha+\beta)}}, \\ p_{-1} &= \frac{2\alpha}{(\alpha + \beta)^2} e^{\frac{2\beta}{(\alpha+\beta)}}, \\ p_{-2} &= -\frac{2\beta}{\alpha(\alpha + \beta)^2} e^{\frac{2\beta}{(\alpha+\beta)}} \end{aligned} \quad (4)$$

The reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_{-1} (b^i Y^j + b^j Y^i) - s_{-2} Y^i Y^j \quad (5)$$

where $b^i = a^{ij} b_j$ and $b^2 = a_{ij} b^i b^j$

$$\begin{aligned} s_0 &= \frac{1}{\tau p} \{p p_0 + (p_0 p_{-2} - p_{-1}^2) \alpha^2\}, \\ s_{-1} &= \frac{1}{\tau p} \{p p_{-1} + (p_0 p_{-2} - p_{-1}^2) \beta\}, \\ s_{-2} &= \frac{1}{\tau p} \{p p_{-2} + (p_0 p_{-2} - p_{-1}^2) b^2\}, \\ \tau &= p(p + p_0 b^2 + p_{-1} \beta) + (p_0 p_{-2} - p_{-1}^2) (\alpha^2 b^2 - \beta^2) \end{aligned} \quad (6)$$

The hv-torsion tensor $\mathcal{C}_{ijk} = \frac{1}{2} \partial_k g_{ij}$ is provided in reference [10].

$$2p \mathcal{C}_{ijk} = p_{-1} (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k \quad (7)$$

where

$$\gamma_1 = p \frac{\partial p_0}{\partial \beta} - 3p_{-1}q_0, \quad m_i = b_i - \alpha^{-2}\beta y_i \quad (8)$$

Here, m_i represents a non-zero covariant vector that is orthogonal to the support element y^i .

Given the components $\{\gamma_{jk}^i\}$ representing the Christoffel symbols of the associated Riemannian space R^n , and using ∇_k to represent the covariant derivative with respect to x^k determined by these Christoffel symbols, we now introduce the following definition:

$$2E_{ij} = b_{ij} + b_{ji}, \quad 2U_{ij} = b_{ij} - b_{ji} \quad (9)$$

where $b_{ij} = \nabla_j b_i$.

The Cartan connection of $\mathcal{U}^n$, represented as $CF = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, \Gamma_{jk}^i)$, defines the special Finsler space. The difference tensor $D_{jk}^i = \Gamma_{jk}^{*i} - \{\gamma_{jk}^i\}$ is given by

$$\begin{aligned} D_{jk}^i = & \mathcal{B}^i E_{jk} + \mathcal{U}_k^i B_j + \mathcal{U}_j^i B_k + \mathcal{B}_j^i b_{0k} + \mathcal{B}_k^i b_{0j} - b_{0m} g^{im} B_{jk} \\ & - C_{jm}^i A_k^m - C_{km}^i A_j^m + C_{jkm} A_s^m g^{is} + \lambda^s (C_{jm}^i C_{sk}^m + \\ & C_{km}^i C_{sj}^m - C_{jk}^m C_{ms}^i) \end{aligned} \quad (10)$$

where

$$\begin{aligned} B_k &= p_0 b_k + p_{-1} y_k, \quad \mathcal{B}^i = g^{ij} B_j, \quad \mathcal{U}_i^k = g^{kj} \mathcal{U}_{ji} \\ B_{ij} &= \frac{\{p_{-1}(a_{ij} - \alpha^{-2} y_i y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}}{2}, \quad \mathcal{B}_i^k = g^{kj} B_{ji} \\ A_k^m &= \mathcal{B}_k^m E_{00} + \mathcal{B}^m E_{k0} + B_k \mathcal{U}_0^m + B_0 \mathcal{U}_k^m \\ \lambda^m &= \mathcal{B}^m E_{00} + 2B_0 \mathcal{U}_0^m, \quad B_0 = B_i y^i \end{aligned} \quad (11)$$

In this context, the symbol '0' represents contraction with y^i , excluding the elements p_0 , q_0 , and s_0 .

3. Cartan Connection for the Hypersurface of a Finsler Space

If $\mathcal{U}^{n-1}$ represents a hypersurface of $\mathcal{U}^n$ defined by $x^i = x^i(u^\alpha)$, with $\{\alpha = 1, 2, 3 \dots (n-1)\}$, and if the supporting element y^i of $\mathcal{U}^n$ is tangent to F^{n-1} [21], then

$$y^i = \mathcal{B}_\alpha^i(u) v^\alpha \quad (12)$$

The metric tensor, the hv-tensor, a unit normal vector, the angular metric tensor, and the connection between projection factors and their inverses for a Finslerian hypersurface F^{n-1} [21] at a point (u^α) are detailed as follows:

$$\begin{aligned} g_{\alpha\beta} &= g_{ij} \mathcal{B}_\alpha^i \mathcal{B}_\beta^j, \quad C_{\alpha\beta\gamma} = C_{ijk} \mathcal{B}_\alpha^i \mathcal{B}_\beta^j \mathcal{B}_\gamma^k, \\ g_{ij} \{x(u, v), y(u, v)\} \mathcal{B}_\alpha^i \mathcal{N}^j &= 0, \quad g_{ij} \{x(u, v), y(u, v)\} \mathcal{N}^i \mathcal{N}^j = 1. \\ h_{\alpha\beta} &= h_{ij} \mathcal{B}_\alpha^i \mathcal{B}_\beta^j, \quad h_{ij} \mathcal{B}_\alpha^i \mathcal{N}^j = 0, \quad h_{ij} \mathcal{N}^i \mathcal{N}^j = 1. \\ \mathcal{B}_i^\alpha &= g^{\alpha\beta} g_{ij} \mathcal{B}_\beta^j, \quad \mathcal{B}_\alpha^i \mathcal{B}_i^\beta = \delta_\alpha^\beta, \quad \mathcal{B}_i^\alpha \mathcal{N}^i = 0, \quad \mathcal{B}_\alpha^i \mathcal{N}_i = 0. \\ \mathcal{N}_i &= g_{ij} \mathcal{N}^j, \quad \mathcal{B}_i^k = g^{kj} \mathcal{B}_{ji}, \quad \mathcal{B}_\alpha^i \mathcal{B}_j^\alpha + \mathcal{N}^i \mathcal{N}_j = \delta_j^i. \end{aligned} \quad (13)$$

The Cartan connection $IC\Gamma = (\Gamma_{\beta\gamma}^{*\alpha}, G_{\beta}^{\alpha}, C_{\beta\gamma}^{\alpha})$ associated with the Finslerian hypersurface F^{n-1} is expressed as

$$\begin{aligned}\Gamma_{\beta\gamma}^{*\alpha} &= \mathcal{B}_i^{\alpha}(\mathcal{B}_{\beta\gamma}^i + \Gamma_{jk}^{*i}\mathcal{B}_{\beta}^j\mathcal{B}_{\gamma}^k) + M_{\beta}^{\alpha}h_{\gamma}, \\ G_{\beta}^{\alpha} &= \mathcal{B}_i^{\alpha}(\mathcal{B}_{0\beta}^i + \Gamma_{0j}^{*i}\mathcal{B}_{\beta}^j), \quad C_{\beta\gamma}^{\alpha} = \mathcal{B}_i^{\alpha}C_{jk}^i\mathcal{B}_{\beta}^j\mathcal{B}_{\gamma}^k,\end{aligned}$$

where

$$M_{\beta\gamma} = \mathcal{N}_i C_{jk}^i \mathcal{B}_{\beta}^j \mathcal{B}_{\gamma}^k, \quad M_{\beta}^{\alpha} = g^{\alpha\gamma} M_{\beta\gamma}, \quad \mathcal{H}_{\beta} = \mathcal{N}_i (\mathcal{B}_{0\beta}^i + \Gamma_{0j}^{*i} \mathcal{B}_{\beta}^j),$$

and

$$\mathcal{B}_{\beta\gamma}^i = \frac{\partial \mathcal{B}_{\beta}^i}{\partial u^{\gamma}}, \quad \mathcal{B}_{0\beta}^i = \mathcal{B}_{\alpha\beta}^i v^{\alpha}.$$

Note: The tensorial quantities $M_{\alpha\beta}$ and $\mathcal{H}_{\alpha}$ are identified as the second fundamental v-tensor and the normal curvature vector, respectively.

Moreover, the second fundamental h-tensor $\mathcal{H}_{\beta\gamma}$ can be represented as, per [21],

$$\mathcal{H}_{\beta\gamma} = \mathcal{N}_i (\mathcal{B}_{\beta\gamma}^i + \Gamma_{jk}^{*i} \mathcal{B}_{\beta}^j \mathcal{B}_{\gamma}^k) + M_{\beta}^{\alpha} \mathcal{H}_{\gamma}, \quad (14)$$

In this context, $M_{\beta} = \mathcal{N}_i C_{jk}^i \mathcal{B}_{\beta}^j \mathcal{N}^k$. Given the above expression, it is evident that the tensorial quantity $\mathcal{H}_{\beta\gamma}$ is non-symmetric, leading to

$$\mathcal{H}_{\beta\gamma} - \mathcal{H}_{\gamma\beta} = M_{\beta}^{\alpha} \mathcal{H}_{\gamma} - M_{\gamma}^{\alpha} \mathcal{H}_{\beta}. \quad (15)$$

The covariant derivatives of the projection factor $\mathcal{B}_{\alpha}^i$ with respect to the h- and v-directions of $IC\Gamma$ can now be articulated as

$$\mathcal{B}_{\alpha|\beta}^i = \mathcal{H}_{\alpha\beta} \mathcal{N}^i, \quad \mathcal{B}_{\alpha}^i|_{\beta} = M_{\alpha\beta} \mathcal{N}^i.$$

When we contract $\mathcal{H}_{\beta\gamma}$ and $\mathcal{H}_{\gamma\beta}$ with v^{β} , the result is

$$\mathcal{H}_{0\gamma} = \mathcal{H}_{\gamma}, \quad \mathcal{H}_{\gamma 0} = \mathcal{H}_{\gamma} + M_{\gamma}^{\alpha} \mathcal{H}_{\alpha}, \quad (16)$$

Hence, the crucial findings for the Finslerian hypersurface [21] that we will utilize in our current study are as follows.

Lemma 1. *The normal curvature tensor becomes zero in all cases if and only if the normal curvature vector vanishes on a Finslerian hypersurface $\mathcal{U}^{(n-1)}$.*

Lemma 2. *In a scenario where $\mathcal{U}^n$ symbolizes a Finsler space and $\mathcal{U}^{(n-1)}$ signifies its hypersurface, the hypersurface $\mathcal{U}^{(n-1)}$ is classified as a hyperplane of the first kind solely when the normal curvature vector completely disappears.*

Lemma 3. *Given a Finsler space denoted by $\mathcal{U}^n$ and its corresponding hypersurface $\mathcal{U}^{(n-1)}$, the hypersurface $\mathcal{U}^{(n-1)}$ is categorized as a hyperplane of the second kind only under the condition that both the normal curvature vector and the second fundamental h-tensor vanish completely.*

Lemma 4. *In the context where $\mathcal{U}^n$ stands for a Finsler space and $\mathcal{U}^{(n-1)}$ denotes its hypersurface, the hypersurface $\mathcal{U}^{(n-1)}$ is classified as a hyperplane of the third kind if and only if the normal curvature vector, the second fundamental h-tensor, and the v-tensor vanish identically.*

4. Hypersurface $\mathcal{U}^{(n-1)}(c)$ of a Finsler Space with Rander's-Type Exponential Form of (α, β) -Metric

In the context of a Finsler space featuring a Rander's-type exponential (α, β) -metric expressed as $\mathcal{G}(\alpha, \beta) = (\alpha + \beta)e^{\frac{\beta}{(\alpha + \beta)}}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ denotes a Riemannian metric and the vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ signifies the gradient of a scalar function $b(x)$, we now explore a hypersurface $\mathcal{U}^{(n-1)}(c)$ determined by the equation $b(x) = c$, where c stands for a constant [10].

Obtained from the parametric representation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we derive

$$\begin{aligned}\frac{\partial b(x)}{\partial u^\alpha} &= 0 \\ \frac{\partial b(x)}{\partial x^i} \frac{\partial x^i}{\partial u^\alpha} &= 0 \\ b_i \mathcal{B}_\alpha^i &= 0\end{aligned}$$

The preceding demonstration illustrates that $b_i(x)$ represents the covariant components of a normal vector field of the hypersurface $\mathcal{U}^{n-1}(c)$. Moreover, we have

$$b_i \mathcal{B}_\alpha^i = 0 \quad \text{and} \quad b_i y^i = 0 \quad \text{i.e.} \quad \beta = 0 \quad (17)$$

and the induced metric $\mathcal{G}(u, v)$ of $\mathcal{U}^{n-1}(c)$ is given by

$$\mathcal{G}(u, v) = a_{\alpha\beta} v^\alpha v^\beta, \quad a_{\alpha\beta} = a_{ij} \mathcal{B}_\alpha^i \mathcal{B}_\beta^j \quad (18)$$

which is a Riemannian metric.

Substituting $\beta = 0$ into Equations (2), (4) and (6) yields

$$\begin{aligned}p &= 1, \quad q_0 = 1, \quad q_{-1} = 0, \quad q_{-2} = -\alpha^{-2} \\ p_0 &= 5, \quad p_{-1} = 2\alpha^{-1}, \quad p_{-2} = 0, \quad \tau = 1 + b^2, \\ s_0 &= \frac{1}{1 + b^2}, \quad s_{-1} = \frac{2}{\alpha(1 + b^2)}, \quad s_{-2} = -\frac{4b^2}{\alpha^2(1 + b^2)}\end{aligned} \quad (19)$$

From Equation (5) we obtain

$$g^{ij} = a^{ij} - \frac{1}{1 + b^2} b^i b^j - \frac{2}{\alpha(1 + b^2)} (b^i y^j + b^j y^i) + \frac{4b^2}{\alpha^2(1 + b^2)} y^i y^j \quad (20)$$

Therefore, traversing the Finslerian hypersurface $\mathcal{U}^{n-1}(c)$ using Equations (20) and (17) results in

$$g^{ij} b_i b_j = \frac{b^2}{(1 + b^2)}$$

Thus, we have

$$b_i(x(u)) = \sqrt{\frac{b^2}{(1 + b^2)}} \mathcal{N}_i, \quad b^2 = a^{ij} b_i b_j \quad (21)$$

where b is the length of the vector b^i .

Once more, by utilizing Equations (20) and (21), we obtain

$$b^i = a^{ij} b_j = b \sqrt{1 + b^2} \mathcal{N}^i + \frac{2b^2 y^i}{\alpha} \quad (22)$$

Thus, we have the following theorem.

Theorem 1. The Riemannian metric induced on a Finsler hypersurface $\mathcal{U}^{(n-1)}(c)$ within a Finsler space $\mathcal{U}^{(n)}(c)$, characterized by the Rander's-type exponential metric described in Equation (1), is delineated in Equation (18). Moreover, the scalar function $b(x)$ is defined by Equations (21) and (22).

