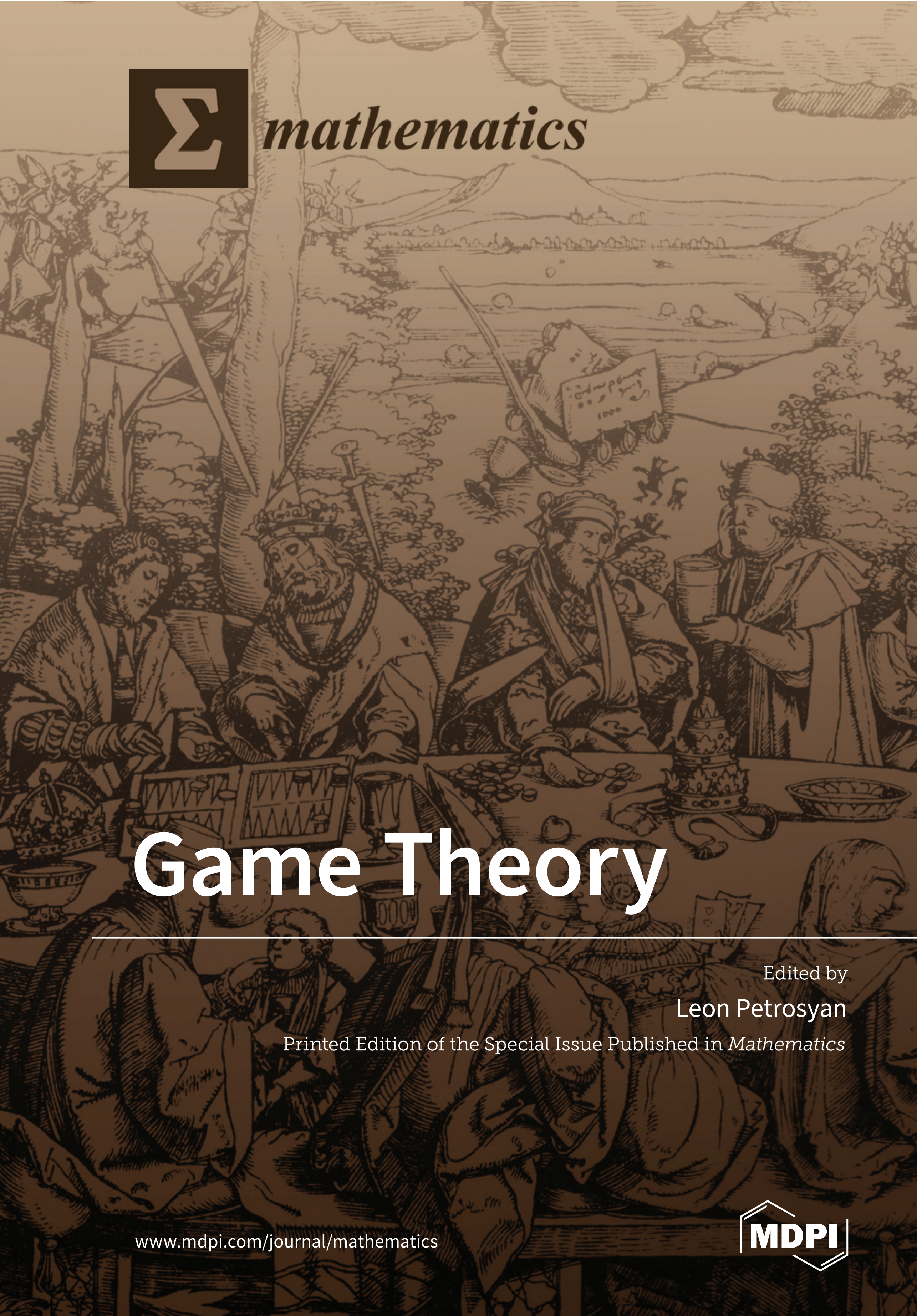

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Game Theory

Edited by

Leon Petrosyan

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Game Theory

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About the Editor

Leon Petrosyan is a professor, chair of the department "Mathematical Game Theory and Statistical Decisions", dean of the faculty of "Applied Mathematics" of Saint Petersburg State University in Saint Petersburg, Russia. He earned his Ph.D. in Mathematics (1995) from Vilnius University (the title of his thesis was "On a Class of Pursuit Games") and Dr.Sci. in Mathematics (1972) from Saint-Petersburg State University (the title of his thesis was "Differential Games of Pursuit"). His research interests include differential and dynamic games, mathematical control theory, operations research, game theory. He was chair of the program and organizing committees of numerous International Conferences in Game theory and Applications and is editor of periodicals in game theory and operations research. He is author of more than 300 publications in mathematical game theory periodicals and author of more than 30 books on this topic. In the years 2008–2012 he was elected as president of the "International Society of Dynamic Games"; he is a fellow member of "Game Theory Society" and a foreign member of "National Academy of Sciences of Armenia".

