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# Integral Transformations, Operational Calculus and Their Applications

Edited by

Hari Mohan Srivastava

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# Integral Transformations, Operational Calculus and Their Applications

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Editor

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# About the Editor

**Hari Mohan Srivastava** was born on 5 July, 1940 in Karon (District Ballia) in the Province of Uttar Pradesh in India. Professor Hari Mohan Srivastava began his university-level teaching career right after having received his M.S. degree in Mathematics in 1959 at the age of 19. He earned his Ph.D. degree in 1965 while he was a full-time member of the teaching faculty at the Jai Narain Vyas University of Jodhpur in India (since 1963). Currently, Professor Srivastava holds the position of Professor Emeritus in the Department of Mathematics and Statistics at the University of Victoria in Canada, having joined the faculty in 1969. Professor Srivastava has held (and continues to hold) numerous Visiting and Chair Professorships at many universities and research institutes in different parts of the world. Having received several D.S. (honoris causa) degrees as well as honorary memberships and fellowships of many scientific academies and scientific societies around the world, he is also actively editorially associated with numerous international scientific research journals as an Honorary or Advisory Editor or as an Editorial Board Member. He has also edited (and is currently editing) many Special Issues of scientific research journals as the Lead or Joint Guest Editor, including (for example) the MDPI journals Axioms, Mathematics, and Symmetry; the Elsevier journals Journal of Computational and Applied Mathematics, Applied Mathematics and Computation, Chaos, Solitons & Fractals, Alexandria Engineering Journal, and Journal of King Saud University—Science, the Wiley journal, Mathematical Methods in Applied Sciences; the Springer journals Advances in Difference Equations, Journal of Inequalities and Applications, Fixed Point Theory and Applications, and Boundary Value Problems; the American Institute of Physics journal Chaos: An Interdisciplinary Journal of Nonlinear Science; the American Institute of Mathematical Sciences journal AIMS Mathematics; the Hindawi journals Advances in Mathematical Physics, International Journal of Mathematics and Mathematical Sciences, and Abstract and Applied Analysis; the De Gruyter (now the Tbilisi Centre for Mathematical Sciences) journal Tbilisi Mathematical Journal; the Yokohama Publisher journal Journal of Nonlinear and Convex Analysis; the University of Nis journal Filomat; the Ministry of Communications and High Technologies (Republic of Azerbaijan) journal Applied and Computational Mathematics: An International Journal, and so on. He is a Clarivate Analytics (Thomson Reuters, Web of Science) Highly-Cited Researcher. Professor Srivastava's research interests include several areas of pure and applied mathematical sciences, such as real and complex analysis, fractional calculus and its applications, integral equations and transforms, higher transcendental functions and their applications, q-series and q-polynomials, analytic number theory, analytic and geometric inequalities, probability and statistics, and inventory modeling and optimization. He has published 36 books, monographs, and edited volumes, 36 book (and encyclopedia) chapters, 48 papers in international conference proceedings, and more than 1350 peer-reviewed international scientific research journal articles, as well as forewords and prefaces to many books and journals. Further details about Professor Srivastava's professional achievements and scholarly accomplishments, as well as honors, awards, and distinctions, can be found at the following web site: http://www.math.uvic.ca/ harimsri/.

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*Editorial*

# Special Issue of Symmetry: “Integral Transformations, Operational Calculus and Their Applications”

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This Special Issue consists of a total of 14 accepted submissions (including several invited feature articles) to the Special Issue of the MDPI’s journal, *Symmetry* on the general subject-area of “Integral Transformations, Operational Calculus and Their Applications” from different parts of the world.

The present Special Issue contains the invited, accepted and published submissions (see [1–14]) to a Special Issue of the MDPI’s journal, *Symmetry*, on the remarkably wide subject-area of “Integral Transformations, Operational Calculus and Their Applications”. Many successful predecessors of this Special Issue happen to be the Special Issues of the MDPI’s journal, *Axioms*, on the subject-areas of “*q*-Series and Related Topics in Special Functions and Analytic Number Theory”, “Mathematical Analysis and Applications" and “Mathematical Analysis and Applications II", the Special Issues of *Mathematics*, on the subject-areas of “Recent Advances in Fractional Calculus and Its Applications”, “Recent Developments in the Theory and Applications of Fractional Calculus”, “Operators of Fractional Calculus and Their Applications” and “Fractional-Order Integral and Derivative Operators and Their Applications", and indeed also the Special Issue of *Symmetry* itself, on the subject-area of “Integral Transforms and Operational Calculus”. In fact, encouraged by the noteworthy successes of this series of Special Issues, as well as of (for example) the two Special Issues of *Axioms*, on the subject-areas of “Mathematical Analysis and Applications” and “Mathematical Analysis and Applications II”, *Axioms* has already started the publication of a Topical Collection, entitled “Mathematical Analysis and Applications” (Collection Editor: H. M. Srivastava), with an open submission deadline. The interested reader should refer to and read the book format of many of these Special Issues (Guest Editor: H. M. Srivastava), which are cited below (see [15–18]).

In recent years, various families of fractional-order integral and derivative operators, such as those named after Riemann-Liouville, Weyl, Hadamard, Grünwald-Letnikov, Riesz, Erdélyi-Kober, Liouville-Caputo and so on, have been found to be remarkably important and fruitful, due mainly to their demonstrated applications in numerous seemingly diverse and widespread areas of the mathematical, physical, chemical, engineering and statistical sciences. Many of these fractional-order operators provide interesting and potentially useful tools for solving ordinary and partial differential equations, as well as integral, differintegral and integro-differential equations; fractional-calculus analogues and extensions of each of these equations; and various other problems involving special functions of mathematical physics and applied mathematics, as well as their extensions and generalizations in one or more variables (see, for details, a widely- and extensively-cited monograph [19]).

As it is known fairly well, investigations involving the theory and applications of integral transformations and operational calculus are remarkably wide-spread in many diverse areas of the mathematical, physical, chemical, engineering and statistical sciences. In this Special Issue, we invited and welcome review, expository and original research articles dealing with the recent state-of-the-art advances on the topics of integral transformations and operational calculus as well as their multidisciplinary applications, together with some relevance to the aspect of symmetry.

The suggested topics of interest for the call of papers for this Special Issue included, but were not limited to, the following keywords:

- Integral Transformations and Integral Equations as well as Other Related Operators Including Their Symmetry Properties and Characteristics
- Applications Involving Mathematical (or Higher Transcendental) Functions Including Their Symmetry Properties and Characteristics
- Applications Involving Fractional-Order Differential and Differintegral Equations and Their Associated Symmetry
- Applications Involving Symmetrical Aspect of Geometric Function Theory of Complex Analysis
- Applications Involving $q$-Series and $q$-Polynomials and Their Associated Symmetry
- Applications Involving Special Functions of Mathematical Physics and Applied Mathematics and Their Symmetrical Aspect
- Applications Involving Analytic Number Theory and Symmetry

Several well-established scientific research journals, which are published by such publishers as (for example) Elsevier Science Publishers, John Wiley and Sons, Hindawi Publishing Corporation, Springer, De Gruyter, MDPI and other publishing houses, have published and continue to publish a number of Special Issues of many of their journals on recent advances on different aspects, especially of the subject of one of the above-mentioned keywords, "Applications Involving Fractional-Order Differential and Differintegral Equations". Many widely-attended international conferences, too, continue to be successfully organized and held world-wide ever since the very first one on this particular subject-area in U.S.A. in the year 1974. In this connection, it seems to be worthwhile to refer the interested readers of this Special Issue to a recently-published survey-cum-expository review article (see [20]) which presented a brief elementary and introductory overview of the theory of the integral and derivative operators of fractional calculus and their applications especially in developing solutions of certain interesting families of ordinary and partial fractional "differintegral" equations. Furthermore, in connection with such works as (for example) [4,7], and indeed also many papers included in the published volumes (see [15–18]), a recent survey-cum-expository review article (see [21]) will be potentially useful in order to motivate further researches and developments involving a wide variety of operators of basic (or $q$-) calculus and fractional $q$-calculus and their widespread applications in Geometric Function Theory of Complex Analysis. In the same survey-cum-expository review article (see [21]), it is also pointed out as to how known results for the $q$-calculus can easily (and possibly trivially) be translated into the corresponding results for the so-called $(p, q)$-calculus (with $0 < q < p \leq 1$) by applying some obvious parametric and argument variations, the additional parameter $p$ being redundant (or superfluous).

Finally, I take this opportunity to thank all of the participating authors, and the referees and the peer-reviewers, for their invaluable contributions toward the remarkable success of each of the above-mentioned Special Issues. I do also greatly appreciate the editorial and managerial help and assistance provided efficiently and generously by Mr. Philip Li, Ms. Linda Cui and Ms. Grace Wang, and also many of their colleagues and associates in the Editorial Office of *Symmetry*.

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*Article*

# Construction of Weights for Positive Integral Operators

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**Abstract:** Let $(X, M, \mu)$ be a $\sigma$-finite measure space and denote by $P(X)$ the $\mu$-measurable functions $f\colon X \to [0,\infty]$, $f < \infty$ $\mu$ ae. Suppose $K\colon X \times X \to [0,\infty)$ is $\mu \times \mu$-measurable and define the mutually transposed operators $T$ and $T'$ on $P(X)$ by $(Tf)(x) = \int_X K(x,y) f(y)\, d\mu(y)$ and $(T'g)(y) = \int_X K(x,y) g(x)\, d\mu(x), f, g \in P(X), x, y \in X$. Our interest is in inequalities involving a fixed (weight) function $w \in P(X)$ and an index $p \in (1,\infty)$ such that: (*): $\int_X [w(x)(Tf)(x)]^p d\mu(x) \lesssim C \int_X [w(y) f(y)]^p d\mu(y)$. The constant $C > 1$ is to be independent of $f \in P(X)$. We wish to construct all $w$ for which (*) holds. Considerations concerning Schur's Lemma ensure that every such $w$ is within constant multiples of expressions of the form $\phi_1^{1/p-1}\phi_2^{1/p}$, where $\phi_1, \phi_2 \in P(X)$ satisfy $T\phi_1 \le C_1\phi_1$ and $T'\phi_2 \le C_2\phi_2$. Our fundamental result shows that the $\phi_1$ and $\phi_2$ above are within constant multiples of (**): $\psi_1 + \sum_{j=1}^{\infty} E^{-j} T^{(j)} \psi_1$ and $\psi_2 + \sum_{j=1}^{\infty} E^{-j} T'^{(j)} \psi_2$respectively; here $\psi_1, \psi_2 \in P(X)$, $E > 1$ and $T^{(j)}, T'^{(j)}$ are the $j$th iterates of $T$ and $T'$. This result is explored in the context of Poisson, Bessel and Gauss–Weierstrass means and of Hardy averaging operators. All but the Hardy averaging operators are defined through symmetric kernels $K(x,y) = K(y,x)$, so that $T' = T$. This means that only the first series in (**) needs to be studied.

**Keywords:** weights; positive integral operators; convolution operators

**MSC:** 2000 Primary 47B34; Secondary 27D10

## 1. Introduction

Consider a $\sigma$-finite measure space $(X, M, \mu)$ and a positive integral operator $T$ defined through a nonnegative kernel $K = K(x,y)$ which is $\mu \times \mu$ measurable on $X \times X$; that is, $T$ is given on the class, $P(X)$, of $\mu$-measurable functions $f : X \to [0,\infty]$, $f < \infty$ $\mu$ ae, by

$$(Tf)(x) = \int_X K(x,y) f(y)\, d\mu(y),\ x \in X.$$

The transpose, $T'$, of $T$ at $g \in P(X)$ is

$$(T'g)(y) = \int_X K(x,y) g(x)\, d\mu(x),\ y \in X;$$

it satisfies

$$\int_X gTf d\mu = \int_X fT'g\, d\mu,\ f, g \in P(X).$$

Our focus will be on inequalities of the form

$$\int_X [uTf]^p\, d\mu \le B^p \int_X [vf]^p\, d\mu, \tag{1}$$

with the index $p$ fixed in $(1, \infty)$ and $B > 0$ independent of $f \in P(X)$; here, $u, v \in P(X)$, $0 \leq u, v < \infty$, $\mu$ *ae*, are so-called weights.

The equivalence need only be proved in one direction. Suppose, then, (1) holds and $g \in P(x)$ satisfies $\int_X [u^{-1}g]^p \, d\mu < \infty$. Then

$$\left[\int_X [v^{-1}T'g]^{p'} \, d\mu\right]^{\frac{1}{p'}} = \sup \int_X f v^{-1} T' g \, d\mu,$$

the supremum being take over $f \in P(X)$ with $\int_X f^p \, d\mu \leq 1$. But, Fubini's Theorem ensures

$$\begin{aligned}\int_X f v^{-1} T' g \, d\mu &= \int_X g T(f v^{-1}) \, d\mu \\ &= \int_X (u^{-1}g) u T(f v^{-1}) \, d\mu \\ &\leq \left[\int_X [u^{-1}g]^{p'} \, d\mu\right]^{\frac{1}{p'}} \left[B^p \int_X [v f v^{-1}]^p \, d\mu\right]^{\frac{1}{p}} \\ &\leq \left[B^{p'} \int_X [u^{-1}g]^{p'} \, d\mu\right]^{\frac{1}{p'}}.\end{aligned}$$

Further, (1) holds if and only if the dual inequality

$$\int_X [v^{-1}T'g]^{p'} \, d\mu \leq B^{p'} \int_X [u^{-1}g]^{p'} \, d\mu, \; p' = \frac{p}{p-1}, \tag{2}$$

does.

Inequality (1) has been studied for various operators $T$ in such papers as [1–9].

In this paper, we are interested in constructing weights $u$ and $v$ for which (1) holds. We restrict attention the case $u = v = w$; the general case will be investigated in the future. Our approach is based on the observation that, implicit in a proof of the converse of Schur's lemma, given in [10], is a method for constructing $w$. An interesting application of Schur's lemma itself to weighted norm inequalities is given in Christ [11].

In Section 2, we prove a number of general results the first of which is the following one.

**Theorem 1.** *Let $(X, M, \mu)$ be a $\sigma$-finite measure space with $u, v \in P(X)$, $0 \leq u, v < \infty$, $\mu$ ae. Suppose that $T$ is a positive integral operator on $P(X)$ with transpose $T'$. Then, for fixed $p$, $1 < p < \infty$, one has (1), with $C > 1$ independent of $f \in P(X)$, if and only if them exists a function $\phi \in P(X)$ and a constant $C > 1$ for which*

$$T(v^{-1}\phi^{p'}) \leq C u^{-1} \phi^{p'} \text{ and } T'(u\phi^p) \leq C v \phi^p. \tag{3}$$

In this case, $B_0$, the smallest $B$ possible in (1) and $Co$, the smallest possible $C$ so that (3) holds for some $\phi$, satisfy

$$B_0 \leq C_0 = \max\left[B_1^p, B_1^{p'}\right],$$

where $B_1 = B_0^{1/p} + B_0^{1/p'}$.

Theorem 1 has the following consequence.

**Corollary 1.** *Under the condition of Theorem 1, (1) holds for $u = v = w$ if and only if $w = \phi_1^{-1/p'} \phi_2^{1/p}$, where $\phi_1, \phi_2$ are functions in $P(X)$ satisfying*

$$T\phi_1 \leq C\phi_1 \text{ and } T'\phi_2 \leq C\phi_2, \tag{4}$$

*for some* $C > 1$.

Though it is often possible to work with the inequalities (4) directly (see Remark 1) it is important to have a general method to construct the functions $\phi_1$ and $\phi_2$. This method is given in our principal result.

**Theorem 2.** *Suppose* $X, \mu$ *and* $T$ *are as in Theorem 1. Let* $\phi \in P(X)$. *Then,* $\phi$ *satisfies an inequality of the form*

$$T\phi \leq C_1\phi,\ C_1 > 0 \text{ constant,} \tag{5}$$

*if and only if there is a constant* $C > 1$ *such that*

$$C^{-1}\phi \leq \psi + \sum_{j=1}^{\infty} C_2^{-j} T^{(j)}\psi \leq C\phi, \tag{6}$$

*where* $\psi \in P(X)$, $C_2 > 1$ *is constant and* $T^{(j)} = T \circ T \cdots \circ T$, $j$ *times.*

The kernels of operator of the form

$$\sum_{j=1}^{\infty} C^{-j} T^{(j)} \text{ and } \sum_{j=1}^{\infty} C^{-j} T'^{(j)}$$

will be called the weight generating kernels of $T$. In Sections 3–6 these kernels will be calculated for particular $T$. All but the Hardy operators considered in Section 6 operate on the class $P(R^n)$ of nonnegative, Lebesgue-measurable functions on $R^n$.

The operators last referred to are, in fact, convolution operators

$$(T_k f)(x) = (k * f)(x) = \int_{R^n} k(x-y) f(y)\, dy,\ x \in R^n,$$

with even integrable kernels $k$, $\int_{R^n} k(y)\, dy = 1$. In particular, the kernel $k(x-y)$ is symmetric, so $T_k' = T_k$, whence only the first series in (**) need be considered.

Further, the convolution kernels are part of an approximate identity $\{k_t\}_{t>0}$ on

$$L^P(R^n) = \left\{ f \text{ Leb. meas: } \left[ \int_{R_n} |f|^p \right]^{1/p} < \infty \right\},$$

see [12]. Thus, it becomes of interest to characterize the weights $w$ for which $\{k_t\}_{l>0}$ is an approximate identity on

$$L^p(w) = L^p(R^n, w) = \left\{ f \text{ Leb. meas: } \|f\|_{p,w} = \left[ \int_{R^n} |wf|^p \right]^{1/p} < \infty \right\};$$

that is $k_t * f \in L^p(w)$ and

$$\lim_{t \to 0+} \|k_t * f - f\|_{p,w} = 0$$

for all $f \in L^p(w)$. It is a consequence of the Banach-Steinhaus Theorem that this will be so if and only if

$$\sup_{0<t<a} \|k_t\| < \infty$$

for some fixed $a > 0$, where $\|k_t\|$ denotes the operator norm of $T_{k_t}$ on $L^p(w)$. We remark here that the operators in Sections 3–5 are bounded on $L^p(w)$ and, indeed, form part of an approximate identity on $L^p(w)$, if $w$ satisfies the $A_p$ condition, namely,

$$\sup \left[\frac{1}{|Q|}\int_Q w^p\right]\left[\frac{1}{|Q|}\int_Q w^{-p'}\right]^{1/p'} < \infty,\ p' = \frac{p}{p-1}, \tag{7}$$

the supremum being taken over all cubes $Q$ in $R^n$ whose sides are parallel to the coordinate axes with $\infty > |Q| =$ Lebesgue measure of $Q$. See ([13], p. 62) and [14].

Finally, all the convolution operators are part of a convolution semigroup $(k_t)_{t>0}$; that is $k_t(x) = t^{-n}k\left(\frac{x}{t}\right)$ and $k_{t_1} * k_{t_2} = k_{t_1+t_2}$, $t_1, t_2 > 0$. The approximate identity result can thus be interpreted as the continuity of the semigroup.

We conclude the introduction with some remarks on terminology and notation. The fact that $T$ is bounded on $L^p(w)$ if and only if $T'$ is bounded on $L^{p'}(w^{-1})$ is called the principle of duality or, simply, duality. Two functions $f, g \in P(X)$ are said to be equivalent if a constant $C > 1$ exists for which

$$C^{-1}g \le f \le Cg. \tag{8}$$

We indicate this by $f \approx g$, with the understanding that $C$ is independent of all parameters appearing, (except dimension) unless otherwise stated. If only one of the inequalities in (8) holds, we use the notation $f \succeq g$ or $f \preceq g$, as appropriate. Lastly, a convolution operator and its kernel are frequently denoted by the same symbol.

## 2. General Results

In this section we give the proofs of the results stated in the Introduction, together with some remarks.

**Proof of Theorem 1.** The conditions (3) are, respectively, equivalent to

$$T' : L^1(u^{-1}\phi^{p'}) \to L^1(v^{-1}\phi^{p'})$$
$$\text{i.e., } T : L^\infty(v\phi^{-p'}) \to L^\infty(u\phi^{-p'})$$

and

$$T : L^1(v\phi^p) \to L^1(u\phi^p).$$

It will suffice to deal with the first condition in (3). So, Fubini's Theorem yields

$$\int_X v^{-1}\phi^{p'}T'f\,d\mu \le C\int_X u^{-1}\phi^{p'}f\,d\mu$$

equivalent to

$$\int_X f\,T(v^{-1}\phi^{p'})\,d\mu \le C\int_X fu^{-1}\phi^{p'}\,d\mu, \quad f \in P(X),$$

and hence to

$$T(v^{-1}\phi^{p'}) \le Cu^{-1}\phi^{p'},$$

since $f$ is arbitrary.

According to the main result of [15], then,

$$T : L^p\left((v\phi^p)^{1/p}(v\phi^{-p'})^{1/p'}\right) \to L^p\left((u\phi^p)^{1/p}(u\phi^{-p'})^{1/p'}\right)$$

i.e., $T : L^p(v) \to L^p(u)$, with norm $\le C$, so that (1) holds with $B \le C$.

Suppose now (1) holds. Following [10], choose $g \in P(X)$ with

$$\int_X g^{pp'}\, d\mu = 1.$$

Let $T_1 g = \left[uT(v^{-1}g^{p'})\right]^{1/p'}$ and $T_2 g = \left[v^{-1}T'(ug^p)\right]^{1/p}$. Set

$$S = T_1 + T_2,\ A = B_0 + \varepsilon \text{ and } \phi = \sum_{j=0}^{\infty} A^{-j} S^{(j)} g.$$

As in [10], conclude $T_1\phi \le A\phi$ and $T_2\phi \le A\phi$, so that (2) is satisfied for $C_0 \le \left[B_1^p, B_1^{p'}\right]$, where $B_1 = B_0^{1/p} + B_0^{1/p'}$. □

**Proof of Corollary 1.** Given (1), one has (2) and Theorem 1 then implies (3), with $T$ replaced by $T'$, namely for $u = v = w$,

$$T(w^{-1}\phi^{p'}) \le Cw^{-1}\phi^{p'} \text{ and } T(w\phi^p) \le Cw\phi^p,$$

whence the inequalities (4) are satisfied by $\phi_1 = w\phi^p$ and $\phi_2 = w^{-1}\psi^{p'}$. Conversely, given (4), and taking $u = v = w = \phi_1^{1/p-1}\phi_2^{1/p}$, one readily obtains (3), with $\psi = (\psi_1\psi_2)^{1/pp'}$. □

**Proof of Theorem 2.** Clearly, if (6) holds,

$$T\phi \le C\left[T\psi + \sum_{j=1}^{\infty} C_2^{-j} T^{(j+1)}\psi\right] = CC_2 \sum_{j=1}^{\infty} C_2^{-j} T^{(j)}\psi \le C^2 C_2 \phi.$$

Suppose $\phi \in P(X)$ satisfies (5). Then,

$$T^{(j)}\phi \le C_1^j \phi_1,\ j = 1, 2, \ldots.$$

It only remains to observe that

$$\left(1 + \frac{C_1}{\varepsilon}\right)^{-1} \phi \le \phi + \sum_{j=1}^{\infty} (C_1 + \varepsilon)^{-j} T^{(j)}\phi \le \phi + \sum_{j=1}^{\infty} \left(\frac{C_1}{C_1 + \varepsilon}\right)^j \phi \le \left(1 + \frac{C_1}{\varepsilon}\right)\phi,$$

for any $\varepsilon > 0$. □

**Remark 1.** *The class of functions $\phi$ determined by the weight-generating operators $\sum_{j=1}^{\infty} C^{-j}T^{(j)}$ effectively remains the same as C increases. Thus, suppose* $0 < C_1 < C_2$, $\psi \in P(X)$ *and* $\phi = \psi + \sum_{j=1}^{\infty} C_1^{-j} T^{(j)}\psi$. *Then, $\phi$ is equivalent to $\psi + \sum_{j=1}^{\infty} C_2^{-j} T^{(j)}\psi$, since*

$$\begin{aligned} \phi \le \phi + \sum_{j=1}^{\infty} C_2^{-j} T^{(j)}\phi &= \sum_{j=0}^{\infty} C_2^{-j} \sum_{k=0}^{\infty} C_1^{-k} T^{(j+k)}\psi = \sum_{l=0}^{\infty} \left( \sum_{j+h=l} C_1^{-k} C_2^{-j} \right) T^{(l)}\psi \\ &= \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{C_1}{C_2}\right)^j C_1^{-l} T^{(l)}\psi = \frac{C_2}{C_2 - C_1} \sum_{l=0}^{\infty} C_1^{-l} T^{(l)}\psi \\ &= \frac{C_2}{C_2 - C_1}\phi. \end{aligned}$$

This means that in dealing with weight-generating operators we need only consider $C > 1$.

We conclude this section with the following observations on approximate identities in weighted Lebesgue spaces.

**Remark 2.** *Suppose* $\{k_t\}_{t>0}$ *is an approximate identity in* $L^p(R^n)$, $1 < p < \infty$. *If the inequalities* (4) *involving* $\phi_1$ *and* $\phi_2$ *can be shown to hold for* $T_{kt}$, $t \in (0, a]$ *for some* $a > 0$, *with* $C > 1$ *independent of such* $t$, *then* $\{k_t\}_{t>0}$ *will also be an approximate identity in* $L^p(w) = L^p(R^n, w)$, $w = \phi_1^{-1/p'}\phi_2^{1/p}$.

**Example 1.** *Let* $k = k(|x|)$ *be any bounded, nonnegative radial function on* $R^n$ *which is a decreasing function of* $|x|$ *and suppose* $\int_{R^n} k(x)\,dx = 1$. *It is well-known, see ([13], p. 63), that* $k_t(x) = t^{-n}k(x/t)$, $x \in R^n$, *is an approximate identity in* $L^p(R^n)$, $1 < p < \infty$.

The weight $w(x) = 1 + |x|^{-n/p}(1 + \log^+(1/|x|))^{-1}$, for fixed $p$, $1 < p < \infty$, has the interesting properly that $T_{k_t} : L^p(w) \to L^p(w)$ for all $t > 0$, yet $\{k_t\}_{t>0}$ is never an approximate identity in $L^p(w)$.

To obtain the boundedness assertion take $\phi_1(x) = 1$ and $\phi_2(x) = 1 + |x|^{-n}\ (1 + \log^+(1/|x|))^{-p}$ in Corollary 1.

Arguments similar to those in [6] show that if $\{k_t\}_{t>0}$ is an approximate identity in $L^p(w)$, then $w$ must satisfy the $A_p$ condition for all cubes $Q$ will sides parallel to the coordinate axes and $|Q| \le a$ for some $a > 0$. However, the weight $w$ does not have this property.

## 3. The Poisson Integral Operators

We recall that for $t > 0$ and $y \in R^n$, the Poisson kernel, $P_t$, is defined by

$$P_t(y) = c_n t(t^2 + |y|^2)^{-(n+1)/2},\ c_n = \pi^{-(n+1)/2}\Gamma((n+1)/2).$$

**Theorem 3.** *The weight-generating kernels for* $P_t$, $t > 0$, *are equivalent to* $P \equiv P_0$. *Indeed, given* $\psi \in P(R^n)$, *with* $P_\psi < \infty$ *a.e.,*

$$C_t^{-1}P_\psi \le \sum_{j=1}^{\infty} C^{-j}P_{jt}\psi \le C_t' P_\psi, \tag{9}$$

*where* $C > 1$, $C_t = C\max[t^{-1}, t^n]$ *and* $C_t' = C_t \sum_{j=1}^{\infty} C^{-j}\max[jt, (jt)^{-n}]$.

**Proof.** Observe that by the semigroup property $P_t^{(j)} = P_{jt}$, $j = 1, 2, \ldots$.

Also,

$$\min[t, t^{-n}]P \le P_t \le \max[t, t^{-n}]P.$$

Now, suppose

$$\psi + \sum_{j=1}^{\infty} C^{-j}P_{jt}\psi \text{ is in } P(R^n),$$

with $C > 1$. Then,

$$\begin{aligned} P_\psi &\le C_t P_t\psi + \sum_{j=1}^{\infty} C^{-j}P_{(j+1)t}\psi \le C_t \sum_{j=1}^{\infty} C^{-j}P_{jt}\psi \le C_t \sum_{j=1}^{\infty} C^{-j}\max[jt, (j)^{-n}]P_\psi \\ &\le C_t' P_\psi. \end{aligned}$$

□

As stated in Section 1, $w \in A_p$ is sufficient for $\{P_t\}_{t>0}$ to be an approximate identify in $L^P(w)$. Moreover, $w \in A_p$ is also necessary for this in the periodic case. See [6,8,16]. It is perhaps surprising then that the class of approximate identity weights is much larger than $A_p$, as is seen in the next result.

**Proposition 1.** *Let $w_\alpha(x) = [1 + |x|]^\alpha$, $\alpha \in R$. Then, for any $t > 0$, $P_t$ is bounded on $L^p(w_\alpha)$ if any only if $-\frac{n}{p} - 1 < \alpha < \frac{n}{p'} + 1$. Moreover, on that range of $\alpha$ one has*

$$\lim_{t \to 0+} \|P_t * f - f\|_{p,\omega_\alpha} = 0, \tag{10}$$

*for all $f \in L^p(\omega_\alpha)$. The set of $\alpha$ for which $w_\alpha \in A_p$, however, is*

$$-\frac{n}{p} < \alpha < \frac{n}{p'}.$$

**Proof.** We omit the easy proof of the assertion concerning the $\alpha$ for which $w_\alpha \in A_p$.

To obtain the "if" part of the other assertion we will show

$$P_t * w_\beta \leq C w_\beta, \ t > 0, \tag{11}$$

if and only if $-n - 1 \leq \beta < 1$, with $C > 1$ independent of both $s$ and $t$, if $t \in (0, 1)$. Corollary 1 and Remark 2, then yield (10) when $-\frac{n}{p} - 1 < \alpha < \frac{n}{p'} + 1$.

Consider, then, fixed $x \in R^n$ and $0 < t < 1$. We have

$$\begin{aligned}(P_t * w_\beta)(x) &= \left( \int_{|y| \leq \frac{|x|}{2}} + \int_{\frac{|x|}{2} < |y| < 2|x|} + \int_{|y| \geq 2|x|} \right) P_t(y) w_\beta(s - t)\, dy \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Now,

$$I_1 \leq w_\beta(x) \int_{|y| < \frac{|x|}{2}} P_t(y)\, dy \leq C w_\beta(x),$$

for all $\beta \in R$.

Again,

$$I_2 \geq c P_t(x) \int_{|x-y| \leq 1} (1 + |x - y|)^\beta\, dy \geq c P_t(x) \geq c|x|^{-n-1},$$

so we require $\beta > n - 1$, if (11) is to hold.

Moreover, for $x \in R^n$ and $0 < t < 1$,

$$\begin{aligned} I_2 &\approx P_t(x) \left[ |x|^n \chi_{|x| \leq 1} + |x|^{\beta+n} \chi_{|x|>1} \right] \\ &\approx \left( \frac{|x|}{t} \right)^n \chi_{|x| \leq 1} + \frac{t}{|x|} \chi_{t \leq |x| \leq 1} + \frac{t}{|x|} |x|^\beta \chi_{|x| \geq 1} \\ &\leq C w_\beta(x). \end{aligned}$$

Next, for $|x| \gg 1$

$$I_3 = \int_{|y| > 2|x|} P_t(y) w_\beta(y)\, dy \preceq t \int_{|y| > 2|x|} |y|^{-n-1+\beta} dy$$

which requires $\beta < 1$ to have $I_3 < \infty$. In that case

$$I_3 \preceq \int_{r > 2|x|} r^{-n-1+\beta} r^{n-1}\, dr \preceq |x|^{\beta - 1} \preceq w_\beta(x).$$

That $P_t$ is not bounded on $L^p(w_\alpha)$ when $\alpha \leq -\frac{n}{p} - 1$ can be seen by noting that, for appropriate $\varepsilon > 0$, the function $f(x) = |x|[\log(1 + |x|)]^{-(1+\varepsilon)/p}$ is in $L^p(w_\alpha)$, while $P_t f \equiv \infty$. The range $\alpha \geq n/p + 1$ is then ruled out by duality. □

## 4. The Bessel Potential Operators

The Bessel kernel, $G_\alpha, \alpha > 0$, can be defined explicitly by

$$G_\alpha(y) = C_\alpha |y|^{(\alpha-n)/2} K_{(n-\alpha)/2}(|y|),\ y \in R^n,$$

where $K_r$ is the modified Bessel function of the third kind and

$$C_\alpha^{-1} = \pi^{n/2} 2^{(n+\alpha-2)} \Gamma(\alpha/2).$$

It is, however, more readily recognized by its Fourier transformation

$$\hat{G}_\alpha(z) = (2\pi)^{-n/2}[1+|z|^2]^{-\alpha/2}.$$

Using the latter formula one picks out the special cases $G_{n-1}$ and $G_{n+1}$ which, except for constant multiplies, are, respectively, $|y|^{-1}e^{-|y|}$ and the Picard kernel $e^{-|y|}$.

The semigroup properly $G_\alpha * G_\beta = G_{\alpha+\beta}$ holds and so the $j$th convolution iterate has kernel $G_{j\alpha}$. Also, $\int_{R^n} G_\alpha(y)\,dy = 1$.

We use the integral representation

$$G_\alpha(y) = g_{\alpha,n}(|y|) = (4\pi)^{-n/2}\Gamma(\alpha/2)^{-1} \int_0^\infty e^{-|y|^2 t/4} e^{-1/t} t^{(n-2)/2} \frac{dt}{t} \tag{12}$$

to show in Lemma 1 below that known estimates [17], are in fact, sharp.

**Lemma 1.** *Suppose $n, \alpha > 0, n \in Z_+$. Set $m = n - \alpha$ and define $r^{-m_+}$ to be $r^{-m}$, $\log_+\left(\frac{2}{r}\right)$ or 1, according as $m > 0, m = 0$ or $m < 0$. Then, a constant $C > 1$ exists, depending on $n$, such that*

$$\begin{aligned} C^{-1} r^{-m+} \le g_{\alpha,n}(r) \le C r^{-m+},\ 0 < r < 1, \\ C^{-1} r^{-(m+1)/2} e^{-r} \le g_{\alpha,n}(r) \le C r^{-(m+1)/2} e^{-r},\ r \ge 1. \end{aligned} \tag{13}$$

**Proof.** As in [17], p. 296

$$g_{\alpha,n}(r) = C_\alpha e^{-r} (\alpha/r)^{m/2} \int_1^\infty e^{-\frac{r}{2}(x+\frac{1}{x}-2)} \left[x^{m/2} + x^{-m/2}\right] \frac{dx}{x}$$

with $C_\alpha = (4\pi)^{-n/2}\Gamma(\alpha/2)^{-1}$. Clearly,

$$g_{\alpha,n}(r) \approx r^{-m/n} e^{-r} \int_1^\infty e^{-\frac{r}{2}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2} x^{|m|/2} \frac{dx}{x}. \tag{14}$$

Let $y = \sqrt{x} - 1/\sqrt{x}$, so that $x = \dfrac{2+y^2+\sqrt{(1+y^2)^2-4}}{2}$ which is essentially 1, when $0 < y < 2$ and $y^2$ when $y > 2$. The integral in (14) is thus equivalent to

$$\int_0^{\sqrt{2}} e^{-\frac{r}{2}y^2} dy + \int_{\sqrt{2}}^\infty e^{-\frac{r}{2}y^2} y^{|m|} \frac{dy}{y}. \tag{15}$$

Next, let $y = \sqrt{2z/t}$ to get (15) equivalent to

$$r^{-1/2} \int_0^r e^{-2} \frac{dz}{\sqrt{z}} + \int_r^\infty e^{-z} |z|^{|m|/2} \frac{dz}{z}. \tag{16}$$

Using L'Hospital's Rule and the asymptotic formula for the incomplete gamma function we find that the expression (16) is effectively $r^{-|m|/2}$ in $(0,0)$ and $r^{-1/2}$ in $(1,\infty)$. This completes the proof when $m \neq 0$. The case $m = 0$ is left to the reader. $\square$

**Remark 3.** *For $p \in (1,\infty)$, let $W_{\alpha,p}$ denote the class of weights $w$ for which $G_\alpha$ is bounded on $L^p(w)$. Then $W_{\alpha,p}$ increases with $\alpha$ and $W_{\alpha,p} = W_{p,p}$, whenever $\alpha, \beta > n$. These facts follow from the semigroup property, the estimates (13) and the inequality $G_{\alpha_t} \leq C G^{1-t}_{\alpha_1}, G^t_{\alpha_2}$ which holds for $\alpha_t = (1-t)_{\alpha_1} + t\alpha_2$, provided $0 < \alpha_1 < \alpha_2$, $0 < t < 1$ and either $\alpha_2 < n$ or $\alpha_1 > n$. However,no two classes $W_{\alpha,p}$ are identical, as is shown in the following proposition.*

**Proposition 2.** *Fix $p \in (1,\infty)$ and $\alpha$, $\beta \in (0,n)$, with $\alpha < \beta/p$. Then, there is a weight $w \in W_{\beta,p} - W_{\alpha,p}$.*

**Proof.** Let $\phi_\gamma(x) = 1 + \sum\limits_{k=1}^{\infty} |x - 4^{-k}|^{-\gamma} \chi_{E_k}(x)$, where

$$E_k = \left\{ x \in R^n : |x - 4^{-k}| \leq \tfrac{1}{2} 4^k \right\}.$$

One readily shows $G_\beta \phi_\gamma \leq C\phi_\gamma$, if $0 < \gamma < \beta$. Hence, taking $w_\gamma = \phi_\gamma^{1/p}$, we have $w_\gamma \in W_{\beta,p}$. For $0 < \delta < n$, $L^p(w_\gamma)$ contains the function

$$f(x) = \sum_{k=1}^{\infty} |x - x_k|^{-\delta/p} \chi_{F_k}$$

where

$$x_k = \tfrac{1}{2}\left[\tfrac{3}{2} \cdot 4^{k+1} + \tfrac{1}{2} \cdot 4^k\right] = 7/4k + 2$$

and

$$F_k = \left\{ x \in R^n : |x - x_k| < \tfrac{1}{2} \cdot 4k + 1 \right\}.$$

We seek conditions on $r$ and $\delta$ so that $w_\gamma \notin W_{\alpha,p}$.
Now, $G_\alpha f = 4^{k[\delta/p - \bar{\alpha}]}$ on $E_k$, so

$$\|G_\alpha f\|^p_{p,w_\gamma} \geq \sum_{k=1}^{\infty} 4^{k[\delta - \alpha p + \gamma - n]} = \infty,$$

if $\delta - \alpha p + \gamma - n \geq 0$. By taking $\gamma$ sufficiently close to $\beta$ and $\delta$ sufficiently closed to $n$, this condition can be met. $\square$

**Theorem 4.** *Suppose $n, \alpha, m$ and $m_+$ are as in Lemma 1. Fix $C > 1$ and set $k = [1 - C^{-2/\gamma}]^{1/2}$. Then, the weight-generating kernel for $G_\alpha$ corresponding to $C$ is equivalent to*

$$|y|^{m_+}, |y| \leq 1,$$

*and*

$$\left[|y|^{-(m+1)/2} + |y|^{(1-n)/2}\right] e^{-k|y|},\ |y| \geq 1.$$

*In particular, for $\alpha \in (0,2]$, the kernel is equivalent to $G_\alpha(ky) + G_2(ky)$.*

**Proof.** In view of (12), the kernel is given by

$$(4\pi)^{-n/2} \int_0^\infty e^{-(r^2/4)t} e^{-1/t} t^{\frac{n}{2}-1} S(t)\, dt,$$

where $r = |y|$ and

$$S(t) = \sum_{j=1}^{\infty} \frac{[C^{-1}t^{-\alpha/2}]^j}{\Gamma(j\alpha/2)},$$

When $C^{-1}t^{-\alpha/2} \leq 1$, that is, $t \geq C^{-2/\alpha} \equiv c$, the sum $S(t)$ is, effectively, $t^{-\alpha/2}$, as is seen from the inequalities

$$\frac{C^{-1}t^{-\alpha/2}}{\Gamma(\alpha/2)} \leq S(t) \leq \frac{C^{-1}t^{-\alpha/2}}{\Gamma(\alpha/2)} \left[1 + \sum_{j=1}^{\infty} \frac{1}{\Gamma(j\alpha/2)}\right].$$

Here, we have used $\Gamma(x+y) \geq \Gamma(x)\Gamma(y)$ when $x, y > 0$.

For $t \leq c$, the asymptotic expression

$$\sum_{j=1}^{\infty} \frac{t^j}{\Gamma(\ell j)} = t^{1/l} e^{t^{1/l}} [1 + 0(t^{-1})], \text{ as } t \to \infty,$$

given in [8], yields

$$S(t) \approx t^{-1} e^{a/t},\ t \leq c.$$

Thus, the kernel is, essentially,

$$\int_0^c e^{-(r^2/4)t} e^{(c-1)/t} t^{(n/2)-2}\, dt + \int_c^{\infty} e^{-(r^2/4)t} e^{-1/t} t^{(n-\alpha)/2} \frac{dt}{t}. \tag{17}$$

Now, the first term in (17) is bounded on $0 \leq r \leq 1$, while the second term is equivalent to $G_\alpha$ for all $r \geq 0$. It only remains to show the first integral, $I$, satisfies $I \approx r^{(1-n)/2} e^{-kr}$ for $r \geq 1$. To this end set $s = rt/2$ in I to obtain

$$I \approx r^{(2-n)/2} e^{-kr} \int_0^{cr/2} e^{-r\left[\sqrt{s} - k/\sqrt{s}\right]^2/2} \cdot S^{\frac{n}{2}-2}\, ds$$

Next, let $y = \sqrt{s} - k/\sqrt{s}$ so that

$$I \approx r^{(2-n)/2} e^{-kr} \int_{-\infty}^{\beta(r)} e^{-ry^2/2} [y + f(y)]^{n-3} [1 + yf(y)^{-1}]\, dy,$$

where $\beta(r) = \sqrt{cr/2} - k\sqrt{2/cr}$ and $f(y) = \sqrt{y^2 + 4l} = \sqrt{s} + \frac{k}{\sqrt{s}}$.

Finally, take $z = \sqrt{r/2}y$ to get

$$I \approx r^{(1-n)/2} e^{-kr} \int_{-\infty}^{\gamma(r)} e^{-z^2} \left[\sqrt{2/r}z + f\left(\sqrt{2/r}z\right)\right]^{n-3} \left[1 + \sqrt{2/r}zf\left(\sqrt{2/r}z\right)^{-1}\right] dz,$$

with $\gamma(r) = \sqrt{c}r/2 - k/\sqrt{c}$. We have now just to observe that when $z \in \mathbb{R}$ and $r \geq 1$

$$0 \leq 1 + \sqrt{2/r}zf\left(\sqrt{2/r}z\right)^{-1} < 2$$

while $\sqrt{2/r}z + f\left(\sqrt{2/r}z\right)$ lies between $2k^{1/2}$ and $\sqrt{2z^2 + 4k}$. $\square$

Typical of $G_\alpha$ weights are the exponential functions $e^{\beta x}$, $-1 < \beta < 1$.

**Proposition 3.** *Suppose $\alpha \in (0, 1/2)$ and $p \in (1, \infty)$. Set $w_\beta(f) = e^{\beta|x|}, x \in R^n$. Then, $G_\alpha$ is bounded on $L^p(w_\beta)$ if and only if $-1 < \beta < 1$. Moreover, on this range of $\beta$, one has*

$$\lim_{\alpha \to 0+} \|G_\alpha * f - f\|_{p,w_\beta} = 0$$

*for all* $f \in L^p(w_\beta)$.

**Proof.** Fix $\beta \in (-1, 1)$. We show $C > 1$ exists, independent of $\alpha \in (0, 1/2)$, such that

$$(G_\alpha w_\beta)(x) \le C w_\beta(x),\ x \in R^n.$$

The "if" part then follows by Remark 2.

Using the simple inequalities $|x + y| \le |x| + |y|$ when $\beta > 0$ and $|x - y| \ge |x| - |y|$ when $\beta < 0$ we obtain

$$(G_\alpha w_\beta)(x) \le w_\beta(x) \int_{R^n} e^{|\beta|\,|y|} G_\alpha(y)\,dy.$$

But, the proof of Lemma 1 shows

$$\begin{aligned}\int_{R^n} e^{|\beta|\,|y|} G_\alpha(y)\,dy &\le \int_{|y|\le 1} e^{|\beta|\,|y|}|y|^{\alpha-n}\,dy + \int_{|y|>1} e^{[|\beta|-1]|y|}|y|^{-\frac{n}{2}-\frac{1}{4}}\,dy \\ &\approx 1,\end{aligned}$$

when $\alpha \in (0, 1)$.

To prove the "only if" part, only the care $\beta = -1$ needs to be considered. We observed that $f(x) = \frac{e^{|x|}}{1+|x|^{n+1}}$ is in $L^p(w_{-1})$ and that $G_\alpha$ bounded on $L^p(w_{-1})$ implies the same of $G_{j\alpha}, j = 2, 3, \ldots$. However, for $j \ge \dfrac{n+3}{\alpha}$, $G_{j\alpha} f \equiv \infty$. □

**Example 2.** *Consider the Bessel potential* $G_2(y)$ *so that the weight-generating kernels are equivalent to* $G_2(ky)$, $0 < k < 1$. *These are especially simple when the dimension,* $n$, *is 1 or 3. In the first case* $G_2(y)$ *is essentially equal to the Picard kernel,* $e^{-|y|}$, *and in the second case to* $|y|^{-1}e^{-|y|}$.

*According to Corollary 1, then,* $T_{G_2}$ *is bounded on* $L^p(e^{k/p'|y|})$ *and* $L^p(e^{-k/p|y|})$ *when* $n = 1$; *on* $L^p\left(|y|^{1/p'}e^{k/p'|y|}\right)$ *and* $L^p\left(|y|^{1/p}e^{-k/p|y|}\right)$ *when* $n = 3$.

## 5. The Gauss–Weierstrass Operators

In this section, we briefly treat the Gauss–Weierstrass kernels, $\{W_t\}_{t>0}$, defined by

$$W_t(y) = (4\pi t)^{-n/2} \exp(-|y|^2/4t),\ y \in R^n.$$

The iterates of $W_t$ satisfy $W_t^{(h)} = W_{ht}, h = 1, 2, \ldots$.

**Proposition 4.** *Fix* $p \in (1, \infty)$ *and set* $w_\beta(x) = e^{\beta|x|}$. *Then,* $W_t$ *is bounded on* $L^p(w_\beta)$ *for all* $\beta \in (-\infty, \infty)$. *Moreover, one has*

$$\lim_{t\to 0_+} \|W_t * f - f\|_{p,w_\beta} = 0, \tag{18}$$

*for every* $f \in L^p(w_\beta)$.

**Proof.** Only $\beta \ge 0$ need by considered, the result for $\beta < 0$ follows by duality.

It will suffice to show that for each $\beta \ge 0$,

$$(W_t * e^{\beta|\cdot|})(x) \le C e^{\beta|x|},$$

with $C > 1$ independent of $x \in R^n$ and $t \in (0, 1)$.

Now,

$$\int_{R^n} W_t(y) e^{\beta|x-y|}\,dy \le \int_{R^n} W_t(y) e^{\beta[|x|+|y|]}\,dy = e^{\beta|x|} \int_{R^n} W_t(y) e^{\beta|y|}\,dy,$$

from which the boundedness assertion follows. Again $W_t(y)$ is an increasing function of $t$ for fixed $y$ with $|y| \geq \sqrt{2nt}$ so,

$$\begin{aligned}
\int_{R^n} W_t(y) e^{\beta|y|}\, dy &= \left( \int_{|y|<\sqrt{2nt}} + \int_{|y|>\sqrt{2nt}} \right) W_t(y) e^{\beta|y|}\, dy \\
&\leq e^{\beta\sqrt{2nt}} \int_{|y|<\sqrt{2nt}} W_t(y)\, dy + \int_{|y|>\sqrt{2nt}} W_1(y) e^{\beta|y|}\, dy \\
&\leq e^{\beta\sqrt{2n}} + (4\pi)^{-n/2} \int_{R_n} \exp(-|y|^2/4) e^{\beta|y|}\, dy
\end{aligned}$$

when $t \in (0,1)$, thereby yielding (18). $\square$

**Theorem 5.** *Fix $C > 1$. Then, the weight-generating kernel for $W_l$ corresponding to $C$ is equivalent to*

$$t^{-\frac{n}{4}-\frac{1}{2}} |y|^{1-n/2} \exp(-t^{-1/2} k|y|),\ k = \sqrt{\log K},\ \textit{for some } K > 1,$$

*with the constants of equivalence independent of $t \in (0,a)$, $|y| > 4ka^{1/2}$, where $0 < a < 1$.*

**Proof.** The desired kernel is

$$\sum_{j=1}^{\infty} C^{-j} (4\pi t j)^{-n/2} \exp(-r^2/4jt) \tag{19}$$

where $r = |y|$.

Let $f(r,t,u) = C^{-u}(4\pi t u)^{-n/2} \exp(-r^2/4ut)$, $u > 0$, and let $\alpha = t^{-1/2}kr$. Denote by $I_1, I_2$ and $I_3$ the intervals $(0, \alpha/4k^2)$, $(\alpha/4k^2, 2\alpha/k^2)$ and $(2\alpha/k^2, \infty)$, respectively. It is easily shown that when $r > 1$ and $t \in (0,1)$, the function $f$, as a function of $u$, increases on $I_1$, decreases on $I_3$ and satisfies $K^{-1} f(r,t,u) \leq f(r,t,u+s) \leq K f(r,t,u)$ for some $K > 1$ and all $u \in I_2$, $s \in (0,1)$. Thus, the study of the sum in (19) amounts to looking at the integrals

$$J_i = \int_{I_i} f(r,t,u)\, du,\ i = 1,2,3.$$

Indeed, $C^{-u} = e^{-k^2 u}$, therefore,

$$\begin{aligned}
C^{-1}(J_1 + J_2 + J_3) &= C^{-1} \left( \int_0^{[\alpha/4h^2]+1} + \int_{[\alpha/4k^2]+1}^{[2\alpha/k^2]} + \int_{[2\alpha/k^2]}^{\infty} \right) f(r,t,u)\, du \\
&\leq \sum_{j=1}^{\infty} f(r,t,j) \\
&= \left( \sum_{j=1}^{[\alpha/4k^2]} + \sum_{j=[\alpha/4k^2]+1}^{[2\alpha/k^2]} + \sum_{j=[2\alpha/k^2]+1}^{\infty} \right) f(r,t,u)\, du \\
&\leq C(J_1 + J_2 + J_3).
\end{aligned}$$

We have

$$\begin{aligned}
J_1 &\leq t^{-n/2} \left( \frac{\alpha}{4k^2} \right)^{-n/2} \exp\left( -k^2 \frac{\alpha}{4k^2} \right) \exp\left( -|y|^2 \Big/ \frac{4\alpha t}{4k^2} \right) \frac{\alpha}{4k^2} \\
&\leq t^{-\frac{n}{4}-\frac{1}{2}} |y|^{1-\frac{n}{2}} \exp\left( -\frac{5}{4} t^{-1/2} k|y| \right)
\end{aligned}$$

Again,

$$J_3 \le t^{-n/2}\left(\frac{2\alpha}{k^2}\right)^{-n/2} \exp\left(-r^2\Big/\frac{4\alpha}{4k^2}t\right)\exp\left(-k^2\frac{\alpha}{4k^2}\right)$$
$$\le t^{-n/4}|y|^{-n/2}\exp\left(-\frac{5}{4}t^{-1/2}k|y|\right) \le J_1.$$

Finally, in $J_2$ take $u = \alpha v/2k^2$ to get

$$J_2 \le t^{-n/4}|y|^{-n/2}\int_{1/2}^{4}\exp\left(-\frac{\alpha}{2}\left[v+\frac{1}{v}\right]\right)v^{-n/2}\,dv$$
$$\le t^{-\frac{n}{4}-\frac{1}{2}}|y|^{1-\frac{n}{2}}\exp\left(-t^{-1/2}k|y|\right).$$

Altogether, then,

$$\int_0^\infty f(|y|,t,u)\,du \le t^{-\frac{n}{4}-\frac{1}{2}}|y|^{1-\frac{n}{2}}\exp\left(-t^{-1/2}k|y|\right).$$

☐

**Remark 4.** *The weight-generating kernels are similar to those of* $G_2$ *on* $R^1$ *and* $R^3$ *(see Example 2), whence the exponential weights of Proposition 4 are in some sense typical. This illustrates a general theorem of Lofstrom, [18], which asserts that no translation-invariant operator is bounded on* $L^p(w)$*, when* $w$ *is a rapidly varying weight such as* $w(\alpha) = \exp(|x|^\alpha), \alpha > 1$.

## 6. The Hardy Averaging Operators

In this section we consider Lebesgue-measurable functions defined on the set

$$R^n_+ = \{y \in R^n : y_i > 0, i = 1, \dots, n\},$$

where, as usual, we write $y = (y_1, \dots, y_n)$. Given $x \in R^n_+$, we define the sets

$$E_n(x) = \{y \in R^n_+ : 0 < y_i < x_i, i = 1, \dots, n\}$$

and

$$F_n(x) = \{y \in R^n_+ : 0 < x_i < y_i, i = 1, \dots, n\}.$$

Finally, we denote the product $x_1^{-1} \dots x_n^{-1}$ by $x^{-1}$ or $\frac{1}{x}$ and the product $(\log \frac{x_1}{y_1}) \dots (\log \frac{x_n}{y_n})$ by $\log \frac{x}{y}$; here, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ belong to $R^n_+$.

The Hardy averaging operators, $P_n$ and $Q_n$, are defined at $f \in P(R^n_+)$, $x \in R^n_+$, by

$$(P_n f)(x) = x^{-1}\int_{E_n(x)} f(y)\,dy$$

and

$$(Q_n f)(x) = \int_{F_n(x)} f(y)\,\frac{dy}{y}.$$

These operators, which are the transposes of one another, are generalizations to $n$-dimensions of the well-known ones, considered in [5] for example. A simple induction argument leads to the following formulas for the iterates of $P_n$ and $Q_n$ :

$$\left(P_n^{(j)} f\right)(x) = \frac{x^{-1}}{\Gamma(j)^n}\int_{F_n(s)} f(y)[\log x/y]^{j-1}\frac{dy}{y},$$

and

$$\left(Q_n^{(j)}f\right)(x) = \frac{1}{\Gamma(j)^n}\int_{F_n(s)} f(y)[\log y/x]^{j-1}\frac{dy}{y},$$

in which $x \in R_+^n$ and $j = 0, 1, \ldots$.

From Theorem 1 of [19], we obtain the representations of the weight-generating kernels of $P_n$ and $Q_n$ described below.

**Theorem 6.** *For $C > 1$ and set $\alpha = nC^{-1/n}$. Then, the weight-generating kernels for $P_n$ and $Q_n$ corresponding to $C$ are equivalent, respectively, to*

$$x^{-1}\left[1 + (\log x/y)^{1/2(n-1)}\exp[\alpha(\log x/y)^{1/n}]\right]\chi_{E_n(x)}(y) \tag{20}$$

*and*

$$y^{-1}\left[1 + (\log y/x)^{1/2(n-1)}\exp[\alpha(\log y/x)^{1/n}]\right]\chi_{F_n(x)}(y). \tag{21}$$

**Proposition 5.** *Let $w_\beta(x) = [1 + |x|]^\beta$, $\beta \in R$. Then $P_n$ is bounded on $L^p(w_\beta)$ if and only if $\beta < 1/p'$; by duality, $Q_n$ is bounded on $L^p(w_\beta)$ of and only if $\beta > -1/p$.*

**Proof.** For simplicity, we consider $n = 2$ only.

Take $\psi = w_\gamma$ and fix $\alpha \in (0, 2)$. Denote by $g$ the weight-generating kernel (20) applied to $\psi$. The change of variable $y_1 = x_1 z_1$, $y_2 = x_2 z_2$ in the integral giving $g(x)$ yields

$$\begin{aligned} g(x) = \int_0^1\int_0^1 &\left[1 + \sqrt{x_1^2 z_1^2 + x_2^2 z_2^2}\right]^\gamma \\ &\left[1 + (\log 1/z_1 \log 1/z_2)^{-1/4} \times \exp\left[\alpha(\log 1/z_1 \log 1/z_2)^{1/2}\right]\right] dz_1\, dz_2 \end{aligned}$$

Hence, when $r > -1$, we find

$$g(x) \approx \begin{cases} 1, & 0 < x_1,\ x_2 \le 1 \\ x_2^\gamma, & 0 < x_1 \le 1,\ x_2 > 1 \\ x_1^\gamma, & x_1 > 1,\ 0 < x_2 \le 1 \\ \max\left[x_1^\gamma, x_2^\gamma\right], & x_1,\ x_2 \ge 1; \end{cases}$$

that is, $g(x) \approx w_\gamma(x)$, provided $r > -1$. This proves the "if" part, since $\beta = -\gamma/p' < 1/p'$.

To see that we must have $\gamma < 1/p'$, note that $h = \chi_{E_2}(\dot{x})$, $\dot{x} = (1, 1)$, is in $L^p(w_\gamma)$ and

$$(P_2 h)(x) = \begin{cases} 1, & 0 < x_1,\ x_2 \le 1 \\ x_2^{-1}, & 0 < x_1 \le 1,\ x_2 > 1 \\ x_1^{-1}, & x_1 > 1,\ 0 < x_2 \le 1 \\ x_1^{-1}x_2^{-1}, & x_1,\ x_2 \ge 1 \end{cases}$$

so

$$\int_{R_+^2} [w_\beta P_2 h]^p = \infty, \text{ if } \beta \ge 1/p'.$$

□

**Theorem 7.** *Denote by $G_1$ and $G_2$ the positive integral operators on $P(R_+^n)$ with kernels (20) and (21), respectively. Suppose $\psi_i \in P(R_+^n)$ is such that $G_i\psi_i < \infty$ ae on $R_+^n$, $i = 1, 2$. Take $\phi_i = \psi_i + G_i\psi_i$, $i = 1, 2$ and set $w = \phi_1^{-\frac{1}{p'}}\phi_2^{\frac{1}{p}}$. Then,*

$$P_n \colon L^p(R_+^n) \to L^p(R_+^n). \tag{22}$$

*Moreover, any weight $w$ satisfying (22) is equivalent to one in the above form.*

**Proof.** This result is a consequence of Corollary 1 and Theorem 2. ☐

**Remark 5.** *When $n = 1$, the functions $x^\beta$, $\beta > -1$, are eigenfunctions of the operator $P$ corresponding to the eigenvalue $(\beta + 1)^{-1}$. As a result, if $\phi(x) = \sum\limits_{k=0}^{\infty} a_k x^k$ converges for all $x$ and if $a_k > 0$, then there exists $\psi \in P(R_+)$ for which $\psi + \sum\limits_{j=1}^{\infty} C^{-j} P^{(j)} \psi \approx \phi$, $C > 1$; namely. $\psi(x) = b_0 + \sum\limits_{k=1}^{\infty} b_k x^k$, where $b_k = a_k \left(1 + \sum\limits_{j=1}^{\infty} \frac{c^{-j}}{(k+1)^j}\right)_k^{-1}$, $k = 0, 1, \ldots$.*

For example, $\phi_1(x) = e^{\beta p' e^x}$, $\beta > 0$, is an entire function with $\phi^{(k)}(0) > 0$, $k = 0, 1, \ldots$. Combining this $\phi_1(x)$ with $\phi_2(x) = x^{\gamma p}$ we obtain the $P$-weight $x^\gamma e^{-\beta e^x}$, $\gamma < 0 < \beta$. Interpolation with change of measure shows one can, in fact, take all $\gamma < 1/p'$.

Similar results are obtained when $\phi(x_1, \ldots, x_n)$ is everywhere on $R^n$ the sum of a power series in $x_1, \ldots, x_n$ with nonnegative coefficients. To take a specific example, consider a power series in one variable, $\sum\limits_{k=0}^{\infty} a_k x^k$, $a_k > 0$, which converges for all $x \in R$. Then, $\phi(x_1, \ldots, x_n) = \sum\limits_{k=0}^{\infty} a_k (x_1 \ldots x_n)^k$ leads to the $P_n$-weights $w(x_1, \ldots, x_n) = x_1^{\gamma_1} \ldots x_n^{\gamma_n}\ \phi(x_1, \ldots, x_n)^{1/p'}$, where $\gamma_i < 1/p'$, $i = 1, \ldots, n$.

Criteria for the boundedness of Hardy operators between weighted Lebesgue spaces with possibly different weights are given in [5] for the case $n = 1$ and in [7] for the case $n = 2$.

**Added in Proof:** While this work was in press the author came across the paper [20]. In it Bloom proves our Theorem 1 using complex interpolation rather than interpolation with change of measure. A (typical) application of his result to the Hardy operators substitutes them in the necessary and sufficient conditions, thereby giving a criterion *for their two* weighted boundedness. This is in contrast to our Theorem 6, in which the explicit form of a *single weight* is given.

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# Some Improvements of the Hermite–Hadamard Integral Inequality

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**Abstract:** We propose several improvements of the Hermite–Hadamard inequality in the form of linear combination of its end-points and establish best possible constants. Improvements of a second order for the class $\Phi(I)$ with applications in Analysis and Theory of Means are also given.

**Keywords:** Convex function; Simpson's rule; differentiable function

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## 1. Introduction

A function $h : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on a non-empty interval $I$ if the inequality

$$h(\frac{x+y}{2}) \le \frac{h(x)+h(y)}{2} \tag{1}$$

holds for all $x, y \in I$.

If the inequality (1) reverses, then $h$ is said to be concave on $I$ [1].

Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval $I$ and $a, b \in I$ with $a < b$. Then

$$h(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b h(t)dt \le \frac{h(a)+h(b)}{2}. \tag{2}$$

This double inequality is well known in the literature as the Hermite–Hadamard (HH) integral inequality for convex functions. It has a plenty of applications in different parts of Mathematics; see [2,3] and references therein.

If $h$ is a concave function on $I$ then both inequalities in (2) hold in the reversed direction.

Our task in this paper is to improve the inequality (2) in a simple manner, i.e., to find some constants $p, q; p+q=1$ such that the relations

$$\frac{1}{b-a}\int_a^b h(t)dt \lesseqgtr p\frac{h(a)+h(b)}{2} + qh(\frac{a+b}{2}), \tag{3}$$

hold for any convex $h$.

It can be easily seen that the condition

$$p+q=1, \tag{4}$$

is necessary for (3) to hold for an arbitrary convex function.

Take, for example, $f(t) = Ct,\ C \in \mathbb{R}$.

Since

$$p(\frac{h(a)+h(b)}{2}) + qh(\frac{a+b}{2}) \le \max\{\frac{h(a)+h(b)}{2}, h(\frac{a+b}{2})\} = \frac{h(a)+h(b)}{2},$$

and, analogously,

$$p(\frac{h(a)+h(b)}{2}) + qh(\frac{a+b}{2}) \ge \min\{\frac{h(a)+h(b)}{2}, h(\frac{a+b}{2})\} = h(\frac{a+b}{2}),$$

it follows that the inequality of the form (3) represents a refinement of Hermite–Hadamard inequality (2) for each $p, q > 0, p + q = 1$.

Note also that the linear form $p\frac{h(a)+h(b)}{2} + qh(\frac{a+b}{2})$ is monotone increasing in $p$. Therefore, if the inequality

$$\frac{1}{b-a}\int_a^b h(t)dt \le p\frac{h(a)+h(b)}{2} + qh(\frac{a+b}{2}),$$

holds for some $p = p_0$, then it also holds for each $p \in [p_0, 1]$.

In the sequel we shall prove that the value $p_0 = 1/2$ is best possible for above inequality to hold for an arbitrary convex function on $I$.

Also, it will be shown that convexity/concavity of the second derivative is a proper condition for inequalities of the form (3) to hold (see Proposition 5 below).

This condition enables us to give refinements of second order and to increase interval of validity to $p_0 = 1/3$ as the best possible constant. In this case, coefficients $p_0 = 1/3, q_0 = 2/3$ are involved in the well-known form of Simpson's rule, which is of great importance in Numerical Analysis. Our results sharply improve Simpson's rule for this class of functions (Proposition 4).

Finally, we give some applications in Analysis and Numerical Analysis. Also, new and precise inequalities between generalized arithmetic means and power-difference means will be proved.

## 2. Results and Proofs

We shall begin with the basic contribution to the problem defined above.

**Theorem 1.** *Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval $I$ and $a, b \in I$. Then*

$$\frac{1}{b-a}\int_a^b h(t)dt \le \frac{1}{2}\frac{h(a)+h(b)}{2} + \frac{1}{2}h(\frac{a+b}{2}). \tag{5}$$

*The constants $p_0 = q_0 = 1/2$ are best possible.*

*If $h$ is a concave function on $I$ then the inequality is reversed.*

**Proof.** We shall derive the proof by Hermite–Hadamard inequality itself. Indeed, applying twice the right part of this inequality, we get

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}} h(t)dt \le \frac{1}{2}(h(a) + h(\frac{a+b}{2})),$$

and

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^b h(t)dt \le \frac{1}{2}(h(\frac{a+b}{2}) + h(b)).$$

Summing up those inequalities the result appears. Therefore, HH inequality has this self-improving property.

That the constants $p_0 = q_0 = 1/2$ are best possible becomes evident by the example $f(t) = |t|, t \in [-a, a]$.

For the second part, note that concavity of $f$ implies convexity of $-f$ on $I$. Hence, applying (5) we get the result. □

For the sake of further refinements, we shall consider in the sequel functions from the class $C^{(m)}(I), m \in \mathbb{N}$ i.e., functions which are continuously differentiable up to m-th order on an interval $I \subset \mathbb{R}$.

Of utmost importance here is the class $\Phi(I)$ of functions which second derivative is convex on $I$. For this class we have the following

**Theorem 2.** *Let $\phi \in \Phi(I)$ and the inequality*

$$\frac{1}{b-a}\int_a^b \phi(t)dt \le p\frac{\phi(a)+\phi(b)}{2} + q\phi(\frac{a+b}{2}), \tag{6}$$

*holds for $a, b \in I$. Then $p \ge p_0 = 1/3$.*

**Proof.** From (6) we have

$$p \ge \frac{\frac{1}{b-a}\int_a^b \phi(t)dt - \phi(\frac{a+b}{2})}{\frac{\phi(a)+\phi(b)}{2} - \phi(\frac{a+b}{2})} =: D_\phi(a,b).$$

Since this inequality should be valid for each $a, b \in I, a < b$, let $b \to a$. We obtain that $\lim_{b \to a} D_\phi(a,b) = 1/3$ almost everywhere on $I$ i.e, whenever $\phi''(a) \ne 0$ or $\phi''(a) = 0, \phi'''(a) \ne 0$.

Indeed, applying L'Hospital's rule 3 and 4 times to the above quotient, we get

$$\lim_{b\to a} D_\phi(a,b) = \lim_{b\to a} \frac{\phi''(b) - \frac{3}{4}\phi''(\frac{a+b}{2}) - \frac{b-a}{8}\phi'''(\frac{a+b}{2})}{\frac{3}{2}\phi''(b) - \frac{3}{4}\phi''(\frac{a+b}{2}) + (b-a)(\frac{1}{2}\phi'''(b) - \frac{1}{8}\phi'''(\frac{a+b}{2}))},$$

and

$$\lim_{b\to a} D_\phi(a,b) = \lim_{b\to a} \frac{\phi'''(b) - \frac{1}{2}\phi'''(\frac{a+b}{2}) - \frac{b-a}{16}\phi^{(4)}(\frac{a+b}{2})}{2\phi'''(b) - \frac{1}{2}\phi'''(\frac{a+b}{2}) + (b-a)(\frac{1}{2}\phi^{(4)}(b) - \frac{1}{16}\phi^{(4)}(\frac{a+b}{2}))}.$$

Therefore, the result follows. □

In the sequel we shall give sharp two-sided bounds of second order for inequalities of the type (3) involving functions from the class $\Phi$ with $p \ge 1/3$.

Main tool in all proofs will be the following relation.

**Lemma 1.** *For an integrable function $\phi : I \to \mathbb{R}$ and arbitrary real numbers $p, q; p + q = 1$, we have the identity*

$$p\frac{\phi(a)+\phi(b)}{2} + q\phi(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b \phi(t)dt = \frac{(b-a)^2}{16}\int_0^1 t(2p-t)(\phi''(x) + \phi''(y))dt,$$

*where $x := a\frac{t}{2} + b(1 - \frac{t}{2}), y := b\frac{t}{2} + a(1 - \frac{t}{2})$.*

**Proof.** It is not difficult to prove this identity by double partial integration of its right-hand side. □

For $t \in [0,1]; a, b \in I, a < b$, denote

$$\xi(a,b;t) := \phi''(a\frac{t}{2} + b(1-\frac{t}{2})) + \phi''(b\frac{t}{2} + a(1-\frac{t}{2}))$$
$$= \phi''(x) + \phi''(y).$$

**Lemma 2.** *If $\phi \in \Phi$ then the function $\xi(a,b;t)$ is monotone decreasing in $t$.*

*Hence,*

$$2\phi''(\frac{a+b}{2}) \leq \phi''(x) + \phi''(y) \leq \phi''(a) + \phi''(b), \tag{7}$$

*for all $t \in [0,1]$.*

**Proof.** Since $\phi''(\cdot)$ is convex, it follows that $\phi'''(\cdot)$ is increasing on $I$.

Also, $x \geq y$ for $t \in [0,1]$ because $x - y = (b-a)(1-t) \geq 0$.

Hence,

$$\xi'(a,b;t) = -\frac{b-a}{2}(\phi'''(x) - \phi'''(y)) \leq 0,$$

and $\xi(a,b;t)$ is decreasing in $t \in [0,1]$.

Therefore,

$$2\phi''(\frac{a+b}{2}) = \xi(a,b;1) \leq \xi(a,b;t) \leq \xi(a,b;0) = \phi''(a) + \phi''(b),$$

which is equivalent with (7).

Note that, if $\phi$ is concave on $I$, then the function $\xi(a,b;t)$ is monotone increasing and the inequality (7) is reversed. □

**Remark 1.** *More general assertion than (7) is contained in [4].*

Main results of this paper are given in the next two assertions.

**Theorem 3.** *Let $\phi \in \Phi(I)$. Then*

$$p\frac{\phi(a)+\phi(b)}{2} + q\phi(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b \phi(t)dt \leq \frac{(b-a)^2}{16}T_\phi(a,b;p),$$

*where*

$$T_\phi(a,b;p) = \begin{cases} \frac{4}{3}p^3(\phi''(a)+\phi''(b)) - \frac{2}{3}(1+p)(2p-1)^2\phi''(\frac{a+b}{2}) & , \frac{1}{3} \leq p \leq \frac{1}{2}; \\ (p-\frac{1}{3})(\phi''(a)+\phi''(b)) & , p \geq \frac{1}{2}. \end{cases}$$

*Also, if $p \leq 0$, we have*

$$p\frac{\phi(a)+\phi(b)}{2} + q\phi(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b \phi(t)dt \leq (p-\frac{1}{3})\frac{(b-a)^2}{8}\phi''(\frac{a+b}{2}).$$

**Proof.** If $p \geq 1/2$ we have that $2p - t \geq 0$. Therefore, applying Lemma 1 and the second part of Lemma 2, we obtain

$$p\frac{\phi(a)+\phi(b)}{2} + q\phi(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b \phi(t)dt = \frac{(b-a)^2}{16}\int_0^1 t(2p-t)(\phi''(x)+\phi''(y))dt$$
$$\leq \frac{(b-a)^2}{16}(\phi''(a)+\phi''(b))\int_0^1 t(2p-t)dt = (p-1/3)\frac{(b-a)^2}{16}(\phi''(a)+\phi''(b)).$$

In the case $1/3 \le p < 1/2$, write

$$\int_0^1 t(2p-t)(\phi''(x)+\phi''(y))dt = \int_0^{2p} t(2p-t)(\cdot)dt - \int_{2p}^1 t(t-2p)(\cdot)dt,$$

and apply Lemma 2 to each integral separately.

It follows that

$$\int_0^1 t(2p-t)(\phi''(x)+\phi''(y))dt \le (\phi''(a)+\phi''(b))\int_0^{2p} t(2p-t)dt - 2\phi''(\frac{a+b}{2})\int_{2p}^1 t(t-2p)dt$$

$$= \frac{4p^3}{3}(\phi''(a)+\phi''(b)) - 2(\frac{1}{3}-p+\frac{4p^3}{3})\phi''(\frac{a+b}{2}),$$

which is equivalent to the stated assertion.

For $p \le 0$ we have that $2p-t \le 0$ and the proof develops in the same manner. □

**Theorem 4.** *If $\phi \in \Phi(I)$, then for $p \ge 1/3$ we get*

$$p\frac{\phi(a)+\phi(b)}{2} + q\phi(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b \phi(t)dt \ge (p-1/3)\frac{(b-a)^2}{8}\phi''(\frac{a+b}{2}),$$

*and*

$$p\frac{\phi(a)+\phi(b)}{2} + q\phi(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b \phi(t)dt \ge (p-1/3)\frac{(b-a)^2}{16}(\phi''(a)+\phi''(b)),$$

*for $p \le 0$.*

**Proof.** By Lemma 1, in terms of Lemma 2, we have

$$p\frac{\phi(a)+\phi(b)}{2} + q\phi(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b \phi(t)dt = \frac{(b-a)^2}{16}\int_0^1 t(2p-t)\xi(a,b;t)dt.$$

By partial integration, we obtain

$$\int_0^1 t(2p-t)\xi(a,b;t)dt = (pt^2 - t^3/3)\xi(a,b;t)|_0^1 - \int_0^1 t^2(p-t/3)\xi'(a,b;t)dt$$

$$\ge 2(p-1/3)\phi''(\frac{a+b}{2}),$$

since $p - t/3 \ge 0$ for $p \ge 1/3$ and, by Lemma 2, $\xi'(a,b;t) \le 0$ for $t \in [0,1]$.

If $p \le 0$ then $2p-t \le 0$ and, applying Lemmas 1 and 2, the result follows. □

Above theorems are the source of a plenty of important inequalities which sharply refine Hermite–Hadamard inequality for this class of functions.

Some of them are listed in the sequel.

**Proposition 1.** *Let $\phi \in \Phi(I)$. Then*

$$\frac{(b-a)^2}{24}\phi''(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b \phi(t)dt - \phi(\frac{a+b}{2}) \le \frac{(b-a)^2}{24}\frac{\phi''(a)+\phi''(b)}{2}.$$

**Proof.** Put $p = 0$ in the above theorems. □

**Proposition 2.** *Let $\phi \in \Phi(I)$. Then*

$$\frac{(b-a)^2}{12}\phi''(\frac{a+b}{2}) \le \frac{\phi(a)+\phi(b)}{2} - \frac{1}{b-a}\int_a^b \phi(t)dt \le \frac{(b-a)^2}{12}\frac{\phi''(a)+\phi''(b)}{2}.$$

**Proof.** This proposition is obtained for $p = 1$. □

The next assertion represents a refinement of Theorem 1 in the case of convex functions.