The angular metric tensor h_{ij} and the fundamental metric tensor g_{ij} of $\mathcal{U}^n$ can be expressed as

$$h_{ij} = a_{ij} + b_i b_j - \frac{1}{\alpha^2} Y_i Y_j \quad \text{and} \quad g_{ij} = a_{ij} + 5b_i b_j + \frac{2}{\alpha} (b_i Y_j + b_j Y_i) \quad (23)$$

By combining Equations (17), (23) and (13), it can be deduced that, if $h_{\alpha\beta}^{(a)}$ represents the angular metric tensor of the Riemannian $a_{ij}(x)$, then, along $\dots \mathcal{U}_{(c)}^{n-1}$, $h_{\alpha\beta} = h_{\alpha\beta}^{(a)}$.

Thus, along $\mathcal{U}_{(c)}^{n-1}$, $\frac{\partial p_0}{\partial \beta} = \frac{4}{\alpha}$.

Deriving from Equation (8), we obtain

$$\gamma_1 = -\frac{2}{\alpha}, \quad m_i = b_i$$

Then, the hv-torsion tensor becomes

$$C_{ijk} = \frac{1}{\alpha} (h_{ij} b_k + h_{jk} b_i + h_{ki} b_j) - \frac{1}{\alpha} b_i b_j b_k \quad (24)$$

In the Rander's-type exponential form of the (α, β) -metric of a Finsler hypersurface $\mathcal{U}_{(c)}^{(n-1)}$, it follows from Equations (13), (14), (16), (17) and (24) that we obtain

$$M_{\alpha\beta} = \frac{1}{\alpha} \sqrt{\frac{b^2}{(1+b^2)}} h_{\alpha\beta} \quad \text{and} \quad M_\alpha = 0 \quad (25)$$

Hence, based on Equation (14), it can be concluded that $\mathcal{H}_{\alpha\beta}$ is symmetric, leading to the following theorem.

Theorem 2. The v -tensor, representing the second fundamental form for the hypersurface $\mathcal{U}^{(n-1)}(c)$ within a Finsler space characterized by the Rander's-type exponential metric described in Equation (1), is given by Equation (25). Simultaneously, the h -tensor $\mathcal{H}_{\alpha\beta}$ is identified as symmetric.

Now, from Equation (17), we have $b_i \mathcal{B}_\alpha^i = 0$. Then, we have

$$b_{i|\beta} \mathcal{B}_\alpha^i + b_i \mathcal{B}_{\alpha|\beta}^i = 0$$

Consequently, by utilizing Equation (16) and the expression $b_{i|\beta} = b_{i|j} \mathcal{B}_\beta^j + b_i |_{\mathcal{J}} \mathcal{N}^j \mathcal{H}_\beta$, we obtain

$$b_{i|j} \mathcal{B}_\alpha^i \mathcal{B}_\beta^j + b_{i|j} \mathcal{B}_\alpha^i \mathcal{N}^j \mathcal{H}_\beta + b_i \mathcal{H}_{\alpha\beta} \mathcal{N}^i = 0 \quad (26)$$

Since $b_i |_{\mathcal{J}} = -b_h C_{ij}^h$, we obtain

$$b_{i|j} \mathcal{B}_\alpha^i \mathcal{N}^j = 0$$

Thus, deriving from Equation (26), we have

$$\sqrt{\frac{b^2}{(1+b^2)}} \mathcal{H}_{\alpha\beta} + b_{i|j} \mathcal{B}_\alpha^i \mathcal{B}_\beta^j = 0 \quad (27)$$

Since b_{ij} is symmetric, upon contracting Equation (27) with v^b and applying Equation (12), we obtain

$$\sqrt{\frac{b^2}{(1+b^2)}} \mathcal{H}_\alpha + b_{ij} \mathcal{B}_\alpha^i y^j = 0 \quad (28)$$

Once more, contracting Equation (28) with v^α and employing Equation (12), we arrive at

$$\sqrt{\frac{b^2}{(1+b^2)}} \mathcal{H}_0 + b_{ij} y^i y^j = 0 \quad (29)$$

From Lemmas 1 and 2, along with Equation (29), it becomes evident that a Finslerian hypersurface $\mathcal{U}^{(n-1)}(c)$ within a Finsler space featuring Rander's-type exponential metric as given in Equation (1) is a first-kind hyperplane if $b_{ij} y^i y^j = 0$.

Given that b_{ij} represents the covariant derivative concerning Γ within the Finsler space $\mathcal{U}^n$ defined over y^i , whereas $b_{ij} = \nabla_j b_i$ denotes the covariant derivative concerning the Riemannian connection $\{\Gamma_{jk}^i\}$, it follows that b_{ij} is independent of y^i . Consequently, we are inclined to examine the disparity $b_{ij} - b_{ji}$, where $b_{ij} = \nabla_j b_i$.

Given that b_i constitutes a gradient vector, we can deduce from Equation (9) that

$$E_{ij} = b_{ij} \quad \mathcal{U}_{ij} = 0 \quad \text{and} \quad \mathcal{U}_j^i = 0$$

Leveraging the aforementioned fact and Equation (10), the difference tensor $D_{jk}^i = \Gamma_{jk}^{*i} - \{\Gamma_{jk}^i\}$ can be articulated as

$$\begin{aligned} D_{jk}^i = & \mathcal{B}^i b_{jk} + \mathcal{B}_j^i b_{0k} + \mathcal{B}_k^i b_{0j} - b_{0m} g^{im} \mathcal{B}_{jk} - C_{jm}^i A_k^m - \\ & C_{km}^i A_j^m + C_{jkm} A_s^m g^{is} + \lambda^s (C_{jm}^i C_{sk}^m + \\ & C_{km}^i C_{sj}^m - C_{jk}^m C_{ms}^i) \end{aligned} \quad (30)$$

where

$$\begin{aligned} \mathcal{B}_i = & 5b_i + 2\alpha^{-1} y_i, \quad \mathcal{B}^i = \frac{5}{(1+b^2)} b^i + \frac{2(1-b^2)}{\alpha(1+b^2)} y^i, \\ \mathcal{B}_i \mathcal{B}^i = & \frac{4+21b^2}{1+b^2}, \quad \lambda^m = \mathcal{B}^m b_{00}, \quad \mathcal{B}_{ij} = \frac{1}{\alpha} (a_{ij} - \frac{y_i y_j}{\alpha^2}) + \frac{2}{\alpha} b_i b_j, \\ \mathcal{B}_j^i = & \frac{1}{\alpha} \{ (\delta_j^i - \alpha^{-2} y^i y_j) + \frac{2}{(1+b^2)} b^i b_j - \frac{4b^2}{\alpha(1+b^2)} b_j y^i \}, \\ A_k^m = & \mathcal{B}_k^m b_{00} + \mathcal{B}^m b_{k0} \end{aligned} \quad (31)$$

Considering Equations (19) and (20), the connection in Equation (11) transforms into $\mathcal{B}_0^i = 0$, $\mathcal{B}_{i0} = 0$, owing to Equation (31), resulting in $A_0^m = \mathcal{B}^m b_{00}$.

Contracting Equation (30) with y^k now yields

$$D_{j0}^i = \mathcal{B}^i b_{j0} + \mathcal{B}_j^i b_{00} - \mathcal{B}^m C_{jm}^i b_{00}$$

Once more, contracting the previous equation with respect to y^j gives

$$D_{00}^i = \mathcal{B}^i b_{00} = \{ \frac{5}{(1+b^2)} b^i + \frac{2(1-b^2)}{\alpha(1+b^2)} y^i \} b_{00}$$

Considering Equation (17) in the context of $\mathcal{U}_{(c)}^{(n-1)}$, we obtain

$$b_i D_{j0}^i = \frac{5b^2}{(1+b^2)} b_{j0} + \frac{1+3b^2}{\alpha(1+b^2)} b_j b_{00} - \frac{5}{1+b^2} b_i b^m C_{jm}^i b_{00} \quad (32)$$

Contracting Equation (32) with y^j , we have

$$b_i D_{00}^i = \frac{5b^2}{1+b^2} b_{00} \quad (33)$$

Given Equations (21), (22) and (25), and $M_\alpha = 0$, we obtain

$$b_i b^m c_{jm}^i B_\alpha^j = \frac{b^2}{(1+b^2)} M_\alpha = 0$$

Therefore, with the relation $b_{i|j} = b_{ij} - b_r D_{ij}^r$, Equations (32) and (33) yield

$$b_{i|j} y^i y^j = b_{00} - b_r D_{00}^r = \frac{1-4b^2}{1+b^2} b_{00}$$

As a result, Equations (28) and (29) can be expressed as

$$\begin{aligned} \sqrt{\frac{b^2}{1+b^2}} \mathcal{H}_\alpha + \frac{1-4b^2}{1+b^2} b_{i0} B_\alpha^i &= 0, \\ \sqrt{\frac{b^2}{1+b^2}} \mathcal{H}_0 + \frac{1-4b^2}{1+b^2} b_{00} &= 0 \end{aligned} \quad (34)$$

Therefore, the condition $\mathcal{H}_0 = 0$ is equivalent to $b_{00} = 0$. Utilizing the fact that $\beta = b_i y^i = 0$, the condition $b_{00} = 0$ can be restated as $b_{ij} y^i y^j = b_i y^i b_j y^j$ for a certain $c_j(x)$. Hence, we can express this as

$$2b_{ij} = b_i c_j + b_j c_i \quad (35)$$

Combining Equations (17) and (35), we obtain

$$b_{00} = 0, \quad b_{ij} B_\alpha^i B_\beta^j = 0, \quad b_{ij} B_\alpha^i y^j = 0$$

Hence, from Equation (34), we obtain $\mathcal{H}_\alpha = 0$; again, from Equations (31) and (35), we obtain $b_{i0} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A_j^i B_\beta^j = 0$, and $B_{ij} B_\alpha^i B_\beta^j = \frac{1}{\alpha} h_{\alpha\beta}$.

Now, employing Equations (20), (21), (22), (25) and (30), we arrive at

$$b_r D_{ij}^r B_\alpha^i B_\beta^j = -\frac{c_0 b^4}{2\alpha(1+b^2)} h_{\alpha\beta} \quad (36)$$

Therefore, Equation (27) simplifies to

$$\mathcal{H}_{\alpha\beta} + \frac{c_0 b^3}{2\alpha(1+b^2)^{3/2}} h_{\alpha\beta} = 0 \quad (37)$$

Therefore, the hypersurface $F_{(c)}^{n-1}$ exhibits umbilic properties.

Theorem 3. *The necessary and sufficient condition for the hypersurface $\mathcal{U}^{(n-1)}(c)$ in a Finsler space $\mathcal{U}^{(n)}(c)$ with a Rander's-type exponential-form metric given by Equation (1) to be a hyperplane of the first kind is precisely defined in Equation (35).*

The following is evident from Equation (35).

Corollary 1. *The second fundamental h-tensor for a Finsler hypersurface $\mathcal{U}^{(n-1)}(c)$ of a Finsler space $\mathcal{U}^{(n)}(c)$ equipped with a Rander's-type exponential-form metric defined in Equation (1) is directly related to its angular metric tensor.*

According to Lemma 3, the hypersurface $\mathcal{U}^{(n-1)}(c)$ qualifies as a hyperplane of the second kind when and only when $\mathcal{H}_\alpha = 0$ and $\mathcal{H}_{\alpha\beta} = 0$. Consequently, deducing from Equation (36), we obtain

$$c_0 = c_i(x)y^i = 0$$

Thus, there exists a function $\psi(x)$ such that

$$c_i(x) = \psi(x)b_i(x)$$

Therefore, from Equation (35), we obtain

$$2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x)$$

This can be expressed as

$$b_{ij} = \psi(x)b_ib_j$$

Theorem 4. *The necessary and sufficient condition for the hypersurface $\mathcal{U}^{(n-1)}(c)$ of a Finsler space $\mathcal{U}^{(n)}(c)$ equipped with a Rander's-type exponential-form metric defined in Equation (1) to be classified as a hyperplane of the second kind is delineated in Equation (37).*

Once more, Lemma 4, in conjunction with Equation (25) and $M_\alpha = 0$, indicates that $\mathcal{U}^{n-1}(c)$ does not form a hyperplane of the third kind. Therefore, the following holds.

Theorem 5. *The hypersurface $\mathcal{U}^{(n-1)}(c)$ within a Finsler space $\mathcal{U}^{(n)}(c)$, distinguished by a Rander's-type exponential metric as specified in Equation (1), is incapable of being a hyperplane of the third kind.*

5. Some Important Results of Hypersurfaces $\mathcal{U}^{(n-1)}(c)$ of a Finsler Space $\mathcal{U}^n(c)$ with Rander's-Type Exponential Form of (α, β) -Metric

The hv-torsion tensor $\mathcal{C}_{ijk}$ of $\mathcal{U}^{(n-1)}(c)$ with a Rander's-type exponential metric of (α, β) , expressed in Equation (24), is given by

$$\mathcal{C}_{ijk} = \frac{1}{\alpha}(\mathbf{h}_{ij}\mathbf{b}_k + \mathbf{h}_{jk}\mathbf{b}_i + \mathbf{h}_{ki}\mathbf{b}_j) - \frac{1}{\alpha}\mathbf{b}_i\mathbf{b}_j\mathbf{b}_k$$

Contracting by $\mathbf{g}^{jk}$, we have

$$\mathcal{C}_i = \mathcal{C}_{ijk}\mathbf{g}^{jk} = \frac{(2+3\mathbf{b}^2)}{\alpha(1+\mathbf{b}^2)}\mathbf{b}_i$$

This indicates that

$$\mathbf{b}_i = \frac{\alpha(1+\mathbf{b}^2)}{(2+3\mathbf{b}^2)}\mathcal{C}_i \quad (38)$$

Therefore, when expressed as Equation (24),

$$\mathcal{C}_{ijk} = \frac{(1+\mathbf{b}^2)}{(2+3\mathbf{b}^2)}(\mathbf{h}_{ij}\mathcal{C}_k + \mathbf{h}_{jk}\mathcal{C}_i + \mathbf{h}_{ki}\mathcal{C}_j) - \frac{\alpha^2(1+\mathbf{b}^2)^3}{(2+3\mathbf{b}^2)^3}\mathcal{C}_i\mathcal{C}_j\mathcal{C}_k \quad (39)$$

Definition 2. *A Finsler Space $\mathcal{U}^n$ is called semi-C-reducible, if the (h) hv-torsion tensor $\mathcal{C}_{ijk}$ is written in the form*

$$\mathcal{C}_{ijk} = \frac{p}{(n+1)}(\mathbf{h}_{ij}\mathcal{C}_k + \mathbf{h}_{jk}\mathcal{C}_i + \mathbf{h}_{ki}\mathcal{C}_j) + \frac{q}{\mathcal{C}^2}\mathcal{C}_i\mathcal{C}_j\mathcal{C}_k. \quad (40)$$

Now, by combining Equations (39) and (40), we obtain

$$p = \frac{(n+1)(1+\mathbf{b}^2)}{(2+3\mathbf{b}^2)} \quad \text{and} \quad q = \frac{-\alpha^2\mathcal{C}^2(1+\mathbf{b}^2)^3}{(2+3\mathbf{b}^2)} \quad (41)$$

Thus, we arrive at the following proposition.