Preface to "Game Theory"

The importance of strategic behavior in the human and social world is increasingly recognized in theory and practice. As a result, game theory has emerged as a fundamental instrument in pure and applied research. It has greatly enhanced our understanding in decision making. The discipline of game theory studies decision making in an interactive environment. It draws on mathematics, statistics, operations research, engineering, biology, economics, political science and other subjects. In canonical form, the game takes place when an individual pursues an objective in a situation in which other individuals concurrently pursue other (possibly conflicting, possibly overlapping) objectives and at the same time the objectives cannot be reached by individual actions of one decision maker. The formulation of optimal behaviors for individuals called players is a fundamental element in the theory of games. The player behaviors satisfying some special principles, called optimality principles, constitute a solution of the game. The problem then is to formulate these principles and, on this basis, define the behavior of each individual (player), investigate the existence of such behavior and, in the case that it exists create analytical or computational methods to find the optimal behavior. Game theory has greatly enhanced our understanding of decision making. As socioeconomics and political problems increase in complexity, further advances in the theory's analytic content, methodology, techniques and applications, as well as case studies and empirical investigations, are urgently required. Theoretical results and applications in games are proceeding apace, in areas ranging from aircraft and missile control to inventory management, market development, natural recourse extraction, competition policy, negotiation techniques, macroeconomic and environmental planning, capital accumulation and investment. In the social sciences, economics and finance are the fields which most vividly display the characteristic of games. Not only would research be directed towards more realistic and relevant analysis of economic and social decision-making, but the game-theoretic approach is likely to reveal new interesting questions and problems specially in pure and applied mathematics.

Leon Petrosyan
Editor

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Article

Characterizations, Potential, and an Implementation of the Shapley-Solidarity Value

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Abstract: In this paper, we provide cooperative and non-cooperative interpretations of the Shapley–Solidarity value for cooperative games with coalition structure. Firstly, we present two new characterizations of this value based on intracoalitional quasi-balanced contributions property. Secondly, we study a potential function of the Shapley–Solidarity value. Finally, we propose a new bidding mechanism for the Solidarity value and then apply the result to the Shapley–Solidarity value.

Keywords: Shapley-Solidarity value; coalition structure; potential; bidding mechanism

1. Introduction

In economic situations, players usually form a coalition with the aim of obtaining more profits. It is an important issue to distribute the surplus of cooperation among the players. Game theory provides general mathematical techniques to analyze such distribution problems. The solution part of cooperative game theory deals with the allocation problem of how to divide the overall earnings among the players in the game. There are various solution concepts in the field of cooperative games.

Probably, the Shapley value [1] and the Solidarity value [2] are the two most well-known solutions in cooperative game theory. They have similar formula, but a different way to value players' contributions to the coalitions. The Shapley value is a marginalist value because it considers the pure marginal contribution of every player, while the Solidarity value behaves more equalitarian than the Shapley value, since it considers the average marginal contribution of every player.

Various authors have proposed many characterizations of these two solutions. Shapley [1] initially proposed an axiomatization of the Shapley value by means of efficiency, additivity, symmetry and the null player property. Among the latter ones, Young's axiomatization [3] stands out by the elegance of its marginality axiom, which replaces additivity and the null player property in the initial characterization of the Shapley value. Myerson [4] proposed an axiom of the balanced contributions property in order to characterize the Shapley value. van den Brink [5] interpreted the Shapley value as the unique solution satisfying fairness, efficiency, and the null player property. Mcquillin and Sugden [6] characterized the Shapley value while using a condition of 'undominated merge-externalities'. Calleja and Llerena [7] provided an axiomatization of the Shapley value by weak self consistency, weak fairness, and the dummy player property. Recently, Alonso-Meijide et al. [8] introduced new properties by considering three special kinds of agents: null, nullifying, and necessary agents, and then obtained new characterizations of the Shapley value. van den Brink et al. [9] discussed two solutions for cooperative transferable utility games, the Shapley value and the proper Shapley value and characterized them by the property of affine invariance.

Moreover, Yokote et al. [10] introduced the balanced contributions property for equal contributors to the γ-egalitarian Shapley values which include many variants of the Shapley value, such as the egalitarian Shapley values and the discounted Shapley values. Casajus [11] proposed the relaxations of symmetry, weak sign symmetry, and characterized the class of the weighted Shapley values, together with efficiency, additivity, and the null player property. Abe and Nakada [12] proposed a new class of allocation rules for TU-games, named the weighted-egalitarian Shapley values, where each rule in this class takes into account each player's contributions and heterogeneity among players in order to determine each player's allocation. They provided an axiomatic foundation for the rules. Choudhury et al. [13] defined the generalized egalitarian Shapley value that gives the planner more flexibility in choosing the level of marginality based on the coalition size, and provided two characterizations of the generalized egalitarian Shapley value. An overview of study that was associated with the Shapley value can be found in literature [14].