**Proposition 3.** *Let $\phi \in \Phi(I)$. Then for each $a, b \in I, a < b$,*

$$\frac{(b-a)^2}{48}\phi''(\frac{a+b}{2}) \le \frac{1}{2}\frac{\phi(a)+\phi(b)}{2} + \frac{1}{2}\phi(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b \phi(t)dt \le \frac{(b-a)^2}{48}\frac{\phi''(a)+\phi''(b)}{2}.$$

*If $\phi''$ is concave on $I$, then*

$$\frac{(b-a)^2}{48}\frac{\phi''(a)+\phi''(b)}{2} \le \frac{1}{2}\frac{\phi(a)+\phi(b)}{2} + \frac{1}{2}\phi(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b \phi(t)dt \le \frac{(b-a)^2}{48}\phi''(\frac{a+b}{2}).$$

**Proof.** Put $p = 1/2$ in Theorems 3 and 4.

The second part follows from a variant of Lemma 2 for concave functions. □

Note that the coefficients $p = 1/3$ and $q = 2/3$ are involved in well-known Simpson's rule which is of importance in numerical integration [5].

The next assertion sharply refines Simpson's rule for this class of functions.

**Proposition 4.** *For $\phi \in \Phi(I)$, we have*

$$\begin{aligned} 0 &\le \frac{1}{3}\frac{\phi(a)+\phi(b)}{2} + \frac{2}{3}\phi(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b \phi(t)dt \\ &\le \frac{(b-a)^2}{162}[\frac{\phi''(a)+\phi''(b)}{2} - \phi''(\frac{a+b}{2})]. \end{aligned}$$

*If $\phi''$ is concave on $I$, then*

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b \phi(t)dt - \frac{1}{6}[\phi(a)+\phi(b)+4\phi(\frac{a+b}{2})] \\ &\le \frac{(b-a)^2}{162}[\phi''(\frac{a+b}{2}) - \frac{\phi''(a)+\phi''(b)}{2}]. \end{aligned}$$

**Proof.** Applying Theorems 3 and 4 with both parts of Lemma 2 for $p = 1/3$, the proof follows. □

The next assertion gives a proper answer to the problem posed in Introduction.

**Proposition 5.** *If $\phi$ is a convex and $\phi''$ is a concave function on $I$, then*

$$\frac{1}{3}\frac{\phi(a)+\phi(b)}{2} + \frac{2}{3}\phi(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b \phi(t)dt \le \frac{1}{2}\frac{\phi(a)+\phi(b)}{2} + \frac{1}{2}\phi(\frac{a+b}{2}).$$

*Analogously, let $\phi$ be concave and $\phi''$ a convex function on $I$, then*

$$\frac{1}{2}\frac{\phi(a)+\phi(b)}{2} + \frac{1}{2}\phi(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b \phi(t)dt \le \frac{1}{3}\frac{\phi(a)+\phi(b)}{2} + \frac{2}{3}\phi(\frac{a+b}{2}).$$

**Proof.** Combining Proposition 4 with the results of Theorem 1, we obtain the proof. □

## 3. Applications in Analysis

Theorems proved above are the source of interesting inequalities from Classical Analysis. As an illustration we shall give here a couple of Cusa-type inequalities.

**Theorem 5.** *The inequality*

$$\frac{1}{2}\cos x+\frac{1}{2}\le\frac{\sin x}{x}\le\frac{1}{3}\cos x+\frac{2}{3},$$

*holds for* $|x|\le\pi/2$.

*Also,*

$$\frac{1}{4}\cosh x+\frac{3}{4}\le\frac{\sinh x}{x}\le\frac{1}{3}\cosh x+\frac{2}{3},$$

*holds for* $|x|\le(3/2)^{3/2}$.

**Proof.** For the first part one should apply Proposition 5 to the function $\phi(t)=\cos t$ on a symmetric interval $t\in[-x,x]\subset[-\pi/2,\pi/2]$.

For the second part, applying Proposition 4 with $\phi(t)=e^t, t\in[-x,x]$, we get

$$0\le\frac{1}{3}\cosh x+\frac{2}{3}-\frac{\sinh x}{x}\le\frac{2x^2}{81}(\cosh x-1).$$

Hence,

$$\frac{\sinh x}{x}\le\frac{1}{3}\cosh x+\frac{2}{3},$$

and

$$\frac{\sinh x}{x}\ge\frac{1}{3}\cosh x+\frac{2}{3}-\frac{2x^2}{81}(\cosh x-1)$$
$$=(\frac{1}{12}-\frac{2x^2}{81})\cosh x+\frac{1}{4}\cosh x+\frac{2}{3}+\frac{2x^2}{81}\ge\frac{1}{4}\cosh x+\frac{3}{4},$$

since $\cosh x\ge 1$ and $1/12-2x^2/81\ge 0$ for $|x|\le(3/2)^{3/2}\approx 1.8371$. □

We give now some numerical examples of the above inequality

$$\frac{1}{2}\cos x+\frac{1}{2}\le\frac{\sin x}{x}\le\frac{1}{3}\cos x+\frac{2}{3},\quad |x|\le\pi/2. \tag{8}$$

Namely, using known formulae

$$\sin\frac{\pi}{2}=1;\ \sin\frac{\pi}{4}=\frac{\sqrt{2}}{2};\ \sin\frac{\pi}{6}=\frac{1}{2};\ \sin\frac{\pi}{12}=\frac{\sqrt{2}}{4}(\sqrt{3}-1)\approx 0.25882;$$

$$\sin\frac{\pi}{24}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}}\approx 0.13053;\ \sin\frac{\pi}{60}=\frac{1}{16}[\sqrt{2}(\sqrt{3}+1)(\sqrt{5}-1)-2(\sqrt{3}-1)\sqrt{5+\sqrt{5}}]\approx 0.052336,$$

and applying inequalities (8), we obtain bounds for the transcendental number $\pi$, as follows

$$x=\frac{\pi}{2}:3<\pi<4;\ x=\frac{\pi}{4}:3.1344<\pi<3.3137;\ x=\frac{\pi}{6}:3.1402<\pi<3.2154;$$

$$x=\frac{\pi}{12}:3.1415<\pi<3.1597;\ x=\frac{\pi}{24}:3.1416<\pi<3.1461;\ x=\frac{\pi}{60}:3.1416<\pi<3.1423.$$

Another application can be obtained by integrating both sides of (8) on the range $x\in[0,a],\ 0<a<\pi/2$.

We get

$$\frac{1}{2}\sin a+\frac{1}{2}a \le \int_0^a \frac{\sin x}{x}dx \le \frac{1}{3}\sin a+\frac{2}{3}a,$$

that is,

$$\frac{a-\sin a}{3} \le a-\int_0^a \frac{\sin x}{x}dx \le \frac{a-\sin a}{2}.$$

By the power series expansion, we know that

$$a-\sin a=\frac{a^3}{3!}-\frac{a^5}{5!}+\frac{a^7}{7!}-\cdots.$$

Hence,

$$\frac{a^3}{18}-\frac{a^5}{360} \le a-\int_0^a \frac{\sin x}{x}dx \le \frac{a^3}{12}.$$

This estimation is effective for small values of $a$.
For example,

$$5.5528\times 10^{-5} \le \frac{1}{10}-\int_0^{1/10} \frac{\sin x}{x}dx \le 8.3333\times 10^{-5}.$$

## 4. Applications in Theory of Means

A *mean* $M(a,b)$ is a map $M:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, with the property

$$\min\{a,b\} \le M(a,b) \le \max\{a,b\},$$

for each $a,b\in\mathbb{R}_+$.

Some refinements of HH inequality by arbitrary means is given in [6].

An ordered set of elementary means is the following family,

$$H \le G \le L \le I \le A \le S,$$

where

$$H=H(a,b)=:2(1/a+1/b)^{-1};\;\; G=G(a,b)=:\sqrt{ab};\;\; L=L(a,b)=:\frac{b-a}{\log b-\log a};$$

$$I=I(a,b)=:\frac{1}{e}(b^b/a^a)^{1/(b-a)};\;\; A=A(a,b)=:\frac{a+b}{2};S=S(a,b)=:a^{\frac{a}{a+b}}b^{\frac{b}{a+b}},$$

are the harmonic, geometric, logarithmic, identric, arithmetic and Gini mean, respectively.

Generalized arithmetic mean $A_\alpha$ is defined by

$$A_\alpha=A_\alpha(a,b)=:\begin{cases}\left(\frac{a^\alpha+b^\alpha}{2}\right)^{1/\alpha} & ,\alpha\ne 0;\\ A_0=G. & \end{cases}$$

Power-difference mean $K_\alpha$ is defined by

$$K_\alpha = K_\alpha(a,b) =: \begin{cases} \frac{\alpha}{\alpha+1}\frac{a^{\alpha+1}-b^{\alpha+1}}{a^\alpha-b^\alpha} & ,\alpha \neq 0,-1; \\ K_0(a,b) = L(a,b); \\ K_{-1}(a,b) = ab/L(a,b). \end{cases}$$

It is well known that both means are monotone increasing with $\alpha$ and, evidently,

$$A_{-1} = H, A_1 = A, K_{-2} = H, K_{-1/2} = G, K_1 = A.$$

As an illustration of our results, we shall give firstly some sharp bounds of power-difference means in terms of the generalized arithmetic mean.

**Theorem 6.** *For $a,b \in \mathbb{R}^+$ and $\alpha \geq 1$, we have*

$$\frac{1}{2}(A(a,b) + A_\alpha(a,b)) \leq K_\alpha(a,b) \leq A_\alpha(a,b). \tag{9}$$

*For $\alpha < 1$ the inequality (9) is reversed.*

**Proof.** Let $g_\alpha(t) = t^{1/\alpha}, \alpha \neq 0$. Since $g_\alpha$ is concave for $\alpha \geq 1$, Theorem 1 combined with the HH inequality gives

$$\begin{aligned} &\frac{1}{2}\Big(\frac{x+y}{2}\Big)^{1/\alpha} + \frac{1}{4}(x^{1/\alpha} + y^{1/\alpha}) \\ \leq &\frac{\alpha}{\alpha+1}\frac{x^{1+1/\alpha} - y^{1+1/\alpha}}{x-y} \leq \Big(\frac{x+y}{2}\Big)^{1/\alpha}. \end{aligned}$$

Now, simple change of variables $x = a^\alpha, y = b^\alpha$ yields the result.
For the second part, note that $g_\alpha$ is convex for $\alpha < 1$ and repeat the procedure. □

The above inequality is refined by the following

**Theorem 7.** *We have,*

$$A_\alpha \leq K_\alpha \leq \frac{1}{3}(A + 2A_\alpha), \quad \alpha \in (-\infty, 1/3) \cup (1/2, 1);$$

$$\frac{1}{3}(A + 2A_\alpha) \leq K_\alpha \leq A_\alpha, \quad \alpha \in [1, \infty);$$

$$\frac{1}{3}(A + 2A_\alpha) \leq K_\alpha \leq \frac{1}{2}(A + A_\alpha), \quad \alpha \in [1/3, 1/2].$$

**Proof.** Observe that $g''_\alpha$ is convex for $\alpha \in (-\infty, 1/3) \cup (1/2, 1)$ and concave for $\alpha \in (1/3, 1/2) \cup (1, \infty)$. Hence, applying Proposition 5 together with the HH inequality, we obtain the result. □

**Remark 2.** *Note that the above inequalities are so precise that in critical points for $\alpha = 1/3, 1/2, 1$ we have equality sign.*

An inequality for the reciprocals follows.

**Theorem 8.** *For $\beta \geq -2$ we have*

$$\frac{1}{A_{\beta+1}} \leq \frac{1}{K_\beta} \leq \frac{1}{2}\Big(\frac{1}{H} + \frac{1}{A_{\beta+1}}\Big).$$

*For $\beta < -2$ the inequality is reversed.*

**Proof.** This is a consequence of Theorem 6. Indeed, putting there $\alpha = -\beta - 1$ and using identities

$$K_\alpha = \frac{ab}{K_\beta}, A_\alpha = \frac{ab}{A_{\beta+1}}, A = \frac{ab}{H},$$

the proof appears. □

Finally, we give a new and precise double inequality for the identric mean $I(a,b)$.

**Theorem 9.** *For arbitrary positive $a, b$ we have*

$$A^{4/3}S^{-1/3}\exp\left(-\frac{4}{81}\frac{(A-H)^2}{AH}\right) \le I \le A^{4/3}S^{-1/3}.$$

**Proof.** We need firstly an auxiliary result.

**Lemma 3.** *For $a, b \in \mathbb{R}^+$, we have*

$$A^{4/3}(a,b)S^{2/3}(a,b)\exp\left(-\frac{4}{81}\frac{(A(a,b)-H(a,b))^2}{A(a,b)H(a,b)}\right) \le I(a^2,b^2) \le A^{4/3}(a,b)S^{2/3}(a,b).$$

**Proof.** Indeed, for $\phi(t) = t\log t$ we get

$$\frac{1}{b-a}\int_a^b \phi(t)dt = \frac{1}{4}\left(\frac{b^2\log b^2 - a^2\log a^2}{b-a} - (a+b)\right) = \frac{a+b}{4}\log I(a^2,b^2).$$

Since $\phi''(t) = 1/t$, Proposition 5 yields

$$\frac{1}{6}(a\log a + b\log b) + \frac{2}{3}A\log A - \frac{(b-a)^2}{324}\left(\frac{1}{a}+\frac{1}{b}-\frac{2}{A}\right)$$
$$\le \frac{a+b}{4}\log I(a^2,b^2) \le \frac{1}{6}(a\log a + b\log b) + \frac{2}{3}A\log A,$$

and the proof follows by dividing the last expression with $(a+b)/4 = A/2$. □

Now, combining this assertion with the identity $I(a^2,b^2) = I(a,b)S(a,b)$, we obtain the desired inequality. □

**Remark 3.** *An equivalent form of the above result is*

$$I^{3/4}S^{1/4} \le A \le I^{3/4}S^{1/4}\exp\left(\frac{(A-H)^2}{27AH}\right),$$

*which refines well-known inequality $I \le A \le S$.*

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# The Principle of Differential Subordination and Its Application to Analytic and $p$-Valent Functions Defined by a Generalized Fractional Differintegral Operator

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**Abstract:** A useful family of fractional derivative and integral operators plays a crucial role on the study of mathematics and applied science. In this paper, we introduce an operator defined on the family of analytic functions in the open unit disk by using the generalized fractional derivative and integral operator with convolution. For this operator, we study the subordination-preserving properties and their dual problems. Differential sandwich-type results for this operator are also investigated.

**Keywords:** analytic function; Hadamard product; differential subordination; differential superordination; generalized fractional differintegral operator

**MSC:** 30C45; 30C50

## 1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the family of analytic functions in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[c, n]$ be the subfamily of $\mathcal{H}(\mathbb{D})$ consisting of functions of the form:

$$f(z) = c + b_n z^n + b_{n+1} z^{n+1} + \cdots \quad (c \in \mathbb{C};\ n \in \mathbb{N} = \{1, 2, \cdots\}).$$

Let $\mathcal{A}(p)$ denote the family of analytic functions in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \ (p \in \mathbb{N};\ f^{(p+1)}(0) \neq 0). \tag{1}$$

For $f, F \in \mathcal{H}(\mathbb{D})$, the function $f(z)$ is said to be subordinate to $F(z)$ or $F(z)$ is superordinate to $f(z)$, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a Schwarz function $\omega(z)$ for $z \in \mathbb{D}$ such that $f(z) = F(\omega(z))$. If $F(z)$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$ (see [1,2]).

Let $\phi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{D}$. If $p(z)$ is analytic in $\mathbb{D}$ and satisfies

$$\phi\left(p(z), zp'(z); z\right) \prec h(z), \tag{2}$$

then $p(z)$ is solution Relation (2). The univalent function $q(z)$ is called a dominant of the solutions of Relation (2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying Relation (2). A univalent dominant $\tilde{q}$ that satisfies

$\tilde{q} \prec q$ for all dominants of Relation (2) is called the best dominant. If $p\left(z\right)$ and $\phi\left(p\left(z\right), zp'\left(z\right); z\right)$ are univalent in $\mathbb{D}$ and if $p(z)$ satisfies

$$h\left(z\right) \prec \phi\left(p\left(z\right), zp'\left(z\right); z\right), \tag{3}$$

then $p\left(z\right)$ is a solution of Relation (3). An analytic function $q\left(z\right)$ is called a subordinant of the solutions of Relation (3) if $q\left(z\right) \prec p\left(z\right)$ for all $p\left(z\right)$ satisfying Relation (3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of Relation (3) is called the best subordinant (see [1,2]).

We now introduce the operator $S_{0,z}^{\lambda,\mu,\eta,p}$ due to Goyal and Prajapat [3] (see also [4]) as follows:

$$S_{0,z}^{\lambda,\mu,\eta,p} f(z) = \begin{cases} \dfrac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z) \ (0 \leq \lambda < \eta + p + 1;\ z \in \mathbb{D}), \\ \dfrac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} I_{0,z}^{-\lambda,\mu,\eta} f(z) \ (-\infty < \lambda < 0;\ z \in \mathbb{D}), \end{cases} \tag{4}$$

where $J_{0,z}^{\lambda,\mu,\eta}$ and $I_{0,z}^{-\lambda,\mu,\eta}$ are the generalized fractional derivative and integral operators, respectively, due to Srivastava et al. [5] (see also [6,7]). For $f \in \mathcal{A}(p)$ of form Equation (1), we have

$$\begin{aligned} S_{0,z}^{\lambda,\mu,\eta,p} f(z) &= z^p {}_3F_2(1, 1+p, 1+p+\eta-\mu; 1+p-\mu, 1+p+\eta-\lambda; z) * f(z) \\ &= z^p + \sum_{n=1}^{\infty} \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n} b_{p+n} z^{p+n} \\ &\quad (p \in \mathbb{N};\ \mu, \eta \in \mathbb{R};\ \mu < p+1;\ -\infty < \lambda < \eta + p + 1), \end{aligned} \tag{5}$$

where ${}_qF_s$ $(q \leq s+1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is the well-known generalized hypergeometric function (for details, see [8,9]), the symbol $*$ stands for convolution of two analytic functions [1] and $(\nu)_n$ is the Pochhammer symbol [8,10].

Setting

$$\begin{aligned} G_{p,\eta,\mu}^{\lambda}(z) &= z^p + \sum_{n=1}^{\infty} \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n} z^{p+n} \\ &\quad (p \in \mathbb{N};\ \mu, \eta \in \mathbb{R};\ \mu < \min\{p+1, p+1+\eta\};\ -\infty < \lambda < \eta + p + 1) \end{aligned} \tag{6}$$

and

$$G_{p,\eta,\mu}^{\lambda}(z) * \left[G_{p,\eta,\mu}^{\lambda,\delta}(z)\right] = \frac{z^p}{(1-z)^{\delta+p}} \ (\delta > -p;\ z \in \mathbb{D}),$$

Tang et al. [11] (see also [12]) defined the operator $H_{p,\eta,\mu}^{\lambda,\delta} : \mathcal{A}(p) \to \mathcal{A}(p)$ by

$$H_{p,\eta,\mu}^{\lambda,\delta} f(z) = \left[G_{p,\eta,\mu}^{\lambda,\delta}(z)\right] * f(z).$$

Then, for $f \in \mathcal{A}(p)$, we have

$$H_{p,\eta,\mu}^{\lambda,\delta} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\delta+p)_n (p+1-\mu)_n (p+1-\lambda+\eta)_n}{(1)_n (p+1)_n (p+1-\mu+\eta)_n} b_{p+n} z^{p+n}. \tag{7}$$

It is easy to verify that

$$z\left(H_{p,\eta,\mu}^{\lambda,\delta} f(z)\right)' = (\delta+p) H_{p,\eta,\mu}^{\lambda,\delta+1} f(z) - \delta H_{p,\eta,\mu}^{\lambda,\delta} f(z), \tag{8}$$

and

$$z\left(H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)\right)' = (p+\eta-\lambda) H_{p,\eta,\mu}^{\lambda,\delta} f(z) - (\eta-\lambda) H_{p,\eta,\mu}^{\lambda+1,\delta} f(z). \tag{9}$$

Making use of the hypergeometric function in the kernel, Saigo [13] proposed generalizations of fractional calculus of both Riemann–Liouville and Weyl types. The general theory of fractional calculus thus developed was applied to the study for several multiplication properties of fractional integrals [14]. In particular, Owa et al. [15] and Srivastava et al. [5] investigated some distortion theorems involving fractional integrals, and sufficient conditions for fractional integrals of analytic functions in the open unit disk to be starlike or convex. Moreover, the theory of fractional calculus is widely applied to not only pure mathematics but also applied science. For some interesting developments in applied science such as bioengineering and applied physics, the readers may be referred to the works of (for examples) Hassan et al. [16], Magin [17], Martínez-García et al. [18] and Othman and Marin [19].

By using the principle of subordination, Miller et al. [20] investigated subordinations-preserving properties for certain integral operators. In addition, Miller and Mocanu [2] studied some important properties on superordinations as the dual problem of subordinations. Furthermore, the study of the subordinaton-preserving properties and their dual problems for various operators is a significant role in pure and applied mathematics. The aim of the present paper, motivated by the works mentioned above, is to systematically investigate the subordination- and superordination-preserving results of the generalized fractional differintegral operator defined Equation (7) with certain differential sandwich-type theorems as consequences of the results presented here. Our results give interesting new properties, and together with other papers that appeared in the last years could emphasize the perspective of the importance of differential subordinations and generalized fractional differintegral operators. We also note that, in recent years, several authors obtained many interesting results involving various linear and nonlinear operators associated with differential subordinations and their dual problrms (for details, see [21–28]).

For the proofs of our main results, we shall need some definitions and lemmas stated below.

**Definition 1** ([1]). *We denote by $\mathcal{Q}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{\mathbb{D}}\backslash E(q)$, where*

$$E(q)=\left\{\zeta\in\partial\mathbb{D}:\lim_{z\to\zeta}q(z)=\infty\right\},$$

*and are $q'(\zeta)\neq 0$ for $\zeta\in\partial\mathbb{D}\backslash E(q)$.*

**Definition 2** ([2]). *A function $\mathcal{I}(z,t)$ $(z\in\mathbb{D},\ t\geq 0)$ is a subordination chain if $\mathcal{I}(.,t)$ is analytic and univalent in $\mathbb{D}$ for all $t\geq 0$, $\mathcal{I}(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z\in\mathbb{D}$ and $\mathcal{I}(z,s)\prec\mathcal{I}(z,t)$ for all $0\leq s\leq t$.*

**Lemma 1** ([29]). *Let $H:\mathbb{C}^2\to\mathbb{C}$ satisfy*

$$\Re\left\{H\left(i\sigma;\tau\right)\right\}\leq 0$$

*for all real $\sigma,\tau$ with $\tau\leq -n\left(1+\sigma^2\right)/2$ and $n\in\mathbb{N}$. If $p(z)=1+p_nz^n+p_{n+1}z^{n+1}+\cdots$ is analytic in $\mathbb{D}$ and*

$$\Re\left\{H\left(p(z);zp'(z)\right)\right\}>0\ \ (z\in\mathbb{D}),$$

*then $\Re\left\{p(z)\right\}>0$ for $z\in\mathbb{D}$.*

**Lemma 2** ([30]). *Let $\kappa,\gamma\in\mathbb{C}$ with $\kappa\neq 0$ and let $h\in H(\mathbb{D})$ with $h(0)=c$. If $\Re\left\{\kappa h(z)+\gamma\right\}>0\,(z\in\mathbb{D})$, then the solution of the differential equation:*

$$q\left(z\right)+\frac{zq'\left(z\right)}{\kappa q(z)+\gamma}=h\left(z\right)\ \ (z\in\mathbb{D};\ q(0)=c)$$

*is analytic in $\mathbb{D}$ and satisfies $\Re\left\{\kappa q(z)+\gamma\right\}>0$ for $z\in\mathbb{D}$.*

**Lemma 3** ([1]). *Suppose that $p \in \mathcal{Q}$ with $q(0) = a$ and $q(z) = a + q_n z^n + q_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{D}$ with $q(z) \neq a$ and $n \geq 1$. If $q(z)$ is not subordinate to $p(z)$, then there exists two points $z_0 = r_0 e^{i\theta} \in \mathbb{D}$ and $\xi_0 \in \partial\mathbb{D} \backslash E(q)$ such that*

$$q(z_0) = p(\xi_0) \text{ and } z_0 q'(z_0) = m\xi_0 p'(\xi_0) \ (m \geq n).$$

**Lemma 4** ([2]). *Let $q \in \mathcal{H}[c, 1]$ and $\varphi : \mathbb{C}^2 \to \mathbb{C}$. In addition, let $\varphi(q(z), zq'(z)) = h(z)$. If $\mathcal{I}(z, t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $q \in \mathcal{H}[c, 1] \cap \mathcal{Q}$, then*

$$h(z) \prec \varphi(p(z), zp'(z)),$$

*implies that $q(z) \prec p(z)$. Moreover, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then $q$ is the best subordinant.*

**Lemma 5** ([31]). *The function $\mathcal{I}(z, t) : \mathbb{D} \times [0, \infty) \longrightarrow \mathbb{C}$ of the form*

$$\mathcal{I}(z, t) = a_1(t) z + \cdots (a_1(t) \neq 0;\ t \geq 0)$$

*and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if*

$$\Re\left\{ \frac{z \frac{\partial \mathcal{I}(z,t)}{\partial z}}{\frac{\partial \mathcal{I}(z,t)}{\partial t}} \right\} > 0 \ (z \in \mathbb{D};\ t \geq 0)$$

*and*

$$|\mathcal{I}(z, t)| \leq K_0 |a_1(t)| \ (t \geq 0)$$

*for constants $K_0 > 0$ and $r_0$ $(|z| < r_0 < 1)$.*

## 2. Main Results

Throughout this paper, we assume that $p \in \mathbb{N}$, $\alpha, \beta > 0$, $\delta > -p$, $\mu, \eta \in \mathbb{R}$, $\mu < \min\{p + 1, p + 1 + \eta\}$, $-\infty < \lambda < \eta + p + 1$, $H^{\lambda,\delta}_{p,\eta,\mu} f(z)/z^p \neq 0$ for $f \in \mathcal{A}(p)$ and all the powers are understood as principal values.

**Theorem 1.** *Suppose that $f, g \in \mathcal{A}(p)$ and*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\rho \tag{10}$$

$$\left( \phi(z) = (1 - \alpha) \left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} g(z)}{z^p} \right]^\beta + \alpha \left[ \frac{H^{\lambda,\delta+1}_{p,\eta,\mu} g(z)}{H^{\lambda,\delta}_{p,\eta,\mu} g(z)} \right] \left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} g(z)}{z^p} \right]^\beta ;\ z \in \mathbb{D} \right),$$

*where $\rho$ is given by*

$$\rho = \frac{\alpha^2 + \beta^2 (\delta + p)^2 - \left| \alpha^2 - \beta^2 (\delta + p)^2 \right|}{4\alpha\beta(\delta + p)}. \tag{11}$$

*Then,*

$$(1 - \alpha) \left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} f(z)}{z^p} \right]^\beta + \alpha \left[ \frac{H^{\lambda,\delta+1}_{p,\eta,\mu} f(z)}{H^{\lambda,\delta}_{p,\eta,\mu} f(z)} \right] \left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} f(z)}{z^p} \right]^\beta \prec \phi(z) \tag{12}$$

*implies that*

$$\left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} f(z)}{z^p} \right]^\beta \prec \left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} g(z)}{z^p} \right]^\beta \tag{13}$$

*and* $\left[\frac{H^{\lambda,\delta}_{p,\eta,\mu}g(z)}{z^p}\right]^{\beta}$ *is the best dominant.*

**Proof.** We define two functions $\Phi(z)$ and $\Psi(z)$ by

$$\Phi(z)=\left[\frac{H^{\lambda,\delta}_{p,\eta,\mu}f(z)}{z^p}\right]^{\beta} \text{ and } \Psi(z)=\left[\frac{H^{\lambda\delta}_{p,\eta,\mu}g(z)}{z^p}\right]^{\beta} \ (z\in\mathbb{D}). \tag{14}$$

Firstly, we will show that, if

$$q(z)=1+\frac{z\Psi''(z)}{\Psi'(z)} \ (z\in\mathbb{D}), \tag{15}$$

then

$$\Re\{q(z)\}>0 \ (z\in\mathbb{D}).$$

From the definitions of $\Psi(z)$ and $\phi(z)$ with Equation (8), we have

$$\phi(z)=\Psi(z)+\frac{\alpha}{\beta(\delta+p)}z\Psi'(z). \tag{16}$$

Differentiation both sides of Equation (16) with respect to $z$ yields

$$\phi'(z)=\Psi'(z)+\frac{\alpha\left[z\Psi''(z)+\Psi'(z)\right]}{\beta(\delta+p)}. \tag{17}$$

From Equations (15) and (17), we easily obtain

$$1+\frac{z\phi''(z)}{\phi'(z)}=q(z)+\frac{zq'(z)}{q(z)+\frac{\beta(\delta+p)}{\alpha}}=h(z) \ (z\in\mathbb{D}). \tag{18}$$

It follows from Relations (10) and (18) that

$$\Re\left\{h(z)+\frac{\beta(\delta+p)}{\alpha}\right\}>0 \ (z\in\mathbb{D}). \tag{19}$$

Furthermore, by means of Lemma 2, we deduce that Equation (18) has a solution $q\in\mathcal{H}(\mathbb{D})$ with $h(0)=q(0)=1$. Let

$$H(u,v)=u+\frac{v}{u+\frac{\beta(\delta+p)}{\alpha}}+\rho, \tag{20}$$

where $\rho$ is given by Equation (11). From Equations (18) and (19), we have

$$\Re\left\{H\left(q(z);zq'(z)\right)\right\}>0 \ (z\in\mathbb{D}).$$

Now, we will show that

$$\Re\{H(i\sigma;\tau)\}\leq 0 \ \left(\sigma\in\mathbb{R};\ \tau\leq-\frac{1+\sigma^2}{2}\right). \tag{21}$$

From Equation (20), we obtain

$$\begin{aligned}\Re\{\mathcal{H}(i\sigma;\tau)\} &= \Re\left\{i\sigma+\frac{\tau}{\frac{\beta(\delta+p)}{\alpha}+i\sigma}+\rho\right\}\\ &= \rho+\frac{\frac{\beta(\delta+p)\tau}{\alpha}}{\left|\frac{\beta(\delta+p)}{\alpha}+i\sigma\right|^2}\leq-\frac{E_\rho(\sigma)}{2\left|\frac{\beta(\delta+p)}{\alpha}+i\sigma\right|^2},\end{aligned}$$

where

$$E_\rho(\sigma)=\left(\frac{\beta(\delta+p)}{\alpha}-2\rho\right)\sigma^2-2\left(\frac{\beta(\delta+p)}{\alpha}\right)^2\rho+\frac{\beta(\delta+p)}{\alpha}. \tag{22}$$

For $\rho$ given by Equation (11), since the coefficient of $\sigma^2$ in $E_\rho(\sigma)$ of Equation (22) is positive or equal to zero and $E_\rho(\sigma)\geq 0$, we obtain that $\Re\{H(i\sigma;\tau)\}\leq 0$ for all $\sigma\in\mathbb{R}$ and $\tau\leq-\frac{1+\sigma^2}{2}$. Thus, by applying Lemma 1, we obtain that

$$\Re\{q(z)\}>0\ \ (z\in\mathbb{D}).$$

Moreover, $\Psi'(0)\neq 0$ since $g^{(p+1)}(0)\neq 0$. Hence, $\Psi(z)$ defined by Equation (14) is convex (univalent) in $\mathbb{D}$. Next, we verify that the Condition (12) implies that

$$\Phi(z)\prec\Psi(z)$$

for $\Phi(z)$ and $\Psi(z)$ given by Equation (14). Without loss of generality, we assume that $\Psi(z)$ is analytic, univalent on $\overline{\mathbb{D}}$ and

$$\Psi'(\xi)\neq 0\ \ (|\xi|=1).$$

Let us consider the function $\mathcal{I}(z,t)$ defined by

$$\mathcal{I}(z,t)=\Psi(z)+\frac{\alpha(1+t)}{\beta(\delta+p)}z\Psi'(z)\ \ (0\leq t<\infty;\ z\in\mathbb{D}). \tag{23}$$

Then, we see easily that

$$\left.\frac{\partial\mathcal{I}(z,t)}{\partial z}\right|_{z=0}=\Psi'(0)\left(1+\frac{\alpha}{\beta(\delta+p)}(1+t)\right)\neq 0\ \ (0\leq t<\infty;\ z\in\mathbb{D}).$$

This shows that

$$\mathcal{I}(z,t)=a_1(t)z+\cdots$$

satisfies the restrictions $\lim\limits_{t\to\infty}|a_1(t)|=\infty$ and $a_1(t)\neq 0\ \ (0\leq t<\infty)$. In addition, we obtain

$$\begin{aligned}\Re\left\{\frac{z\frac{\partial\mathcal{I}(z,t)}{\partial z}}{\frac{\partial\mathcal{I}(z,t)}{\partial t}}\right\} &= \Re\left\{\frac{\beta(\delta+p)}{\alpha}+(1+t)\left(1+\frac{z\Psi''(z)}{\Psi'(z)}\right)\right\}>0\\ &\quad (0\leq t<\infty;\ z\in\mathbb{D}),\end{aligned}$$

since $\Psi(z)$ is convex and $\Re\left(\frac{\beta(\delta+p)}{\alpha}\right)>0$. Moreover, we have

$$\left|\frac{\mathcal{I}(z,t)}{a_1(t)}\right|=\left|\frac{\Psi(z)+\frac{\alpha(1+t)}{\beta(\delta+p)}z\Psi'(z)}{\Psi(0)\left(1+\frac{\alpha(1+t)}{\beta(\delta+p)}\right)}\right| \tag{24}$$

and also the function $\Psi(z)$ may be written by

$$\Psi(z) = \Psi(0) + \Psi'(0)\psi(z) \quad (z \in \mathbb{D}), \tag{25}$$

where $\psi(z)$ is a normalized univalent function in $\mathbb{D}$. We note that, for the function $\psi(z)$, we have the following sharp growth and distortion results [32]:

$$\frac{r}{(1+r)^2} \leq |\psi(z)| \leq \frac{r}{(1-r)^2} \quad (|z| = r < 1) \tag{26}$$

and

$$\frac{1-r}{(1+r)^3} \leq \psi'(z) \leq \frac{1+r}{(1-r)^3} \quad (|z| = r < 1). \tag{27}$$

Hence, by applying Equations (25), (26) and (27) to Equation (24), we can find easily an upper bound for the right-hand side of Equation (24). Thus, the function $\mathcal{I}(z,t)$ satisfies the second condition of Lemma 5, which proves that $\mathcal{I}(z,t)$ is a subordination chain. From the definition of subordination chain, we note that

$$\phi(z) = \Psi(z) + \frac{\alpha}{\beta(\delta+p)} z\Psi'(z) = \mathcal{I}(z,0)$$

and

$$\mathcal{I}(z,0) \prec \mathcal{I}(z,t) \ \ (0 \leq t < \infty),$$

which implies that

$$\mathcal{I}(\xi,t) \notin \mathcal{I}(\mathbb{D},0) = \phi(\mathbb{D}) \ \ (0 \leq t < \infty;\ \xi \in \partial\mathbb{D}). \tag{28}$$

If $\Phi(z)$ is not subordinate to $\Psi(z)$, by Lemma 3, we see that there exist two points $z_0 \in \mathbb{D}$ and $\xi_0 \in \partial\mathbb{D}$ satisfying

$$\phi(z_0) = \Psi(\xi_0) \text{ and } z_0\Phi'(z_0) = (1+t)\,\xi_0\Psi'(\xi_0) \ \ (0 \leq t < \infty). \tag{29}$$

Hence, by using Relations (12), (14), (23) and (29), we obtain

$$\begin{aligned}
\mathcal{I}(\xi_0,t) &= \Psi(\xi_0) + \frac{\alpha}{\beta(\delta+p)}(1+t)\,\xi_0\Psi'(\xi_0) \\
&= \Phi(z_0) + \frac{\alpha}{\beta(\delta+p)} z_0\Phi'(z_0) \\
&= (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z_0)}{z_0^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f(z_0)}{H_{p,\eta,\mu}^{\lambda,\delta} f(z_0)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z_0)}{z_0^p}\right]^{\beta} \in \phi(\mathbb{D}).
\end{aligned}$$

This Contradicts (28). Thus, we conclude that $\Phi(z) \prec \Psi(z)$. If we consider $\Phi = \Psi$, then we know that $\Psi$ is the best dominant. Therefore, we complete the proof of Theorem 1. □

**Remark 1.** *The function $\Psi'(z) \neq 0$ for $z \in \mathbb{D}$ in Theorem 1 under the assumption*

$$\Re\{q(z)\} = 1 + \Re\left\{\frac{z\Psi''(z)}{\Psi'(z)}\right\} > 0 \ \ (z \in \mathbb{D}). \tag{30}$$

*In fact, if $\Psi'(z)$ has a zero of order $m$ at $z = z_1 \in \mathbb{D}\backslash\{0\}$, then we may write*

$$\Psi(z) = (z - z_1)^m \Psi_1(z) \quad (m \in \mathbb{N}),$$

*where $\Psi_1(z)$ is analytic in $\mathbb{D}\backslash\{0\}$ and $\Psi_1(z_1) \neq 0$. Then, we have*

$$q(z) = 1 + \frac{z\Psi''(z)}{\Psi'(z)} = 1 + \frac{mz}{z - z_1} + \frac{z\Psi_1'(z)}{\Psi_1(z)}. \tag{31}$$

*Thus, choosing* $z \to z_1$ *suitably, the real part of the right-hand side of Equation* (31) *can take any negative infinite values, which contradicts hypothesis Equation* (30)*. In addition, it is obvious that* $\Psi'(0) \neq 0$ *since* $g^{(p+1)}(0) \neq 0$.

Using similar methods given in the proof of Theorem 1, we have the following result.

**Theorem 2.** *Suppose that* $f, g \in \mathcal{A}(p)$ *and*

$$\Re\left\{1+\frac{z\psi''(z)}{\psi'(z)}\right\} > -\sigma \tag{32}$$

$$\left(\psi(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta}g(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g(z)}{z^p}\right]^{\beta};\ z\in\mathbb{D}\right),$$

*where* $\sigma$ *is given by*

$$\sigma = \frac{\alpha^2+\beta^2(p+\eta-\lambda)^2-\left|\alpha^2-\beta^2(p+\eta-\lambda)^2\right|}{4\alpha\beta(p+\eta-\lambda)}. \tag{33}$$

*Then,*

$$(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^{\beta} \prec \psi(z) \tag{34}$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^{\beta} \prec \left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g(z)}{z^p}\right]^{\beta} \tag{35}$$

*and* $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g(z)}{z^p}\right]^{\beta}$ *is the best dominant.*

Next, we derive the dual result of Theorem 1.

**Theorem 3.** *Suppose that* $f, g \in \mathcal{A}(p)$ *and*

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\rho$$

$$\left(\phi(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}g(z)}{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}{z^p}\right]^{\beta};\ z\in\mathbb{D}\right),$$

*where* $\rho$ *is given by Equation* (11)*. If*

$$(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta}$$

*is univalent in* $\mathbb{D}$ *and* $\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta} \in \mathcal{H}[1,1]\cap\mathcal{Q}$, *then*

$$\phi(z) \prec (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta} \tag{36}$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}{z^{p}}\right]^{\beta} \prec \left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^{p}}\right]^{\beta} \tag{37}$$

*and* $\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}{z^{p}}\right]^{\beta}$ *is the best subordinant.*

**Proof.** By using the functions $\Phi(z)$, $\Psi(z)$ and $q(z)$ given by Equations (14) and (15), we have

$$\phi(z) = \Psi(z) + \frac{\alpha}{\beta(\delta+p)} z\Psi'(z) = \varphi\left(\Psi(z), z\Psi'(z)\right) \tag{38}$$

and

$$\Re\{q(z)\} > 0 \ (z \in \mathbb{D}).$$

Next, we will show that $\Psi(z) \prec \Phi(z)$. To derive this, we consider the function $\mathcal{I}(z,t)$ defined by

$$\mathcal{I}(z,t) = \Psi(z) + \frac{\alpha}{\beta(\delta+p)} tz\Psi'(z) \quad (0 \leq t < \infty;\ z \in \mathbb{D}).$$

Then, we see that

$$\left.\frac{\partial\mathcal{I}(z,t)}{\partial z}\right|_{z=0} = \Psi'(0)\left(1 + \frac{\alpha}{\beta(\delta+p)} t\right) \neq 0 \ (0 \leq t < \infty;\ z \in \mathbb{D}),$$

which shows that

$$\mathcal{I}(z,t) = a_1(t) z + \cdots$$

satisfies $\lim_{t\to\infty} |a_1(t)| = \infty$ and $a_1(t) \neq 0 \ (0 \leq t < \infty)$. Furthermore, we obtain

$$\begin{aligned} \Re\left\{\frac{z\frac{\partial\mathcal{I}(z,t)}{\partial z}}{\frac{\partial\mathcal{I}(z,t)}{\partial t}}\right\} &= \Re\left\{\frac{\beta(\delta+p)}{\alpha} + t\left(1 + \frac{z\Psi''(z)}{\Psi'(z)}\right)\right\} > 0 \\ &\quad (0 \leq t < \infty;\ z \in \mathbb{D}). \end{aligned}$$

By using a similar method as in the proof of Theorem 1, we can prove the second inequality of Lemma 5. Hence, $\mathcal{I}(z,t)$ is a subordination chain. Therefore, by means of Lemma 4, we see that Relation (36) must imply given by Relation (37). Moreover, since Equation (38) has a univalent solution $\Psi$, it is the best subordinant. Therefore, we complete the proof. ☐

Using similar techniques given in the proof of Theorem 3, we have the following result.

**Theorem 4.** *Suppose that* $f, g \in \mathcal{A}(p)$ *and*

$$\Re\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\sigma$$

$$\left(\psi(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g(z)}{z^{p}}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta}g(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g(z)}{z^{p}}\right]^{\beta};\ z \in \mathbb{D}\right),$$

*where* $\sigma$ *is given by Equation* (33). *If*

$$(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^{p}}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^{p}}\right]^{\beta}$$

*is univalent in* $\mathbb{D}$ *and* $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^\beta \in \mathcal{H}[1,1] \cap \mathcal{Q}$*, then*

$$\psi(z) \prec (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^\beta + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^\beta \tag{39}$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g(z)}{z^p}\right]^\beta \prec \left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^\beta \tag{40}$$

*and* $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g(z)}{z^p}\right]^\beta$ *is the best subordinant.*

If we combine Theorems 1 and 3, and Theorems 2 and 4, then we have the unified sandwich-type results, respectively.

**Theorem 5.** *Suppose that* $f, g_j \in \mathcal{A}(p)\ \ (j = 1,\ 2)$ *and*

$$\Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\rho \tag{41}$$

$$\left(\phi_j(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_j(z)}{z^p}\right]^\beta + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}g_j(z)}{H_{p,\eta,\mu}^{\lambda,\delta}g_j(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_j(z)}{z^p}\right]^\beta;\ z \in \mathbb{D}\right),$$

*where* $\rho$ *is given by Equation* (11)*. If*

$$(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^\beta + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^\beta$$

*is univalent in* $\mathbb{D}$ *and* $\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^\beta \in \mathcal{H}[1,1] \cap \mathcal{Q}$*, then*

$$\phi_1(z) \prec (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^\beta + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^\beta \prec \phi_2(z) \tag{42}$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_1(z)}{z^p}\right]^\beta \prec \left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^\beta \prec \left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_2(z)}{z^p}\right]^\beta. \tag{43}$$

*Moreover,* $\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_1(z)}{z^p}\right]^\beta$ *and* $\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_2(z)}{z^p}\right]^\beta$ *are the best subordinant and the best dominant, respectively.*

**Theorem 6.** *Suppose that* $f, g_j \in \mathcal{A}(p)\ \ (j = 1,\ 2)$ *and*

$$\Re\left\{1+\frac{z\psi_j''(z)}{\psi_j'(z)}\right\} > -\sigma \tag{44}$$

$$\left(\psi_j(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g_j(z)}{z^p}\right]^\beta + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_j(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta}g_j(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g_j(z)}{z^p}\right]^\beta;\ z \in \mathbb{D}\right),$$

*where $\sigma$ is given by Equation (33). If*

$$(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^{p}}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^{p}}\right]^{\beta}$$

*is univalent in $\mathbb{D}$ and $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^{p}}\right]^{\beta}\in\mathcal{H}[1,1]\cap\mathcal{Q}$, then*

$$\psi_1(z)\prec(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^{p}}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^{p}}\right]^{\beta}\prec\psi_2(z) \tag{45}$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g_1(z)}{z^{p}}\right]^{\beta}\prec\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^{p}}\right]^{\beta}\prec\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g_2(z)}{z^{p}}\right]^{\beta}. \tag{46}$$

*Moreover, $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g_1(z)}{z^{p}}\right]^{\beta}$ and $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g_2(z)}{z^{p}}\right]^{\beta}$ are the best subordinant and the best dominant, respectively.*

We note that the assumption of Theorem 5, which states that

$$(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^{p}}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^{p}}\right]^{\beta}\text{ and }\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^{p}}\right]^{\beta}$$

needs to be univalent in $\mathbb{D}$, may be exchanged by a different condition.

**Corollary 1.** *Suppose that $f,g_j\in\mathcal{A}(p)\ \ (j=1,\,2)$ and*

$$\Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\}>-\rho$$

$$\left(\phi_j(z)=(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_j(z)}{z^{p}}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}g_j(z)}{H_{p,\eta,\mu}^{\lambda,\delta}g_j(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_j(z)}{z^{p}}\right]^{\beta};\ z\in\mathbb{D}\right)$$

*and*

$$\Re\left\{1+\frac{z\chi''(z)}{\chi'(z)}\right\}>-\rho, \tag{47}$$

$$\left(\chi(z)=(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^{p}}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^{p}}\right]^{\beta};\ z\in\mathbb{D}\right),$$

*where $\rho$ is given by Equation (11). Then,*

$$\phi_1(z)\prec(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^{p}}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^{p}}\right]^{\beta}\prec\phi_2(z)$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_1(z)}{z^{p}}\right]^{\beta}\prec\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^{p}}\right]^{\beta}\prec\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_2(z)}{z^{p}}\right]^{\beta}.$$

**Proof.** To derive Corollary 1, we need to show that the Restriction (47) implies the univalence of $\chi(z)$. Noting that $0\le\rho<1/2$, it follows that $\chi(z)$ is close-to-convex function in $\mathbb{D}$ (see [33]) and so $\chi(z)$

is univalent in $\mathbb{D}$. In addition, by applying the similar methods given in the proof of Theorem 1, we see that the function $\Phi(z)$ defined by Equation (14) is convex (univalent) in $\mathbb{D}$. Therefore, by using Theorem 5, we get the desired result. $\square$

Using similar methods given in the proof of Corollary 1 with Theorem 6, we obtain the following corollary.

**Corollary 2.** *Suppose that* $f, g_j \in \mathcal{A}(p)$ $(j = 1,\ 2)$ *and*

$$\Re\left\{1 + \frac{z\psi_j''(z)}{\psi_j'(z)}\right\} > -\sigma$$

$$\left(\psi_j(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} g_j(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_j(z)}{H_{p,\eta,\mu}^{\lambda,\delta} g_j(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_j(z)}{z^p}\right]^{\beta};\ z \in \mathbb{D}\right)$$

*and*

$$\Re\left\{1 + \frac{z\Upsilon''(z)}{\Upsilon'(z)}\right\} > -\rho,$$

$$\left(\Upsilon(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta};\ z \in \mathbb{D}\right),$$

*where* $\sigma$ *is given by* (33). *Then,*

$$\psi_1(z) \prec (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta} \prec \psi_2(z)$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_1(z)}{z^p}\right]^{\beta} \prec \left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^{\beta} \prec \left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_2(z)}{z^p}\right]^{\beta}.$$

## 3. Conclusions

Various applications of fractional calculus have an immense impact on the study of pure mathematic and applied science. In the present paper, we obtain new results on subordinations and superordinations for a wide class of operators defined by generalized fractional derivative operators and generalized fractional integral operators. Furthermore, the differential sandwich-type theorems are also discussed for these operators.

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symmetry

*Article*

# Class of Analytic Functions Defined by $q$-Integral Operator in a Symmetric Region

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**Abstract:** The aim of the present paper is to introduce a new class of analytic functions by using a $q$-integral operator in the conic region. It is worth mentioning that these regions are symmetric along the real axis. We find the coefficient estimates, the Fekete–Szegö inequality, the sufficiency criteria, the distortion result, and the Hankel determinant problem for functions in this class. Furthermore, we study the inverse coefficient estimates for functions in this class.

**Keywords:** analytic functions; $q$-integral operator; conic region

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## 1. Introduction

Let $\mathcal{A}$ denote the class of functions $f$ of the form:

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad z \in \mathbb{D}. \tag{1}$$

which are analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S}$ denotes a subclass of $\mathcal{A}$, which contains univalent functions in $\mathbb{D}$. Let $f$ be a univalent function in $\mathbb{D}$. Then, its inverse function $f^{-1}$ exists in some disc $|w| < r \leq 1/4$, of the form:

$$f^{-1}(w) = w + B_2 w^2 + B_3 w^3 + \cdots . \tag{2}$$

For any analytic functions $f$ of the form (1) and $g$ of the form:

$$g(z) = z + \sum_{m=2}^{\infty} b_m z^m, \qquad z \in \mathbb{D}, \tag{3}$$

the convolution (Hadamard product) is given as:

$$(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m, \ (z \in \mathbb{D}).$$

Let $f$ and $g$ be analytic functions in $\mathbb{D}$. Then, $f$ is said to be subordinate to $g$, written as $f(z) \prec g(z)$, if there exists a function $w$ analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Moreover, if $g$ is univalent in $\mathbb{D}$, then the following equivalent relation holds:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

The classes of $k$-uniformly starlike and $k$-uniformly convex functions were introduced by Kanas and Wiśniowska [1,2]. A function $f \in \mathcal{S}$ is in $k - \mathcal{ST}$, if and only if:

$$\Re \frac{zf'(z)}{f(z)} > k \left| \frac{zf'(z)}{f(z)} - 1 \right|,$$

where $k \in [0, \infty)$ and $z \in \mathbb{D}$. Similarly, for $k \in [0, \infty)$, a function $f \in \mathcal{S}$ is in $k - \mathcal{UCV}$, if and only if:

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right|.$$

In particular, the classes $0 - \mathcal{ST} = \mathcal{ST}$ and $0 - \mathcal{UCV} = \mathcal{UCV}$ are the familiar classes of uniformly-starlike and uniformly-convex functions, respectively. These classes have been studied extensively. For some details, see [1–5].

Recently, a vivid interest has been shown by many researchers in quantum calculus due to its wide-spread applications in many branches of sciences especially in mathematics and physics. Among the contributors to the study, Jackson was the first to provide the basic notions and established results for the theory of $q$-calculus [6,7]. The idea of the $q$-derivative was first time used by Ismail et al. [8], and they introduced the $q$-extension of the class of starlike functions. A remarkable usage of the $q$-calculus in the context of geometric function theory was basically furnished, and the basic (or $q$-) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, p. 347 of [9]). The idea of $q$-starlikeness was further extended to certain subclasses of $q$-starlike functions. Recently, the $q$-analogue of the Ruscheweyh operator was introduced in [4], and it was studied in [10]. Many researchers contributed to the development of the theory by introducing certain classes with the help of $q$-calculus. For some details about these contributions, see [11–25]. We contribute to the subject by studying the $q$-integral operator in the conic region.

Now, we write some notions and basic concepts of $q$-calculus, which will be useful in our discussions. Throughout our discussion, we suppose that $q \in (0,1)$, $\mathbb{N} = \{1, 2, 3, \cdots\}$, and $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$, unless otherwise mentioned.

**Definition 1.** *Let $q \in (0,1)$. Then, the q-number $[t]_q$ is defined as:*

$$[t]_q = \begin{cases} \frac{1-q^t}{1-q}, & t \in \mathbb{C}, \\ \sum\limits_{j=0}^{m-1} q^j = 1 + q + q^2 + \cdots + q^{m-1}, & t = m \in \mathbb{N}. \end{cases}$$

**Definition 2.** *Let $q \in (0,1)$. Then, the q-factorial $[m]_q!$ is defined as:*

$$[m]_q! = \begin{cases} 1, & m = 0, \\ \prod\limits_{j=1}^{m} [j]_q, & m \in \mathbb{N}. \end{cases}$$

**Definition 3.** *Let $q \in (0,1)$. Then, the q-Pochhammer symbol $[t]_{m,q}$, $(z \in \mathbb{C},\ m \in \mathbb{N}_0)$ is defined as:*

$$[t]_{m,q} = \frac{(q^t; q)_m}{(1-q)^m} = \begin{cases} 1, & m = 0, \\ [t]_q [t+1]_q [t+2]_q \cdots [t+m-1]_q & m \in \mathbb{N}. \end{cases}$$

Furthermore, the gamma function in the $q$-analogue is defined by the following relation:

$$\Gamma_q(1) = 1 \text{ and } \Gamma_q(t+1) = [t]_q \Gamma_q(t).$$

**Definition 4.** *Let $q \in (0,1)$. Then, the $q$-derivative $D_q$ of a function $f$ is defined as:*

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0) & z = 0 \end{cases} \tag{4}$$

*provided that $f'(0)$ exists.*

We observe that:

$$\lim_{q \to 1^-} D_q f(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{z(1-q)} = f'(z).$$

From Definition 4 and (1), it is clear that:

$$D_q f(z) = 1 + \sum_{m=2}^{\infty} [m]_q a_m z^{m-1}.$$

Now, take the function:

$$F_{q,\mu+1}(z) = z + \sum_{m=2}^{\infty} \Lambda_m z^m, \tag{5}$$

where $\mu > -1$, $\Lambda_m = \frac{[\mu+1]_{m-1,q}}{[m-1]_q!}$ and $z \in \mathbb{D}$. Now, consider a function $F^{(-1)}_{q,\mu+1}$ by:

$$F^{(-1)}_{q,\mu+1}(z) * F_{q,\mu+1}(z) = zD_q f(z),$$

then the $q$-Noor integral operator is define by:

$$\mathrm{I}^{\mu}_q f(z) = F^{(-1)}_{q,\mu+1}(z) * f(z) = z + \sum_{m=2}^{\infty} \Phi_{m-1} a_m z^m, \ (\mu > -1, \ z \in \mathbb{D}), \tag{6}$$

where:

$$\Phi_{m-1} = \frac{[m]_q!}{[\mu+1]_{m-1,q}}. \tag{7}$$

It is clear that $\mathrm{I}^0_q f(z) = zD_q f(z)$ and $\mathrm{I}^1_q f(z) = f(z)$. From (6), we obtain:

$$[\mu+1, q]\mathrm{I}^{\mu}_q f(z) = [\mu, q]\mathrm{I}^{\mu+1}_q f(z) + q^{\mu} z D_q \left(\mathrm{I}^{\mu+1}_q f(z)\right). \tag{8}$$

The $q$-Noor integral operator was recently defined by Arif et al. [26]. By taking $q \to 1^-$, the operator defined in (6) coincides with the Noor integral operator defined in [27,28]. For some details about the $q$-analogues of various differential operators, see [29–33]. The main aim of the current paper is to study the $q$-Noor integral operator by defining a class of analytic functions. Now, we introduce it as follows:

**Definition 5.** *A function $f$ belongs to the class $\mathcal{K} - \mathcal{UST}^{\mu}_q(\gamma)$, $\gamma \in \mathbb{C} - \{0\}$, if:*

$$\Re\left\{\frac{1}{\gamma}\left(\frac{zD_q\mathrm{I}^{\mu}_q f(z)}{\mathrm{I}^{\mu}_q f(z)} - 1\right) + 1\right\} > k\left|\frac{1}{\gamma}\left(\frac{zD_q\mathrm{I}^{\mu}_q f(z)}{\mathrm{I}^{\mu}_q f(z)} - 1\right)\right|, \ \mu > -1, \ k \in [0,\infty), \ z \in \mathbb{D}. \tag{9}$$

**Geometric Interpretation**

Let $f \in \mathcal{K} - \mathcal{UST}^{\mu}_q(\gamma)$. Then, $\frac{zD_q\mathrm{I}^{\mu}_q f(z)}{\mathrm{I}^{\mu}_q f(z)}$ assumes all the values in the domain $\Delta_{k,\gamma} = h_{k,r}(\mathbb{D})$ such that:

$$\Delta_{k,\gamma} = \gamma\Delta_k + (1-\gamma),$$

where:

$$\Delta_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\},$$

or equivalently,

$$\frac{zD_q \mathrm{I}_q^{\mu} f(z)}{I_q^{\mu} f(z)} \prec h_{k,\gamma}(z). \tag{10}$$

The boundary $\partial\Delta_{k,\gamma}$ of the above region is the imaginary axis when $k = 0$. It is a hyperbola in the case of $k \in (0,1)$. When $k \in [0,1)$, we have:

$$h_{k,\gamma}(z) = 1 + \frac{2\gamma}{1-k^2} \left\{ \left( \frac{2}{\pi} arccosk \right) arctanh\sqrt{z} \right\}, \quad z \in \mathbb{D}.$$

In the case of $k = 1$, $\partial\Delta_{k,\gamma}$ is a parabola, and in this case:

$$h_{1,\gamma}(z) = 1 + \frac{2\gamma}{\pi} \left( log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, \quad z \in \mathbb{D}.$$

When $k > 1$, $\partial\Delta_{k,\gamma}$ is an ellipse and:

$$h_{k,\gamma}(z) = 1 + \frac{2\gamma}{k^2-1} sin \left( \frac{\pi}{2\mathcal{F}(s)} \int_0^{v(z)/\sqrt{s}} \left(1-y^2\right)^{-1/2} \left(1-(sy)^2\right)^{-1/2} dy \right) + \frac{2\gamma}{1-k^2},$$

where $v(z) = \frac{z-\sqrt{s}}{1-\sqrt{sz}}$, $0 < s < 1$, $z \in \mathbb{D}$, and $z$ is selected so that $k = \cosh\left(\frac{\pi\mathcal{F}'(s)}{4\mathcal{F}(s)}\right)$, where $\mathcal{F}$ is the first kind of Legendre's complete elliptic integral and $\mathcal{F}'$ is the complementary integral of $\mathcal{F}$; see [1,2]. Kanas and Wiśniowska [1,2] showed that the function $h_{k,\gamma}(\mathbb{D})$ is convex and univalent. All the curves discussed above have a vertex at $(k+\gamma)/(k+1)$. Now, it is clear that the domain $\Delta_{k,\gamma}$ is the right half plane for $k = 0$, hyperbolic for $k \in (0,1)$, parabolic when $k = 1$, and elliptic when $k > 1$. It is worth mentioning that the domain $\Delta_{k,\gamma}$ is symmetric with respect to the real axis. The function $h_{k,\gamma}(\mathbb{D}) = \Delta_{k,\gamma}$ is the extremal function in many problems for the classes of uniformly-starlike and uniformly-convex functions. For more about the conic domain; see [3,34].

Let $\mathcal{P}$ denote the class of functions $h$ of the form:

$$h(z) = 1 + \sum_{m=1}^{\infty} c_m z^m, \quad z \in \mathbb{D}, \tag{11}$$

which are analytic with a positive real part in $\mathbb{D}$. If $k \in [0,\infty)$, $\gamma \in \mathbb{C} - \{0\}$, then the class $\mathcal{P}\left(h_{k,\gamma}\right)$ can be defined as:

$$\mathcal{P}(h_{k,\gamma}) = \left\{ h \in \mathcal{P} : h(\mathbb{D}) \subset \Delta_{k,\gamma} \right\}.$$

**Lemma 1** ([35])**.** *Let $k \in [0,\infty)$ and $h_{k,\gamma}$ be introduced above. If:*

$$h_{k,\gamma}(z) = 1 + \sum_{m=1}^{\infty} Q_m z^m, \tag{12}$$

*then:*

$$Q_1 = \begin{cases} \frac{2\gamma A^2}{1-k^2}, & 0 \leq k < 1, \\ \frac{8\gamma}{\pi^2}, & k = 1, \\ \frac{\pi^2\gamma}{4\sqrt{s}(k^2-1)R^2(s)(1+s)}, & k > 1, \end{cases} \tag{13}$$

*and:*

$$Q_2 = \begin{cases} \frac{A^2+2}{3} Q_1 & 0 \leq k < 1, \\ \frac{2}{3} Q_1 & k = 1, \\ \frac{4R^2(s)\left(s^2+6s+1\right)-\pi^2}{24\sqrt{s}R^2(s)(1+s)} Q_1 & k > 1, \end{cases} \tag{14}$$

*where:*

$$A = \frac{2\cos^{-1} k}{\pi},$$

*and* $0 < s < 1$, *which is selected so that* $k = cosh\left(\frac{\pi \mathcal{F}'(s)}{\mathcal{F}(s)}\right)$.

Let:

$$f_{k,\gamma}(z) = z + \sum_{m=2}^{\infty} A_m z^m$$

be the extremal function in class $\mathcal{K} - \mathcal{UST}_q^{\mu}(\gamma)$ and $h_{k,\gamma}$ be of the form (12). Then, these functions can be related by the relation:

$$\frac{z D_q \mathrm{I}_q^{\mu} f_{k,\gamma}(z)}{\mathrm{I}_q^{\mu} f_{k,\gamma}(z)} = h_{k,\gamma}(z). \tag{15}$$

From (15) , we have:

$$z D_q \mathrm{I}_q^{\mu} f_{k,\gamma}(z) = p_{k,\gamma}(z) \mathrm{I}_q^{\mu} f_{k,\gamma}(z).$$

Furthermore:

$$z + \sum_{m=2}^{\infty} [m]_q \Phi_{m-1} A_m z^m = \left(\sum_{m=0}^{\infty} Q_m z^m\right)\left(z + \sum_{m=2}^{\infty} \Phi_{m-1} A_m z^m\right).$$

Equating the coefficients of $z^m$ in the above relation, we obtain:

$$[m]_q \Phi_{m-1} A_m = \Phi_{m-1} A_m + \sum_{j=1}^{m-1} \Phi_{j-1} A_j Q_{m-j}$$

and:

$$A_m = \frac{1}{q\,[m-1]_q \Phi_{m-1}} \sum_{j=1}^{m-1} \Phi_{j-1} A_j Q_{m-j}. \tag{16}$$

This implies that:

$$A_2 = \frac{Q_1}{q\Phi_1}, \tag{17}$$

$$A_3 = \frac{Q_1^2 + qQ_2}{q^2(1+q)\Phi_2}, \tag{18}$$

$$A_4 = \frac{1}{(1+q+q^2)\,q\Phi_3}\left\{Q_3 + \frac{Q_1 Q_2}{q} + \frac{Q_1^3 + qQ_1Q_2}{q^2(1+q)}\right\}. \tag{19}$$

**Lemma 2** ([36]). *If* $h \in \mathcal{P}$ *satisfies* (11) *, then:*

$$|c_2 - vc_1^2| \leq 2\ \max\{1; |2v-1|\} \quad (v \in \mathbb{C}).$$

**Lemma 3** ([37]). *If* $h \in \mathcal{P}$ *satisfies* (11) *, then:*

$$|c_n - c_{n-m}c_m| < 2, n > \ m,\ n = 1, 2, 3, \cdots.$$

**Lemma 4** ([38]). *If $h \in \mathcal{P}$ satisfies (11), then:*

$$|c_3 - 2c_1c_2 + c_1^3| \leq 2.$$

## 2. Main Results

**Theorem 1.** *If $f \in \mathcal{K} - \mathcal{UST}_q^{\mu}(\gamma)$, then:*

$$|a_2| \leq A_2, \quad |a_3| \leq A_3, \tag{20}$$

*and:*

$$|a_4| \leq \frac{Q_1}{4q\,[3]_q\,\Phi_3}\{|F| + |(E-2F)| + |(F-E+4)|\}, \tag{21}$$

*where:*

$$E = 4 - \frac{4Q_2}{Q_1} - \frac{2Q_1}{q} - \frac{2Q_1}{q[2]_q}, \tag{22}$$

*with:*

$$F = 1 + \frac{Q_3}{Q_1} - \frac{2Q_2}{Q_1} + \frac{1+[2]_q}{q[2]_q}(Q_2 - Q_1) + \frac{Q_1^2}{q[2]_q}. \tag{23}$$

**Proof.** Suppose that:

$$\frac{zD_q\mathrm{I}_q^{\mu}f(z)}{\mathrm{I}_q^{\mu}f(z)} = p(z), \tag{24}$$

where $p$ is analytic in $\mathbb{D}$. Then, from (24), we have:

$$zD_q\mathrm{I}_q^{\mu}f(z) = p(z)\mathrm{I}_q^{\mu}f(z).$$

Consider:

$$p(z) = 1 + \sum_{m=1}^{\infty} p_m z^m \tag{25}$$

and $\mathrm{I}_q^{\mu}f(z)$ is given in the relation (6). Then:

$$z + \sum_{m=2}^{\infty}[m]_q\Phi_{m-1}a_m z^m = \left(\sum_{m=0}^{\infty} p_m z^m\right)\left(z + \sum_{m=2}^{\infty}\Phi_{m-1}a_m z^m\right).$$

It follows from the above relation that:

$$[m]_q\Phi_{m-1}a_m = \Phi_{m-1}a_m + \sum_{j=1}^{m-1}\Phi_{j-1}a_j p_{m-j}$$

and:

$$a_m = \frac{1}{q[m-1]_q\Phi_{m-1}}\sum_{j=1}^{m-1}\Phi_{j-1}a_j p_{m-j}. \tag{26}$$

Furthermore, consider the function:

$$h(z) = (1 + w(z))(1 - w(z))^{-1} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{27}$$

Then, $h$ is analytic in $\mathbb{D}$ with $Re(h(z)) > 0$. By using (12) and (27) , we have:

$$\begin{aligned} p(z) &= p_{k,\gamma}\left(\frac{-1+h(z)}{1+h(z)}\right) = 1+\frac{1}{2}c_1Q_1z+\left(\frac{1}{2}c_2Q_1+\frac{1}{4}c_1^2(Q_2-Q_1)\right)z^2 \\ &\quad +\left\{\frac{1}{8}(Q_1-2Q_2+Q_3)c_1^3+\frac{1}{2}(Q_2-Q_1)c_2c_1+\frac{1}{2}Q_1c_3\right\}z^3+\cdots. \end{aligned} \tag{28}$$

Now, from (26) and (28) , we obtain:

$$a_2 = \frac{p_1}{q\Phi_1} = \frac{c_1Q_1}{2q\Phi_1}. \tag{29}$$

Now, using the fact that $|c_m| \leq 2$, we get:

$$|a_2| = \left|\frac{p_1}{q\Phi_1}\right| = \left|\frac{c_1Q_1}{2q\Phi_1}\right| \leq \frac{|Q_1|}{q\Phi_1} = \frac{Q_1}{q\Phi_1} = A_2.$$

Similarly:

$$a_3 = \frac{1}{q[2]_q\Phi_2}\left\{p_2+p_1a_2\Phi_1\right\} = \frac{qp_2+p_1^2}{(1+q)\,q^2\Phi_2}. \tag{30}$$

In view of the relation $|p_1|^2+|p_2| \leq Q_1^2+Q_2$ (see [5]) and (17), we obtain:

$$\begin{aligned} |a_3| = \frac{|qp_2+p_1^2|}{(1+q)\,q^2\Phi_2} &\leq \frac{q\left(|p_2|+|p_1^2|\right)+(1-q)\,|p_1^2|}{(1+q)\,q^2\Phi_2} \\ &\leq \frac{q\left(|Q_2|+|Q_1^2|\right)+(1-q)\,|Q_1^2|}{(1+q)\,q^2\Phi_2} \\ &\leq \frac{q\,|Q_2|+|Q_1^2|}{(1+q)\,q^2\Phi_2} = A_3, \end{aligned}$$

which implies the required result. Now, equating the coefficients of $z^3$, we have:

$$a_4 = \frac{Q_1}{8\,[3]_q\,q\Phi_3}(4c_3-Ec_1c_2+Fc_1^3), \tag{31}$$

where $E$ and $F$ are given by (22) and (23), respectively. This implies that:

$$\begin{aligned} |a_4| &= \frac{Q_1}{8q\,[3]_q\,\Phi_3}|F(c_3-2c_1c_2+c_1^3)+(E-2F)(c_3-c_1c_2)+(F-E+4)c_3| \\ &\leq \frac{Q_1}{8q\,[3]_q\,\Phi_3}|F(c_3-2c_1c_2+c_1^3)|+|(E-2F)(c_3-c_1c_2)|+|(F-E+4)c_3| \\ &\leq \frac{Q_1}{2q\,[3]_q\,\Phi_3}|F|+|(E-2F)|+|(F-E+4)|, \end{aligned}$$

where we have used Lemmas 3 and 4. □

**Theorem 2.** *Let $0 \leq k < \infty$, $q \in (0,1)$, and $\gamma \in \mathbb{C}-\{0\}$. If $f \in \mathcal{K}-\mathcal{UST}_q^{\mu}(\gamma)$ of the form (1), then:*

$$|a_m| \leq \frac{Q_1\,(Q_1+q)\left(Q_1+q\,[2]_q\right)\ldots\left(Q_1+q\,[m-2]_q\right)}{q^{m-1}\Phi_{m-1}\prod\left(1+q+\ldots+q^{k-1}\right)}, \quad m \geq 2.$$

**Proof.** The result is clearly true for $m = 2$. That is:

$$|a_2| \leq \frac{Q_1}{q} = A_2.$$

Let $m \geq 2$, and suppose that the relation is true for $j \leq m-1$, then we obtain:

$$\begin{aligned}
|a_m| &= \frac{1}{q\,[m-1]_q\,\Phi_{m-1}} \left| p_{m-1} + \sum_{j=2}^{m-1} \Phi_{j-1} a_j p_{m-j} \right| \\
&\leq \frac{1}{q\,[m-1]_q\,\Phi_{m-1}} \left\{ Q_1 + \sum_{j=2}^{m-1} \Phi_{j-1} |a_j| Q_1 \right\} \\
&\leq \frac{1}{q\,[m-1]_q\,\Phi_{m-1}} Q_1 \left\{ 1 + \sum_{j=2}^{m-1} \Phi_{j-1} |a_j| \right\} \\
&\leq \frac{1}{q\,[m-1]_q\,\Phi_{m-1}} Q_1 \left\{ 1 + \sum_{j=2}^{m-1} \Phi_{j-1} \frac{Q_1\,(Q_1+q)\left(Q_1+q\,[2]_q\right)\ldots\left(Q_1+q\,[j-2]_q\right)}{q^{j-1}\Phi_{j-1}\prod\left(1+q+\ldots+q^{k-1}\right)} \right\},
\end{aligned}$$

where we applied the induction hypothesis to $|a_j|$ and the Rogosinski result $|p_m| \leq Q_1$ (see [39]). This implies that:

$$|a_m| \leq \frac{1}{q\,[m-1]_q\,\Phi_{m-1}} Q_1 \left\{ 1 + \sum_{j=2}^{m-1} \frac{Q_1\,(Q_1+q)\left(Q_1+q\,[2]_q\right)\ldots\left(Q_1+q\,[j-2]_q\right)}{q^{j-1}\prod\left(1+q+\ldots+q^{k-1}\right)} \right\}.$$

Applying the principal of mathematical induction, we find:

$$\begin{aligned}
&1 + \sum_{j=2}^{m-1} \frac{Q_1\,(Q_1+q)\left(Q_1+q\,[2]_q\right)\ldots\left(Q_1+q\,[j-2]_q\right)}{q^{j-1}\prod\left(1+q+\ldots+q^{k-1}\right)} \\
&= \frac{Q_1\,(Q_1+q)\left(Q_1+q\,[2]_q\right)\ldots\left(Q_1+q\,[m-2]_q\right)}{q^{m-2}\prod\left(1+q+\ldots+q^{k-2}\right)}.
\end{aligned}$$

Hence, the desired result. □

**Theorem 3.** *If $f \in \mathcal{A}$ is given in (1) and the inequality:*

$$\sum_{m=2}^{\infty} \left\{ q[m-1]_q(k+1) + |\gamma| \right\} \Phi_{m-1} |a_m| \leq |\gamma| \tag{32}$$

*holds true for some $0 \leq k < \infty$, $q \in (0,1)$ and $\gamma \in \mathbb{C} - \{0\}$, then $f \in \mathcal{K} - \mathcal{UST}_q^{\mu}(\gamma)$.*

**Proof.** Using (9), we have:

$$k \left| \frac{1}{\gamma} \left( \frac{zD_q\mathrm{I}_q^{\mu} f(z)}{\mathrm{I}_q^{\mu} f(z)} - 1 \right) \right| - \Re \left\{ \frac{1}{\gamma} \left( \frac{zD_q\mathrm{I}_q^{\mu} f(z)}{\mathrm{I}_q^{\mu} f(z)} - 1 \right) \right\} < 1.$$

This implies that:

$$\begin{aligned}
&k\left|\frac{1}{\gamma}\left(\frac{zD_q\mathrm{I}_q^{\mu}f\left(z\right)}{\mathrm{I}_q^{\mu}f\left(z\right)}-1\right)\right|-\Re\left\{\frac{1}{\gamma}\left(\frac{zD_q\mathrm{I}_q^{\mu}f\left(z\right)}{\mathrm{I}_q^{\mu}f\left(z\right)}-1\right)\right\}\\
&\leq\frac{k}{|\gamma|}\left|\frac{zD_q\mathrm{I}_q^{\mu}f\left(z\right)}{\mathrm{I}_q^{\mu}f\left(z\right)}-1\right|+\frac{1}{|\gamma|}\left|\frac{zD_q\mathrm{I}_q^{\mu}f\left(z\right)}{\mathrm{I}_q^{\mu}f\left(z\right)}-1\right|\\
&\leq\frac{(k+1)}{|\gamma|}\left|\frac{zD_q\mathrm{I}_q^{\mu}f\left(z\right)}{\mathrm{I}_q^{\mu}f\left(z\right)}-1\right|.
\end{aligned}$$

We see that:

$$\begin{aligned}
\left|\frac{zD_q\mathrm{I}_q^{\mu}f\left(z\right)}{\mathrm{I}_q^{\mu}f\left(z\right)}-1\right| &= \left|\frac{z+\sum_{m=2}^{\infty}\left[m\right]_q\Phi_{m-1}a_mz^m-z-\sum_{m=2}^{\infty}\Phi_{m-1}a_mz^m}{z+\sum_{m=2}^{\infty}\Phi_{m-1}a_mz^m}\right|\\
&=\left|\frac{\sum_{m=2}^{\infty}q[m-1]_q\Phi_{m-1}a_mz^m}{z+\sum_{m=2}^{\infty}\Phi_{m-1}a_mz^m}\right|\\
&\leq\frac{\sum_{m=2}^{\infty}q[m-1]_q\Phi_{m-1}\left|a_m\right|}{1-\sum_{m=2}^{\infty}\Phi_{m-1}\left|a_m\right|}.
\end{aligned}$$

From the above, we have:

$$\begin{aligned}
&k\left|\frac{1}{\gamma}\left(\frac{zD_q\mathrm{I}_q^{\mu}f\left(z\right)}{\mathrm{I}_q^{\mu}f\left(z\right)}-1\right)\right|-\Re\left\{\frac{1}{\gamma}\left(\frac{zD_q\mathrm{I}_q^{\mu}f\left(z\right)}{\mathrm{I}_q^{\mu}f\left(z\right)}-1\right)\right\}\\
&\leq\frac{(k+1)}{|\gamma|}\frac{\sum_{m=2}^{\infty}q[m-1,q]\Phi_{m-1}\left|a_m\right|}{1-\sum_{m=2}^{\infty}\Phi_{m-1}\left|a_m\right|}\\
&\leq 1.
\end{aligned}$$

This completes the proof. □

**Theorem 4.** *If $f\in\mathcal{K}-\mathcal{UST}_q^{\mu}(\gamma)$, then $f(\mathbb{D})$ contains an open disk of radius:*

$$\frac{q\left(1+q\right)}{q\left|Q_1\right|\left[\mu+1\right]_q+2q\left(1+q\right)},$$

*where $Q_1$ is defined by (11).*

**Proof.** Let $w_0\in\mathbb{C}$ and $w_0\neq 0$ with $f(z)\neq w_0$ in $\mathbb{D}$. Then:

$$f_1(z)=w_0f\left(z\right)\left(w_0-f\left(z\right)\right)^{-1}=z+\left(\frac{1}{w_0}+a_2\right)z^2+....$$

Since $f_1\in\mathcal{S}$,

$$\left|\frac{1}{w_0}+a_2\right|\leq 2.$$

Now, by applying Theorem 1, we obtain:

$$|\frac{1}{w_0}|\leq 2+\frac{2\left|Q_1\right|\left[\mu+1\right]_q}{q\left(1+q\right)}.$$

Hence:

$$|w_0|\geq\frac{q\left(1+q\right)}{\left|Q_1\right|q[\mu+1]_q+2q\left(1+q\right)}.$$

□

**Theorem 5.** *If* $f \in \mathcal{K}-\mathcal{UST}_q^{\mu}(\gamma)$, *then:*

$$\mathrm{I}_q^{\mu} f(z) \prec zexp \int_0^z \frac{h_{k,\gamma}(w(\xi))-1}{\xi} d\xi, \tag{33}$$

*where* $w$ *is analytic in* $\mathbb{D}$ *with* $w(0)=0$ *and* $|w(z)|<1$. *Moreover, for* $|z|=\rho$, *we have:*

$$\left(exp \int_0^1 \frac{h_{k,\gamma}(-\rho)-1}{\rho} d\rho\right) \leq \left|\frac{\mathrm{I}_q^{\mu} f(z)}{z}\right| \leq \left(exp \int_0^1 \frac{h_{k,\gamma}(\rho)-1}{\rho} d\rho\right),$$

*where* $h_{k,\gamma}$ *is given in* (10).