Proposition 1. *The Finslerian hypersurface $\mathcal{U}^{(n-1)}(c)$ of Finsler space $\mathcal{U}^n$ equipped with a Rander's-type exponential-form metric defined in Equation (1) is always a semi-C-reducible Finsler space if Equation (41) satisfied.*

Moreover, by contracting Equation (24) with $\mathcal{B}_\alpha^i$ and utilizing Equation (17), we can derive

$$\mathcal{C}_{ijk}\mathcal{B}_\alpha^i = \frac{1}{\alpha}(\mathfrak{h}_{ij}\mathfrak{b}_k\mathcal{B}_\alpha^i + \mathfrak{h}_{ki}\mathfrak{b}_j\mathcal{B}_\alpha^i) \quad (42)$$

By contracting Equation (42) with $\mathcal{B}_\beta^j$ and applying Equation (17), we obtain

$$\mathcal{C}_{ijk}\mathcal{B}_\alpha^i\mathcal{B}_\beta^j = \frac{1}{\alpha}\mathfrak{h}_{ij}\mathfrak{b}_k\mathcal{B}_\alpha^i\mathcal{B}_\beta^j$$

This suggests that

$$\mathcal{C}_{ijk}\mathcal{B}_\alpha^i\mathcal{B}_\beta^j = \frac{1}{\alpha}\mathfrak{h}_{\alpha\beta}\mathfrak{b}_k \quad (43)$$

By contracting Equation (43) and applying Equation (17), we find

$$\mathcal{C}_{ijk}\mathcal{B}_\alpha^i\mathcal{B}_\beta^j\mathcal{B}_\gamma^k = \mathcal{C}_{\alpha\beta\gamma} = 0 \quad (44)$$

Proposition 2. *The Finslerian hypersurface $\mathcal{U}^{(n-1)}(c)$ of Finsler space $\mathcal{U}^n$ equipped with a Rander's-type exponential-form metric defined in Equation (1) is a C-reducible Finsler space if Equation (44) satisfied.*

Definition 3. *A Finsler space $\mathcal{U}^n$ is called C2-like if the (h) hv-tortion tensor $\mathcal{C}_{ijk}$ is written in the form*

$$\mathcal{C}_{ijk} = \frac{1}{\mathcal{C}^2}\mathcal{C}_i\mathcal{C}_j\mathcal{C}_k$$

The main scalar I [22] of a two-dimensional Finsler space is defined as

$$LC_{ijk} = Im_im_jm_k$$

Since $m_i = b_i$, then we have

$$LC_{ijk} = Ib_ib_jb_k$$

Contracting g^{jk} , we have

$$LC_i = \frac{Ib^2}{(1+b^2)}b_i \quad (45)$$

Now, the main scalar I of a two-dimensional Finsler space is

$$\mathcal{C}_{ijl} = \frac{L^2(1+b^2)^3}{I^3b^6}\mathcal{C}_i\mathcal{C}_j\mathcal{C}_k \quad (46)$$

Therefore, utilizing the definition above and Equation (46), we find

$$\mathcal{C} = \frac{I^{\frac{3}{2}}\mathfrak{b}^3}{L(1+\mathfrak{b}^2)^{\frac{3}{2}}} \quad (47)$$

Proposition 3. *The Finslerian hypersurface $\mathcal{U}^{(n-1)}(c)$ in a two-dimensional Finsler space $\mathcal{U}^n$, which is endowed with Rander's-type exponential-form metric as defined in Equation (1), will consistently represent a C2-like Finsler space given that Equation (47) holds true.*

6. Conclusions

Following an in-depth exploration of a distinct Finsler space defined by a Rander's-type exponential-form metric with the expression $\mathcal{G}(\alpha, \beta) = (\alpha + \beta)e^{\frac{\beta}{(\alpha + \beta)}}$, where α represents the Riemannian metric and β denotes the 1-form metric, this study has delved into the intrinsic properties of this specialized geometric space.

The research has primarily focused on investigating the behavior of hypersurfaces within this Finsler space and their resemblance to hyperplanes categorized into the first, second, and third kinds. By scrutinizing the conditions under which these hypersurfaces exhibit such characteristics, we have unveiled significant insights into the interplay between the metric structure and the geometric properties of the space.

Furthermore, our study has examined the reducibility of the Cartan tensor $\mathcal{C}$ for these hypersurfaces in diverse forms, unveiling further layers of complexity within the geometric framework.

By shedding light on the nuanced relationship between the metric structure and the geometric features of Finsler spaces, this research contributes to a deeper understanding of the behavior of these spaces under specific conditions. The findings presented in this study pave the way for further exploration and research in the realm of Finsler geometry, offering a promising avenue for uncovering additional intriguing facets and applications within this field.

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Article

Curves in Multiplicative Equiaffine Space

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Abstract: In this study, the geometric properties of curves in multiplicative equiaffine space are investigated using multiplicative calculus. Fundamental geometric concepts such as multiplicative arc length, multiplicative equiaffine curvature, and torsion are introduced. This study derives the multiplicative Frenet frame and associated Frenet equations, providing a systematic framework for describing the geometric behavior of multiplicative equiaffine curves. Additionally, curves with constant multiplicative curvature and torsion are characterized and supported with illustrative examples.

Keywords: affine space; equiaffine space curves; multiplicative calculus; multiplicative derivative; multiplicative integral

MSC: 53A04; 53A15; 11A05

1. Introduction

The foundational principles of classical analysis, a pivotal element of modern mathematical theory, were laid out by Leibniz and Newton in the late 17th century, primarily focusing on the notation of differential and integral calculus. Rooted in trigonometry, algebra, and analytic geometry, classical analysis incorporates concepts like limits, differentiation, integration, and series expansions. These operations, often regarded as fundamental as those of basic arithmetic, have earned classical analysis the alternative name “summational analysis”. Its versatility has led to widespread use across various disciplines that require precise mathematical modeling and optimization.

However, classical analysis faces limitations in certain contexts, prompting the development of alternative methodologies. Among these, Volterra proposed a novel analytical approach in 1887 called Volterra-type or multiplicative analysis, which prioritizes multiplication over addition as its core operation [1]. In this paradigm, multiplication and division replace addition and subtraction, introducing a new structural framework for mathematical modeling. Expanding on Volterra’s foundation, Grossman and Katz, between 1972 and 1983, developed what is now recognized as non-Newtonian analysis. This framework introduced novel concepts and definitions, revolutionizing mathematical operations across diverse domains [2,3]. These methods, encompassing geometric, bigeometric, and anageometric analysis, offer innovative tools for problems unsolvable by classical techniques.

Multiplicative analysis, as a counterpart to classical analysis, has gained substantial attention and has encouraged further research. Numerous studies, including [4–8], have significantly contributed to its advancement. Recent works in differential geometry also integrate multiplicative analysis. For instance, Georgiev’s contributions [9–11], particularly [10], are considered foundational texts in multiplicative geometry, establishing new definitions and theorems linking geometric structures with multiplicative principles. These

studies emphasize the interplay between multiplicative geometry and other mathematical disciplines, broadening its scope and applications.

Advances in geometric analysis include Nurkan et al. application of geometric calculus to derive Gram–Schmidt vectors [12]. Similarly, ref. [13] examined spherical indicatrices and helices in non-Newtonian Euclidean spaces, providing fresh characterizations and examples. Aydın et al. classified and visualized rectifying curves within multiplicative Euclidean space using multiplicative spherical curves [14], while [15] extended these concepts to tube surfaces with multiplicative quaternions. Moreover, several intriguing studies in differential geometry have focused on curves. Broscăţeanu et al. [16] analyzed the infinitesimal bending of rectifying curves, Burlacu and Mihai [17] investigated applications of curve theory in road design, and Jianu et al. [18] explored a surface associated with the Catalan triangle.

Further investigations into non-Newtonian geometry include Has, Yılmaz, and Yıldırım’s exploration of Lorentz–Minkowski space through non-Newtonian approaches [19], and Özdemir and Ceyhan’s study on multiplicative hyperbolic split quaternions and their role in hyperbolic rotation matrices [20]. Has and Yılmaz also analyzed non-Newtonian conics through multiplicative analytic geometry [21]. Additionally, Es’s research [22] utilized homothetic multiplicative calculus to derive kinematic expressions for plane motion. Expanding on these, Has and Yılmaz examined magnetic curves within multiplicative Riemannian manifolds using non-Newtonian methods [23]. Moreover, Aydın has studied the multiplicative rectifying submanifolds of multiplicative Euclidean space and made important contributions to this area [24]. Ceyhan has worked on a non-Newtonian approach to the geometric phase through optical fibers via multiplicative quaternions [25].

In the realm of affine geometry, extensive research has illuminated the properties of affine spaces. Works like [26–29] provide comprehensive discussions on both foundational and advanced aspects of affine geometry, bridging classical and contemporary perspectives.

This paper is organized as follows. It first introduces the key concepts of multiplicative calculus. Subsequently, the essential elements of equiaffine geometry relevant to the analysis are discussed. Finally, the curvature of a multiplicative plane curve is computed.

2. Preliminaries

This section delves into two fundamental topics. The initial part discusses the foundational aspects of equiaffine plane curves, emphasizing the determinant in the affine plane and the concept of equiaffine arc-length. The latter part introduces the framework of multiplicative analysis, examining multiplicative operations and structures such as addition, multiplication, and derivatives in the multiplicative context. Furthermore, the multiplicative determinant is presented as a logarithmic counterpart to the classical determinant.

2.1. Basics of Equiaffine Space Curves

Equiaffine spaces are a subspace of affine spaces, possessing a fixed volume form, which provides them with additional geometric structure. Equiaffine geometry was considered in the paper due to its intrinsic properties that provide a natural framework for analyzing curves in an affine space without relying on an external Euclidean metric. Unlike classical differential geometry, which depends on metric-based notions of distance and angle, equiaffine geometry allows for the study of curves and surfaces using volume-preserving transformations. This makes it particularly suitable for applications in multiplicative calculus, where traditional metric structures are replaced by multiplicative analogs. By integrating equiaffine concepts with multiplicative calculus, the paper aims to develop a systematic approach to defining arc length, curvature, and torsion in a manner

that is invariant under multiplicative affine transformations. Now, this section provides an overview of the key principles of equiaffine space curves, avoiding in-depth explanations.

Let $\mathbf{R}^3$ be the affine space with the determinant

$$[\mathbf{u} \ \mathbf{v} \ \mathbf{w}] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1), \quad (1)$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3) \in \mathbf{R}^3$.

Let $\varphi : I \rightarrow \mathbf{R}^3$ represent a regular curve in $\mathbf{R}^3$. Typically, for equiaffine regular curves, the condition of non-degeneracy $[\dot{\varphi}(t) \ \ddot{\varphi}(t) \ \ddot{\ddot{\varphi}}(t)] \neq 0$ for every $t \in I$ is assumed, where

$$\dot{\varphi}(t) = \frac{d\varphi}{dt}(t), \quad \ddot{\varphi}(t) = \frac{d^2\varphi}{dt^2}(t), \quad \ddot{\ddot{\varphi}}(t) = \frac{d^3\varphi}{dt^3}(t). \quad (2)$$

Given this, the equiaffine arc length of φ , measured from $\varphi(t_0)$ ($t_0 \in I$), is given as

$$s(t) = \int_{t_0}^t [\dot{\varphi}(t) \ \ddot{\varphi}(t) \ \ddot{\ddot{\varphi}}(t)]^{\frac{1}{6}} dt. \quad (3)$$

Let

$$\varphi'(s) = \frac{d\varphi}{ds}(s), \quad \varphi''(s) = \frac{d^2\varphi}{ds^2}(s), \quad \varphi'''(s) = \frac{d^3\varphi}{ds^3}(s). \quad (4)$$

A regular curve φ is parameterized by the equiaffine arc length if it satisfies

$$\left[\frac{d\varphi}{ds}(s) \ \frac{d^2\varphi}{ds^2}(s) \ \frac{d^3\varphi}{ds^3}(s) \right] = 1. \quad (5)$$

for all s .

The set $\left\{ \frac{d\varphi}{ds}(s), \frac{d^2\varphi}{ds^2}(s), \frac{d^3\varphi}{ds^3}(s) \right\}$ is referred to as the *equiaffine Frenet frame* of $\varphi(s)$. When we take the derivative of (5) with respect to the parameter s , it is apparent that

$$\left[\frac{d\varphi}{ds}(s) \ \frac{d^2\varphi}{ds^2}(s) \ \frac{d^4\varphi}{ds^4}(s) \right] = 0, \quad (6)$$

where the set is linearly dependent for every $s \in J$, given by

$$\left\{ \frac{d\varphi}{ds}(s), \frac{d^2\varphi}{ds^2}(s), \frac{d^4\varphi}{ds^4}(s) \right\}. \quad (7)$$

Therefore, this implies the existence of smooth functions $\kappa(s)$ and $\tau(s)$ on J such that

$$\frac{d^4\varphi}{ds^4}(s) + \kappa(s) \frac{d\varphi}{ds}(s) + \tau(s) \frac{d^2\varphi}{ds^2}(s) = 0, \quad (8)$$

where

$$\kappa(s) = - \left[\frac{d^2\varphi}{ds^2}(s) \ \frac{d^3\varphi}{ds^3}(s) \ \frac{d^4\varphi}{ds^4}(s) \right], \quad (9)$$

and

$$\tau(s) = \left[\frac{d\varphi}{ds}(s) \ \frac{d^3\varphi}{ds^3}(s) \ \frac{d^4\varphi}{ds^4}(s) \right]. \quad (10)$$

The functions $\kappa(s)$ and $\tau(s)$ are called *equiaffine curvature* and *equiaffine torsion* of the curve $\varphi(s)$, respectively. The equiaffine curvature and equiaffine torsion are the equiaffine invariants in $\mathbf{R}^3$.

Let φ be a non-degenerate smooth curve in $\mathbf{R}^3$ parameterized by equiaffine arc length s . As a result, the equiaffine equations of Frenet type are presented in matrix form as

$$\begin{bmatrix} \frac{d^2\varphi}{ds^2}(s) \\ \frac{d^3\varphi}{ds^3}(s) \\ \frac{d^4\varphi}{ds^4}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\kappa(s) & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \frac{d\varphi}{ds}(s) \\ \frac{d^2\varphi}{ds^2}(s) \\ \frac{d^3\varphi}{ds^3}(s) \end{bmatrix}. \quad (11)$$

2.2. Basics of Multiplicative Analysis

This section introduces the fundamental definitions and key principles of multiplicative analysis, drawing inspiration from the contributions of Georgiev [9–11]. Subsequently, the discussion transitions to multiplicative equiaffine space curves.