**Proof.** From (10), we obtain:

$$\frac{D_q \mathrm{I}_q^{\mu} f(z)}{\mathrm{I}_q^{\mu} f(z)} = \frac{h_{k,\gamma}(w(z))-1}{z} + \frac{1}{z},$$

for a function $w$, which is analytic in $\mathbb{D}$ with $w(0)=0$ and $|w(z)|<1$. Integrating the above relation with respect to $z$, we have:

$$\mathrm{I}_q^{\mu} f(z) \prec zexp \int_0^z \frac{h_{k,\gamma}(w(\xi))-1}{\xi} d\xi. \tag{34}$$

Since the function $h_{k,\gamma}$ is univalent and maps the disk $|z|<\rho (0<\rho\leq 1)$ onto a convex and symmetric region with respect to the real axis,

$$\frac{k+\gamma}{\gamma+1} < h_{k,\gamma}(-\rho|z|) \leq \Re\{h_{k,\gamma}(w(\rho z))\} \leq h_{k,\gamma}(\rho|z|). \tag{35}$$

Using the above inequality, we have:

$$\int_o^1 \frac{h_{k,\gamma}(-\rho|z|)-1}{\rho} d\rho \leq \Re \int_o^1 \frac{h_{k,\gamma} w(\rho z)-1}{\rho} d\rho \leq \int_o^1 \frac{h_{k,\gamma}(\rho|z|)-1}{\rho} d\rho, \quad z \in \mathbb{D}.$$

Consequently, the subordination (24) implies that:

$$\int_0^1 \frac{h_{k,\gamma}(-\rho|z|)-1}{\rho} d\rho \leq log \left|\frac{\mathrm{I}_q^{\mu} f(z)}{z}\right| \leq \int_0^1 \frac{h_{k,\gamma}(\rho|z|)-1}{\rho} d\rho.$$

Furthermore, the relations $h_{k,\gamma}(-\rho) \leq h_{k,\gamma}(-\rho|z|)$, $h_{k,\gamma}(\rho|z| \leq h_{k,\gamma}(\rho)$ leads to:

$$\left(exp \int_0^1 \frac{h_{k,\gamma}(-\rho|z|)-1}{\rho} d\rho\right) \leq \left|\frac{\mathrm{I}_q^{\mu} f(z)}{z}\right| \leq \left(exp \int_0^1 \frac{h_{k,\gamma}(\rho|z|)-1}{\rho} d\rho\right).$$

This completes the proof. □

**Theorem 6.** *Let* $k \in [0,\infty)$ *and* $f \in \mathcal{K}-\mathcal{UST}_q^{\mu}(\gamma)$ *of the form* (1). *Then:*

$$|a_3 - \sigma a_2^2| \leq \frac{|Q_1|}{2q\,[2]_q\,\Phi_2}\, max\{1; |2v-1|\}, \quad \sigma \in \mathbb{C},$$

*where:*

$$v=\frac{1}{2}\left(1-\frac{Q_2}{Q_1}-\frac{Q_1}{q}+\frac{\sigma Q_1\Phi_2\,(1+q)}{q\Phi_1^2}\right). \tag{36}$$

*The values of $Q_1$ and $Q_2$ are given by (13) and (14), respectively, and that of $\Phi_2$ is given in (7).*

**Proof.** If $f\in\mathcal{K}-\mathcal{UST}_q^{\mu}(\gamma)$, then using (29) and (30), we have:

$$\begin{aligned}a_2&=\frac{Q_1c_1}{2q\Phi_1},\\ a_3&=\frac{1}{4q[2]_q\Phi_2}\left\{2c_2Q_1+c_1^2(Q_2-Q_1)+\frac{Q_1^2c_1^2}{q}\right\},\end{aligned}$$

which together imply that:

$$\begin{aligned}\left|a_3-\sigma a_2^2\right|&=\frac{1}{4q[2]_q\Phi_2}\left|\left\{\left(2c_2Q_1+c_1^2(Q_2-Q_1)\right)+\frac{Q_1^2c_1^2}{q}\right\}-\frac{\sigma Q_1^2c_1^2}{4q^2\Phi_1^2}\right|\\ &=\frac{Q_1}{4q[2]_q\Phi_2}\left|c_2-vc_1^2\right|,\end{aligned}$$

where $v$ is defined by (36). Applying Lemma 2, we have the desired result. □

**Theorem 7.** *If $f\in\mathcal{K}-\mathcal{UST}_q^{\mu}(\gamma)$ is given in (1), then:*

$$|a_2a_3-a_4|\leq\frac{|Q_1|}{4q\,[3]_q\,\Phi_3}\left\{|A|+|(B-2A)|+|A-B+4|\right\},$$

*where:*

$$B=E+\frac{2Q_1\Phi_3\,[3]_q}{q\,[2]_q\,\Phi_1\Phi_2},\quad A=F+\frac{Q_1\Phi_3\,[3]_q}{q\,[2]_q\,\Phi_1\Phi_2}\left(Q_2-Q_1+\frac{Q_1^2}{q}\right),$$

*with E and F given in (22) and (23), respectively.*

**Proof.** By using (29)–(31), it is easy to see that:

$$\begin{aligned}|a_2a_3-a_4|&=\frac{|-Q_1|}{8q\,[3]_q\,\Phi_3}\left|4c_3-Bc_1c_2+Ac_1^3\right|\\ &=\frac{|Q_1|}{8q\,[3]_q\,\Phi_3}\left|(A-B+4)c_3+(B-2A)(c_3-c_1c_2)+A(c_3-2c_1c_2+c_1^3)\right|\\ &\leq\frac{|Q_1|}{4q\,[3]_q\,\Phi_3}\left\{|A|+|(B-2A)|+|A-B+4|\right\},\end{aligned}$$

where we used Lemmas 3 and 4. This completes the proof. □

**Theorem 8.** *If $k\in[0,\infty)$ and letting $f\in\mathcal{K}-\mathcal{UST}_q^{\mu}(\gamma)$ and having the inverse coefficients of the form (2), then the following results hold:*

$$|B_2|\leq\frac{|Q_1|}{q\Phi_1},$$

$$|B_3|\leq\frac{|Q_1|}{q\,[2]_q\,\Phi_2}max\left\{1;\left|\frac{Q_1H}{q}+\frac{Q_2}{Q_1}\right|\right\},$$

*and:*

$$H=\frac{2[2]_q\Phi_2}{\Phi_1^2}-1. \tag{37}$$

**Proof.** Since $f(f^{-1}(\omega))=\omega$; therefore, using (2) , we have:

$$B_2=-a_2, \quad B_3=2a_2^2-a_3.$$

Putting the value of $a_2$ and $a_3$ in the above relation, it follows easily that:

$$B_2=-a_2=-\frac{c_1Q_1}{2q\Phi_1}. \tag{38}$$

Using the coefficient bound $|c_1|\leq 2$, we can write:

$$|B_2|=\left|\frac{-c_1Q_1}{2q\Phi_1}\right|\leq\frac{|Q_1|}{q\Phi_1}. \tag{39}$$

Now with the help of Lemma 2, we obtain:

$$\begin{aligned}
B_3&=2a_2^2-a_3\\
&=-\frac{Q_1}{2q\,[2]_q\,\Phi_2}\left\{c_2-\frac{c_1^2}{2}\left(1-\frac{Q_2}{Q_1}-\frac{Q_1}{q}\right)-\frac{c_1^2Q_1}{q\Phi_1^2}[2]_q\Phi_2\right\}\\
&=-\frac{Q_1}{2q\,[2]_q\,\Phi_2}\left\{c_2-\frac{c_1^2}{2}\left(1-\frac{Q_2}{Q_1}-\frac{Q_1}{q}\left(\frac{2[2]_q\Phi_2}{\Phi_1^2}-1\right)\right)\right\}\\
&=-\frac{Q_1}{2q\,[2]_q\,\Phi_2}\left\{c_2-\frac{c_1^2}{2}\left(1-\frac{Q_2}{Q_1}-\frac{Q_1H}{q}\right)\right\}.
\end{aligned} \tag{40}$$

Taking the absolute value of the above relation, we have:

$$\begin{aligned}
|B_3|&\leq\frac{|Q_1|}{q\,[2]_q\,\Phi_2}\left|c_2-\frac{c_1^2}{2}\left(1-\frac{Q_2}{Q_1}-\frac{Q_1H}{q}\right)\right|\\
&\leq\frac{|Q_1|}{q\,[2]_q\,\Phi_2}max\left\{1;\left|\frac{Q_1H}{q}+\frac{Q_2}{Q_1}\right|\right\}.
\end{aligned}$$

□

**Theorem 9.** *If $f\in\mathcal{K}-\mathcal{UST}_q^{\mu}(\gamma)$ with inverse coefficients given by (2), then for a complex number $\lambda$, we have:*

$$|B_3-\lambda B_2^2|\leq\frac{|Q_1|}{q\,[2]_q\,\Phi_2}max\left\{1;\left|\left(\frac{(2-\lambda)\,[2]_q\,\Phi_2Q_1}{q\Phi_1^2}-1\right)\frac{Q_1}{q}+\frac{Q_2}{Q_1}\right|\right\}.$$

**Proof.** From (38) and (40), we have:

$$\begin{aligned}
B_3-\lambda B_2^2&=\frac{c_1^2Q_1^2}{2q^2\Phi_1^2}-\frac{Q_1}{2q\,[2]_q\,\Phi_2}\left(c_2-\frac{c_1^2}{2}\left(1-\frac{Q_2}{Q_1}-\frac{Q_1}{q}\right)\right)-\frac{\lambda c_1^2Q_1^2}{4q^2\Phi_1^2}\\
&=\frac{c_1^2Q_1^2}{4q^2\Phi_1^2}(2-\lambda)-\frac{Q_1}{2q\,[2]_q\,\Phi_2}\left(c_2-\frac{c_1^2}{2}\left(1-\frac{Q_2}{Q_1}-\frac{Q_1}{q}\right)\right)\\
&=-\frac{Q_1}{2q\,[2]_q\,\Phi_2}\left\{c_2-\frac{c_1^2}{2}\left(1-\frac{Q_2}{Q_1}-\frac{Q_1}{q}\left(\frac{(2-\lambda)\,[2]_q\,\Phi_2Q_1}{q\Phi_1^2}-1\right)\right)\right\}.
\end{aligned}$$

Now, by applying Lemma 2, the absolute value of the above equation becomes:

$$\begin{aligned}|B_3-\lambda B_2^2| &\leq \frac{|Q_1|}{2q\,[2]_q\,\Phi_2}\left|c_2-\frac{c_1^2}{2}\left(1-\frac{Q_2}{Q_1}-\frac{Q_1}{q}\left(\frac{(2-\lambda)\,[2]_q\,\Phi_2 Q_1}{q\Phi_1^2}-1\right)\right)\right| \\ &\leq \frac{|Q_1|}{q\,[2]_q\,\Phi_2} max\left\{1;\left|\left(\frac{(2-\lambda)\,[2]_q\,\Phi_2 Q_1}{q\Phi_1^2}-1\right)\frac{Q_1}{q}+\frac{Q_2}{Q_1}\right|\right\}.\end{aligned}$$

This completes the proof. □

## 3. Future Work

The idea presented in this paper can easily be implemented to introduce some more subfamilies of analytic and univalent functions connected with different image domains.

## 4. Conclusions

In this article, we defined a new class of analytic functions by using the $q$-Noor integral operator. We investigated some interesting properties, which are useful to study the geometry of the image domain. We found the coefficient estimates, the Fekete–Szegö inequality, the sufficiency criteria, the distortion result, and the Hankel determinant problem for this class.

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# New Symmetric Differential and Integral Operators Defined in the Complex Domain

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**Abstract:** The symmetric differential operator is a generalization operating of the well-known ordinary derivative. These operators have advantages in boundary value problems, statistical studies and spectral theory. In this effort, we introduce a new symmetric differential operator (SDO) and its integral in the open unit disk. This operator is a generalization of the Sàlàgean differential operator. Our study is based on geometric function theory and its applications in the open unit disk. We formulate new classes of analytic functions using SDO depending on the symmetry properties. Moreover, we define a linear combination operator containing SDO and the Ruscheweyh derivative. We illustrate some inclusion properties and other inequalities involving SDO and its integral.

**Keywords:** univalent function; symmetric differential operator; unit disk; analytic function; subordination

**MSC:** 30C45

## 1. Introduction

Investigation of the theory of operators (differential, integral, mixed, convolution and linear) has been a capacity of apprehension for numerous scientists in all fields of mathematical sciences, such as mathematical physics, mathematical biology and mathematical computing. An additional definite field is the study of inequalities in the complex domain. Works' review shows masses of studies created by the classes of analytic functions. The relationship of geometry and analysis signifies a very central feature in geometric function theory in the open unit disk. This fast development is directly connected to the existence between analysis, construction and geometric performance [1]. In 1983, Sàlàgean introduced his famous differential operator of normalized analytic functions in the open unit disk [2]. This operator is generalized and extended to many classes of univalent functions. It plays a significant tool to develop the geometric structure of many analytic functions by suggesting different classes. Later this operator has been generalized and motivated by many researchers, for example, the Al-Oboudi differential operator [3]. Recently, a new study is presented by using the Sàlàgean operator [4]. Our research is to formulate a new symmetric differential operator and its integral by utilizing the concept of the symmetric derivative of complex variables. This concept is an operation, extending the original derivative. Note that its practical use in the the symmetry models in math modeling remains open. For example, for application in mathematical physics it is critical to employ group analysis methods. Such methods enable methods for branching solutions construction using group symmetry [5,6].

## 2. Preparatory

We shall need the following basic definitions throughout this paper. A function $\phi \in \Lambda$ is said to be univalent in $\mathbb{U}$ if it never takes the same value twice; that is, if $z_1 \neq z_2$ in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ then $\phi(z_1) \neq \phi(z_2)$ or equivalently, if $\phi(z_1) = \phi(z_2)$ then $z_1 = z_2$. Without loss of generality, we can use the notion $\Lambda$ for our univalent functions taking the expansion

$$\phi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n, \quad z \in \mathbb{U}. \tag{1}$$

We let $\mathcal{S}$ denote the class of such functions $\phi \in \Lambda$ that are univalent in $\mathbb{U}$.

A function $\phi \in \mathcal{S}$ is said to be starlike with respect to origin in $\mathbb{U}$ if the linear segment joining the origin to every other point of $\phi(z : |z| = r < 1)$ lies entirely in $\phi(z : |z| = r < 1)$. In more picturesque language, the requirement is that every point of $\phi(z : |z| = r < 1)$ be visible from the origin. A function $\phi \in \mathcal{S}$ is said to be convex in $\mathbb{U}$ if the linear segment joining any two points of $\phi(z : |z| = r < 1)$ lies entirely in $\phi(z : |z| = r < 1)$. In other words, a function $\phi \in \mathcal{S}$ is said to be convex in $\mathbb{U}$ if it is starlike with respect to each and every of its points. We denote the class of functions $\phi \in \mathcal{S}$ that are starlike with respect to origin by $\mathcal{S}^*$ and convex in $\mathbb{U}$ by $\mathcal{C}$.

Neatly linked to the classes $\mathcal{S}^*$ and $\mathcal{C}$ is the class $\mathcal{P}$ of all functions $\phi$ analytic in $\mathbb{U}$ and having positive real part in $\mathbb{U}$ with $\phi(0) = 1$. In fact $f \in \mathcal{S}^*$ if and only if $z\phi'(z)/\phi(z) \in \mathcal{P}$ and $\phi \in \mathcal{C}$ if and only if $1 + z\phi''(z)/\phi'(z) \in \mathcal{P}$. In general, for $\epsilon \in [0, 1)$ we let $\mathcal{P}(\epsilon)$ consist of functions $\phi$ analytic in $\mathbb{U}$ with $\phi(0) = 1$ so that $\Re(\phi(z)) > \epsilon$ ('$\Re$' represents to the real part) for all $z \in \mathbb{U}$. Note that $\mathcal{P}(\epsilon_2) \subset \mathcal{P}(\epsilon_1) \subset \mathcal{P}(0) \equiv \mathcal{P}$ for $0 < \epsilon_1 < \epsilon_2$ (e.g., see Duren [1]).

For functions $\phi$ and $\psi$ in $\Lambda$ we say that $\phi$ is subordinate to $\psi$, denoted by $\phi \prec \psi$, if there exists a Schwarz function $\omega$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ so that $\phi(z) = \psi(\omega(z))$ for all $z \in \mathbb{U}$ (see [7]). Evidently $\phi(z) \prec \psi(z)$ is equivalent to $\phi(0) = \psi(0)$ and $\phi(\mathbb{U}) \subset \psi(\mathbb{U})$. We request the following results, which can be located in [7].

**Lemma 1.** *For $a \in \mathbb{C}$ and positive integer $n$ let $\mathfrak{H}[a, n] = \{\varrho : \varrho(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\}$.*

*i. If $\gamma \in \mathbb{R}$ then $\Re\Big(\varrho(z) + \gamma z\varrho'(z)\Big) > 0 \Longrightarrow \Re\big(\varrho(z)\big) > 0$. Moreover, if $\gamma > 0$ and $\varrho \in \mathfrak{H}[1, n]$, then there are constants $\lambda > 0$ and $\beta > 0$ with $\beta = \beta(\gamma, \lambda, n)$ so that*

$$\varrho(z) + \gamma z\varrho'(z) \prec \left[\frac{1+z}{1-z}\right]^{\beta} \Rightarrow \varrho(z) \prec \left[\frac{1+z}{1-z}\right]^{\lambda}.$$

*ii. If $\delta \in [0, 1)$ and $\varrho \in \mathfrak{H}[1, n]$ then there is a constant $\lambda > 0$ with $\lambda = \lambda(\alpha, n)$ so that*

$$\Re\Big(\varrho^2(z) + 2\varrho(z).z\varrho'(z)\Big) > \delta \Rightarrow \Re(\varrho(z)) > \lambda.$$

*iii. If $\varrho \in \mathfrak{H}[a, n]$ with $\Re a > 0$ then $\Re\Big(\varrho(z) + z\varrho'(z) + z^2\varrho''(z)\Big) > 0$ or for $\vartheta : \mathbb{U} \to \mathbb{R}$ with $\Re\Big(\varrho(z) + \vartheta(z)\dfrac{z\varrho'(z)}{\varrho(z)}\Big) > 0$ then $\Re(\varrho(z)) > 0$.*

**Lemma 2.** *Let $h$ be a convex function with $h(0) = a$, and let $\mu \in \mathbb{C} \setminus \{0\}$ be a complex number with $\Re\gamma \geq 0$. If $\varrho \in \mathfrak{H}[a, n]$, and $\varrho(z) + (1/\mu)z\varrho'(z) \prec h(z), \quad z \in U$, then $\varrho(z) \prec \iota(z) \prec h(z)$, where*

$$\iota(z) = \frac{\mu}{nz^{\mu/n}} \int_0^z h(t) t^{\frac{\mu}{(n-1)}} dt, \quad z \in U.$$

## 3. Formulas of Symmetric Operators

Let $\phi \in \Lambda$, taking the power series (1). For a function $\phi(z)$ and a constant $\alpha \in [0,1]$, we formulate the SDO as follows:

$$\begin{aligned}
\mathcal{M}_\alpha^0\phi(z) &= \phi(z)\\
\mathcal{M}_\alpha^1\phi(z) &= \alpha z\phi'(z) - (1-\alpha)z\phi'(-z)\\
&= \alpha\left(z+\sum_{n=2}^{\infty} n\varphi_n z^n\right) - (1-\alpha)\left(-z+\sum_{n=2}^{\infty} n(-1)^n\varphi_n z^n\right)\\
&= z+\sum_{n=2}^{\infty}[n\,(\alpha-(1-\alpha)(-1)^n)]\,\varphi_n z^n\\
\mathcal{M}_\alpha^2\phi(z) &= \mathcal{M}_\alpha^1[\mathcal{M}_\alpha^1\phi(z)] = z+\sum_{n=2}^{\infty}[n\,(\alpha-(1-\alpha)(-1)^n)]^2\,\varphi_n z^n\\
&\vdots\\
\mathcal{M}_\alpha^k\phi(z) &= \mathcal{M}_\alpha^1[\mathcal{M}_\alpha^{k-1}\phi(z)] = z+\sum_{n=2}^{\infty}[n\,(\alpha-(1-\alpha)(-1)^n)]^k\,\varphi_n z^n.
\end{aligned}\tag{2}$$

It is clear that when $\alpha = 1$, we have Sàlàgean differential operator [2] $\mathcal{S}^k\phi(z) = z+\sum_{n=2}^{\infty} n^k\,\varphi_n z^n$. We may say that SDO (2) is the symmetric Sàlàgean differential operator in the open unit disk. In the same manner of the formula of Sàlàgean integral operator, we consume that for a function $\phi \in \Lambda$, the symmetric integral operator $\mathcal{J}_\alpha^k$ satisfies

$$\mathcal{J}_\alpha^k\phi(z) = z+\sum_{n=2}^{\infty}\frac{1}{[n\,(\alpha-(1-\alpha)(-1)^n)]^k}\,\varphi_n z^n \in \Lambda.$$

Similarly, when $\alpha = 1$, we have Sàlàgean integral operator [2], Remark 5. Furthermore, we conclude the relation $\mathcal{M}_\alpha^k\left(\mathcal{J}_\alpha^k\phi(z)\right) = \mathcal{J}_\alpha^k\left(\mathcal{M}_\alpha^k\phi(z)\right) = \phi(z)$.

Next, we proceed to formulate a linear combination operator involving SDO and the Ruscheweyh derivative. For a function $\phi \in \Lambda$, the Ruscheweyh derivative achieves the formula

$$\mathcal{R}^k\phi(z) = z+\sum_{n=2}^{\infty} C_{k+n-1}^k\,\varphi_n z^n,$$

where the term $C_{k+n-1}^k$ is the combination coefficients. In this note, we introduce a new operator combining $R^k$ and $\mathcal{M}_\alpha^k$ as follows:

$$\begin{aligned}
\mathbf{C}_{\alpha,\kappa}^k\phi(z) &= (1-\kappa)\mathcal{R}^k\phi(z)+\kappa\mathcal{M}_\alpha^k\phi(z)\\
&= z+\sum_{n=2}^{\infty}\left((1-\kappa)C_{k+n-1}^k+\kappa[n\,(\alpha-(1-\alpha)(-1)^n)]^k\right)\varphi_n z^n.
\end{aligned}\tag{3}$$

**Remark 1.**

- $k = 0 \Longrightarrow \mathbf{C}_{\alpha,\kappa}^0\phi(z) = \phi(z)$;
- $\alpha = 1 \Longrightarrow \mathbf{C}_{1,\kappa}^k\phi(z) = \mathcal{L}_\kappa^k\phi(z)$; *[8] (Lupas operator)*
- $\kappa = 0 \Longrightarrow \mathbf{C}_{\alpha,0}^k\phi(z) = \mathcal{R}^k\phi(z)$;
- $\alpha = 1, \kappa = 1 \Longrightarrow \mathbf{C}_{1,1}^k\phi(z) = \mathcal{S}^k\phi(z)$;
- $\kappa = 1 \Longrightarrow \mathbf{C}_{\alpha,\kappa}^k\phi(z) = \mathcal{M}_\alpha^k\phi(z)$.

We shall deal with the following classes

$$S_k^{*\alpha}(h) = \Big\{\phi \in \Lambda : \frac{z(\mathcal{M}_\alpha^k \phi(z))'}{\mathcal{M}_\alpha^k \phi(z)} \prec h(z),\, h \in \mathcal{C}\Big\}.$$

Obviously, the subclass $S_0^*(h) = \mathcal{S}^*(h)$.

**Definition 1.** *If $\phi \in \Lambda$, then $\phi \in \mathbb{J}_\alpha^b(A, B, k)$ if and only if*

$$1 + \frac{1}{b}\Big(\frac{2\mathcal{M}_\alpha^{k+1}\phi(z)}{\mathcal{M}_\alpha^k\phi(z) - \mathcal{M}_\alpha^k\phi(-z)}\Big) \prec \frac{1+Az}{1+Bz},$$

$$\Big(z \in \mathbb{U},\ -1 \leq B < A \leq 1,\ k = 1, 2, ...,\ b \in \mathbb{C} \setminus \{0\},\ \alpha \in [0,1]\Big).$$

- $\alpha = 1 \Longrightarrow$ [9];
- $\alpha = 1, B = 0 \Longrightarrow$ [10];
- $\alpha = 1, A = 1, B = -1, b = 2 \Longrightarrow$ [11].

**Definition 2.** *Let $\epsilon \in [0,1), \alpha \in [0,1], \kappa \geq 0$, and $k \in \mathbb{N}$. A function $\phi \in \Lambda$ is said to be in the set $T_k(\alpha, \kappa, \epsilon)$ if and only if*

$$\Re\Big((\mathbf{C}_{\alpha,\kappa}^k \phi(z))'\Big) > \epsilon, \quad z \in U.$$

## 4. Geometric Results

In this section, we utilize the above constructions of the symmetric operators to get some geometric fulfillment.

**Theorem 1.** *For $\phi \in \Lambda$ if one of the following facts holds*

- *The operator $\mathcal{M}_\alpha^k\phi(z)$ in (2) is of bounded boundary rotation;*
- *$\phi$ achieves the subordination inequality*

$$(\mathcal{M}_\alpha^k\phi(z))' \prec \left(\frac{1+z}{1-z}\right)^\beta, \quad \beta > 0,\ z \in \mathbb{U}, \quad \alpha \in [0, \infty);$$

- *$f$ satisfies the inequality*

$$\Re\Big((\mathcal{M}_\alpha^k\phi(z))'\frac{\mathcal{M}_\alpha^k\phi(z)}{z}\Big) > \frac{\delta}{2}, \quad \delta \in [0,1),\ z \in \mathbb{U},$$

- *$\phi$ admits the inequality*

$$\Re\Big(z\mathcal{M}_\alpha^k\phi(z))'' - \mathcal{M}_\alpha^k\phi(z))' + 2\frac{\mathcal{M}_\alpha^k\phi(z))}{z}\Big) > 0,$$

- *$\phi$ confesses the inequality*

$$\Re\Big(\frac{z\mathcal{M}_\alpha^k\phi(z))'}{\mathcal{M}_\alpha^k\phi(z))} + 2\frac{\mathcal{M}_\alpha^k\phi(z)}{z}\Big) > 1,$$

*then $\frac{\mathcal{M}_\alpha^k\phi(z)}{z} \in \mathcal{P}(\epsilon)$ for some $\epsilon \in [0,1)$.*

**Proof.** Define a function $\varrho$ as follows

$$\varrho(z) = \frac{\mathcal{M}_\alpha^k \phi(z)}{z} \Rightarrow z\varrho'(z) + \varrho(z) = (\mathcal{M}_\alpha^k \phi(z))'. \tag{4}$$

By the first fact, $\mathcal{M}_\alpha^k \phi(z)$ is of bounded boundary rotation, it implies that $\Re(z\varrho'(z) + \varrho(z)) > 0$. Thus, by Lemma 1.i, we obtain $\Re(\varrho(z)) > 0$ which yields the first part of the theorem.

In view of the second fact, we have the following subordination relation

$$(\mathcal{M}_\alpha^k \phi(z))' = z\varrho'(z) + \varrho(z) \prec [\frac{1+z}{1-z}]^\beta.$$

Now, according to Lemma 1.i, there is a constant $\gamma > 0$ with $\beta = \beta(\gamma)$ such that

$$\frac{\mathcal{M}_\alpha^k \phi(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^\gamma.$$

This implies that $\Re(\mathcal{M}_\alpha^k \phi(z)/z) > \epsilon$, for some $\epsilon \in [0,1)$.

Finally, consider the third fact, a simple computation yields

$$\Re\Big(\varrho^2(z) + 2\varrho(z).z\varrho'(z)\Big) = 2\Re\Big((\mathcal{M}_\alpha^k \phi(z))' \frac{\mathcal{M}_\alpha^k \phi(z)}{z}\Big) > \delta. \tag{5}$$

In virtue of Lemma 1.ii, there is a constant $\lambda > 0$ such that $\Re(\varrho(z)) > \lambda$ which implies that $\varrho(z) = \frac{\mathcal{M}_\alpha^k \phi(z)}{z} \in \mathcal{P}(\epsilon)$ for some $\epsilon \in [0,1)$. It follows from (5) that $\Re\Big(\mathcal{M}_\alpha^k \phi(z))'\Big) > 0$ and thus by Noshiro-Warschawski and Kaplan Theorems, $\mathcal{M}_\alpha^k \phi(z)$ is univalent and of bounded boundary rotation in $\mathbb{U}$.

By differentiating (4) and taking the real, we have

$$\Re\Big(\varrho(z) + z\varrho'(z) + z^2\varrho''(z)\Big) = \Re\Big(z(\mathcal{M}_\alpha^k \phi(z))'' - (\mathcal{M}_\alpha^k \phi(z))' + 2\frac{\mathcal{M}_\alpha^k \phi(z)}{z}\Big) > 0.$$

Thus, in virtue of Lemma 1.ii, we obtain $\Re(\frac{\mathcal{M}_\alpha^k \phi(z)}{z}) > 0$.

By logarithmic differentiation (4) and taking the real, we have

$$\Re\Big(\varrho(z) + \frac{z\varrho'(z)}{\varrho(z)} + z^2\varrho''(z)\Big) = \Re\Big(\frac{z(\mathcal{M}_\alpha^k \phi(z))'}{\mathcal{M}_\alpha^k \phi(z)} + 2\frac{\mathcal{M}_\alpha^k \phi(z)}{z} - 1\Big) > 0.$$

Hence, in virtue of Lemma 1.iii, with $\vartheta(z) = 1$, we conclude that $\Re(\frac{\mathcal{M}_\alpha^k \phi(z)}{z}) > 0$. This completes the proof. □

**Theorem 2.** *Let $\phi \in S_k^{*\alpha}(h)$, where $h(z)$ is convex univalent function in $\mathbb{U}$. Then*

$$\mathcal{M}_\alpha^k \phi(z) \prec z \exp\Big(\int_0^z \frac{h(\omega(\xi)) - 1}{\xi} d\xi\Big),$$

*where $\omega(z)$ is analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$. Furthermore, for $|z| = \eta$, $\mathcal{M}_\alpha^k \phi(z)$ achieves the inequality*

$$\exp\Big(\int_0^1 \frac{h(\omega(-\eta)) - 1}{\eta}\Big) d\eta \le \Big|\frac{\mathcal{M}_\alpha^k \phi(z)}{z}\Big| \le \exp\Big(\int_0^1 \frac{h(\omega(\eta)) - 1}{\eta}\Big) d\eta.$$

**Proof.** Since $\phi \in S_k^{*\alpha}(h)$, we have

$$\Big(\frac{z(\mathcal{M}_\alpha^k\phi(z))'}{\mathcal{M}_\alpha^k\phi(z)}\Big) \prec h(z), \quad z \in \mathbb{U},$$

which means that there exists a Schwarz function with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\Big(\frac{z(\mathcal{M}_\alpha^k\phi(z))'}{\mathcal{M}_\alpha^k\phi(z)}\Big) = h(\omega(z)), \quad z \in \mathbb{U},$$

which implies that

$$\Big(\frac{(\mathcal{M}_\alpha^k\phi(z))'}{\mathcal{M}_\alpha^k\phi(z)}\Big) - \frac{1}{z} = \frac{h(\omega(z)) - 1}{z}.$$

Integrating both sides, we have

$$\log \mathcal{M}_\alpha^k\phi(z) - \log z = \int_0^z \frac{h(\omega(\xi)) - 1}{\xi} d\xi.$$

Consequently, this yields

$$\log \frac{\mathcal{M}_\alpha^k\phi(z)}{z} = \int_0^z \frac{h(\omega(\xi)) - 1}{\xi} d\xi. \tag{6}$$

By using the definition of subordination, we get

$$\mathcal{M}_\alpha^k\phi(z) \prec z \exp\Big(\int_0^z \frac{h(\omega(\xi)) - 1}{\xi} d\xi\Big).$$

In addition, we note that the function $h(z)$ maps the disk $0 < |z| < \eta < 1$ onto a region which is convex and symmetric with respect to the real axis, that is

$$h(-\eta|z|) \le \Re(h(\omega(\eta z))) \le h(\eta|z|), \quad \eta \in (0, 1),$$

which yields the following inequalities:

$$h(-\eta) \le h(-\eta|z|), \quad h(\eta|z|) \le h(\eta)$$

and

$$\int_0^1 \frac{h(\omega(-\eta|z|)) - 1}{\eta} d\eta \le \Re\Big(\int_0^1 \frac{h(\omega(\eta)) - 1}{\eta} d\eta\Big) \le \int_0^1 \frac{h(\omega(\eta|z|)) - 1}{\eta} d\eta.$$

By using the above relations and Equation (6), we conclude that

$$\int_0^1 \frac{h(\omega(-\eta|z|)) - 1}{\eta} d\eta \le \log\Big|\frac{\mathcal{M}_\alpha^k\phi(z)}{z}\Big| \le \int_0^1 \frac{h(\omega(\eta|z|)) - 1}{\eta} d\eta.$$

This equivalence to the inequality

$$\exp\Big(\int_0^1 \frac{h(\omega(-\eta|z|)) - 1}{\eta} d\eta\Big) \le \Big|\frac{\mathcal{M}_\alpha^k\phi(z)}{z}\Big| \le \exp\Big(\int_0^1 \frac{h(\omega(\eta|z|)) - 1}{\eta} d\eta\Big).$$

Thus, we obtain

$$\exp\Big(\int_0^1 \frac{h(\omega(-\eta)) - 1}{\eta}\Big) d\eta \le \Big|\frac{\mathcal{M}_\alpha^k\phi(z)}{z}\Big| \le \exp\Big(\int_0^1 \frac{h(\omega(\eta)) - 1}{\eta}\Big) d\eta.$$

This completes the proof. □

**Theorem 3.** *Consider the class* $\mathbb{J}_\alpha^b(A,B,k)$ *in Definition 1. If* $\phi \in \mathbb{J}_\alpha^b(A,B,k)$ *then the odd function*

$$\mathfrak{O}(z) = \frac{1}{2}[\phi(z) - \phi(-z)], \quad z \in \mathbb{U}$$

*achieves the following inequality*

$$1 + \frac{1}{b}\Big(\frac{\mathcal{M}_\alpha^{k+1}\mathfrak{O}(z)}{\mathcal{M}_\alpha^{k}\mathfrak{O}(z)} - 1\Big) \prec \frac{1+Az}{1+Bz},$$

*and*

$$\Re\Big(\frac{z\mathfrak{O}(z)'}{\mathfrak{O}(z)}\Big) \geq \frac{1-r^2}{1+r^2}, \quad |z| = r < 1,$$

$$\Big(z \in \mathbb{U},\ -1 \leq B < A \leq 1,\ k = 1,2,...,\ b \in \mathbb{C} \setminus \{0\},\ \alpha \in [0,1]\Big).$$

**Proof.** Since $\phi \in \mathbb{J}_\alpha^b(A,B,k)$ then there is a function $P \in \mathbb{J}(A,B)$ such that

$$b(P(z) - 1) = \Big(\frac{2\mathcal{M}_\alpha^{k+1}\phi(z)}{\mathcal{M}_\alpha^{k}\phi(z) - \mathcal{M}_\alpha^{k}\phi(-z)}\Big)$$

and

$$b(P(-z) - 1) = \Big(\frac{-2\mathcal{M}_\alpha^{k+1}\phi(-z)}{\mathcal{M}_\alpha^{k}\phi(z) - \mathcal{M}_\alpha^{k}\phi(-z)}\Big).$$

This implies that

$$1 + \frac{1}{b}\Big(\frac{\mathcal{M}_\alpha^{k+1}\mathfrak{O}(z)}{\mathcal{M}_\alpha^{k}\mathfrak{O}(z)} - 1\Big) = \frac{P(z) + P(-z)}{2}.$$

Also, since

$$P(z) \prec \frac{1+Az}{1+Bz}$$

where $\frac{1+Az}{1+Bz}$ is univalent then by the definition of the subordination, we obtain

$$1 + \frac{1}{b}\Big(\frac{\mathcal{M}_\alpha^{k+1}\mathfrak{O}(z)}{\mathcal{M}_\alpha^{k}\mathfrak{O}(z)} - 1\Big) \prec \frac{1+Az}{1+Bz}.$$

Moreover, the function $\mathfrak{O}(z)$ is starlike in $\mathbb{U}$ which implies that

$$\frac{z\mathfrak{O}(z)'}{\mathfrak{O}(z)} \prec \frac{1-z^2}{1+z^2}$$

that is, there exists a Schwarz function $\wp \in \mathbb{U}, |\wp(z)| \leq |z| < 1, \wp(0) = 0$ such that

$$\Phi(z) := \frac{z\mathfrak{O}(z)'}{\mathfrak{O}(z)} \prec \frac{1-\wp(z)^2}{1+\wp(z)^2}$$

which yields that there is $\xi, |\xi| = r < 1$ such that

$$\wp^2(\xi) = \frac{1-\Phi(\xi)}{1+\Phi(\xi)}, \quad \xi \in \mathbb{U}.$$

A calculation gives that

$$\left|\frac{1-\Phi(\xi)}{1+\Phi(\xi)}\right| = |\wp(\xi)|^2 \le |\xi|^2.$$

Hence, we have the following conclusion

$$\left|\Phi(\xi) - \frac{1+|\xi|^4}{1-|\xi|^4}\right|^2 \le \frac{4|\xi|^4}{(1-|\xi|^4)^2}$$

or

$$\left|\Phi(z) - \frac{1+|\xi|^4}{1-|\xi|^4}\right| \le \frac{2|\xi|^2}{(1-|\xi|^4)}.$$

This implies that

$$\Re(\Phi(z)) \ge \frac{1-r^2}{1+r^2}, \quad |\xi| = r < 1.$$

□

Next consequence result of Theorem 3 can be found in [9,11] respectively.

**Corollary 1.** *Let* $\alpha = 1$ *in Theorem 3. Then*

$$1 + \frac{1}{b}\Big(\frac{\mathcal{M}_1^{k+1}\mathfrak{D}(z)}{\mathcal{M}_1^{k}\mathfrak{D}(z)} - 1\Big) \prec \frac{1+Az}{1+Bz}.$$

**Corollary 2.** *Let* $\alpha = 1, k = 1$ *in Theorem 3. Then*

$$1 + \frac{1}{b}\Big(\frac{\mathcal{M}_1^{2}\mathfrak{D}(z)}{\mathcal{M}_1\mathfrak{D}(z)} - 1\Big) \prec \frac{1+Az}{1+Bz}.$$

**Theorem 4.** *The set* $T_k(\alpha,\kappa,\epsilon)$ *in Definition 2 is convex.*

**Proof.** Let $\phi_i, i = 1,2$ be two functions in the set $T_k(\alpha,\kappa,\epsilon)$ satisfying $\phi_1(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\phi_2(z) = z + \sum_{n=2}^{\infty} b_n z^n$. It is sufficient to prove that the function

$$H(z) = c_1\phi_1(z) + c_2\phi_2(z), \quad z \in \mathbb{U}$$

is in $T_k(\alpha,\kappa,\epsilon)$, where $c_1 > 0, c_2 > 0$ and $c_1 + c_2 = 1$. By the definition of $H(z)$, a calculation implies that

$$H(z) = z + \sum_{n=2}^{\infty}(c_1 a_n + c_2 b_n) z^n$$

then under the operator $\mathbf{C}^k_{\alpha,\kappa}$, we obtain

$$\mathbf{C}^k_{\alpha,\kappa}H(z) = z + \sum_{n=2}^{\infty}(c_1 a_n + c_2 b_n)$$

$$\times[(1-\kappa)C^k_{k+n-1} + \kappa\big(n[\alpha-(1-\alpha)(-1)^n]\big)^k]z^n.$$

By taking the derivative for the last equation and following by the real, we have

$$\begin{aligned}
&\Re\Big\{(\mathbf{C}^k_{\alpha,\kappa}H(z))'\Big\}\\
&= 1 + c_1\Re\Big\{\sum_{n=2}^{\infty} n[(1-\kappa)C^k_{k+n-1} + \kappa\big(n[\alpha-(1-\alpha)(-1)^n]\big)^k a_n z^{n-1}\Big\}\\
&+ c_2\Re\Big\{\sum_{n=2}^{\infty} n[(1-\kappa)C^k_{k+n-1} + \kappa\big(n[\alpha-(1-\alpha)(-1)^n]\big)^k b_n z^{n-1}\Big\}\\
&> 1 + c_1(\epsilon-1) + c_2(\epsilon-1)\\
&= \epsilon.
\end{aligned}$$

This completes the proof. □

Next consequence result of Theorem 4 can be found in [8].

**Corollary 3.** *Let $\alpha = 1$ in Theorem 4. Then the set $T_k(1,\kappa,\epsilon)$ is convex.*

**Theorem 5.** *Let $\phi \in T_k(\alpha,\kappa,\epsilon)$, and let $\varphi$ be convex. Then for a function*

$$F(z) = \frac{2+c}{z^{1+c}}\int_0^z t^c\phi(t)dt, \quad z \in U$$

*the subordination*

$$\Big(\mathbf{C}^k_{\alpha,\kappa}\phi(z)\Big)' \prec \varphi(z) + \frac{(z\varphi'(z))}{2+c}, \quad c > 0,$$

*implies*

$$\Big(\mathbf{C}^k_{\alpha,\kappa}F(z)\Big)' \prec \varphi(z),$$

*and this result is sharp.*

**Proof.** Our aim is to apply Lemma 2. By the definition of $F(z)$, we obtain

$$\Big(\mathbf{C}^k_{\alpha,\kappa}F(z)\Big)' + \frac{\Big(\mathbf{C}^k_{\alpha,\kappa}F(z)\Big)''}{2+c} = \Big(\mathbf{C}^k_{\alpha,\kappa}\phi(z)\Big)'.$$

By the assumption, we get

$$\Big(\mathbf{C}^k_{\alpha,\kappa}F(z)\Big)' + \frac{\Big(\mathbf{C}^k_{\alpha,\kappa}F(z)\Big)''}{2+c} \prec \varphi(z) + \frac{(z\varphi'(z))}{2+c}.$$

By letting

$$\varrho(z) := \Big(\mathbf{C}^k_{\alpha,\kappa}F(z)\Big)',$$

one can find

$$\varrho(z) + \frac{(z\varrho'(z))}{2+c} \prec \varphi(z) + \frac{(z\varphi'(z))}{2+c}.$$

In virtue of Lemma 2, we have

$$\Big(\mathbf{C}^k_{\alpha,\kappa}F(z)\Big)' \prec \varphi(z),$$

and $\varphi$ is the best dominant. □

**Theorem 6.** *Let $\varphi$ be convex achieving $\varphi(0) = 1$. If*

$$\left(\mathbf{C}^k_{\alpha,\kappa}\phi(z)\right)' \prec \varphi(z) + z\varphi'(z), \quad z \in U,$$

*then*

$$\frac{\mathbf{C}^k_{\alpha,\kappa}\phi(z)}{z} \prec \varphi(z),$$

*and this result is sharp.*

**Proof.** Our aim is to apply Lemma 1. Define the function

$$\varrho(z) := \frac{\mathbf{C}^k_{\alpha,\kappa}\phi(z)}{z} \in \mathfrak{H}[1,1] \tag{7}$$

By this assumption, yields

$$\mathbf{C}^k_{\alpha,\kappa}\phi(z) = z\varrho(z) \Longrightarrow \left(\mathbf{C}^k_{\alpha,\kappa}\phi(z)\right)' = \varrho(z) + z\varrho'(z).$$

Thus, we deduce the following subordination:

$$\varrho(z) + z\varrho'(z) \prec \varphi(z) + z\varphi'(z).$$

In view of Lemma 1, we receive

$$\frac{\mathbf{C}^k_{\alpha,\kappa}\phi(z)}{z} \prec \varphi(z),$$

and $\varphi$ is the best dominant. □

**Theorem 7.** *If $\phi \in \Lambda$ satisfies the subordination relation*

$$(\mathbf{C}^k_{\alpha,\kappa}\phi(z))' \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \quad z \in \mathbb{U},\ \beta > 0,$$

*then*

$$\Re\Big(\frac{\mathbf{C}^k_{\alpha,\kappa}\phi(z)}{z}\Big) > \epsilon$$

*for some $\epsilon \in [0,1)$.*

**Proof.** Define a function $\varrho$ as in (7). Then, by subordination properties, we have

$$(\mathbf{C}^k_{\alpha,\kappa}\phi(z))' = z\varrho'(z) + \varrho(z) \prec [\frac{1+z}{1-z}]^{\beta}.$$

Now, in view of Lemma 1.i, there is a constant $\gamma > 0$ with $\beta = \beta(\gamma)$ such that

$$\frac{\mathbf{C}^k_{\alpha,\kappa}\phi(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}.$$

This implies that $\Re(\mathbf{C}^k_{\alpha,\kappa}\phi(z)/z) > \epsilon$, for some $\epsilon \in [0,1)$. □

**Theorem 8.** *If $\phi \in \Lambda$ satisfies the inequality*

$$\Re\Big((\mathbf{C}^k_{\alpha,\kappa}\phi(z))'\frac{\mathbf{C}^k_{\alpha,\kappa}\phi(z)}{z}\Big) > \frac{\alpha}{2}, \quad z \in U,\ \alpha \in [0,1)$$

*then* $\mathbf{C}^k_{\alpha,\kappa}\phi(z) \in T_k(\alpha,\kappa,\epsilon)$ *for some* $\epsilon \in [0,1)$. *Furthermore, it is univalent and of bounded boundary rotation in* $U$.

We inform the readers that in virtue of Noshiro-Warschawski Theorem (Duren [1], p. 47) if a function $\phi$ is analytic in the simply connected complex domain $\mathbb{U}$ and $\Re\{\phi'(z)\} > 0$ in $\mathbb{U}$ then $\phi$ is univalent in $\mathbb{U}$ and in view of Kaplan's Theorem (Duren [1], p. 48) such functions $\phi$ is of bounded boundary rotation.

**Proof.** Define a function $\varrho$ as in (7). A simple computation yields

$$\Re\Big(\varrho^2(z) + 2\varrho(z).z\varrho'(z)\Big) = 2\Re\Big(\mathbf{C}^k_{\alpha,\kappa}\phi(z))'\frac{\mathbf{C}^k_{\alpha,\kappa}\phi(z)}{z}\Big) > \alpha. \tag{8}$$

By virtue of Lemma 1.ii, there is a constant $\lambda$ depending on $\alpha$ such that $\Re(\varrho(z)) > \lambda$, which implies that $\Re(\varrho(z)\ ) > \epsilon$ for some $\epsilon \in [0,1)$. It follows from (8) that $\Re\Big(\mathbf{C}^k_{\alpha,\kappa}\phi(z))'\Big) > \epsilon$ and thus by Noshiro-Warschawski and Kaplan Theorems, $\mathbf{C}^k_{\alpha,\kappa}\phi(z)$ is univalent and of bounded boundary rotation in $\mathbb{U}$. □

**Example 1.** *We have the following data:* $\phi(z) = z/(1-z)$, $\alpha = 0.25$. *A calculation brings*

$$\begin{aligned}\mathcal{M}^1_\alpha\phi(z) &= \alpha z\phi'(z) - (1-\alpha)z\phi'(-z)\\ &= \frac{0.25z}{(1-z)^2} + \frac{0.75z}{(1+z)^2} = \frac{z(z^2-z+1)}{(1-z)^2(1+z)^2}\\ &= z - z^2 + 3z^3 - 2z^4 + 5z^5 + o(z^6)\end{aligned} \tag{9}$$

*with*

$$\begin{aligned}&\Re\left((\mathcal{M}^1_\alpha\phi(z))'\frac{\mathcal{M}^1_\alpha\phi(z)}{z}\right)\\ &= \Re\left(\frac{(-z^4+2z^3-6z^2+2z-1)\left(\frac{0.25z}{(1-z)^2}+\frac{0.75z}{(1+z)^2}\right)}{z(z^2-1)^3}\right)\\ &> 0,\end{aligned} \tag{10}$$

*when* $z \to 1$. *Hence, in view of Theorem 1,* $\frac{\mathcal{M}^1_\alpha\phi(z)}{z} \in \mathcal{P}(\epsilon)$ *for some* $\epsilon \in [0,1)$.

## 5. Conclusions and Future Works

Motivated by this method, in the recent investigation we have presented new classes of univalent functions that connect to a symmetric differential operator in the open unit disk. We have obtained sufficient and necessary conditions in relation to these subclasses. Linear combinations, operator and other properties are also explored. For further research, we indicate to study the certain new classes related to other types of analytic functions such as meromorphic, harmonic and $p$-valent functions with respect to symmetric points associated with SDO.

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# Sufficiency Criterion for A Subfamily of Meromorphic Multivalent Functions of Reciprocal Order with Respect to Symmetric Points

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**Abstract:** In the present research paper, our aim is to introduce a new subfamily of meromorphic $p$-valent (multivalent) functions. Moreover, we investigate sufficiency criterion for such defined family.

**Keywords:** meromorphic multivalent starlike functions; subordination

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## 1. Introduction

Let the notation $\Omega_p$ be the family of meromorphic $p$-valent functions $f$ that are holomorphic (analytic) in the region of punctured disk $\mathbb{E} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and obeying the following normalization

$$f(z) = \frac{1}{z^p} + \sum_{j=1}^{\infty} a_{j+p}\, z^{j+p} \ \ (z \in \mathbb{E})\,. \tag{1}$$

In particular $\Omega_1 = \Omega$, the familiar set of meromorphic functions. Further, the symbol $\mathcal{MS}^*$ represents the set of meromorphic starlike functions which is a subfamily of $\Omega$ and is given by

$$\mathcal{MS}^* = \left\{ f : f(z) \in \Omega \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) < 0 \ \ (z \in \mathbb{E}) \right\}.$$

Two points $p$ and $p'$ are said to be symmetrical with respect to $o$ if $o'$ is the midpoint of the line segment $pp'$. This idea was further nourished in [1,2] by introducing the family $\mathcal{MS}_s^*$ which is defined in set builder form as;

$$\mathcal{MS}_s^* = \left\{ f : f(z) \in \Omega \text{ and } \Re\left(\frac{-2zf'(z)}{f(-z) - f(z)}\right) < 0 \ \ (z \in \mathbb{E}) \right\}.$$

Now, for $-1 \leq t < s \leq 1$ with $s \neq 0 \neq t$, $0 < \xi < 1$, $\lambda$ is real with $|\lambda| < \frac{\pi}{2}$ and $p \in \mathbb{N}$, we introduce a subfamily of $\Omega_p$ consisting of all meromorphic $p$-valent functions of reciprocal order $\xi$, denoted by $\mathcal{NS}_p^{\lambda}(s,t,\xi)$, and is defined by

$$\mathcal{NS}_p^{\lambda}(s,t,\xi) = \left\{ f : f(z) \in \Omega_p \text{ and } \Re\left( e^{-i\lambda} \frac{ps^p t^p}{s^p - t^p} \frac{f(sz) - f(tz)}{zf'(z)} \right) > \xi \cos\lambda \ \ (z \in \mathbb{E}) \right\}.$$

We note that for $p = s = 1$ and $t = -1$, the class $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ reduces to the class $\mathcal{NS}_1^{\lambda}(1,-1,\xi) = \mathcal{NS}_*^{\lambda}(\xi)$ and is represented by

$$\mathcal{NS}_*^{\lambda}(\xi) = \left\{ f : f(z) \in \Omega \text{ and } \Re\left( e^{-i\lambda} \frac{f(-z) - f(z)}{2zf'(z)} \right) > \xi \cos\lambda \ \ (z \in \mathbb{E}) \right\}.$$

For detail of the related topics, see the work of Al-Amiri and Mocanu [3], Rosihan and Ravichandran [4], Aouf and Hossen [5], Arif [6], Goyal and Prajapat [7], Joshi and Srivastava [8], Liu and Srivastava [9], Raina and Srivastava [10], Sun et al. [11], Shi et al. [12] and Owa et al. [13], see also [14–16].

For simplicity and ignoring the repetition, we state here the constraints on each parameter as $0 < \xi < 1$, $-1 \leq t < s \leq 1$ with $s \neq 0 \neq t$, $\lambda$ is real with $|\lambda| < \frac{\pi}{2}$ and $p \in \mathbb{N}$.

We need to mention the following lemmas which will use in the main results.

**Lemma 1.** *"Let $H \subset \mathbb{C}$ and let $\Phi : \mathbb{C}^2 \times \mathbb{E}^* \to \mathbb{C}$ be a mapping satisfying $\Phi(ia, b : z) \notin H$ for $a, b \in \mathbb{R}$ such that $b \leq -n\frac{1+a^2}{2}$. If $p(z) = 1 + c_n z^n + \cdots$ is regular in $\mathbb{E}^*$ and $\Phi(p(z), zp'(z) : z) \in H \ \forall\, z \in \mathbb{E}^*$, then $\Re(p(z)) > 0$."*

**Lemma 2.** *"Let $p(z) = 1 + c_1 z + \cdots$ be regular in $\mathbb{E}^*$ and $\eta$ be regular and starlike univalent in $\mathbb{E}^*$ with $\eta(0) = 0$. If $zp'(z) \prec \eta(z)$, then*

$$p(z) \prec 1 + \int_0^z \frac{\eta(t)}{t} dt.$$

*This result is the best possible."*

## 2. Sufficiency Criterion for the Family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$

In this section, we investigate the sufficiency criterion for any meromorphic $p$-valent functions belonging to the introduced family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ :

Now, we obtain the necessary and sufficient condition for the p-valent function $f$ to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ as follows:

**Theorem 1.** *Let the function $f(z)$ be the member of the family $\Omega_p$. Then*

$$f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi) \Leftrightarrow \left| \frac{e^{i\lambda}}{\mathcal{G}(z)} - \frac{1}{2\xi\cos\lambda} \right| < \frac{1}{2\xi\cos\lambda}, \tag{2}$$

*where*

$$\mathcal{G}(z) = \frac{p\, s^p t^p}{(s^p - t^p)} \frac{f(sz) - f(tz)}{zf'(z)}. \tag{3}$$

**Proof.** Suppose that inequality (2) holds. Then, we have

$$\begin{aligned}
\left|\frac{2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}\left(z\right)}{2\xi\cos\lambda e^{-i\lambda}\mathcal{G}\left(z\right)}\right| &< \frac{1}{2\xi\cos\lambda} \\
&\Leftrightarrow \left|\frac{2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}\left(z\right)}{2\xi\cos\lambda e^{-i\lambda}\mathcal{G}\left(z\right)}\right|^2 < \frac{1}{4\xi^2\cos^2\lambda} \\
&\Leftrightarrow \left(2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}\left(z\right)\right)\left(\overline{2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}\left(z\right)}\right) < \left(e^{i\lambda}\overline{\mathcal{G}(z)}\right)e^{-i\lambda}\mathcal{G}(z) \\
&\Leftrightarrow 4\xi^2\cos^2\lambda - 2\xi\cos\lambda\left(e^{i\lambda}\overline{\mathcal{G}(z)} + e^{-i\lambda}\mathcal{G}\left(z\right)\right) < 0 \\
&\Leftrightarrow 2\xi\cos\lambda - 2\Re\left(e^{-i\lambda}\mathcal{G}\left(z\right)\right) < 0 \\
&\Leftrightarrow \Re\left(e^{-i\lambda}\mathcal{G}\left(z\right)\right) > \xi\cos\lambda,
\end{aligned}$$

and hence the result follows. □

Next, we investigate the sufficient condition for the p-valent function $f$ to be in the family $\mathcal{NS}_p^{\lambda}\left(s,t,\xi\right)$ in the following theorem:

**Theorem 2.** *If $f(z)$ belongs to the family $\Omega_p$ of meromorphic p-valent functions and obeying*

$$\sum_{n=p+1}^{\infty}\left|\left(\frac{s^n-t^n}{s^p-t^p}s^pt^p - \frac{n\beta\cos\lambda}{p}e^{i\lambda}\right)\right|\left|a_n\right| < \frac{1}{2}\left(1-\left|1-2\beta\cos\lambda e^{i\lambda}\right|\right), \tag{4}$$

*then $f\left(z\right) \in \mathcal{NS}_p^{\lambda}\left(s,t,\xi\right)$.*

**Proof.** To prove the required result we only need to show that

$$\left|\frac{2e^{i\lambda}\xi\cos\lambda zf'\left(z\right)/p - \frac{s^pt^p}{(t^p-s^p)}\left(f\left(tz\right)-f\left(sz\right)\right)}{\frac{s^pt^p}{(t^p-s^p)}\left(f\left(tz\right)-f\left(sz\right)\right)}\right| < 1. \tag{5}$$

Now consider the left hand side of (5), we get

$$\begin{aligned}
LHS &= \left|\frac{2e^{i\lambda}\xi\cos\lambda zf'\left(z\right)/p - \frac{s^pt^p}{(t^p-s^p)}\left(f\left(tz\right)-f\left(sz\right)\right)}{\frac{s^pt^p}{(t^p-s^p)}\left(f\left(tz\right)-f\left(sz\right)\right)}\right| \\
&= \left|\frac{\left(2e^{i\lambda}\xi\cos\lambda-1\right) + \sum\limits_{n=p+1}^{\infty}\left(\frac{s^n-t^n}{s^p-t^p}s^pt^p - \frac{2n\xi\cos\lambda}{p}e^{i\lambda}\right)a_nz^{n+p}}{1+\sum\limits_{n=p+1}^{\infty}\left(\frac{s^n-t^n}{s^p-t^p}\right)s^pt^pa_nz^{n+p}}\right| \\
&\leq \frac{\left|2e^{i\lambda}\xi\cos\lambda-1\right| + \sum\limits_{n=p+1}^{\infty}\left|\left(\frac{s^n-t^n}{s^p-t^p}s^pt^p - 2\beta\cos\lambda e^{i\lambda}\frac{n}{p}\right)\right|\left|a_n\right|\left|z^{n+p}\right|}{1-\sum\limits_{n=p+1}^{\infty}\left|\left(\frac{s^n-t^n}{s^p-t^p}\right)s^pt^p\right|\left|a_n\right|\left|z^{n+p}\right|} \\
&\leq \frac{\left|2e^{i\lambda}\xi\cos\lambda-1\right| + \sum\limits_{n=p+1}^{\infty}\left|\left(\frac{s^n-t^n}{s^p-t^p}s^pt^p - 2\beta\cos\lambda e^{i\lambda}\frac{n}{p}\right)\right|\left|a_n\right|}{1-\sum\limits_{n=p+1}^{\infty}\left|\left(\frac{s^n-t^n}{s^p-t^p}\right)s^pt^p\right|\left|a_n\right|}.
\end{aligned}$$

By virtue of inequality (4), we at once get the desired result. □

Also, we obtain another sufficient condition for the p-valent function $f$ to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ by using Lemma 1, in the following theorem:

**Theorem 3.** *If* $f(z) \in \Omega_p$ *satisfies*

$$\Re\left\{e^{-i\lambda}\left(\alpha z\frac{\mathcal{G}'(z)}{\mathcal{G}(z)}+1\right)\mathcal{G}(z)\right\} > \beta\cos\lambda - \frac{n}{2}\left((1-\beta)\alpha\cos\lambda\right),$$

*then* $f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi)$, *where* $\mathcal{G}(z)$ *is defined in Equation* (3).

**Proof.** Let we choose the function $q(z)$ by

$$q(z) = \frac{e^{-i\lambda}\mathcal{G}(z) - \beta\cos\lambda + i\sin\lambda}{(1-\beta)\cos\lambda}, \tag{6}$$

then Equation (6) shows that $q(z)$ is holomorphic in $\mathbb{E}$ and also normalized by $q(0) = 1$.

From Equation (6), we can easily obtain that

$$e^{-i\lambda}\mathcal{G}(z)\left(1+\alpha z\frac{\mathcal{G}'(z)}{\mathcal{G}(z)}\right) = \Phi\left(q(z), zq'(z), z\right),$$

where

$$\Phi\left(q(z), zq'(z), z\right) = \left[(1-\beta)\alpha zq'(z) + (1-\beta)q(z) + \beta\right]\cos\lambda - i\sin\lambda.$$

Now for all $a, b \in \mathbb{R}$ satisfying $2y \leq -n\left(1+a^2\right)$, we have

$$\begin{aligned}\Re\left\{\Phi(ia,b,z)\right\} &\leq \beta\cos\lambda - \frac{n}{2}\left(1+a^2\right)(1-\beta)\alpha\cos\lambda \\ &\leq \beta\cos\lambda - \frac{n}{2}(1-\beta)\alpha\cos\lambda.\end{aligned}$$

Now, let us define a set as

$$H = \left\{\zeta : \Re(\zeta) > \beta\cos\lambda - \frac{n}{2}\left((1-\beta)\alpha\cos\lambda\right)\right\},$$

then, we see that $\Phi(ia,b,z) \notin H$ and $\Phi\left(q(z), zq'(z), z\right) \in H$. Therefore, by using Lemma 1, we obtain that $\Re\left(q(z)\right) > 0$.
□

Further, in the next theorem, we obtain the sufficient condition for the p-valent function $f$ to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ by using Lemma 2.

**Theorem 4.** *If* $f(z)$ *is a member of the family* $\Omega_p$ *of meromorphic p-valent functions and satisfies*

$$\left|\frac{e^{i\lambda}}{\mathcal{G}(z)}\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right)\right| < \frac{1}{\beta\cos\lambda} - 1, \tag{7}$$

*then* $f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi)$, *where* $\mathcal{G}(z)$ *is given by Equation* (3).

**Proof.** In order to prove the required result, we need to define the following function

$$q(z)\cos\lambda = e^{-i\lambda}\mathcal{G}(z) + i\sin\lambda,$$

then, Equation (6) shows that th function $q(z)$ is holomorphic in $\mathbb{E}$ and also normalized by $q(0) = 1$.

Now, by routine computations, we get

$$\frac{zq'(z)}{q(z) - i\tan\lambda} = \frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}.$$

Now, let us consider $z\left(\frac{1}{q(z)\cos\lambda - i\sin\lambda}\right)'$ and then by using inequality (7), we have

$$\left|z\left(\frac{1}{q(z)\cos\lambda - i\sin\lambda}\right)'\right| = \left|\frac{e^{i\lambda}}{\mathcal{G}(z)}\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right)\right| < \frac{1}{\beta\cos\lambda} - 1,$$

therefore

$$z\left(\frac{1}{q(z)\cos\lambda - i\sin\lambda}\right)' \prec \frac{(1-\beta\cos\lambda)z}{\beta\cos\lambda}.$$

Using Lemma 2, we have

$$\frac{1}{(q(z) - i\tan\lambda)\cos\lambda} \prec 1 + \frac{(1-\beta\cos\lambda)}{\beta\cos\lambda}z,$$

equivalently

$$(q(z) - i\tan\lambda)\cos\lambda \prec \frac{\beta\cos\lambda}{\beta\cos\lambda + (1-\beta\cos\lambda)z} = H(z)\,(say)\,. \tag{8}$$

After simplifications, we get

$$1 + \Re\left(\frac{zH''(z)}{H'(z)}\right) = 2\beta\cos\lambda - 1 > 0, \quad for\ \frac{1}{2} < \beta < 1.$$

The region $H(\mathbb{E})$ shows that it is symmetric about the real axis and also $H(z)$ is convex. Hence

$$\Re\ (\mathcal{G}(z)) \geq H(1) > 0,$$

or

$$\Re\ (q(z)\cos\lambda - i\sin\lambda) > \beta\cos\lambda,$$

or

$$\Re\ \left(e^{-i\lambda}\mathcal{G}(z)\right) > \beta\cos\lambda,\ for\ \frac{1}{2} < \beta < 1.$$

□

Finally, we investigate the sufficient condition for the p-valent function $f$ to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ in the following theorem:

**Theorem 5.** *If $f(z) \in \Omega_p$ satisfies*

$$\left|\left(\frac{2\beta\cos\lambda e^{i\lambda}}{\mathcal{G}(z)} - 1\right)'\right| \leq \eta\,|z|^{\gamma}\,,\ for\ \ 0 < \eta \leq \gamma + 1, \tag{9}$$

*then $f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi)$, where $\mathcal{G}(z)$ is defined in Equation (3).*

**Proof.** Let us put

$$G(z) = z\left(\frac{2\beta\cos\lambda e^{i\lambda}}{\mathcal{G}(z)} - 1\right).$$

Then $G(0) = 0$ and $G(z)$ is analytic in $\mathbb{E}$. Using inequality (9), we can write

$$\left|\left(\frac{G(z)}{z}\right)'\right| = \left|\left(\frac{2\beta\cos\lambda e^{i\lambda}}{\mathcal{G}(z)} - 1\right)'\right| \leq \eta\,|z|^{\gamma}.$$

Now,

$$\left|\left(\frac{G(z)}{z}\right)\right| = \left|\int_0^z \left(\frac{G(t)}{t}\right)' dt\right| \leq \int_0^{|z|} \left|\left(\frac{G(t)}{t}\right)'\right| dt \leq \int_0^{|z|} \eta\,|t|^{\gamma}\,dt = \frac{\eta\,|z|^{\gamma+1}}{\gamma+1} < 1,$$

and this implies that

$$\left|\frac{2\beta\cos\lambda e^{i\lambda}}{\mathcal{G}(z)} - 1\right| < 1.$$

Now by using Theorem 1, we get the result which we needed. □

## 3. Conclusions

In our results, a new subfamily of meromorphic $p$-valent (multivalent) functions were introduced. Further, various sufficient conditions for meromorphic $p$-valent functions belonging to these subfamilies were obtained and investigated.

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# Geometric Properties of Certain Classes of Analytic Functions Associated with a $q$-Integral Operator

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**Abstract:** This article presents certain families of analytic functions regarding $q$-starlikeness and $q$-convexity of complex order $\gamma$ ($\gamma \in \mathbb{C} \setminus \{0\}$). This introduced a $q$-integral operator and certain subclasses of the newly introduced classes are defined by using this $q$-integral operator. Coefficient bounds for these subclasses are obtained. Furthermore, the $(\delta, q)$-neighborhood of analytic functions are introduced and the inclusion relations between the $(\delta, q)$-neighborhood and these subclasses of analytic functions are established. Moreover, the generalized hyper-Bessel function is defined, and application of main results are discussed.

**Keywords:** Geometric Function Theory; $q$-integral operator; $q$-starlike functions of complex order; $q$-convex functions of complex order; $(\delta, q)$-neighborhood

**MSC:** 30C15; 30C45

## 1. Introduction

Recently, many researchers have focused on the study of $q$-calculus keeping in view its wide applications in many areas of mathematics, e.g., in the $q$-fractional calculus, $q$-integral calculus, $q$-transform analysis and others (see, for example, [1,2]). Jackson [3] was the first to introduce and develop the $q$-derivative and $q$-integral. Purohit [4] was the first one to introduce and analyze a class in open unit disk and he used a certain operator of fractional q-derivative. His remarkable contribution was to give q-extension of a number of results that were already known in analytic function theory. Later, the $q$-operator was studied by Mohammed and Darus regarding its geometric properties on certain analytic functions, see [5]. A very significant usage of the $q$-calculus in the context of Geometric Function Theory was basically furnished and the basic (or $q$-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [6] pp. 347 et seq.;

see also [7]). Earlier, a class of $q$-starlike functions were introduced by Ismail et al. [8]. These are the generalized form of the known starlike functions by using the $q$-derivatives. Sahoo and Sharma [9] obtained many results of $q$-close-to-convex functions. Also, some recent results and investigations associated with the $q$-derivatives operator have been in [6,10–13].

It is worth mentioning here that the ordinary calculus is a limiting case of the quantum calculus. Now, we recall some basic concepts and definitions related to $q$-derivative, to be used in this work. For more details, see References [3,14–16].

The quantum derivative (named as $q$-derivative) of function $f$ is defined as:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \qquad (z \neq 0;\ 0 < q < 1).$$

We note that $D_q f(z) \longrightarrow f'(z)$ as $q \longrightarrow 1-$ and $D_q f(0) = f'(0)$, where $f'$ is the ordinary derivative of $f$.

In particular, $q$-derivative of $h(z) = z^n$ is as follows :

$$D_q h(z) = [n]_q z^{n-1}, \tag{1}$$

where $[n]_q$ denotes $q$-number which is given as:

$$[n]_q = \frac{1-q^n}{1-q} \qquad (0 < q < 1). \tag{2}$$

Since we see that $[n]_q \longrightarrow n$ as $q \longrightarrow 1-$, therefore, in view of Equation (1), $D_q h(z) \longrightarrow h'(z)$ as $q \longrightarrow 1-$, where $h'$ represents ordinary derivative of $h$.

The $q$-gamma function $\Gamma_q$ is defined as:

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+t}} \qquad (t > 0;\ 0 < q < 1), \tag{3}$$

which has the following properties:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \tag{4}$$

and

$$\Gamma_q(t+1) = [t]_q!\,, \tag{5}$$

where $t \in \mathbb{N}$ and $[.]_q!$ denotes the $q$-factorial and defined as:

$$[t]_q! = \begin{cases} [t]_q [t-1]_q \ldots [2]_q [1]_q, & t = 1,2,3,\ldots; \\ 1, & t = 0. \end{cases} \tag{6}$$

Also, the $q$-beta function $B_q$ is defined as:

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x \qquad (t,s > 0;\ 0 < q < 1), \tag{7}$$

which has the following property:

$$B_q(t,s) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}, \tag{8}$$

where $\Gamma_q$ is given by Equation (3).

Furthermore, $q$-binomial coefficients are defined as [17]:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_n![n-k]_q!}, \tag{9}$$

where $[.]_q!$ is given by Equation (6).

We consider the class $\mathcal{A}$ comprising the functions that are analytic in open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and are of the form given as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{10}$$

Using Equation (1), the $q$-derivative of $f$, defined by Equation (10) is as follows:

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \qquad (z \in \mathbb{U};\ 0 < q < 1), \tag{11}$$

where $[n]_q$ is given by Equation (2).

The two important subsets of the class $\mathcal{A}$ are the families $\mathcal{S}^*$ consisting of those functions that are starlike with reference to origin and $\mathcal{C}$ which is the collection of convex functions. A function $f$ is from $S^*$if for each point $x \in f(\mathbb{U})$ the linear segment between 0 and $x$ is contained in $f(\mathbb{U})$ . Also, a function $f \in \mathcal{C}$ if the image $f(\mathbb{U})$ is a convex subset of complex plane $\mathbb{C}$, i.e., $f(\mathbb{U})$ must have every line segment that joins its any two points.

Nasr and Aouf [18] defined the class of those functions which are starlike and are of complex order $\gamma$ $(\gamma \in \mathbb{C}\backslash \{0\})$, denoted by $\mathcal{S}^*(\gamma)$ and Wiatrowski [19] gave the class of similar type convex functions i.e., of complex order $\gamma$ $(\gamma \in \mathbb{C}\backslash \{0\})$, denoted by $\mathcal{C}(\gamma)$ as:

$$\mathcal{S}^*(\gamma) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) > 0 \ \ (z \in \mathbb{U}; \gamma \in \mathbb{C}\backslash \{0\}) \right\} \tag{12}$$

and

$$\mathcal{C}(\gamma) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right) > 0 \ \ (z \in \mathbb{U}; \gamma \in \mathbb{C}\backslash \{0\}) \right\}, \tag{13}$$

respectively.

From Equations (12) and (13), it is clear that $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ are subclasses of the class $\mathcal{A}$.

The class denoted by ${\mathcal{S}^*}_q(\mu)$ of such $q$-starlike functions that are of order $\mu$ is defined as:

$${\mathcal{S}^*}_q(\mu) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zD_q f(z)}{f(z)}\right) > \mu \ \ (z \in \mathbb{U}; 0 \le \mu < 1) \right\}. \tag{14}$$

Also, the class $\mathcal{C}_q(\mu)$ of $q$-convex functions of order $\mu$ is defined as:

$$\mathcal{C}_q(\mu) = \left\{ f \in \mathcal{A} : \Re\left(\frac{D_q(zD_q f(z))}{D_q f(z)}\right) > \mu \ \ (z \in \mathbb{U}; 0 \le \mu < 1) \right\}. \tag{15}$$

For more detail, see [20]. From Equations (14) and (15), it is clear that $\mathcal{S}_q^*(\mu)$ and $\mathcal{C}_q(\mu)$ are subclasses of the class $\mathcal{A}$.

Next, we recall that the $\delta$-neighborhood of the function $f(z) \in \mathcal{A}$ is defined as [21]:

$$\mathcal{N}_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \middle| \sum_{n=2}^{\infty} n\,|a_n - b_n| \le \delta \right\} \qquad (\delta \ge 0). \tag{16}$$

In particular, the $\delta$-neighborhood of the identity function $p(z) = z$ is defined as [21]:

$$\mathcal{N}_\delta(p) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \middle| \sum_{n=2}^{\infty} n\,|b_n| \le \delta \right\} \qquad (\delta \ge 0). \tag{17}$$

Finally, we recall that the Jung-Kim-Srivastava integral operator $\mathcal{Q}_\beta^\alpha : \mathcal{A} \to \mathcal{A}$ are defined as [22]:

$$\begin{aligned} \mathcal{Q}_\beta^\alpha f(z) =& \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^\beta} \int_0^z t^{\beta-1} \left(1 - \frac{t}{z}\right)^{\alpha-1} f(t)dt \\ =& z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n z^n \qquad (\beta > -1;\ \alpha > 0;\ f \in \mathcal{A}). \end{aligned} \tag{18}$$

The Bessel functions are associated with a wide range of problems in important areas of mathematical physics and Engineering. These functions appear in the solutions of heat transfer and other problems in cylindrical and spherical coordinates. Rainville [23] discussed the properties of the Bessel function.

The generalized Bessel functions $w_{\nu,b,d}(z)$ are defined as [24]:

$$w_{\nu,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d)^n}{n!\Gamma\left(\nu + n + \dfrac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu}, \tag{19}$$

where $\nu, b, d, z \in \mathbb{C}$.

Orhan, Deniz and Srivastava [25] defined the function $\varphi_{\nu,b,d}(z) : \mathbb{U} \to \mathbb{C}$ as:

$$\varphi_{\nu,b,d}(z) = 2^\nu \Gamma\left(\nu + \frac{b+1}{2}\right) z^{-\frac{\nu}{2}} w_{\nu,b,d}(\sqrt{z}), \tag{20}$$

by using the Generalized Bessel function $w_{\nu,b,d}(z)$, given by Equation (12).

The power series representation for the function $\varphi_{\nu,b,d}(z)$ is as follows [25]:

$$\varphi_{\nu,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d/4)^n}{(c)_n n!} z^n, \tag{21}$$

where $c = \nu + \dfrac{b+1}{2} > 0, \nu, b, d \in \mathbb{R}$ and $z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

The hyper-Bessel function is defined as [26]:

$$J_{\alpha_d}(z) = \sum_{n=0}^{\infty} \frac{(z/d+1)^{\alpha_1+\dots\alpha_d}}{\Gamma(\alpha_1+1)\dots\Gamma(\alpha_d+1)} {}_0F_d \left( -, (\alpha_d+1); -\left(\frac{z}{d+1}\right)^{d+1} \right), \tag{22}$$

where the hypergeometric function ${}_pF_q$ is defined by:

$${}_pF_q\left((\beta_p); (\eta_q); x\right) = \sum_{n=0}^{\infty} \frac{(\beta_1)_n (\beta_2)_n \dots (\beta_p)_n}{(\alpha_1)_n (\alpha_2)_n \dots (\alpha_q)_n} \frac{x^n}{n!}, \tag{23}$$

using above Equation (23) in Equation (22), then the function $J_{\alpha_d}(z)$ has the following power series:

$$J_{\alpha_d}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\alpha_1+n+1)\Gamma(\alpha_2+n+1)\dots\Gamma(\alpha_d+n+1)} \left(\frac{z}{d+1}\right)^{n(d+1)+\alpha_1+\dots\alpha_d}, \tag{24}$$

By choosing $d = 1$ and putting $\alpha_1 = \nu$, we get the classical Bessel function

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(\nu + n + 1\right)} z^{2n+\nu}. \tag{25}$$

In the next section, we introduce the classes of $q$-starlike functions that are of complex order $\gamma$ $(\gamma \in \mathbb{C}\setminus\{0\})$ and similarly, $q$-convex functions that are of complex order $\gamma$ $(\gamma \in \mathbb{C}\setminus\{0\})$, which are denoted by $\mathcal{S}_q^*(\gamma)$ and $\mathcal{C}_q(\gamma)$, respectively. Also, we define a $q$-integral operator and define the subclasses $\mathcal{S}_q(\alpha, \beta, \gamma)$ and $\mathcal{C}_q(\alpha, \beta, \gamma)$ of the class $\mathcal{A}$ by using this $q$-integral operator. Then, we find the coefficient bounds for these subclasses.

First, we define the $q$-starlike function of complex order $\gamma$ $(\gamma \in \mathbb{C}\setminus\{0\})$, denoted by $\mathcal{S}_q^*(\gamma)$ and the $q$-convex function of complex order $\gamma$ $(\gamma \in \mathbb{C}\setminus\{0\})$, denoted by $\mathcal{C}_q(\gamma)$ by taking the $q$-derivative in place of ordinary derivatives in Equations (12) and (13), respectively.

The respective definitions of the classes $\mathcal{S}_q^*(\gamma)$ and $\mathcal{C}_q(\gamma)$ are as follows:

**Definition 1.** *The function $f \in \mathcal{A}$ will belong to the class $\mathcal{S}_q^*(\gamma)$ if it satisfies the following inequality:*

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{zD_q f(z)}{f(z)} - 1\right)\right) > 0 \qquad (\gamma \in \mathbb{C}\setminus\{0\},\ 0 < q < 1). \tag{26}$$

**Definition 2.** *The function $f \in \mathcal{A}$ will belong to the class $\mathcal{C}_q(\gamma)$ if it satisfies the following inequality:*

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right)\right) > 0 \qquad (\gamma \in \mathbb{C}\setminus\{0\}, 0 < q < 1). \tag{27}$$

**Remark 1.** *(i) If $\gamma \in \mathbb{R}$ and $\gamma = 1 - \mu$ $(0 \leq \mu < 1)$, then the subclasses $\mathcal{S}_q^*(\gamma)$ and $\mathcal{C}_q(\gamma)$ give the sub classes $\mathcal{S}_q^*(\mu)$ and $\mathcal{C}_q(\mu)$, respectively.*

*(ii) Using the fact that $\lim_{q\to 1-} D_q f(z) = f'(z)$, we get that $\lim_{q\to 1-} \mathcal{S}_q^*(\gamma) = \mathcal{S}^*(\gamma)$ and $\lim_{q\to 1-} \mathcal{C}_q(\gamma) = \mathcal{C}(\gamma)$.*

Now, we introduce the $q$-integral operator $\chi_{\beta,q}^{\alpha}$ as:

$$\chi_{\beta,q}^{\alpha} f(z) = \binom{\alpha+\beta}{\beta}_q \frac{[\alpha]_q}{z^\beta} \int_0^z t^{\beta-1}\left(1 - \frac{qt}{z}\right)_q^{\alpha-1} f(t) d_q t$$
$$(\alpha > 0;\ \beta > -1;\ 0 < q < 1;\ |z| < 1;\ f \in \mathcal{A}). \tag{28}$$

It is clear that $\chi_{\beta,q}^{\alpha} f(z)$ is analytic in open disc $\mathbb{U}$.

Using Equations (4), (5) and (7)–(9), we get the following power series for the function $\chi_{\beta,q}^{\alpha} f$ in $\mathbb{U}$:

$$\chi_{\beta,q}^{\alpha} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} a_n z^n \qquad (\alpha > 0;\ \beta > -1;\ 0 < q < 1;\ f \in \mathcal{A}). \tag{29}$$

**Remark 2.** *For $q \longrightarrow 1-$, Equation (29), gives the Jung-Kim-Srivastava integral operator $\mathcal{Q}_\beta^\alpha$, given by Equation (18).*

**Remark 3.** *Taking $\alpha = 1$ in Equation (28) and using Equations (4), (5) and (9), we get the q-Bernardi integral operator, defined as [27]:*

$$\mathcal{F}(z) = \frac{[1+\beta]_q}{z^\beta} \int_0^z t^{\beta-1} f(t) d_q t \qquad \beta = 1, 2, 3, \ldots$$

Next, in view of the Definitions 1 and 2 and the fact that $\Re(z) < |z|$, we introduce the subclasses $\mathcal{S}_q(\alpha, \beta, \gamma)$ and $\mathcal{C}_q(\alpha, \beta, \gamma)$ of the classes $\mathcal{S}_q^*(\gamma)$ and $\mathcal{C}_q(\gamma)$, respectively, by using the operator $\chi_{\beta,q}^{\alpha}$, as:

**Definition 3.** *The function $f \in \mathcal{A}$ will belong to $\mathcal{S}_q(\alpha, \beta, \gamma)$ if it satisfies the following inequality:*

$$\left| \frac{zD_q(\chi_{\beta,q}^{\alpha} f(z))}{\chi_{\beta,q}^{\alpha} f(z)} - 1 \right| < |\gamma|, \tag{30}$$

*where $\alpha > 0;\ \beta > -1;\ 0 < q < 1;\ \gamma \in \mathbb{C} \backslash \{0\}$.*

**Definition 4.** *The function $f \in \mathcal{A}$ will belong to $\mathcal{C}_q(\alpha, \beta, \gamma)$ if it satisfies the following inequality:*

$$\left| \frac{D_q\left(zD_q\chi_{\beta,q}^{\alpha} f(z)\right)}{D_q\chi_{\beta,q}^{\alpha} f(z)} \right| < |\gamma|, \tag{31}$$

*where $\alpha > 0;\ \beta > -1;\ 0 < q < 1; \gamma \in \mathbb{C} \backslash \{0\}$.*

Now, we establish the following result, which gives the coefficient bound for the subclass $\mathcal{S}_q(\alpha, \beta, \gamma)$:

**Lemma 1.** *If $f$ is an analytic function such that it belongs to the class $\mathcal{S}_q(\alpha, \beta, \gamma)$, then*

$$\sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} \left([n]_q - |\gamma| - 1\right) a_n < |\gamma| \quad (\alpha > 0;\ \beta > -1;\ 0 < q < 1; \gamma \in \mathbb{C} \backslash \{0\}), \tag{32}$$

*where $\Gamma_q$ and $[n]_q$ are given by Equations (3) and (2), respectively.*

**Proof.** Let $f \in \mathcal{A}$, then using Equations (11) and (29), we have

$$\left| \frac{zD_q(\chi_{\beta,q}^{\alpha} f(z))}{\chi_{\beta,q}^{\alpha} f(z)} - 1 \right| = \left| \frac{z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} [n]_q a_n z^n}{z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} a_n z^n} - 1 \right|. \tag{33}$$

If $f \in \mathcal{S}_q(\alpha, \beta, \gamma)$, then in view of Definition 3 and Equation (33), we have

$$\left| \frac{z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} [n]_q a_n z^n}{z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} a_n z^n} - 1 \right| < |\gamma|,$$

which, on simplifying, gives

$$\left| \frac{\sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} \left([n]_q - 1\right) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} a_n z^{n-1}} \right| < |\gamma|. \tag{34}$$

Now, using the fact that $\Re(z) < |z|$ in the Inequality (34), we get

$$\Re\left(\frac{\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}\left([n]_q-1\right)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}a_nz^{n-1}}\right) < |\gamma|. \tag{35}$$

Since $\chi_{\beta,q}^{\alpha}f(z)$ is analytic in $\mathbb{U}$, therefore taking limit $z \to 1$—through real axis, Inequality (35), gives the Assertion (32). □

Also, we establish the following result, which gives the coefficient bound for the subclass $\mathcal{C}_q(\alpha,\beta,\gamma)$:

**Lemma 2.** *If $f$ is an analytic function such that it belongs to the class $\mathcal{C}_q(\alpha,\beta,\gamma)$ and $|\gamma| \geq 1$ then*

$$\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}\left([n]_q\left([n]_q-|\gamma|\right)\right)a_n < |\gamma|-1 \quad (\alpha>0;\ \beta>-1;\ 0<q<1; \gamma\in\mathbb{C}\backslash\{0\}), \tag{36}$$

*where $\Gamma_q$ and $[n]_q$ are given by Equations (3) and (2), respectively.*

**Proof.** Let $f \in \mathcal{A}$, then using Equations (11) and (29), we get

$$\left|\frac{D_q\left(zD_q\chi_{\beta,q}^{\alpha}f(z)\right)}{D_q\chi_{\beta,q}^{\alpha}f(z)}\right| = \left|\frac{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}([n]_q)^2a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}[n]_qa_nz^{n-1}}\right|. \tag{37}$$

If $f \in \mathcal{C}_q(\alpha,\beta,\gamma)$, then in view of Definition 4 and Equation (37), we have

$$\left|\frac{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}([n]_q)^2a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}[n]_qa_nz^{n-1}}\right| < |\gamma| \tag{38}$$

Now, using the fact that $\Re(z) < |z|$ in Inequality (38), we get

$$\Re\left(\frac{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}([n]_q)^2a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}[n]_qa_nz^{n-1}}\right) < |\gamma| \tag{39}$$

Since $\chi_{\beta,q}^{\alpha}f(z)$ is analytic in $\mathbb{U}$, therefore taking limit $z \to 1-$ through real axis, Inequality (39) gives the Assertion (36). □

In the next section, we define $(\delta,q)$-neighborhood of the function $f \in \mathcal{A}$ and establish the inclusion relations of the subclasses $\mathcal{S}_q(\alpha,\beta,\gamma)$ and $\mathcal{C}_q(\alpha,\beta,\gamma)$ with the $(\delta,q)$-neighborhood of the identity function $p(z) = z$.

## 2. The Classes $\mathcal{N}_{\delta,q}(f)$ and $\mathcal{N}_{\delta,q}(p)$

In view of Equation (16), we define the $(\delta,q)$-neighborhood of the function $f \in \mathcal{A}$ as:

$$\mathcal{N}_{\delta,q}(f) = \left\{g(z) = z + \sum_{n=2}^{\infty}b_nz^n \middle| \sum_{n=2}^{\infty}[n]_q\left|a_n - b_n\right| \leq \delta\right\} \quad (\delta \geq 0,\ 0<q<1), \tag{40}$$

where $[n]_q$ is given by Equation (2).

In particular, the $(\delta, q)$-neighborhood of the identity function $p(z) = z$, defined as:

$$\mathcal{N}_{\delta,q}(p) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \middle| \sum_{n=2}^{\infty} [n]_q \, |b_n| \leq \delta \right\} \qquad (\delta \geq 0,\ 0 < q < 1). \tag{41}$$

Since $[n]_q$ approaches $n$ as $q$ approaches $1-$, therefore, from Equations (16) and (40), we note that $\lim_{q\to 1-} \mathcal{N}_{\delta,q}(f) = \mathcal{N}_{\delta}(f)$, where $\mathcal{N}_{\delta}(f)$ is defined by Equation (16). In particular, $\lim_{q\to 1-} \mathcal{N}_{\delta,q}(p) = \mathcal{N}_{\delta}(p)$.

Now, we establish the following inclusion relation between the class $\mathcal{S}_q(\alpha, \beta, \gamma)$ and $(\delta, q)$-neighborhood $\mathcal{N}_{\delta,q}(p)$ of identity function $p$ for the specified range of values of $\delta$:

**Theorem 1.** *If* $-1 < \beta \leq 0$, $|\gamma| \leq [n]_q - 1 \;\; (n = 2, 3, \ldots)$ *and*

$$\delta \geq \frac{|\gamma| [2]_q \Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)}{([2]_q - |\gamma| - 1) \Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}, \tag{42}$$

*then*

$$\mathcal{S}_q(\alpha, \beta, \gamma) \subset \mathcal{N}_{\delta,q}(p) \qquad (\gamma \in \mathbb{C} \backslash \{0\}\,;\ \alpha > 0;\ 0 < q < 1). \tag{43}$$

**Proof.** Let $f \in \mathcal{S}_q(\alpha, \beta, \gamma)$, then, in view of Lemma 1, Inequality (32) holds. Since for $\alpha > 0$, $-1 < \beta \leq 0$, the sequence $\left\{ \dfrac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \right\}_{n=2}^{\infty}$ is non-decreasing, therefore, we have

$$\frac{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \left( [2]_q - |\gamma| - 1 \right) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta + n) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + n) \Gamma_q(\beta + 1)} \left( [n]_q - |\gamma| - 1 \right) a_n,$$

which in view of Inequality (32), gives

$$\frac{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \left( [2]_q - |\gamma| - 1 \right) \sum_{n=2}^{\infty} a_n < |\gamma|, \tag{44}$$

or, equivalently,

$$\sum_{n=2}^{\infty} a_n < \frac{|\gamma| \Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)}{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1) \left( [2]_q - |\gamma| - 1 \right)}. \tag{45}$$

Again, using the fact that the sequence $\left\{ \dfrac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \right\}_{n=2}^{\infty}$ is non-decreasing for $\alpha > 0$ and $-1 < \beta \leq 0$, Inequality (32), gives

$$\frac{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \sum_{n=2}^{\infty} ([n]_q - |\gamma| - 1) a_n < |\gamma|,$$

or, equivalently,

$$\frac{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \sum_{n=2}^{\infty} [n]_q a_n < |\gamma| + \frac{(1 + |\gamma|) \Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \sum_{n=2}^{\infty} a_n, \tag{46}$$

which on using the Inequality (45), gives

$$\frac{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \sum_{n=2}^{\infty} [n]_q a_n < |\gamma| + \frac{(1 + |\gamma|) |\gamma|}{[2]_q - |\gamma| - 1},$$

or, equivalently,

$$\sum_{n=2}^{\infty}[n]_q\, a_n < \frac{|\gamma|[2]_q\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{([2]_q-|\gamma|-1)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}. \tag{47}$$

Now, if we take $\delta \geq \dfrac{|\gamma|[2]_q\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{([2]_q-|\gamma|-1)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}$, then in view of Equation (41) and Inequality (47), we obtain that $f(z) \in \mathcal{N}_{\delta,q}(p)$, which proves the inclusion Relation (43). □

Next, we establish the following inclusion relation between the class $\mathcal{C}_q(\alpha,\beta,\gamma)$ and $(\delta,q)$-neighborhood $\mathcal{N}_{\delta,q}(p)$ of identity function $p$ for the specified range of values of $\delta$:

**Theorem 2.** *If* $-1<\beta\leq 0$, $|\gamma|\geq 1$ *and*

$$\delta \geq \frac{(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{([2]_q-|\gamma|)\,\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}, \tag{48}$$

*then*

$$\mathcal{C}_q(\alpha,\beta,\gamma) \subset \mathcal{N}_{\delta,q}(p) \qquad (\alpha>0;\ \gamma\in\mathbb{C}\setminus\{0\};\ 0<q<1). \tag{49}$$

**Proof.** Let $f \in \mathcal{C}_q(\alpha,\beta,\gamma)$, then, in view of Lemma 2, Inequality (36) holds. Since for $\alpha>0,\ -1<\beta\leq 0$, the sequence $\left\{\dfrac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\right\}_{n=2}^{\infty}$ is non-decreasing, therefore we have

$$\frac{\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}\left([2]_q-|\gamma|\right)\sum_{n=2}^{\infty}[n]_q a_n \leq \sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}([n]_q\left([n]_q-|\gamma|\right))a_n,$$

which, in view of Inequality (36), gives

$$\frac{\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}\left([2]_q-|\gamma|\right)\sum_{n=2}^{\infty}[n]_q a_n < |\gamma|-1, \tag{50}$$

or, equivalently

$$\sum_{n=2}^{\infty}[n]_q a_n < \frac{(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{([2]_q-|\gamma|)\,\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}. \tag{51}$$

Now, if we take $\delta \geq \dfrac{(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{([2]_q-|\gamma|)\,\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}$, then in view of Equation (41) and Inequality (51), we obtain that $f(z) \in \mathcal{N}_{\delta,q}(p)$, which proves the inclusion Relation (49). □

## 3. The Classes $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ and $\mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma)$

In this section, the classes $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ and $\mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma)$ are defined. Then, we establish the inclusion relations between the neighborhood of a function belonging to $\mathcal{S}_q(\alpha,\beta,\gamma)$ and $\mathcal{C}_q(\alpha,\beta,\gamma)$ with $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ and $\mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma)$, respectively. First, we define the class $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ as follows.

**Definition 5.** *The function* $f\in\mathcal{A}$, *belongs to* $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ $(\alpha>0;\ -1<\beta;\ \gamma\in\mathbb{C}\setminus\{0\};\ 0<q<1;\ 0\leq\eta<1)$ *if there exists a function* $g(z)\in\mathcal{S}_q(\alpha,\beta,\gamma)$ *that satisfies*

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-\eta, \tag{52}$$

*where*

$$g(z) = z+\sum_{n=2}^{\infty} b_n z^n. \tag{53}$$

Similarly, we define the class $\mathcal{S}_q^{(\eta)}(\alpha, \beta, \gamma)$ as:

**Definition 6.** *The function* $f \in \mathcal{A}$*, belongs to* $\mathcal{C}_q^{(\eta)}(\alpha, \beta, \gamma)$ $(\alpha > 0;\ -1 < \beta;\ \gamma \in \mathbb{C}\setminus\{0\};\ 0 < q < 1;\ 0 \leq \eta < 1)$ *if there exists a function* $g$*, given by Equation (53), in the class* $\mathcal{C}_q(\alpha, \beta, \gamma)$*, satisfying the Inequality (52).*

Now, we establish the following inclusion relation between a neighborhood $\mathcal{N}_{\delta,q}(g)$ of any function $g \in \mathcal{S}_q(\alpha, \beta, \gamma)$ and the class $\mathcal{S}_q^{(\eta)}(\alpha, \beta, \gamma)$ for the specified range of values of $\eta$:

**Theorem 3.** *Let the function* $g$*, given by Equation (53), belongs to the class* $\mathcal{S}_q(\alpha, \beta, \gamma)$ *and*

$$\eta < 1 - \frac{\delta\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)\left([2]_q - |\gamma| - 1\right)}{[2]_q\left(\left([2]_q - |\gamma| - 1\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1) - |\gamma|\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}, \tag{54}$$

*then*

$$\mathcal{N}_{\delta,q}(g) \subset \mathcal{S}_q^{(\eta)}(\alpha, \beta, \gamma), \tag{55}$$

*where* $\alpha > 0;\ -1 < \beta \leq 0;\ \gamma \in \mathbb{C}\setminus\{0\};\ \delta \geq 0;\ 0 < q < 1;\ 0 \leq \eta < 1.$

**Proof.** We assume that $f \in \mathcal{N}_{\delta,q}(g)$, then in view of Relation (40), we have

$$\sum_{n=2}^{\infty}[n]_q\left|a_n - b_n\right| \leq \delta. \tag{56}$$

Since $\left\{[n]_q\right\}_{n=2}^{\infty}$ is non-decreasing sequence, therefore

$$\sum_{n=2}^{\infty}[2]_q\left|a_n - b_n\right| \leq \sum_{n=2}^{\infty}[n]_q\left|a_n - b_n\right|,$$

This implies that

$$[2]_q\sum_{n=2}^{\infty}\left|a_n - b_n\right| \leq \sum_{n=2}^{\infty}[n]_q\left|a_n - b_n\right|,$$

which in view of Inequality (56) gives

$$[2]_q\sum_{n=2}^{\infty}\left|a_n - b_n\right| \leq \delta,$$

or, equivalently

$$\sum_{n=2}^{\infty}\left|a_n - b_n\right| \leq \frac{\delta}{[2]_q} \qquad (0 < q < 1;\ \delta \geq 0). \tag{57}$$

Since $-1 < \beta \leq 0$, therefore, for the function $g$, given by Equation (53), in the class $\mathcal{S}_q(\alpha, \beta, \gamma)$, using Inequality (45), we get

$$\sum_{n=2}^{\infty} b_n \leq \frac{|\gamma|\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)\left([2]_q + |\gamma| - 1\right)}. \tag{58}$$

Using Equations (10), (53) and the fact that $|z| < 1$, we get

$$\left|\frac{f(z)}{g(z)} - 1\right| = \left|\frac{\sum_{n=2}^{\infty}\left(a_n - b_n\right)z^{n-1}}{1 + \sum_{n=2}^{\infty} b_n z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty}\left|a_n - b_n\right|}{1 - \sum_{n=2}^{\infty}\left|b_n\right|} \leq \frac{\sum_{n=2}^{\infty}\left|a_n - b_n\right|}{1 - \sum_{n=2}^{\infty} b_n}, \tag{59}$$

Now, using Inequalities (57) and (58) in Inequality (59), we get

$$\left|\frac{f(z)}{g(z)}-1\right| \leq \frac{\delta\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)\left([2]_q-|\gamma|-1\right)}{[2]_q\left(\left([2]_q-|\gamma|-1\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)-|\gamma|\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}. \tag{60}$$

If we take $\eta < 1-\dfrac{\delta\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)\left([2]_q-|\gamma|-1\right)}{[2]_q\left(\left([2]_q-|\gamma|-1\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)-|\gamma|\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}$, then in view of Definition 5 and Inequality (60), we obtain that $f \in \mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$, which proves the inclusion Relation (55). □

Next, we establish the following inclusion relation between a neighborhood $\mathcal{N}_{\delta,q}(g)$ of any function $g \in \mathcal{C}_q(\alpha,\beta,\gamma)$ and the class $\mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma)$ for the specified range of values of $\eta$:

**Theorem 4.** *Let the function g, given by Equation (53), belongs to the class $\mathcal{C}_q(\alpha,\beta,\gamma)$ and*

$$\eta < 1-\frac{\delta[2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}{[2]_q\left([2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)-(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}, \tag{61}$$

*then*

$$\mathcal{N}_{\delta,q}(g) \subset \mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma), \tag{62}$$

*where* $|\gamma|>1,\ \alpha>0;\ -1<\beta\leq 0;\ \gamma\in\mathbb{C}\backslash\{0\};\ 0<q<1;\ \delta\geq 0;\ 0\leq\eta<1.$

**Proof.** If we take any $f \in \mathcal{N}_{\delta,q}(g)$, then Inequality (57) holds.

Now, since $-1 < \beta \leq 0$, therefore, for any function $g$, given by Equation (53), in the class $\mathcal{C}_q(\alpha,\beta,\gamma)$, using Inequality (51) and the fact that the sequence $\{[n]_q\}_{n=2}^{\infty}$ is non-decreasing, we get

$$\sum_{n=2}^{\infty} b_n < \frac{(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{[2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}. \tag{63}$$

Using Inequalities (57) and (63) in Inequality (59), we get

$$\left|\frac{f(z)}{g(z)}-1\right| \leq \frac{\delta[2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}{[2]_q\left([2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)-(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}. \tag{64}$$

If we take $\eta < 1-\dfrac{\delta[2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}{[2]_q\left([2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)-(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}$, then in view of Definition 6 and Inequality (64), we obtain that $f \in \mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma)$, which proves the Assertion (61). □

## 4. Application

First, we define the generalized hyper-Bessel function $w_{c,b,\alpha_d}(z)$ as :

$$w_{c,b,\alpha_d}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\prod_{i=1}^{d}\Gamma\left(\alpha_i+n+\dfrac{b+1}{2}\right)}\left(\frac{z}{d+1}\right)^{n(d+1)+\sum_{i=1}^{d}\alpha_i} \tag{65}$$

where $\nu, b, d, z \in \mathbb{C}$.

Second, we define the function $\varphi_{\alpha_d,b,c}(z) : \mathbb{U} \to \mathbb{C}$ as:

$$\varphi_{\alpha_d,b,c}(z) = (d+1)^{\sum_{i=1}^{d} \alpha_i} \prod_{i=1}^{d} \Gamma\left(\alpha_i + \frac{b+1}{2}\right) z^{1-\frac{\sum_{i=1}^{d} \alpha_i}{d+1}} w_{\alpha_d,b,c}(z^{1/d+1}), \tag{66}$$

by using Equation (65) in Equation (66), we get

$$\begin{aligned} \varphi_{c,b,\alpha_d}(z) &= \sum_{n=0}^{\infty} \frac{(-c)^n}{n! \prod_{i=1}^{d} \left(\alpha_i + \dfrac{b+1}{2}\right)_n (d+1)^{n(d+1)}} z^{n+1} \\ &= z + \sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{(n-1)! \prod_{i=1}^{d} \left(\alpha_i + \dfrac{b+1}{2}\right)_{n-1} (d+1)^{(n-1)(d+1)}} z^n \end{aligned} \tag{67}$$

by choosing $d = 1$ and $\alpha_1 = \nu$, then the functions $w_{c,b,\alpha_d}(z)$ and $\varphi_{\alpha_d,b,c}(z)$ are reduce to $w_{\nu,b,d}(z)$ and $\phi_{\nu,b,d}(z)$, respectively.

Third, we applying the introduced function $\varphi_{c,b,\alpha_d}(z)$, given by Equation (67) in the results of Lemma 1 and Lemma 2, we get the conditions for that function $\varphi_{c,b,\alpha_d}(z)$ to be in the classes $\mathcal{S}_q(\alpha, \beta, \gamma)$ and $\mathcal{C}_q(\alpha, \beta, \gamma)$ in the following corollaries, respectively:

**Corollary 1.** *If $\varphi_{c,b,\alpha_d}(z)$ is an analytic function such that it belongs to the class $\mathcal{S}_q(\alpha, \beta, \gamma)$, then*

$$\sum_{n=2}^{\infty} \frac{(-c)^{n-1} \Gamma_q(\beta+n) \Gamma_q(\alpha+\beta+1)}{(n-1)! \prod_{i=1}^{d} \left(\alpha_i + \dfrac{b+1}{2}\right)_{n-1} (d+1)^{(n-1)(d+1)} \Gamma_q(\alpha+\beta+n) \Gamma_q(\beta+1)} \times \left([n]_q - |\gamma| - 1\right) < |\gamma| \quad (\alpha > 0;\ \beta > -1;\ 0 < q < 1; \gamma \in \mathbb{C} \setminus \{0\}),$$

*where $\Gamma_q$ and $[n]_q$ are given by Equations (2) and (3), respectively.*

**Corollary 2.** *If $\varphi_{c,b,\alpha_d}(z)$ is an analytic function such that it belongs to the class $\mathcal{C}_q(\alpha, \beta, \gamma)$ and $|\gamma| \geq 1$ then*

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{(-c)^{n-1} \Gamma_q(\beta+n) \Gamma_q(\alpha+\beta+1)}{(n-1)! \prod_{i=1}^{d} \left(\alpha_i + \dfrac{b+1}{2}\right)_{n-1} (d+1)^{(n-1)(d+1)} \Gamma_q(\alpha+\beta+n) \Gamma_q(\beta+1)} \\ &\times \left([n]_q \left([n]_q - |\gamma|\right)\right) a_n < |\gamma| - 1 \quad (\alpha > 0;\ \beta > -1;\ 0 < q < 1; \gamma \in \mathbb{C} \setminus \{0\}), \end{aligned}$$

*where $\Gamma_q$ and $[n]_q$ are given by Equations (2) and (3), respectively.*

## 5. Discussion of Results and Future Work

The concept of $q$-derivatives has so far been applied in many areas of not only mathematics but also physics, including fractional calculus and quantum physics. However, research on $q$-calculus is in connection with function theory and especially geometric properties of analytic functions such as starlikeness and convexity, which is fairly familiar on this topic. Finding sharp coefficient bounds for analytic functions belonging to Classes of starlikeness and convexity defined by $q$-calculus operators is of particular importance since any information can shed light on the study of the geometric properties of such functions. Our results are applicable by using any analytic functions.

## 6. Conclusions

In this paper, we have used $q$-calculus to introduce a new $q$-integral operator which is a generalization of the known Jung-Kim-Srivastava integral operator. Also, a new subclass involving the $q$-integral operator introduced has been defined. Some interesting coefficient bounds for these subclasses of analytic functions have been studied. Furthermore, the $(\delta, q)$-neighborhood of analytic functions and the inclusion relation between the $(\delta, q)$-neighborhood and the subclasses involving the $q$-integral operator have been derived. The ideas of this paper may stimulate further research in this field.

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# Existence of Solution for Non-Linear Functional Integral Equations of Two Variables in Banach Algebra

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**Abstract:** The aim of this article is to establish the existence of the solution of non-linear functional integral equations $x(l,h) = \left(U(l,h,x(l,h)) + F\left(l,h,\int_0^l\int_0^h P(l,h,r,u,x(r,u))drdu, x(l,h)\right)\right) \times G\left(l,h,\int_0^a\int_0^a Q\left(l,h,r,u,x(r,u)\right)drdu, x(l,h)\right)$ of two variables, which is of the form of two operators in the setting of Banach algebra $C\left([0,a]\times[0,a]\right), a > 0$. Our methodology relies upon the measure of noncompactness related to the fixed point hypothesis. We have used the measure of noncompactness on $C\left([0,a]\times[0,a]\right)$ and a fixed point theorem, which is a generalization of Darbo's fixed point theorem for the product of operators. We additionally illustrate our outcome with the help of an interesting example.

**Keywords:** functional integral equations; Banach algebra; fixed point theorem; measure of noncompactness

**MSC:** (2010): 45G15; 47H10

## 1. Introduction

Many real-life problems in which we go over the investigation of various branches of mathematical physics, for example, gas kinetic theory, radiation, and neutron transportation, can be depicted and demonstrated by methods of non-linear functional integral equations (for example, we refer to [1–4]). Banaś and Lecko [5] introduced the concept of fixed points of product operators in Banach algebra. Dhage [6,7] used the concept of the fixed point theorem to find the solution of functional integral equations in Banach algebra. Banaś and Olszowy [8] used the class of measures of noncompactness to obtain the existence of solutions of nonlinear integral equations in Banach algebra. Deepmala and Pathak [9] studied the existence of the solution of nonlinear functional integral equations of a single variable in Banach algebra $C[a,b]$ of all real-valued continuous functions on the interval $[a,b]$ equipped with the maximum norm.

Kuratowski [10], in the year 1930, first introduced the idea of the measure of noncompactness (denoted by "$\alpha$"). For any bounded subset $A$ of a metric space $X$,

$$\alpha\left(A\right)=\inf\left\{\delta>0: A\subset\bigcup_{j=1}^{m}A_j, A_j\subset X, \operatorname{diam}\left(A_j\right)<\delta\ (j=1,...,m), m\in\mathbb{N}\right\},$$

where:

$$\operatorname{diam}\left(A_j\right)=\sup\left\{d(a_1,a_2): a_1,a_2\in A_j\right\}.$$

Using this idea, Darbo [11] exhibited a fixed point theorem that plays a very significant role in the finding of existence theorems. In the recent past, there have been a few fruitful endeavours to apply the idea of the measure of noncompactness in the investigation of the existence of solutions for various kinds of differential and integral equations, for example one can refer to [12–20].

In many physical problems, we come across nonlinear integral equations. The fixed point theory plays a significant role to obtain the solutions of such equations. Deepmala and Pathak, in [9], studied the following nonlinear functional integral equation, which can be considered as a particular case of many nonlinear functional integral equations that are applicable in mechanics, physics, economics, etc.,

$$\begin{aligned}x(t)=&\left(u(t,x(t))+f\left(t,\int_0^t p(t,s,x(s))ds, x(\alpha(t))\right)\right)\\&\times g\left(t,\int_0^a q\left(t,s,x(s)\right)ds, x(\beta(t))\right)\ \text{for } t\in[0,a].\end{aligned}\tag{1}$$

The authors of [9] used the measure of noncompactness to obtain the existence of the solution of the integral Equation (1) in Banach algebra $C[0,a]$ with the help of the fixed point theorem.

Motivated by the work of [9], in this article, we study the solvability of non-linear functional integral equations of two variables, which we come across in various branches of nonlinear analysis. We consider an integral equation in the following form:

$$\begin{aligned}x(l,h)=&\left(U(l,h,x(l,h))+F\left(l,h,\int_0^l\int_0^h P(l,h,r,u,x(r,u))drdu, x(l,h)\right)\right)\\&\times G\left(l,h,\int_0^a\int_0^a Q\left(l,h,r,u,x(r,u)\right)drdu, x(l,h)\right)\ \text{for } l,h\in[0,a].\end{aligned}\tag{2}$$

The right-hand side of the above integral equation that we are considering is the product of two functional operators involving integral operators and applying a fixed point theorem, which is a generalization of Darbo's fixed point theorem for the product of operators to check the existence of the solution of the integral equation in Banach algebra. It can be seen that Equation (2) is a generalization of Equation (1) in two variables. Here, we used a fixed point theorem associated with Darbo's condition of the measure of noncompactness in Banach algebra of continuous functions in $[0,a]\times[0,a]$ to establish the solvability of Equation (2). Furthermore, we used the modified homotopy perturbation analytic method to find the solution of Equation (2).

## 2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, and write $\mathbb{R}_+=[0,\infty)$. Suppose $\bar{E}$ is a real Banach space with the norm $\|\ .\ \|$, and let $X(\neq\phi)\subseteq\bar{E}$. The closure and convex closure of $X$ will be denoted by $\bar{X}$ and conv$X$, respectively. The convex closure of a set $X$ of points in the Euclidean plane or in a Euclidean space over the reals is the smallest convex set that contains $X$. A closed ball in $\bar{E}$ centred at $a$ and with radius $b$ is denoted by $B(a,b)$. In addition, we use the symbol $\mathcal{M}_{\bar{E}}$ to denote the family of all non-empty and bounded subsets of $\bar{E}$ and use $\mathcal{N}_{\bar{E}}$ to denote its subfamily consisting of all relatively compact sets.

**Definition 1.** *Let $X$ be a linear space over $\mathbb{R}$. A norm on $X$ is a function from $X$ to $\mathbb{R}_+$, commonly denoted $||.||$ such that:*

*(N1)* $||x|| \geq 0$ *and* $||x|| = 0 \iff x = 0$;
*(N2)* $||\alpha x|| = |\alpha|||x||$;
*(N3)* $||x+y|| \leq ||x|| + ||y||$ *for all* $x, y \in X$ *and* $\alpha \in \mathbb{R}$.

*The pair* $(X, ||.||)$ *is called a normed space. A complete normed space is called a Banach space.*

**Definition 2.** *An algebra $A$ is a vector space $A$ over a field $K$ such that for each ordered pair of elements $x, y \in A$, a unique product $xy \in A$ is defined with the properties:*

*(A1)* $(xy)z = x(yz)$,
*(A2)* $x(y+z) = xy + xz$,
*(A3)* $(x+y)z = xz + yz$,
*(A4)* $\alpha(xy) = (\alpha x)y = x(\alpha y)$ *for all* $x, y, z \in A$ *and scalars* $\alpha$.

*A normed algebra $A$ is normed space, which is an algebra such that for all $x, y \in A$:*

$$\| xy \| \leq \| x \| \| y \|$$

*and if $A$ has an identity $e$, then $\| e \| = 1$.*

A Banach algebra is a normed algebra that is complete, considered as a normed space.

The notion of the measure of noncompactness due to Banaś and Goebel [21] is as follows:

**Definition 3.** *A function $\mu : \mathcal{M}_{\bar{E}} \to [0, \infty)$ is said to be a measure of noncompactness in $\bar{E}$ if:*

*(i)* *for all* $X \in \mathcal{M}_{\bar{E}}$, *we have that* $\mu(X) = 0$ *implies that* $X$ *is precompact.*
*(ii)* *the family* $\ker \mu = \{X \in \mathcal{M}_{\bar{E}} : \mu(X) = 0\}$ *is non-empty, and* $\ker \mu \subset \mathcal{N}_{\bar{E}}$.
*(iii)* $X \subseteq Z \implies \mu(X) \leq \mu(Z)$.
*(iv)* $\mu(\bar{X}) = \mu(X)$.
*(v)* $\mu(convX) = \mu(X)$ *where* $convX$ *is the convex closure of set* $X$.
*(vi)* $\mu(\lambda X + (1-\lambda) Z) \leq \lambda\mu(X) + (1-\lambda)\mu(Z)$ *for* $\lambda \in [0,1]$.
*(vii)* *if* $X_n \in \mathcal{M}_{\bar{E}}, X_n = \bar{X}_n$, $X_{n+1} \subset X_n$ *for* $n = 1, 2, 3, ...$ *and* $\lim\limits_{n \to \infty} \mu(X_n) = 0$, *then* $\bigcap_{n=1}^{\infty} X_n \neq \phi$.

The family $\ker \mu$ is called the *kernel of measure* $\mu$. Note that the intersection set $X_\infty$ from the above condition (vii) is a member of the family $\ker \mu$. Since $\mu(X_\infty) \leq \mu(X_n)$ for any $n$, we deduce $\mu(X_\infty) = 0$. Consequently, $X_\infty \in \ker\mu$.

For given subsets $X, Y$ in a Banach algebra $E$, the product $XY$ defined by:

$$XY = \{xy : x \in X, y \in Y\}.$$

In [8], Banaś and Olszowy defined the measure of noncompactness $\mu$ on the Banach algebra $E$, which *satisfies condition* $(m)$ if for arbitrary sets $X, Y \in \mathcal{M}_E$ such that:

$$\mu(XY) \leq ||X||\mu(Y) + ||y||\mu(X).$$

Deepmala and Pathak [9] used this concept of measure of noncompactness and obtained the existence of the solution of Equation (1).

**Definition 4** ([21])**.** *Let $E$ be a Banach space. Consider a non-empty subset $X$ of $E$ and a continuous operator $T : X \to E$ transforming the bounded subset of $X$ to the bounded ones. We say that $T$ satisfies the Darbo condition with a constant $k$ with respect to measure $\mu$ provided $\mu(TY) \leq k\mu(Y)$ for each $Y \in \mathcal{M}_E$ such that $Y \subset X$. If $k < 1$, then $T$ is called a contraction with respect to $\mu$.*

**Remark 1.** *The Darbo condition has many applications, particularly in fixed point theorems, which can be applied to check the existence of the solution of different types of integral, differential, and integro differential equations. The Darbo condition can be potentially applied to extend the linear space in the work of Shang [22]. The assumptions (1)–(4) of the next section have been utilized in the study of consensus problems (see [23,24]).*

We recall the following important theorems:

**Theorem 1** ([11]). *Assume that $Z$ is a non-empty, closed, bounded, and convex subset of a Banach space $\bar{E}$. Let $S : Z \to Z$ be a continuous mapping. Suppose that there is a constant $k \in [0, 1)$ such that:*

$$\mu(SM) \leq k\mu(M),\ M \subseteq Z.$$

*Then, $S$ has a fixed point.*

**Theorem 2** ([8]). *Suppose that $X$ is a non-empty, bounded, convex, and closed subset of a Banach algebra $E$, and the operators $P$ and $T$ transform continuously the set $X$ into $E$ such that $P(X)$ and $T(X)$ are bounded. Furthermore, suppose that the operator $S = P.T$ transforms $X$ into itself. If $P$ and $T$ satisfy on the set $X$ the Darbo condition with respect to the measure of noncompactness $\mu$ with the constants $k_1$ and $k_2$, respectively, then $S$ satisfies on $X$ the Darbo condition with constant $\parallel P(X) \parallel k_2 + \parallel T(X) \parallel k_1$. Particularly, if:*

$$\parallel P(X) \parallel k_2 + \parallel T(X) \parallel k_1 < 1,$$

*then $S$ is a contraction with respect to the measure of noncompactness $\mu$ and has at least one fixed point in $X$.*

We consider the space $E = C([0, a] \times [0, a])$, which consists of the set of real-valued continuous functions on $[0, a] \times [0, a]$. It is obvious that $E$ is the vector space over the field of scalars $\mathbb{R}$ with the following operations:

$$(x + y)(t, s) = x(t, s) + y(t, s)$$

and:

$$(\alpha x)(t, s) = \alpha x(t, s),$$

where $x, y \in E,\ \alpha \in \mathbb{R}$ and $t, s \in [0, a]$. Since $x, y \in E$, i.e., both $x, y$ are real-valued continuous functions on $[0, a] \times [0, a]$ and the product of two real-valued continuous functions is also a real-valued continuous function, therefore $xy \in E$, where:

$$(xy)(t, s) = x(t, s)y(t, s),\ t, s \in [0, a].$$

Let $z \in E$. For all $t, s \in [0, a]$,

$$\begin{aligned} ((xy)\, z)\, (t, s) &= (xy)\, (t, s)z(t, s) \\ &= x(t, s)y(t, s)z(t, s) \\ &= x(t, s)\, (yz)\, (t, s) \\ &= (x\, (yz))\, (t, s). \end{aligned}$$

Since $t, s$ are arbitrary, therefore $(xy)z = x(yz)$.

Similarly, it can be shown that:

$$x(y + z) = xy + xz,$$

$$(x + y)z = xz + yz$$

and:

$$\alpha\, (xy) = (\alpha x)\, y = x\, (\alpha y)\, .$$

Therefore $E$ is an algebra.

The space $E$ is also a normed space with the norm:

$$\| x \| = \sup \{ |x(l,h)| : l,h \in [0,a],\ a > 0 \},\ x \in E.$$

For all $x, y \in E$ and $l, h \in [0,a]$,

$$|(xy)(l,h)| = |x(l,h)y(l,h)| = |x(l,h)|\,|y(l,h)|$$

and so:

$$\sup_{l,h\in[0,a]} |(xy)(l,h)| \leq \sup_{l,h\in[0,a]} |x(l,h)| \sup_{l,h\in[0,a]} |y(l,h)|,$$

i.e.,

$$\| xy \| \leq \| x \| \| y \| .$$

Thus, $E$ is a normed algebra.

Let $(x_n(t,s))_{n=1}^{\infty}$ be a Cauchy sequence in $E$ where $x_n(t,s) \in \mathbb{R} \times \mathbb{R}$ for all $n \in \mathbb{N}$ and $t, s \in [0,a]$. Then:

$$\| x_n - x_m \| \to 0 \quad (n, m \to \infty).$$

Therefore, for all $t, s \in [0,a]$, we get:

$$|x_n(t,s) - x_m(t,s)| \to 0 \quad (n, m \to \infty).$$

For fixed $t, s \in [0,a]$, the sequence $(x_n(t,s))$ is a Cauchy sequence of real numbers, so it is a convergent sequence and converging to $x_0(t,s) \in E$ (say) as the limit of the continuous function is also continuous. Therefore, for all $t, s \in [0,a]$:

$$|x_n(t,s) - x_0(t,s)| \to 0 \quad (n, m \to \infty)$$

which yields:

$$\sup_{t,s\in[0,a]} |x_n(t,s) - x_0(t,s)| \to 0 \quad (n, m \to \infty)$$

Thus:

$$\| x_n - x_0 \| \to 0 \quad (n \to \infty)$$

which proves that $E$ is complete normed space. Hence, we conclude that the space $E$ has the Banach algebra structure.

Let $X$ be a fixed non-empty and bounded subset of $E = C([0,a] \times [0,a])$, and for $x \in X$ and $\epsilon > 0$, the modulus of the continuity function (denoted by $\omega(x,\epsilon)$) is given by the formula:

$$\omega(x,\epsilon) = \sup \{ |x(l,h) - x(v,w)| : l,h,v,w \in [0,a], |l - v| \leq \epsilon, |h - w| \leq \epsilon \}.$$

Further, we define:

$$\omega(X,\epsilon) = \sup \{ \omega(x,\epsilon) : x \in X \},\ \omega_0(X) = \lim_{\epsilon\to 0} \omega(x,\epsilon).$$

Similar to [5], it can be shown that the function $\omega_0(X)$ is a regular measure of non-compactness in the space $C([0,a] \times [0,a])$. Apart from this, it is easy to check that the measure $\omega_0(X)$ satisfies condition $(m)$.

## 3. Main Result

In this section, we study the existence of solutions of the integral Equation (2). We consider the following assumptions:

(1) The functions $U : [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$, $F : [0,a] \times [0,a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $G : [0,a] \times [0,a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, and there exist nonnegative constants $L, M$ such that:

$$|U(l,h,0)| \leq L, \quad |F(l,h,M_1,0)| \leq M \text{ and } |F(l,h,M_2,0)| \leq M,$$

where $M_1, M_2 \in \mathbb{R}$.

(2) Let $A_i : [0,a] \times [0,a] \to \mathbb{R}_+$ $(i = 1,2,3,4,5)$ be continuous functions such that:

$$|U(l,h,x_1) - U(l,h,x_2)| \leq A_1(l,h)\,|x_1 - x_2|\,,$$

$$|F(l,h,y,x_1) - F(l,h,y,x_2)| \leq A_2(l,h)\,|x_1 - x_2|\,,$$

$$|G(l,h,y,x_1) - G(l,h,y,x_2)| \leq A_3(l,h)\,|x_1 - x_2|\,,$$

$$|F(l,h,y_1,x) - F(l,h,y_2,x)| \leq A_4(l,h)\,|y_1 - y_2|$$

and:

$$|G(l,h,y_1,x) - G(l,h,y_2,x)| \leq A_5(l,h)\,|y_1 - y_2|\,,$$

where $l, h \in [0,a]$ and $x, x_1, x_2, y, y_1, y_2 \in \mathbb{R}$. Furthermore, let:

$$K = \max\{A_i(l,h) : i = 1,2,3,4,5; l,h \in [0,a]\},$$

where $K \geq 0$.

(3) The functions $P,\ Q$ are continuous functions from $[0,a] \times [0,a] \times [0,a] \times [0,a] \times \mathbb{R}$ to $\mathbb{R}$.

(4) Furthermore, $4\alpha\beta < 1$ for $\alpha = 2k,\ \beta = L + M$.

**Theorem 3.** *Under the hypotheses (1)–(4), Equation (2) has at least one solution in* $E = C(I \times I)$, *where* $I = [0,a]$.

**Proof.** Let us consider the operators $\hat{F}$ and $\hat{G}$ defined on $E$ by:

$$(\hat{F}x)(l,h) = U(l,h,x(l,h)) + F\left(l,h, \int_0^l \int_0^h P(l,h,r,u,x(r,u))drdu, x(l,h)\right)$$

and:

$$(\hat{G}x)(l,h) = G\left(l,h, \int_0^a \int_0^a Q\left(l,h,r,u,x(r,u)\right)drdu, x(l,h)\right), \text{ where } l,h \in [0,a].$$

From Assumptions (1)–(3), we get that $\hat{F}$ and $\hat{G}$ map $C(I \times I)$ into itself. Furthermore, let us define another operator $\hat{T}$ on $C(I \times I)$ as follows:

$$\hat{T}x = (\hat{F}x)(\hat{G}x).$$

It is obvious that $\hat{T}$ maps $C(I \times I)$ into itself.

Let:

$$I_1(x) = \int_0^l \int_0^h P(l,h,r,u,x(r,u))drdu$$

and:

$$I_2(x) = \int_0^a \int_0^a Q\left(l,h,r,u,x(r,u)\right)drdu.$$

Let $x \in C(I \times I)$ be fixed and $l, h \in I$. We get:

$$\begin{aligned}
\left|(\hat{T}x)(l,h)\right| &= \left|(\hat{F}x)(l,h)\right| . \left|(\hat{G}x)(l,h)\right| \\
&= |U(l,h,x(l,h)) + F\,(l,h,I_1(x),x(l,h))| \times |G\,(l,h,I_2(x),x(l,h))| \\
&\leq (|U(l,h,x(l,h)) - U(l,h,0)| + |U(l,h,0)| + |F\,(l,h,I_1(x),x(l,h)) \\
&\quad -F\,(l,h,I_1(x),0)\,| + |F\,(l,h,I_1(x),0)|) \times (|G(l,h,I_2(x),x(l,h)) \\
&\quad -G(l,h,I_2(x),0)| + |G(l,h,I_2(x),0)|) \\
&\leq (A_1(l,h)\,|x(l,h)| + L + A_2(l,h)\,|x(l,h)| + M) \times (A_3(l,h)\,|x(l,h)| + M) \\
&\leq (2K \parallel x \parallel + L + M)\,(K \parallel x \parallel + M) \\
&\leq (2K \parallel x \parallel + L + M)^2 .
\end{aligned}$$

Let $\alpha = 2k$, $\beta = L + M$. Then, we have:

$$\parallel \hat{F}x \parallel \leq \alpha \parallel x \parallel + \beta,$$

$$\parallel \hat{G}x \parallel \leq \alpha \parallel x \parallel + \beta$$

and:

$$\parallel \hat{T}x \parallel \leq (\alpha \parallel x \parallel + \beta)^2 \tag{3}$$

for $x \in C(I \times I)$.

From (3), we have that the operator $\hat{T}$ maps $B_d \subset C(I \times I)$ into $B_d$, where:

$$B_d = \{x(l,h) \in I : \parallel x(l,h) \parallel \leq d\}$$

for $d_2 \leq d \leq d_1$, where:

$$d_1 = \frac{1 - 2\alpha\beta - \sqrt{1 - 4\alpha\beta}}{2\alpha^2}$$

and:

$$d_2 = \frac{1 - 2\alpha\beta + \sqrt{1 - 4\alpha\beta}}{2\alpha^2}.$$

Furthermore, we have:

$$\parallel \hat{F}B_d \parallel \leq \alpha d + \beta \tag{4}$$

and:

$$\parallel \hat{G}B_d \parallel \leq \alpha d + \beta. \tag{5}$$

Let $\epsilon > 0$ be fixed and $x(l,h), y(l,h) \in B_d$ such that:

$$\parallel x - y \parallel \leq \epsilon, \quad (l, h \in I).$$

Then, we have:

$$\begin{aligned}
\left|(\hat{F}x)(l,h)-(\hat{F}y)(l,h)\right| &= \Big|U\,(l,h,x(l,h))+F\,(l,h,I_1(x),x(l,h)) \\
&\quad -U\,(l,h,y(l,h))-F\,(l,h,I_1(y),y(l,h))\Big| \\
&\le |U\,(l,h,x(l,h))-U\,(l,h,y(l,h))| \\
&\quad +|F\,(l,h,I_1(x),x(l,h))-F\,(l,h,I_1(x),y(l,h))| \\
&\quad +|F\,(l,h,I_1(x),y(l,h))-F\,(l,h,I_1(y),y(l,h))| \\
&\le A_1(l,h)\,|x(l,h)-y(l,h)|+A_2(l,h)\,|x(l,h)-y(l,h)| \\
&\quad +A_4(l,h)\,|I_1(x)-I_1(y)| \\
&\le 2K\parallel x-y\parallel \\
&\quad +K\int_0^l\int_0^h|P(l,h,r,u,x(r,u))-P(l,h,r,u,y(r,u))|\,drdu \\
&\le 2K\parallel x-y\parallel+Ka^2\omega(P,\epsilon),
\end{aligned}$$

where:

$$\omega(P,\epsilon)=\sup\left\{\begin{array}{c}|P(l,h,r,u,x(r,u))-P(l,h,r,u,y(r,u))|:l,h,r,u\in I,\\ x,y\in[-d,d],\parallel x-y\parallel<\epsilon\end{array}\right\}.$$

Since $P$ is continuous, so it is uniformly continuous on the compact set $I\times I\times I\times I\times[-d,d]$; therefore:

$$\omega(P,\epsilon)\to 0\ \text{ as }\ \epsilon\to 0.$$

Thus, $\hat{F}$ is continuous on $B_d$. Similarly, one can prove that $\hat{G}$ is continuous on $B_d$. Thus, we can conclude that $\hat{T}$ is continuous on $B_d$.

Let us consider a non-empty subset $X$ of $B_d$ and $x \in X$. Then, for a fixed $\epsilon > 0$ and $l_1, l_2, h_1, h_2 \in I$ such that $l_1 \leq l_2$, $h_1 \leq h_2$, $l_1 - l_2 \leq \epsilon$, $h_1 - h_2 \leq \epsilon$, one obtains:

$$
\begin{aligned}
&\left|(\hat{F}x)(l_2,h_2) - (\hat{F}x)(l_1,h_1)\right| \\
&= \Bigg| U\left(l_2,h_2,x(l_2,h_2)\right) + F\left(l_2,h_2,\int_0^{l_2}\int_0^{h_2} P(l_2,h_2,r,u,x(r,u))drdu, x(l_2,h_2)\right) \\
&\quad - U\left(l_1,h_1,x(l_1,h_1)\right) - F\left(l_1,h_1,\int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu, x(l_1,h_1)\right)\Bigg| \\
&\leq \left|U\left(l_2,h_2,x(l_2,h_2)\right) - U\left(l_2,h_2,x(l_1,h_1)\right)\right| + \left|U\left(l_2,h_2,x(l_1,h_1)\right) - U\left(l_1,h_1,x(l_1,h_1)\right)\right| \\
&\quad + \Bigg|F\left(l_2,h_2,\int_0^{l_2}\int_0^{h_2} P(l_2,h_2,r,u,x(r,u))drdu, x(l_2,h_2)\right) \\
&\quad - F\left(l_2,h_2,\int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu, x(l_2,h_2)\right)\Bigg| \\
&\quad + \Bigg|F\left(l_2,h_2,\int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu, x(l_2,h_2)\right) \\
&\quad - F\left(l_1,h_1,\int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu, x(l_2,h_2)\right)\Bigg| \\
&\quad + \Bigg|F\left(l_1,h_1,\int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu, x(l_2,h_2)\right) \\
&\quad - F\left(l_1,h_1,\int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu, x(l_1,h_1)\right)\Bigg| \\
&\leq A_1(l,h)\left|x(l_2,h_2) - x(l_1,h_1)\right| + \left|U\left(l_2,h_2,x(l_1,h_1)\right) - U\left(l_1,h_1,x(l_1,h_1)\right)\right| \\
&\quad + A_4(l,h)\left|\int_0^{l_2}\int_0^{h_2} P(l_2,h_2,r,u,x(r,u))drdu - \int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu\right| \\
&\quad + \Bigg|F\left(l_2,h_2,\int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu, x(l_2,h_2)\right) \\
&\quad - F\left(l_1,h_1,\int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu, x(l_2,h_2)\right)\Bigg| \\
&\quad + A_2(l,h)\left|x(l_2,h_2) - x(l_1,h_1)\right|
\end{aligned}
$$

which yields:

$$
\begin{aligned}
&\left|(\hat{F}x)(l_2,h_2) - (\hat{F}x)(l_1,h_1)\right| \leq 2K\left|x(l_2,h_2) - x(l_1,h_1)\right| \\
&+ \left|U\left(l_2,h_2,x(l_1,h_1)\right) - U\left(l_1,h_1,x(l_1,h_1)\right)\right| \\
&+ K\Bigg|\int_0^{l_2}\int_0^{h_2} P(l_2,h_2,r,u,x(r,u))drdu
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu\Bigg| \\
&+ \left|F\left(l_2,h_2,\int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu, x(l_2,h_2)\right)\right. \\
&\left. - F\left(l_1,h_1,\int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu, x(l_2,h_2)\right)\right|.
\end{aligned}
$$

Let:

$$\omega(U,\epsilon) = \sup\left\{ \begin{array}{c} |U(l_2,h_2,x(l_2,h_2)) - U(l_1,h_1,x(l_1,h_1))| : l_1,l_2,h_1,h_2 \in I, \\ |l_2-l_1| \le \epsilon, |h_2-h_1| \le \epsilon, x \in [-d,d] \end{array} \right\},$$

$$\omega(P,\epsilon) = \sup\left\{ \begin{array}{c} |P(l_2,h_2,r,u,x(r,u)) - P(l_1,h_1,r,u,x(r,u))| : l_1,l_2,h_1,h_2,r,u \in I, \\ |l_2-l_1| \le \epsilon, |h_2-h_1| \le \epsilon, x \in [-d,d] \end{array} \right\},$$

$$\bar{k} = \sup\Big\{ |P(l,h,r,u,x(r,u))| : l,h,r,u \in I, x \in [-d,d]\Big\}$$

and:

$$\omega(F,\epsilon) = \sup\left\{ \begin{array}{c} |F(l_2,h_2,z,x(l_2,h_2)) - F(l_1,h_1,z,x(l_1,h_1))| : l_1,l_2,h_1,h_2 \in I, \\ |l_2-l_1| \le \epsilon, |h_2-h_1| \le \epsilon, x \in [-d,d], z \in [-\bar{k}a^2,\bar{k}a^2] \end{array} \right\}.$$

Furthermore:

$$
\begin{aligned}
&\left|\int_0^{l_2}\int_0^{h_2} P(l_2,h_2,r,u,x(r,u))drdu - \int_0^{l_1}\int_0^{h_1} P(l_1,h_1,r,u,x(r,u))drdu\right| \\
&\le \left|\int_0^{l_2}\int_0^{h_2} \left(P(l_2,h_2,r,u,x(r,u)) - P(l_1,h_1,r,u,x(r,u))\right)drdu\right| \\
&+ \left|\int_{l_1}^{l_2}\int_{h_1}^{h_2} P(l_1,h_1,r,u,x(r,u))drdu\right| \\
&\le a^2\omega(P,\epsilon) + \bar{k}\epsilon^2.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\left|(\hat{F}x)(l_2,h_2) - (\hat{F}x)(l_1,h_1)\right| \quad &\le \quad 2K|x(l_2,h_2) - x(l_1,h_1)| + \omega(U,\epsilon) \\
&\quad +K\left(a^2\omega(P,\epsilon) + \bar{k}\epsilon^2\right) + \omega(F,\epsilon).
\end{aligned}
$$

This gives:

$$\omega(\hat{F}x,\epsilon) \le 2K\omega(x,\epsilon) + \omega(U,\epsilon) + K\left[a^2\omega(P,\epsilon) + \bar{k}a^2\right] + \omega(F,\epsilon).$$

Since $U$ and $F$ are continuous on $I \times I \times \mathbb{R}$ and $I \times I \times \mathbb{R} \times \mathbb{R}$, respectively, therefore we get:

$$\omega(U,\epsilon) \to 0,\ \omega(P,\epsilon) \to 0 \text{ and } \omega(F,\epsilon) \to 0 \text{ as } \epsilon \to 0.$$

Thus:

$$\omega_0(\hat{F}X) \le 2K\omega_0(X). \tag{6}$$

Similarly, we can show that:

$$\omega_0(\hat{G}X) \le 2K\omega_0(X). \tag{7}$$

From (4)–(7) and Theorem 2 (for the details of this theorem, we refer to [8]), we get that $\hat{T}$ satisfies the Darbo condition on $B_d$ with respect to measure $\omega_0$ with constant:

$$\begin{aligned} 2K(\alpha d+\beta)+2K(\alpha d+\beta) &= 4K(\alpha d+\beta) \\ &= 4K(\alpha d_1+\beta) \\ &= 4K\left[\alpha\left(\frac{1-2\alpha\beta-\sqrt{1-4\alpha\beta}}{2\alpha^2}\right)+\beta\right] \\ &= 2K\left(\frac{1-\sqrt{1-4\alpha\beta}}{\alpha}\right) \\ &< 1. \end{aligned}$$

This implies that $\hat{T}$ is a contraction operator on $B_d$ with respect to $\omega_0$. Thus, by Theorem 2, we have that $\hat{T}$ has at least one fixed point in $B_d$. Hence, Equation (2) has at least one solution in $B_d \subset C([0,a]\times[0,a])$. This completes the proof. □

## 4. An Illustrative Example

We construct the following example to illustrate the obtained result in the previous section.

**Example 1.** *Consider the following integral equation:*

$$x(l,h)=\left(\frac{1}{6}\cos\left(\frac{l+h}{2}\right)+\frac{1}{9}\int_0^l\int_0^h\frac{rue^{-lh}}{3+x^2(r,u)}drdu\right)\left(\frac{1}{8}\int_0^1\int_0^1\frac{lh}{6+|x(r,u)|}drdu\right) \quad (8)$$

*for $l,h\in[0,1]=I$. Here, we have:*

$$U(l,h,x(l,h))=\frac{1}{6}\cos\left(\frac{l+h}{2}\right),$$

$$F(l,h,y,x(l,h))=\frac{y}{9},$$

$$G(l,h,y,x(l,h))=\frac{y}{8},$$

$$P(l,h,r,u,x(r,u))=\frac{rue^{-lh}}{3+x^2(r,u)},$$

$$Q(l,h,r,u,x(r,u))=\frac{lh}{6+|x(r,u)|}$$

*and $a=1$; $x,y\in\mathbb{R}$.*

*It is obvious that all the functions $U, F, G, P$, and $Q$ are continuous. We have:*

$$|U(l,h,x_1)-U(l,h,x_2(l,h))|=0.\,|x_1(l,h)-x_2(l,h)|,$$

$$|F(l,h,y,x_1(l,h))-F(l,h,y,x_2(l,h))|=0.\,|x_1(l,h)-x_2(l,h)|,$$

$$|G(l,h,y,x_1(l,h))-G(l,h,y,x_2(l,h))|=0.\,|x_1(l,h)-x_2(l,h)|,$$

$$|F(l,h,y_1,x(l,h))-F(l,h,y_2,x(l,h))|=\frac{1}{9}|y_1-y_2|,$$

*and:*

$$|G(l,h,y_1,x(l,h))-G(l,h,y_2,x(l,h))|=\frac{1}{8}|y_1-y_2|.$$

*It follows that:*

$$A_1(l,h) = A_2(l,h) = A_3(l,h) = 0,\ A_4(l,h) = \frac{1}{9} \text{ and } A_5(l,h) = \frac{1}{8}.$$

*Consequently, we get* $K = \frac{1}{8}$.

*Furthermore,*

$$|U(l,h,0)| \leq \frac{1}{6},$$

$$|F(l,h,y_1,0)| \leq \frac{1}{27e^2} \approx \frac{1}{198.29}$$

*and:*

$$|G(l,h,y_2,0)| \leq \frac{1}{48}.$$

*Thus:*

$$M = \frac{1}{48},\ \ L = \frac{1}{6}$$

*and:*

$$4\alpha\beta = \frac{9}{48} < 1.$$

*Hence, all the assumption from (1)–(4) are satisfied. Thus, by applying Theorem 3, we conclude that Equation (8) has at least one solution in the Banach algebra* $C([0,1]\times[0,1])$.

## 5. An Iterative Algorithm Created by a Coupled Semi-Analytic Method to Find the Solution of the Integral Equation

To find an approximation of solution for Equation (8), we make an iterative algorithm by a coupled method created by modified homotopy perturbation and the Adomian decomposition method in the case of two-dimensional functions. Applications of the modified homotopy perturbation method to solve nonlinear integral equations, nonlinear singular integral equations, and nonlinear differential equations can be seen in [25–27], respectively. The Adomian decomposition method to solve physical problems was used in [28] and also to solve integro-differential equations system in [29]. However, in this article, we introduce a modified homotopy perturbation method in terms of a function with two variables, and for simplification of nonlinear terms, we use the Adomian decomposition method in the suitable form; therefore, we make an effective algorithm by the above process. Equation (8) can be shown in a general form of the two-dimensional nonlinear problem:

$$A\left(x(l,h)\right) - f(l,h) = 0$$

with $(l,h) \in I \times I$, where $A$ is a general nonlinear operator and $f$ is a known analytic function. Similar to [26,27], we divide the general operator $A$ into two nonlinear operators as $M_1$ and $M_2$. Of course, $M_1$ or $M_2$ can be linear operators in the special case that also $f$ is converted to $f_1$ and $f_2$ functions; in other words, we have:

$$M_1\left(x(l,h)\right) - f_1(l,h) + M_2\left(x(l,h)\right) - f_2(l,h) = 0.$$

A modified homotopy perturbation for the above problem can be introduced as follows:

$$H\left(u(l,h),p\right) = M_1\left(u(l,h)\right) - f_1(l,h) + p\left[M_2\left(u(l,h)\right) - f_2(l,h)\right] = 0,\ \ p \in [0,1], \tag{9}$$

where $p$ is an embedding parameter and $u$ is an approximation of $x$. According to the variations of $p = 0$ to $p = 1$, it can be observed that $M_1\left(u(l,h)\right) = f_1(l,h)$ to $A\left(u(l,h)\right) = f(l,h)$. This implies that for $p = 1$ in (9), we get the solution of (5).

We consider the above solution as the series:

$$x(l,h) \simeq u(l,h) = \sum_{k=0}^{\infty} p^k u_k(l,h) \tag{10}$$

and:

$$x(l,h) = \lim_{p\to 1} u(l,h). \tag{11}$$

To solve Equation (8), $M_1, M_2$ and $f$ can be defined as follows:

$$M_1(x(l,h)) = x(l,h), \tag{12}$$

$$M_2(x(l,h)) = -\left(\frac{1}{6}\cos\left(\frac{l+h}{2}\right) + \frac{1}{9}\int_0^l \int_0^h \frac{rue^{-lh}}{3+x^2(r,u)}drdu\right)\left(\frac{1}{8}\int_0^1 \int_0^1 \frac{lh}{6+|x(r,u)|}drdu\right), \tag{13}$$

and:

$$f(l,h) = f_1(l,h) + f_2(l,h). \tag{14}$$

Since in (8) $f(l,h) = 0$, therefore $f_1(l,h) = f_2(l,h) = 0$. From (9)–(14), we have:

$$\begin{aligned}&\sum_{k=0}^{\infty} p^k u_k(l,h)\\ &-p\left(\frac{1}{6}\cos\left(\frac{l+h}{2}\right) + \frac{1}{9}\int_0^l \int_0^h \frac{rue^{-lh}}{3+\left(\sum\limits_{k=0}^{\infty} p^k u_k(r,u)\right)^2}drdu\right)\\ &\left(\frac{1}{8}\int_0^1 \int_0^1 \frac{lh}{6+\left|\sum\limits_{k=0}^{\infty} p^k u_k(r,u)\right|}drdu\right) = 0.\end{aligned}$$

Now, we use Adomain polynomials for simplicity for the nonlinear terms:

$$\frac{1}{6}\cos\left(\frac{l+h}{2}\right) + \frac{1}{9}\int_0^l \int_0^h \frac{rue^{-lh}}{3+\left(\sum\limits_{k=0}^{\infty} p^k u_k(r,u)\right)^2}drdu = \sum_{k=0}^{\infty} p^k A_k(l,h)$$

and:

$$\frac{1}{8}\int_0^1 \int_0^1 \frac{lh}{6+\left|\sum\limits_{k=0}^{\infty} p^k u_k(r,u)\right|}drdu = \sum_{k=0}^{\infty} p^k \hat{A}_k(l,h),$$

where the Adomain polynomials are given by:

$$A_k(l,h) = \frac{1}{k!}\left[\frac{d^k}{dp^k}\left\{\frac{1}{6}\cos\left(\frac{l+h}{2}\right) + \frac{1}{9}\int_0^l \int_0^h \frac{rue^{-lh}}{3+\left(\sum\limits_{k=0}^{\infty} p^k u_k(r,u)\right)^2}drdu\right\}\right]_{p=0}$$

and:

$$\hat{A}_k(l,h) = \frac{1}{k!}\left[\frac{d^k}{dp^k}\left\{\frac{1}{8}\int_0^1 \int_0^1 \frac{lh}{6+\left|\sum\limits_{k=0}^{\infty} p^k u_k(r,u)\right|}drdu\right\}\right]_{p=0}$$

Therefore, we have:

$$\sum_{k=0}^{\infty} p^k u_k(l,h) - p\left\{\sum_{k=0}^{\infty} p^k A_k(l,h)\right\}\left\{\sum_{k=0}^{\infty} p^k \hat{A}_k(l,h)\right\} = 0. \tag{15}$$

By rearranging the terms in powers in $p$ of (15) and using modified homotopy perturbation (9), the coefficients of $p$ powers must be equal to zero, so we obtain an iterative algorithm (Algorithm 1) to solve for the numerical solution of Equation (8).

**Algorithm 1. Algorithm of calculating $u_k(l,h)$**

```
$u_0(l,h) = 0,$
$u_1(l,h) = A_0(l,h)\hat{A}_0(l,h),$
$u_k(l,h) = \sum_{i=0}^{k-1} A_i(l,h)\hat{A}_{k-1-i}(l,h),\ k = 2, 3, \cdots .$
```

Calculating the sequence $\{u_0(l,h), u_1(l,h), ...\}$, we can obtain a closed form of the solution for (8) using the above algorithm.

We compute the Adomain polynomial for $k = 0$,

$$\begin{aligned} &A_0(l,h) \\ &= \frac{1}{6}\cos\left(\frac{l+h}{2}\right) + \frac{1}{9}\int_0^l \int_0^h \frac{rue^{-lh}}{3+u_0^2(r,u)}drdu \\ &= \frac{1}{6}\cos\left(\frac{l+h}{2}\right) + \frac{l^2h^2e^{-lh}}{108} \end{aligned}$$

and:

$$\hat{A}_0(l,h) = \frac{1}{8}\int_0^1 \int_0^1 \frac{lh}{6+|u_0(r,u)|}drdu = \frac{lh}{48}.$$

Therefore, we obtain by the algorithm:

$$u_1(l,h) = \left\{\frac{1}{6}\cos\left(\frac{l+h}{2}\right) + \frac{l^2h^2e^{-lh}}{108}\right\}\frac{lh}{48}.$$

We use (10) to approximate $x(l,h)$ by a few term of $u_k(l,h)$ as follows:

$$x_1(l,h) = u_0(l,h) + u_1(l,h) = \left\{\frac{1}{6}\cos\left(\frac{l+h}{2}\right) + \frac{l^2h^2e^{-lh}}{108}\right\}\frac{lh}{48}.$$

## 6. Conclusions

In our present investigation, we have established the existence of the solution of a functional integral equation of two variables, which is of the form of the product of two operators in the Banach algebra $C([0,a] \times [0,a])$, $a > 0$ and illustrated our results with the help of an example. We also constructed an iteration algorithm to get the solution of Equation (8). Further, one can solve Equation (8) using different numerical, as well as analytical methods in the setting of Banach sequence spaces and Banach algebra. Moreover, due our existence theorem for Equation (8) of two variables, we therefore conclude that our existence result is more general than the one obtained earlier by Deepmala and Pathak [9].

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# Generalized Mittag-Leffler Input Stability of the Fractional Differential Equations

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**Abstract:** The behavior of the analytical solutions of the fractional differential equation described by the fractional order derivative operators is the main subject in many stability problems. In this paper, we present a new stability notion of the fractional differential equations with exogenous input. Motivated by the success of the applications of the Mittag-Leffler functions in many areas of science and engineering, we present our work here. Applications of Mittag-Leffler functions in certain areas of physical and applied sciences are also very common. During the last two decades, this class of functions has come into prominence after about nine decades of its discovery by a Swedish Mathematician Mittag-Leffler, due to the vast potential of its applications in solving the problems of physical, biological, engineering, and earth sciences, to name just a few. Moreover, we propose the generalized Mittag-Leffler input stability conditions. The left generalized fractional differential equation has been used to help create this new notion. We investigate in depth here the Lyapunov characterizations of the generalized Mittag-Leffler input stability of the fractional differential equation with input.