Let $\mathbf{R}_* = (0, \infty)$, where we define the operations of multiplicative addition, multiplication, subtraction, and division as follows:

$$x +_* y = e^{\log x + \log y} = x \cdot y, \quad (12)$$

$$x \cdot_* y = e^{(\log x)(\log y)} = x^{\log y}, \quad (13)$$

$$x -_* y = e^{\log x - \log y} = \frac{x}{y}, \quad (14)$$

$$x /_* y = e^{\frac{\log x}{\log y}} = x^{\frac{1}{\log y}} \text{ for } y \neq 1. \quad (15)$$

These operations define the multiplicative structure $(\mathbf{R}_*, +_*, -_*)$. Each x_* in $\mathbf{R}_*$ is called a multiplicative number, where $x_* = e^x$. The multiplicative zero and unit are $0_* = 1$ and $1_* = e$, respectively.

For $k \in \mathbf{N}$, the power k_* of a multiplicative number x is defined as $x^{k_*} = x \cdot_* x \cdots \cdot_* x$ (repeated k times), which simplifies to $e^{(\log x) \cdot k}$.

The multiplicative sine and cosine functions are defined as:

$$\sin_* x = e^{\sin(\log x)} \quad (16)$$

and

$$\cos_* x = e^{\cos(\log x)}. \quad (17)$$

Similarly, the multiplicative hyperbolic sine and cosine functions are given by

$$\sinh_* x = e^{\sinh(\log x)} \quad (18)$$

and

$$\cosh_* x = e^{\cosh(\log x)}. \quad (19)$$

Let $A \in \mathbf{R}_*$ and f be a first-order continuously differentiable function. The first-order multiplicative derivative of f , denoted by $f^*(x)$, is defined as

$$f^*(x) = \lim_{h \rightarrow 0_*} \frac{f(x +_* h) -_* f(x)}{h} = e^{\frac{xf'(x)}{f(x)}}, \quad (20)$$

where f' represents the classical derivative of f .

From now on, the multiplicative derivative given in Equation (20) will also be denoted by $(\dots)^*$.

For $a, b \in \mathbf{R}_*$ and $f : \mathbf{R}_* \rightarrow \mathbf{R}_*$, the multiplicative indefinite integral and the multiplicative Cauchy integral are defined as

$$\int_* f(x) \cdot_* d_* x = e^{\int \frac{1}{x} \log f(x) dx} \quad (21)$$

and

$$\int_a^b f(x) \cdot_* d_* x = e^{\int_a^b \frac{1}{x} \log f(x) dx}. \quad (22)$$

Finally, for three vectors $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, and $\mathbf{z} = (z_1, z_2, z_3)$ the multiplicative determinant is defined as

$$\begin{aligned} [\mathbf{x} \mathbf{y} \mathbf{z}]_* &= x_1 \cdot_* y_2 \cdot_* z_3 +_* y_1 \cdot_* z_2 \cdot_* x_3 +_* z_1 \cdot_* x_2 \cdot_* y_3 \\ &\quad -_* z_1 \cdot_* y_2 \cdot_* x_3 -_* x_1 \cdot_* z_2 \cdot_* y_3 -_* y_1 \cdot_* x_2 \cdot_* z_3, \end{aligned} \quad (23)$$

which can be expressed in logarithmic terms as

$$[\mathbf{x} \mathbf{y} \mathbf{z}]_* = \frac{e^{(\log x_1 \cdot \log y_2 \cdot \log z_3 + \log y_1 \cdot \log z_2 \cdot \log x_3 + \log z_1 \cdot \log x_2 \cdot \log y_3)}}{e^{(\log z_1 \cdot \log y_2 \cdot \log x_3 + \log x_1 \cdot \log z_2 \cdot \log y_3 + \log y_1 \cdot \log x_2 \cdot \log z_3)}}. \quad (24)$$

This serves as the multiplicative analogue of the classical determinant in $\mathbf{R}_*^3$.

3. Multiplicative Equiaffine Space Curves

In this section, we extend the theory of multiplicative equiaffine space curves by introducing the concept of multiplicative affine space and the multiplicative determinant, which serves as a generalization for volume computation. Additionally, we define key notions such as multiplicative linear independence, the multiplicative equiaffine group, and the multiplicative arc length. The discussion further includes the development of the multiplicative Frenet apparatus and the corresponding Frenet equations, which characterize the curvature and frame transformations of these curves.

The *multiplicative affine space* $\mathbf{R}_*^3$ is defined as a three-dimensional affine space with a multiplicative structure replacing the conventional additive framework. Within this space, a constant volume form is maintained, acting as a generalized determinant to compute the “volume” formed by three vectors. This volume form is expressed multiplicatively as follows:

$$[\mathbf{x} \mathbf{y} \mathbf{z}]_* = x_1 \cdot_* (y_2 \cdot_* z_3 -_* y_3 \cdot_* z_2) -_* x_2 \cdot_* (y_1 \cdot_* z_3 -_* y_3 \cdot_* z_1) +_* x_3 \cdot_* (y_1 \cdot_* z_2 -_* y_2 \cdot_* z_1), \quad (25)$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, and $\mathbf{z} = (z_1, z_2, z_3)$ are vectors in $\mathbf{R}_*^3$.

Proposition 1. In $\mathbf{R}_*^3$, for vectors $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, and $\mathbf{z} = (z_1, z_2, z_3)$

$$([\mathbf{x} \mathbf{y} \mathbf{z}]_*)^* = [\mathbf{x}^* \mathbf{y} \mathbf{z}]_* +_* [\mathbf{x} \mathbf{y}^* \mathbf{z}]_* +_* [\mathbf{x} \mathbf{y} \mathbf{z}^*]_*, \quad (26)$$

equality is satisfied.

Proof. Let us assume that

$$[\mathbf{x} \mathbf{y} \mathbf{z}]_* = x_1 \cdot_* (y_2 \cdot_* z_3 -_* y_3 \cdot_* z_2) -_* x_2 \cdot_* (y_1 \cdot_* z_3 -_* y_3 \cdot_* z_1) +_* x_3 \cdot_* (y_1 \cdot_* z_2 -_* y_2 \cdot_* z_1), \quad (27)$$

Now, consider $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ as matrices with the vectors as columns

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \quad (28)$$

Then, the determinant of the matrix formed by $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ denoted as $[\mathbf{x} \ \mathbf{y} \ \mathbf{z}]_*$, is given by

$$[\mathbf{x} \ \mathbf{y} \ \mathbf{z}]_* = \det_* \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \quad (29)$$

$$\begin{aligned} &= x_1 \cdot_* (y_2 \cdot_* z_3 -_* y_3 \cdot_* z_2) \\ &-_* x_2 \cdot_* (y_1 \cdot_* z_3 -_* y_3 \cdot_* z_1) \\ &+_* x_3 \cdot_* (y_1 \cdot_* z_2 -_* y_2 \cdot_* z_1) \end{aligned} \quad (30)$$

or

$$\begin{aligned} &= e^{\log x_1 \cdot \log y_2 \cdot \log z_3 + \log x_2 \cdot \log y_3 \cdot \log z_1 + \log x_3 \cdot \log y_1 \cdot \log z_2} \\ &\cdot e^{-(\log x_1 \cdot \log y_3 \cdot \log z_2 + \log x_2 \cdot \log y_1 \cdot \log z_3 + \log x_3 \cdot \log y_2 \cdot \log z_1)} \end{aligned} \quad (31)$$

Now, let us multiplicatively differentiate this last expression, then

$$\begin{aligned} ([\mathbf{x} \ \mathbf{y} \ \mathbf{z}]_*)^* &= \left(e^{(\log x_1 \cdot \log y_2 \cdot \log z_3 + \log x_2 \cdot \log y_3 \cdot \log z_1 + \log x_3 \cdot \log y_1 \cdot \log z_2)} \right. \\ &\left. \cdot e^{-(\log x_1 \cdot \log y_3 \cdot \log z_2 + \log x_2 \cdot \log y_1 \cdot \log z_3 + \log x_3 \cdot \log y_2 \cdot \log z_1)} \right)^* \end{aligned} \quad (32)$$

is obtained. Using the multiplicative derivative rule, we have

$$\begin{aligned} &= e^{x \left((\log x_1)^* \cdot (\log y_2 \cdot \log z_3 - \log y_3 \cdot \log z_2) + (\log x_2)^* \cdot (\log y_3 \cdot \log z_1 - \log y_1 \cdot \log z_3) + (\log x_3)^* \cdot (\log y_1 \cdot \log z_2 - \log y_2 \cdot \log z_1) \right)} \\ &+_* e^{x \left((\log y_1)^* \cdot (\log x_3 \cdot \log z_2 - \log x_2 \cdot \log z_3) + (\log y_2)^* \cdot (\log x_1 \cdot \log z_3 - \log x_3 \cdot \log z_1) + (\log y_3)^* \cdot (\log x_2 \cdot \log z_1 - \log x_1 \cdot \log z_2) \right)} \\ &+_* e^{x \left((\log z_1)^* \cdot (\log x_2 \cdot \log y_3 - \log x_3 \cdot \log y_2) + (\log z_2)^* \cdot (\log x_1 \cdot \log y_3 - \log x_3 \cdot \log y_1) + (\log z_3)^* \cdot (\log x_1 \cdot \log y_2 - \log x_2 \cdot \log y_1) \right)}. \end{aligned} \quad (33)$$

Therefore, we have proved that

$$([\mathbf{x} \ \mathbf{y} \ \mathbf{z}]_*)^* = [\mathbf{x}^* \ \mathbf{y} \ \mathbf{z}]_* +_* [\mathbf{x} \ \mathbf{y}^* \ \mathbf{z}]_* +_* [\mathbf{x} \ \mathbf{y} \ \mathbf{z}^*]_*, \quad (34)$$

completing the proof. $\square$

Definition 1. A set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ of vectors in $\mathbf{R}_*^3$ is said to be multiplicatively linearly independent if the equation

$$\lambda_1 \cdot_* \mathbf{x} +_* \lambda_2 \cdot_* \mathbf{y} +_* \lambda_3 \cdot_* \mathbf{z} = 0_* = 1 \quad (35)$$

holds only when $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Otherwise, if there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{R}_*$, not all equal to 1, such that Equation (35) holds, then $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are said to be multiplicatively linearly dependent.

$SL(3, \mathbf{R}_*^{3 \times 3}) = \{A \in \mathbf{R}_*^{3 \times 3} : |A|_* = e\}$. The multiplicative equiaffine group of $\mathbf{R}_*^3$ is produced by the combined action of the multiplicative rotation group and the multiplicative translation group. A representation of a multiplicative equiaffine transformation of $\mathbf{R}_*^3$ can be expressed in the form

$$\bar{\mathbf{C}} = A \cdot_* \mathbf{C} +_* \mathbf{B}, \quad (36)$$

where $\bar{\mathbf{C}}, \mathbf{C}, \mathbf{B} \in \mathbf{R}_*^{3 \times 1}$. Note that the transformation given in Equation (36) preserves the volume of a parallelepiped and hence is called a *volume-preserving multiplicative affine transformation*.

Definition 2. Let $\varphi = \varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$, where $t \in I \subset \mathbf{R}_*$, is a multiplicative non-degenerate smooth parametric curve in $\mathbf{R}_*^3$. This means that

$$[\varphi^* \ \varphi^{**} \ \varphi^{***}]_* \neq 0_* \text{ (or } 1), \quad (37)$$

for any t .

Definition 3. The multiplicative equiaffine arc length function ρ for the multiplicative equiaffine curve φ in $\mathbf{R}_*$ is defined as

$$\rho(t) = \int_{*t_0}^t ([\varphi^* \varphi^{**} \varphi^{***}]_*)^{\frac{1}{6}*} \cdot_* d_* u = e^{\int_{t_0}^t \frac{1}{u} (\log([\varphi^* \varphi^{**} \varphi^{***}]_*)) \frac{1}{6} du}, \quad (38)$$

where $[\varphi^* \varphi^{**} \varphi^{***}]_*$ is the multiplicative determinant of the first, second, and third multiplicative derivatives of the curve φ . The parameter ρ is the multiplicative equiaffine arc length parameter.

Proposition 2. Let φ be a multiplicative non-degenerate smooth parametric curve in $\mathbf{R}^3$ parameterized by the multiplicative equiaffine arc length ρ . Then, we have

$$[(d_* \varphi /_* d_* \rho \ d_*^2 \varphi /_* d_* \rho^{2*} \ d_*^3 \varphi /_* d_* \rho^{3*})_*] = 1_*.$$

Proof. Using the chain rule, we write

$$d_* \varphi /_* d_* \rho = (d_* \varphi /_* d_* t) \cdot_* (d_* t /_* d_* \rho)$$

or

$$d_* \varphi /_* d_* \rho = \varphi^* \cdot_* (d_* t /_* d_* \rho). \quad (39)$$

If we take the multiplicative derivative of (39), then

$$d_*^2 \varphi /_* d_* \rho^{2*} = \varphi^{**} \cdot_* (d_* t /_* d_* \rho)^{2*} +_* \varphi^* \cdot_* (d_*^2 t /_* d_* \rho^{2*}). \quad (40)$$

Again, take the multiplicative derivative of (40), then

$$\begin{aligned} d_*^3 \varphi /_* d_* \rho^{3*} &= \varphi^{***} \cdot_* (d_* t /_* d_* \rho)^{3*} \\ &+_* \varphi^{**} \cdot_* 2 \cdot_* (d_* t /_* d_* \rho) \cdot_* (d_*^2 t /_* d_* \rho^{2*}) \\ &+_* \varphi^{**} \cdot_* (d_* t /_* d_* \rho) \cdot_* (d_*^2 t /_* d_* \rho^{2*}) \\ &+_* \varphi^* \cdot_* (d_*^3 t /_* d_* \rho^{3*}) \end{aligned}$$

Thus, we have

$$[(d_* \varphi /_* d_* \rho \ d_*^2 \varphi /_* d_* \rho^{2*} \ d_*^3 \varphi /_* d_* \rho^{3*})_*] = (d_* t /_* d_* \rho)^{6*} \cdot_* [\varphi^* \varphi^{**} \varphi^{***}]_*.$$

According to the Mean Value Theorem in analysis, since

$$[\varphi^* \varphi^{**} \varphi^{***}]_* = (d_* \rho /_* d_* t)^{6*}$$

from (38), we get

$$[(d_* \varphi /_* d_* \rho \ d_*^2 \varphi /_* d_* \rho^{2*} \ d_*^3 \varphi /_* d_* \rho^{3*})_*] = 1_*.$$

This completes the proof. $\square$

Definition 4. Let φ be a multiplicative non-degenerate smooth curve in $\mathbf{R}_*^3$ parameterized by the multiplicative equiaffine arc length s . The multiplicative equiaffine Frenet equations are given in matrix form as

$$\begin{bmatrix} d_*^2 \varphi / d_* s^{2*} \\ d_*^3 \varphi / d_* s^{3*} \\ d_*^4 \varphi / d_* s^{4*} \end{bmatrix} = \begin{bmatrix} 0_* & 1_* & 0_* \\ 0_* & 0_* & 1_* \\ -\kappa_*(s) & -\tau_*(s) & 0_* \end{bmatrix} \begin{bmatrix} d_* \varphi / d_* s \\ d_*^2 \varphi / d_* s^{2*} \\ d_*^3 \varphi / d_* s^{3*} \end{bmatrix}, \quad (41)$$

where the functions $\kappa_*(s)$ and $\tau_*(s)$ are called the multiplicative equiaffine curvature and multiplicative equiaffine torsion of the curve $\varphi(s)$, respectively.