**Keywords:** fractional differential equations with input; Mittag-Leffler stability; left generalized fractional derivative; $\rho$-Laplace transforms

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## 1. Introduction

The behavior of the analytical solutions of the fractional differential equation described by the fractional order derivative operators is the main subject in stability problems [1]. There exist many stability notions introduced in fractional calculus. Some examples are asymptotic stability, global asymptotic uniform stability, synchronization problems, stabilization problems, Mittag-Leffler stability and fractional input stability. In this paper, we extend the Mittag-Leffler input stability in the context of the fractional differential equations described by the left generalized fractional derivative. We note here that the left generalized fractional derivative is the generalization of the Liouville-Caputo fractional derivative and the Riemann-Liouville fractional derivative [2]. There exist many works related to stability problems. In [3], Souahi et al. propose some new Lyapunov characterizations of fractional differential equations described by the conformable fractional derivative. In [4], Sene proposes a new stability notion and introduce the Lyapunov characterization of the conditional asymptotic stability. In [5,6], Sene proposes some applications of the fractional input stability to the electrical circuits described the Liouville-Caputo fractional derivative and the Riemann-Liouville fractional derivative. In [7], Li et al. introduce the Mittag-Leffler stability of the fractional differential equations described by the Liouville-Caputo fractional derivative [8]. In [9], Song et al. analyze the stability of the fractional differential equations with time variable impulses. In [10], Tuan et al. propose a novel methodology for studying the stability of the fractional differential equations using the Lyapunov direct method. In [11], Makhlouf studies the stability with respect to part of the variables of nonlinear Caputo fractional differential equations. In [12], Alidousti et al. propose a new stability analysis of the fractional differential equation described by the Liouville-Caputo fractional derivative. Many other works related to the stability analysis exist in literature, we direct our readers to the References section for more related literature.

The generalized Mittag-Leffler input stability is a new stability notion. This new stability notion studies the behavior of the analytical solution of the fractional differential equations with exogenous input described by the left generalized fractional derivative [13]. We know from previous work in stability problems, it is not trivial to get analytical solutions. The issue is to propose a method to analyze the stability of the fractional differential equations with exogenous input. Classically, the most popular method is the Lyapunov direct method as given in [14–18]. We propose the Lyapunov characterization of the generalized Mittag-Leffler input stability here in this work. As we will be able to show, the generalized Mittag-Leffler input stability generates three properties:

- the converging-input converging-state
- the bounded-input bounded-state
- the uniform global asymptotic stability of the trivial solution of the unforced fractional differential equation (fractional differential equation without exogenous input).

We note here that the fractional differential equation with exogenous input is said to be generalized Mittag-Leffler input stable when the Euclidean norm of its solution is bounded, by a generalized Mittag-Leffler function, plus a quantity which is proportional to the exogenous input bounded when the input is bounded and converging when the input converges in time. The fractional input stability and its consequences are a good compromise in stability problems of the fractional differential equations described by the fractional order derivative operators.

We organize the rest of the paper as follows. In Section 2, we recall the definition of the fractional derivative operators with or without singular kernels. In Section 3, we propose our motivations for studying the generalized Mittag-Leffler input stability. In Section 4, we give the Lyapunov characterizations for the generalized Mittag-Leffler input stability of the fractional differential equations with exogenous inputs. In Section 5, we provide numerical examples for illustrating the main results of this paper. Finally, we finish with some concluding remarks in Section 6.

## 2. Background on Fractional Derivatives

Let us first recall the fractional derivative operators and the comparison functions [19]. We will use them throughout this paper. There exist many fractional derivative operators in fractional calculus. There exist two types of fractional derivative operators. The first is fractional derivatives with singular kernels and the second is fractional derivatives without singular kernels. With regards to fractional derivatives with singular kernels, we cite the Riemann-Liouville fractional derivative [2], the Liouville-Caputo fractional derivative [2], the Hilfer fractional derivative [20], the Hadamard fractional derivative [2], and Erdélyi-Kober fractional derivative [21]. We note here that all previous fractional derivatives are associated to their fractional integrals [2,20]. As fractional derivatives without singular kernels we cite the Atangana-Baleanu-Liouville-Caputo derivative [22], the Caputo-Fabrizio fractional derivative [23], and the Prabhakar fractional derivative [24]. We note here that all previous fractional derivatives are associated to their fractional integrals [21–24]. Recently, the generalization of the Riemann-Liouville and the Liouville-Caputo fractional derivative were introduced in the literature by Udita [25]. Namely, the generalized fractional derivative and the Liouville-Caputo generalized fractional derivative. Let us now observe the comparison functions used in this paper.

**Definition 1.** *The class $\mathcal{PD}$ function denotes the set of all continuous functions $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\alpha(0) = 0$, and $\alpha(s) > 0$ for all $s > 0$. A class $\mathcal{K}$ function is an increasing $\mathcal{PD}$ function. The class $\mathcal{K}_\infty$ represents the set of all unbounded $\mathcal{K}$ functions [17].*

**Definition 2.** *A continuous function* $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is said to be of class* $\mathcal{L}$ *if* $\beta$ *is non-increasing and tends to zero as its arguments tend to infinity [17].*

**Definition 3.** *Let the function* $f : [0, +\infty[ \longrightarrow \mathbb{R}$*, the Liouville-Caputo derivative of the function* $f$ *of order* $\alpha$ *is expressed in the form*

$$D^c_\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds, \tag{1}$$

*for all* $t > 0$*, where the order* $\alpha \in (0,1)$ *and* $\Gamma(.)$ *is the gamma function [2,26–29].*

**Definition 4.** *Let the function* $f : [0, +\infty[ \longrightarrow \mathbb{R}$*, the Riemann-Liouville derivative of the function* $f$ *of order* $\alpha$ *is expressed in the form*

$$D^{RL}_\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\alpha} ds, \tag{2}$$

*for all* $t > 0$*, where the order* $\alpha \in (0,1)$ *and* $\Gamma(.)$ *is the gamma function [2,26–30].*

**Definition 5.** *Let the function* $f : [0, +\infty[ \longrightarrow \mathbb{R}$*, the Liouville-Caputo generalized derivative of the function* $f$ *of order* $\alpha$ *is expressed in the form*

$$\left(D^{\alpha,\rho}_c f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{-\alpha} f'(s) \frac{ds}{s^{1-\rho}}, \tag{3}$$

*for all* $t > 0$*, where the order* $\alpha \in (0,1)$ *and* $\Gamma(.)$ *is the gamma function [2,26,28,29,31].*

**Definition 6.** *Let the function* $f : [0, +\infty[ \longrightarrow \mathbb{R}$*, the left generalized derivative of the function* $f$ *of order* $\alpha$ *is expressed in the form*

$$\left(D^{\alpha,\rho} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt}\right) \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{-\alpha} f(s) \frac{ds}{s^{1-\rho}}, \tag{4}$$

*for all* $t > 0$*, where the order* $\alpha \in (0,1)$ *and* $\Gamma(.)$ *is the gamma function [2,26,28,29,31].*

**Definition 7.** *Let us take the function* $f : [0, +\infty[ \longrightarrow \mathbb{R}$*, the Caputo-Fabrizio fractional derivative of the function* $f$ *of order* $\alpha$ *is expressed in the form*

$$D^{CF}_\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t f'(s) \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds, \tag{5}$$

*for all* $t > 0$*, where the order* $\alpha \in (0,1)$ *and* $\Gamma(.)$ *is the gamma function [22].*

**Definition 8.** *Let the function* $f : [0, +\infty[ \longrightarrow \mathbb{R}$*, the Caputo-Fabrizio fractional derivative of the function* $f$ *of order* $\alpha$ *is expressed in the form*

$$D^{ABC}_\alpha f(t) = \frac{AB(\alpha)}{1-\alpha} \int_0^t f'(s) E_\alpha \left(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\right) ds, \tag{6}$$

*for all* $t > 0$*, where the order* $\alpha \in (0,1)$ *and* $\Gamma(.)$ *is the gamma function [22,30].*

**Definition 9.** *Let us consider the function* $f : [0, +\infty[ \longrightarrow \mathbb{R}$*, the Erdélyi-Kober fractional integral of the function* $f$ *of order* $\alpha > 0$*,* $\eta > 0$ *and* $\gamma \in \mathbb{R}$ *is expressed in the form*

$$I^{\gamma,\alpha}_\eta f(t) = \frac{t^{-\eta(\gamma+\alpha)}}{\Gamma(\alpha)} \int_0^t \tau^{\eta\gamma} \left(t^\eta - \tau^\eta\right)^{\alpha-1} f(\tau) d(\tau^\eta), \tag{7}$$

*for all* $t > 0$, *and* $\Gamma(.)$ *is the gamma function [21].*

**Definition 10.** *Let us consider the function* $f : [0, +\infty[ \longrightarrow \mathbb{R}$, *the Erdélyi-Kober fractional derivative of the function* $f$ *of order* $\alpha > 0$, $\eta > 0$ *and* $\gamma \in \mathbb{R}$ *is expressed in the form*

$$D_{\eta}^{\gamma,\alpha} f(t) = \prod_{j=1}^{n} \left( \gamma + j + \frac{1}{\eta} \frac{d}{dt} \right) \left( I_{\eta}^{\gamma+\alpha, n-\mu} f(t) \right), \tag{8}$$

*for all* $t > 0$, *and where* $n - 1 < \alpha \leq n$ *[21].*

Some special cases can be recovered with the above definitions. In Definition 8, when $\rho = 1$, we recover the Liouville-Caputo fractional derivative. In Definition 9, when $\rho = 1$, we recover the Riemann-Liouville fractional derivative. In Definition 10, when $\gamma = -\alpha$ and $\eta = 1$, we obtain the relation existing between Erdélyi-Kobar fractional derivative and Riemann-Liouville fractional derivative expressed in the form $D_{1}^{-\alpha,\alpha} f(t) = t^{\alpha} D^{\alpha,1} f(t)$.

The Laplace transform will be used for solving a class of the fractional differential equations. The $\rho$-Laplace transform was recently introduced by Fahd et al. in order to solve differential equations in the frame of conformable derivatives to extend the possibility of working in a large class of functions [2]. We encourage readers to refer to [2] for more detailed information about $\rho$-Laplace transforms and their applications.

The $\rho$-Laplace transform of the generalized fractional derivative in the Liouville-Caputo sense is expressed in the following form

$$\mathcal{L}_{\rho} \left\{ \left( D_{c}^{\alpha,\rho} f \right) (t) \right\} = s^{\alpha} \mathcal{L}_{\rho} \left\{ f(t) \right\} - s^{\alpha-1} f(0), \tag{9}$$

The $\rho$-Laplace transform of the function $f$ is given in the form

$$\mathcal{L}_{\rho} \left\{ f(t) \right\} (s) = \int_{0}^{\infty} e^{-s\frac{t^{\rho}}{\rho}} f(t) \frac{dt}{t^{1-\rho}}. \tag{10}$$

**Definition 11.** *The Mittag–Leffler function with two parameters is defined as the following series*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + \beta)} \tag{11}$$

*where* $\alpha > 0$, $\beta \in \mathbb{R}$ *and* $z \in \mathbb{C}$. *The classical exponential function is obtained with* $\alpha = \beta = 1$. *Here we see that when* $\alpha$ *and* $\beta$ *are strictly positive, the series is convergent [14].*

## 3. New Stability Notion of the Fractional Differential Equations

In this section, we introduce a new stability notion for the fractional differential equation with exogenous input described by the left generalized fractional derivative. Historically, the fractional input stability and the Mittag-Leffler input stability of the fractional differential equation represented by the Liouville-Caputo fractional derivative were stated in previous works [5,18]. Moreover, the idea of a discrete version of fractional derivatives is studied in the seminal work [32]. The Lyapunov characterizations of these new stability notions have been provided in [15,18]. In this section, we extend the Mittag-Leffler input stability involving the left generalized fractional derivative. We provide some modifications in the structure of the definitions, however the idea is not modified. The new stability notion addressed in this paper is called the generalized Mittag-Leffler input stability. In the literature there exist many stability notions related to the fractional differential equations without exogenous inputs such as the asymptotic stability [7,14], the practical stability [12,33], the Mittag-Leffler stability [7] and many others notions. Let us consider the fractional differential equations with exogenous inputs. In fractional calculus, we have not seen a lot of work related to the stability of the fractional differential equations with inputs. The stabilization problems [3] of the fractional differential equations with exogenous inputs is one of the most popular notion existing in the known literature. The challenge consists of finding possible values of the input under which the trivial solution of the obtained fractional differential equation is asymptotically stable. In this paper, we adopt a new method. Let us consider the fractional differential equation with exogenous input described by the left generalized fractional derivative

$$D^{\alpha,\rho}x = Ax + Bu, \tag{12}$$

where $x \in \mathbb{R}^n$ is a state variable, the matrix $A \in \mathbb{R}^{n\times n}$ satisfies the property $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$, the matrix $B \in \mathbb{R}^{n\times n}$ and $u \in \mathbb{R}^n$ represents the exogenous input. The initial boundary condition is defined by $\left(I^{1-\alpha,\rho}x\right)(0) = x_0$. Firstly, we give the analytical solution of the fractional differential equation with exogenous input described by the left generalized fractional derivative defined by Equation (12). Applying the $\rho$-Laplace transform to both sides of Equation (12), we obtain

$$\begin{aligned}
\mathcal{L}_\rho\left(D^{\alpha,\rho}x(t)\right) - \left(I^{1-\alpha,\rho}x\right)(0) &= A\mathcal{L}_\rho\left(x(t)\right) + \mathcal{L}_\rho\left(Bu\right) \\
s^\alpha\bar{x}(s) - x_0 &= A\bar{x}(s) + B\bar{u}(s) \\
\bar{x}(s) - x_0\left(s^\alpha I_n - A\right)^{-1} &= \left(s^\alpha I_n - A\right)^{-1}B\bar{u}(s),
\end{aligned} \tag{13}$$

where $\bar{x}$ denotes the Laplace transform of the function $x$ and $\bar{u}$ denotes the Laplace transform of the function $u$. Applying the inverse of the $\rho$-Laplace transform to both sides of Equation (13), we obtain

$$\begin{aligned}
x(t) = {} & x_0\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\alpha\right) \\
& + \int_{t_0}^{t}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\alpha\right)Bu(s)\frac{ds}{s^{1-\rho}}.
\end{aligned} \tag{14}$$

Applying the Euclidean norm to both sides of Equation (14), we obtain the following relationship

$$\begin{aligned}
\|x(t)\| \leq {} & \|x_0\|\left\|\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\alpha\right)\right\| \\
& + \|B\|\,\|u\|\int_{t_0}^{t}\left\|\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\alpha\right)\frac{ds}{s^{1-\rho}}\right\|.
\end{aligned} \tag{15}$$

From assumption $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$, there exist a positive number $M > 0$ [4,18,34] such that, we have

$$\int_{t_0}^{t}\left\|\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\alpha\right)\frac{ds}{s^{1-\rho}}\right\| \leq M. \tag{16}$$

This inequality is a classic condition in stability analysis of fractional derivatives shown in [34]. Finally, the solution of the fractional differential Equation (12) described by the left generalized fractional derivative with exogenous input satisfies the following relationship

$$\|x(t)\| \leq \|x_0\|\left\|\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\alpha\right)\right\| + \|B\|\,\|u\|\,M. \tag{17}$$

We first notice, when the exogenous input of the fractional differential Equation (12) described by the left generalized fractional derivative is null $\|u\| = 0$. The solution obtained in Equation (17) becomes

$$\|x(t)\| \leq \|x_0\| \left\| \left( \frac{t^{\rho} - t_0^{\rho}}{\rho} \right)^{\alpha - 1} E_{\alpha,\alpha} \left( A \left( \frac{t^{\rho} - t_0^{\rho}}{\rho} \right)^{\alpha} \right) \right\|. \tag{18}$$

It corresponds to the classical Mittag-Leffler stability of the trivial solution of the fractional differential equation without input $D^{\alpha,\rho}x = Ax$ described by the left generalized fractional derivative.

Secondly, let us consider the exogenous input converging to zero when $t$ tends to infinity. We know when the identity $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$ is held, we have

$$E_{\alpha,\alpha} \left( A \left( \frac{t^{\rho} - t_0^{\rho}}{\rho} \right)^{\alpha} \right) \longrightarrow 0. \tag{19}$$

From which we obtain $\|x(t)\| \longrightarrow 0$. Summarizing, we have the following

$$\|u\| \longrightarrow 0 \Longrightarrow \|x(t)\| \longrightarrow 0. \tag{20}$$

In other words, a converging input generates a converging state. This property is called the CICS property, derived in [15,18].

Finally, let us consider the exogenous input bounded ($\|u\| \leq \eta$). The solution of the fractional differential Equation (12) described by the left generalized fractional derivative satisfies the following relationship

$$\|x(t)\| \leq \|x_0\| \left\| \left( \frac{t^{\rho} - t_0^{\rho}}{\rho} \right)^{\alpha - 1} E_{\alpha,\alpha} \left( A \left( \frac{t^{\rho} - t_0^{\rho}}{\rho} \right)^{\alpha} \right) \right\| + \|B\| \eta M. \tag{21}$$

Furthermore, we consider the function $\left( \frac{t^{\rho} - t_0^{\rho}}{\rho} \right)^{\alpha - 1} E_{\alpha,\alpha} \left( A \left( \frac{t^{\rho} - t_0^{\rho}}{\rho} \right)^{\alpha} \right) \in \mathcal{L}$, thus there exist $\sigma > 0$ such that we have the following relationship

$$\left( \frac{t^{\rho} - t_0^{\rho}}{\rho} \right)^{\alpha - 1} E_{\alpha,\alpha} \left( A \left( \frac{t^{\rho} - t_0^{\rho}}{\rho} \right)^{\alpha} \right) \leq \sigma. \tag{22}$$

Thus Equation (21) can be expressed in the following form

$$\|x(t)\| \leq \|x_0\| \sigma + \|B\| \eta M. \tag{23}$$

Thus, the solution of the fractional differential given in Equation (12) described by the left generalized fractional derivative is bounded as well. A bounded input for Equation (12), we obtain a bounded state for Equation (12). This property is called the BIBS property, created in [15,18]. The objective in this paper is to introduce a new stability notion taking into account a few things; namely the converging input, the converging state, the bounded input bounded state and the generalized Mittag-Leffler stability of the trivial solution of the unforced fractional differential equation. This stability notion we refer to as the generalized Mittag-Leffler input stability. In other words, the fractional differential equation described by the Left generalized fractional derivative is said generalized Mittag-Leffler stable, when its solution is bounded by a class $\mathcal{KL}$ function (contain a Mittag-Leffler function) plus a class $\mathcal{K}_{\infty}$ function proportional to the input of the given fractional differential equation. A similar derivation leading to Equation (23) has also recently been applied to the study of fixed-time stability in [35].

In this section, we introduce new stability notion in the context of the fractional differential equations described by the left generalized fractional derivative operator. The fractional differential equation under consideration is expressed in the following form

$$D^{\alpha,\rho}x = f(t,x,u) \tag{24}$$

where the function $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous locally Lipschitz function, the function $x \in \mathbb{R}^n$ is a state variable, and furthermore the condition $f(t,0,0) = 0$ is held. Given an initial condition $x_0 \in \mathbb{R}^n$, the solution of the fractional differential Equation (24) starting at $x_0$ at time $t_0$ is represented by $x(.) = x(.,x_0,u)$.

**Definition 12.** *The solution $x = 0$ of the fractional differential equation described by the left generalized fractional derivative defined by Equation (24) is said to be generalized Mittag-Leffler stable if, for any initial condition $\|x_0\|$ and initial time $t_0$, its solution satisfies the following condition*

$$\|x(t,\|x_0\|)\| \leq \left[ m(\|x_0\|) \left( \frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha}\left( \eta \left( \frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha} \right) \right]^a, \tag{25}$$

*where $a > 0$, $\eta < 0$ and the function $m$ is locally Lipschitz on a domain contained in $\mathbb{R}^n$ and satisfies $m(0) = 0$ [7,14].*

In the following definition, we introduce the definition of the generalized Mittag-Leffler input stability in the context of the fractional differential equation described by the left generalized fractional derivative operator.

**Definition 13.** *The fractional differential equation described by the left generalized fractional derivative defined by Equation (24) is said to be generalized Mittag-Leffler input stable if, there exist a class $\gamma \in \mathcal{K}_\infty$ function such that for any initial condition $\|x_0\|$, its solution satisfies the following condition*

$$\|x(t,\|x_0\|)\| \leq \left[ m(\|x_0\|) \left( \frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha}\left( \eta \left( \frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha} \right) \right]^a + \gamma\left(\|u\|_\infty\right), \tag{26}$$

*where $a > 0$ and $\eta < 0$.*

From the condition $\gamma \in \mathcal{K}_\infty$, we get $\gamma(0) = 0$. We recover Definition 13. That is, the generalized Mittag-Leffler input stability of the fractional differential given in Equation (24) implies the generalized Mittag-Leffler stability of the trivial solution of the fractional differential equation with no input defined by $D^{\alpha,\rho}x = f(t,x,0)$. From the fact $\gamma \in \mathcal{K}_\infty$, when the input is bounded implies the function $\gamma\left(\|u\|_\infty\right)$ is bounded as well. Thus the state of the fractional differential Equation (24) is bounded too. We thus recover BIBS. From the fact $\gamma \in \mathcal{K}_\infty$, a converging input causes $\gamma\left(\|u\|_\infty\right)$ to converge. Thus the state of the fractional differential Equation (24) is converging as well. We thus recover CICS. In conclusion we can say that Definition 12 is well posed.

## 4. Lyapunov Characterizations of the Generalized Mittag-Leffler input Stability

In this section, we give the Lyapunov characterization of the generalized Mittag-Leffler input stability of the fractional differential equation. We know, it is not always trivial to get the analytical solution of the fractional differential equation with exogenous inputs. An alternative is to propose a method of analyzing the Mittag-Leffler input stability. The method consist of calculating the fractional energy of the fractional differential equation along the trajectories. In other words, we use the Lyapunov direct method.

**Theorem 1.** *Let us consider that there exists a positive function* $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *continuous and differentiable, and a class* $\mathcal{K}_\infty$ *function* $\chi_1$ *and class* $\mathcal{K}$ *functions* $\chi_2$*,* $\chi_3$ *satisfying the following assumptions:*

1. $\|x\|^a \leq V(t,x) \leq \chi_1\left(\|x\|\right).$
2. *If for any* $\|x\| \geq \chi_2((|u|)) \Longrightarrow D_c^{\alpha,\rho}V(t,x) \leq -\chi_3((\|x\|)).$

*Then the fractional differential Equation (24) described by the left generalized fractional derivative is generalized Mittag-Leffler input stable.*

**Proof.** Summarizing [18], combining Assumption (1) and Assumption (2), we have

$$\begin{aligned} \|x\|^a &\leq V(x,t) \leq \alpha_1 \circ \alpha_2\left(\|u\|\right) \\ \|x\| &\leq \left(\alpha_1 \circ \alpha_2\left(\|u\|\right)\right)^{1/a} \\ \|x\| &\leq \gamma\left(\|u\|\right), \end{aligned} \tag{27}$$

where the function $\gamma\left(\|u\|\right) = \left(\alpha_1 \circ \alpha_2\left(\|u\|\right)\right)^{1/a} \in \mathcal{K}_\infty$.

From Assumption (2), using an exponential form of the Lyapunov function in, there exist positive constant such that, we have

$$\begin{aligned} \|x\| \geq \chi_2((|u|)) &\Longrightarrow D_c^{\alpha,\rho}V(t,x) \leq -\chi_3((\|x\|)) \\ &\Longrightarrow D_c^{\alpha,\rho}V(t,x) \leq -\chi_3((\|x\|)) \leq -kV(x,t). \end{aligned} \tag{28}$$

It follows from Equation (28), the following inequality

$$\begin{aligned} \|x\|^a &\leq V(t,x) \leq V(\|x_0\|)\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-k\left(\frac{t^\rho - t_0^\rho}{\rho}\right)\right) \\ \|x\| &\leq \left\{V(\|x_0\|)\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-k\left(\frac{t^\rho - t_0^\rho}{\rho}\right)\right)\right\}^{1/a}. \end{aligned} \tag{29}$$

Finally, combining Equations (27) and (29), we obtain

$$\|x\| \leq \max\left\{\left\{V(\|x_0\|)\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-k\left(\frac{t^\rho - t_0^\rho}{\rho}\right)\right)\right\}^{1/a}; \gamma\left(\|u\|\right)\right\}. \tag{30}$$

Thus the fractional differential equation defined by Equation (24) is generalized Mittag-Leffler input stable. □

The second characterization is a consequence of the first theorem. It is more simplest to be applied in many cases. We have the following characterization.

**Theorem 2.** *Let there exist a positive function* $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *continuous and differentiable, and a class* $\mathcal{K}_\infty$ *of functions* $\chi_1$ *and class* $\mathcal{K}_\infty$ *function* $\gamma$ *satisfying the following assumption:*

1. $\|x\|^a \leq V(t,x) \leq \chi_1\left(\|x\|\right).$
2. $D_c^{\alpha,\rho}V(t,x) \leq -kV(x,t) + \gamma\left(\|u\|\right).$

*Then fractional differential Equation (24) described by the left generalized fractional derivative is generalized Mittag-Leffler input stable stable.*

**Proof.** From Assumption (2), we have the following relationships

$$\begin{aligned} D_c^{\alpha,\rho}V(t,x) &\leq -kV(x,t)+\gamma\left(\|u\|\right) \\ D_c^{\alpha,\rho}V(t,x) &\leq -\left(1-\theta\right)kV(x,t)-\theta kV(x,t)+\gamma\left(\|u\|\right), \end{aligned} \tag{31}$$

where $\theta\in(0,1)$. We have

$$\begin{aligned} -\theta kV(x,t)+\gamma\left(\|u\|\right)\leq 0 &\implies D_c^{\alpha,\rho}V(t,x)\leq-\left(1-\theta\right)kV(x,t) \\ V(x,t)\geq\frac{\gamma(\|u\|)}{\theta k} &\implies D_c^{\alpha,\rho}V(t,x)\leq-\left(1-\theta\right)kV(x,t). \end{aligned} \tag{32}$$

From first assumption, it yields that

$$\theta k\chi_1\left(\|x\|\right)\geq\gamma(\|u\|)\implies D_c^{\alpha,\rho}V(t,x)\leq-\left(1-\theta\right)kV(x,t).$$

Thus the fractional differential equation described by the left generalized fractional derivative is Mittag-Leffler input stable. □

## 5. Practical Applications

In this section, we give many practical applications of the Mittag-Leffler input stability of the fractional differential equation described by the generalized fractional derivative using the Lyapunov characterizations.

Let us consider the fractional differential equation described by the left generalized fractional differential equation defined by

$$\begin{cases} D_c^{\alpha,\rho}x_1=-x_1+\frac{1}{2}x_2+\frac{1}{2}u_1 \\ D_c^{\alpha,\rho}x_2=-x_2+\frac{1}{2}u_2 \end{cases} \tag{33}$$

where $x=(x_1,x_2)\in\mathbb{R}^2$ and $u=(u_1,u_2)\in\mathbb{R}^2$ represents the exogenous input. Let us take the Lyapunov function defined by $V(x)=\frac{1}{2}\left(x_1^2+x_2^2\right)$. The left generalized fractional derivative of the Lyapunov function along the trajectories is given by

$$\begin{aligned} D_c^{\alpha,\rho}V(t,x) &= -x_1^2+\frac{1}{2}x_1x_2+\frac{1}{2}x_1u_1-x_2^2+\frac{1}{2}x_2u_2 \\ &\leq -\frac{1}{2}x_1^2-\frac{1}{2}x_2^2+\frac{1}{4}\|u\|^2 \\ &\leq -V(x)+\frac{1}{4}\|u\|^2 \end{aligned} \tag{34}$$

Consider $\gamma(\|u\|)=\frac{1}{4}\|u\|^2\in\mathcal{K}_\infty$. It follows from Theorem 2, the fractional differential equation described by the left generalized fractional derivative given in Equation (33) is Mittag-Leffler input stable. Thus, the origin of the unforced fractional differential equation obtained with $u=(u_1,u_2)=(0,0)$

$$\begin{cases} D_c^{\alpha,\rho}x_1=-x_1+\frac{1}{2}x_2 \\ D_c^{\alpha,\rho}x_2=-x_2 \end{cases} \tag{35}$$

where $x=(x_1,x_2)\in\mathbb{R}^2$, is Mittag-Leffler stable.

Let us consider the fractional differential equation described by the left generalized fractional differential equation defined by

$$\begin{cases} D_c^{\alpha,\rho}x_1=-x_1+x_2+u_1 \\ D_c^{\alpha,\rho}x_2=-x_2+u_2 \end{cases}, \tag{36}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $u = (u_1, u_2) \in \mathbb{R}^2$ represents the exogenous input. Let the Lyapunov function defined by $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$. The left generalized fractional derivative of the Lyapunov function along the trajectories is given by

$$\begin{aligned} D_c^{\alpha,\rho} V(t,x) &= -x_1^2 + x_1x_2 + x_1u_1 - x_2^2 + x_2u_2 \\ &\leq -x_1^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_1^2 - x_2^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_2^2 + \|u\|^2 \\ &\leq \|u\|^2 . \end{aligned} \tag{37}$$

Let $\gamma(\|u\|) = \frac{1}{4}\|u\|^2 \in \mathcal{K}_\infty$. It follows from Theorem 2, the fractional differential equation described by the left generalized fractional derivative in Equation (35) is bounded as well [36].

Let us consider the electrical RL circuit described by the left generalized fractional differential equation defined by

$$D_c^{\alpha,\rho} x = -\frac{\sigma^{1-\alpha}R}{L}x + u \tag{38}$$

with the initial boundary condition defined by $x(0) = x_0$. The parameter $\sigma$ is associated with the temporal components in the differential equation. $u$ represents the exogenous input. Let us take the Lyapunov function defined by $V(x) = \frac{1}{2}\|x\|^2$. The left generalized fractional derivative of the Lyapunov function along the trajectories is given by

$$\begin{aligned} D_c^{\alpha,\rho} V(t,x) &= \frac{\sigma^{1-\alpha}R}{L}x^2 + xu \\ &\leq -\frac{\sigma^{1-\alpha}R}{L}\|x\|^2 + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|u\| \\ &\leq -\left(\frac{\sigma^{1-\alpha}R}{L} - \frac{1}{2}\right)\|x\|^2 + \frac{1}{2}\|u\| . \end{aligned} \tag{39}$$

Let us consider $k = \frac{\sigma^{1-\alpha}R}{L} - \frac{1}{2}$ and $\theta \in (0,1)$. We have the following relationship

$$D_c^{\alpha,\rho} V(t,x) \leq -(1-\theta)k\|x\|^2 + k\theta\|x\|^2 + \frac{1}{2}\|u\| \tag{40}$$

From Theorem 1, if $\|x\| \geq \frac{\|u\|}{2k\theta}$, we have $D_c^{\alpha,\rho} V(t,x) \leq -(1-\theta)k\|x\|^2$. Thus, the electrical RL circuit (36) is Mittag-Leffler input stable form.

Let us consider the fractional differential equation described in [4] by the left generalized fractional differential equation defined by

$$D_c^{\alpha,\rho} x = -x + xu, \tag{41}$$

where $x \in \mathbb{R}^n$ is a state variable. $u$ represents the exogenous input. Let's the Lyapunov function defined by $V(x) = \frac{1}{2}\|x\|^2$. The left generalized fractional derivative of the Lyapunov function along the trajectories is given by

$$\begin{aligned} D_c^{\alpha,\rho} V(t,x) &= -x^2 + x^2u \\ &\leq -\|x\|^2 + \|x\|^2\|u\| \\ &\leq -(1 - \|u\|)\|x\|^2 . \end{aligned} \tag{42}$$

We can observe, when we pick $\|u\| > 1$, using $\alpha$-integration, the state $x$ of Equation (42) diverge as $t$ tends to infinity. Then the fractional differential equation is not BIBS. Thus, the fractional differential Equation (41) is not, in general, Mittag-Leffler input stable.

## 6. Conclusions

In this paper, the Mittag-Leffler input stability has been thoroughly investigated. We have tried to motivate this study with its connection to many real world applications known to use Mittag-Leffler functions. We also address the Lyapunov characterization of the fractional differential equations. In doing so, we have created a further Lyapunov characterization which is more useful. Finally, we give some numerical examples to help illustrate the work that was accomplished in this paper. Analyzing the generalized Mittag-Leffer input stability of the fractional differential equations without decomposing it can be non trivial. The possible issue is to decompose it as a cascade of triangular equations and to find a method to analyze the generalized Mittag-Leffer input stability of the obtained fractional differential equation. In other words, finding the conditions for the generalized Mittag-Leffer input stability of the fractional differential cascade equations will be subject of future works.

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symmetry

Article

# An Investigation of the Third Hankel Determinant Problem for Certain Subfamilies of Univalent Functions Involving the Exponential Function

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**Abstract:** In the current article, we consider certain subfamilies $\mathcal{S}_e^*$ and $\mathcal{C}_e$ of univalent functions associated with exponential functions which are symmetric along real axis in the region of open unit disk. For these classes our aim is to find the bounds of Hankel determinant of order three. Further, the estimate of third Hankel determinant for the family $\mathcal{S}_e^*$ in this work improve the bounds which was investigated recently. Moreover, the same bounds have been investigated for 2-fold symmetric and 3-fold symmetric functions.

**Keywords:** subordinations; exponential function; Hankel determinant

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## 1. Introduction and Definitions

Let the collection of functions $f$ that are holomorphic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by conditions $f(0) = f'(0) - 1 = 0$ be denoted by the symbol $\mathcal{A}$. Equivalently; if $f \in \mathcal{A}$, then the Taylor-Maclaurin series representation has the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta). \tag{1}$$

Further, let we name by the notation $\mathcal{S}$ the most basic sub-collection of the set $\mathcal{A}$ that are univalent in $\Delta$. The familiar coefficient conjecture for the function $f \in \mathcal{S}$ of the form (1) was first presented by Bieberbach [1] in 1916 and proved by de-Branges [2] in 1985. In 1916-1985, many mathematicians struggled to prove or disprove this conjecture and as result they defined several subfamilies of the set $\mathcal{S}$ of univalent functions connected with different image domains. Now we mention some of them, that is; let the notations $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$, shows the families of starlike, convex and close-to-convex functions respectively and are defined as:

$$\begin{aligned}
\mathcal{S}^* &= \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \ (z \in \Delta) \right\}, \\
\mathcal{C} &= \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} \prec \frac{1+z}{1-z}, \ (z \in \Delta) \right\}, \\
\mathcal{K} &= \left\{ f \in \mathcal{S} : \frac{f'(z)}{g'(z)} \prec \frac{1+z}{1-z}, \text{ for } g(z) \in \mathcal{C}, \ (z \in \Delta) \right\},
\end{aligned}$$

where the symbol " $\prec$ " denotes the familiar subordinations between analytic functions and is define as; the function $h_1$ is subordinate to a function $h_2$, symbolically written as $h_1 \prec h_2$ or $h_1(z) \prec h_2(z)$, if we can find a function $w$, which is holomorphic in $\Delta$ with $w(0) = 0$ & $|w(z)| < 1$ such that $h_1(z) = h_2(w(z))$ $(z \in \Delta)$. Thus, $h_1(z) \prec h_2(z)$ implies $h_1(\Delta) \subset h_2(\Delta)$. In case of univalency of $h_1$ in $\Delta$, then the following relation holds:

$$h_1(z) \prec h_2(z) \quad (z \in \Delta) \quad \Longleftrightarrow \quad h_1(0) = h_2(0) \quad \text{and} \quad h_1(\Delta) \subset h_2(\Delta).$$

In [3], Padmanabhan and Parvatham in 1985 defined a unified families of starlike and convex functions using familiar convolution with the function $z/(1-z)^a$, for all $a \in \mathbb{R}$. Later on, Shanmugam [4] generalized the idea of paper [3] and introduced the set

$$\mathcal{S}_h^*(\phi) = \left\{ f \in \mathcal{A} : \frac{z(f*h)'}{(f*h)} \prec \phi(z), \quad (z \in \Delta) \right\},$$

where "$*$" stands for the familiar convolution, $\phi$ is a convex and $h$ is a fixed function in $\mathcal{A}$. We obtain the families $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ when taking $z/(1-z)$ and $z/(1-z)^2$ instead of $h$ in $\mathcal{S}_h^*(\phi)$ respectively. In 1992, Ma and Minda [5] reduced the restriction to a weaker supposition that $\phi$ is a function, with $Re\phi > 0$ in $\Delta$, whose image domain is symmetric about the real axis and starlike with respect to $\phi(0) = 1$ with $\phi'(0) > 0$ and discussed some properties. The set $\mathcal{S}^*(\phi)$ generalizes various subfamilies of the set $\mathcal{A}$, for example:

1. If $\phi(z) = \frac{1+Az}{1+Bz}$ with $-1 \le B < A \le 1$, then $\mathcal{S}^*[A,B] := \mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right)$ is the set of Janowski starlike functions, see [6]. Further, if $A = 1-2\alpha$ and $B = -1$ with $0 \le \alpha < 1$, then we get the set $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha$.
2. The class $\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z})$ was introduced by Sokól and Stankiewicz [7], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$.
3. For $\phi(z) = 1 + \sin z$, the class $\mathcal{S}^*(\phi)$ lead to the class $\mathcal{S}_{\sin}^*$, introduced in [8].
4. The family $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$ was introduced by Mediratta et al. [9] given as:

$$\mathcal{S}_e^* = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec e^z, \ (z \in \Delta) \right\}, \tag{2}$$

or, equivalently

$$\mathcal{S}_e^* = \left\{ f \in \mathcal{S} : \left| \log \frac{zf'(z)}{f(z)} \right| < 1, \ (z \in \Delta) \right\}. \tag{3}$$

They investigated some interesting properties and also links these classes to the familiar subfamilies of the set $\mathcal{S}$. In [9], the authors choose the function $f(z) = z + \frac{1}{4}z^2$ (Figure 1) and then sketch the following figure of the function class $\mathcal{S}_e^*$ by using the form (3) as:

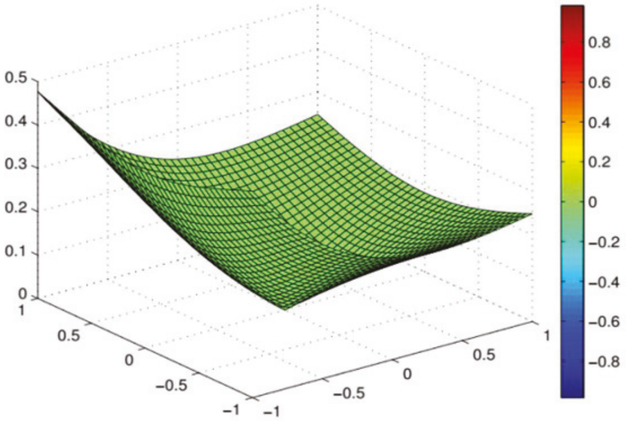

**Figure 1.** The figure of the function class $\mathcal{S}_1^*$ for $f(z) = z + \frac{1}{4}z^2$.

Similarly, by using Alexandar type relation in [9], we have;

$$\mathcal{C}_e = \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} \prec e^z, \ (z \in \Delta) \right\}. \tag{4}$$

From the above discussion, we conclude that the families $\mathcal{S}_e^*$ and $\mathcal{C}_e$ considered in this paper are symmetric about the real axis.

For given parameters $q, n \in \mathbb{N} = \{1, 2, \ldots\}$, the Hankel determinant $H_{q,n}(f)$ was defined by Pommerenke [10,11] for a function $f \in \mathcal{S}$ of the form (1) as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix}. \tag{5}$$

The concept of Hankel determinant is very useful in the theory of singularities [12] and in the study of power series with integral coefficients. For deep insight, the reader is invited to read [13–15]. Specifically, the absolute sharp bound of the functional $H_{2,2}(f) = a_2a_4 - a_3^2$ for each of the sets $\mathcal{S}^*$ and $\mathcal{C}$ were proved by Janteng et al. [16,17] while the exact estimate of this determinant for the family of close-to-convex functions is still unknown (see, [18]). On the other side for the set of Bazilevič functions, the sharp estimate of $|H_{2,2}(f)|$ was given by Krishna et al. [19]. Recently, Srivastava and his coauthors [20] found the estimate of second Hankel determinant for bi-univalent functions involving symmetric $q$-derivative operator while in [21], the authors discussed Hankel and Toeplitz determinants for subfamilies of $q$-starlike functions connected with a general form of conic domain. For more literature see [22–29]. The determinant with entries from (1)

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

is known as Hankel determinant of order three and the estimation of this determinant $|H_{3,1}(f)|$ is very hard as compared to derive the bound of $|H_{2,2}(f)|$. The very first paper on $H_{3,1}(f)$ visible in 2010 by

Babalola [30] in which he got the upper bound of $H_{3,1}(f)$ for the families of $\mathcal{S}^*$ and $\mathcal{C}$. Later on, many authors published their work regarding $|H_{3,1}(f)|$ for different sub-collections of univalent functions, see [8,31–36]. In 2017, Zaprawa [37] upgraded the results of Babalola [30] by giving

$$|H_{3,1}(f)| \leq \begin{cases} 1, & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{49}{540}, & \text{for} \quad f \in \mathcal{C}, \end{cases}$$

and claimed that these bounds are still not best possible. Further for the sharpness, he examined the subfamilies of $\mathcal{S}^*$ and $\mathcal{C}$ consisting of functions with $m$-fold symmetry and obtained the sharp bounds. Moreover this determinant was further improved by Kwon et al. [38] and proved $|H_{3,1}(f)| \leq 8/9$ for $f \in \mathcal{S}^*$, yet not best possible. The authors in [39–41] contributed in similar direction by generalizing different classes of univalent functions with respect to symmetric points. In 2018, Kowalczyk et al. [42] and Lecko et al. [43] got the sharp inequalities

$$|H_{3,1}(f)| \leq 4/135, \quad \text{and} \quad |H_{3,1}(f)| \leq 1/9,$$

for the recognizable sets $\mathcal{K}$ and $\mathcal{S}^*(1/2)$ respectively, where the symbol $\mathcal{S}^*(1/2)$ indicates to the family of starlike functions of order $1/2$. Also we would like to cite the work done by Mahmood et al. [44] in which they studied third Hankel determinant for a subset of starlike functions in $q$-analogue. Additionally Zhang et al. [45] studied this determinant for the set $\mathcal{S}_e^*$ and obtained the bound $|H_{3,1}(f)| \leq 0.565$.

In the present article, our aim is to investigate the estimate of $|H_{3,1}(f)|$ for both the above defined classes $\mathcal{S}_e^*$ and $\mathcal{C}_e$. Moreover, we also study this problem for $m$-fold symmetric starlike and convex functions associated with exponential function.

## 2. A Set of Lemmas

Let $\mathcal{P}$ denote the family of all functions $p$ that are analytic in $\mathbb{D}$ with $\Re(p(z)) > 0$ and has the following series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \,(z \in \Delta). \tag{6}$$

**Lemma 1.** *If $p \in \mathcal{P}$ and has the form , then*

$$\begin{aligned} |c_n| &\leq 2 \text{ for } n \geq 1, && (7)\\ |c_{n+k} - \mu c_n c_k| &< 2, \text{ for } 0 \leq \mu \leq 1, && (8)\\ |c_m c_n - c_k c_l| &\leq 4 \text{ for } m+n = k+l, && (9)\\ \left|c_{n+2k} - \mu c_n c_k^2\right| &\leq 2(1+2\mu); \text{ for } \mu \in \mathbb{R}, && (10)\\ \left|c_2 - \frac{c_1^2}{2}\right| &\leq 2 - \frac{|c_1|^2}{2}, && (11) \end{aligned}$$

*and for complex number $\lambda$, we have*

$$\left|c_2 - \lambda c_1^2\right| \leq 2\max\{1, |2\lambda - 1|\}. \tag{12}$$

*For the inequalities (7), (11), (8), (10), (9) see [46] and (12) is given in [47].*

## 3. Improved Bound of $|H_{3,1}(f)|$ for the Set $\mathcal{S}_e^*$

**Theorem 1.** *If $f$ belongs to $\mathcal{S}_e^*$, then*

$$|H_{3,1}(f)| \leq 0.50047781.$$

**Proof.** Let $f \in \mathcal{S}_e^*$. Then we can write (2), in terms of Schwarz function as

$$\frac{zf'(z)}{f(z)} = e^{w(z)}.$$

If $h \in \mathcal{P}$, then it can be written in form of Schwarz function as

$$h(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots .$$

From above, we can get

$$w(z) = \frac{h(z)-1}{h(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}.$$

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= 1 + a_2 z + \left(2a_3 - a_2^2\right) z^2 + \left(3a_4 - 3a_2 a_3 + a_2^3\right) z^3 \\ &\quad + \left(4a_5 - 2a_3^2 - 4a_2 a_4 + 4a_2^2 a_3 - a_2^4\right) z^4 = 1 + p_1 z + p_2 z^2 + \cdots . \end{aligned} \tag{13}$$

and from the series expansion of $w$ along with some calculations, we have

$$e^{w(z)} = 1 + w(z) + \frac{(w(z))^2}{2!} + \frac{(w(z))^3}{3!} + \frac{(w(z))^4}{4!} + \frac{(w(z))^5}{5!} + \cdots .$$

After some computations and rearranging, it yields

$$\begin{aligned} e^{w(z)} &= 1 + \frac{1}{2} c_1 z + \left(\frac{c_2}{2} - \frac{c_1^2}{8}\right) z^2 + \left(\frac{c_1^3}{48} + \frac{c_3}{2} - \frac{c_1 c_2}{4}\right) z^3 \\ &\quad + \left(\frac{1}{384} c_1^4 + \frac{1}{2} c_4 - \frac{1}{8} c_2^2 + \frac{1}{16} c_1^2 c_2 - \frac{1}{4} c_1 c_3\right) z^4 + \cdots . \end{aligned} \tag{14}$$

Comparing (13) and (14), we have

$$a_2 = \frac{c_1}{2}, \tag{15}$$

$$a_3 = \frac{1}{4}\left(c_2 + \frac{c_1^2}{4}\right), \tag{16}$$

$$a_4 = \frac{1}{6}\left(c_3 + \frac{c_1 c_2}{4} - \frac{c_1^3}{48}\right), \tag{17}$$

$$a_5 = \frac{1}{4}\left(\frac{c_1^4}{288} + \frac{c_4}{2} + \frac{c_1 c_3}{12} - \frac{c_1^2 c_2}{24}\right). \tag{18}$$

From (5), the Third Hankel determinant can be written as

$$H_{3,1}(f) = -a_2^2 a_5 + 2a_2 a_3 a_4 - a_3^3 + a_3 a_5 - a_4^2.$$

Using (15), (16), (17) and (18), we get

$$H_{3,1}(f) = \frac{35}{27648} c_1^4 c_2 + \frac{53}{6912} c_1^3 c_3 + \frac{c_2 c_4}{32} + \frac{19}{576} c_1 c_2 c_3 - \frac{211}{331776} c_1^6 - \frac{c_2^3}{64} - \frac{3}{128} c_1^2 c_4 - \frac{13}{2304} c_1^2 c_2^2 - \frac{c_3^2}{36}.$$

After rearranging, it yields

$$\begin{aligned} H_{3,1}(f) &= \frac{211}{165888}c_1^4\left(c_2-\frac{c_1^2}{2}\right)+\frac{3}{64}c_4\left(c_2-\frac{c_1^2}{2}\right)-\frac{c_1c_3}{96}\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{165888}c_1^3\left(c_3-c_1c_2\right) \\ &\quad +\frac{407}{165888}c_1^2\left(c_1c_3-c_2^2\right)-\frac{c_3}{36}\left(c_3-c_1c_2\right)-\frac{c_2}{64}\left(c_4-c_1c_3\right)-\frac{529}{165888}c_1^2c_2^2-\frac{c_2^3}{64}. \end{aligned}$$

Using triangle inequality along with (7), (11), (8) and (9), provide us

$$\begin{aligned} \left|H_{3,1}(f)\right| &\leq \frac{211}{165888}\left|c_1\right|^4\left(2-\frac{\left|c_1\right|^2}{2}\right)+\frac{3}{32}\left(2-\frac{\left|c_1\right|^2}{2}\right)+\frac{\left|c_1\right|}{48}\left(2-\frac{\left|c_1\right|^2}{2}\right)+\frac{1}{82944}\left|c_1\right|^3 \\ &\quad +\frac{407}{41472}\left|c_1\right|^2+\frac{1}{9}+\frac{1}{16}+\frac{529}{41472}\left|c_1\right|^2+\frac{1}{8}. \end{aligned}$$

If we substitute $|c_1| = x \in [0,2]$, we obtain a function of variable $x$. Therefore, we can write

$$\begin{aligned} \left|H_{3,1}(f)\right| &\leq \frac{211}{165888}x^4\left(2-\frac{x^2}{2}\right)+\frac{3}{32}\left(2-\frac{x^2}{2}\right)+\frac{x}{48}\left(2-\frac{x^2}{2}\right)+\frac{1}{82944}x^3 \\ &\quad +\frac{407}{41472}x^2+\frac{1}{9}+\frac{1}{16}+\frac{529}{41472}x^2+\frac{1}{8}. \end{aligned}$$

The above function attains its maximum value at $x = 0.64036035$, which is

$$\left|H_{3,1}(f)\right| \leq 0.50047781.$$

Thus, the proof is completed. ☐

## 4. Bound of $|H_{3,1}(f)|$ for the Set $\mathcal{C}_e$

**Theorem 2.** *Let $f$ has the form (1) and belongs to $\mathcal{C}_e$. Then*

$$\begin{aligned} |a_2| &\leq \frac{1}{2}, && (19) \\ |a_3| &\leq \frac{1}{4}, && (20) \\ |a_4| &\leq \frac{17}{144}, && (21) \\ |a_5| &\leq \frac{7}{96}. && (22) \end{aligned}$$

*The first three inequalities are sharp.*

**Proof.** If $f \in \mathcal{C}_e$, then we can write (4), in form of Schwarz function as

$$1+\frac{zf''(z)}{f'(z)} = e^{w(z)}.$$

From (1), we can write

$$\begin{aligned} 1+\frac{zf''(z)}{f'(z)} &= 1+2a_2z+\left(6a_3-4a_2^2\right)z^2+\left(12a_4-18a_2a_3+8a_2^3\right)z^3 \\ &\quad +\left(20a_5-18a_3^2-32a_2a_4+48a_2^2a_3-16a_2^4\right)z^4+\cdots. \end{aligned} \tag{23}$$

By comparing (23) and (14), we get

$$a_2 = \frac{c_1}{4}, \tag{24}$$

$$a_3 = \frac{1}{12}\left(c_2 + \frac{c_1^2}{4}\right), \tag{25}$$

$$a_4 = \frac{1}{24}\left(\frac{c_1 c_2}{4} + c_3 - \frac{c_1^3}{48}\right), \tag{26}$$

$$a_5 = \frac{1}{20}\left(\frac{c_1^4}{288} + \frac{c_4}{2} + \frac{c_1 c_3}{12} - \frac{c_1^2 c_2}{24}\right). \tag{27}$$

Implementing (7), in (24) and (25), we have

$$|a_2| \leq \tfrac{1}{2} \quad \text{and} \quad |a_3| \leq \tfrac{1}{4}.$$

Reshuffling (26), we have

$$|a_4| = \frac{1}{24}\left|\frac{5}{24}c_1 c_2 + \frac{c_1}{24}\left(c_2 - \frac{c_1^2}{2}\right) + c_3\right|.$$

Application of triangle inequality and (7) and (11) leads us to

$$|a_4| \leq \frac{1}{24}\left\{\frac{5}{12}|c_1| + \frac{|c_1|}{24}\left(2 - \frac{|c_1|^2}{2}\right) + 2\right\}.$$

If we insert $|c_1| = x \in [0,2]$, then we get

$$|a_4| \leq \frac{1}{24}\left\{\frac{5}{12}x + \frac{x}{24}\left(2 - \frac{x^2}{2}\right) + 2\right\}.$$

The overhead function has a maximum value at $x = 2$, thus

$$|a_4| \leq \frac{17}{144}.$$

Reordering (27), we have

$$|a_5| = \frac{1}{20}\left|\frac{1}{2}\left(c_4 - \frac{c_1^2 c_2}{48}\right) - \frac{c_1^2}{96}\left(c_2 - \frac{c_1^2}{3}\right) + \frac{c_1}{12}\left(c_3 - \frac{c_1 c_2}{4}\right)\right|.$$

By using triangle inequality along with (7), and (8), we get

$$|a_5| \leq \frac{7}{96}.$$

Equalities are obtain if we take

$$f(z) = \int_0^z e^{J(t)}dt = z + \frac{1}{2}z^2 + \frac{1}{4}z^3 + \frac{17}{144}z^4 + \frac{19}{360}z^5 + \cdots \tag{28}$$

where

$$J(t) = \int_0^t \frac{e^x - 1}{x}dx.$$

□

**Theorem 3.** *If $f$ is of the form (1) belongs to $\mathcal{C}_e$, then*

$$\left|a_3 - \gamma a_2^2\right| \leq \frac{1}{6} \max\left\{1, \frac{3}{2}\left|\gamma - 1\right|\right\}, \tag{29}$$

*where $\gamma$ is a complex number.*

**Proof.** From (24) and (25), we get

$$\left|a_3 - \gamma a_2^2\right| = \left|\frac{c_2}{12} + \frac{c_1^2}{48} - \frac{\gamma}{16}c_1^2\right|.$$

By reshuffling it, provides

$$\left|a_3 - \gamma a_2^2\right| = \frac{1}{12}\left|\left(c_2 - \frac{1}{2}\left(\frac{3\gamma - 1}{2}\right)c_1^2\right)\right|.$$

Application of (12), leads us to

$$\left|a_3 - \gamma a_2^2\right| \leq \max\left\{\frac{1}{6}, \frac{1}{12}\left|3\gamma - 3\right|\right\}.$$

□

Substituting $\gamma = 1$, we obtain the following inequality.

**Corollary 1.** *If $f \in \mathcal{C}_e$ and has the series represntaion (1), then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{6}. \tag{30}$$

**Theorem 4.** *If $f$ has the form (1) belongs to $\mathcal{C}_e$, then*

$$\left|a_2 a_3 - a_4\right| \leq \frac{31}{288}. \tag{31}$$

**Proof.** Using (24), (25) and (26), we have

$$\left|a_2 a_3 - a_4\right| = \left|\frac{c_1 c_2}{96} + \frac{7}{1152}c_1^3 - \frac{c_3}{24}\right|.$$

By rearranging it, gives

$$\left|a_2 a_3 - a_4\right| = \left|-\frac{1}{48}\left(c_3 - \frac{c_1 c_2}{2}\right) - \frac{1}{48}\left(c_3 - \frac{7}{24}c_1^3\right)\right|.$$

By applying triangle inequality plus (8) and (10), we get

$$\left|a_2 a_3 - a_4\right| \leq \left\{\frac{1}{24} + \frac{19}{288}\right\} = \frac{31}{288}.$$

□

**Theorem 5.** *Let $f \in \mathcal{C}_e$ be of the form (1). Then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{3}{64}. \tag{32}$$

**Proof.** From (24), (25) and (26), we have

$$\left|a_2a_4-a_3^2\right|=\left|\frac{c_1c_3}{96}-\frac{c_1^4}{1536}-\frac{c_1^2c_2}{1152}-\frac{c_2^2}{144}\right|.$$

By reordering it, yields

$$\left|a_2a_4-a_3^2\right|=\left|\frac{c_1}{576}\left(c_3-\frac{c_1c_2}{2}\right)+\frac{c_1}{576}\left(c_3-\frac{3}{8}c_1^3\right)+\frac{1}{144}\left(c_1c_3-c_2^2\right)\right|.$$

Application of triangle inequality plus (7), (11), (10) and (9), we obtain

$$\left|a_2a_4-a_3^2\right|\leq\frac{4}{576}+\frac{7}{576}+\frac{4}{144}=\frac{3}{64}.$$

□

**Theorem 6.** *If $f\in\mathcal{C}_e$ and has the form (1), then*

$$\left|H_{3,1}\left(f\right)\right|\leq 0.0234598.$$

**Proof.** Using (5), the Hankel determinant of order three can be formed as;

$$H_{3,1}\left(f\right)=-a_2^2a_5+2a_2a_3a_4-a_3^3+a_3a_5-a_4^2.$$

Using (24), (25), (26) and (27), gives us

$$H_{3,1}\left(f\right)=\frac{7}{5760}c_1c_2c_3-\frac{c_3^2}{576}-\frac{c_2^3}{1728}-\frac{173}{6635520}c_1^6+\frac{23}{276480}c_1^4c_2+\frac{c_2c_4}{480}-\frac{13}{46980}c_1^2c_2^2-\frac{c_1^2c_4}{960}+\frac{23}{69120}c_1^3c_3.$$

Now, rearranging it provides

$$\begin{aligned}H_{3,1}\left(f\right) &= \frac{173}{3317760}c_1^4\left(c_2-\frac{c_1^2}{2}\right)-\frac{103}{1658880}c_1^2c_2\left(c_2-\frac{c_1^2}{2}\right)+\frac{c_4}{480}\left(c_2-\frac{c_1^2}{2}\right)\\ &\quad+\frac{11}{17280}c_1c_2\left(c_3-\frac{365}{1056}c_1c_2\right)+\frac{c_2}{1728}\left(c_1c_3-c_2^2\right)-\frac{c_3}{576}\left(c_3-\frac{23}{120}c_1^3\right).\end{aligned}$$

Application of triangle inequality plus (7), (11), (8), (10) and (9), leads us to

$$\left|H_{3,1}\left(f\right)\right|\leq\frac{173}{3317760}\left|c_1\right|^4\left(2-\frac{\left|c_1\right|^2}{2}\right)+\frac{103}{829440}\left|c_1\right|^2\left(2-\frac{\left|c_1\right|^2}{2}\right)+\frac{1}{4320}\left|c_1\right|+\frac{83}{8640}+\frac{1}{216}.$$

Now, replacing $\left|c_1\right|=x\in[0,2]$, then, we can write

$$\left|H_{3,1}\left(f\right)\right|\leq\frac{173}{3317760}x^4\left(2-\frac{x^2}{2}\right)+\frac{103}{829440}x^2\left(2-\frac{x^2}{2}\right)+\frac{1}{240}\left(2-\frac{x^2}{2}\right)+\frac{11}{4320}x+\frac{41}{2880}.$$

The above function gets its maximum at $x=0.7024858$, Therefore, we have

$$\left|H_{3,1}\left(f\right)\right|\leq 0.02345979.$$

Thus the proof is completed. □

## 5. Bounds of $|H_{3,1}(f)|$ for 2-Fold and 3-Fold Functions

Let $m \in \mathbb{N} = \{1, 2, \ldots\}$. If a rotation $\triangle$ about the origin through an angle $2\pi/m$ carries $\triangle$ on itself, then such a domain $\triangle$ is called $m$-fold symmetric. An analytic function $f$ is $m$-fold symmetric in $\Delta$, if

$$f\left(e^{2\pi i/m}z\right) = e^{2\pi i/m} f(z), \; (z \in \Delta).$$

By $\mathcal{S}^{(m)}$, we define the set of $m$-fold univalent functions having the following Taylor series form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \; (z \in \Delta). \tag{33}$$

The sub-families $\mathcal{S}_e^{*(m)}$ and $\mathcal{C}_e^{(m)}$ of $\mathcal{S}^{(m)}$ are the sets of $m$-fold symmetric starlike and convex functions respectively associated with exponential functions. More intuitively, an analytic function $f$ of the form (33), belongs to the families $\mathcal{S}_e^{*(m)}$ and $\mathcal{C}_e^{(m)}$, if and only if

$$\frac{zf'(z)}{f(z)} = \exp\left(\frac{p(z)-1}{p(z)+1}\right), \; p \in \mathcal{P}^{(m)}, \tag{34}$$

$$1 + \frac{zf''(z)}{f'(z)} = \exp\left(\frac{p(z)-1}{p(z)+1}\right), \; p \in \mathcal{P}^{(m)}. \tag{35}$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk}, \; (z \in \Delta) \right\}. \tag{36}$$

Here we prove some theorems related to 2-fold and 3-fold symmetric functions.

**Theorem 7.** *If $f \in \mathcal{S}_e^{*(2)}$ and has the form (33), then*

$$|H_{3,1}(f)| \leq \frac{1}{8}.$$

**Proof.** Let $f \in \mathcal{S}_e^{*(2)}$. Then, there exists a function $p \in \mathcal{P}^{(2)}$, such that

$$\frac{zf'(z)}{f(z)} = \exp\left(\frac{p(z)-1}{p(z)+1}\right).$$

Using the series form (33) and (36), when $m = 2$ in the above relation, we can get

$$a_3 = \frac{c_2}{4}, \tag{37}$$

$$a_5 = \frac{c_4}{8}. \tag{38}$$

Now,

$$H_3(f) = a_3 a_5 - a_3^3.$$

Utilizing (37) and (38), we get

$$H_{3,1}(f) = -\frac{c_2^3}{64} + \frac{c_2 c_4}{32}.$$

By rearranging, it yields

$$H_{3,1}(f) = \frac{c_2}{32}\left(c_4 - \frac{c_2^2}{2}\right).$$

Using triangle inequality long with (8) and (7), gives us

$$\left|H_{3,1}\left(f\right)\right| \leq \frac{1}{8}.$$

Hence, the proof is done. □

**Theorem 8.** *If $f \in \mathcal{S}_e^{*(3)}$ and has the series form (33), then*

$$\left|H_{3,1}\left(f\right)\right| \leq \frac{1}{9}.$$

*This result is sharp for the function*

$$f\left(z\right) = \exp\left(\int_0^z \frac{e^{x^3}}{x}dx\right) = z + \frac{1}{3}z^4 + \frac{5}{36}z^7 + \cdots \tag{39}$$

**Proof.** As, $f \in \mathcal{S}_e^{*(3)}$, therefore there exists a function $p \in \mathcal{P}^{(3)}$, such that

$$\frac{zf'(z)}{f(z)} = \exp\left(\frac{p\left(z\right)-1}{p\left(z\right)+1}\right).$$

Utilizing the series form (33) and (36), when $m = 3$ in the above relation, we can obtain

$$a_4 = \frac{c_3}{6}.$$

Then,

$$H_{3,1}\left(f\right) = -a_4^2 = -\frac{c_3^2}{36}.$$

Utilizing (7) and triangle inequality, we have

$$\left|H_{3,1}\left(f\right)\right| \leq \frac{1}{9}.$$

Thus the proof is ended. □

**Theorem 9.** *Let $f \in \mathcal{C}_e^{(2)}$ and has the form given in (33). Then*

$$\left|H_{3,1}\left(f\right)\right| \leq \frac{1}{120}.$$

**Proof.** As, $f \in \mathcal{C}_e^{(2)}$, then there exists a function $p \in \mathcal{P}^{(2)}$, such that

$$1 + \frac{zf''(z)}{f'(z)} = \exp\left(\frac{p\left(z\right)-1}{p\left(z\right)+1}\right).$$

Utilizing the series form (33) and (36), when $m = 2$ in the above relation, we can obtain

$$a_3 = \frac{c_2}{12}, \tag{40}$$

$$a_5 = \frac{c_4}{40}. \tag{41}$$

$$H_{3,1}\left(f\right) = a_3a_5 - a_3^3.$$

Using (40) and (41), we have

$$H_{3,1}(f) = -\frac{c_2^3}{1728} + \frac{c_2 c_4}{480}.$$

Now, reordering the above equation, we obtain

$$H_3(f) = \frac{c_2}{480}\left(c_4 - \frac{5}{18}c_2^2\right).$$

Application of (7), (8) and triangle inequality, leads us to

$$|H_{3,1}(f)| \leq \frac{1}{120}.$$

Thus, the required result is completed. □

**Theorem 10.** *If $f \in \mathcal{C}_e^{(3)}$ and has the form given in (33), then*

$$|H_{3,1}(f)| \leq \frac{1}{144}. \tag{42}$$

*This result is sharp for the function*

$$f(z) = \int_0^z e^{I(t)} dt = z + \frac{1}{12}z^4 + \frac{5}{252}z^7 + \cdots \tag{43}$$

*where*

$$I(t) = \int_0^t \frac{e^{x^3} - 1}{x} dx.$$

**Proof.** Let, $f \in \mathcal{C}_e^{(3)}$. Then there exists a function $p \in \mathcal{P}^{(3)}$, such that

$$1 + \frac{zf''(z)}{f'(z)} = \exp\left(\frac{p(z) - 1}{p(z) + 1}\right).$$

Utilizing the series form (33) and (36), when $m = 3$ in the above relation, we can obtain

$$a_4 = \frac{c_3}{24}.$$

Then,

$$H_{3,1}(f) = -\frac{c_3^2}{576}.$$

Implementing (7) and triangle inequality, we have

$$|H_{3,1}(f)| \leq \frac{1}{144}.$$

Hence, the proof is done. □

## 6. Conclusions

In this article, we studied Hankel determinant $H_{3,1}(f)$ for the families $\mathcal{S}_e^*$ and $\mathcal{C}_e$ whose image domain are symmetric about the real axis. Furthermore, we improve the bound of third Hankel determinant for the family $\mathcal{S}_e^*$. These bounds are also discussed for 2-fold symmetric and 3-fold symmetric functions.

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Article

# On Periodic Solutions of Delay Differential Equations with Impulses

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**Abstract:** The purpose of this paper is to study the nonlinear distributed delay differential equations with impulses effects in the vectorial regulated Banach spaces $\mathcal{R}([-r,0],\mathbb{R}^n)$. The existence of the periodic solution of impulsive delay differential equations is obtained by using the Schäffer fixed point theorem in regulated space $\mathcal{R}([-r,0],\mathbb{R}^n)$.

**Keywords:** delay differential equations; integral operator; periodic solutions

**MSC:** primary 06F30; 46B20; 47E10; 34K13; 34K05

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## 1. Introduction

In this paper, we will investigate the existence of periodic solutions for vectorial distributed delay differential equations with impulses in regulated Banach spaces. More precisely, the prototype of this delay differential equations with impulses, is of the form

$$\begin{aligned} \frac{dx(t)}{dt} &= -\lambda x(t) + f(t,x_t), a.e.\ t \in [0,\omega+\tau], \lambda > 0, \omega > 0, && (1)\\ x(t_j) &= x(t_j^-), \text{ and } x(t_j^+) - x(t_j) = h_j(x(t_j)), \forall j = 1,\ldots,l, && (2)\\ x_0(\theta) &= \varphi(\theta),\ \theta \in [-\tau,0], && (3) \end{aligned}$$

with $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau,0]$, $\tau > 0$ and where $x$ and $\varphi$ are $\mathbb{R}^n$-valued functions on $[-\tau,\omega]$, and $[-\tau,0]$, respectively. The Equation (1) is a nonlinear delay differential equation. More details about this type of equations can be found in [1]. Moreover, we assume that

(i) $h_j \in C(\mathbb{R}^n,\mathbb{R}^n)$, $j = 1,\ldots,l$,
(ii) $\{t_1,t_2,\cdots,t_l\}$ is an increasing family of strictly positive real numbers,
(iii) there exist $\delta > 0$ and $T < \infty$, such that for any $j = 1,\ldots,l-1$, we have

$$0 < \delta \leq t_{j+1} - t_j \leq T < \infty.$$

We call (2) the impulses equation where, $x(t_j^-)$ (resp. $x(t_j^+)$) denotes the limit from the left (resp. from the right) of $x(t)$, as $t$ tends to $t_j$. This type of differential equations without delay was initiated in 1960's by Milman and Myshkis [2,3]. This problem started to be popular mostly in Eastern Europe in the years 1960–1970, with special attention during the seventies of the last century. Later on, several investigations and important monographs appeared with more details, which show the importance of studying such systems, see for example [4–11]. In recent years, many investigations have arisen with applications to life sciences, such that the periodic treatment of some biomedical applications, where the impulses correspond to administration of a drug treatment at certain given times [12–15]. However, comparatively speaking, not much has been done in the study of impulsive functional differential equations in regulated vectorial space, taking into account the general theory of functional analysis and having an acceptable hypothesis that can be used in real life applications, see [12] for more details.

Let us first introduce for each $\tau > 0$, the regulated Banach space $\mathcal{R} = \mathcal{R}([-\tau, 0], \mathbb{R}^n)$, given by:

$$\mathcal{R} = \left\{ \ \varphi : [-\tau, 0] \to \mathbb{R}^n : \ \varphi \text{ has left and right limits at every points of } [-\tau, 0] \ \right\},$$

endowed with the following norm

$$\|\varphi\|_{\mathcal{R}} = \sup_{\theta \in [-\tau, 0]} \|\varphi(\theta)\|.$$

We will make the following assumptions

(I) The map $f : [0, \omega + \tau] \times \mathbb{R}^n \to \mathbb{R}^n$, $\omega > 0$, satisfies

- $\|f(t, \varphi) - f(t, \psi)\|_{\mathcal{R}} \leq K\|\varphi - \psi\|_{\mathcal{R}}, \forall t \in [0, \omega + \tau], \varphi, \psi \in \mathcal{R},$
- $\exists M > 0, \|f(t, 0)\|_{\mathcal{R}} \leq M, \forall t \in [0, \omega + \tau].$

(II) For each regulated map $x : [a, b] \to \mathbb{R}^n$, with $b - a > \tau$, we assume that the map $t \to f(t, x_t)$ is measurable over $[a + \tau, b]$.

(III) For each $j = 1, \dots, l$, $h_j : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map.

We set the initial value problem as follows

**Problem 1.** *Let $\varphi$ be an element of $\mathcal{R}$. We want to find a function $x$ defined on $[-\tau, \omega + \tau]$ such that $x$ satisfies (1)–(3).*

We consider the nonlinear impulsive delay differential equation in $\mathcal{R}$ as

$$\begin{cases} \dfrac{dx(t)}{dt} &= -\lambda x(t) + f(t, x_t), a.e.\ t \in [0, \omega + \tau], \lambda > 0, \omega > 0, \\ x(t_j) &= x(t_j^-), \text{ and } x(t_j^+) - x(t_j) = h_j(x(t_j)), \forall j = 1, \dots, l, \\ x_0(\theta) &= \varphi(\theta), \theta \in [-\tau, 0] \text{ and } x(0^+) = \xi \in \mathbb{R}^n. \end{cases}$$

The aim of this paper is to extend the main results related to the existence of the $\omega$-periodic solutions for ordinary differential equations with impulses presented by Li et al. [16] and Nieto [17]. These papers contain references which provide additional reading on this topic, i.e., differential equations with impulses by using the fixed point theory.

## 2. Existence and Uniqueness of Solution

Let us start first by introducing some related definitions and lemmas.

**Definition 1.** *A function $x : [-\tau, \omega + \tau] \to \mathbb{R}^n$ is called a solution of (1)–(3) if:*

1. *$x$ is absolutely continuous with respect to the Lebesgue measure;*
2. *$x$ is differentiable on the complement of a countable subset of $[0, \omega + \tau]$, and satisfies Equation (1) whenever $\dfrac{dx(t)}{dt}$ and the right hand side of (1) are defined on $[0, \omega + \tau]$;*
3. *$x$ satisfies (2) at each point $t_j, t_j \geq 0, \forall j = 1, \dots, l$, and the initial value function satisfies (3).*

**Lemma 1.** *Let $f : [0, \omega + \tau] \times \mathcal{R} \to \mathbb{R}^n$ be a map satisfying (I) and (II) and $t_1 \in [0, \omega + \tau]$. Then, for each $(\varphi, \xi) \in \mathcal{R} \times \mathbb{R}^n$, the problem*

$$\frac{dx(t)}{dt} = -\lambda x(t) + f(t, x_t), a.e.\ t \in [0, t_1] \tag{4}$$

$$(x_0, x(0^+)) = (\varphi, \xi) \in \mathcal{R} \times \mathbb{R}^n, \tag{5}$$

*has a unique solution.*

**Proof.** We set $S = \{y \in C([0,t_1],\mathbb{R}^n), y(0) = x(0^+) = \xi\}$. Let us define the operator $T$ by

$$T(x)(t) \quad = \quad \xi + \int_0^t \Big(f(s,x_s) - \lambda x(s)\Big)ds, 0 \le t \le t_1. \tag{6}$$

For each $y \in S$, we consider the Nemytski operator $F$, defined by

$$F(y)(t) \quad = \quad f(t,z_t), \tag{7}$$

where

$$z_t(\theta) \quad = \begin{cases} y(t+\theta), & \text{if } t+\theta \ge 0, \\ \varphi(t+\theta), & \text{if } t+\theta \le 0. \end{cases} \tag{8}$$

Then, we get

$$T(y)(t) \quad = \quad \xi + \int_0^t \Big(F(y)(s) - \lambda y(s)\Big)ds. \tag{9}$$

Define, the norm of any function $y$ in $S$ by

$$\|y\|_S \quad = \quad \sup_{0\le t\le t_1} \Big\{\|y(t)\|e^{-\rho t}\Big\}, \tag{10}$$

where $\rho$ is a fixed positive constant greater than $K+\lambda$. We have for each $y_1(t)$ and $y_2(t)$ in $S$,

$$\begin{aligned}
\| T(y_1)(t) - T(y_2)(t) \| &\le (K+\lambda)\int_0^t \| y_1(s) - y_2(s) \| \, ds, \\
&\le (K+\lambda)\int_0^t e^{\rho s - \rho s} \| y_1(s) - y_2(s) \| \, ds, \\
&\le (K+\lambda) \| y_1 - y_2 \|_S \int_0^t e^{\rho s} ds, \\
&\le \frac{(K+\lambda)}{\rho} \| y_1 - y_2 \|_S \, e^{\rho t},
\end{aligned}$$

and hence

$$\| T(y_1) - T(y_2) \|_S \quad \le \quad \frac{(K+\lambda)}{\rho} \| y_1 - y_2 \|_S \, .$$

Since $\frac{K+\lambda}{\rho} < 1$, then, $T$ is a contraction on $S$, and the result follows immediately. □

**Lemma 2.** *[18] Let $f : [0,\omega+\tau] \times \mathcal{R} \to \mathbb{R}^n$ be a map satisfying (I) and (II) and $h_j$, for $j = 1, \cdots, l$, satisfy the condition (III). Then the problem (1)–(3) has a unique solution.*

**Proof.** The proof follows by using the last lemma. □

**Lemma 3.** *[18] Under the assumptions (I) and (II), if $x(\varphi)(t)$ is the unique solution of (4) and (5), then one has:*

$$\| x(\varphi)(t) \| \quad \le \quad e^{Kt}\Big( \| \varphi \| + \int_0^{\omega} \| f(s,0) \| \, ds\Big). \tag{11}$$

The next Lemma, gives a similar, key representation formula for the solutions of the delay differential equations with impulses (1)–(3) in regulated Banach space $\mathcal{R}$, see [4] for more details.

**Lemma 4.** *The problem (1)–(3) can be written as*

$$\begin{aligned} x_t &= \varphi_t^0 + H_t^0 \otimes ((\xi e^{-\lambda \max(0,\bullet)})_t - \varphi(0)) \\ &\quad + \Big( \int_0^{\max(0,\bullet)} f(s,x_s) e^{-\lambda(\bullet - s)} ds + \sum_{0 \le t_j < \bullet} e^{-\lambda(\bullet - t_j)} u_j \Big)_t, \end{aligned}$$

*where*

$$\varphi^0(\theta) = \begin{cases} \varphi(\theta), & \text{if } \theta \le 0, \\ \varphi(0), & \text{if } \theta > 0, \end{cases} \tag{12}$$

*$H^0$ is the Heaviside function*

$$H^0(\theta) = \begin{cases} 0, & \text{if } \theta \le 0, \\ 1, & \text{if } \theta > 0, \end{cases} \tag{13}$$

*and the sequence*

$$u_k = x(t_k^+) - x(t_k), k \ge 1$$

*is determined by the following non-autonomous recurrence equation*

$$u_k = h_k\Big(\xi e^{-\lambda t_k} + \int_0^{t_k} f(s,x_s) e^{-\lambda(t_k - s)} ds + \sum_{0 \le t_j < t_k} e^{-\lambda(t_k - t_j)} u_j\Big), k \ge 1,$$

*starting from*

$$u_1 = h_1\Big(\xi e^{-\lambda t_1} + \int_0^{t_1} f(s,x_s) e^{-\lambda(t_1 - s)} ds\Big).$$

**Proof.** Let us consider $z(t) = e^{\lambda t} x(t), \forall t \in [0, \omega + \tau]$, then the problem (1)–(3) becomes

$$\frac{dz(t)}{dt} = f(t, e^{-\lambda(t+\theta)} z_t) e^{\lambda t}, a.e.\ t \in [0, \omega + \tau], \lambda > 0, \omega > 0, \tag{14}$$

$$z(t_j) = z(t_j^-), \text{ and } z(t_j^+) - z(t_j) = e^{\lambda t_j} h_j(e^{-\lambda t_j} z(t_j)), \forall j = 1, \dots, l, \tag{15}$$

$$z_0(\theta) = e^{\lambda \theta} \varphi(\theta) = \widetilde{\varphi}(\theta), \theta \in [-\tau, 0], \text{ and } z(0^+) = \xi \in \mathbb{R}^n. \tag{16}$$

Let us consider $t \in [t_j, t_{j+1}), j = 1, \dots, l-1$, with $t_0 = 0$, then we get

$$z(t) = z(t_j^+) + \int_{t_j}^t f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds.$$

By passing to the limit as $t$ goes to $t_j^-$, and by solving the delay differential Equation (14) on the interval $[t_{j-1}, t_j)$, we have

$$z(t_j) = z(t_{j-1}^+) + \int_{t_{j-1}}^{t_j} f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds.$$

Then, by taking into account the impulses condition (15), we have

$$z(t) = z(t_{j-1}^+) + \int_{t_{j-1}}^t f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds + e^{\lambda t_j} h_j(e^{-\lambda t_j} z(t_j)),$$

for all $t \in [t_j, t_{j+1})$, for $j = 1, \cdots, l-1$. Consequently, we can rewrite the last equations in more general form for all $t > 0$

$$z(t) = \xi + \int_0^t f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds + \sum_{0 \le t_j < t} e^{\lambda t_j} u_j, t \notin \{t_k\}_{k \ge 1}, \tag{17}$$

where $z(0^+) = x(0^+) = \xi$, and

$$u_k = z(t_k^+) - z(t_k) = h_k(e^{-\lambda t_k} z(t_k)), k \ge 1. \tag{18}$$

Now, we will try to involve the $u_k's$. To this end, we will take the limit from the left of the Formula (17) as $t$ tends to $t_k > 0$, we obtain

$$z(t_k) = \xi + \int_0^{t_k} f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds + \sum_{0 \le t_j < t_k} e^{\lambda t_j} u_j.$$

Substituting the last expression into (18), we have

$$u_k = h_k\Big(e^{-\lambda t_k}\xi + \int_0^{t_k} f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda(s-t_k)} ds + \sum_{0 \le t_j < t_k} e^{\lambda(t_j - t_k)} u_j\Big).$$

In particular, we have $\{j : 0 \le t_j < t_1\} = \emptyset$, and therefore

$$u_1 = h_1\Big(e^{-\lambda t_1}\xi + \int_0^{t_1} f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds\Big).$$

By using, the Equation (16), we can rewrite the Equation (17) as

$$\begin{aligned} z_t(\theta) &= \xi + \int_0^{t+\theta} f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds \\ &\quad + \sum_{0 \le t_j < t+\theta} e^{\lambda t_j} u_j, t+\theta \notin \{t_k\}_{k \ge 0}, \text{ and } t+\theta \ge 0, \end{aligned} \tag{19}$$

and by using $x(t) = e^{-\lambda t} z(t)$, we have for $t + \theta \notin \{t_k\}_{k \ge 1}$, and $t + \theta \ge 0$

$$x_t(\varphi(\theta)) = \xi e^{-\lambda(t+\theta)} + \int_0^{t+\theta} f(s, x_s) e^{-\lambda(t+\theta-s)} ds + \sum_{0 \le t_j < t+\theta} e^{-\lambda(t+\theta-t_j)} u_j.$$

Using (12) and (13), we get

$$\begin{aligned} x_t(\varphi) &= \varphi_t^0 + H_t^0 \otimes ((\xi e^{-\lambda \max(0,\bullet)})_t - \varphi(0)) \\ &\quad + \Big(\int_0^{\max(0,\bullet)} f(s, x_s) e^{-\lambda(\bullet - s)} ds + \sum_{0 \le t_j < \bullet} e^{-\lambda(\bullet - t_j)} u_j\Big)_t, \end{aligned}$$

where

$$u_k = h_k\Big(\xi e^{-\lambda t_k} + \int_0^{t_k} f(s, x_s) e^{-\lambda(t_k - s)} ds + \sum_{0 \le t_j < t_k} e^{-\lambda(t_k - t_j)} u_j\Big), k \ge 1.$$

starting from

$$u_1 = h_1\Big(\xi e^{-\lambda t_1} + \int_0^{t_1} f(s, x_s) e^{-\lambda(t_1 - s)} ds\Big).$$

□

**Remark 1.** *Taking into account the conditions (II)–(III), we have* $u_t \in \mathcal{R}$*,* $\forall t \in [0, \omega + \tau]$*, and* $t \to x_t$ *is a regulated function, because the functions* $t \to \varphi_t^0$*, and* $t \to H_t^0$ *are regulated.*

In the next section, we will investigate the existence of the periodic solution(s) for the delay differential equation with impulses (1)–(3) using Schäffer's fixed point theorem [19].

## 3. Existence of Periodic Solutions

Let us consider the Poincaré operator, given by:

$$\begin{array}{rcl} J : \mathcal{R} & \to & \mathcal{R} \\ \varphi & \to & x_\omega(\varphi), \end{array}$$

where $x_\omega(\varphi)$ is the solution of the delay differential equation with impulses (1)–(3). It is clear that if the Poincaré operator $J$ admit a fixed point, then (1)–(3) has a $\omega$-periodic solution. The following lemma is useful to prove the main theorem.

**Lemma 5.** *The problem (1)–(3) has a* $\omega$*-periodic solution in* $\mathcal{R}$ *if and only if the integral equation*

$$x_t(\varphi)(\theta) = \begin{cases} e^{-\lambda\theta} \int_{t+\theta}^{t+\omega+\theta} G(t,s) f(s, x_s) ds + e^{-\lambda\theta} \sum_{t+\theta \le t_j < t+\omega+\theta} G(t,t_j) u_j, & \text{if } 0 \le t+\theta \le \omega, \\ \varphi(t+\theta), & \text{if } -\tau \le t+\theta \le 0, \end{cases}$$

*has a solution* $\forall t \in [0, \omega + \tau]$ *and* $\omega \ge \tau$*, where*

$$G(t,s) = \frac{e^{-\lambda(t-s)}}{e^{\lambda\omega} - 1}, \tag{20}$$

*and the sequence*

$$u_k = x(t_k^+) - x(t_k), k \ge 1$$

*is determined by the following non-autonomous recurrence equation*

$$u_k = h_k\Big( \int_{t_k}^{t_k+\omega} G(t,s) f(s, x_s) ds + \sum_{t_k \le t_j < t_k+\omega} G(t,t_j) u_j \Big), k \ge 1,$$

*starting from*

$$u_1 = h_1\Big( \xi e^{-\lambda t_1} + \int_0^{t_1} f(s, x_s) e^{-\lambda(t_1 - s)} ds \Big).$$

**Proof.** Using the expression (19) for $t + \omega + \theta$, where $t \ge 0$, and $\omega \ge \tau$, we have for all $t + \theta \ge 0$

$$\begin{aligned} z_{t+\omega}(\theta) &= \xi + \int_0^{t+\omega+\theta} f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds + \sum_{0 \le t_j < t+\omega+\theta} e^{\lambda t_j} u_j, \\ &= \xi + \int_0^{t+\theta} f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds + \sum_{0 \le t_j < t+\theta} e^{\lambda t_j} u_j \\ &\quad + \int_{t+\theta}^{t+\omega+\theta} f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds + \sum_{t+\theta \le t_j < t+\omega+\theta} e^{\lambda t_j} u_j, \\ &= z_t(\theta) + \int_{t+\theta}^{t+\omega+\theta} f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds + \sum_{t+\theta \le t_j < t+\omega+\theta} e^{\lambda t_j} u_j, \end{aligned}$$

and, by using the $\omega$-periodic condition $z_{t+\omega}(\theta) = e^{\lambda\omega} z_t(\theta)$, we get

$$z_t(\theta) = \frac{1}{e^{\lambda\omega}-1}\int_{t+\theta}^{t+\omega+\theta} f(s, e^{-\lambda(s+\theta)} z_s) e^{\lambda s} ds + \frac{1}{e^{\lambda\omega}-1}\sum_{t+\theta\le t_j < t+\omega+\theta} e^{\lambda t_j} u_j.$$

Therefore, using $z_t(\theta) = e^{\lambda(t+\theta)} x_t(\theta)$, we have

$$x_t(\theta) = e^{-\lambda\theta}\int_{t+\theta}^{t+\omega+\theta} G(t,s) f(s, x_s) ds + e^{-\lambda\theta}\sum_{t+\theta\le t_j < t+\omega+\theta} G(t, t_j) u_j,$$

where

$$G(t,s) = \frac{e^{-\lambda(t-s)}}{e^{\lambda\omega}-1}. \tag{21}$$

Then

$$u_k = h_k\Big(\int_{t_k}^{t_k+\omega} G(t,s) f(s,x_s) ds + \sum_{t_k\le t_j < t_k+\omega} G(t,t_j) u_j\Big), k \ge 1,$$

starting from

$$u_1 = h_1\Big(\xi e^{-\lambda t_1} + \int_0^{t_1} f(s, x_s) e^{-\lambda(t_1 - s)} ds\Big).$$

□

**Example 1.** *Let us consider the scalar delay differential equation with impulses:*

$$\frac{dx(t)}{dt} = -\lambda x(t) + f(t, x(t-\tau)), a.e.\ t \in [0, 2\tau], \tag{22}$$

$$x(\tau) = x(\tau^-),\ and\ x(\tau^+) - x(\tau) = c x(\tau), \tag{23}$$

$$x(\theta) = \varphi(\theta), \theta \in [-\tau, 0], \tag{24}$$

*where $f : [0, 2\tau] \times \mathcal{R} \to \mathbb{R}^n$ is a map satisfying (II). Let us investigate the existence of the $\tau$-periodic solution of (22)–(24) such that $x_{t+\tau}(-\tau) = x_t(-\tau), \tau \le t \le 2\tau$. The solution of the delay differential Equations (22)–(24), can be written as*

$$x(t) = \begin{cases} \varphi(\theta), & if -\tau \le t \le 0, \\ \varphi(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} f(s, x(s-\tau)) ds, & if\, 0 \le t \le \tau, \\ x(\tau^+) e^{-\lambda(t-\tau)} + \int_\tau^t e^{-\lambda(t-s)} f(s, x(s-\tau)) ds, & if\, \tau < t \le 2\tau. \end{cases} \tag{25}$$

*Using (23), we get*

$$\begin{aligned} x(\tau^+) &= x(\tau) + c x(\tau), \\ &= (c+1) x(0) e^{-\lambda\tau} + (c+1)\int_0^\tau e^{-\lambda(\tau-s)} f(s, x(s-\tau)) ds. \end{aligned}$$

*Therefore, if $\tau \le t < 2\tau$, we have*

$$\begin{aligned} x(t) &= \Big((c+1)\varphi(0)e^{-\lambda\tau} + (c+1)\int_0^\tau e^{-\lambda(\tau-s)}f(s,x(s-\tau))ds\Big)e^{-\lambda(t-\tau)} \\ &\quad + \int_\tau^t e^{-\lambda(t-s)}f(s,x(s-\tau))ds, \\ &= (c+1)\varphi(0)e^{-\lambda\tau}e^{-\lambda(t-\tau)} + (c+1)e^{-\lambda\tau}\int_0^{t-\tau} e^{-\lambda(t-\tau-s)}f(s,x(s-\tau))ds \\ &\quad +(c+1)e^{-\lambda\tau}\int_{t-\tau}^{\tau} e^{-\lambda(t-\tau-s)}f(s,x(s-\tau))ds + \int_\tau^t e^{-\lambda(t-s)}f(s,x(s-\tau))ds, \\ &= (c+1)e^{-\lambda\tau}\Big(x(t-\tau) + \int_{t-\tau}^{\tau} e^{-\lambda(t-\tau-s)}f(s,x(s-\tau))ds\Big) + \int_\tau^t e^{-\lambda(t-s)}f(s,x(s-\tau))ds, \end{aligned}$$

*which implies*

$$\begin{aligned} x_{t+\tau}(-\tau) &= (c+1)e^{-\lambda\tau}\Big(x_t(-\tau) + \int_{t-\tau}^{\tau} e^{-\lambda(t-\tau-s)}f(s,x(s-\tau))ds\Big) \\ &\quad + \int_\tau^t e^{-\lambda(t-s)}f(s,x(s-\tau))ds. \end{aligned}$$

*Then, we have three cases.*

***(1)*** *If* $1-(c+1)e^{-\lambda\tau} \neq 0$*, then, we have the existence and uniqueness of a* $\tau$*-periodic solution.*

***(2)*** *If* $1-(c+1)e^{-\lambda\tau} = 0$*, and*

$$\int_{t-\tau}^{\tau} e^{-\lambda(t-\tau-s)}f(s,x(s-\tau))ds + \int_\tau^t e^{-\lambda(t-s)}f(s,x(s-\tau))ds = 0,$$

*then, we have the existence of infinitely many* $\tau$*-periodic solutions.*

***(3)*** *If* $1-(c+1)e^{-\lambda\tau} = 0$*, and*

$$\int_{t-\tau}^{\tau} e^{-\lambda(t-\tau-s)}f(s,x(s-\tau))ds + \int_\tau^t e^{-\lambda(t-s)}f(s,x(s-\tau))ds \neq 0,$$

*then, there exists no* $\tau$*-periodic solution.*

Now, we can consider for each $t \geq -\tau$ and $\omega \geq \tau$, the Poincaré operator $J : \mathcal{R} \to \mathcal{R}$ defined by

$$J\varphi = \Big(e^{-\lambda(\bullet-t)}\int_\bullet^{\bullet+\omega} G(t,s)f(s,\varphi)ds + e^{-\lambda(\bullet-t)}\sum_{\bullet\leq t_j<\bullet+\omega} G(t,t_j)u_j\Big)_t,$$

where

$$u_k = h_k\Big(\int_{t_k}^{t_k+\omega} G(t,s)f(s,x_s)ds + \sum_{t_k\leq t_j<t_k+\omega} G(t,t_j)u_j\Big), k\geq 2,$$

and, starting from

$$u_1 = h_1\Big(\xi e^{-\lambda t_1} + \int_0^{t_1} f(s,x_s)e^{-\lambda(t_1-s)}ds\Big).$$

It is clear, that, the $\omega$-periodic solutions in $\mathcal{R}$ of (1)–(3) are exactly the fixed points of the Poincaré operator $J$, i.e., $J\varphi = \varphi$.

The following theorem, is known as the Schäffer's fixed point theorem [19], which can be found for example in Deimling's book [20].

**Theorem 1.** *[19–22] Let $X$ be a normed space, $\mathcal{F}$ a continuous mapping of $X$ into $X$, such that the closure of $\mathcal{F}(B)$ is compact for any bounded subset $B$ of $X$. Then either:*

*(i)* *the equation $x = \lambda \mathcal{F} x$ has a solution for $\lambda = 1$, or*
*(ii)* *the set of all such solutions $x$, for $0 < \lambda < 1$, is unbounded.*

Before, we state the main theorem of our work, we will need the following lemma.

**Lemma 6.** *Let $f : [0, \omega + \tau] \times \mathcal{R} \to \mathbb{R}^n$ be a map satisfying (I) and (II), where $\omega \geq \tau$, and $h_j, j = 1, \dots, l$ are bounded and satisfy the condition (III). Then, the Poincaré operator $J : \mathcal{R} \to \mathcal{R}$ is completely continuous.*

**Proof.** Let $B \subset \mathcal{R}$ be a bounded set and $\varphi \in B$. Then by using the condition (I), we have

$$\|f(t,\varphi)\|_{\mathcal{R}} \leq \|f(t,0)\|_{\mathcal{R}} + \|f(t,\varphi) - f(t,0)\|_{\mathcal{R}} \leq M + K\|\varphi\|_{\mathcal{R}} < \infty.$$

Therefore, there exist two constants $\widetilde{M}$ and $\overline{M}$ such that

$$\begin{aligned} \parallel J\varphi(\theta) \parallel &= \parallel e^{-\lambda\theta} \int_{t+\theta}^{t+\theta+\omega} f(s,\varphi) G(t,s) ds + e^{-\lambda\theta} \sum_{t+\theta \leq t_j < t+\theta+\omega} G(t,t_j) u_j \parallel, \\ &\leq \parallel e^{-\lambda\theta} \int_{t+\theta}^{t+\theta+\omega} f(s,\varphi) G(t,s) ds \parallel + e^{\lambda r} \parallel \sum_{t+\theta \leq t_j < t+\theta+\omega} G(t,t_j) u_j \parallel, \\ &\leq e^{\lambda\tau} \omega \widetilde{M} + e^{\lambda r} \sum_{t+\theta \leq t_j < t+\theta+\omega} \overline{M}, \end{aligned} \tag{26}$$

where

$$\|u_k\| = \left\| h_k \Big( \int_{t_k}^{t_k+\omega} G(t,s) f(s,x_s) ds + \sum_{t_k \leq t_j < t_k+\omega} G(t,t_j) u_j \Big) \right\| < \infty, k \geq 2,$$

and starting from

$$\|u_1\| = \left\| h_1 \Big( \xi e^{-\lambda t_1} + \int_0^{t_1} f(s,x_s) e^{-\lambda(t_1 - s)} ds \Big) \right\| < \infty,$$

and, we have

$$\parallel J\varphi \parallel_{\mathcal{R}} \leq e^{\lambda\tau} \omega \widetilde{M} + e^{\lambda r} \overline{M} \sum_{t+\theta \leq t_j < t+\theta+\omega} 1,$$

which imply that $J(B)$ is uniformly bounded. For each $t \geq 0$, there exists $n \in \mathbb{N}^*$ such that $t \in [t_n, t_{n+1})$, and for any $\theta, \widetilde{\theta} \in [-r, 0]$, one can obtain for any $\varphi \in B$

$$\begin{aligned} \parallel J\varphi(\theta) - J\varphi(\widetilde{\theta}) \parallel &\leq \parallel e^{-\lambda\theta} \int_{t+\theta}^{t+\theta+\omega} f(s,\varphi) G(t,s) ds - e^{-\lambda\theta} \int_{t+\widetilde{\theta}}^{t+\widetilde{\theta}+\omega} f(s,\varphi) G(t,s) ds \parallel \\ &\quad + \parallel e^{-\lambda\theta} \sum_{t+\theta \leq t_j < t+\theta+\omega} G(t,t_j) u_j - e^{-\lambda\theta} \sum_{t+\widetilde{\theta} \leq t_j < t+\widetilde{\theta}+\omega} G(t,t_j) u_j \parallel, \\ &\leq \frac{e^{\lambda(r+\omega)} \Big( M + K\|\varphi\| \Big)}{1 - e^{-\lambda\omega}} \parallel \int_{t+\theta}^{t+\theta+\omega} e^{-\lambda(t-s)} ds - \int_{t+\widetilde{\theta}}^{t+\widetilde{\theta}+\omega} e^{-\lambda(t-s)} ds \parallel \\ &\quad + \frac{e^{\lambda(r+\omega)}}{1 - e^{-\lambda\omega}} \parallel \sum_{t+\theta \leq t_j < t+\theta+\omega} u_j - \sum_{t+\widetilde{\theta} \leq t_j < t+\widetilde{\theta}+\omega} u_j \parallel . \end{aligned}$$

Therefore, for each $t \in [t_n, t_{n+1})$, we will have as $\mid \theta - \widetilde{\theta} \mid$ goes to 0, $\| J\varphi(\theta) - J\varphi(\widetilde{\theta}) \|$ goes to 0, which imply that the Poincaré operator $J(B)$ is equicontinuous. Using Arzelà-Ascoli's theorem, we conclude that the Poincaré operator $J$ is completely continuous. □

Now, we are ready to state the main result of our work, related to the existence of $\omega$-periodic solution(s) of (1)–(3).

**Theorem 2.** *Let $f : [0, \omega + \tau] \times \mathcal{R} \to \mathbb{R}^n$ be a map satisfying (I) and (II), where $\omega \geq \tau$, and $h_j, j = 1, \ldots, l$ are bounded and satisfy the condition (III). Then, the nonlinear impulsive problem (1)–(3), has at least one $\omega$-periodic solution in $\mathcal{R}$.*

**Proof.** Let us define $H(\varphi, \mu) : \mathcal{R} \times [0, 1] \longrightarrow \mathcal{R}$ by

$$H(\varphi, \mu) \quad = \quad \mu J\varphi. \tag{27}$$

Then, by using (26), we have

$$\|H(\varphi, \mu)\|_{\mathcal{R}} \quad \leq \quad \mu\Big(e^{\lambda\tau}\omega\widetilde{M} + e^{\lambda r} \sum_{t+\theta \leq t_j < t+\theta+\omega} \overline{M}\Big).$$

Then, for each $\mu \in (0, 1)$ the set $S = \{\varphi : \varphi = H(\varphi, \mu)\}$ is bounded. Since $J$ is completely continuous, then by using Schäffer's fixed point theorem, the Poincaré operator $J$ admits a fixed point. □

Next, we give the conditions of the existence and uniqueness of a $\omega$-periodic solution of (1)–(3).

**Theorem 3.** *Let $f : [0, \omega + \tau] \times \mathcal{R} \to \mathbb{R}^n$ be a map satisfying (I) and (II), where $\omega \geq \tau$, and $h_j, j = 1, \ldots, l$ are bounded and satisfy the condition (III), and there exist constants $\overline{H}_j$, $j = 1, \ldots, l$, such that*

$$\| h_j(\varphi(0)) - h_j(\psi(0)) \| \quad \leq \quad \overline{H}_j \| \varphi - \psi \|_{\mathcal{R}} \, .$$

*If, there exists a constant $C < 1$, such that*

$$\frac{K\omega e^{\lambda r}}{1 - e^{-\lambda\omega}} + \frac{e^{\lambda r}}{1 - e^{-\lambda\omega}} \sum_{t-r+\omega \leq t_j < t+\omega} \overline{H}_j \quad \leq \quad C,$$

*then, the nonlinear impulsive problem (1)–(3), has a unique $\omega$-periodic solution in $\mathcal{R}$.*

**Proof.** Let $\varphi, \psi \in \mathcal{R}$ be two solutions of (1)–(3), i.e., $J\varphi = \varphi$ and $J\psi = \psi$. Assume $\phi \neq \psi$. We have

$$\begin{aligned} \| \varphi(\theta) - \psi(\theta) \| \quad &= \quad \| J\varphi(\theta) - J\psi(\theta) \|, \\ &\leq \quad e^{\lambda r} \int_{t+\theta}^{t+\theta+\omega} \mid G(t,s) \mid\| f(s,\varphi) - f(s,\psi) \|_{\mathcal{R}} \, ds + \\ &\quad\;\; e^{\lambda r} \sum_{t-r+\omega \leq t_j < t+\omega} \mid G(t,t_j) \mid\| h_j(\varphi(0) - h_j(\psi(0)) \|, \\ &\leq \quad \Big(\frac{K\omega e^{\lambda r}}{1 - e^{-\lambda\omega}} + \frac{e^{\lambda r}}{1 - e^{-\lambda\omega}} \sum_{t-r+\omega \leq t_j < t+\omega} \overline{H}_j\Big) \| \varphi - \psi \|_{\mathcal{R}}, \\ &\leq \quad C \| \varphi - \psi \|_{\mathcal{R}} \, . \end{aligned}$$

Hence

$$\begin{aligned} \| \varphi - \psi \|_{\mathcal{R}} &\leq C \| \varphi - \psi \|_{\mathcal{R}}, \\ &< \| \varphi - \psi \|_{\mathcal{R}} . \end{aligned} \tag{28}$$

This contradiction implies, the uniqueness of the $\omega$-periodic solution of (1)–(3). □

## 4. Conclusions

The method described in this work presents new challenges for more investigation on more realistic models; such as the extension of the ascorbic acid model [12] and $HIV$ model [13,14]. Taking into account the delay effect on respective compartments [23–25]. This kind of work, will need more investigation on modeling validation effort, keeping a close eye on the real life data in order to have a more realistic model. The explicit solutions presented in the technical Lemma 4 and methods of proving the existence of periodic solutions are very useful for further future investigations.

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# Collaborative Content Downloading in VANETs with Fuzzy Comprehensive Evaluation

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**Abstract:** Vehicle collaborative content downloading has become a hotspot in current vehicular ad-hoc network (VANET) research. However, in reality, the highly dynamic nature of VANET makes users lose resources easily, and the transmission of invalid segment data also wastes valuable bandwidth and storage of the users' vehicles. In addition, the individual need of each customer vehicle should also be taken into consideration when selecting an agent vehicle for downloading. In this paper, a novel scheme is proposed for vehicle selection in the download of cooperative content from the Internet, by considering the basic evaluation information of the vehicle. To maximize the overall throughput of the system, a collaborative content downloading algorithm is proposed, which is based on fuzzy evaluation and a customer's own expectations, in order to solve the problems of agent vehicle selection. With the premise of ensuring successful downloading and the selection preferences of customer vehicles, linear programming is used to optimize the distribution of agent vehicles and maximize customer's satisfaction. Simulation results show that the proposed scheme works well in terms of average quality of service, average bandwidth efficiency, failure frequency, and average consumption.

**Keywords:** vehicle collaborative content downloading; fuzzy comprehensive evaluation; VANET

## 1. Introduction

With the rapid development of the network, the demand of the network extends to all aspects of people's lives. As a platform, which provides a specific network service, the vehicular ad-hoc network (VANET) brings new technical challenges to the transmission of information while providing information services, including: how to improve the efficiency of the vehicle network and how to meet the continuous improvement of the users' needs [1,2].

Scholars have discussed different ways to modify VANET, in order to improve the performance of vehicle networking and meet the growing demand of users. In terms of enhancing the performance of the vehicle-to-infrastructure (V2I) connection in VANET, the possibility of constructing a network with the TV white space geolocation database for vehicle networking was discussed by some scholars. Then vehicular communication architectures were proposed to mitigate the resulting high spectrum demands and provide vehicular connectivity with wider communication range, higher transmission rate, and lower data transfer cost [3,4]. By analyzing the end-to-end transmission performance from individual vehicles to a road side unit (RSU), an efficient message routing scheme was put forward to balance the data traffic across the network and improve the network throughput [5]. In Reference [6], a collaborative download algorithm, namely maximum throughput and minimum delay collaborative download (MMCD) was proposed, which minimizes the average transmission delay of each user's request and maximizes the number of packets downloaded from an RSU. Reference [7] mainly studies the cost of minimizing the download of a hybrid vehicle ad hoc network, and proposes the basic satisfaction algorithm (BMA) and heuristic algorithm (TSA) to solve the huge download delay caused by vehicle mobility in VANET. In order to solve the frequent collision among agent vehicles and customer vehicles, a transmission scheduling method was put forward to adjust the relationship between link routing and transmission time [8].

In terms of enhancing the vehicle-to-vehicle (V2V) collaborative download performance in VANET: ECDS gives an efficient collaborative downloading solution to popular content distribution in urban vehicle networks. Furthermore, a cross-domain relay selection strategy was proposed to build a peer-to-peer (P2P) network, which helps strengthen information dissemination [9]. In Reference [10], to solve the problems of popular content distribution (PCD) in a highway scene, the author modeled the problem as a coalition formation game with transferable utilities, and proposed a coalition formation algorithm that converges into a Nash-stable partition, adapting to environmental changes as a result of the VANET's rapid and unpredictable topological changes. In Reference [11], the design incentive mechanism is employed to propose a collaborative downloading method, which encourages cooperation between vehicles and helps users effectively obtain the required resources. The author designed a server-assisted key management scheme that promotes cooperation and ensures fairness and efficiency. In the scheme, vehicles with common interests form a cluster and take turns as the cluster head, which directly downloads data packets from the Internet and V2V shares the content with surrounding vehicles [12]. A delicate linear cluster formation scheme is proposed and applied to significantly enhance the probability of a successful file download in VANET [13]. In Reference [14], the author proposed a security incentive program (SIRC) to achieve reliable, fair, and secure collaborative downloading in VANET. SIRC stimulates vehicle users to help each other download and forward packets, encourages cooperation between users, and also punishes malicious vehicles to ensure the safety of vehicles. Efficient privacy-preserving cooperative data downloading for value-added services is used to solve the problems of limited communication range and high dynamics, which gains the access control in VANET [15].

The methods of improving the performance of the vehicle network are also discussed from other aspects. Digital fountain code (DFC) is proposed and applied in the field of cooperative downloading for VANET. As long as enough data packets encoded by hierarchical fountain code are available, the client can recover the raw data and avoid data transmission interruption [16,17]. In Reference [18], a fuzzy logic-based resource management (FLRM) scheme was proposed, and the lifetime of each storage resource was defined by the proposed fuzzy logic-based popularity evaluation algorithm.

### 1.1. Related Work

The agent vehicle selection method and vehicle distribution scheme are important links to achieve collaborative downloading. A fuzzy logic-based cooperative file transfer scheme (FL-CFT) was proposed to optimally select relays to help transfer the file and ensure the file integrity, in which the relative velocity, distance, and predicted connection time among vehicles were considered [19]. To solve the problems of the low utilization of spatiotemporal resources in DA and an unbalanced service of cooperative downloading, a balanced cooperative downloading method was proposed, which dynamically uses the Euclidean and Manhattan distances in order to select the vehicles according to the number of clients [20]. In Reference [21], a k-hop bandwidth aggregation scheme was proposed to select agent vehicles, to help download and forward videos and more effectively send video streams to requesters through DSRC VANET. In Reference [22], a preferential response incentive mechanism (PRIM) was proposed to motivate vehicles to participate in collaborative downloading, and game theory was used to analyze a vehicle's behavior in order to find the optimal strategy for each collaborator, reduce repeated downloads, and promote V2I cooperation to reduce delays and expenses. In Reference [23], a security collaboration data download framework for paid services in VANET was proposed. An application layer data sharing protocol was developed to coordinate vehicle data sharing according to its location. The seed screening scheme SIEVE was proposed in

Reference [24], using users' interest and near-term contact predictions to select the best vehicle node (vehicle) to download the object (via the cellular network) and propagate the object (via the RSU). In order to effectively characterize users' preferences and network performance, previous authors use parameters such as energy efficiency, signal intensity, network cost, delay, and bandwidth to construct utility functions. Then, these utility functions and multi-criteria utility theory are used to construct an energy-efficient network selection approach and a joint multi-criteria utility function for network selection of the appropriate access network [25].

### *1.2. Motivation and Contributions*

In fact, the goal of a cooperative downloading method is to ensure more efficient data transmission, provide balanced services, and meet the requirements of all customers on the agent vehicles, so that the customers' cooperative unloading requirements can be satisfied. Based on the ideas above, this paper proposes a vehicle selection algorithm for the vehicle network agent based on fuzzy comprehensive evaluation. This algorithm takes the basic parameters of the customer vehicle, the agent vehicle, and the relationship between them into account, and improves the average throughput and customer satisfaction under the condition of satisfying a customer vehicle's information data requests. Compared with the previous articles, the contributions of this paper are in four aspects:

- We provide a fuzzy evaluation method based on the relationship between the agent vehicle and the customer vehicle, and evaluate the agent vehicle synthetically. In our opinion, we can judge whether the vehicle is suitable for cooperation by its relevant attributes. These attributes include computing capability, bandwidth, unit cost, credibility, and path consistency between vehicles, which are meaningful data for vehicle selection. Therefore, using this information as the evaluation factor for the fuzzy comprehensive evaluation, corresponding agent vehicles for each customer vehicle are scored, and the vehicles with higher scores are selected as the priority.
- In order to satisfy the requests of more customer vehicles and maximize resource utilization, this paper proposes an agent vehicle distribution strategy based on the maximization of service quality. Our approach allocates a certain number of agent vehicle resources to each customer vehicle, and takes the bandwidth limitation of the agent vehicle into consideration, so as to select the most suitable agent vehicle for the customer vehicle and maximize overall resource utilization.
- In a simulation, the performance of the proposed algorithm is compared with other schemes. The simulation results show that the proposed algorithm can gain significant performance achievements, which demonstrates the superiority of the scheme.
- By analyzing the fuzzy relationship between multiple constraints on the target, the fuzzy comprehensive evaluation method quantifies and unifies the relationship as an index to realize vehicle selection. This method woks well in dealing with the problems of fuzzification that are constrained by many factors. Additionally, it can be used as a reference for the solutions of multi-factor constraint model problems such as mobile vehicle network selection problems, vehicle routing problems in complex environments, and so on.

The organization of this paper is as follow. Section 2 describes the system model used by this scenario. Section 3 explains, in detail, the vehicle network cooperation content downloading method, based on fuzzy comprehensive evaluation, proposed in this paper. Section 4 shows our simulation results and discussion. Finally, Section 5 summarizes the method of this paper and points out the future work.

## 2. System Model

In this section, the model is first introduced, and some parameters are defined, including the vehicle evaluation index and the format of the data packet.

### *2.1. System Model*

In the system model shown in Figure 1, the vehicles are grouped as customer vehicles and agent vehicles. Customer vehicles send download requests to the local server (RC), and agent vehicles are responsible for helping them download the requested content. The customer and agent vehicles together form a VANET.

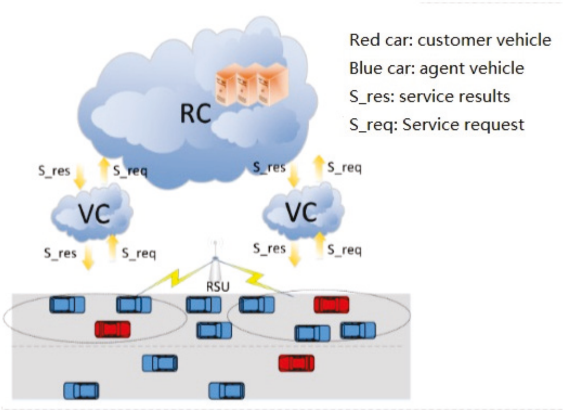

**Figure 1.** Comprehensive evaluation index system.

When passing by an RSU, a vehicle in the vehicle cloud downloads the response file (vehicle-to-infrastructure, V2I); when leaving the RSU affected area, the vehicles in the VANET share the downloaded files (vehicle-to-vehicle, V2V). The V2I and V2V process forms a circulation, and many such circulations have to be gone through to complete the download of a large file.

In this paper, we make the following assumptions:

- The customer vehicle which requests cooperation, selects an agent vehicle only once every period. When the agent vehicle is selected, its map route must be consistent with the customer vehicle.
- The local server can obtain the vehicle's navigation information (that is, the driving route of each vehicle on a map), in order to allocate an agent vehicle traveling on the same road section as the customer vehicle and reduce the waste of resources. The vehicle uploads any relevant information to the local server. The local server selects the agent vehicle for the customer vehicle according to the scheme proposed herein.
- On the basis of content consistency, the local server counts the request information of the customer vehicle and the service information of the agent vehicle. There are two forms of vehicle computing capability. The first form is collaborative computing. In this case, the computing capability of the agent vehicle is determined by the hardware of the vehicle itself, which represents the total amount of data that the agent vehicle needs to receive and send in the service. In the second form, the request of the customer vehicle is content downloading. It is set that the storage and removal of files in the vehicle are in chronological order. In this case, the computing capability of the customer vehicle is a request for the files that have not been downloaded yet, which can be part of a file or an entire file. The computing capability of the agent vehicle is the part of the file reserved in the current storage, which can be part of the file or the entire file. In this paper, the method for obtaining data from the agent vehicle will not be discussed and we assume that the agent vehicle has had the corresponding computing capability before providing services to the customer vehicle.

Failure of a customer vehicle's request will occur due to the following:

- If there is only one agent vehicle for more than one customer vehicle, the selected agent vehicle cannot serve more than one vehicle, and it will be allocated to the customer vehicle with the greatest satisfaction. The other customer vehicles' requests will fail.
- According to Algorithm 1, a request fails if the customer vehicle cannot find an agent vehicle that meets its requirements.

This paper mainly studies the content downloading through V2V in VANET, and focuses on how to choose the best cooperators for a customer vehicle. According to the relevant data of the vehicles, under the premise of satisfying the cooperation standard expected by the customer vehicle, the overall rating of the agent vehicle is maximized, so each customer obtains a satisfactory downloading experience.

### *2.2. Definitions*

In order to record the data set, the packet format for the customer vehicle (CV) and the agent vehicle (AV) are defined respectively as:

- Request package of customer vehicle:
    - CV-ID: Customer vehicle's ID;
    - CV-computing: Computing capability of customer vehicle request;
    - CV-bandwidth: Customer vehicle's bandwidth;
    - CV-path: The travel route of the customer vehicle in the process of the data request;
    - CV-position: Customer vehicle's position;
    - CV-speed: Customer vehicle's speed;
- Service package of agent vehicle:
    - AV-ID: Agent vehicle's ID;
    - AV-computing: Agent vehicle's computing capability;
    - AV-bandwidth: Agent vehicle's bandwidth;
    - AV-path: The travel route of the agent vehicle in the process of the data service;
    - AV-credit: Agent vehicle's credit;
    - AV-position: Agent vehicle's position;
    - AV-cost: Service cost of agent vehicle in unit time;
    - AV-speed: Agent vehicle's speed;
- The format of the reply message of the local server is as follow:
    - Server-ID: ID of the local server that communicates with the current vehicle;
    - Reply (N = AV-ID): if the reply message is 0, the local server finds the agent vehicle. If the reply message is a series of numbers (which are defined as positive integers), they represent the IDs of all the agent vehicles assigned to it by the local server;

Therefore, the vehicle and the local server use the information as an evaluation factor in the communication process to complete the evaluation of the vehicle. A comprehensive evaluation index system is designed, as shown in Table 1.

**Table 1.** Comprehensive evaluation index system.

| Target Layer | Factor Layer |
|---|---|
| Vehicle selection result | Computing capability<br>Bandwidth<br>Unit cost<br>Credibility<br>Path consistency |

The computing capability is determined by the customer vehicle's requirement data and agent vehicle's service data. Bandwidth is determined by the hardware properties of the vehicle. Unit cost is the remuneration to be paid per unit time when the service is provided by the agent vehicle. Credibility is the score given on cooperation in the vehicle's historical records, which is evaluated in VANET. If it can serve the customer vehicle very well every time, the score will be high; if there is a malicious termination of the cooperation, the behavioral reputation value will be correspondingly reduced. Path consistency represents the proportion of path that maintains communication between an agent and customer vehicle in the total path.

In addition, due to the mobility of the vehicle, datagrams will be updated every time period to ensure good transmission. In the next section, the vehicle selection method based on fuzzy comprehensive evaluation will be introduced in detail.

## 3. Vehicle Network Collaborative Content Downloading Method Based on Fuzzy Comprehensive Evaluation

In this section, we describe the specific method for the local server to select an agent vehicle for a customer vehicle, in detail. The fuzzy comprehensive evaluation model is also introduced to make a fuzzy comprehensive evaluation of the factors affecting the vehicle selection in agent vehicle unloading. The choice of vehicles tends to be optimal.

The detailed communication process of finding agent vehicles is as follow:

- Several customer vehicles send request packets to a local server. A request packet contains the requirements for an agent vehicle and the relevant information of the customer vehicle itself.
- After the local server receives the message, it uses the fuzzy comprehensive evaluation method proposed in this paper to analyze the request packet of the customer vehicle and the service packet of an agent vehicle. Then it forms the distribution plan of the agent vehicle for the customer vehicle, and sends a response message back to them.
- Response message. If the message is 0, it means that the local server did not find an agent vehicle and the customer vehicle needs to wait for the next assignment. If the message is a series of numbers (which are defined as positive integers, indicating the IDs of all the agent vehicles assigned by the local server), it means that the distribution of agent vehicles was successful, and the local server notifies the agent vehicle to serve the corresponding customer vehicle according to the allocation plan.
- After the entire communication is over, the local server records the evaluation of the agent vehicle, to update the credibility of the agent vehicle. A penalty mechanism is established to punish a vehicle which is rated poorly by the customer vehicle in this cooperation. A punished vehicle is unable to participate in the next cooperation and cannot obtain the expected rewards.

This section mainly evaluates objective ratings and customers' satisfaction for agent vehicles in the decision domain based on certain fuzzy constraints. Agent vehicles with higher scores in comprehensive evaluation should be given priority, while those with lower scores should be given a second thought, when selecting vehicles based on the demand.

### *3.1. Pre-Selection of Agent Vehicles*

To find an appropriate agent vehicle for the customer vehicle from a large number of vehicles, in order to meet their information requests in the process of routing, we need to establish an information selecting mechanism. In the mechanism, the relationships between a customer vehicle's and an agent vehicle's information are compared and analyzed, to meet the customer's data requests. Alternative vehicles should meet the following requirements:

- Computing capability $c$: Computing capability is the main content of requests for customer vehicles. For the agent vehicle, it decides whether it can serve the customer vehicle or not. Computing capability $c_j$ provided by the agent vehicle $j$ should be better than or equal to the computing capability $c_i$ requested by the customer vehicle $i$; so as to meet the demand of the customer vehicle:

$$c_j \geq c_i.$$

- Bandwidth $b$: Bandwidth determines the fluency of a customer vehicle's data request. The bandwidth $b_j$ provided by the agent vehicle $j$ should be better than or equal to the bandwidth $b_i$ requested by the customer vehicle $i$, so as to meet the need of the customer vehicle:

$$b_j \geq b_i.$$

- Agent vehicle $j$ should satisfy customer $i's$ requests for computing capability within the time of collaboration between the two vehicles. $L_a$ is the effective distance between the customer and agent vehicles, and if the distance between them exceeds $L_a$, then the connection will fail. $L$ is the path length. $v_i$ is the average speed of the customer vehicle. $v_j$ is the average driving speed of the agent vehicle. Thus:

$$\frac{c_i}{b_i} \leq \min\left(\frac{L_a}{\left|v_i - v_j\right|}, \frac{L}{v_i}\right).$$

- Path consistency determines the time length of the service that a customer vehicle obtains from the agent vehicle. It indicates whether the customer vehicle can get complete service from the agent vehicle or not. The path consistency $pc_{ij}$ is calculated to express the ratio of the effective signal path to the whole path when the agent vehicle provides data service to the customer vehicle:

$$pc_{ij} = \frac{L_a v_i}{\left|v_i - v_j\right| L}.$$

Based on the requirements above, we filter the agent vehicles according to Algorithm 1, and record the information of the selected agent vehicles for each customer vehicle.

**Algorithm 1** Attaining the Available Agent Vehicle List

**Input:** Customer vehicle request package; agent vehicle service package; signal effective distance $L_a$; path length $L$;
**Output:** Available agent vehicle (AV) list $N_i$ and path consistency $pc_{ij}$, for each customer vehicle $CV_i$

```
for each $CV_i$, $i \in [1, n]$ do
    for each $AV_j$, $j \in [1, m]$ do
        if $c_i \leq c_j$ & $b_i \leq b_j$ & $\frac{c_i}{b_i} \leq \min\left(\frac{L_a}{|v_i - v_j|}, \frac{L}{v_i}\right)$ then
            write $AV_j$ into the list $N_i$
            $pc_{ij} = \frac{L_a v_i}{|v_i - v_j| L}$
        end if
    end for
end for
```

## 3.2. Comprehensive Evaluation of Customer Satisfaction

### 3.2.1. The Determination of the Domain and Various Factors of Agent Vehicles:

Based on the illustration above, the factor domain of agent vehicles is recorded as: $U = \{u_1, u_2, u_3, u_4, u_5\}$ ,where $u_1$ is the computing capability; $u_2$ is the bandwidth; $u_3$ is the unit cost; $u_4$ is the credibility; $u_5$ is the path consistency;

Among them, each factor belongs to a different domain, i.e.,

$$\mu_1 \in D_1 = (\chi_1, \psi_1);$$
$$\mu_2 \in D_2 = (\chi_2, \psi_2);$$
$$\mu_3 \in D_3 = (\chi_3, \psi_3);$$
$$\mu_4 \in D_4 = (\chi_4, \psi_4);$$
$$\mu_5 \in D_5 = (\chi_5, \psi_5);$$

The local server divides the data sets of factors $u_1, u_2, u_3, u_4$, and $u_5$ into three categories: low, medium, and high. These are represented by $V_1$, $V_2$, and $V_3$, respectively, and the level domain of each factor is $V = \{V_1, V_2, V_3\}$ , which corresponds to the numerical values $\{1, 2, 3\}$, in order.

If the fuzzy experiment determines the first division of the factor $u_i$ on the domain $(\chi_i, \psi_i)$, one pair can be determined for each division:$(\xi_{u_i}, \eta_{u_i})$, where $\xi_{u_i}$ is the demarcated point between $V_{1\_u_i}$ and $V_{2\_u_i}$ and $\eta_{u_i}$ is the demarcated point between $V_{2\_u_i}$ and $V_{3\_u_i}$.

On the contrary, if $(\xi, \eta)$ is given, the mapping $e$ is also determined, and $V_{1\_u_i}, V_{2\_u_i}, V_{3\_u_i}$ are separated, thus the fuzzy concept is clarified.

The interval of $V_{1\_u_i}, V_{2\_u_i}, V_{3\_u_i}$ is a random interval, and so $\xi_{u_i}$ and $\eta_{u_i}$ are random variables. They follow characteristic normal distributions, as shown in Figure 2, namely: $\xi_{u_i} : N(\alpha_{1\_u_i}, \sigma^2_{1\_u_i})$; $\eta_{u_i} : N(\alpha_{2\_u_i}, \sigma^2_{2\_u_i})$

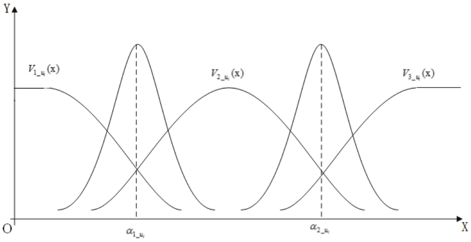

**Figure 2.** The normal distribution properties.

Based on the definition of each factor, the values $\alpha_{1\_u_i}, \alpha_{2\_u_i}, \sigma^2_{1\_u_i}$, and $\sigma^2_{2\_u_i}$ are determined.

For factor $u_1, u_2, u_4$, and $u_5$ (i.e., computing capability, bandwidth, credibility, and path consistency), the bigger they are, the better it is for the vehicle cooperative downloading. Thus they are defined as:

$$\alpha_{1\_u_i} = \omega(u_{i\_\min} + u_{i\_ave}), 0 < \omega < 1$$

$$\alpha_{2\_u_i} = \vartheta(u_{i\_\max} + u_{i\_ave}), 0 < \vartheta < 1$$

For factor $u_3$ (unit cost), the smaller the better in consideration of a user's benefits. Thus, they are defined as:

$$\alpha_{1\_u_i} = \omega(u_{i\_\max} + u_{i\_ave}), 0 < \omega < 1$$

$$\alpha_{2\_u_i} = \vartheta(u_{i\_\min} + u_{i\_ave}), 0 < \vartheta < 1$$

In order to make the distribution of demarcated points relatively centralized:

$$0 < \sigma^2_{1\_u_i} \le 1, 0 < \sigma^2_{2\_u_i} \le 1$$

$$u_{i\_\max} = \max(u_{i1}, u_{i2}, \ldots, u_{in}), u_{il} \in u_i$$

$$u_{i\_\min} = \min(u_{i1}, u_{i2}, \ldots, u_{in}), u_{il} \in u_i$$

$$u_{\text{ave}} = \frac{u_{i1}, u_{i2}, \ldots, u_{in}}{n}, u_{il} \in u_i$$

### 3.2.2. Determination of Membership Functions

The membership function of factors $u_i \in (\chi_i, \psi_i)$ with the three levels $V_{1\_u_i}, V_{2\_u_i}$, and $V_{3\_u_i}$ is determined by the three division method. The three division method is a fuzzy statistical method that determines the membership function with three levels of fuzzy concepts. The basic principles of this method are as follow.

From above, we know that the partition $(\xi_{u_i}, \eta_{u_i})$ of the factor $u_i$ on the universe $(\chi_i, \psi_i)$ obeys a normal distribution: $\xi_{u_i}$ obeys $N(\alpha_{1\_u_i}, \sigma^2_{1\_u_i})$, and $\eta_{u_i}$ obeys $N(\alpha_{2\_u_i}, \sigma^2_{2\_u_i})$.

Furthermore, the number of $(\xi, \eta)$ determines the mapping $e\,(\xi_i, \eta_i) : U \to \{V_{1\_u_1}, V_{2\_u_2}, V_{3\_u_3}\}$, which is:

$$e\,(\xi_{u_i}, \eta_{u_i})\,(x) = \begin{cases} V_{1\_u_i}(x) & , x \le \xi_{u_i} \\ V_{2\_u_i}(x), \xi_{u_i} < x < \eta_{u_i} \\ V_{3\_u_i}(x) & , \eta_{u_i} < x \end{cases} \tag{1}$$

The value $P\left(x \le \xi_{u_i}\right)$ is the probability that the random variable $x$ falls in the interval $[x, b)$. If $x$ increases, $[x, b)$ becomes smaller, and the probability of falling in the interval $[x, b)$ also becomes smaller. This character of probability $P\left(x \le \xi_{u_i}\right)$ is the same as the "low" fuzzy set $V_{1\_u_i}$, so $V_{1\_u_i}(x) = P\{x \le \xi_{u_i}\} = \int_x^\infty P_{\xi_{u_i}}(x)dx$. Similarly $V_{3\_u_i}(x) = P\{\eta_{u_i} < x\} = \int_x^\infty P_{\eta_{u_i}}(x)dx$. In these expressions $P_{\xi_{u_i}}(x)$ and $P_{\eta_{u_i}}(x)$ are the probability densities of the random variable $\xi_{u_i}$ and $\eta_{u_i}$ respectively, and $V_{2\_u_i}(x) = 1 - V_{1\_u_i}(x) - V_{3\_u_i}(x)$.

Calculated in the probabilistic method, the membership function of each level can be obtained:

$$V_{1\_u_i}(x) = 1 - \Phi\left(\frac{x - a_{1\_u_i}}{\sigma_{1\_u_i}}\right) \tag{2}$$

$$V_{3\_u_i}(x) = \Phi\left(\frac{x - a_{2\_u_i}}{\sigma_{2\_u_i}}\right) \tag{3}$$

$$V_{2\_u_i}(x) = \Phi\left(\frac{x - a_{1\_u_i}}{\sigma_{1\_u_i}}\right) - \Phi\left(\frac{x - a_{2\_u_i}}{\sigma_{2\_u_i}}\right), \tag{4}$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$.

However, for the convenience of presentation we still use $V_{1\_u_i}, V_{2\_u_i}$, and $V_{3\_u_i}$ to represent the three level membership function of a factor $u_i \in (\chi_i, \psi_i)$.

### 3.2.3. Constructing the Fuzzy Evaluation Matrix

From the above, the membership function of each factor $u_i$ can be obtained. Bringing the data of the five factors of an agent vehicle $j$ into the corresponding membership functions, the relationship between the five factors $u_i$ and the grading of V can be obtained as:

$$(V^j_{1\_u_i}(x), V^j_{2\_u_i}(x), V^j_{3\_u_i}(x))$$

Thus a fuzzy relation matrix for vehicle j can be obtained:

$$\mathrm{R}^j = \begin{bmatrix} V^j_{1\_u_1}(x) & V^j_{2\_u_1}(x) & V^j_{3\_u_1}(x) \\ V^j_{1\_u_2}(x) & V^j_{2\_u_2}(x) & V^j_{3\_u_2}(x) \\ V^j_{1\_u_3}(x) & V^j_{2\_u_3}(x) & V^j_{3\_u_3}(x) \\ V^j_{1\_u_4}(x) & V^j_{2\_u_4}(x) & V^j_{3\_u_4}(x) \\ V^j_{1\_u_5}(x) & V^j_{2\_u_5}(x) & V^j_{3\_u_5}(x) \end{bmatrix} = \begin{bmatrix} r^j_{1,1} & r^j_{1,2} & r^j_{1,3} \\ r^j_{2,1} & r^j_{2,2} & r^j_{2,3} \\ r^j_{3,1} & r^j_{3,2} & r^j_{3,3} \\ r^j_{4,1} & r^j_{4,2} & r^j_{4,3} \\ r^j_{5,1} & r^j_{5,1} & r^j_{5,3} \end{bmatrix}. \tag{5}$$

Assuming that there are $m$ agent vehicle participating in the evaluation, $m$ fuzzy relation matrices will be obtained for each customer vehicle $i$: $R^1{}_i, R^2{}_i, \cdots, R^m{}_i$. In Algorithm 1, we select the agent vehicles that meet the customer vehicle's needs.

### 3.2.4. Determination of the Weight $a_k$

The importances of the five factors in the comprehensive evaluation system are not the same. If the status is important, it should be given a greater weight; otherwise, it should be given a smaller weight.

Assume that the weight set is $A = \{a_1, a_2, ..., a_5\}$, where $\sum\limits_{\mathrm{i}=1}^{5} a_i = 1$.

### 3.2.5. Fuzzy Comprehensive Evaluation

$A$ and $R^j$ are used in a fuzzy synthesis operation: $A \circ R^j = B^j$ to obtain a comprehensive evaluation $B^j = (b_1^j, b_2^j, b_3^j)$ for the agent vehicle $j$. Here, $B^j = A^\circ R^j = \min(1, \sum\limits_{i=1}^{n} a_i \cdot r_{ij})$ considers the degree of subordination of the agent vehicle $j$. Then according to the principle of maximum subordination, we can get the evaluation level of agent vehicle $j$:

$$B^j = A \circ R^j = (a_1, a_2, a_3, a_4, a_5) \begin{bmatrix} r_{1,1}^j & r_{1,2}^j & r_{1,3}^j \\ r_{2,1}^j & r_{2,2}^j & r_{2,3}^j \\ r_{3,1}^j & r_{3,2}^j & r_{3,3}^j \\ r_{4,1}^j & r_{4,2}^j & r_{4,3}^j \\ r_{5,1}^j & r_{5,1}^j & r_{5,3}^j \end{bmatrix} = (b_1^j, b_2^j, b_3^j). \tag{6}$$

Here $b_k^j = (a_1 \bullet r_{1k}) \oplus (a_2 \bullet r_{2k}) \oplus ... \oplus (a_5 \bullet r_{5k})$. Additionally, the fuzzy synthesis operator "$\circ$" selects the fuzzy operator $M(\bullet, \oplus)$. In the fuzzy operator $M(\bullet, \oplus)$, $\bullet$ is defined as multiplication, and $\oplus$ is defined as the operation $x \oplus y = \min(1, x + y)$. Thus, $B^j = A \circ R^j = \min(1, \sum\limits_{i=1}^{n} a_i \cdot r_{ij})$ .

The following normalization is performed on B:

$$B^j = \left( \frac{b_1^j}{\sum\limits_{i=1}^{3} b_i^j}, \frac{b_2^j}{\sum\limits_{i=1}^{3} b_i^j}, \frac{b_3^j}{\sum\limits_{i=1}^{3} b_i^j} \right) \overset{\Delta}{=} (C_1^j\%, C_2^j\%, C_3^j\%) \tag{7}$$

An understanding of $B$ can be achieved through the following example: for an agent vehicle $j$, its comprehensive evaluation $B^j = (10\%, 50\%, 40\%)$ indicates that taking the five factors of the agent vehicle $j$ into consideration, 10% of vehicles evaluate it as "low", 50% of vehicles evaluate it as "medium", and 40% of vehicles evaluate it as "high". According to the principle of maximum degree of membership, the evaluation level of agent vehicle $j$ is "medium".

Next, based on the quantized value of the fuzzy comment set, that is:

$$V = \{V_1, V_2, V_3\} = \{1, 2, 3\},$$

the overall rating of the agent vehicle $j$ is:

$$E_j = B^j V^T = (B^1, B^2, B^3, B^4, B^5)(V_1, V_2, V_3)^T = (B^1, B^2, B^3, B^4, B^5)(1, 2, 3)^T. \tag{8}$$

In this way, we can get the comment sets of several agent vehicles from the customers who participate in the evaluation:

$$E_i = (E_{i1}, E_{i2}, \cdots, E_{im}). \tag{9}$$

Then the comments on agent vehicles from customers are expressed as follows, where $n$ represents the number of customer vehicles, and $m$ represents the number of agent vehicles:

$$E = (E_1, E_2, \cdots, E_n) = \begin{bmatrix} E_{11}E_{12}\cdots\cdots E_{1m} \\ E_{21}E_{22}\cdots\cdots E_{2m} \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ E_{n1}E_{n2}\cdots\cdots E_{nm} \end{bmatrix}. \tag{10}$$

**Algorithm 2** Fuzzy Comprehensive Evaluation Algorithm

```
Input: Customer vehicle request package; agent vehicle service package; signal effective distance $L_a$; path length $L$; available agent vehicle list $N_i$ for each $CV_i$
Output: Available QoS (quality of service) for each agent vehicle $j$; list $E_{ij}$ for customer vehicle $i$
for each $CV_i$ $i \in [1, n]$ do
    for each $AV_j$ in $N_i$ do
        compute path consistency $pc_{ij} = \frac{L_a v_i}{|v_i - v_j| L}$
    end for
    for each $AV_j$ in $N_i$ do
        for each element k of the $AV_j$ do
            $R(j,k,1) = 1 - \Phi\left(\frac{x - a_{1\_u_1}}{\sigma_{1\_u_1}}\right); R(j,k,2) = \Phi\left(\frac{x - a_{1\_u_i}}{\sigma_{1\_u_i}}\right) - \Phi\left(\frac{x - a_{2\_u_i}}{\sigma_{2\_u_i}}\right); R(j,k,3)(x) = \Phi\left(\frac{x - a_{2\_u_i}}{\sigma_{2\_u_i}}\right);$
        end for
    end for
    for each $AV_j$ in $N_i$ do
        $B(j) = A \circ R(j,:,:); B(j) = \frac{1}{\sum_{i=1}^{3} b_i^j} B(j); E_{ij} = B(j) \cdot (1,2,3)^T;$
    end for
end for
return $E$
```

## 3.3. Optimization

### 3.3.1. Comprehensive Vehicle Evaluation

To satisfy the requests of more customer vehicles and enable the agent vehicles to provide more effective service, taking the bandwidth limitation of the agent vehicles and the comprehensive scores given by the customer vehicles into account, this section distributes the agent vehicle resources and chooses the most suitable vehicle for customers. According to the discussion above, we propose the following access selection model:

$$\text{QoS} = \max\left(\sum_{i=1}^{n}\sum_{j=1}^{m} E_{ij}x_{ij}\right) \tag{11}$$

s.t.

$$\begin{array}{l} \sum\limits_{i=1}^{n} x_{ij} = 1 \\ \sum\limits_{i=1}^{n} x_{ij} \times b_i \le b_j \\ c_i \le c_j \\ \frac{c_i}{b_i} \le \min\left(\frac{L_a}{|v_i - v_j|}, \frac{L}{v_i}\right), \end{array}$$

where: $\sum_{i=1}^{n} x_{ij} = 1$ means that each customer vehicle can only be connected to one agent vehicle at the same time; $\sum_{i=1}^{n} x_{ij} \times b_i \leq b_j$ means that the bandwidth sum of customer vehicles served by the same proxy vehicle should not exceed its bandwidth capacity; $c_i \leq c_j$ means that the computing capability provided by the agent vehicle shall be no less than the computing capability of the customer vehicle's requirements; and $\frac{c_i}{b_i} \leq \min\left(\frac{L_a}{|v_i - v_j|}, \frac{L}{v_i}\right)$ indicates the computing capability that the customer vehicle should meet to satisfy the requirement of collaborative download within the service time of the agent vehicle in the path, so as to ensure the integrity of data transmission.

### 3.3.2. Agent Vehicle Resource Allocation Algorithm

**Algorithm 3** Agent Vehicle Distribution Optimization Algorithm

```
Input: The comments on agent vehicles for each customer vehicle $E$
Output: Collaborative offload distribution scheme $X$
1: Construct the formula of overall customer satisfaction by maximizing the customer satisfaction: $QoS = \max\left(\sum_{i=1}^{n}\sum_{j=1}^{m} E_{ij}x_{ij}\right)$
2: The bandwidth sum of customer vehicles served by the same proxy vehicle should not exceed its bandwidth capacity: $\sum_{i=1}^{n} x_{ij} \times b_i \leq b_j$
3: Each customer vehicle can only be connected to one agent vehicle at the same time: $\sum_{i=1}^{n} x_{ij} = 1$
4: Determine the distribution of agent vehicles that maximize customer satisfaction by using linear programming.
5: Output the agent vehicles' distribution $X$.
```

## 4. Performance Evaluation

In this section, we use the proposed FCE (fuzzy comprehensive evaluation) algorithm to construct a series of experiments for the V2V agent vehicle selection problem based on the MATLAB platform. The experimental parameters are shown in Table 2. We compare the performance of the FCE algorithm with the FL-CFT [19] and RSB (random selection based on computing capability and bandwidth) algorithms under different numbers of customer requests. In order to realize the comparison between the FCE algorithm and the FL-CFT algorithm, we quantify the index obtained by FL-CFT using the process after the second step 13 of the FL-CFT algorithm. The comparative performance is as follows.

**Table 2.** The basic parameters of the simulation.

| Parameter | Number | Unit | Information Description |
|---|---|---|---|
| $L$ | 3000 | m | Path length |
| $L_a$ | 200 | m | Effective distance between the customer vehicle and the agent vehicle |
| $n$ | 50 | | Number of customer vehicles requesting data |
| $m$ | 300 | | Number of agent vehicles providing data services |
| $c_i$ | 20–80 | Mb | Computing capability of customer vehicle *i*'s request |
| $b_i$ | 3–12 | Mbps | Bandwidth of customer vehicle *i* |
| $v_i$ | 20-35 | m/s | Speed of customer vehicle *i* |
| $c_j$ | 20–80 | Mb | Computing capability of agent vehicle *j* |
| $b_j$ | 3–12 | Mbps | Bandwidth of agent vehicle *j* |
| $co_j$ | 0–3 | | Service cost of agent vehicle *j* in unit time |
| $cr_j$ | 0–1 | | Accumulated credit ratio of agent vehicle *j* |
| $v_j$ | 20–35 | m/s | Speed of agent vehicle *j* |
| $pc_{ij}$ | 0–1 | | Path consistency between customer vehicle *i* and agent vehicle *j* |
| $N_i$ | | | The list of agent vehicles available for customer vehicle *i* |
| $x_{ij}$ | | | Connection status between customer vehicle *i* and agent vehicle *j* |
| $E_{ij}$ | | | Available QoS list of agent vehicle *j* to customer vehicle *i* |

### 4.1. Experimental Setup

In the experiment, we consider that cooperative uninstallation occurs in the area without network coverage between two RSUs. The information requested between vehicles can only be shared through the information sharing mechanism between V2V. Vehicles apply to the vehicle cloud (VC) before arriving in the region. The vehicle cloud aggregates vehicle information, and uses the FCE algorithm proposed in this paper to analyze the information of customer vehicles and agent vehicles, so as to provide an agent vehicle allocation scheme that maximizes customer satisfaction. The following assumptions are employed in our simulations:

- Set the same driving path between the customer vehicle and the agent vehicle.
- Equip each vehicle (including the customer and agent vehicles) with an OBU, which can receive information and transmit information to the surrounding vehicles, and set the effective communication range of the vehicle.
- There are only two forms of data transmission between a customer vehicle and an agent vehicle: completion and failure.
- Each vehicle can act as a customer vehicle when requesting data and an agent vehicle when providing data service, but it can only be one in a period.

### 4.2. Performance Analysis

In this paper, we analyze the performance of the algorithm in four aspects: quality of service, average throughput, number of request failures, and average consumption. Quality of service is a comprehensive evaluation index under multi-factor consideration. It is the standard to verify the performance of the algorithm. Average throughput is the data transmission volume per unit time, which is the main factor to ensure the fluency of a customer vehicle's data requests. The number of request failures is the number of times that the agent vehicle cannot provide complete data transmission for the customer vehicle, which shows the stability of data transmission. Cost is an important reference factor for each customer vehicle in choosing agent vehicle service. We discuss the impact of the number of customer vehicles on the quality of service, average throughput, number of request failures, and average consumption in the process of collaboration between customer and agent vehicles.

Figure 3 shows that the average customer satisfaction curve obtained by the FCE algorithm is higher than that of the FL-CFT and RSB algorithms when changing the number of customer vehicle requests. The RSB curve has the worst performance. This is because the FCE algorithm considers the computing capability, bandwidth, unit cost, credibility, and path consistency of the agent vehicle in the process of selection; while FL-CFT only considers the velocity, distance, and connection of the agent vehicle; and RSB only considers the bandwidth and path consistency. Figure 3 shows that the FCE algorithm has better average customer satisfaction performance than the FL-CFT and RSB algorithms.

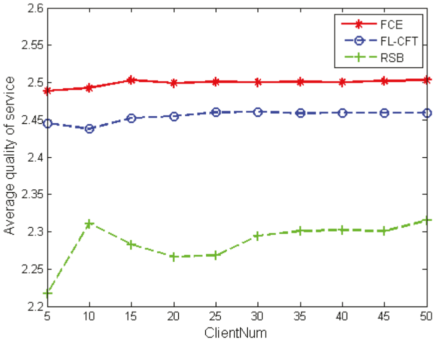

**Figure 3.** Average quality of service of the three algorithms.

Figure 4 shows that in the process of changing the number of customer vehicle requests, the effective bandwidth ratio of the FCE algorithm to the RSB Algorithm is 1, and the performance of the FL-CFT method is the worst. This is because the FCE and RSB algorithms take the bandwidth as an important index to evaluate the agent vehicle selection process, while FL-CFT does not consider this index. Figure 4 shows that the FCE and RSB algorithms have better average bandwidth utilization than the FL-CFT algorithm. This index also shows whether the selected agent vehicle can meet the customer's bandwidth requirements. The FCE and RSB algorithms can provide a better data fluency experience for a customer vehicle.

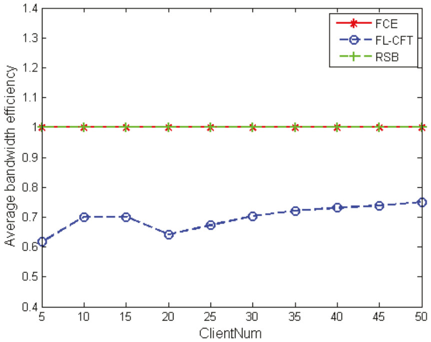

**Figure 4.** Average bandwidth efficiency of the three algorithms.

In order to verify the correctness and stability of the algorithm, we run a model experiment with 500 customer vehicles and 3000 agent vehicles, and count the failure times of customer requests under the experimental conditions. Figure 5 shows that in the process of changing the number of customer vehicle requests, the FCE algorithm does not fail, while the failure rates of the FL-CFT and RSB algorithms increase with an increase in the number of customer requests. This is because the FCE algorithm takes into account the interaction of many factors, and takes the path matching degree of customer vehicles and agent vehicles and the reputation of customer vehicles as important indicators. Figure 5 shows that the FCE algorithm has better selectivity and stability than the FL-CFT and RSB algorithms.

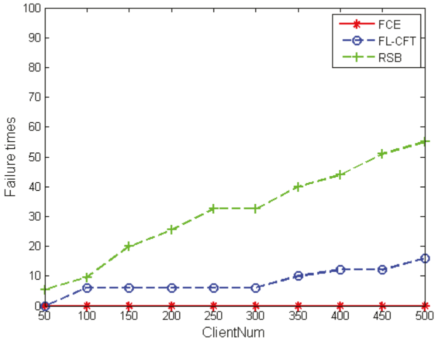

**Figure 5.** Number of failures for the three algorithms.

Customer consumption is always an important indicator of customer vehicles in the selection of agent vehicles. Figure 6 shows that the average consumption of the FCE algorithm is less than that of the FL-CFT and RSB algorithms with increasing customer vehicle requests. The main reason is that the FCE algorithm considers all the required vehicle information when selecting the agent vehicle, and seeks an optimal selection strategy. The FL-CFT and RSB algorithms ignore the interaction of these factors. Figure 6 shows that the FCE algorithm is more economical than the FL-CFT and RSB algorithms.

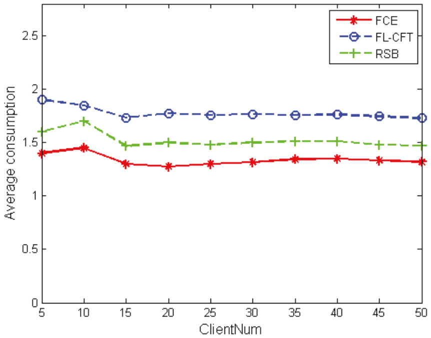

**Figure 6.** Average consumption of the three algorithms.

## 5. Conclusions

In this paper, the problem of agent vehicle selection for V2V collaborative unloading in a vehicle network is studied. A method of joint selection in a V2V network, based on fuzzy comprehensive evaluation is proposed. Different from the previous articles, many factors such as computing capability, bandwidth, consumption index, reputation, and path matching of the vehicles are considered in this paper. At the same time, we take the relationship between the customer vehicle demand and the agent vehicle as tan index item. In this paper, we propose a fuzzy comprehensive evaluation method to evaluate customer and agent vehicles, and give a multi-constrained optimization model to describe the agent vehicle allocation scheme. The simulation results show that the proposed vehicle selection algorithm has good prospects of usability and application.

**Author Contributions:** These authors contributed equally to this work. All authors discussed the required algorithm to complete the manuscript. T.H., X.G. and Z.C. conceived the paper. T.H. and Y.Z. did the experiments and wrote the paper. X.G. and Z.C. checked typographical errors.

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# Statistically and Relatively Modular Deferred-Weighted Summability and Korovkin-Type Approximation Theorems

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**Abstract:** The concept of statistically deferred-weighted summability was recently studied by Srivastava et al. (Math. Methods Appl. Sci. **41** (2018), 671–683). The present work is concerned with the deferred-weighted summability mean in various aspects defined over a modular space associated with a generalized double sequence of functions. In fact, herein we introduce the idea of relatively modular deferred-weighted statistical convergence and statistically as well as relatively modular deferred-weighted summability for a double sequence of functions. With these concepts and notions in view, we establish a theorem presenting a connection between them. Moreover, based upon our methods, we prove an approximation theorem of the Korovkin type for a double sequence of functions on a modular space and demonstrate that our theorem effectively extends and improves most (if not all) of the previously existing results. Finally, an illustrative example is provided here by the generalized bivariate Bernstein–Kantorovich operators of double sequences of functions in order to demonstrate that our established theorem is stronger than its traditional and statistical versions.

**Keywords:** statistical convergence; $P$-convergent; statistically and relatively modular deferred-weighted summability; relatively modular deferred-weighted statistical convergence; Korovkin-type approximation theorem; modular space; convex space; $\mathcal{N}$-quasi convex modular; $\mathcal{N}$-quasi semi-convex modular

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## 1. Introduction, Preliminaries, and Motivation

The gradual evolution on sequence spaces results in the development of statistical convergence. It is more general than the ordinary convergence in the sense that the ordinary convergence of a sequence requires that almost all elements are to satisfy the convergence condition, that is, every element of the sequence needs to be in some neighborhood (arbitrarily small) of the limit. However, such restriction is relaxed in statistical convergence, where set having a few elements that are not in the neighborhood of the limit is discarded subject to the condition that the natural density of the set is zero, and at the same time the condition of convergence is valid for the other majority of the elements. In the year 1951, Fast [1] and Steinhaus [2] independently studied the term statistical convergence for single real sequences; it is a generalization of the concept of ordinary convergence. Actually, a root of the notion of statistical convergence can be detected by Zygmund (see [3], p. 181), where he used the term "almost convergence", which turned out to be equivalent to the concept of statistical convergence. We also find such concepts in random graph theory (see [4,5]) in the sense that almost convergence means convergence with probability 1, whereas in statistical convergence the probability is not necessarily 1. Mathematically, a sequence of random variables $\{X_n\}$ is statistically convergent (converges in probability) to a random variable $X$ if $\lim_{n\to\infty} P(|Xn - X| \geqq \epsilon) = 0$, for all $\epsilon > 0$ (arbitrarily small); and almost convergent to $X$ if $P(\lim_{n\to\infty} X_n = X) = 1$.

For different results concerning statistical versions of convergence as well as of the summability of single sequences, we refer to References [1,2,6].

Let $\mathbb{N}$ be the set of natural numbers and let $\mathcal{H} \subseteq \mathbb{N}$. Also let

$$\mathcal{H}_n = \{k : k \leqq n, \text{ and } k \in \mathcal{H}\}$$

and suppose that $|\mathcal{H}_n|$ is the cardinality of $\mathcal{H}_n$. Then, the *natural density* of $\mathcal{H}$ is defined by

$$\delta(\mathcal{H}) = \lim_{n\to\infty} \frac{|\mathcal{H}_n|}{n} = \lim_{n\to\infty} \frac{1}{n}\{k : k \leqq n \text{ and } k \in \mathcal{H}\},$$

provided that the limit exists.