Therefore, this implies the existence of smooth functions $\kappa_*(s)$ and $\tau_*(s)$ such that

$$d_*^4 \varphi / d_* s^{4*} + \kappa_*(s) \cdot d_* \varphi / d_* s + \tau_*(s) \cdot d_*^2 \varphi / d_* s^{2*} = 0_*, \quad (42)$$

where

$$\kappa_*(s) = -[d_*^2 \varphi / d_* s^{2*} \quad d_*^3 \varphi / d_* s^{3*} \quad d_*^4 \varphi / d_* s^{4*}]_* \quad (43)$$

and

$$\tau_*(s) = [d_* \varphi / d_* s \quad d_*^3 \varphi / d_* s^{3*} \quad d_*^4 \varphi / d_* s^{4*}]_*. \quad (44)$$

The multiplicative equiaffine curvature $\kappa_*(s)$ and torsion $\tau_*(s)$ are multiplicative equiaffine invariants in $\mathbf{R}_*^3$.

Theorem 1. Let φ be a curve in $\mathbf{R}_*^3$ parameterized by multiplicative equiaffine arc length. Then the curve φ and $A \cdot \varphi + B$ have the same multiplicative equiaffine curvature and torsion, where $A \in SL$ and $B \in \mathbf{R}_*^{3 \times 1}$.

Proof. Let β be a curve other than φ and let it be represented as follows:

$$\beta = A \cdot \varphi + B. \quad (45)$$

If the derivatives are taken in the multiplicative sense, the following equalities are obtained:

$$d_* \beta / d_* s = A \cdot d_* \varphi / d_* s, \quad (46)$$

$$d_*^2 \beta / d_* s^{2*} = A \cdot d_*^2 \varphi / d_* s^{2*}, \quad (47)$$

$$d_*^3 \beta / d_* s^{3*} = A \cdot d_*^3 \varphi / d_* s^{3*}, \quad (48)$$

$$d_*^4 \beta / d_* s^{4*} = A \cdot d_*^4 \varphi / d_* s^{4*}. \quad (49)$$

Moreover, from the properties of the multiplicative determinant, we obtain the following expressions:

$$\begin{aligned} & [d_* \beta / d_* s \quad d_*^2 \beta / d_* s^{2*} \quad d_*^3 \beta / d_* s^{3*}]_* \\ &= [A \cdot d_* \varphi / d_* s \quad A \cdot d_*^2 \varphi / d_* s^{2*} \quad A \cdot d_*^3 \varphi / d_* s^{3*}]_* \\ &= [d_* \varphi / d_* s \quad d_*^2 \varphi / d_* s^{2*} \quad d_*^3 \varphi / d_* s^{3*}]_* \\ &= 1_* \cdot (\text{ or } e) \end{aligned} \quad (50)$$

This last equation implies that β is parameterized by multiplicative equiaffine arc length.

If Definition 4 is considered, the following equalities are found between the curvatures and torsions of these two curves:

$$\begin{aligned}\kappa_*^\beta(s) &= -_* \begin{bmatrix} d_*^2 \beta / d_* s^2 & d_*^3 \beta / d_* s^3 & d_*^4 \beta / d_* s^4 \end{bmatrix}_* \\ &= -_* \begin{bmatrix} d_*^2 \varphi / d_* s^2 & d_*^3 \varphi / d_* s^3 & d_*^4 \varphi / d_* s^4 \end{bmatrix}_* \\ &= \kappa_*^\varphi(s),\end{aligned}\quad (51)$$

and

$$\begin{aligned}\tau_*^\beta(s) &= \begin{bmatrix} d_* \beta / d_* s & d_*^2 \beta / d_* s^2 & d_*^3 \beta / d_* s^3 \end{bmatrix}_* \\ &= \begin{bmatrix} d_* \varphi / d_* s & d_*^2 \varphi / d_* s^2 & d_*^3 \varphi / d_* s^3 \end{bmatrix}_* \\ &= \tau_*^\varphi(s).\end{aligned}\quad (52)$$

The proof is thus completed. $\square$

Now, based on the above explanations, we can provide the following three examples in the multiplicative equiaffine space.

First, consider the curve $\psi(s)$ in the multiplicative equiaffine space, given by multiplicative operations as

$$\psi(s) = \left(s, e^{1/2} \cdot_* e^{(\log s)^2}, e^{1/6} \cdot_* e^{(\log s)^3} \right), \quad (53)$$

or by classical operations as

$$\psi(s) = \left(s, e^{1/2(\log s)^2}, e^{1/6(\log s)^3} \right). \quad (54)$$

The curvatures $\kappa_*(s)$ and $\tau_*(s)$ of this curve are identically equal to 0_* or 1. The graph of this curve can be seen in Figure 1.

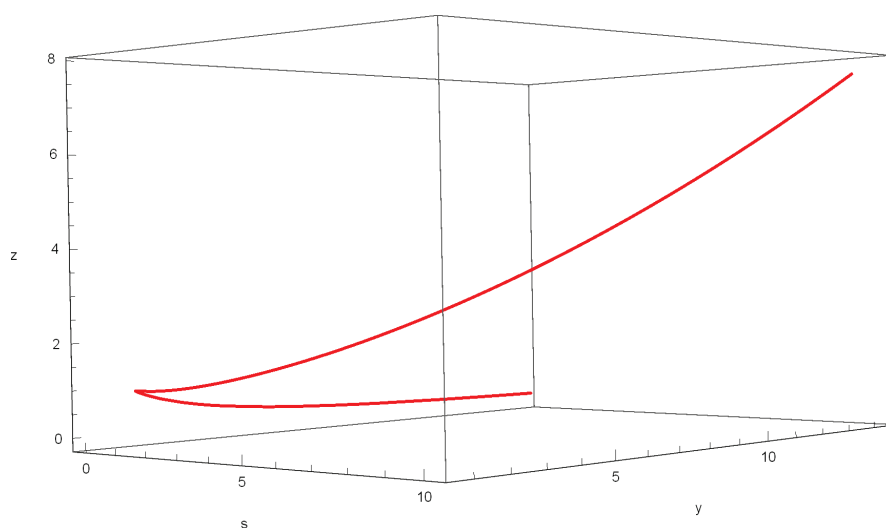


Figure 1. Graph of Equation (54).

Next, consider the curve $\Phi(s)$ given by

$$\Phi(s) = (-_* s, \sin_* s, \cos_* s), \quad (55)$$

or

$$\Phi(s) = \left(1/s, e^{\log(\sin s)}, e^{\log(\cos s)}\right), \quad (56)$$

where $\kappa_*(s) = 0_* = 1$ and $\tau_*(s) = 1_* = e$. The graph of this curve can be seen in Figure 2.

Lately, consider the curve $\Psi(s)$ given by

$$\Psi(s) = \left(e^{5/216} \cdot_* e^{1/s}, e^{\sqrt{s}} \cdot_* e^{\log(\sin \sqrt{s})}, e^{\sqrt{s}} \cdot_* e^{\log(\cos \sqrt{s})}\right), \quad (57)$$

or

$$\Psi(s) = \left(e^{\frac{5}{216s}}, e^{\sqrt{s} \log(\sin(\sqrt{s}))}, e^{\sqrt{s} \log(\cos(\sqrt{s}))}\right) \quad (58)$$

where $\kappa_*(s) = -_* e^{2/27} = e^{-2/27}$ and $\tau_*(s) = -_* e^{1/3} = e^{-1/3}$. The graph of this curve can be seen in Figure 3.

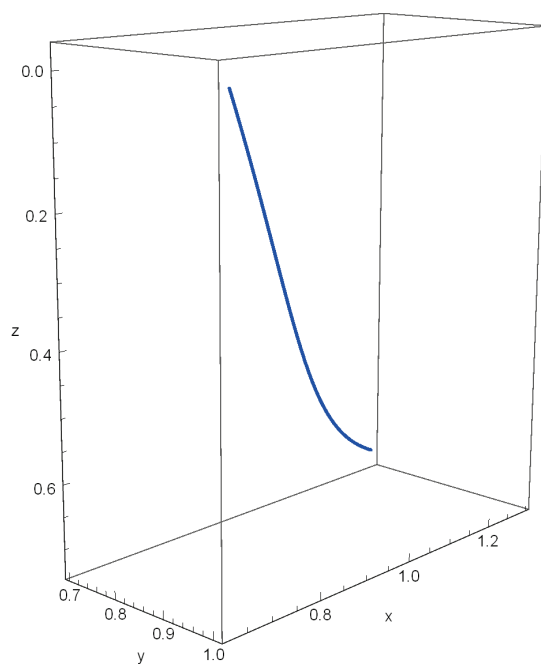


Figure 2. Graph of Equation (56).

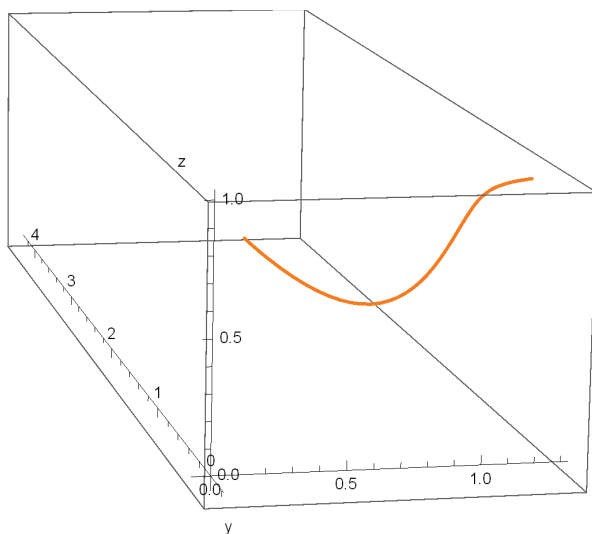


Figure 3. Graph of Equation (58).

4. Conclusions

This study introduces a comprehensive framework for analyzing curves in multiplicative equiaffine space by integrating concepts from multiplicative calculus and equiaffine differential geometry. Key contributions include the definitions of multiplicative equiaffine arc length, curvature, and torsion, alongside the development of the multiplicative Frenet apparatus and corresponding Frenet equations. Through these tools, the study systematically classifies space curves with constant multiplicative curvature and torsion, providing new insights into the geometric behavior of these curves.

The presented examples demonstrate how multiplicative calculus extends classical geometric concepts, showcasing its potential to generalize and enhance traditional methodologies. These advancements contribute not only to the theoretical development of affine geometry but also highlight potential applications in fields such as physics, engineering, and mathematical modeling, particularly in scenarios involving exponential growth, scaling phenomena, or non-Newtonian frameworks.

Future research could explore the broader applicability of this framework, examining its integration with other geometric and physical systems and its potential to address complex real-world problems.

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Article

Invariant Geometric Objects of the Equitorsion Canonical Biholomorphically Projective Mappings of Generalized Riemannian Space in the Eisenhart Sense

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Abstract: The study of the equitorsion biholomorphically projective mappings between two generalized Riemannian spaces in the sense of Eisenhart's definition is continued. Some new invariant geometric objects of an equitorsion canonical biholomorphically projective mapping are found, as well as some relations between these objects. At the end, the linear independence of the obtained invariants is examined.

Keywords: equitorsion quasi-canonical biholomorphically projective mapping; generalized Riemannian space; invariant geometric object; linear independence

MSC: 53B05; 53B20; 53C15

1. Introduction

Differentiable manifolds $\mathbb{G}\mathbb{R}_N$ with a non-symmetric metric tensor, $\mathbb{G}\mathbb{A}_N$, and non-symmetric affine connection and their mappings are of interest to many scientists [1–10]. Although the notion of non-symmetric affine connection was used in several works before A. Einstein, for example in [11], the use of non-symmetric connection became especially evident after the appearance of the works of A. Einstein, relating to create his Unified Field Theory [12,13], where the symmetric part of a basic tensor is related to gravitation and the anti-symmetric one to electromagnetism. Since 1951, Eisenhart was occupied with problems of spaces with non-symmetric basic tensors and non-symmetric connection in several works [14]. He defined a generalized Riemannian space as “a space of coordinates with which is associated a non-symmetric tensor g_{ij} ”, and the connection coefficients are defined by g_{ij} . In 1952, Eisenhart obtained two curvature tensors in generalized Riemannian space, using the fact that the connection coefficients are non-symmetric.

Geometric mappings are interesting and significant, both theoretically and practically. Geodesic and almost geodesic lines are important for geometry and physics. Geodesic mappings and their generalizations, like F -planar and holomorphically projective mappings, have been considered in many papers [1,4,6,9,10,15–17]. As a kind of generalization of holomorphically projective mappings, biholomorphically projective mappings were defined and considered in the paper [18].

In the sense of Eisenhart's definition, an N -dimensional manifold $\mathcal{M}_N$ equipped with a non-symmetric metric tensor $\hat{g}$, whose components are g_{ij} such that $g_{ij} \neq g_{ji}$ for at least

one pair of indices (i, j) , is a generalized (pseudo-)Riemannian space $\mathbb{G}\mathbb{R}_N$. The symmetric and anti-symmetric parts of the metric $\hat{g}$ are

$$g_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) \quad \text{and} \quad g_{ij} = \frac{1}{2}(g_{ij} - g_{ji}). \quad (1)$$

We also assume that $\det [g_{ij}] \neq 0$, which we use to define a contravariant symmetric metric tensor by components such as $[g^{ij}] = [g_{ij}]^{-1}$. The affine connection coefficients of the space $\mathbb{G}\mathbb{R}_N$ are generalized Christoffel symbols Γ_{jk}^i given by [18],

$$\Gamma_{jk}^i = g^{ip} \Gamma_{p,jk},$$

where $\Gamma_{i,jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j})$, where, for example, $g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}$. Generally, $\Gamma_{jk}^i \neq \Gamma_{kj}^i$, and the symmetric and anti-symmetric parts of Γ_{jk}^i , respectively, are given by the formulas

$$\Gamma_{jk}^i = \frac{1}{2}(\Gamma_{jk}^i + \Gamma_{kj}^i), \quad \Gamma_{jk}^i = \frac{1}{2}(\Gamma_{jk}^i - \Gamma_{kj}^i).$$

The magnitude Γ_{jk}^i is the torsion tensor of the space $\mathbb{G}\mathbb{R}_N$.