A sequence $(x_n)$ is *statistically convergent* to $\ell$ if for every $\epsilon > 0$,

$$\mathcal{H}_\epsilon = \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - \ell| \geqq \epsilon\}$$

has zero natural (asymptotic) density (see [1,2]). That is, for every $\epsilon > 0$,

$$\delta(\mathcal{H}_\epsilon) = \lim_{n\to\infty} \frac{|\mathcal{H}_\epsilon|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k : k \leqq n \quad \text{and} \quad |x_k - \ell| \geqq \epsilon\}| = 0.$$

Here, we write

$$\text{stat} \lim_{n\to\infty} x_n = \ell.$$

As an extension of statistical versions of convergence, the idea of weighted statistical convergence of single sequences was presented by Karakaya and Chishti [7], and it has been further generalized by various authors (see [8–12]). Moreover, the concept of deferred weighted statistical convergence was studied and introduced by Srivastava et al. [13] (see also [14–19]).

In the year 1900, Pringsheim [20] studied the convergence of double sequences. Recall that a double sequence $(x_{m,n})$ is convergent (or $P$-convergent) to a number $\ell$ if for given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_{m,n} - \ell| < \epsilon$, whenever $m, n \geqq n_0$ and is written as $P \lim x_{m,n} = \ell$. Likewise, $(x_{m,n})$ is bounded if there exists a positive number $\mathcal{K}$ such that $|x_{m,n}| \leqq \mathcal{K}$. In contrast to the case of single sequences, here we note that a convergent double sequence is not necessarily bounded. We further recall that, a double sequence $(x_{m,n})$ is non-increasing in *Pringsheim's sense* if $x_{m+1,n} \leqq x_{m,n}$ and $x_{m,n+1} \leqq x_{m,n}$.

Let $\mathcal{H} \subset \mathbb{N} \times \mathbb{N}$ be the set of integers and let $\mathcal{H}(i,j) = \{(m,n) : m \leqq i \text{ and } n \leqq j\}$. The *double natural density* of $\mathcal{H}$ denoted by $\delta(\mathcal{H})$ is given by

$$\delta(\mathcal{H}) = P \lim_{i,j} \frac{1}{ij}|\mathcal{H}(i,j)|,$$

provided the limit exists. A double sequence $(x_{m,n})$ of real numbers is statistically convergent to $\ell$ in the *Pringsheim sense* if, for each $\epsilon > 0$

$$\delta(\mathcal{H}_\epsilon(i,j)) = 0,$$

where

$$\delta(\mathcal{H}_\epsilon(i,j)) = \frac{1}{ij}\{(m,n) : m \leqq i, n \leqq j \text{ and } |x_{m,n} - \ell| \geqq \epsilon\}.$$

Here, we write

$$\mathrm{stat}^2 \lim_{m,n} x_{m,n} = \ell.$$

Note that every $P$-convergent double sequence is stat$^2$-convergent to the same limit, but the converse is not necessarily true.

**Example 1.** *Suppose we consider a double sequence* $x = (x_{m,n})$ *as*

$$x_{m,n} = \begin{cases} \sqrt{nm} & (m = k^2,\ n = l^2;\ \forall\, k, l \in \mathbb{N}), \\ \\ \frac{1}{nm} & \textit{otherwise}. \end{cases}$$

*It is trivially seen that, in the ordinary sense* $(x_{m,n})$ *is not P-convergent; however,* 0 *is its statistical limit.*

Let $\mathcal{I} = [0, \infty) \subseteq \mathbb{R}$, and let the Lebesgue measure $v$ be defined over $\mathcal{I}$. Let $\mathcal{I}^2 = [0, \infty) \times [0, \infty)$ and suppose that $X(\mathcal{I}^2)$ is the space of all measurable real-valued functions defined over $\mathcal{I}^2$ equipped with the equality almost everywhere. Also, let $C(\mathcal{I}^2)$ be the space of all continuous real-valued functions and suppose that $C^\infty(\mathcal{I}^2)$ is the space of all functions that are infinitely differentiable on $\mathcal{I}^2$. We recall here that a functional $\omega : X(\mathcal{I}^2) \to [0, \infty)$ is a *modular* on $X(\mathcal{I}^2)$ such that it satisfies the following conditions:

(i) $\omega(f) = 0$ if and only if $f = 0$, almost everywhere in $\mathcal{I}$ $(\forall\, f \in \mathcal{I}')$,
(ii) $\omega(\alpha f + \beta g) \leqq \omega(f) + \omega(g)$, $\forall\, f, g \in X(\mathcal{I}^2)$ and for any $\alpha, \beta \geqq 0$ with $\alpha + \beta = 1$,
(iii) $\omega(-f) = \omega(f)$, for each $f \in X(\mathcal{I}^2)$, and
(iv) $\omega$ is continuous on $[0, \infty)$.

Also, we further recall that a modular $\omega$ is

- $\mathcal{N}$-*Quasi convex* if there exists a constant $\mathcal{N} \geqq 1$ satisfying

  $$\omega(\alpha f + \beta g) \leqq \mathcal{N}\alpha\omega(\mathcal{N} f) + \mathcal{N}\beta\omega(\mathcal{N} g)$$

  for every $f, g \in X(\mathcal{I}^2)$, $\alpha, \beta \geqq 0$ such that $\alpha + \beta = 1$. Also, in particular, for $\mathcal{N} = 1$, $\omega$ is simply called *convex*; and
- $\mathcal{N}$-*Quasi semi-convex* if there exists a constant $\mathcal{N} \geqq 1$ such that

  $$\omega(\lambda f) \leqq \mathcal{N}\lambda\omega(\mathcal{N} f)$$

  holds for all $f \in X(\mathcal{I}^2)$ and $\lambda \in (0, 1]$.

Also, it is trivial that every $\mathcal{N}$-Quasi semi-convex modular is $\mathcal{N}$-Quasi convex. The above concepts were initially studied by Bardaro et al. [21,22].

We now appraise some suitable subspaces of vector space $X(\mathcal{I}^2)$ under the modular $\omega$ as follows:

$$L^\omega(\mathcal{I}^2) = \{f \in X(\mathcal{I}^2) : \lim_{\lambda \to 0^+} \omega(\lambda f) = 0\}$$

and

$$E^\omega(\mathcal{I}^2) = \{f \in L^\omega(\mathcal{I}^2) : \omega(\lambda f) < +\infty,\ \ \forall\ \lambda > 0\}.$$

Here, $L^\omega(\mathcal{I}^2)$ is known as the modular space generated by $\omega$ and $E^\omega(\mathcal{I}^2)$ is known as the space of the finite elements of $L^\omega(\mathcal{I}^2)$. Also, it is trivial that whenever $\omega$ is $\mathcal{N}$-Quasi semi-convex,

$$\{f \in X(\mathcal{I}^2) : \omega(\lambda f) < +\infty,\ \ \forall\ \lambda > 0\}$$

coincides with $L^{\omega}(\mathcal{I}^2)$. Moreover, for a *convex modular* $\omega$ in $X(\mathcal{I}^2)$, the *F-norm* is given by the formula:

$$\|f\|_{\omega} = \inf\left\{\lambda > 0 : \omega\left(\frac{f}{\lambda}\right) \leqq 1\right\}.$$

The notion of modular was introduced in [23] and also widely discussed in [22].

In the year 1910, Moore [24] introduced the idea of the relatively uniform convergence of a sequence of functions. Later, along similar lines it was modified by Chittenden [25] for a sequence of functions defined over a closed interval $I = [a, b] \subseteq \mathbb{R}$.

We recall here the definition of uniform convergence relative to a scale function as follows.

A sequence of functions $(f_n)$ defined over $[a, b]$ is *relatively uniformly convergent* to a limit function $f$ if there exists a non-zero scale function $\sigma$ defined over $[a, b]$, such that for each $\epsilon > 0$ there exists an integer $n_\epsilon$ and for every $n > n_\epsilon$,

$$\left|\frac{f_n(x) - f(x)}{\sigma(x)}\right| \leqq \epsilon$$

holds uniformly for all $x \in [a, b] \subseteq \mathbb{R}$.

Now, to see the importance of relatively uniform convergence (ordinary and statistical) over classical uniform convergence, we present the following example.

**Example 2.** *For all $n \in \mathbb{N}$, we define $f_n : [0, 1] \to \mathbb{R}$ by*

$$f_n(x) = \begin{cases} \frac{nx}{1+n^2x^2} & (0 < x \leqq 1), \\ 0 & (x = 0). \end{cases}$$

*It is not difficult to see that the sequence $(f_n)$ of functions is neither classically nor statistically uniformly convergent in $[0, 1]$; however, it is convergent uniformly to $f = 0$ relative to a scale function*

$$\sigma(x) = \begin{cases} \frac{1}{x} & (0 < x \leqq 1) \\ 0 & (x = 0) \end{cases}$$

*on $[0, 1]$. Here, we write*

$$f_n \rightrightarrows f = 0 \quad ([0, 1]; \sigma).$$

In the middle of the twentieth century, H. Bohman [26] and P. P. Korovkin [27] established some approximation results by using positive linear operators. Later, some Korovkin-type approximation results with different settings were extended to several functional spaces, such as Banach space and Musielak–Orlicz space etc. Bardaro, Musielak, and Vinti [22] studied generalized nonlinear integral operators in connection with some approximation results over a modular space. Furthermore, Bardaro and Mantellini [28] proved some approximation theorems defined over a modular space by positive linear operators. They also established a conventional Korovkin-type theorem in a multivariate modular function space (see [21]). In the year 2015, Orhan and Demirci [29] established a result on statistical approximation by double sequences of positive linear operators on modular space. Demirci and Burçak [30] introduced the idea of $A$-statistical relative modular convergence of positive linear operators. Moreover, Demirci and Orhan [31] established some results on statistically relatively approximation on modular spaces. Recently, Srivastava et al. [13] established some approximation results on Banach space by using deferred weighted statistical convergence. Subsequently, they also introduced deferred weighted equi-statistical convergence to prove some approximation theorems (see [17]). Very recently, Md. Nasiruzzaman et al. [32] proved Dunkl-type generalization of Szász-Kantorovich operators via post-quantum calculus, and consequently, Srivastava et al. [33]

established the construction of Stancu-type Bernstein operators based on Bézier bases with shape parameter $\lambda$.

Motivated essentially by the above-mentioned results, in this paper we introduce the idea of relatively modular deferred-weighted statistical convergence and statistically as well as relatively modular deferred-weighted summability for double sequences of functions. We also establish an inclusion relation between them. Moreover, based upon our proposed methods, we prove a Korovkin-type approximation theorem for a double sequence of functions defined over a modular space and demonstrate that our result is a non-trivial generalization of some well-established results.

## 2. Relatively Modular Deferred-Weighted Mean

Let $(a_n)$ and $(b_n)$ be sequences of non-negative integers satisfying the conditions: (i) $a_n < b_n$ $(n \in \mathbb{N})$ and (ii) $\lim_{n\to\infty} b_n = \infty$. Note that (i) and (ii) are the regularity conditions for the proposed deferred weighted mean (see Agnew [34]). Now, for the double sequence $(f_{m,n})$ of functions, we define the deferred weighted summability mean $(N_D(f_{m,n}))$ as

$$N_D(f_{m,n}) = \frac{1}{T_m S_n} \sum_{u,v=a_n+1}^{b_m,b_n} t_u s_v f_{u,v}(x), \tag{1}$$

where $(s_n)$ and $(t_n)$ are the sequences of non-negative real numbers satisfying

$$S_n = \sum_{v=a_n+1}^{b_n} s_v \text{ and } T_m = \sum_{u=a_n+1}^{b_m} t_v.$$

**Definition 1.** *A double sequence $(f_{m,n})$ of functions belonging to $L^{\omega}(\mathcal{I}^2)$ is relatively modular deferred weighted $(N_D(f_{m,n}))$-summable to a function $f$ on $L^{\omega}(\mathcal{I}^2)$ if and only if there exists a non-negative scale function $\sigma \in X(\mathcal{I}^2)$ such that*

$$P \lim_{m,n\to\infty} \omega\left(\lambda\left(\frac{N_D(f_{m,n}) - f}{\sigma}\right)\right) = 0 \textit{ for some } \lambda_0 > 0.$$

*Here, we write*

$$\mathcal{N}_{\mathcal{D}} \lim_{m,n} \left\|\frac{f_{m,n} - f}{\sigma}\right\|_{\omega} = 0 \textit{ for some } \lambda_0 > 0.$$

**Definition 2.** *A double sequence $(f_{m,n})$ of functions belonging to $L^{\omega}(\mathcal{I}^2)$ is relatively F-norm (locally convex) deferred weighted summable (or relatively strong deferred weighted summable) to $f$ if and only if*

$$P \lim_{m,n\to\infty} \omega\left(\lambda\left(\frac{N_D(f_{m,n}) - f}{\sigma}\right)\right) = 0 \textit{ for some } \lambda > 0.$$

*Here, we write*

$$\mathcal{F}\,\mathcal{N}_{\mathcal{D}} \lim_{m,n} \left\|\frac{f_{m,n} - f}{\sigma}\right\|_{\omega} = 0 \textit{ for some } \lambda_0 > 0.$$

It can be promptly seen that, Definitions 1 and 2 are identical if and only if the modular $\omega$ fairly holds the $\Delta_2$-condition, that is, there exists a constant $\mathcal{M} > 0$ such that $\omega(2f) \leqq \mathcal{M}\omega(f)$ for every $f \in X(\mathcal{I}^2)$. Precisely, relatively strong summability of the double sequence $(f_{m,n})$ to $f$ is identical to the condition

$$P \lim_{m,n} \omega\left(2^n \lambda\left(\frac{N_D(f_{m,n}) - f}{\sigma}\right)\right) = 0,$$

$\forall \quad n \in \mathbb{N}$ and some $\lambda > 0$. Thus, if $(f_{m,n})$ is relatively modular deferred weighted $(N_D(f_{m,n}))$-summable to $f$, then by Definition 1 there exists a $\lambda > 0$ such that

$$P \lim_{m,n\to\infty} \omega\left(\lambda\left(\frac{N_D(f_{m,n})-f}{\sigma}\right)\right) = 0.$$

Clearly, under $\Delta_2$-condition, we have

$$\omega\left(2^n\lambda\left(\frac{N_D(f_{m,n})-f}{\sigma}\right)\right) \leqq \mathcal{M}^n \omega\left(\lambda\left(\frac{N_D(f_{m,n})-f}{\sigma}\right)\right).$$

This implies that

$$P \lim_{m,n} \omega\left(2^n\lambda\left(\frac{N_D(f_{m,n})-f}{\sigma}\right)\right) = 0.$$

**Definition 3.** *A double sequence $(f_{m,n})$ of functions belonging to $L^{\omega}(\mathcal{I}^2)$ is relatively modular deferred-weighted $(N_D(f_{m,n}))$ statistically convergent to a function $f \in L^{\omega}(\mathcal{I}^2)$ if there exists a non-zero scale function $\sigma \in X(\mathcal{I}^2)$ such that, for every $\epsilon > 0$, the following set:*

$$P \lim_{m,n} \frac{1}{T_m S_n}\left\{(u,v) : u \leqq T_m, v \leqq S_m \text{ and } \omega\left(\lambda_0\left(\frac{t_u s_v |f_{u,v}-f|}{\sigma}\right)\right) \geqq \epsilon\right\} \text{ for some } \lambda_0 > 0$$

*has zero relatively deferred-weighted density, that is,*

$$P \lim_{m,n} \frac{1}{T_m S_n}\left|\left\{(u,v) : u \leqq T_m, v \leqq S_m \text{ and } \omega\left(\lambda_0\left(\frac{t_u s_v |f_{u,v}-f|}{\sigma}\right)\right) \geqq \epsilon\right\}\right| = 0 \text{ for some } \lambda_0 > 0.$$

*Here, we write*

$$stat_{N_D} \lim_{m,n} \left\|\frac{f_{m,n}-f}{\sigma}\right\|_{\omega} = 0.$$

*Moreover, $(f_{m,n})$ is relatively F-norm (locally convex) deferred-weighted $(N_D(f_{m,n}))$ statistically convergent (or relatively strong deferred-weighted $(N_D(f_{m,n}))$ statistically convergent) to a function $f \in X(\mathcal{I}^2)$ if and only if*

$$P \lim_{m,n} \frac{1}{T_m S_n}\left|\left\{(u,v) : u \leqq T_m, v \leqq S_m \text{ and } \omega\left(\lambda_0\left(\frac{t_u s_v |f_{u,v}-f|}{\sigma}\right)\right) \geqq \epsilon\right\}\right| = 0 \text{ for some } \lambda > 0,$$

*where $\sigma \in X(\mathcal{I}^2)$ is a non-zero scale function and $\epsilon > 0$.*

*Here, we write*

$$\mathcal{F}\ stat_{N_D} \lim_{m,n} \left\|\frac{f_{m,n}-f}{\sigma}\right\|_{\omega} = 0.$$

**Definition 4.** *A double sequence $(f_{m,n})$ of functions belonging to $L^{\omega}(\mathcal{I}^2)$ is statistically and relatively modular deferred-weighted $(N_D(f_{m,n}))$-summable to a function $f \in L^{\omega}(\mathcal{I}^2)$ if there exists a non-zero scale function $\sigma \in X(\mathcal{I}^2)$ such that, for every $\epsilon > 0$, the following set:*

$$P \lim_{m,n} \frac{1}{m,n}\left\{(u,v) : u \leqq m, v \leqq m \text{ and } \omega\left(\lambda_0\left(\frac{N_D(f_{m,n})-f}{\sigma}\right)\right) \geqq \epsilon\right\} \text{ for some } \lambda_0 > 0$$

*has zero relatively deferred-weighted density, that is,*

$$P \lim_{m,n} \frac{1}{mn}\left|\left\{(u,v) : u \leqq m, v \leqq n \text{ and } \omega\left(\lambda_0\left(\frac{N_D(f_{m,n})-f}{\sigma}\right)\right) \geqq \epsilon\right\}\right| = 0 \text{ for some } \lambda_0 > 0.$$

*Here, we write*

$$N_D stat \lim_{m,n} \left\| \frac{f_{m,n} - f}{\sigma} \right\|_{\omega} = 0.$$

*Furthermore,* $(f_{m,n})$ *is statistically and relatively F-norm (locally convex) deferred-weighted* $(N_D(f_{m,n}))$*-summable (or statistically and relatively strong deferred-weighted* $(N_D(f_{m,n}))$*-summable) to a function* $f \in X(\mathcal{I}^2)$ *if and only if*

$$P \lim_{m,n} \frac{1}{m,n} \left| \left\{ (u,v) : u \leqq m, v \leqq n \text{ and } \omega\left( \lambda_0 \left( \frac{N_D(f_{m,n}) - f}{\sigma} \right) \right) \geqq \epsilon \right\} \right| = 0 \text{ for some } \lambda > 0,$$

*where* $\sigma \in X(\mathcal{I}^2)$ *is a non-zero scale function and* $\epsilon > 0$*.*

*Here, we write*

$$\mathcal{F}\ N_D stat \lim_{m,n} \left\| \frac{f_{m,n} - f}{\sigma} \right\|_{\omega} = 0.$$

**Remark 1.** *If we put* $a_n = 0$*,* $b_n = n$*,* $b_m = m$*, and* $t_m = s_n = 1$ *in Definition 3, then it reduces to relatively modular statistical convergence (see [31]).*

Next, for our present study on a modular space we have the assumptions as follows:

- If $\omega(f) \leqq \omega(g)$ for $|f| \leqq |g|$, then $\omega$ is monotone;
- If $\chi \in L^{\omega}(\mathcal{I}^2)$ with $\mu(A) < \infty$, where $A$ is a measurable subset of $\mathcal{I}^2$, then $\omega$ is finite;
- If $\omega$ is finite and for each $\epsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ and $\omega(\lambda \chi_B) < \epsilon$ for any measurable subset $B \subset \mathcal{I}^2$ such that $\mu(B) < \delta$, then $\omega$ is absolutely finite;
- If $\chi_{\mathcal{I}^2} \in E^{w}(\mathcal{I}^2)$, then $\omega$ is strongly finite;
- If for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\omega(\alpha f \chi_B) < \epsilon$ $(\alpha > 0)$, where $B$ is a measurable subset of $\mathcal{I}^2$ with $\mu(B) < \delta$ and for each $f \in X(\mathcal{I}^2)$ with $\omega(f) < +\infty$, then $\omega$ is absolutely continuous.

It is clearly observed from the above assumptions that if a modular $\omega$ is finite and monotone, then $C(\mathcal{I}^2) \subset L^{\omega}(\mathcal{I}^2)$. Also, if $\omega$ is strongly finite and monotone, then $C(\mathcal{I}^2) \subset E^{\omega}(\mathcal{I}^2)$. Furthermore, if $\omega$ is absolutely continuous, monotone, and absolutely finite, then $\overline{C^{\infty}(\mathcal{I}^2)} = L^{\omega}(\mathcal{I}^2)$, where the closure $\overline{C^{\infty}(\mathcal{I}^2)}$ is compact over the modular space.

Now we establish the following theorem by demonstrating an inclusion relation between relatively deferred-weighted statistical convergence and statistically as well as relatively deferred-weighted summability over a modular space.

**Theorem 1.** *Let* $\omega$ *be a strongly finite, monotone, and* $\mathcal{N}$*-Quasi convex modular on* $L^{\omega}(\mathcal{I}^2)$*. If a double sequence* $(f_{m,n})$ *of functions belonging to* $L^{\omega}(\mathcal{I}^2)$ *is bounded and relatively modular deferred-weighted statistically convergent to a function* $f \in L^{\omega}(\mathcal{I}^2)$*, then it is statistically and relatively modular deferred weighted summable to the function* $f$*, but not conversely.*

**Proof.** Assume that $(f_{m,n}) \in L^{\omega}(\mathcal{I}^2) \cap \ell_{\infty}$. Let us set

$$\mathcal{H}_{\epsilon} = \left\{ (u,v) : u \leqq m, v \leqq n \text{ and } \omega\left( \lambda_0 \left( \frac{f_{u,v} - f}{\sigma} \right) \right) \geqq \epsilon \text{ for some } \lambda_0 > 0 \right\}$$

and

$$\mathcal{H}_{\epsilon}^{c} = \left\{ (u,v) : u \leqq m, v \leqq n \text{ and } \omega\left( \lambda_0 \left( \frac{f_{u,v} - f}{\sigma} \right) \right) > \epsilon \text{ for some } \lambda_0 > 0 \right\}.$$

From the regularity condition of our proposed mean, we have

$$P \lim_{u,v} \frac{1}{T_m S_n} \sum_{u,v=a_n+1}^{b_m,b_n} t_u s_v = 0. \tag{2}$$

Thus, we obtain

$$\begin{aligned}
\omega\left(\lambda_0\left(N_D\left(\frac{f_{m,n}-f}{\sigma}\right)\right)\right) &= \omega\left(\lambda_0\left(\frac{1}{T_mS_n}\sum_{u,v=a_n+1}^{b_m,b_n} t_us_v\left(\frac{f_{u,v}-f}{\sigma}\right)\right)\right)\\
&\leqq \omega\Bigg(\frac{\lambda_0}{T_mS_n}\sum_{\substack{u,v=a_n+1,\\(u,v)\in\mathcal{H}_\epsilon}}^{b_m,b_n} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right| + \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=0,v=b_n+1,\\(u,v)\in\mathcal{H}_\epsilon}}^{b_m,\infty} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right|\\
&+ \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=b_m+1,v=0,\\(u,v)\in\mathcal{H}_\epsilon}}^{\infty,b_n} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right| + \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=b_m+1,v=b_n+1,\\(u,v)\in\mathcal{H}_\epsilon}}^{\infty,\infty} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right|\\
&+ \omega\Bigg(\frac{\lambda_0}{T_mS_n}\sum_{\substack{u,v=a_n+1,\\(u,v)\in\mathcal{H}_\epsilon^c}}^{b_m,b_n} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right| + \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=0,v=b_n+1,\\(u,v)\in\mathcal{H}_\epsilon^c}}^{b_m,\infty} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right|\\
&+ \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=b_m+1,v=0;\\(u,v)\in\mathcal{H}_\epsilon^c}}^{\infty,b_n} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right| + \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=b_m+1,v=b_n+1\\(u,v)\in\mathcal{H}_\epsilon^c}}^{\infty,\infty} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right|\\
&+ \mathcal{K}\left|\frac{1}{T_mS_n}\sum_{u,v=a_n+1}^{\infty,\infty} t_us_v - 1\right|\Bigg),
\end{aligned}$$

where

$$\mathcal{K} = \sup_{x,y}\left|\frac{f(x,y)}{\sigma}\right|.$$

Further, $\omega$ being $\mathcal{N}$-Quasi convex modular, monotone, and strongly finite on $L^\omega(\mathcal{I}^2)$, it follows that

$$\begin{aligned}
\omega\left(\lambda_0\left(N_D\left(\frac{f_{m,n}-f}{\sigma}\right)\right)\right) &\leqq 3\omega\left(\frac{9\lambda_0|\mathcal{H}_\epsilon|G}{T_mS_n}\sum_{\substack{u,v=a_n+1,\\(u,v)\in\mathcal{H}_\epsilon}}^{b_m,b_n} t_us_v\right)\\
&+ \epsilon\omega\left(\frac{9\lambda_0|\mathcal{H}_\epsilon|}{T_mS_n}\sum_{\substack{u,v=a_n+1,\\(u,v)\in\mathcal{H}_\epsilon}}^{b_m,b_n} t_us_v\right) + \omega\left(\frac{9\lambda_0Gb_mb_n}{T_mS_n}\sum_{u,v=a_n+1}^{b_m,b_n} t_us_v\right)\\
&+ \omega\left(\frac{9\lambda_0Gb_m}{T_mS_n}\sum_{u=0,v=a_n+1}^{b_m,\infty} t_us_v\right) + \omega\left(\frac{9\lambda_0Gb_n}{T_mS_n}\sum_{u=a_n+1,v=0}^{\infty,b_m} t_us_v\right)\\
&+ \epsilon\omega\left(\frac{9\lambda_0}{T_mS_n}\sum_{u,v=a_n+1}^{\infty,\infty} t_us_v\right) + \omega\left(\frac{9\lambda_0\mathcal{K}}{T_mS_n}\sum_{u,v=a_n+1}^{\infty,\infty} t_us_v - 1\right),
\end{aligned}$$

where $G = \max\left|\frac{f_{u,v}-f(x,y)}{\sigma}\right|, \forall\, u,v \in \mathbb{N}$ and $(x,y) \in \mathcal{I}^2$. In the last inequality, considering $P$ limit as $m,n \to \infty$ under the regularity conditions of deferred weighted mean and by using (2), we obtain

$$P\lim_{m,n}\omega\left(\lambda_0\left(\frac{N_D(f_{m,n})-f}{\sigma}\right)\right) = 0.$$

This implies that $(f_{m,n})$ is relatively modular deferred weighted $N_D(f_{m,n})$-summable to a function $f$. Hence,

$$P\lim_{m,n}\frac{1}{m,n}\left|\left\{(u,v) : u \leqq m, v \leqq m \text{ and } \omega\left(\lambda_0\left(\frac{N_D(f_{m,n})-f}{\sigma}\right)\right) \geqq \epsilon\right\}\right| = 0 \text{ for some } \lambda_0 > 0.$$

Next, to see that the converse part of the theorem is not necessarily true, we consider the following example.

**Example 3.** *Suppose that* $\mathcal{I} = [0,1]$ *and let* $\varphi : [0,\infty) \to [0,\infty)$ *be a continuous function with* $\varphi(0) = 0$, $\varphi(u) > 0$ *for* $u > 0$ *and* $\lim_{u\to\infty}\varphi(u) = \infty$. *Let* $f \in X(\mathcal{I}^2)$ *be a measurable real-valued function, and consider the functional* $\omega^{\varphi}$ *on* $X(\mathcal{I}^2)$ *defined by*

$$\omega^{\varphi}(f) = \int_0^1 \int_0^1 \varphi(|f_{m,n}(x,y)|)dxdy \quad (f \in X(\mathcal{I}^2).$$

$\varphi$ *being convex,* $\omega^{\varphi}$ *is modular convex on* $X(\mathcal{I}^2)$*, which satisfies the above assumptions. Consider* $L^{\omega}_{\varphi}(\mathcal{I}^2)$ *as the Orlicz space produced by* $\varphi$ *of the form:*

$$L^{\omega}_{\varphi}(\mathcal{I}^2) = \{f \in X(\mathcal{I}^2) : \omega^{\varphi}(\lambda(f)) < +\infty \text{ for some } \lambda > 0\}.$$

*For all* $m,n \in \mathbb{N}$*, we consider a double sequence of functions* $f_{m,n} : [0,1] \times [0,1] \to \mathbb{R}$ *defined by*

$$f_{m,n}(x,y) = \begin{cases} 1, & (m,n) \in \mathfrak{U}\times\mathfrak{U} \text{ and } (x,y) \in (0,\frac{1}{m}] \times (0,\frac{1}{n}], \\ 0, & \{(m,n) \in \mathfrak{V}\times\mathfrak{V} \text{ and } (x,y) \in (\frac{1}{m},0] \times (\frac{1}{n},1]; \\ & (m,n) \in \mathfrak{U}\times\mathfrak{V} \text{ or } (m,n) \in \mathfrak{V}\times\mathfrak{U} \text{ or } (x,y) \in (0,0)\}, \end{cases}$$

*where the set of all odd and even numbers are* $\mathfrak{U}$ *and* $\mathfrak{V}$*, respectively.*

*We have*

$$\omega\lambda(N_D(f_{m,n})) = \omega\left(\frac{\lambda_0}{S_m T_n}\sum_{u,v=a_n+1}^{b_m,b_n} t_u s_v\right),$$

*and this implies*

$$\omega\lambda(N_D(f_{m,n})) = \lambda_0 \begin{cases} \int_0^{1/b_m}\int_0^{1/b_n} dxdy, & (m,n) \in \mathfrak{U}\times\mathfrak{U} \text{ and } (x,y) \in (0,\frac{1}{m}] \times (0,\frac{1}{n}], \\ 0, & \{(m,n) \in \mathfrak{V}\times\mathfrak{V} \text{ and } (x,y) \in (\frac{1}{m},0] \times (\frac{1}{n},1]; \\ & (m,n) \in \mathfrak{U}\times\mathfrak{V} \text{ or } (m,n) \in \mathfrak{V}\times\mathfrak{U} \text{ or } (x,y) \in (0,0)\}. \end{cases}$$

*Clearly,* $(f_{m,n})$ *is relatively modular deferred weighted summable to* $f = 0$*, with respect to a non-zero scale function* $\sigma(x,y)$ *such that*

$$\sigma(x,y) = \begin{cases} 1, & (x,y) = (0,0) \\ \frac{1}{xy}, & (x,y) \in (0,1] \times (0,1]. \end{cases}$$

*That is,*

$$P\lim_{m,n}\omega\left(\lambda_0\left(\frac{N_D(f_{m,n}) - f}{\sigma}\right)\right) = 0 \text{ for some } \lambda_0 > 0.$$

*Thus, we have*

$$P\lim_{m,n}\frac{1}{m,n}\left|\left\{(u,v) : u \leqq m, v \leqq m \text{ and } \omega\left(\lambda_0\left(\frac{N_D(f_{m,n}) - f}{\sigma}\right)\right) \geqq \epsilon\right\}\right| = 0 \text{ for some } \lambda_0 > 0.$$

*On the other hand, it is not relatively modular deferred-weighted statistically convergent to the function* $f = 0$*, that is,*

$$P \lim_{m,n} \frac{1}{T_m S_n} \left| \left\{ (u,v) : u \leqq T_m, v \leqq S_m \text{ and } \omega \left( \lambda_0 \left( \frac{t_u s_v |f_{u,v} - f|}{\sigma} \right) \right) \geqq \epsilon \right\} \right| \neq 0 \text{ for some } \lambda_0 > 0.$$

□

## 3. A Korovkin-Type Theorem in Modular Space

In this section, we extend here the result of Demirci and Orhan [31] by using the idea of the statistically and relatively modular deferred-weighted summability of a double sequence of positive linear operators defined over a modular space.

Let $\omega$ be a finite modular and monotone over $X(\mathcal{I}^2)$. Suppose $E$ is a set such that $C^\infty(\mathcal{I}^2) \subset E \subset L^\omega(\mathcal{I}^2)$. We can construct such a subset $E$ when $\omega$ is monotone and finite. We also assume $L = \{\mathcal{L}_{m,n}\}$ as the sequence of positive linear operators from $E$ in to $X(\mathcal{I}^2)$, and there exists a subset $X_L \subset E$ containing $C^\infty(\mathcal{I}^2)$. Let $\sigma \in X(\mathcal{I}^2)$ be an unbounded function with $|\sigma(x,y)| \neq 0$, and $R$ is a positive constant such that

$$N_D\text{stat} \limsup_{m,n} \omega \left( \lambda \left( \frac{\Upsilon_{m,n}(f)}{\sigma} \right) \right) \leqq R\omega(\lambda f) \tag{3}$$

holds for each $f \in X_L, \lambda > 0$ and

$$\Upsilon_{m,n}(f;x,y) = \frac{1}{T_m S_n} \sum_{u,v=a_n+1}^{b_m,b_n} t_u s_v \mathcal{T}_{m,n}(f;x,y).$$

We denote here the value of $\mathcal{L}_{m,n}(f)$ at a point $(x,y) \in \mathcal{I}^2$ by $\mathcal{L}_{m,n}(f(x^*,y^*);x,y)$, or briefly by $\mathcal{L}_{m,n}(f;x,y)$. We now prove the following theorem.

**Theorem 2.** *Let $(a_n)$ and $(b_n)$ be the sequences of non-negative integers and let $\omega$ be an $\mathcal{N}$-Quasi semi-convex modular, absolutely continuous, strongly finite, and monotone on $X(\mathcal{I}^2)$. Assume that $L = \{\mathcal{L}_{m,n}\}$ is a double sequence of positive linear operators from $E$ in to $X(\mathcal{I}^2)$ that satisfy the assumption (3) for every $f \in X_L$ and suppose that $\sigma_i(x,y)$ is an unbounded function such that $|\sigma_i(x,y)| \geqq u_i > 0$ $(i = 0,1,2,3)$. Assume further that*

$$N_D\text{stat} \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(f_i;x,y) - f(x,y)}{\sigma} \right\|_\omega = 0 \text{ for each } \lambda > 0 \text{ and } i = 0,1,2,3, \tag{4}$$

*where*

$$f_0(x,y) = 1, \ f_1(x,y) = x, \ f_2(x,y) = y \text{ and } f_3(x,y) = x^2 + y^2.$$

*Then, for every $f \in L^\omega(\mathcal{I}^2)$ and $g \in C^\infty(\mathcal{I}^2)$ with $f - g \in X_L$,*

$$N_D\text{stat} \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(f;x,y) - f(x,y)}{\sigma} \right\|_\omega = 0 \text{ for every } \lambda_0 > 0, \tag{5}$$

*where $\sigma(x,y) = \max\{|\sigma_i(x,y)| : i = 0,1,2,3\}$.*

**Proof.** First we claim that,

$$N_D\text{stat} \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(g;x,y) - g(x,y)}{\sigma} \right\|_\omega = 0 \text{ for every } \lambda_0 > 0. \tag{6}$$

In order to justify our claim, we assume that $g \in C(\mathcal{I}^2) \cap E$. Since $g$ is continuous on $\mathcal{I}^2$, for given $\epsilon > 0$, there exists a number $\delta > 0$ such that for every $(x^*,y^*),(x,y) \in \mathcal{I}^2$ with $|x^* - x| < \delta$ and $|y^* - y| < \delta$, we have

$$|g(x^*,y^*) - g(x,y)| < \epsilon. \tag{7}$$

Also, for all $(x^*, y^*), (x, y) \in \mathcal{I}^2$ with $|x^* - x| > \delta$ and $|x^* - x| > \delta$, we have

$$|g(x^*, y^*) - g(x, y)| < \frac{2\mathcal{A}}{\delta^2}\left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2\right), \tag{8}$$

where

$$\varphi_1(x^*, x) = (x^* - x), \quad \varphi_2(y^*, y) = (y^* - y), \quad \text{and} \quad \mathcal{A} = \sup_{x,y \in \mathcal{I}^2} |g(x, y)|.$$

From Equations (7) and (8), we obtain

$$|g(x^*, y^*) - g(x, y)| < \epsilon + \frac{2\mathcal{A}}{\delta^2}\left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2\right).$$

This implies that

$$-\epsilon - \frac{2\mathcal{A}}{\delta^2}\left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2\right) < g(x^*, y^*) - g(x, y) < \epsilon + \frac{2\mathcal{A}}{\delta^2}\left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2\right). \tag{9}$$

Now $\mathcal{L}_{m,n}(g_0; x, y)$ being linear and monotone, by applying the operator $\mathcal{L}_{m,n}(g_0; x, y)$ to this inequality (9), we fairly have

$$\begin{aligned} \mathcal{L}_{m,n}(g_0; x, y)\left(-\epsilon - \frac{2\mathcal{A}}{\delta^2}\left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2\right)\right) &< \mathcal{L}_{m,n}(g_0; x, y)(g(x^*, y^*) - g(x, y)) \\ &< \mathcal{L}_{m,n}(g_0; x, y)\left(\epsilon + \frac{2\mathcal{A}}{\delta^2}\left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2\right)\right). \end{aligned} \tag{10}$$

Note that $x, y$ is fixed, and so also $g(x, y)$ is a constant number. This implies that

$$\begin{aligned} -\epsilon\mathcal{L}_{m,n}(g_0; x, y) - \frac{2\mathcal{A}}{\delta^2}\mathcal{L}_{m,n}\left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2; x, y\right) &< \mathcal{L}_{m,n}(g; x, y) - g(x, y)\mathcal{L}_{m,n}(g_0; x, y) \\ &< \epsilon\mathcal{L}_{m,n}(g_0; x, y) + \frac{2\mathcal{A}}{\delta^2}\mathcal{L}_{m,n}([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2; x, y). \end{aligned} \tag{11}$$

However,

$$\mathcal{L}_{m,n}(g; x, y) - g(x, y) = [\mathcal{L}_{m,n}(g; x, y) - g(x, y)\mathcal{L}_{m,n}(g_0; x, y)] + g(x, y)[\mathcal{L}_{m,n}(g_0; x, y) - g_0(x, y)]. \tag{12}$$

Now, using (11) and (12), we have

$$\begin{aligned} |\mathcal{L}_{m,n}(g; x, y) - g(x, y)| \leqq &\left|\epsilon\mathcal{L}_{m,n}(g_0; x, y) + \frac{2\mathcal{A}}{\delta^2}\mathcal{L}_{m,n}\left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2; x, y\right)\right| \\ &+ \mathcal{A}[\mathcal{L}_{m,n}(g_0; x, y) - g_0(x, y)]. \end{aligned} \tag{13}$$

Next,

$$\begin{aligned} |\mathcal{L}_{m,n}(g; x, y) - g(x, y)| = \epsilon + (\epsilon + \mathcal{A})[\mathcal{L}_{m,n}(g_0; x, y) - g_0(x, y)] - \frac{4\mathcal{A}}{\delta^2}|g_1(x, y)|[\mathcal{L}_{m,n}(g_1; x, y) - g_1(x, y)] \\ + \frac{2\mathcal{A}}{\delta^2}[\mathcal{L}_{m,n}(g_3; x, y) - g_3(x, y)] - \frac{4\mathcal{A}}{\delta^2}|g_2(x, y)|[\mathcal{L}_{m,n}(g_2; x, y) - g_2(x, y)] \\ + \frac{2\mathcal{A}}{\delta^2}|g_3(x, y)|[\mathcal{L}_{m,n}(g_0; x, y) - g_0(x, y)]. \end{aligned}$$

Since the choice of $\epsilon$ is arbitrarily small, we can easily write

$$\begin{aligned}|\mathcal{L}_{m,n}(g;x,y)-g(x,y)| \leqq{}& \epsilon+\left(\epsilon+\frac{2\mathcal{A}}{\delta^2}+\mathcal{A}\right)|\mathcal{L}_{m,n}(g_0;x,y)-g_0(x,y)| \\ &+\frac{4\mathcal{A}}{\delta^2}|g_1(x,y)||\mathcal{L}_{m,n}(g_1;x,y)-g_1(x,y)|+\frac{2\mathcal{A}}{\delta^2}|\mathcal{L}_{m,n}(g_3;x,y)-g_3(x,y)| \quad (14)\\ &-\frac{4\mathcal{A}}{\delta^2}|g_2(x,y)||\mathcal{L}_{m,n}(g_2;x,y)-g_2(x,y)|.\end{aligned}$$

Now multiplying $\frac{1}{\sigma(x,y)}$ to both sides of (14), we have, for any $\lambda > 0$

$$\begin{aligned}\lambda\left|\frac{\mathcal{L}_{m,n}(g;x,y)-g(x,y)}{\sigma(x,y)}\right| \leqq{}& \frac{\lambda\epsilon}{\sigma(x,y)}+\lambda\mathcal{B}\Bigg\{\left|\frac{\mathcal{L}_{m,n}(g_0;x,y)-g_0(x,y)}{\sigma(x,y)}\right| \\ &+\left|\frac{\mathcal{L}_{m,n}(g_1;x,y)-g_1(x,y)}{\sigma(x,y)}\right|+\left|\frac{\mathcal{L}_{m,n}(g_3;x,y)-g_3(x,y)}{\sigma(x,y)}\right| \quad (15)\\ &-\left|\frac{\mathcal{L}_{m,n}(g_2;x,y)-g_2(x,y)}{\sigma(x,y)}\right|\Bigg\},\end{aligned}$$

where $\mathcal{B}=\max\left(\epsilon+\frac{2\mathcal{A}}{\delta^2}+\mathcal{A}, \frac{4\mathcal{A}}{\delta^2}, \frac{2\mathcal{A}}{\delta^2}\right)$ and $g_1(x,y)$, $g_2(x,y)$ are constants for $\forall\ (x,y)$.

Next, applying the modular $\omega$ to the above inequality, also $\omega$ being $\mathcal{N}$-Quasi semi-convex, strongly finite, monotone, and $\sigma(x,y)=\max\{|\sigma_i(x,y)\ \ (i=0,1,2,3)|\}$, we have

$$\begin{aligned}\omega\left(\lambda\left(\frac{\mathcal{L}_{m,n}(g;x,y)-g(x,y)}{\sigma(x,y)}\right)\right) \leqq{}& \omega\left(\frac{5\lambda\epsilon}{\sigma(x,y)}\right)+\omega\left(5\lambda\mathcal{B}\left(\frac{\mathcal{L}_{m,n}(g_0;x,y)-g_0(x,y)}{\sigma_0(x,y)}\right)\right) \\ &+\omega\left(5\lambda\mathcal{B}\left(\frac{\mathcal{L}_{m,n}(g_1;x,y)-g_1(x,y)}{\sigma_1(x,y)}\right)\right) \\ &+\omega\left(5\lambda\mathcal{B}\left(\frac{\mathcal{L}_{m,n}(g_3;x,y)-g_3(x,y)}{\sigma_2(x,y)}\right)\right) \quad (16)\\ &-\omega\left(5\lambda\mathcal{B}\left(\frac{\mathcal{L}_{m,n}(g_2;x,y)-g_2(x,y)}{\sigma_3(x,y)}\right)\right).\end{aligned}$$

Now, replacing $\mathcal{L}_{m,n}(f;x,y)$ by

$$\frac{1}{S_m T_n}\sum_{u,v=a_n+1}^{b_m,b_n} s_u t_v \mathcal{T}_{u,v}(g;x,y)=\Upsilon_{m,n}(f;x,y)$$

and then by $\Psi(f;x,y)$ in (16), for a given $\kappa > 0$ there exists $\epsilon > 0$, such that $\omega\left(\frac{5\lambda\epsilon}{\sigma}\right) < \kappa$. Then, by setting

$$\Psi=\left\{(m,n):\omega\left(\lambda\left(\frac{\Upsilon_{m,n}(g)-g}{\sigma}\right)\right)\geqq\kappa\right\}$$

and for $i=0,1,2,$

$$\Psi_i=\left\{(m,n):\omega\left(\lambda\left(\frac{\Upsilon_{m,n}(g_i)-g}{\sigma_i}\right)\right)\geqq\frac{\kappa-\omega\left(\frac{5\lambda\epsilon}{\sigma}\right)}{4\mathcal{B}}\right\},$$

we obtain

$$\Psi\leqq\sum_{i=0}^{3}\Psi_i.$$

Clearly,

$$\frac{\|\Psi\|_{\omega}}{mn} \leqq \sum_{i=0}^{3} \frac{\|\Psi_i\|_{\omega}}{mn}. \tag{17}$$

Now, by the assumption under (4) as well as by Definition 4, the right-hand side of (17) tends to zero as $m, n \to \infty$. Clearly, we get

$$\lim_{m,n\to\infty} \frac{\|\Psi\|_{\omega}}{mn} = 0 \ (\kappa > 0),$$

which justifies our claim (6). Hence, the implication (6) is fairly obvious for each $g \in C^{\infty}(\mathcal{I}^2)$.

Now let $f \in L^{\omega}(\mathcal{I}^2)$ such that $f - g \in X_L$ for every $g \in C^{\infty}(\mathcal{I}^2)$. Also, $\omega$ is absolutely continuous, monotone, strongly and absolutely finite on $X(\mathcal{I}^2)$. Thus, it is trivial that the space $C^{\infty}(\mathcal{I}^2)$ is modularly dense in $L^{\omega}(\mathcal{I}^2)$. That is, there exists a sequence $(g_{i,j}) \in C^{\infty}(\mathcal{I}^2)$ provided that $\omega(3\lambda_0^* g) < +\infty$ and

$$P \lim_{i,j} \omega(3\lambda_0^*(g_{i,j} - f)) = 0 \ \text{ for some } \ \lambda_0^*. \tag{18}$$

This implies that for each $\epsilon > 0$ there exist two positive integers $\bar{i}$ and $\bar{j}$ such that

$$\omega(3\lambda_0^*(g_{i,j} - f)) < \epsilon \ \text{ whenever } \ i \geqq \bar{i} \text{ and } j \geqq \bar{j}.$$

Further, since the operators $\Upsilon_{m,n}$ are positive and linear, we have that

$$\begin{aligned} \lambda_0^*|\Upsilon_{m,n}(f;x,y) - f(x,y)| \leqq \lambda_0^*|\Upsilon_{m,n}(f - g_{\bar{i},\bar{j}};x,y)| &+ \lambda_0^*|\Upsilon_{m,n}(g_{\bar{i},\bar{j}};x,y) - g_{\bar{i},\bar{j}}(x,y)| \\ &+ \lambda_0^*|g_{\bar{i},\bar{j}}(x,y) - f(x,y)| \end{aligned}$$

holds true for each $m, n \in \mathbb{N}$ and $x, y \in \mathcal{I}$. Applying the monotonicity of modular $\omega$ and further multiplying $\frac{1}{\sigma(x,y)}$ to both sides of the above inequality, we have

$$\begin{aligned} \omega\left(\lambda_0^*\left(\frac{\Upsilon_{m,n}(f;x,y) - f(x,y)}{\sigma}\right)\right) &\leqq \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(f - g_{\bar{i},\bar{j}})}{\sigma}\right)\right) \\ &\quad + \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}}) - g_{\bar{i},\bar{j}}}{\sigma}\right)\right) + \omega\left(3\lambda_0^*\left(\frac{g_{\bar{i},\bar{j}} - f}{\sigma}\right)\right). \end{aligned}$$

Thus, for $|\sigma(x,y)| \geqq M > 0$ ($M = \max\{M_i : i = 0, 1, 2, 3\}$), we can write

$$\begin{aligned} \omega\left(\lambda_0^*\left(\frac{\Upsilon_{m,n}(f) - f}{\sigma}\right)\right) &\leqq \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(f - g_{\bar{i},\bar{j}})}{\sigma}\right)\right) \\ &\quad + \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}}) - g_{\bar{i},\bar{j}}}{\sigma}\right)\right) + \omega\left(\frac{3\lambda_0^*}{M}\left(g_{\bar{i},\bar{j}} - f\right)\right). \end{aligned} \tag{19}$$

Then, it follows from (18) and (19) that

$$\omega\left(\lambda_0^*\left(\frac{\Upsilon_{m,n}(f) - f}{\sigma}\right)\right) \leqq \epsilon + \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(f - g_{\bar{i},\bar{j}})}{\sigma}\right)\right) + \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}}) - g_{\bar{i},\bar{j}}}{\sigma}\right)\right). \tag{20}$$

Now, taking statistical limit superior as $m, n \to \infty$ on both sides of (20) and also using (3), we deduce that

$$
\begin{aligned}
P\ \limsup_{m,n} \omega\left(\lambda_0^*\left(\frac{\Upsilon_{m,n}(f)-f}{\sigma}\right)\right) \leqq \epsilon + R\omega\left(3\lambda_0^*(f-g_{\bar{i},\bar{j}})\right) \\
+ P\ \limsup_{m,n} \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}})-g_{\bar{i},\bar{j}}}{\sigma}\right)\right).
\end{aligned}
$$

Thus, it implies that

$$
P\ \limsup_{m,n} \omega\left(\lambda_0^*\left(\frac{\Upsilon_{m,n}(f)-f}{\sigma}\right)\right) \leqq \epsilon + \epsilon R + P\ \limsup_{m,n} \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}})-g_{\bar{i},\bar{j}}}{\sigma}\right)\right). \tag{21}
$$

Next, by (4), for some $\lambda_0^* > 0$, we obtain

$$
P\ \limsup_{m,n} \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}})-g_{\bar{i},\bar{j}}}{\sigma}\right)\right) = 0. \tag{22}
$$

Clearly from (21) and (22), we get

$$
P\ \limsup_{m,n} \omega\left(\lambda_0^*\left(\frac{\Upsilon_{m,n}(f)-f}{\sigma}\right)\right) \leqq \epsilon(1+R).
$$

Since $\epsilon > 0$ is arbitrarily small, the right-hand side of the above inequality tends to zero. Hence,

$$
P\ \limsup_{m,n} \omega\left(\lambda_0^*\left(\frac{\Upsilon_{m,n}(f)-f}{\sigma}\right)\right) = 0,
$$

which completes the proof. □

Next, one can get the following theorem as an immediate consequence of Theorem 2 in which the modular $\omega$ satisfies the $\Delta_2$-condition.

**Theorem 3.** *Let $(\mathcal{L}_{m,n})$, $(a_n)$, $(b_n)$, $\sigma$ and $\omega$ be the same as in Theorem 2. If the modular $\omega$ satisfies the $\Delta_2$-condition, then the following assertions are identical:*

*(a)* $N_D stat\ \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(f_i;x,y)-f(x,y)}{\sigma} \right\|_{\omega} = 0$ *for each* $\lambda > 0$ *and* $i = 0,1,2,3;$

*(b)* $N_D stat\ \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(f;x,y)-f(x,y)}{\sigma} \right\|_{\omega} = 0$ *for each* $\lambda > 0$ *such that any function* $f \in L^{\omega}(\mathcal{I}^2)$ *provided that* $f - g \in X_L$ *for each* $g \in C^{\infty}(\mathcal{I}^2)$.

Next, by using the definitions of relatively modular deferred-weighted statistical convergence given in Definition 3 and statistically as well as relatively modular deferred-weighted summability given in Definition 4, we present the following corollaries in view of Theorem 2.

Let $a_n = 0$ and $b_n = n,\ b_m = m$, then Equation (3) reduces to

$$
\text{stat}_N\ \limsup_{m,n} \omega\left(\lambda\left(\frac{\mathfrak{L}_{m,n}(f)}{\sigma}\right)\right) \leqq R\omega(\lambda f) \tag{23}
$$

for each $f \in X_L$ and $\lambda > 0$, where $R$ is a constant.

Moreover, if we replace $\text{stat}_N$ limit by $N$stat limit, then Equation (3) reduces to

$$
N\text{stat}\ \limsup_{m,n} \omega\left(\lambda\left(\frac{\Omega_{m,n}(f)}{\sigma}\right)\right) \leqq R\omega(\lambda f). \tag{24}
$$

**Corollary 1.** *Let $\omega$ be an $\mathcal{N}$-Quasi semi-convex modular, strongly finite, monotone, and absolutely continuous on $X(\mathcal{I}^2)$. Also, let $(\mathfrak{L}_{m,n})$ be a double sequence of positive linear operators from E in to $X(\mathcal{I}^2)$ satisfying the* *assumption* (23) *for every* $X_L$ *and* $\sigma_i(x,y)$ *be an unbounded function such that* $|\sigma_i(x,y)| \geqq u_i > 0 \quad (i = 0,1,2,3)$. *Suppose that*

$$stat_N \lim_{m,n} \left\| \frac{\mathfrak{L}_{m,n}(f_i;x,y) - f(x,y)}{\sigma} \right\|_{\omega} = 0 \text{ for each } \lambda > 0 \text{ and } i = 0,1,2,3,$$

*where*

$$f_0(x,y) = 1, \ f_1(x,y) = x, \ f_2(x,y) = y \text{ and } f_3(x,y) = x^2 + y^2.$$

*Then, for every* $f \in L^{\omega}(\mathcal{I}^2)$ *and* $g \in C^{\infty}(\mathcal{I}^2)$ *with* $f - g \in X_L$,

$$stat_N \lim_{m,n} \left\| \frac{\mathfrak{L}_{m,n}(f;x,y) - f(x,y)}{\sigma} \right\|_{\omega} = 0 \text{ for each } \lambda_0 > 0,$$

*where*

$$\sigma(x,y) = \max\{|\sigma_i(x,y)| : i = 0,1,2,3\}. \tag{25}$$

**Corollary 2.** *Let* $\omega$ *be an* $\mathcal{N}$*-Quasi semi-convex modular, absolutely continuous, monotone, and strongly finite on* $X(\mathcal{I}^2)$. *Also, let* $\Omega_{m,n}$ *be a double sequence of positive linear operators from E in to* $X(\mathcal{I}^2)$ *satisfying the assumption* (24) *for every* $X_L$ *and* $\sigma_i(x,y)$ *be an unbounded function such that* $|\sigma_i(x,y)| \geqq u_i > 0 \quad (i = 0,1,2,3)$. *Suppose that*

$$Nstat \lim_{m,n} \left\| \frac{\Omega_{m,n}(f_i;x,y) - f(x,y)}{\sigma} \right\|_{\omega} = 0 \text{ for each } \lambda > 0 \text{ and } i = 0,1,2,3,$$

*where*

$$f_0(x,y) = 1, \ f_1(x,y) = x, \ f_2(x,y) = y \text{ and } f_3(x,y) = x^2 + y^2.$$

*Then, for every* $f \in L^{\omega}(\mathcal{I}^2)$ *and* $g \in C^{\infty}(\mathcal{I}^2)$ *with* $f - g \in X_L$,

$$Nstat \lim_{m,n} \left\| \frac{\Omega_{m,n}(f;x,y) - f(x,y)}{\sigma} \right\|_{\omega} = 0 \text{ for every } \lambda_0 > 0,$$

*where* $\sigma$ *is given by* (25).

Note that for $a_n = 0$, $b_n = n$, $b_m = m$, and $s_m = 1 = t_n$, Equation (3) reduces to

$$\text{stat} \limsup_{m,n} \omega\left(\lambda\left(\mathfrak{L}^{*}_{m,n}(f)\right)\right) \leqq R\omega(\lambda f) \tag{26}$$

for each $f \in X_L$ and $\lambda > 0$, where $R$ is a positive constant.

Also, if we replace statistically convergent limit by the statistically summability limit, then Equation (3) reduces to

$$\text{stat} \limsup_{m,n} \omega\left(\lambda\left(\Lambda_{m,n}(f)\right)\right) \leqq R\omega(\lambda f). \tag{27}$$

Now, we present the following corollaries in view of Theorem 2 as the generalization of the earlier results of Demirci and Orhan [31].

**Corollary 3.** *Let* $\omega$ *be an* $\mathcal{N}$*-Quasi semi-convex modular, absolutely continuous, monotone, and strongly finite on* $X(\mathcal{I}^2)$. *Also, let* $(\mathfrak{L}^{*}_{m,n})$ *be a double sequence of positive linear operators from E in to* $X(\mathcal{I}^2)$ *satisfying the assumption* (26) *for every* $X_L$ *and* $\sigma_i(x,y)$ *be an unbounded function such that* $|\sigma_i(x,y)| \geqq u_i > 0 \quad (i = 0,1,2,3)$. *Suppose that*

$$\mathit{stat}\lim_{m,n}\left\|\frac{\mathfrak{L}^*_{m,n}(f_i;x,y)-f(x,y)}{\sigma}\right\|_\omega = 0 \text{ for every } \lambda>0 \text{ and } i=0,1,2,3,$$

*where*

$$f_0(x,y)=1,\ \ f_1(x,y)=x,\ \ f_2(x,y)=y \text{ and } f_3(x,y)=x^2+y^2.$$

*Then, for every* $f\in L^\omega(\mathcal{I}^2)$ *and* $g\in C^\infty(\mathcal{I}^2)$ *with* $f-g\in X_L$,

$$\mathit{stat}\lim_{m,n}\left\|\frac{\mathfrak{L}^*_{m,n}(f;x,y)-f(x,y)}{\sigma}\right\|_\omega = 0 \text{ for every } \lambda_0>0,$$

*where* $\sigma$ *is given by* (25).

**Corollary 4.** *Let* $\omega$ *be an* $\mathcal{N}$*-Quasi semi-convex modular, monotone, absolutely continuous, and strongly finite on* $X(\mathcal{I}^2)$*. Also, let* $(\Lambda_{m,n})$ *be a double sequence of positive linear operators from* $E$ *in to* $X(\mathcal{I}^2)$ *satisfying the assumption* (27) *for every* $X_L$ *and* $\sigma_i(x,y)$ *be an unbounded function such that* $|\sigma_i(x,y)|\geqq u_i>0\ (i=0,1,2,3)$*. Suppose that*

$$\mathit{stat}\lim_{m,n}\left\|\frac{\Lambda_{m,n}(f_i;x,y)-f(x,y)}{\sigma}\right\|_\omega = 0 \text{ for every } \lambda>0 \text{ and } i=0,1,2,3,$$

*where*

$$f_0(x,y)=1,\ \ f_1(x,y)=x,\ \ f_2(x,y)=y \text{ and } f_3(x,y)=x^2+y^2.$$

*Then, for every* $f\in L^\omega(\mathcal{I}^2)$ *and* $g\in C^\infty(\mathcal{I}^2)$ *with* $f-g\in X_L$,

$$\mathit{stat}\lim_{m,n}\left\|\frac{\Lambda_{m,n}(f;x,y)-f(x,y)}{\sigma}\right\|_\omega = 0 \text{ for every } \lambda_0>0,$$

*where* $\sigma$ *is given by* (25).

## 4. Application of Korovkin-Type Theorem

In this section, by presenting a further example, we demonstrate that our proposed Korovkin-type approximation results in modular space are stronger than most (if not all) of the previously existing results in view of the corollaries provided in this paper.

Let $\mathcal{I}=[0,1]$ and $\varphi$, $\omega^\varphi$, and $L^\omega_\varphi(\mathcal{I}^2)$ be as given in Example 3. Also, recall the *bivariate Bernstein–Kantorovich* operators (see [35]), $\mathbb{B}=\{B_{m,n}\}$ on the space $L^\omega_\varphi(\mathcal{I}^2)$ given by

$$B_{m,n}(f;x,y)=\sum_{i,j=0}^{m,n}p^{(m,n)}_{i,j}(x,y)(m+1)(n+1)\times\int_{\frac{i}{m+1}}^{\frac{i+1}{m+1}}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}f(s,t)dsdt \tag{28}$$

for $x,y\in\mathcal{I}$ and

$$p^{(m,n)}_{i,j}(x,y)=\binom{m}{i}\binom{n}{j}x^iy^j(1-x)^{m-i}(1-y)^{n-j}.$$

Also, we have

$$\sum_{i,j=0}^{m,n}p^{(m,n)}_{i,j}(x,y)=1. \tag{29}$$

Clearly, we observe that

$$\begin{aligned} B_{m,n}(1;x,y) &= 1, \\ B_{m,n}(s;x,y) &= \frac{mx}{m+1} + \frac{1}{2(m+1)}, \\ B_{m,n}(t;x,y) &= \frac{ny}{n+1} + \frac{1}{2(n+1)} \end{aligned}$$

and

$$\begin{aligned} B_{m,n}(t^2+s^2;x,y) = {} & \frac{m(m-1)x^2}{(m+1)^2} + \frac{2mx}{(m+1)^2} \\ & + \frac{1}{3(m+1)^2}\frac{n(n-1)y^2}{(n+1)^2} + \frac{2ny}{(n+1)^2} + \frac{1}{3(n+1)^2}. \end{aligned}$$

It is further observed that $B_{m,n} : L^{\omega}_{\varphi}(\mathcal{I}^2) \to L^{\omega}_{\varphi}(\mathcal{I}^2)$. Recall [28] (Lemma 5.1) and [29] (Example 1). Now because of (29), we have from *Jensen inequality*, for each $f \in L^{\omega}_{\varphi}(\mathcal{I}^2)$ and $m, n \in \mathbb{N}$, there exists a constant $M$ such that

$$\omega^{\varphi}\left(\frac{B_{m,n}(f;x,y)}{\sigma}\right) \leqq M\omega^{\varphi}(f).$$

We now present an illustrative example for the validity of the operators $(\mathcal{L}_{m,n})$ for our Theorem 2.

**Example 4.** *Let* $\mathcal{L}_{m,n} : L^{\omega}(\mathcal{I}^2) \to L^{\omega}(\mathcal{I}^2)$ *be defined by*

$$\mathcal{L}_{m,n}(f;x,y) = (1+f_{m,n})B_{m,n}(f;x,y), \tag{30}$$

*where* $(f_{m,n})$ *is a sequence defined as in Example 3. Then, we have*

$$\begin{aligned} \mathcal{L}_{m,n}(1;x,y) &= 1 + f_{m,n}(x,y), \\ \mathcal{L}_{m,n}(1;x,y) &= 1 + f_{m,n}(x,y) \cdot \left[\frac{mx}{m+1} + \frac{1}{2(m+1)}\right], \\ \mathcal{L}_{m,n}(1;x,y) &= 1 + f_{m,n}(x,y) \cdot \left[\frac{ny}{n+1} + \frac{1}{2(n+1)}\right] \end{aligned}$$

*and*

$$\begin{aligned} \mathcal{L}_{m,n}(1;x,y) = 1 + f_{m,n}(x,y) & \\ \cdot \left[\frac{m(m-1)x^2}{(m+1)^2} + \frac{2mx}{(m+1)}^2 + \frac{1}{3(m+1)^2}\frac{n(n-1)y^2}{(n+1)^2} + \frac{2ny}{(n+1)^2} + \frac{1}{3(n+1)^2}\right]. \end{aligned}$$

*We thus obtain*

$$\begin{aligned} N_D stat \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(1;x,y) - 1}{\sigma} \right\|_{\omega} &= 0, \\ N_D stat \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(s;x,y) - s}{\sigma} \right\|_{\omega} &= 0, \\ N_D stat \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(t;x,y) - t}{\sigma} \right\|_{\omega} &= 0, \\ N_D stat \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(s^2+t^2;x,y) - s^2 + t^2}{\sigma} \right\|_{\omega} &= 0. \end{aligned}$$

*This means that the operators $\mathcal{L}_{m,n}(f;x,y)$ fulfil the conditions (4). Hence, by Theorem 2 we have*

$$N_D stat \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(f;x,y) - f(x,y)}{\sigma} \right\|_{\omega} = 0 \text{ for every } \lambda_0 > 0.$$

However, since $(f_{m,n})$ is not relatively modular weighted statistically convergent, the result of Demirci and Orhan ([31], p. 1173, Theorem 1) is not fairly true under the operators defined by us in (30). Furthermore, since $(f_{m,n})$ is statistically and relatively modular deferred-weighted summable, we therefore conclude that our Theorem 2 works for the operators which we have considered here.

## 5. Concluding Remarks and Observations

In the concluding section of our study, we put forth various supplementary remarks and observations concerning several outcomes which we have established here.

**Remark 2.** *Let $(f_{m,n})_{m,n\in\mathbb{N}}$ be a sequence of functions given in Example 3. Then, since*

$$N_D stat \lim_{m\to\infty} f_{m,n} = 0 \text{ on } [0,1]\times[0,1],$$

*we have*

$$N_D stat \lim_{m\to\infty} \|\mathcal{L}_{m,n}(f_i;x,y) - f_i(x,y)\|_{\omega} = 0 \quad (i = 0,1,2,3). \tag{31}$$

*Thus, we can write (by Theorem 2)*

$$N_D stat \lim_{m\to\infty} \|\mathcal{L}_{m}(f;x,y) - f(x,y)\|_{\omega} = 0, \ (i = 0,1,2,3), \tag{32}$$

*where*

$$f_0(x,y) = 1, \ f_1(x,y) = x, \ f_2(x,y) = y \text{ and } f_3(x,y) = x^2 + y^2.$$

*Moreover, as $(f_{m,m})$ is not classically convergent it therefore does not converge uniformly in modular space. Thus, the traditional Korovkin-type approximation theorem will not work here under the operators defined in (30). Therefore, this application evidently demonstrates that our Theorem 2 is a non-trivial extension of the conventional Korovkin-type approximation theorem (see [27]).*

**Remark 3.** *Let $(f_{m,n})_{m,n\in\mathbb{N}}$ be a sequence as considered in Example 3. Then, since*

$$N_D stat \lim_{m\to\infty} f_{m,n} = 0 \text{ on } [0,1]\times[0,1],$$

*(31) fairly holds true. Now under condition (31) and by applying Theorem 2, we have that the condition (32) holds true. Moreover, since $(f_{m,n})$ is not relatively modular statistically Cesàro summable, Theorem 1 of Demirci and Orhan (see [31], p. 1173, Theorem 1) does not hold fairly true under the operators considered in (30). Hence, our Theorem 2 is a non-trivial generalization of Theorem 1 of Demirci and Orhan (see [31], p. 1173, Theorem 1) (see also [29]). Based on the above facts, we conclude here that our proposed method has effectively worked for the operators considered in (30), and therefore it is stronger than the traditional and statistical versions of the Korovkin-type approximation theorems established earlier in References [27,29,31].*

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# Construction of Stancu-Type Bernstein Operators Based on Bézier Bases with Shape Parameter $\lambda$

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**Abstract:** We construct Stancu-type Bernstein operators based on Bézier bases with shape parameter $\lambda \in [-1, 1]$ and calculate their moments. The uniform convergence of the operator and global approximation result by means of Ditzian-Totik modulus of smoothness are established. Also, we establish the direct approximation theorem with the help of second order modulus of smoothness, calculate the rate of convergence via Lipschitz-type function, and discuss the Voronovskaja-type approximation theorems. Finally, in the last section, we construct the bivariate case of Stancu-type $\lambda$-Bernstein operators and study their approximation behaviors.

**Keywords:** Stancu-type Bernstein operators; Bézier bases; Voronovskaja-type theorems; modulus of continuity; rate of convergence; bivariate operators; approximation properties

**MSC:** 41A25; 41A35

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## 1. Introduction

A famous mathematician Bernstein [1] constructed polynomials nowadays called Bernstein polynomials, which are familiar and widely investigated polynomials in theory of approximation. Bernstein gave a simple and very elegant way to obtain Weierstrass approximation theorem with the help of his newly constructed polynomials. For any continuous function $f(x)$ defined on $C[0,1]$, Bernstein polynomials of order $n$ are given by

$$B_n(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) b_{n,i}(x) \qquad (x \in [0,1]), \tag{1}$$

where the Bernstein basis functions $b_{n,i}(x)$ are defined by

$$b_{n,i}(x) = \binom{n}{i} x^i \, (1-x)^{n-i} \qquad\qquad (i = 0, \ldots, n).$$

Stancu [2] presented a generalization of Bernstein polynomials with the help of two parameters $\alpha$ and $\beta$ such that $0 \leq \alpha \leq \beta$, as follows:

$$S_{n,\alpha,\beta}(f;x) = \sum_{i=0}^{n} f\left(\frac{i+\alpha}{n+\beta}\right) \binom{n}{i} x^i \, (1-x)^{n-i} \qquad (x \in [0,1]). \tag{2}$$

If we take both the parameters $\alpha = \beta = 0$, then we get the classical Bernstein polynomials. The operators defined by (2) are called Bernstein–Stancu operators. For some recent work, we refer to [3–6].

In the recent past, Cai et al. [7] presented a new construction of Bernstein operators with the help of Bézier bases with shape parameter $\lambda$ and called it $\lambda$-Bernstein operators, which are defined by

$$B_n^{\lambda}(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \tilde{b}_{n,i}(\lambda;x) \qquad (n \in \mathbb{N}) \tag{3}$$

where $\tilde{b}_{n,i}(\lambda;x)$ are Bézier bases with shape parameter $\lambda$ (see [8]), defined by

$$\begin{aligned} \tilde{b}_{n,0}(\lambda;x) &= b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\ \tilde{b}_{n,i}(\lambda;x) &= b_{n,i}(x) + \frac{n-2i+1}{n^2-1}\lambda b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1}\lambda b_{n+1,i+1}(x), \quad i = 1,2\ldots,n-1, \\ \tilde{b}_{n,n}(\lambda;x) &= b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x), \end{aligned} \tag{4}$$

in this case $\lambda \in [-1,1]$ and $b_{n,i}(x)$ are the Bernstein basis functions. By taking the above operators into account, they established various approximation results, namely, Korovkin- and Voronovskaja-type theorems, rate of convergence via Lipschitz continuous functions, local approximation and other related results. In the same year, Cai [9] generalized $\lambda$-Bernstein operators by constructing the Kantorovich-type $\lambda$-Bernstein operators, as well as its Bézier variant, and studied several approximation results. Later, various approximation properties and asymptotic type results of the Kantorovich-type $\lambda$-Bernstein operators have been studied by Acu et al. [10]. Very recently, Özger [11] obtained statistical approximation for $\lambda$-Bernstein operators including a Voronovskaja-type theorem in statistical sense. In the same article, he also constructed bivariate $\lambda$-Bernstein operators and studied their approximation properties.

The Bernstein operators are some of the most studied positive linear operators which were modified by many authors, and we are mentioning some of them and other related work [12–23].

We are now ready to construct our new operators as follows: Suppose that $\alpha$ and $\beta$ are two non-negative parameters such that $0 \leq \alpha \leq \beta$. Then, the Stancu-type modification of $\lambda$-Bernstein operators $B^{\lambda}_{n,\alpha,\beta}(f;x) : C[0,1] \longrightarrow C[0,1]$ is defined by

$$B^{\lambda}_{n,\alpha,\beta}(f;x) = \sum_{i=0}^{n} f\left(\frac{i+\alpha}{n+\beta}\right) \tilde{b}_{n,i}(\lambda;x) \tag{5}$$

for any $n \in \mathbb{N}$ and we call it Stancu-type $\lambda$-Bernstein operators or $\lambda$-Bernstein–Stancu operators, where Bézier bases $\tilde{b}_{n,i}(\lambda;x)$ are defined in (4).

**Remark 1.** *We have the following results for Stancu-type $\lambda$-Bernstein operators:*

- (i) *If we take $\lambda = 0$ in (5), then Stancu-type $\lambda$-Bernstein Stancu operators reduce to the classical Bernstein–Stancu operators defined in [2].*
- (ii) *The choice of $\alpha = \beta = 0$ in (5) gives $\lambda$-Bernstein operators defined by Cai et al. [7].*
- (iii) *If we choose $\alpha = \beta = \lambda = 0$, then (5) reduces to the classical Bernstein operators defined in [1].*

The rest of the paper is organized as follows: In Section 2, we calculate the moments of (5) and prove global approximation formula in terms of Ditzian–Totik uniform modulus of smoothness of first and second order. The local direct estimate of the rate of convergence by Lipschitz-type function involving two parameters for $\lambda$-Bernstein–Stancu operators is investigated. In Section 3, we establish quantitative Voronovskaja-type theorem for our operators. The final section of the paper is devoted to study the bivariate case of $\lambda$-Bernstein–Stancu operators .

## 2. Some Auxiliary Lemmas and Approximation by Stancu-Type $\lambda$-Bernstein Operators

In this section, we first prove some lemma which will be used to study the approximation results of (5).

**Lemma 1.** *For $x \in [0,1]$, the moments of Stancu-type $\lambda$-Bernstein operators are given as:*

$$
\begin{aligned}
B^{\lambda}_{n,\alpha,\beta}(1;x) &= 1;\\
B^{\lambda}_{n,\alpha,\beta}(t;x) &= \frac{\alpha+nx}{n+\beta} + \lambda\left[\frac{1-2x+x^{n+1}+(\alpha-1)(1-x)^{n+1}}{(n+\beta)(n-1)} + \frac{\alpha x(1-x)^n}{n+\beta}\right];\\
B^{\lambda}_{n,\alpha,\beta}(t^2;x) &= \frac{1}{(n+\beta)^2}\left\{n(n-1)x^2+(1+2\alpha)nx+\alpha^2\right\}\\
&\quad + \lambda\left[\frac{2nx-1-4nx^2+(2n+1)x^{n+1}+(1-x)^{n+1}}{(n+\beta)^2(n-1)} + \frac{\alpha^2-4\alpha x}{(n+\beta)^2(n-1)}\right.\\
&\quad \left. + \frac{2\alpha n-2\alpha(\alpha+n)(x^{n+1}+(1-x)^n)+\alpha^2 x(n^2+1)(1-x)^n}{(n+\beta)^2(n^2-1)}\right].
\end{aligned}
$$

**Proof.** Using the definition of operators (5) and Bézier–Bernstein bases $\tilde{b}_{n,i}(\lambda;x)$ (4), we write

$$
\begin{aligned}
B^{\lambda}_{n,\alpha,\beta}(t;x) &= \sum_{i=0}^{n}\frac{i+\alpha}{n+\beta}\,\tilde{b}_{n,i}(\lambda;x) = \frac{\alpha}{n+\beta}\,b_{n,0}(x) - \frac{\alpha}{n+\beta}\frac{\lambda}{n+1}\,b_{n+1,1}(x)\\
&\quad + \sum_{i=1}^{n-1}\frac{i+\alpha}{n+\beta}\left[b_{n,i}(x)+\lambda\left(\frac{n-2i+n+1}{n^2-1}\,b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1}b_{n+1,i+1}(x)\right)\right]\\
&\quad + \frac{n+\alpha}{n+\beta}b_{n,n}(x) - \frac{n+\alpha}{n+\beta}\frac{\lambda}{n+1}b_{n+1,n}(x)\\
&= \sum_{i=0}^{n}\frac{i+\alpha}{n+\beta}b_{n,i}(x) + \lambda\left(\theta_1(n,\alpha,\beta,x)-\theta_2(n,\alpha,\beta,x)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_1(n,\alpha,\beta,x) &= \sum_{i=0}^{n}\frac{i+\alpha}{n+\beta}\frac{n-2i+1}{n^2-1}b_{n+1,i}(x);\\
\theta_2(n,\alpha,\beta,x) &= \sum_{i=1}^{n-1}\frac{i+\alpha}{n+\beta}\frac{n-2i-1}{n^2-1}b_{n+1,i+1}(x).
\end{aligned}
$$

Now, we compute the expressions $\theta_1(n,\alpha,\beta,x)$ and $\theta_2(n,\alpha,\beta,x)$. Since the Bernstein–Stancu operators are linear, and Bernstein–Stancu operators and fundamental Bernstein bases satisfy the following equality:

$$
\sum_{i=1}^{n}\frac{i+\alpha}{n+\beta}b_{n,i}(x) = \frac{nx}{n+\beta} + \frac{\alpha}{n+\beta},
$$

one writes

$$
\begin{aligned}
\theta_1(n,\alpha,\beta,x) &= \frac{1}{n-1}\sum_{i=0}^{n}\frac{i+\alpha}{n+\beta}\,b_{n+1,i}(x) - \frac{2}{n^2-1}\sum_{i=0}^{n}\frac{i^2+\alpha i}{n+\beta}\,b_{n+1,i}(x)\\
&= \frac{1}{n-1}\sum_{i=0}^{n}\frac{i}{n+\beta}\,b_{n+1,i}(x) + \frac{1}{n-1}\sum_{i=0}^{n}\frac{\alpha}{n+\beta}\,b_{n+1,i}(x)\\
&\quad - \frac{2}{n^2-1}\sum_{i=1}^{n}\frac{i^2}{n+\beta}\,b_{n+1,i}(x) - \frac{2}{n^2-1}\sum_{i=1}^{n}\frac{\alpha i}{n+\beta}\,b_{n+1,i}(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(n+1)x-2x}{(n+\beta)(n-1)} \sum_{i=0}^{n-1} b_{n,i}(x) + \frac{\alpha}{(n+\beta)(n-1)} \sum_{i=0}^{n} b_{n+1,i}(x) \\
&\quad - \frac{2\alpha x}{(n+\beta)(n-1)} \sum_{i=0}^{n-1} b_{n,i}(x) - \frac{2nx^2}{(n+\beta)(n-1)} \sum_{i=0}^{n-2} b_{n-1,i}(x) \\
&= \frac{x-x^{n+d+1}}{n+\beta} - \frac{2nx^2-2nx^{n+1}}{(n+\beta)(n-1)} + \frac{x-x^{n+d+1}}{n+\beta} - \frac{\alpha-2\alpha x+\alpha x^{n+1}}{(n+\beta)(n-1)}
\end{aligned}
$$

$$
\begin{aligned}
\theta_2(n,\alpha,\beta,x) &= \frac{1}{n+1} \sum_{i=1}^{n-1} \frac{i+\alpha}{n+\beta}\, b_{n+1,i+1}(x) - \frac{2}{n^2-1} \sum_{i=1}^{n-1} \frac{i^2+\alpha i}{n+\beta}\, b_{n+1,i+1}(x) \\
&= \frac{1}{n+1} \sum_{i=1}^{n-1} \frac{i}{n+\beta}\, b_{n+1,i+1}(x) + \frac{1}{n+1} \sum_{i=1}^{n-1} \frac{\alpha}{n+\beta}\, b_{n+1,i+1}(x) \\
&\quad - \frac{2}{n^2-1} \sum_{i=1}^{n-1} \frac{i^2}{n+\beta}\, b_{n+1,i+1}(x) - \frac{2}{n^2-1} \sum_{i=1}^{n-1} \frac{\alpha i}{n+\beta}\, b_{n+1,i+1}(x) \\
&= \frac{x}{n+\beta} \sum_{i=1}^{n-1} b_{n,i}(x) - \frac{1}{(n+\beta)(n+1)} \sum_{i=1}^{n-1} b_{n+1,i+1}(x) \\
&\quad - \frac{2nx^2}{(n+\beta)(n+1)} \sum_{i=0}^{n-2} b_{n-1,i}(x) + \frac{2x}{(n+\beta)(n+1)} \sum_{i=1}^{n-1} b_{n,i}(x) \\
&\quad - \frac{2}{(n+\beta)(n^2-1)} \sum_{i=1}^{n-1} b_{n+1,i+1}(x) + \frac{\alpha}{(n+\beta)(n+1)} \sum_{i=1}^{n-1} b_{n+1,i+1}(x) \\
&\quad - \frac{2\alpha x}{(n+\beta)(n-1)} \sum_{i=1}^{n-1} b_{n,i}(x) + \frac{2\alpha}{(n+\beta)(n^2-1)} \sum_{i=1}^{n-1} b_{n+1,i+1}(x) \\
&= \frac{x-x^{n+1}}{n+\beta} - \frac{x(1-x)^n}{n+\beta} - \frac{1-(1-x)^{n+1}-x(n+1)(1-x)^n-x^{n+1}}{(n+\beta)(n+1)} \\
&\quad - \frac{2-(1-x)^{n+1}-2x(n+1)(1-x)^n-2x^{n+1}}{(n+\beta)(n^2-1)} + \frac{\alpha-\alpha(1-x)^{n+1}-\alpha x^{n+1}}{(n+\beta)(n+1)} \\
&\quad + \frac{2x-2x(1-x)^n-2x^{n+1}}{(n+\beta)(n-1)} - \frac{2nx^2-2nx^{n+1}}{(n+\beta)(n-1)} - \frac{\alpha x(1-x)^n}{n+\beta} \\
&\quad - \frac{2\alpha x-2\alpha x^{n+1}}{(n+\beta)(n+1)} + \frac{2\alpha-2\alpha(1-x)^{n+1}-2\alpha x^{n+1}}{(n+\beta)(n^2-1)}.
\end{aligned}
$$

We get the desired result for $B_{n,\alpha,\beta}^{\lambda}(t;x)$ by combining the results obtained for $\theta_1(n,\alpha,\beta,x)$ and $\theta_2(n,\alpha,\beta,x)$.

Again, by using the following identity;

$$
\sum_{i=1}^{n} \frac{(i+\alpha)^2}{(n+\beta)^2} b_{n,i}(x) = \frac{1}{(n+\beta)^2} \left\{ n(n-1)x^2 + (1+2\alpha)nx + \alpha^2 \right\}
$$

together with (4) and (5), we can write

$$
\begin{aligned}
B_{n,\alpha,\beta}^{\lambda}(t^2;x) = \sum_{i=0}^{n} \frac{(i+\alpha)^2}{(n+\beta)^2} \tilde{b}_{n,i}(\lambda;x) &= \frac{\alpha^2}{(n+\beta)^2} b_{n,0}(x) - \frac{\alpha^2}{(n+\beta)^2} \frac{\lambda}{n+1} b_{n+1,1}(x) \\
&\quad + \sum_{i=1}^{n-1} \frac{(i+\alpha)^2}{(n+\beta)^2} \left[ b_{n,i}(x) + \lambda \left( \frac{n-2i+1}{n^2-1} b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1} b_{n+1,i+1}(x) \right) \right] \\
&\quad + \frac{(n+\alpha)^2}{(n+\beta)^2} b_{n,n}(x) - \frac{(n+\alpha)^2}{(n+\beta)^2} \frac{\lambda}{n+1} b_{n+1,n}(x)
\end{aligned}
$$

$$= \sum_{i=0}^{n} \frac{(i+\alpha)^2}{(n+\beta)^2} b_{n,i}(x) + \lambda \left(\theta_3(n,\alpha,\beta,x) - \theta_4(n,\alpha,\beta,x)\right),$$

where

$$\theta_3(n,\alpha,\beta,x) = \sum_{i=0}^{n} \frac{(i+\alpha)^2}{(n+\beta)^2} \frac{n-2i+1}{n^2-1} b_{n+1,i}(x);$$
$$\theta_4(n,\alpha,\beta,x) = \sum_{i=1}^{n-1} \frac{(i+\alpha)^2}{(n+\beta)^2} \frac{n-2i-1}{n^2-1} b_{n+1,i+1}(x).$$

We now compute the expressions $\theta_3(n,\alpha,\beta,x)$ and $\theta_4(n,\alpha,\beta,x)$ as follows:

$$\begin{aligned}
\theta_3(n,\alpha,\beta,x) &= \frac{1}{n-1} \sum_{i=0}^{n} \frac{(i+\alpha)^2}{(n+\beta)^2} b_{n+1,i}(x) - \frac{2}{n^2-1} \sum_{i=0}^{n} \frac{(i+\alpha)^2 i}{(n+\beta)^2} b_{n+1,i}(x) \\
&= \frac{1}{n-1} \sum_{i=0}^{n} \frac{i^2}{(n+\beta)^2} b_{n+1,i}(x) + \frac{2\alpha}{n-1} \sum_{i=0}^{n} \frac{i}{(n+\beta)^2} b_{n+1,i}(x) \\
&\quad + \frac{\alpha^2}{n-1} \sum_{i=0}^{n} b_{n+1,i}(x) - \frac{2}{n^2-1} \sum_{i=0}^{n} \frac{i^3}{(n+\beta)^2} b_{n+1,i}(x) \\
&\quad - \frac{4\alpha}{n^2-1} \sum_{i=0}^{n} \frac{i^2}{(n+\beta)^2} b_{n+1,i}(x) - \frac{2\alpha^2}{n^2-1} \sum_{i=0}^{n} \frac{i}{(n+\beta)^2} b_{n+1,i}(x) \\
&= \frac{n(n+1)x^2}{(n+\beta)^2(n-1)} \sum_{i=0}^{n-2} b_{n-1,i}(x) + \frac{(n+1)x}{(n+\beta)^2(n-1)} \sum_{i=0}^{n-1} b_{n,i}(x) \\
&\quad - \frac{2nx^3}{(n+\beta)^2} \sum_{i=0}^{n-3} b_{n-2,i}(x) - \frac{6nx^2}{(n+\beta)^2(n-1)} \sum_{i=0}^{n-2} b_{n-1,i}(x) \\
&\quad - \frac{x}{(n+\beta)^2(n-1)} \sum_{i=0}^{n-1} b_{n,i}(x) + \frac{2\alpha x(n+1)}{(n+\beta)^2(n-1)} \sum_{i=0}^{n-1} b_{n,i}(x) \\
&\quad + \frac{\alpha^2}{(n+\beta)^2(n-1)} \sum_{i=0}^{n} b_{n+1,i}(x) - \frac{4\alpha n x^2}{(n+\beta)^2(n-1)} \sum_{i=0}^{n-2} b_{n-1,i}(x) \\
&\quad - \frac{4\alpha x}{(n+\beta)^2(n-1)} \sum_{i=0}^{n-1} b_{n,i}(x) - \frac{2\alpha^2 x}{(n+\beta)^2(n-1)} \sum_{i=0}^{n-1} b_{n,i}(x) \\
&= \frac{2n(x^{n+1}-x^3)}{(n+\beta)^2} + \frac{x-x^{n+1}}{(n+\beta)^2} + \frac{(n^2-5n)(x^2-x^{n+1})}{(n+\beta)^2(n-1)} \\
&\quad + \frac{2\alpha(n+1)x^{n+1} + \alpha^2(1-x+x^{n+1}) - 4\alpha n x^2}{(n+\beta)^2(n-1)} + \frac{2\alpha x}{(n+\beta)^2}.
\end{aligned}$$

$$\begin{aligned}
\theta_4(n,\alpha,\beta,x) &= \frac{1}{n+1} \sum_{i=1}^{n-1} \frac{(i+\alpha)^2}{(n+\beta)^2} b_{n+1,i+1}(x) - \frac{2}{n^2-1} \sum_{i=1}^{n-1} \frac{(i+\alpha)^2 i}{(n+\beta)^2} b_{n+1,i+1}(x) \\
&= \frac{1}{n+1} \sum_{i=1}^{n-1} \frac{i^2}{(n+\beta)^2} b_{n+1,i+1}(x) + \frac{2\alpha}{n+1} \sum_{i=1}^{n-1} \frac{i}{(n+\beta)^2} b_{n+1,i+1}(x) \\
&\quad + \frac{\alpha^2}{n+1} \sum_{i=1}^{n-1} \frac{1}{(n+\beta)^2} b_{n+1,i+1}(x) - \frac{2}{n^2-1} \sum_{i=1}^{n-1} \frac{i^3}{(n+\beta)^2} b_{n+1,i+1}(x) \\
&\quad - \frac{4\alpha}{n^2-1} \sum_{i=1}^{n-1} \frac{i^2}{(n+\beta)^2} b_{n+1,i+1}(x) - \frac{2\alpha^2}{n^2-1} \sum_{i=1}^{n-1} \frac{i}{(n+\beta)^2} b_{n+1,i+1}(x)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{nx^2}{(n+\beta)^2}\sum_{i=0}^{n-2} b_{n-1,i}(x) + \frac{1}{(n+\beta)^2(n-1)}\sum_{i=1}^{n-1} b_{n+1,i+1}(x) \\
&\quad - \frac{2nx^3}{(n+\beta)^2}\sum_{i=0}^{n-3} b_{n-2,i}(x) - \frac{-2x}{(n+\beta)^2(n-1)}\sum_{i=1}^{n-1} b_{n,i}(x) \\
&\quad + \frac{2}{(n+\beta)^2(n-1)}\sum_{i=1}^{n-1} b_{n+1,i+1}(x) - \frac{x}{(n+\beta)^2}\sum_{i=1}^{n-1} b_{n,i}(x) \\
&\quad + \frac{2\alpha x}{(n+\beta)^2}\sum_{i=1}^{n-1} b_{n,i}(x) - \frac{2\alpha}{(n+\beta)^2(n+1)}\sum_{i=1}^{n-1} b_{n+1,i+1}(x) \\
&\quad + \frac{\alpha^2}{(n+\beta)^2(n+1)}\sum_{i=1}^{n-1} b_{n+1,i+1}(x) - \frac{4\alpha n x^2}{(n+\beta)^2(n-1)}\sum_{i=0}^{n-2} b_{n-1,i}(x) \\
&\quad + \frac{4\alpha x}{(n+\beta)^2(n-1)}\sum_{i=1}^{n-1} b_{n,i}(x) - \frac{4\alpha}{(n+\beta)^2(n^2-1)}\sum_{i=1}^{n-1} b_{n+1,i+1}(x) \\
&\quad - \frac{2\alpha^2 x}{(n+\beta)^2(n-1)}\sum_{i=1}^{n-1} b_{n,i}(x) + \frac{2\alpha}{(n+\beta)^2(n^2-1)}\sum_{i=1}^{n-1} b_{n+1,i+1}(x) \\
&= \frac{nx^2+(n+1)x^{n+1}-x-2nx^3}{(n+\beta)^2} + \frac{1-(1-x)^{n+1}-x^{n+1}}{(n+\beta)^2(n+1)} \\
&\quad + \frac{2x^{n+1}-2x}{(n+\beta)^2(n-1)} + \frac{2-2(1-x)^{n+1}-2x^{n+1}}{(n+\beta)^2(n^2-1)} \\
&\quad + \frac{2\alpha x+2\alpha x^{n+1}-\alpha^2 x(1-x)^n}{(n+\beta)^2} + \frac{\alpha(\alpha-2)(1-x^{n+1}-(1-x)^{n+1})}{(n+\beta)^2(n+1)} \\
&\quad + \frac{2\alpha(\alpha-2)x((1-x)^{n+1}-1)+2\alpha^2 x^{n+1}}{(n+\beta)^2(n-1)} + \frac{2\alpha(x^{n+1}+(1-x)^{n+1}-1)}{(n+\beta)^2(n^2-1)},
\end{aligned}
$$

which completes the result for $B^{\lambda}_{n,\alpha,\beta}(t^2;x)$ by combining the results obtained for $\theta_3(n,\alpha,\beta,x)$ and $\theta_4(n,\alpha,\beta,x)$. □

**Corollary 1.** *The following relations hold:*

$$
\begin{aligned}
B^{\lambda}_{n,\alpha,\beta}(t-x;x) &= \sum_{i=0}^{n}\frac{i+\alpha}{n+\beta}\tilde{b}_{n,i}(\lambda;x) - x\sum_{i=0}^{n}\tilde{b}_{n,i}(\lambda;x) \\
&= \frac{\alpha-\beta x}{n+\beta} + \lambda\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{(n+\beta)(n-1)} \\
&\quad + \lambda\frac{\alpha x(1-x)^n}{n+\beta} + \lambda\frac{\alpha(1-x)^{n+1}}{(n+\beta)(n-1)}; \\
B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x) &= \sum_{i=0}^{n}\left(\frac{i+\alpha}{n+\beta}\right)^2\tilde{b}_{n,i}(\lambda;x) - 2x\sum_{i=0}^{n}\frac{i+\alpha}{n+\beta}\tilde{b}_{n,i}(\lambda;x) + x^2\sum_{i=0}^{n}\tilde{b}_{n,i}(\lambda;x) \\
&= \frac{nx(1-x)+(\beta x-\alpha)^2}{(n+\beta)^2} \\
&\quad + \lambda\left[\frac{4x^2-2x-2x^{n+2}-2(\alpha-1)x(1-x)^{n+1}}{(n+\beta)(n-1)} - \frac{2\alpha x^2(1-x)^n}{n+\beta}\right] \\
&\quad + \lambda\frac{2nx-1-4nx^2+(2n+1)x^{n+1}+(1-x)^{n+1}+\alpha^2-4\alpha x}{(n+\beta)^2(n-1)} \\
&\quad + \lambda\frac{2\alpha n-2\alpha(\alpha+n)(x^{n+1}+(1-x)^n)+\alpha^2 x(n^2+1)(1-x)^n}{(n+\beta)^2(n^2-1)}.
\end{aligned}
$$

**Corollary 2.** *The following identities hold:*

$$\lim_{n\to\infty} n\, B^{\lambda}_{n,\alpha,\beta}(t-x;x;) = \alpha-\beta x;$$
$$\lim_{n\to\infty} n\, B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x) = x(1-x).$$

We obtain the uniform convergence of operators $B^{\lambda}_{n,\alpha,\beta}(f;x)$ by applying well-known Bohman–Korovkin–Popoviciu theorem.

**Theorem 1.** *Let* $C[0,1]$ *denote the space of all real-valued continuous functions on* $[0,1]$ *endowed with the supremum norm. Then*

$$\lim_{n\to\infty} B^{\lambda}_{n,\alpha,\beta}(f;x) = f(x) \quad (f\in C[0,1])$$

*uniformly in* $[0,1]$.

**Proof.** It is sufficient to show that

$$\lim_{n\to\infty} \|B^{\lambda}_{n,\alpha,\beta}(t^j;x)-t^j\|_{C[0,1]} = 0, \qquad j=0,1,2$$

as stated in Bohman–Korovkin–Popoviciu theorem. We have the following relations by Lemma 1:

$$\lim_{n\to\infty} \|B^{\lambda}_{n,\alpha,\beta}(t^0;x)-t^0\|_{C[0,1]} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|B^{\lambda}_{n,\alpha,\beta}(t;x)-t\|_{C[0,1]} = 0.$$

It is easy to show

$$\begin{aligned} B^{\lambda}_{n,\alpha,\beta}(t^2;x) \le{} & \frac{n(n+1)x^2+(1+2\alpha)nx+\alpha^2}{(n+\beta)^2} \\ & +\lambda\left[\frac{2nx+1+4nx^2+(2n+1)x^{n+1}+(1-x)^{n+1}}{(n+\beta)^2(n-1)}+\frac{\alpha^2+4\alpha x}{(n+\beta)^2(n-1)}\right. \\ & \left.+\frac{2\alpha n+2\alpha(\alpha+n)(x^{n+1}+(1-x)^n)+\alpha^2(n^2+1)x(1-x)^n}{(n+\beta)^2(n^2-1)}\right] \end{aligned}$$

and hence

$$\lim_{n\to\infty} \|B^{\alpha,\beta}_{n}(t^2;x;\lambda)-t^2\|_{C[0,1]} = 0.$$

This implies $B^{\lambda}_{n,\alpha,\beta}(f;x)$ converge uniformly to $f$ on $[0,1]$. □

Recall that the first and second order Ditzian–Totik uniform modulus of smoothness are given by

$$\omega_{\xi}(f,\delta) := \sup_{0<|h|\le\delta}\ \sup_{x,x+h\xi(x)\in[0,1]} \{|f(x+h\xi(x))-f(x)|\}$$

and

$$\omega_2^{\phi}(f,\delta) := \sup_{0<|h|\le\delta}\ \sup_{x,x\pm h\phi(x)\in[0,1]} \{|f(x+h\phi(x))-2f(x)+f(x-h\phi(x))|\},$$

respectively, where $\phi$ is an admissible step-weight function on $[a,b]$, that is, $\phi(x)=[(x-a)(b-x)]^{1/2}$ if $x\in[a,b]$ (see [24]). Let

$$K_{2,\phi(x)}(f,\delta) = \inf_{g\in W^2(\phi)} \{||f-g||_{C[0,1]}+\delta||\phi^2 g''||_{C[0,1]} : g\in C^2[0,1]\} \qquad (\delta>0)$$

be the corresponding $K$-functional, where

$$W^2(\phi) = \{g \in C[0,1] : g' \in AC[0,1],\ \phi^2 g'' \in C[0,1]\}$$

and

$$C^2[0,1] = \{g \in C[0,1] : g', g'' \in C[0,1]\}.$$

In this case, $g' \in AC[0,1]$ means that $g'$ is absolutely continuous on $[0,1]$. It is known by [25] that there exists an absolute constant $C > 0$, such that

$$C^{-1}\omega_2^{\phi}(f, \sqrt{\delta}) \leq K_{2,\phi(x)}(f, \delta) \leq C\omega_2^{\phi}(f, \sqrt{\delta}). \tag{6}$$

We are now ready to obtain global approximation theorem.

**Theorem 2.** *Let $\lambda \in [-1,1]$ and $f \in C[0,1]$. Suppose that $\phi(\neq 0)$ such that $\phi^2$ is concave. Then*

$$|B_{n,\alpha,\beta}^{\lambda}(f;x) - f(x)| \leq C\omega_2^{\phi}\left(f, \frac{\delta_n(\alpha,\beta,\lambda;x)}{2\phi(x)}\right) + \omega_{\xi}\left(f, \frac{\mu_n(\alpha,\beta,\lambda;x)}{\xi(x)}\right)$$

*for $x \in [0,1]$ and $C > 0$, where $\mu_n(\alpha,\beta,\lambda;x) = B_{n,\alpha,\beta}^{\lambda}(t-x;x)$, $\delta_n(\alpha,\beta,\lambda;x) = \big(\nu_n(\alpha,\beta,\lambda;x) + \mu_n^2(\alpha,\beta,\lambda;x)(x)\big)^{\frac{1}{2}}$ and $\nu_n(\alpha,\beta,\lambda;x)(x) = B_{n,\alpha,\beta}^{\lambda}((t-x)^2;x)$.*

**Proof.** Consider the operators

$$\begin{aligned} \tilde{B}_{n,\alpha,\beta}^{\lambda}(f;x) &= B_{n,\alpha,\beta}^{\lambda}(f;x) + f(x) \\ &\quad - f\Big(\frac{\alpha-\beta x}{n+\beta} + \lambda\frac{\alpha x(1-x)^n}{n+\beta} + \lambda\frac{1-2x+x^{n+1}+(\alpha-1)(1-x)^{n+1}}{(n+\beta)(n-1)}\Big) \end{aligned} \tag{7}$$

for $\lambda \in [-1,1]$, $x \in [0,1]$. We observe that $\tilde{B}_{n,\alpha,\beta}^{\lambda}(1;x) = 1$ and $\tilde{B}_{n,\alpha,\beta}^{\lambda}(t;x) = x$, that is $\tilde{B}_{n,\alpha,\beta}^{\lambda}(t-x;x) = 0$.

Let $u = \rho x + (1-\rho)t$, $\rho \in [0,1]$. Since $\phi^2$ is concave on $[0,1]$, we have $\phi^2(u) \geq \rho\phi^2(x) + (1-\rho)\phi^2(t)$ and hence

$$\frac{|t-u|}{\phi^2(u)} \leq \frac{\rho|x-t|}{\rho\phi^2(x) + (1-\rho)\phi^2(t)} \leq \frac{|t-x|}{\phi^2(x)}. \tag{8}$$

So

$$\begin{aligned} |\tilde{B}_{n,\alpha,\beta}^{\lambda}(f;x) - f(x)| &\leq |\tilde{B}_{n,\alpha,\beta}^{\lambda}(f-g;x)| + |\tilde{B}_{n,\alpha,\beta}^{\lambda}(g;x) - g(x)| + |f(x) - g(x)| \\ &\leq 4\|f-g\|_{C[0,1]} + |\tilde{B}_{n,\alpha,\beta}^{\lambda}(g;x) - g(x)|. \end{aligned} \tag{9}$$

We obtain the following relations by applying the Taylor's formula:

$$\begin{aligned} &|\tilde{B}_{n,\alpha,\beta}^{\lambda}(g;x) - g(x)| \\ &\leq B_{n,\alpha,\beta}^{\lambda}\left(\left|\int_x^t |t-u|\,|g''(u)|du\right|;x\right) + \left|\int_x^{x+\mu_n} |x+\mu_n(\alpha,\beta,\lambda;x) - u|\,|g''(u)|\,du\right| \\ &\leq \|\phi^2 g''\|_{C[0,1]} B_{n,\alpha,\beta}^{\lambda}\left(\left|\int_x^t \frac{|t-u|}{\phi^2(u)}du\right|;x\right) + \|\phi^2 g''\|_{C[0,1]}\left|\int_x^{x+\mu_n} \frac{|x+\mu_n(\alpha,\beta,\lambda;x) - u|}{\phi^2(u)}\,du\right| \\ &\leq \phi^{-2}(x)\|\phi^2 g''\|_{C[0,1]} B_{n,\alpha,\beta}^{\lambda}((t-x)^2;x) + \phi^{-2}(x)\|\phi^2 g''\|_{C[0,1]}\beta_n^2(x). \end{aligned} \tag{10}$$

By using the definition of $K$-functional together with (6) and the inequalities (9) and (10), we have

$$\begin{aligned}|\tilde{B}^{\lambda}_{n,\alpha,\beta}(f;x)-f(x)| &\leq \phi^{-2}(x)\|\phi^2 g''\|_{C[0,1]}\left(\nu_n(\alpha,\beta,\lambda;x)+\mu_n^2(\alpha,\beta,\lambda;x)\right)+4\|f-g\|_{C[0,1]}\\ &\leq C\omega_2^{\phi}\left(f,\frac{\left(\nu_n(\alpha,\beta,\lambda;x)+\mu_n^2(\alpha,\beta,\lambda;x)\right)^{\frac{1}{2}}}{2\phi(x)}\right).\end{aligned}$$

Also, by first order Ditzian–Totik uniform modulus of smoothness, we have

$$\begin{aligned}|f(x+\mu_n)-f(x)| &= \left|f\left(x+\xi(x)\frac{\mu_n(\alpha,\beta,\lambda;x)}{\xi(x)}\right)-f(x)\right|\\ &\leq \omega_{\xi}\left(f,\frac{\mu_n(\alpha,\beta,\lambda;x)}{\xi(x)}\right).\end{aligned}$$

Therefore, the following inequalities hold:

$$\begin{aligned}|B^{\lambda}_{n,\alpha,\beta}(f;x)-f(x)| &\leq |\tilde{B}^{\lambda}_{n,\alpha,\beta}(f;x)-f(x)|+|f(x+\mu_n(\alpha,\beta,\lambda;x))-f(x)|\\ &\leq C\omega_2^{\phi}\left(f,\frac{\delta_n(\alpha,\beta,\lambda;x)}{2\phi(x)}\right)+\omega_{\xi}\left(f,\frac{\mu_n(\alpha,\beta,\lambda;x)}{\xi(x)}\right),\end{aligned}$$

which completes the proof. □

In order to obtain next result, we first recall some concepts and results concerning modulus of continuity and Peetre's $K$-functional. For $\delta > 0$, the modulus of continuity $w(f,\delta)$ of $f \in C[a,b]$ is given by

$$w(f,\delta) := \sup\{|f(x)-f(y)| : \ x,y \in [a,b],\ |x-y| \leq \delta\}.$$

It is also well known that, for any $\delta > 0$ and each $x \in [a,b]$,

$$|f(x)-f(y)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta}+1\right). \tag{11}$$

For $f \in C[0,1]$, the second-order modulus of smoothness is given by

$$w_2(f,\sqrt{\delta}) := \sup_{0<h\leq\sqrt{\delta}} \ \sup_{x,x+2h\in[0,1]} \{|f(x+2h)-2f(x+h)+f(x)|\},$$

and the corresponding Peetre's $K$-functional [26] is

$$K_2(f,\delta) = \inf\{||f-g||_{C[0,1]}+\delta||g''||_{C[0,1]} : g \in W^2[0,1]\},$$

where

$$W^2[0,1] = \{g \in C[0,1] : g', g'' \in C[0,1]\}.$$

It is well-known that the inequality

$$K_2(f,\delta) \leq Cw_2(f,\sqrt{\delta}) \qquad (\delta > 0) \tag{12}$$

holds in which the absolute constant $C > 0$ is independent of $\delta$ and $f$ (see [25]).

We are now ready to establish a direct local approximation theorem for operators $B^{\lambda}_{n,\alpha,\beta}(f;x)$ via second order modulus of smoothness and usual modulus of continuity.

**Theorem 3.** *Assume that $f \in C[0,1]$ and $x \in [0,1]$. Then there exists an absolute constant $C$ such that*

$$|B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x)| \leq C\, w_2\Big(f, \frac{1}{2}\delta_n(\alpha,\beta,\lambda;x)\Big) + w(f, \mu_n(\alpha,\beta,\lambda;x))$$

*for the operators $B^{\lambda}_{n,\alpha,\beta}(f;x)$, where $\mu_n(\alpha,\beta,\lambda;x)$ and $\delta_n(\alpha,\beta,\lambda;x)$ are given in Theorem 2.*

**Proof.** Consider the operators $\tilde{B}^{\lambda}_{n,\alpha,\beta}(f;x)$ as defined in Theorem 2. Assume that $t,x \in [0,1]$ and $g \in W^2[0,1]$. The following equality yields by Taylor's expansion formula:

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du. \tag{13}$$

If we apply $\tilde{B}^{\lambda}_{n,\alpha,\beta}(\cdot;x)$ to both sides of (13) and keeping in mind these operators preserve constants and linear functions, we obtain

$$\begin{aligned}\tilde{B}^{\lambda}_{n,\alpha,\beta}(g;x) - g(x) &= g'(x)\tilde{B}^{\lambda}_{n,\alpha,\beta}(t-x;x) + \tilde{B}^{\lambda}_{n,\alpha,\beta}\Big(\int_x^t (t-u)g''(u)du;x\Big)\\ &= B^{\lambda}_{n,\alpha,\beta}\Big(\int_x^t (t-u)g''(u)du;x\Big) - \int_x^{x+\mu_n} (x+\mu_n(\alpha,\beta,\lambda;x)-u)g''(u)du.\end{aligned}$$

Therefore,

$$\begin{aligned}|\tilde{B}^{\lambda}_{n,\alpha,\beta}(g;x) - g(x)| &\leq B^{\lambda}_{n,\alpha,\beta}\Big(\Big|\int_x^t |t-u|\,|g''(u)|du\Big|;x\Big)\\ &\quad - \int_x^{x+\mu_n} \big|x+\mu_n(\alpha,\beta,\lambda;x)-u\big|\,|g''(u)|\,du\\ &\leq \|g''\|_{C[0,1]}\big(B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x) + \big(B^{\lambda}_{n,\alpha,\beta}(t-x;x)\big)^2\big).\end{aligned}$$

With the help of (7), one obtains

$$\begin{aligned}\|\tilde{B}^{\lambda}_{n,\alpha,\beta}(g;x)\|_{C[0,1]} &\leq \|B^{\lambda}_{n,\alpha,\beta}(g;x)\|_{C[0,1]} + \|g(x)\|_{C[0,1]} + \|g(x+\mu_n(\alpha,\beta,\lambda;x))\|_{C[0,1]}\\ &\leq \|3g\|_{C[0,1]}.\end{aligned} \tag{14}$$

Now, for $f \in C[0,1]$ and $g \in W^2[0,1]$, using (7) and (14), we get

$$\begin{aligned}|B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x)| &\leq |\tilde{B}^{\lambda}_{n,\alpha,\beta}(f-g;x)| + |\tilde{B}^{\lambda}_{n,\alpha,\beta}(g;x) - g(x)|\\ &\quad + |g(x) - f(x)| + |f(x+\mu_n(\alpha,\beta,\lambda;x)) - f(x)|\\ &\leq \delta_n^2(\alpha,\beta,\lambda;x)\|g''\|_{C[0,1]} + w(f,\mu_n(\alpha,\beta,\lambda;x)) + 4\|f-g\|_{C[0,1]}.\end{aligned}$$

Finally, by assuming the infimum on the right-hand side of the above inequality over all $g \in W^2[0,1]$ togrther with inequality (12), we obtain

$$\begin{aligned}|B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x)| &\leq 4K_2\Big(f, \frac{\delta_n^2(\alpha,\beta,\lambda;x)}{4}\Big) + w(f,\mu_n(\alpha,\beta,\lambda;x))\\ &\leq C\, w_2\Big(f, \frac{1}{2}\delta_n(\alpha,\beta,\lambda;x)\Big) + w(f,\mu_n(\alpha,\beta,\lambda;x)),\end{aligned}$$

which completes the proof. □

In the following theorem, we obtain a local direct estimate of the rate of convergence via Lipschitz-type function involving two parameters for the operators $B^{\lambda}_{n,\alpha,\beta}$. Before proceeding further, let us recall that

$$Lip_M^{(k_1,k_2)}(\eta) := \left\{ f \in C[0,1] : |f(t) - f(x)| \leq M \frac{|t-x|^{\eta}}{(k_1x^2 + k_2x + t)^{\frac{\eta}{2}}};\ x \in (0,1], t \in [0,1] \right\}$$

for $k_1 \geq 0, k_2 > 0$, where $\eta \in (0,1]$ and $M$ is a positive constant (see [27]).

**Theorem 4.** *If $f \in Lip_M^{(k_1,k_2)}(\eta)$, then*

$$|B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x)| \leq M \sqrt{\frac{\nu_n^{\eta}(\alpha,\beta,\lambda;x)}{(k_1x^2 + k_2x)^{\eta}}}$$

*for all $\lambda \in [-1,1]$, $x \in (0,1]$ and $\eta \in (0,1]$, where $\nu_n(\alpha,\beta,\lambda;x)$ is defined in Theorem 2.*

**Proof.** Let $f \in Lip_M^{(k_1,k_2)}(\eta)$ and $\eta \in (0,1]$. First, we are going to show that statement is true for $\eta = 1$. We write

$$\begin{aligned} |B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x)| &\leq |B^{\lambda}_{n,\alpha,\beta}(|f(t) - f(x)|;x)| + f(x)\ |B^{\lambda}_{n,\alpha,\beta}(1;x) - 1| \\ &\leq \sum_{i=0}^{n} \left| f\left(\frac{i+\alpha}{n+\beta}\right) - f(x) \right| \tilde{b}_{n,i}(x;\lambda) \\ &\leq M \sum_{i=0}^{n} \frac{|\frac{i+\alpha}{n+\beta} - x|}{(k_1x^2 + k_2x + t)^{\frac{1}{2}}}\ \tilde{b}_{n,i}(x;\lambda) \end{aligned}$$

for $f \in Lip_M^{(k_1,k_2)}(1)$. By using the relation

$$(k_1x^2 + k_2x + t)^{-1/2} \leq (k_1x^2 + k_2x)^{-1/2} \quad (k_1 \geq 0, k_2 > 0)$$

and applying Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x)| &\leq M(k_1x^2 + k_2x)^{-1/2} \sum_{i=0}^{n} |\frac{i+\alpha}{n+\beta} - x|\ \tilde{b}_{n,i}(x;\lambda) \\ &= M(k_1x^2 + k_2x)^{-1/2} |B^{\lambda}_{n,\alpha,\beta}(t-x;x)| \\ &\leq M|\nu_n(\alpha,\beta,\lambda;x)|^{1/2}(k_1x^2 + k_2x)^{-1/2}. \end{aligned}$$

Hence, the statement is true for $\eta = 1$. By the monotonicity of $B^{\lambda}_{n,\alpha,\beta}(f;x)$ and applying Hölder's inequality two times with $a = 2/\eta$ and $b = 2/(2-\eta)$, we can see that the statement is true for $\eta \in (0,1]$ as follows:

$$\begin{aligned} \left| B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x) \right| &\leq \sum_{i=0}^{n} \left| f\left(\frac{i+\alpha}{n+\beta}\right) - f(x) \right| \tilde{b}_{n,i}(x;\lambda) \\ &\leq \left( \sum_{i=0}^{n} \left| f\left(\frac{i+\alpha}{n+\beta}\right) - f(x) \right|^{\frac{2}{\eta}} \tilde{b}_{n,i}(x;\lambda) \right)^{\frac{\eta}{2}} \left( \sum_{i=0}^{n} \tilde{b}_{n,i}(x;\lambda) \right)^{\frac{2-\eta}{2}} \\ &\leq M \left( \sum_{i=0}^{n} \frac{\left(\frac{i+\alpha}{n+\beta} - x\right)^2 \tilde{b}_{n,i}(x;\lambda)}{\frac{i+\alpha}{n+\beta} + k_1x^2 + k_2x} \right)^{\frac{\eta}{2}} \end{aligned}$$

$$\leq M(k_1x^2+k_2x)^{-\eta/2}\left\{\sum_{i=0}^{n}\left(\frac{i+\alpha}{n+\beta}-x\right)^2\tilde{b}_{n,i}(x;\lambda)\right\}^{\frac{\eta}{2}}$$

$$\leq M(k_1x^2+k_2x+t)^{-\eta/2}\left[B_n^{\alpha,\beta}((t-x)^2;x;\lambda)\right]^{\frac{\eta}{2}}$$

$$= M\sqrt{\frac{\nu_n^{\eta}(\alpha,\beta,\lambda;x)}{(k_1x^2+k_2x)^{\eta}}}.$$

□

**Theorem 5.** *The following inequality holds:*

$$|B^{\lambda}_{n,\alpha,\beta}(f;x)-f(x)|\leq|\mu_n(\alpha,\beta,\lambda;x)|\,|f'(x)|+2\sqrt{\nu_n(\alpha,\beta,\lambda;x)}w(f',\sqrt{\nu_n(\alpha,\beta,\lambda;x)}\,)$$

*for* $f\in C^1[0,1]$ *and* $x\in[0,1]$*, where* $\mu_n(\alpha,\beta,\lambda;x)$ *and* $\nu_n(\alpha,\beta,\lambda;x)$ *are defined in Theorem 2.*

**Proof.** We have

$$f(t)-f(x)=(t-x)f'(x)+\int_x^t(f'(u)-f'(x))du \tag{15}$$

for any $t\in[0,1]$ and $x\in[0,1]$. By applying the operators $B^{\lambda}_{n,\alpha,\beta}(\cdot;x)$ to both sides of (15), we have

$$B^{\lambda}_{n,\alpha,\beta}(f(t)-f(x);x)=f'(x)B^{\lambda}_{n,\alpha,\beta}(t-x;x)+B^{\lambda}_{n,\alpha,\beta}\left(\int_x^t(f'(u)-f'(x))du;x\right).$$

The following inequality holds for any $\delta>0$, $u\in[0,1]$ and $f\in C[0,1]$:

$$|f(u)-f(x)|\leq w(f,\delta)\left(\frac{|u-x|}{\delta}+1\right).$$

Thus, we obtain

$$\left|\int_x^t(f'(u)-f'(x))du\right|\leq w(f',\delta)\left(\frac{(t-x)^2}{\delta}+|t-x|\right).$$

Hence

$$\begin{aligned}|B^{\lambda}_{n,\alpha,\beta}(f;x)-f(x)|&\leq|f'(x)|\,|B^{\lambda}_{n,\alpha,\beta}(t-x;x)|\\&\quad+w(f',\delta)\left\{\frac{1}{\delta}B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x)+B^{\lambda}_{n,\alpha,\beta}(t-x;x)\right\}.\end{aligned} \tag{16}$$

By applying Cauchy–Schwarz inequality on the right hand side of last inequality (16), we have

$$\begin{aligned}|B^{\lambda}_{n,\alpha,\beta}(f;x)-f(x)|&\leq|f'(x)|\,|\mu_n(\alpha,\beta,\lambda;x)|\\&\quad+w(f',\delta)\left\{\frac{1}{\delta}\sqrt{B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x)}+1\right\}\sqrt{B^{\lambda}_{n,\alpha,\beta}(|t-x|;x)}.\end{aligned}$$

Consequently, we obtain the desired result if we choose $\delta$ as $\nu_n^{1/2}(\alpha,\beta,\lambda;x)$. □

## 3. Voronovskaja-Type Theorems

Here, we prove the following Voronovskaja-type theorems by $B^{\lambda}_{n,\alpha,\beta}(f;x)$.

**Theorem 6.** *Let $f, f', f'' \in C_B[0,1]$, where $C_B[0,1]$ is the set of all real-valued bounded and continuous functions defined on $[0,1]$. Then, for each $x \in [0,1]$, we have*

$$\lim_{n\to\infty} n\{B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x)\} = (\alpha - \beta x)\, f'(x) + \frac{x(1-x)}{2} f''(x)$$

*uniformly on $[0,1]$.*

**Proof.** We first write the following equality by Taylor's expansion theorem of function $f(x)$ in $C_B[0,1]$:

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + (t-x)^2\, r_x(t), \tag{17}$$

where $r_x(t)$ is Peano form of the remainder, $r_x \in C[0,1]$ and $r_x(t) \to 0$ as $t \to x$. Applying the operators $B^{\lambda}_{n,\alpha,\beta}(\cdot;x)$ to identity (17), we have

$$B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x) = f'(x)B^{\lambda}_{n,\alpha,\beta}(t-x;x) + \frac{f''(x)}{2} B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x) + B^{\lambda}_{n,\alpha,\beta}((t-x)^2 r_x(t);x).$$

Using Cauchy–Schwarz inequality, we have

$$B^{\lambda}_{n,\alpha,\beta}((t-x)^2 r_x(t);x) \leq \sqrt{B^{\lambda}_{n,\alpha,\beta}((t-x)^4;x)}\sqrt{B^{\lambda}_{n,\alpha,\beta}(r_x^2(t);x)}. \tag{18}$$

We observe that $\lim_n B^{\lambda}_{n,\alpha,\beta}(r_x^2(t);x) = 0$ and hence

$$\lim_{n\to\infty} n\{B^{\lambda}_{n,\alpha,\beta}((t-x)^2 r_x(t);x)\} = 0.$$

Thus

$$\begin{aligned}\lim_{n\to\infty} n\{B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x)\} = \lim_{n\to\infty} n\{&B^{\lambda}_{n,\alpha,\beta}(t-x;x)f'(x) + \frac{f''(x)}{2}B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x) \\ &+ B^{\lambda}_{n,\alpha,\beta}((t-x)^2 r_x(t);x)\}.\end{aligned}$$

The result follows immediately by applying the Corollaries 1 and 2. □

For $f \in C[0,1]$ and $\delta > 0$, the Ditzian–Totik modulus of smoothness is given by

$$\omega_{\phi}(f,\delta) := \sup_{0<|h|\leq\delta} \left\{ \left| f\left(x + \frac{h\phi(x)}{2}\right) - f\left(x - \frac{h\phi(x)}{2}\right) \right|, x \pm \frac{h\phi(x)}{2} \in [0,1] \right\},$$

where $\phi(x) = (x(1-x))^{1/2}$, and let

$$K_{\phi}(f,\delta) = \inf_{g\in W_{\phi}[0,1]} \{||f-g|| + \delta||\phi g'|| : g \in C^1[0,1]\}$$

be the corresponding Peetre's $K$-functional, where

$$W_{\phi}[0,1] = \{g :\ g \in AC_{loc}[0,1],\ \|\phi g'\| < \infty\}$$

and $AC_{loc}[0,1]$ denotes the class of absolutely continuous functions defined on $[a,b] \subset [0,1]$. There exists a constant $C > 0$ such that $K_{\phi}(f,\delta) \leq C\, \omega_{\phi}(f,\delta)$.

Next, we give a quantitative Voronovskaja-type result for $B_n^{\alpha,\beta}(f;x;\lambda)$.

**Theorem 7.** *Suppose that $f \in C[0,1]$ such that $f', f'' \in C[0,1]$. Then*

$$\begin{aligned} &\left| B^{\lambda}_{n,\alpha,\beta}(f;x)f(x) - f(x) - \mu_n(\alpha,\beta,\lambda;x)f'(x) - \{\nu_n(\alpha,\beta,\lambda;x)+1\}\frac{f''(x)}{2} \right| \\ &\leq \frac{C}{n}\phi^2(x)\omega_\phi\left(f'', \frac{1}{\sqrt{n}}\right). \end{aligned} \tag{19}$$

*for every $x \in [0,1]$ and sufficiently large $n$, where $C$ is a positive constant, $\mu_n(\alpha,\beta,\lambda;x)$ and $\nu_n(\alpha,\beta,\lambda;x)$ are defined in Theorem 2.*

**Proof.** Consider the following equality

$$f(t) - f(x) - (t-x)f'(x) = \int_x^t (t-u)f''(u)du$$

for $f \in C[0,1]$. It follows that

$$f(t) - f(x) - (t-x)f'(x) - \frac{f''(x)}{2}((t-x)^2+1) \leq \int_x^t (t-u)[f''(u) - f''(x)]du. \tag{20}$$

Applying $B^{\lambda}_{n,\alpha,\beta}(\cdot;x)$ to both sides of (20), we obtain

$$\begin{aligned} &\left| B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x) - B^{\lambda}_{n,\alpha,\beta}((t-x);x)f'(x) - \frac{f''(x)}{2}(B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x) + B^{\lambda}_{n,\alpha,\beta}(1;x)) \right| \\ &\leq B^{\lambda}_{n,\alpha,\beta}\left( \left| \int_x^t |t-u|\, |f''(u) - f''(x)|\, du \right| ; x \right). \end{aligned} \tag{21}$$

The quantity in the right hand side of (21) can be estimated as

$$\left| \int_x^t |t-u|\, |f''(u) - f''(x)|\, du \right| \leq 2\|f''-g\|(t-x)^2 + 2\|\phi g'\|\phi^{-1}(x)|t-x|^3, \tag{22}$$

where $g \in W_\phi[0,1]$. There exists $C > 0$ such that

$$B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x) \leq \frac{C}{2n}\phi^2(x) \quad \text{and} \quad B^{\lambda}_{n,\alpha,\beta}((t-x)^4;x) \leq \frac{C}{2n^2}\phi^4(x) \tag{23}$$

for sufficiently large $n$. By taking (21)–(23) into our account and using Cauchy–Schwarz inequality, we have

$$\begin{aligned} &\left| B^{\lambda}_{n,\alpha,\beta}(f;x) - f(x) - B^{\lambda}_{n,\alpha,\beta}((t-x);x)f'(x) - \frac{f''(x)}{2}(B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x) + B^{\lambda}_{n,\alpha,\beta}(1;x)) \right| \\ &\leq 2\|f''-g\|B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x) + 2\|\phi g'\|\phi^{-1}(x)B^{\lambda}_{n,\alpha,\beta}(|t-x|^3;x) \\ &\leq \frac{C}{n}x(1-x)\|f''-g\| + 2\|\phi g'\|\phi^{-1}(x)\{B^{\lambda}_{n,\alpha,\beta}((t-x)^2;x)\}^{1/2}\{B^{\lambda}_{n,\alpha,\beta}((t-x)^4;x)\}^{1/2} \\ &\leq \frac{C}{n}\phi^2(x)\left\{\|f''-g\| + n^{-1/2}\|\phi g'\|\right\}. \end{aligned}$$

Finally, by taking infimum over all $g \in W_\phi[0,1]$, this last inequality leads us to the assertion (19) of Theorem 7. □

As an immediate consequence of Theorem 7, we have the following result.

**Corollary 3.** *If $f \in C[0,1]$ such that $f', f'' \in C[0,1]$, then*

$$\lim_{n\to\infty} n\left| B^{\lambda}_{n,\alpha,\beta}(f;x)f(x) - f(x) - \mu_n(\alpha,\beta,\lambda;x)f'(x) - \{\nu_n(\alpha,\beta,\lambda;x)+1\}\frac{f''(x)}{2} \right| = 0,$$

*where $\mu_n(\alpha,\beta,\lambda;x)$ and $\nu_n(\alpha,\beta,\lambda;x)$ are defined in Theorem 2.*

## 4. The Bivariate Case of the Operators $B^{\lambda}_{n,\alpha,\beta}(f;x)$

We construct bivariate version of Stancu-type $\lambda$-Bernstein operators defined which was defined in the first section of this manuscript as (5) and study their approximation properties.

For $0 \le \alpha_i \le \beta_i$ $(i = 1,2)$, we defined the bivariate version of Stancu-type $\lambda$-Bernstein operators by

$$B^{\lambda,\alpha,\beta}_{n,m}(f;x,y) = \sum_{i_1=0}^{n}\sum_{i_2=0}^{m} f\left(\frac{i_1+\alpha_1}{n+\beta_1}, \frac{i_2+\alpha_2}{m+\beta_2}\right)\tilde{b}_{n,i_1}(\lambda_1;x)\tilde{b}_{m,i_2}(\lambda_2;y) \tag{24}$$

for $(x,y) \in I$ and $f \in C(I)$, where $I = [0,1]\times[0,1]$ and $\tilde{b}_{n,i_1}(\lambda_1;x)$ and $\tilde{b}_{m,i_2}(\lambda_2;x)$ are Bézier bases defined in (4).

We remark that if we take $\lambda_1 = \lambda_2 = 0$ in bivariate $\lambda$-Bernstein–Stancu operators, then (24) reduces to the classical bivariate Bernstein–Stancu operators defined in [28]. Also, for $\alpha_1 = \beta_1 = \lambda_1 = 0$ and $\alpha_2 = \beta_2 = \lambda_2 = 0$, the bivariate $\lambda$-Bernstein–Stancu operators (24) reduce to classical bivariate Bernstein operators defined in [29].

**Lemma 2.** *The following equalities hold for bivariate $\lambda$-Bernstein–Stancu operators:*

$$\begin{aligned}
B^{\lambda,\alpha,\beta}_{n,m}(1;x,y) &= 1;\\
B^{\alpha,\beta}_{n,m}(s;x,y) &= \frac{\alpha_1+nx}{n+\beta_1} + \lambda_1\left[\frac{1-2x+x^{n+1}+(\alpha_1-1)(1-x)^{n+1}}{(n+\beta_1)(n-1)} + \frac{\alpha_1 x(1-x)^n}{n+\beta_1}\right];\\
B^{\lambda,\alpha,\beta}_{n,m}(t;x,y) &= \frac{\alpha_2+my}{m+\beta_2} + \lambda_2\left[\frac{1-2y+y^{m+1}+(\alpha_2-1)(1-y)^{m+1}}{(m+\beta_2)(m-1)} + \frac{\alpha_2 y(1-y)^m}{m+\beta_2}\right];\\
B^{\lambda,\alpha,\beta}_{n,m}(s^2;x,y) &= \frac{1}{(n+\beta_1)^2}\left\{n(n-1)x^2+(1+2\alpha_1)nx+\alpha_1^2\right\}\\
&\quad + \lambda_1\left[\frac{2nx-1-4nx^2+(2n+1)x^{n+1}+(1-x)^{n+1}}{(n+\beta_1)^2(n-1)} + \frac{\alpha_1^2-4\alpha_1 x}{(n+\beta_1)^2(n-1)}\right.\\
&\quad \left. + \frac{2\alpha_1 n - 2\alpha_1(\alpha_1+n)(x^{n+1}+(1-x)^n)+\alpha_1^2 x(n^2+1)(1-x)^n}{(n+\beta_1)^2(n^2-1)}\right];\\
B^{\lambda,\alpha,\beta}_{n,m}(t^2;x,y) &= \frac{1}{(m+\beta_2)^2}\left\{m(m-1)y^2+(1+2\alpha_2)my+\alpha_2^2\right\}\\
&\quad + \lambda_2\left[\frac{2my-1-4my^2+(2m+1)y^{m+1}+(1-y)^{m+1}}{(m+\beta_2)^2(m-1)} + \frac{\alpha_2^2-4\alpha_2 y}{(n+\beta_2)^2(m-1)}\right.\\
&\quad \left. + \frac{2\alpha_2 m - 2\alpha_2(\alpha_2+m)(y^{m+1}+(1-y)^m)+\alpha_2^2 y(m^2+1)(1-y)^m}{(m+\beta_2)^2(m^2-1)}\right].
\end{aligned}$$

**Theorem 8.** *Let $e_{ij}(x,y) = x^i y^j$, where $0 \le i+j \le 2$. Then, the sequence $B^{\lambda,\alpha,\beta}_{n,m}(f;x,y)$ of operators converges uniformly to $f$ on $I$ for each $f \in C(I)$.*

**Proof.** It is enough to prove the following condition

$$\lim_{n,m\to\infty} B^{\lambda,\alpha,\beta}_{n,m}(e_{ij};x,y) = e_{ij}$$

converges uniformly on $I$. With the help of Lemma 2, one can see that

$$\lim_{m,n\to\infty} B_{n,m}^{\lambda,\alpha,\beta}\left(e_{00};x,y\right)=e_{00},$$

$$\lim_{n,m\to\infty} B_{n,m}^{\lambda,\alpha,\beta}\left(e_{10};x,y\right)=e_{10},\qquad \lim_{n,m\to\infty} B_{n,m}^{\lambda,\alpha,\beta}\left(e_{01};x,y\right)=e_{01}$$

and

$$\lim_{n,m\to\infty} B_{n,m}^{\lambda,\alpha,\beta}\left(e_{02}+e_{20};x,y\right)=e_{02}+e_{20}.$$

Keeping in mind the above conditions and Korovkin type theorem established by Volkov [30], we obtain

$$\lim_{m,n\to\infty} B_{n,m}^{\lambda,\alpha,\beta}\left(f;x,y\right)=f$$

converges uniformly. □

Now, we compute the rate of convergence of operators (24) by means of the modulus of continuity. Recall that the modulus of continuity for bivariate case is defined as

$$\omega(f,\delta)=\sup\left\{|f(s,t)-f(x,y)|:\sqrt{(s-x)^2+(t-y)^2}\leq\delta\right\}$$

for $f\in C(I_{ab})$ and for every $(s,t),(x,y)\in I_{ab}=[0,a]\times[0,b]$. The partial moduli of continuity with respect to $x$ and $y$ are defined by

$$\begin{aligned}\omega_1(f,\delta) &= \sup\left\{|f(x_1,y)-f(x_2,y)|: y\in[0,a] \text{ and } |x_1-x_2|\leq\delta\right\},\\ \omega_2(f,\delta) &= \sup\left\{|f(x,y_1)-f(x,y_2)|: x\in[0,b] \text{ and } |y_1-y_2|\leq\delta\right\}.\end{aligned}$$

Peetre's $K$-functional is given by

$$K(f,\delta)=\inf_{g\in C^2(I_{ab})}\left\{\|f-g\|_{C(I_{ab})}+\delta\|g\|_{C^2(I_{ab})}\right\}$$

for $\delta>0$, where $C^2(I_{ab})$ is the space of functions of $f$ such that $f$, $\frac{\partial^j f}{\partial x^j}$ and $\frac{\partial^j f}{\partial y^j}$ $(j=1,2)$ in $C(I_{ab})$ [26]. We now give an estimate of the rates of convergence of operators $B_{n,m}^{\lambda,\alpha,\beta}\left(f;x,y\right)$.

**Theorem 9.** *Let $f\in C\left(I\right)$. Then*

$$\left|B_{n,m}^{\lambda,\alpha,\beta}\left(f;x,y\right)-f\left(x,y\right)\right|\leq 4\omega\left(f;\nu_n^{1/2}(\alpha,\beta,\lambda;x),\nu_m^{1/2}(\alpha,\beta,\lambda;y)\right)$$

*for all $x\in I$, where*

$$\nu_n(\alpha,\beta,\lambda;x)=B_{n,m}^{\lambda,\alpha,\beta}\left((s-x)^2;x,y\right)\ \text{ and }\ \nu_m(\alpha,\beta,\lambda;y)=B_{n,m}^{\lambda,\alpha,\beta}\left((t-y)^2;x,y\right).$$

**Proof.** Since (24) is linear and positive, we have

$$\begin{aligned}\left|B_{n,m}^{\lambda,\alpha,\beta}\left(f;x,y\right)-f(x,y)\right| &\leq B_{n,m}^{\lambda,\alpha,\beta}\left(|f(s,t)-f(x,y)|;x,y\right)\\ &\leq B_{n,m}^{\lambda,\alpha,\beta}\left(\omega\left(f;\sqrt{(s-x)^2+(t-y)^2}\right);x,y\right)\\ &\leq \omega\left(f;\sqrt{\nu_n(\alpha,\beta,\lambda;x)},\sqrt{\nu_m(\alpha,\beta,\lambda;y)}\right)\end{aligned}$$

$$\times\left[\frac{1}{\sqrt{\nu_n(\alpha,\beta,\lambda;x)\nu_m(\alpha,\beta,\lambda;y)}}B_{n,m}^{\lambda,\alpha,\beta}\left(\sqrt{(s-x)^2+(t-y)^2};x,y\right)\right].$$

The Cauchy–Schwartz inequality gives that

$$\begin{aligned}
&|B_{n,m}^{\lambda,\alpha,\beta}(f;x,y)-f(x,y)|\\
&\leq\omega\left(f;\sqrt{\nu_n(\alpha,\beta,\lambda;x)},\sqrt{\nu_m(\alpha,\beta,\lambda;y)}\right)\\
&\times\left[1+\frac{1}{\sqrt{\nu_n(\alpha,\beta,\lambda;x)\nu_m(\alpha,\beta,\lambda;y)}}\left\{B_{n,m}^{\lambda,\alpha,\beta}\left((s-x)^2;x,y\right)B_{n,m}^{\lambda,\alpha,\beta}\left((t-y)^2;x,y\right)\right\}^{1/2}\right.\\
&\left.+\frac{\sqrt{B_{n,m}^{\lambda,\alpha,\beta}((s-x)^2;x,y)}}{\sqrt{\nu_n(\alpha,\beta,\lambda;x)}}+\frac{\sqrt{B_{n,m}^{\lambda,\alpha,\beta}((t-y)^2;x,y)}}{\sqrt{\nu_m(\alpha,\beta,\lambda;y)}}\right].
\end{aligned}$$

If we choose

$$\nu_n(\alpha,\beta,\lambda;x)=B_{n,m}^{\lambda,\alpha,\beta}\left((s-x)^2;x,y\right)\quad\text{and}\quad\nu_m(\alpha,\beta,\lambda;y)=B_{n,m}^{\lambda,\alpha,\beta}\left((t-y)^2;x,y\right)$$

for all $(x,y)\in I$ we complete the proof, where

$$\begin{aligned}
B_{n,m}^{\lambda,\alpha,\beta}\left((s-x)^2;x,y\right) &= B_{n,m}^{\lambda,\alpha,\beta}\left(s^2;x,y\right)-2xB_{n,m}^{\lambda,\alpha,\beta}(s;x,y)+x^2B_{n,m}^{\lambda,\alpha,\beta}(1;x,y)\\
&= \frac{nx(1-x)+(\beta_1x-\alpha_1)^2}{(n+\beta_1)^2}\\
&\quad+\lambda_1\left[\frac{4x^2-2x-2x^{n+2}-2(\alpha_1-1)x(1-x)^{n+1}}{(n+\beta_1)(n-1)}-\frac{2\alpha_1x^2(1-x)^n}{n+\beta_1}\right]\\
&\quad+\lambda_1\frac{2nx-1-4nx^2+(2n+1)x^{n+1}+(1-x)^{n+1}+\alpha_1^2-4\alpha_1x}{(n+\beta_1)^2(n-1)}\\
&\quad+\lambda_1\frac{2\alpha_1n-2\alpha_1(\alpha_1+n)(x^{n+1}+(1-x)^n)+\alpha_1^2x(n^2+1)(1-x)^n}{(n+\beta_1)^2(n^2-1)};\\
B_{n,m}^{\lambda,\alpha,\beta}\left((t-y)^2;x,y\right) &= \frac{my(1-y)+(\beta_2y-\alpha_2)^2}{(m+\beta_2)^2}\\
&\quad+\lambda_2\left[\frac{4y^2-2y-2y^{m+2}-2(\alpha_2-1)y(1-y)^{m+1}}{(m+\beta_2)(m-1)}-\frac{2\alpha_2y^2(1-y)^n}{m+\beta_2}\right]\\
&\quad+\lambda_2\frac{2my-1-4my^2+(2m+1)y^{m+1}+(1-y)^{m+1}+\alpha_2^2-4\alpha_2y}{(m+\beta_2)^2(m-1)}\\
&\quad+\lambda_2\frac{2\alpha_2m-2\alpha_2(\alpha_2+m)(y^{m+1}+(1-y)^m)+\alpha_2^2y(m^2+1)(1-y)^m}{(m+\beta_2)^2(m^2-1)}.
\end{aligned}$$

□

**Theorem 10.** *Let $f\in C(I)$. Then, the following inequality holds:*

$$\left|B_{n,m}^{\lambda,\alpha,\beta}(f;x,y)-f(x,y)\right|\leq 2\left[\omega_1\left(f;\nu_n^{1/2}(\alpha,\beta,\lambda;x)\right)+\omega_2\left(f;\nu_n^{1/2}(\alpha,\beta,\lambda;y)\right)\right],$$

*where $\nu_n(\alpha,\beta,\lambda;x)$ and $\nu_m(\alpha,\beta,\lambda;y)$ are defined in Theorem 9.*

**Proof.** By using the definition of partial modulus of continuity and Cauchy–Schwartz inequality, we have

$$\begin{aligned}
&|B_{n,m}^{\lambda,\alpha,\beta}(f;x,y)-f(x,y)| \\
&\quad\le B_{n,m}^{\lambda,\alpha,\beta}(|f(s,t)-f(x,y)|;x,y) \\
&\quad\le B_{n,m}^{\lambda,\alpha,\beta}(|f(s,t)-f(x,t)|;x,y)+B_{n,m}^{\lambda,\alpha,\beta}(|f(x,t)-f(x,y)|;x,y) \\
&\quad\le B_{n,m}^{\lambda,\alpha,\beta}(|\omega_1(f;|s-x|)|;x,y)+B_{n,m}^{\lambda,\alpha,\beta}(|\omega_2(f;|t-y|)|;x,y) \\
&\quad\le \omega_1(f,\nu_n(\alpha,\beta,\lambda;x))\left[1+\frac{1}{\nu_n(\alpha,\beta,\lambda;x)}B_{n,m}^{\lambda,\alpha,\beta}(|s-x|;x,y)\right] \\
&\qquad+\omega_2(f,\nu_m(\alpha,\beta,\lambda;y))\left[1+\frac{1}{\nu_m(\alpha,\beta,\lambda;y)}B_{n,m}^{\lambda,\alpha,\beta}(|t-y|;x,y)\right] \\
&\quad\le \omega_1(f,\nu_n^{1/2}(\alpha,\beta,\lambda;x))\left[1+\frac{1}{\nu_n^{1/2}(\alpha,\beta,\lambda;x)}\left(B_{n,m}^{\lambda,\alpha,\beta}\left((s-x)^2;x,y\right)\right)^{1/2}\right] \\
&\qquad+\omega_2(f,\nu_n^{1/2}(\alpha,\beta,\lambda;x))\left[1+\frac{1}{\nu_m^{1/2}(\alpha,\beta,\lambda;y)}\left(B_{n,m}^{\lambda,\alpha,\beta}\left((t-y)^2;x,y\right)\right)^{1/2}\right].
\end{aligned}$$

Finally, by choosing $\nu_n(\alpha,\beta,\lambda;x)$ and $\nu_m(\alpha,\beta,\lambda;y)$ as defined in Theorem 9, we obtain desired result. □

We recall that the Lipschitz class $Lip_M(\widehat{\beta}_1,\widehat{\beta}_2)$ for the bivariate is given by

$$|f(s,t)-f(x,y)|\le M\,|s-x|^{\widehat{\beta}_1}\,|t-y|^{\widehat{\beta}_2}$$

for $\widehat{\beta}_1,\widehat{\beta}_2\in(0,1]$ and $(s,t),(x,y)\in I_{ab}$.

**Theorem 11.** *Let $f\in Lip_M(\widehat{\beta}_1,\widehat{\beta}_2)$. Then, for all $(x,y)\in I_{ab}$, we have*

$$|B_{n,m}^{\lambda,\alpha,\beta}(f;x,y)-f(x,y)|\le M\nu_n^{\widehat{\beta}_1/2}(\alpha,\beta,\lambda;x)\nu_m^{\widehat{\beta}_2/2}(\alpha,\beta,\lambda;y),$$

*where $\nu_n(\alpha,\beta,\lambda;x)$ and $\nu_m(\alpha,\beta,\lambda;y)$ are defined in Theorem 9.*

**Proof.** We have

$$\begin{aligned}
|B_{n,m}^{\lambda,\alpha,\beta}(f;x,y)-f(x,y)| &\le B_{n,m}^{\lambda,\alpha,\beta}(|f(s,t)-f(x,y)|;x,y) \\
&\le MB_{n,m}^{\lambda,\alpha,\beta}(|s-x|^{\widehat{\beta}_1}|t-y|^{\widehat{\beta}_2};x,y)
\end{aligned}$$

$$= MB_{n,m}^{\lambda,\alpha,\beta}(|s-x|^{\widehat{\beta}_1}|;x,y)B_{n,m}^{\lambda,\alpha,\beta}(|t-y|^{\widehat{\beta}_2};x,y)$$

since $f\in Lip_M(\widehat{\beta}_1,\widehat{\beta}_2)$. Then, by applying the Hölder's inequality for

$$\widehat{p}_1=\frac{2}{\widehat{\beta}_1},\widehat{q}_1=\frac{2}{2-\widehat{\beta}_1}$$

and

$$\widehat{p}_2=\frac{1}{\widehat{\beta}_2},\widehat{q}_2=\frac{2}{2-\widehat{\beta}_2},$$

we obtain

$$\begin{aligned}|B_{n,m}^{\lambda,\alpha,\beta}(f;x,y)-f(x,y)| &\leq M\{B_{n,m}^{\lambda,\alpha,\beta}(|s-x|^2;x,y)\}^{\widehat{\beta}_1/2}\{B_{n,m}^{\lambda,\alpha,\beta}(1;x,y)\}^{\widehat{\beta}_1/2}\\ &\quad\times\{B_{n,m}^{\lambda,\alpha,\beta}(|t-y|^2;x,y)\}^{\widehat{\beta}_2/2}\{B_{n,m}^{\lambda,\alpha,\beta}(1;x,y)\}^{\widehat{\beta}_2/2}\\ &= M\nu_n(\alpha,\beta,\lambda;x)^{\widehat{\beta}_1/2}\nu_m(\alpha,\beta,\lambda;y)^{\widehat{\beta}_2/2}.\end{aligned}$$

This completes the proof. □

**Theorem 12.** *For $f\in C^1(I)$, the following inequality holds:*

$$|B_{n,m}^{\lambda,\alpha,\beta}(f;x,y)-f(x,y)| \leq \parallel f_x\parallel_{C(I)}\nu_n^{1/2}(\alpha,\beta,\lambda;x)+\parallel f_y\parallel_{C(I)}\nu_m^{1/2}(\alpha,\beta,\lambda;y),$$

*where $\nu_n(\alpha,\beta,\lambda;x)$ and $\nu_m(\alpha,\beta,\lambda;y)$ are defined in Theorem 9.*

**Proof.** We have

$$f(t)-f(s)=\int_x^t f_u(u,s)du+\int_y^s f_v(x,v)du$$

for $(s,t)\in I$. Thus, by applying the operators defined in (24) to the above equality, we obtain

$$\begin{aligned}&|B_{n,m}^{\lambda,\alpha,\beta}(f;x,y)-f(x,y)|\\ &\leq B_{n,m}^{\lambda,\alpha,\beta}\left(\left|\int_x^t f_u(u,s)du\right|;x,y\right)+B_{n,m}^{\lambda,\alpha,\beta}\left(\left|\int_y^s f_v(x,v)du\right|;x,y\right).\end{aligned}$$

By taking the following relations into our consideration

$$\left|\int_x^t f_u(u,s)du\right|\leq\parallel f_x\parallel_{C(I_{ab})}|s-x|$$

and

$$\left|\int_y^s f_v(x,v)du\right|\leq f_y\parallel_{C(I_{ab})}|t-y|,$$

one obtains

$$\begin{aligned}&|B_{n,m}^{\lambda,\alpha,\beta}(f;x,y)-f(x,y)|\\ &\leq\parallel f_x\parallel_{C(I)}B_{n,m}^{\lambda,\alpha,\beta}(|s-x|;x,y)+\parallel f_y\parallel_{C(I)}B_{n,m}^{\lambda,\alpha,\beta}(|t-y|;x,y).\end{aligned}$$

Using Cauchy–Schwarz inequality, we have

$$\begin{aligned}&|B_{n,m}^{\lambda,\alpha,\beta}(f;x,y)-f(x,y)|\\ &\leq\parallel f_x\parallel_{C(I)}\{B_{n,m}^{\lambda,\alpha,\beta}\left((s-x)^2;x,y\right)\}^{1/2}\{B_{n,m}^{\lambda,\alpha,\beta}(1;x,y)\}^{1/2}\\ &\quad+\parallel f_y\parallel_{C(I)}\{B_{n,m}^{\lambda,\alpha,\beta}\left((t-y)^2;x,y\right)\}^{1/2}\{B_{n,m}^{\lambda,\alpha,\beta}(1;x,y)\}^{1/2}.\end{aligned}$$

□

Finally, we presents a Voronovskaja-type theorem for $B_{n,n}^{\lambda,\alpha,\beta}(f;x,y)$.

**Theorem 13.** *Let $f \in C^2(I)$. Then*

$$\begin{aligned} \lim_{n\to\infty} n\left[B_{n,n}^{\lambda,\alpha,\beta}(f;x,y) - f(x,y)\right] &= (\alpha_1 - \beta_1 x)f_x + (\alpha_2 - \beta_2 y)f_y \\ &+\frac{x(1-x)}{2}f_{xx} + \frac{y(1-y)}{2}f_{yy}. \end{aligned}$$

**Proof.** Let $(x,y) \in I$ and write the Taylor's formula of $f(s,t)$ as

$$\begin{aligned} f(s,t) &= f(x,y) + f_x(s-x) + f_y(t-y) \\ &+\frac{1}{2}\left\{f_{xx}(s-x)^2 + 2f_{xy}(s-x)(t-y) + f_{yy}(t-y)^2\right\} \\ &+\varepsilon(s,t)\left((s-x)^2 + (t-y)^2\right), \end{aligned} \tag{25}$$

where $(s,t) \in I$ and $\varepsilon(s,t) \longrightarrow 0$ as $(s,t) \longrightarrow (x,y)$. If we apply sequence of operators $B_{n,n}^{\lambda,\alpha,\beta}(\cdot;x,y)$ on (25) keeping in mind linearity of operator, we have

$$\begin{aligned} & B_{n,n}^{\lambda,\alpha,\beta}(f;s,t) - f(x,y) \\ &= f_x(x,y)B_{n,n}^{\lambda,\alpha,\beta}((s-x);x,y) + f_y(x,y)B_{n,n}^{\lambda,\alpha,\beta}((t-y);x,y) \\ &\quad+\frac{1}{2}\Big\{f_{xx}B_{n,n}^{\lambda,\alpha,\beta}((s-x)^2;x,y) + 2f_{xy}B_{n,n}^{\lambda,\alpha,\beta}((s-x)(t-y);x,y) \\ &\quad+f_{yy}B_{n,n}^{\lambda,\alpha,\beta}((t-y)^2;x,y)\Big\} + B_{n,n}^{\lambda,\alpha,\beta}\left(\varepsilon(s,t)\left((s-x)^2 + (t-y)^2\right);x,y\right). \end{aligned}$$

Applying limit to both sides of the last equality as $n \to \infty$, we have

$$\begin{aligned} & \lim_{n\to\infty} n(B_{n,n}^{\lambda,\alpha,\beta}(f;s,t) - f(x,y)) \\ &= \lim_{n\to\infty} n\left\{f_x(x,y)B_{n,n}^{\lambda,\alpha,\beta}((s-x);x,y) + f_y(x,y)B_{n,n}^{\lambda,\alpha,\beta}((t-y);x,y)\right\} \\ &\quad+\lim_{n\to\infty}\frac{n}{2}\Big\{f_{xx}B_{n,n}^{\lambda,\alpha,\beta}((s-x)^2;x,y) \\ &\quad+2f_{xy}B_{n,n}^{\lambda,\alpha,\beta}((s-x)(t-y);x,y) + f_{yy}B_{n,n}^{\lambda,\alpha,\beta}((t-y)^2;x,y)\Big\} \\ &\quad+\lim_{n\to\infty} nB_{n,n}^{\lambda,\alpha,\beta}\left(\varepsilon(s,t)\left((s-x)^2 + (t-y)^2\right);x,y\right). \end{aligned}$$

Using Hölder inequality for the last term of above equality, we have

$$\begin{aligned} & B_{n,n}^{\lambda,\alpha,\beta}\left(\varepsilon(s,t)\left((s-x)^2 + (t-y)^2\right);x,y\right) \\ &\le \sqrt{2}\sqrt{B_{n,n}^{\lambda,\alpha,\beta}\left(\varepsilon^2(s,t);x,y\right)} \\ &\quad\times\sqrt{B_{n,n}^{\lambda,\alpha,\beta}\left(\varepsilon(s,t)\left((s-x)^4 + (t-y)^4\right);x,y\right)}. \end{aligned}$$

Since

$$\lim_{n\to\infty} B_{n,n}^{\lambda,\alpha,\beta}\left(\varepsilon^2(s,t);x,y\right) = \varepsilon^2(x,y) = 0$$

we have

$$\lim_{n\to\infty} n\, B_{n,n}^{\lambda,\alpha,\beta}\left(\varepsilon(s,t)\left((s-x)^4 + (t-y)^4\right);x,y\right) = 0. \tag{26}$$

Consequently, we obtain

$$\lim_{n\to\infty} n\, B_{n,n}^{\lambda,\alpha,\beta}((s-x);x,y) = \alpha_1 - \beta_1 x, \tag{27}$$

$$\lim_{n\to\infty} n\, B_{n,n}^{\lambda,\alpha,\beta}((t-y);x,y) = \alpha_2 - \beta_2 y, \tag{28}$$

$$\lim_{n\to\infty} n\, B_{n,n}^{\lambda,\alpha,\beta}((s-x)^2;x,y) = x(1-x), \tag{29}$$

$$\lim_{n\to\infty} n\, B_{n,n}^{\lambda,\alpha,\beta}((t-y)^2;x,y) = y(1-y). \tag{30}$$

Combining (26)–(30), we deduce the desired result. □

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