In the case of generalized Riemannian space $\mathbb{G}\mathbb{R}_N$, the following curvature tensors are linearly independent [19]:

$$R_{1jmn}^i = R_{jmn}^i + \Gamma_{jm|n}^i - \Gamma_{jn|m}^i + \Gamma_{jm}^p \Gamma_{pn}^i - \Gamma_{jn}^p \Gamma_{pm}^i, \quad (2)$$

$$R_{2jmn}^i = R_{jmn}^i - \Gamma_{jm|n}^i + \Gamma_{jn|m}^i + \Gamma_{jm}^p \Gamma_{pn}^i - \Gamma_{jn}^p \Gamma_{pm}^i, \quad (3)$$

$$R_{3jmn}^i = R_{jmn}^i + \Gamma_{jm|n}^i + \Gamma_{jn|m}^i - \Gamma_{jm}^p \Gamma_{pn}^i + \Gamma_{jn}^p \Gamma_{pm}^i - 2\Gamma_{mn}^p \Gamma_{pj}^i, \quad (4)$$

$$R_{4jmn}^i = R_{jmn}^i + \Gamma_{jm|n}^i + \Gamma_{jn|m}^i - \Gamma_{jm}^p \Gamma_{pn}^i + \Gamma_{jn}^p \Gamma_{pm}^i + 2\Gamma_{mn}^p \Gamma_{pj}^i, \quad (5)$$

$$R_{5jmn}^i = R_{jmn}^i - \Gamma_{jm}^p \Gamma_{pn}^i + \Gamma_{jn}^p \Gamma_{pm}^i, \quad (6)$$

$$R_{6jmn}^i = R_{jmn}^i + \Gamma_{jm}^p \Gamma_{pn}^i + \Gamma_{jn}^p \Gamma_{pm}^i, \quad (7)$$

where “|” denotes a covariant derivative with respect to the symmetric affine connection Γ_{jk}^i , whose curvature tensor is R_{jmn}^i .

The family of curvature tensors of the space $\mathbb{G}\mathbb{R}_N$ is

$$K_{jmn}^i = R_{jmn}^i + u\Gamma_{jm|n}^i + u'\Gamma_{jn|m}^i + v\Gamma_{jm}^p \Gamma_{pn}^i + v'\Gamma_{jn}^p \Gamma_{pm}^i + w\Gamma_{mn}^p \Gamma_{pj}^i, \quad (8)$$

for scalars u, u', v, v' , and w .

The linearly independent curvature tensors (2)–(7) are extracted from the family (8) by selecting the following constants: (u, u', v, v', w) :

$$(u, u', v, v', w) \in \{(1, -1, 1, -1, 0), (-1, 1, 1, -1, 0), (1, 1, -1, 1, -2), \\ (1, 1, -1, 1, 2), (1, 0, 0, -1, 1, 0), (1, 0, 0, 1, 1, 0)\}. \quad (9)$$

Remark 1. Let us obtain invariants for a geodesic mapping $f: \mathbb{R}_N \rightarrow \bar{\mathbb{R}}_N$ in the manner proposed by H. Weyl [20]. The basic equation of this mapping is

$$\bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \psi_k \delta_j^i + \psi_j \delta_k^i. \quad (10)$$

From this basic equation, one finds the transformation rule of curvature tensor R_{jmn}^i to $\bar{R}_{jmn}^i$,

$$\bar{R}_{jmn}^i = R_{jmn}^i + \delta_j^i (\psi_{m|n} - \psi_{n|m}) + \delta_m^i \psi_{j|n} - \delta_n^i \psi_{j|m} - \delta_m^i \psi_j \psi_n + \delta_n^i \psi_j \psi_m. \quad (11)$$

If the exchange $\psi_{ij} = \psi_{i|j} - \psi_i \psi_j$ is involved in Equation (11), we will transform it to

$$\bar{R}_{jmn}^i = R_{jmn}^i + \delta_j^i (\psi_{mn} - \psi_{nm}) + \delta_m^i \psi_{jn} - \delta_n^i \psi_{jm}. \quad (12)$$

After contracting the relation (12) by i and j , and by i and m , and with respect to the symmetry $R_{ij} = R_{ji}$ in Riemannian space, such as $R_{pmn}^p = R_{nm} - R_{mn}$, we obtain

$$\psi_{ij} = \psi_{ji} = -\frac{1}{N-1} (\bar{R}_{ij} - R_{ij}). \quad (13)$$

After substituting the expression (13) into (12), we obtain

$$\bar{R}_{jmn}^i = R_{jmn}^i - \frac{1}{N-1} (\delta_m^i \bar{R}_{jn} - \delta_n^i \bar{R}_{jm}) + \frac{1}{N-1} (\delta_m^i R_{jn} - \delta_n^i R_{jm}), \quad (14)$$

From Equation (14), we obtain that the Weyl projective tensor

$$W_{jmn}^i = R_{jmn}^i + \frac{1}{N-1} (\delta_m^i R_{jn} - \delta_n^i R_{jm})$$

is an invariant for the geodesic mapping f . The traces of W_{jmn}^i are $W_{pmn}^p = 0$, $W_{jpn}^p = 0$, and $W_{jnp}^p = 0$.

For this reason, there are two problems:

1. We do not have an invariant for the geodesic mapping that does not neglect the Ricci tensor R_{ij} after contraction.
2. If there are not Kronecker delta-symbols in the transformation rule of the curvature tensor under an analyzed mapping, we are not able to apply Weyl's method to obtain an invariant for the studied mapping.

These problems were solved in the paper [21] by obtaining general formulas for basic and derived invariants of a mapping between affine connection spaces. We will apply this methodology in this research and obtain invariants for a special mapping of generalized Riemannian space. Because the corresponding deformation tensor will not be a function of a Kronecker delta-symbol, we will be able to obtain just one invariant for the studied mapping, which is a function of the curvature tensor R_{jmn}^i . Unlike the Weyl projective tensor, the traces of the invariant that will be obtained in this paper will not vanish.

2. Review of Basic Invariants

One of the most important current problems in differential geometry is finding new invariant geometric objects. T. Y. Thomas [22] and H. Weyl [20] started the research about invariants of special diffeomorphisms between spaces with symmetric affine connection. Many authors have continued Thomas's and Weyl's works, for example V. Berezovski, J. Mikeš, E. Stepanova, etc. [1,2,5,15]. Also, some invariant geometrical objects for diffeomorphisms of spaces with non-symmetric affine connection are obtained in many papers, for example [9,16,17,23].

In this section, we will refer to the results presented in [21]. Namely, if $f : \mathbb{R}_N \rightarrow \bar{\mathbb{R}}_N$ is a mapping of Riemannian space $\mathbb{R}_N$ whose deformation tensor is

$$\bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \bar{\omega}_{jk}^i - \omega_{jk}^i, \quad (15)$$

for geometric objects $\bar{\omega}_{jk}^i$ and ω_{jk}^i symmetric by indices j and k , it is obtained that $\bar{\mathcal{T}}_{jk}^i = \mathcal{T}_{jk}^i$ and $\bar{\mathcal{W}}_{jmn}^i = \mathcal{W}_{jmn}^i$ for

$$\mathcal{T}_{jk}^i = \Gamma_{jk}^i + \omega_{jk}^i, \quad (16)$$

$$\mathcal{W}_{jmn}^i = R_{jmn}^i + \omega_{jm|n}^i - \omega_{jn|m}^i + \omega_{jm}^p \omega_{pn}^i - \omega_{jn}^p \omega_{pm}^i, \quad (17)$$

and the corresponding $\bar{\mathcal{T}}_{jk}^i$ and $\bar{\mathcal{W}}_{jmn}^i$. The geometrical objects $\mathcal{T}_{jk}^i$ and $\mathcal{W}_{jmn}^i$ are named the associated basic invariants of Thomas and Weyl type for the mapping f .

For an equitorsion mapping $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\bar{\mathbb{R}}_N$, the families curvature tensors (8) and

$$\bar{K}_{jmn}^i = \bar{R}_{jmn}^i + u \bar{\Gamma}_{jm|n}^i + u' \bar{\Gamma}_{jn|m}^i + v \bar{\Gamma}_{jm}^p \bar{\Gamma}_{pn}^i + v' \bar{\Gamma}_{jn}^p \bar{\Gamma}_{pm}^i + w \bar{\Gamma}_{mn}^p \bar{\Gamma}_{pj}^i,$$

for the covariant derivatives with respect to torsion-free affine connections Γ_{jk}^i and $\bar{\Gamma}_{jk}^i$, respectively, denoted by “|” and “||”, satisfy the relation

$$\begin{aligned} \bar{K}_{jmn}^i - K_{jmn}^i &= \bar{R}_{jmn}^i - R_{jmn}^i + u(\bar{\Gamma}_{pn}^i \bar{\Gamma}_{jm}^p - \Gamma_{pn}^i \Gamma_{jm}^p) - u(\bar{\Gamma}_{jn}^p \bar{\Gamma}_{pm}^i - \Gamma_{jn}^p \Gamma_{pm}^i) \\ &\quad - (u + u')(\bar{\Gamma}_{mn}^p \bar{\Gamma}_{jp}^i - \Gamma_{mn}^p \Gamma_{jp}^i) + u'(\bar{\Gamma}_{pm}^p \bar{\Gamma}_{jn}^i - \Gamma_{pm}^p \Gamma_{jn}^i) \\ &\quad - u'(\bar{\Gamma}_{jm}^p \bar{\Gamma}_{pn}^i - \Gamma_{jm}^p \Gamma_{pn}^i). \end{aligned} \quad (18)$$

Because the mapping f is equitorsion, the equality $\bar{\Gamma}_{jk}^i = \Gamma_{jk}^i$ holds. With this in mind, and the invariance of geometrical objects $\mathcal{T}_{jk}^i$ and $\mathcal{W}_{jmn}^i$, the equality $\bar{\mathcal{W}}_{jmn}^i = \mathcal{W}_{jmn}^i$ is satisfied for

$$\bar{\mathcal{W}}_{jmn}^i = \mathcal{W}_{jmn}^i + u \bar{\mathcal{T}}_{pn}^i \bar{\Gamma}_{jm}^p - u \bar{\mathcal{T}}_{jn}^p \bar{\Gamma}_{pm}^i - (u + u') \bar{\mathcal{T}}_{mn}^p \bar{\Gamma}_{jp}^i + u' \bar{\mathcal{T}}_{pm}^p \bar{\Gamma}_{jn}^i - u' \bar{\mathcal{T}}_{jm}^p \bar{\Gamma}_{pn}^i, \quad (19)$$

$$\bar{\mathcal{W}}_{jmn}^i = \mathcal{W}_{jmn}^i + u \mathcal{T}_{pn}^i \Gamma_{jm}^p - u \mathcal{T}_{jn}^p \Gamma_{pm}^i - (u + u') \mathcal{T}_{mn}^p \Gamma_{jp}^i + u' \mathcal{T}_{pm}^p \Gamma_{jn}^i - u' \mathcal{T}_{jm}^p \Gamma_{pn}^i. \quad (20)$$

After substituting the expressions of $\mathcal{T}_{jk}^i$ and $\mathcal{W}_{jmn}^i$ given by (16) and (17) and the corresponding expressions of $\bar{\mathcal{T}}_{jk}^i$ and $\bar{\mathcal{W}}_{jmn}^i$ into the equalities (19) and (20), respectively, one obtains

$$\begin{aligned} \bar{\mathcal{W}}_{jmn}^i &= \bar{R}_{jmn}^i + \bar{\omega}_{jm|n}^i - \bar{\omega}_{jn|m}^i + \bar{\omega}_{jm}^p \bar{\omega}_{pn}^i - \bar{\omega}_{jn}^p \bar{\omega}_{pm}^i \\ &\quad + u \bar{\Gamma}_{pn}^i \bar{\Gamma}_{jm}^p - u \bar{\Gamma}_{jn}^p \bar{\Gamma}_{pm}^i - (u + u') \bar{\Gamma}_{mn}^p \bar{\Gamma}_{jp}^i + u' \bar{\Gamma}_{pm}^p \bar{\Gamma}_{jn}^i - u' \bar{\Gamma}_{jm}^p \bar{\Gamma}_{pn}^i \\ &\quad + u \bar{\Gamma}_{jm}^p \bar{\omega}_{pn}^i - u \bar{\Gamma}_{pn}^p \bar{\omega}_{jm}^i - (u + u') \bar{\Gamma}_{jp}^p \bar{\omega}_{mn}^i + u' \bar{\Gamma}_{jn}^p \bar{\omega}_{pm}^i - u' \bar{\Gamma}_{pn}^p \bar{\omega}_{jm}^i, \end{aligned} \quad (21)$$

$$\begin{aligned} \bar{\mathcal{W}}_{jmn}^i &= R_{jmn}^i + \omega_{jm|n}^i - \omega_{jn|m}^i + \omega_{jm}^p \omega_{pn}^i - \omega_{jn}^p \omega_{pm}^i \\ &\quad + u \Gamma_{pn}^i \Gamma_{jm}^p - u \Gamma_{jn}^p \Gamma_{pm}^i - (u + u') \Gamma_{mn}^p \Gamma_{jp}^i + u' \Gamma_{pm}^p \Gamma_{jn}^i - u' \Gamma_{jm}^p \Gamma_{pn}^i \\ &\quad + u \Gamma_{jm}^p \omega_{pn}^i - u \Gamma_{pn}^p \omega_{jm}^i - (u + u') \Gamma_{jp}^p \omega_{mn}^i + u' \Gamma_{jn}^p \omega_{pm}^i - u' \Gamma_{pn}^p \omega_{jm}^i. \end{aligned} \quad (22)$$

The difference of geometrical objects $\widetilde{\mathcal{W}}_{jmn}^i$ and $\widetilde{\mathcal{W}}_{jmn}^i$ given by Formulas (21) and (22) is

$$\begin{aligned} 0 = \widetilde{\mathcal{W}}_{jmn}^i - \widetilde{\mathcal{W}}_{jmn}^i &= \bar{R}_{jmn}^i - R_{jmn}^i + u(\bar{\Gamma}_{\underline{pn}}^i \bar{\Gamma}_{jm}^p - \Gamma_{\underline{pn}}^i \Gamma_{jm}^p) - u(\bar{\Gamma}_{jn}^p \bar{\Gamma}_{pm}^i - \Gamma_{jn}^p \Gamma_{pm}^i) \\ &\quad - (u + u')(\bar{\Gamma}_{\underline{mn}}^p \bar{\Gamma}_{jp}^i - \Gamma_{\underline{mn}}^p \Gamma_{jp}^i) + u'(\bar{\Gamma}_{\underline{pm}}^i \bar{\Gamma}_{jn}^p - \Gamma_{\underline{pm}}^i \Gamma_{jn}^p) \\ &\quad - u'(\bar{\Gamma}_{jm}^p \bar{\Gamma}_{pn}^i - \Gamma_{jm}^p \Gamma_{pn}^i) + \bar{\omega}_{j[m|n]}^i - \omega_{j[m|n]}^i + \bar{\omega}_{j[m}^p \bar{\omega}_{pn]}^i - \omega_{j[m}^p \omega_{pn]}^i \\ &\quad + u\bar{\Gamma}_{jm}^p \bar{\omega}_{pn}^i - u\bar{\Gamma}_{pm}^p \bar{\omega}_{jn}^i - (u + u')\bar{\Gamma}_{jp}^i \bar{\omega}_{mn}^p + u'\bar{\Gamma}_{jn}^p \bar{\omega}_{pm}^i - u'\bar{\Gamma}_{pn}^i \bar{\omega}_{jm}^p \\ &\quad - u\Gamma_{jm}^p \omega_{pn}^i + u\Gamma_{pm}^p \omega_{jn}^i + (u + u')\Gamma_{jp}^i \omega_{mn}^p - u'\Gamma_{jn}^p \omega_{pm}^i + u'\Gamma_{pn}^i \omega_{jm}^p, \end{aligned} \quad (23)$$

where $[mn]$ is an anti-symmetrization without division by indices m and n .

The first seven summands in Equation (23) coincide with the right side of equality (18). Hence, the forthcoming equality holds,

$$\begin{aligned} 0 = \widetilde{\mathcal{W}}_{jmn}^i - \widetilde{\mathcal{W}}_{jmn}^i &= \bar{K}_{jmn}^i - K_{jmn}^i + \bar{\omega}_{j[m|n]}^i - \omega_{j[m|n]}^i + \bar{\omega}_{j[m}^p \bar{\omega}_{pn]}^i - \omega_{j[m}^p \omega_{pn]}^i \\ &\quad + u\bar{\Gamma}_{jm}^p \bar{\omega}_{pn}^i - u\bar{\Gamma}_{pm}^p \bar{\omega}_{jn}^i - (u + u')\bar{\Gamma}_{jp}^i \bar{\omega}_{mn}^p + u'\bar{\Gamma}_{jn}^p \bar{\omega}_{pm}^i - u'\bar{\Gamma}_{pn}^i \bar{\omega}_{jm}^p \\ &\quad - u\Gamma_{jm}^p \omega_{pn}^i + u\Gamma_{pm}^p \omega_{jn}^i + (u + u')\Gamma_{jp}^i \omega_{mn}^p - u'\Gamma_{jn}^p \omega_{pm}^i + u'\Gamma_{pn}^i \omega_{jm}^p, \end{aligned} \quad (24)$$

for the families K_{jmn}^i and $\bar{K}_{jmn}^i$ of the curvature tensors of the spaces $\mathbb{GR}_N$ and $\mathbb{GR}_N$.

From the equality (24), we obtain the expressions of the geometrical objects $\widetilde{\mathcal{W}}_{jmn}^i$ and $\widetilde{\mathcal{W}}_{jmn}^i$ as follows:

$$\begin{aligned} \widetilde{\mathcal{W}}_{jmn}^i &= \bar{K}_{jmn}^i + \bar{\omega}_{jm|n}^i - \bar{\omega}_{jn|m}^i + \bar{\omega}_{jm}^p \bar{\omega}_{pn}^i - \bar{\omega}_{jn}^p \bar{\omega}_{pm}^i \\ &\quad + u\bar{\Gamma}_{jm}^p \bar{\omega}_{pn}^i - u\bar{\Gamma}_{pm}^p \bar{\omega}_{jn}^i - (u + u')\bar{\Gamma}_{jp}^i \bar{\omega}_{mn}^p + u'\bar{\Gamma}_{jn}^p \bar{\omega}_{pm}^i - u'\bar{\Gamma}_{pn}^i \bar{\omega}_{jm}^p, \end{aligned} \quad (25)$$

$$\begin{aligned} \widetilde{\mathcal{W}}_{jmn}^i &= K_{jmn}^i + \omega_{jm|n}^i - \omega_{jn|m}^i + \omega_{jm}^p \omega_{pn}^i - \omega_{jn}^p \omega_{pm}^i \\ &\quad - u\Gamma_{jm}^p \omega_{pn}^i + u\Gamma_{pm}^p \omega_{jn}^i + (u + u')\Gamma_{jp}^i \omega_{mn}^p - u'\Gamma_{jn}^p \omega_{pm}^i + u'\Gamma_{pn}^i \omega_{jm}^p. \end{aligned} \quad (26)$$

The next theorem holds.

Theorem 1. Let $f : \mathbb{GR}_N \rightarrow \mathbb{GR}_N$ be an equitorsion mapping of a generalized Riemannian space $\mathbb{GR}_N$. The geometrical objects $\widetilde{\mathcal{W}}_{jmn}^i$ given by Equation (22) are invariants for the mapping f expressed in terms of the curvature tensor of the associated space $\mathbb{R}_N$.

The geometrical objects $\widetilde{\mathcal{W}}_{jmn}^i$ given by Equation (26) are invariants for the mapping f expressed in terms of the curvature tensors of the space $\mathbb{GR}_N$.

The geometrical objects $\widetilde{\mathcal{W}}_{jmn}^i$ are equitorsion invariants for the mapping f .

Because there are six linearly independent curvature tensors [19], the next corollaries are satisfied.

Corollary 1. Let $f : \mathbb{GR}_N \rightarrow \mathbb{GR}_N$ be an equitorsion mapping of Riemannian space $\mathbb{GR}_N$. Three of the invariants for this mapping, given by Equation (22), are linearly independent.

Corollary 2. Let $f : \mathbb{GR}_N \rightarrow \mathbb{GR}_N$ be an equitorsion mapping of Riemannian space $\mathbb{GR}_N$. Six of the invariants for this mapping, given by Equation (26), are linearly independent.

Corollary 3. Let $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\mathbb{R}_N$ be an equitorsion mapping of Riemannian space $\mathbb{G}\mathbb{R}_N$. The geometrical objects $\widetilde{\mathcal{W}}_{ij} = \widetilde{\mathcal{W}}_{ij}^p$, whose components are

$$\begin{aligned} \widetilde{\mathcal{W}}_{ij} = & R_{ij} + \omega_{ij|p}^p - \omega_{ip|j}^p + \omega_{ij}^q \omega_{qp}^p - \omega_{iq}^p \omega_{jp}^q \\ & + u \Gamma_{pq}^q \Gamma_{ij}^p + u \Gamma_{ip}^q \Gamma_{jq}^p - (u + u') \Gamma_{jp}^q \Gamma_{iq}^p + u' \Gamma_{jp}^q \Gamma_{iq}^p \\ & + u \Gamma_{ij}^p \omega_{pq}^q + u \Gamma_{jq}^p \omega_{ip}^q - (u + u') \Gamma_{iq}^p \omega_{jp}^q + u' \Gamma_{iq}^p \omega_{jp}^q, \end{aligned} \quad (27)$$

are invariants for the mapping f .

Remark 2. When we express the invariant $\widetilde{\mathcal{W}}_{jmn}^i$ in terms of curvature tensors of the space $\mathbb{G}\mathbb{R}_N$, we involve summands like $\Gamma_{jk}^i \Gamma_{qr}^p$. These summands are invariants for the equitorsion mapping f , and they make six linearly independent invariants. These summands are extracted from the invariant $\widetilde{\mathcal{W}}_{jmn}^i$ expressed in terms of the curvature tensor of the associated space $\mathbb{R}_N$ and other non-invariant objects. For this reason, we will obtain invariants without including other invariants as separate summands.

3. Invariants for Equitorsion Canonical Biholomorphically Projective Mappings

Let $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\mathbb{R}_N$ be two generalized Riemannian spaces. We will observe these spaces in the common system of coordinates defined by the mapping $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\mathbb{R}_N$. If Γ_{jk}^i and $\bar{\Gamma}_{jk}^i$ are connection coefficients of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\mathbb{R}_N$, respectively, then $P_{jk}^i = \bar{\Gamma}_{jk}^i - \Gamma_{jk}^i$ is the deformation tensor of the connection for a mapping f [18].

In the paper [18], we defined biholomorphically projective mappings between two generalized Riemannian spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\mathbb{R}_N$ with almost complex structures that are equal in a common system of coordinates defined by the mapping $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\mathbb{R}_N$. We considered a generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ with a non-symmetric metric tensor g_{ij} and almost complex structure F_j^i such that $F_j^i \neq a \delta_j^i$, where a is a scalar invariant, and we defined a biholomorphically projective curve of the kind θ ($\theta = 1, 2$) and a biholomorphically projective mapping of the kind θ ($\theta = 1, 2$).

Definition 1 ([18]). In the space $\mathbb{G}\mathbb{R}_N$, a curve l given in parametric form

$$x^i = x^i(t), \quad (i = 1, \dots, N),$$

is said to be biholomorphically projective of the kind θ ($\theta = 1, 2$) if it satisfies the following equation:

$$\lambda_{|p}^h(t) \lambda^p(t) = a(t) \lambda^h(t) + b(t) F_p^h \lambda^p(t) + c(t) \bar{F}_p^h \lambda^p(t),$$

where a, b, c are functions of parameter t , where $\lambda^i = \frac{dx^i}{dt}$, $\bar{F}_p^h = F_q^h F_p^q$.

Definition 2 ([18]). A diffeomorphism $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\mathbb{R}_N$ is a biholomorphically projective mapping of the kind θ ($\theta = 1, 2$) if biholomorphically projective curves of the kind θ ($\theta = 1, 2$) of the space $\mathbb{G}\mathbb{R}_N$ are mapped to the biholomorphically projective curves of the kind θ of the space $\mathbb{G}\mathbb{R}_N$.

After defining, we concluded that the biholomorphically projective curves of the first kind and the biholomorphically projective curves of the second kind match, so we simply named them the biholomorphically projective curves, and the corresponding mappings are named biholomorphically projective mappings.

Also, the mapping $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{R}}_N$ is a biholomorphically projective mapping if in the common coordinate system the connection coefficients Γ_{ij}^h and $\overline{\Gamma}_{ij}^h$ satisfy the relation [18]

$$\overline{\Gamma}_{jk}^i = \Gamma_{jk}^i + \psi_{(j}\delta_{k)}^i + \sigma_{(j}F_{k)}^i + \tau_{(j}\overline{F}_{k)}^i + \xi_{jk}^i,$$

and the deformation tensor has the form [18]

$$P_{jk}^i = \psi_{(j}\delta_{k)}^i + \sigma_{(j}F_{k)}^i + \tau_{(j}\overline{F}_{k)}^i + \xi_{jk}^i, \quad (28)$$

where (ij) is a symmetrization without division by indices i, j , ψ_i , σ_i , and τ_i are vectors, $\overline{F}_j^i = F_q^i F_j^q$, and ξ_{jk}^i is an anti-symmetric tensor. Inspired by the form of the deformation tensor (28), we have defined another type of mapping.

Let F_j^i and $\overline{F}_j^i$ be almost complex structures of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$, respectively, where $F_j^i = \overline{F}_j^i$ in the common system of coordinates defined by the mapping $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{R}}_N$, and assume that it holds $F_j^i \neq a\delta_j^i$, where a is a scalar invariant. The mapping $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{R}}_N$ is a canonical biholomorphically projective mapping if in the common coordinate system the connection coefficients Γ_{jk}^i and $\overline{\Gamma}_{jk}^i$ satisfy the relation [24]

$$\overline{\Gamma}_{jk}^i = \Gamma_{jk}^i + \sigma_{(j}F_{k)}^i + \tau_{(j}\overline{F}_{k)}^i + \xi_{jk}^i, \quad (29)$$

where (jk) is a symmetrization without division by indices j and k , σ_j and τ_j are vectors, $\overline{F}_j^i = F_p^i F_j^p$, and ξ_{jk}^i is an anti-symmetric tensor. If P_{jk}^i is a deformation tensor with respect to the canonical biholomorphically projective mapping $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{R}}_N$, then we have [24]

$$P_{jk}^i = \sigma_{(j}F_{k)}^i + \tau_{(j}\overline{F}_{k)}^i + \xi_{jk}^i. \quad (30)$$

The mapping $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{R}}_N$ is an equitortion canonical biholomorphically projective mapping if the torsion tensors of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ are equal in a common coordinate system after the mapping f . Then, from Ref. [24],

$$\xi_{jk}^i = 0. \quad (31)$$

In this case, the relation (30) becomes [24]

$$P_{jk}^i = \sigma_{(j}F_{k)}^i + \tau_{(j}\overline{F}_{k)}^i. \quad (32)$$

As a special case of equitortion canonical biholomorphically projective mappings, the equitortion mapping $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{R}}_N$, whose deformation tensor is

$$P_{jk}^i = \overline{\Gamma}_{jk}^i - \Gamma_{jk}^i = -\tau_p F_j^p F_k^i - \tau_p F_k^p F_j^i + \tau_j \overline{F}_k^i + \tau_k \overline{F}_j^i, \quad (33)$$

was studied. In this case, an invariant of Thomas type [24]

$$\mathcal{CH}\mathcal{T}_{jk}^i = \Gamma_{jk}^i - \frac{1}{e} \left(\Gamma_{pq}^q F_j^p F_k^i + \Gamma_{pq}^q F_k^p F_j^i - \Gamma_{jp}^p \bar{F}_k^i - \Gamma_{kp}^p \bar{F}_j^i \right), \quad (34)$$

is obtained for $e = \pm 1$, assuming that δ_j^i , F_j^i , and $\bar{F}_j^i$ are linearly independent.

The invariant $\mathcal{CH}\mathcal{T}_{jk}^i$ is the associated basic invariant for mapping f . From this invariant, one reads

$$\omega_{jk}^i = -\frac{1}{e} \left(\Gamma_{pq}^q F_j^p F_k^i + \Gamma_{pq}^q F_k^p F_j^i - \Gamma_{jp}^p \bar{F}_k^i - \Gamma_{kp}^p \bar{F}_j^i \right). \quad (35)$$

We will express the previous ω_{jk}^i in some other form, better for further computing

$$\omega_{jk}^i = -\frac{1}{e} \Gamma_{pq}^q \left(F_j^p F_k^i + F_k^p F_j^i - \delta_j^p \bar{F}_k^i - \delta_k^p \bar{F}_j^i \right) = -\frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{jk}^{pi}, \quad (36)$$

for

$$\mathcal{F}_{jk}^{pi} = F_j^p F_k^i + F_k^p F_j^i - \delta_j^p \bar{F}_k^i - \delta_k^p \bar{F}_j^i. \quad (37)$$

After substituting the expression (36) into Equations (17) and (22), assuming that F is covariantly constant, we obtain the following results:

$$\mathcal{W}_{jmn}^i = R_{jmn}^i + \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{jn}^{pi} + \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{jm}^{pi} + \Gamma_{aq}^q \Gamma_{bs}^s \mathcal{F}_{j[m}^{ap} \mathcal{F}_{pn]}^{bi} \quad (38)$$

$$\begin{aligned} \mathcal{W}_{ij} = R_{ij} + \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{ir}^{pr} - \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{ij}^{pr} + \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{ij|r}^{pr} - \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{ir|j}^{pr} \\ + \Gamma_{aq}^q \Gamma_{bs}^s \mathcal{F}_{ij}^{ap} \mathcal{F}_{pr}^{br} - \Gamma_{aq}^q \Gamma_{bs}^s \mathcal{F}_{ir}^{ap} \mathcal{F}_{pj}^{br}, \end{aligned} \quad (39)$$

$$\begin{aligned} \widetilde{\mathcal{W}}_{jmn}^i = R_{jmn}^i + \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{jn}^{pi} + \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{jm}^{pi} + \Gamma_{aq}^q \Gamma_{bs}^s \mathcal{F}_{j[m}^{ap} \mathcal{F}_{pn]}^{bi} \\ + u \Gamma_{pn}^i \Gamma_{jm}^p - u \Gamma_{jn}^p \Gamma_{pm}^i - (u + u') \Gamma_{mn}^p \Gamma_{jp}^i + u' \Gamma_{pm}^i \Gamma_{jn}^p - u' \Gamma_{jm}^p \Gamma_{pn}^i - \frac{u}{e} \Gamma_{jm}^p \Gamma_{rq}^q \mathcal{F}_{pn}^{ri} \\ + \frac{u}{e} \Gamma_{pm}^i \Gamma_{rq}^q \mathcal{F}_{jn}^{rp} + \frac{u + u'}{e} \Gamma_{jp}^i \Gamma_{rq}^q \mathcal{F}_{mn}^{rp} - \frac{u'}{e} \Gamma_{jn}^p \Gamma_{rq}^q \mathcal{F}_{pm}^{ri} + \frac{u'}{e} \Gamma_{pn}^i \Gamma_{rq}^q \mathcal{F}_{jm}^{rp}, \end{aligned} \quad (40)$$

$$\begin{aligned} \widetilde{\mathcal{W}}_{ij} = R_{ij} + \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{ir}^{pr} - \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{ij}^{pr} + \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{ij|r}^{pr} - \frac{1}{e} \Gamma_{pq}^q \mathcal{F}_{ir|j}^{pr} \\ + \Gamma_{aq}^q \Gamma_{bs}^s \mathcal{F}_{ij}^{ap} \mathcal{F}_{pr}^{br} - \Gamma_{aq}^q \Gamma_{bs}^s \mathcal{F}_{ir}^{ap} \mathcal{F}_{pj}^{br} + u \Gamma_{pq}^q \Gamma_{ij}^p + u \Gamma_{iq}^p \Gamma_{jp}^q - u \Gamma_{jq}^p \Gamma_{ip}^q \\ - \frac{u}{e} \Gamma_{ij}^p \Gamma_{rq}^q \mathcal{F}_{ps}^{rs} + \frac{u}{e} \Gamma_{pj}^s \Gamma_{rq}^q \mathcal{F}_{is}^{rp} + \frac{u + u'}{e} \Gamma_{ip}^s \Gamma_{rq}^q \mathcal{F}_{js}^{rp} - \frac{u'}{e} \Gamma_{is}^p \Gamma_{rq}^q \mathcal{F}_{pj}^{rs}. \end{aligned} \quad (41)$$

Accordingly, we conclude that the following theorem is valid.

Theorem 2. Let $f : \mathbb{GR}_N \rightarrow \mathbb{GR}_N$ be an equitortion canonical biholomorphically projective mapping that preserves the structure F , which is covariantly constant according to the covariant derivative with respect to torsion-free affine connection. The geometrical objects $\mathcal{W}_{jmn}^i$ and $\widetilde{\mathcal{W}}_{jmn}^i$, given by (38) and (40), respectively, are the associated basic invariant and equitortion invariant for the mapping f .

Corollary 4. The invariants $\mathcal{W}_{jmn}^i$ and $\widetilde{\mathcal{W}}_{jmn}^i$ satisfy the equation

$$\begin{aligned}\widetilde{\mathcal{W}}_{jmn}^i &= \mathcal{W}_{jmn}^i + u\Gamma_{pn}^i\Gamma_{jm}^p - u\Gamma_{jn}^p\Gamma_{pm}^i - (u + u')\Gamma_{mn}^p\Gamma_{jp}^i + u'\Gamma_{pm}^i\Gamma_{jn}^p - u'\Gamma_{jm}^p\Gamma_{pn}^i \\ &\quad - \frac{u}{e}\Gamma_{jm}^p\Gamma_{rq}^q\mathcal{F}_{pn}^{ri} + \frac{u}{e}\Gamma_{pm}^i\Gamma_{rq}^q\mathcal{F}_{jn}^{rp} + \frac{u + u'}{e}\Gamma_{jp}^i\Gamma_{rq}^q\mathcal{F}_{mn}^{rp} - \frac{u'}{e}\Gamma_{jn}^p\Gamma_{rq}^q\mathcal{F}_{pm}^{ri} + \frac{u'}{e}\Gamma_{pn}^i\Gamma_{rq}^q\mathcal{F}_{jm}^{rp}.\end{aligned}\quad (42)$$

The invariants $\mathcal{W}_{ij}$ and $\widetilde{\mathcal{W}}_{ij}$ satisfy the equation

$$\begin{aligned}\widetilde{\mathcal{W}}_{ij} &= \mathcal{W}_{ij} + \Gamma_{aq}^q\Gamma_{bs}^s\mathcal{F}_{ij}^{ap}\mathcal{F}_{pr}^{br} - \Gamma_{aq}^q\Gamma_{bs}^s\mathcal{F}_{ir}^{ap}\mathcal{F}_{pj}^{br} + u\Gamma_{pq}^q\Gamma_{ij}^p + u\Gamma_{iq}^p\Gamma_{jp}^q - u\Gamma_{jq}^p\Gamma_{ip}^q \\ &\quad - \frac{u}{e}\Gamma_{ij}^p\Gamma_{rq}^q\mathcal{F}_{ps}^{rs} + \frac{u}{e}\Gamma_{pj}^s\Gamma_{rq}^q\mathcal{F}_{is}^{rp} + \frac{u + u'}{e}\Gamma_{ip}^s\Gamma_{rq}^q\mathcal{F}_{js}^{rp} - \frac{u'}{e}\Gamma_{is}^p\Gamma_{rq}^q\mathcal{F}_{pj}^{rs}.\end{aligned}\quad (43)$$

The invariants $\widetilde{\mathcal{W}}_{jmn}^i$, $\mathcal{W}_{jmn}^i$, $\widetilde{\mathcal{W}}_{ij}$, and $\mathcal{W}_{ij}$ are tensors.

Linearly Independent Invariants

The family of invariants $\widetilde{\mathcal{W}}_{jmn}^i$ given by (42) depends u and u' , and none of the scalars present in the family of curvature tensors (8) feature. After changing the corresponding scalars u and u' from linearly independent curvature tensors into the Equation (41), we will obtain the following invariants:

$$\begin{aligned}\widetilde{\mathcal{W}}_1^i &= R_{jmn}^i + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jm}^{pi} + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jn}^{pi} + \Gamma_{aq}^q\Gamma_{bs}^s\mathcal{F}_{j[m}^{ap}\mathcal{F}_{pn]}^{bi} \\ &\quad + \Gamma_{pn}^i\Gamma_{jm}^p - \Gamma_{jn}^p\Gamma_{pm}^i - \Gamma_{pm}^i\Gamma_{jn}^p + \Gamma_{jm}^p\Gamma_{pn}^i - \frac{1}{e}\Gamma_{jm}^p\Gamma_{rq}^q\mathcal{F}_{pn}^{ri} \\ &\quad + \frac{1}{e}\Gamma_{pm}^i\Gamma_{rq}^q\mathcal{F}_{jn}^{rp} + \frac{1}{e}\Gamma_{jn}^p\Gamma_{rq}^q\mathcal{F}_{pm}^{ri} - \frac{1}{e}\Gamma_{pn}^i\Gamma_{rq}^q\mathcal{F}_{jm}^{rp},\end{aligned}\quad (44)$$

$$\begin{aligned}\widetilde{\mathcal{W}}_2^i &= R_{jmn}^i + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jm}^{pi} + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jn}^{pi} + \Gamma_{aq}^q\Gamma_{bs}^s\mathcal{F}_{j[m}^{ap}\mathcal{F}_{pn]}^{bi} \\ &\quad - \Gamma_{pn}^i\Gamma_{jm}^p + \Gamma_{jn}^p\Gamma_{pm}^i + \Gamma_{pm}^i\Gamma_{jn}^p - \Gamma_{jm}^p\Gamma_{pn}^i + \frac{1}{e}\Gamma_{jm}^p\Gamma_{rq}^q\mathcal{F}_{pn}^{ri} \\ &\quad - \frac{1}{e}\Gamma_{pm}^i\Gamma_{rq}^q\mathcal{F}_{jn}^{rp} - \frac{1}{e}\Gamma_{jn}^p\Gamma_{rq}^q\mathcal{F}_{pm}^{ri} + \frac{1}{e}\Gamma_{pn}^i\Gamma_{rq}^q\mathcal{F}_{jm}^{rp},\end{aligned}\quad (45)$$

$$\begin{aligned}\widetilde{\mathcal{W}}_3^i &= R_{jmn}^i + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jm}^{pi} + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jn}^{pi} + \Gamma_{aq}^q\Gamma_{bs}^s\mathcal{F}_{j[m}^{ap}\mathcal{F}_{pn]}^{bi} \\ &\quad + \Gamma_{pn}^i\Gamma_{jm}^p - \Gamma_{jn}^p\Gamma_{pm}^i - 2\Gamma_{mn}^p\Gamma_{jp}^i + \Gamma_{pm}^i\Gamma_{jn}^p - \Gamma_{jm}^p\Gamma_{pn}^i - \frac{1}{e}\Gamma_{jm}^p\Gamma_{rq}^q\mathcal{F}_{pn}^{ri} \\ &\quad + \frac{1}{e}\Gamma_{pm}^i\Gamma_{rq}^q\mathcal{F}_{jn}^{rp} + \frac{2}{e}\Gamma_{jp}^i\Gamma_{rq}^q\mathcal{F}_{mn}^{rp} - \frac{1}{e}\Gamma_{jn}^p\Gamma_{rq}^q\mathcal{F}_{pm}^{ri} + \frac{1}{e}\Gamma_{pn}^i\Gamma_{rq}^q\mathcal{F}_{jm}^{rp},\end{aligned}\quad (46)$$

$$\begin{aligned}\widetilde{\mathcal{W}}_4^i &= R_{jmn}^i + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jm}^{pi} + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jn}^{pi} + \Gamma_{aq}^q\Gamma_{bs}^s\mathcal{F}_{j[m}^{ap}\mathcal{F}_{pn]}^{bi} \\ &\quad + \Gamma_{pn}^i\Gamma_{jm}^p - \Gamma_{jn}^p\Gamma_{pm}^i - 2\Gamma_{mn}^p\Gamma_{jp}^i + \Gamma_{pm}^i\Gamma_{jn}^p - \Gamma_{jm}^p\Gamma_{pn}^i - \frac{1}{e}\Gamma_{jm}^p\Gamma_{rq}^q\mathcal{F}_{pn}^{ri} \\ &\quad + \frac{1}{e}\Gamma_{pm}^i\Gamma_{rq}^q\mathcal{F}_{jn}^{rp} + \frac{2}{e}\Gamma_{jp}^i\Gamma_{rq}^q\mathcal{F}_{mn}^{rp} - \frac{1}{e}\Gamma_{jn}^p\Gamma_{rq}^q\mathcal{F}_{pm}^{ri} + \frac{1}{e}\Gamma_{pn}^i\Gamma_{rq}^q\mathcal{F}_{jm}^{rp},\end{aligned}\quad (47)$$

$$\widetilde{\mathcal{W}}_5^i = R_{jmn}^i + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jm}^{pi} + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jn}^{pi} + \Gamma_{aq}^q\Gamma_{bs}^s\mathcal{F}_{j[m}^{ap}\mathcal{F}_{pn]}^{bi},\quad (48)$$

$$\widetilde{\mathcal{W}}_6^i = R_{jmn}^i + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jm}^{pi} + \frac{1}{e}\Gamma_{pq}^q\mathcal{F}_{jn}^{pi} + \Gamma_{aq}^q\Gamma_{bs}^s\mathcal{F}_{j[m}^{ap}\mathcal{F}_{pn]}^{bi}.\quad (49)$$

Theorem 3. Three of the invariants (44)–(49) are linearly independent.

Proof. The family of invariants $\widetilde{\mathcal{W}}_{jmn}^i$ given by (40) is isomorphic to the vector space, whose elements are $v = (1, u, -u, -u - u', u', -u', u, u, u + u', -u', u')$. The fifths (u, u', v, v', w) , which generate six linearly independent curvature tensors, are given by (9). After taking the values of u and u' from these fifths and substituting them into the vectors v , we obtain the following vectors:

$$v_1 = (1, 1, -1, 0, -1, 1, 1, 1, 0, 1, -1), \quad (50)$$

$$v_2 = (1, -1, 1, 0, 1, -1, -1, -1, 0, -1, 1), \quad (51)$$

$$v_3 = (1, 1, -1, -2, 1, -1, 1, 1, 2, -1, 1), \quad (52)$$

$$v_4 = (1, 1, -1, -2, 1, -1, 1, 1, 2, -1, 1), \quad (53)$$

$$v_5 = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \quad (54)$$

$$v_6 = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0). \quad (55)$$

The rank of matrix $\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix}$ is 3, which completes the proof of this theorem. $\square$

Corollary 5. Three linearly independent invariants are $\widetilde{\mathcal{W}}_1^{ijn}$, $\widetilde{\mathcal{W}}_2^{ijn}$, and $\widetilde{\mathcal{W}}_3^{ijn}$.

Corollary 6. Three of the invariants $\widetilde{\mathcal{W}}_{ij}$ are linearly independent. Invariants $\widetilde{\mathcal{W}}_1^{ijn}$, $\widetilde{\mathcal{W}}_2^{ijn}$, $\widetilde{\mathcal{W}}_3^{ijn}$ are linearly independent.

4. Discussion

Invariants for different mappings in the case of spaces with symmetric and non-symmetric affine connection have been obtained by different authors. The generalizations of the Weyl conformal, the Weyl projective tensor, and the Thomas projective parameters are objects that have been generalized in many different papers regarding invariants for different types of geometric mappings. This paper is a natural continuation of the research discussed in papers [21,24]. The study of the equitortion biholomorphically projective mapping between two generalized Riemannian spaces in the sense of Eisenhart's definition has been continued by finding new invariant geometric objects for the special case of the equitortion canonical biholomorphically projective mapping. Also, the linear independence of the obtained invariants was examined. Considering the diversity of types of biholomorphically projective mappings, the goal of further research will be to find new invariant geometric objects, both in the case of the considered equitortion canonical biholomorphically projective mapping and in the case of some other types of biholomorphically projective mappings. The results of this paper motivate us to ask ourselves whether there is an interpretation from a physical point of view, as well as what the geometrical significance of the obtained objects is.

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