Nowak and Radzik [2] characterized the Solidarity value by introducing the A-null player property instead of the null player property in the axiomatization of the Shaplay value [1]. Casajus [15] proposed a differential version of Young's marginality axiom. Differential marginality, together with efficiency and the (A-)null player property, allows for a direct proof of additivity, entailing characterizations of the Shapley value and the Solidarity value. Gutiérrez-López [16] defined the class of all egalitarian solidarity values, which are convex combinations of the solidarity value and the equal division solution, then provided two alternative characterizations of the corresponding values. There are also more various axiomatizations of the Shapley value and the Solidarity value in literature [17–19].

The coalition structure is used to model the situation where the players form groups for bargaining payoffs in cooperative games. The Owen value [20] is a well-known value for games with coalition structure. The Owen value adopts the Shapley value both inside the unions and among the unions. It behaves as a pure marginalist solution even inside the union, where the player's behaviour should be united. Alonso-Meijide et al. [21] extended the equal division and the equal surplus division values to the more general setup of cooperative games with a coalition structure, and provided axiomatic characterizations of the values. Hu [22] studied the weighted Shapley-egalitarian value and the collective value, and parallel axiomatizations of them were proposed by replacing the collective balanced contributions axiom with two intuitive axioms. Zou et al. [23] defined an extended Shapley value for generalized cooperative games under precedence constraints and provided two axiomatic characterizations of this value.

Calvo and Gutiérrez [24] combined the Shapley value and the Solidarity value together and defined a value named the Shapley–Solidarity value for games with coalition structure. They thought that the players within a union were more willing to show their solidarity and each union was more inclined to protect its revenue. Therefore, the Solidarity value is applied in order to obtain the payoffs of the players inside each union while the Shapley value is applied in order to compute the total payoffs of each union.

Calvo and Gutiérrez [24] characterized the Shapley-Solidarity value using efficiency, coalitional balanced contributions and intracoalitional equal averaged gains. Intracoalitional equal averaged gains is a coalitional version of the equal averaged gains property, which is used to characterize the Solidarity value [24]. Recently, Hu and Li [25] gave another characterization of the Shapley-Solidarity value by five axioms, including efficiency, additivity, intracoalitional symmetry, coalitional symmetry, and the partial A-null player property. The partial A-null player property states that a partial A-null player should receive zero in the game. The partial A-null player property is an extension of the A-null player property [2].

The intracoalitional balanced contributions property was introduced by Lorenzo–Freire [26,27] to characterize the Owen value and the Banzhaf-Owen value. In this paper, we propose a similar axiom, the intracoalitional quasi-balanced contributions property, in order to characterize the Shapley-Solidarity value.

Hart and Mas-Colell [28] introduced a way of potential function to characterize the Shapley value. They defined a potential function assigning to every cooperative game a real number and proved that the marginal contribution vector of the function coincides with the Shapley value. Winter [29] extended this way of characterization to games with coalition structure where the Owen value [20], the AD value [30] were considered. Xu et al. [31] adjusted the original potential function for the Shapley value, so that they got a potential function for the Solidarity value. For the Shapley–Solidarity value, we can obtain the potential function from the potential functions of the Owen and Solidarity value.

Another approach to study a cooperative game solution is mechanism design which can be viewed as a part of the Nash program [32] to implement cooperative game solutions through non-cooperative game theory. Bidding mechanism, first introduced by Pérez-Castrillo and Wettstein [33], is used to implement the Shapley value. Vidal-Puga and Bergantiños [34] implemented the Owen value by a two-rounds coalitional bidding mechanism. Sikker [35] studied the non-cooperative foundations of network allocation rules, including several classical solutions, such as the Myerson value and the position value, which are implemented by bidding mechanism. Ju and Wettstein [36] introduced a generalized bidding approach based on the original bidding approach in Pérez-Castrillo and Wettstein. Addtitionally, the Shapley, consensus [37] and equal surplus values [38] are implemented. van den Brink et al. [39] implemented the egalitarian Shapley value based on Pérez-Castrillo and Wettstein's bidding approach. For the discounted Shapley value, van den Brink and Funaki [40] implemented it in a similar approach.

There is no present research for studying the non-cooperative implementation of the Shapley–Solidarity value. If we want to implement the Shapley-Solidarity value we should replace the first round in the coalitional bidding mechanism in Vidal-Puga and Bergantiños [34] with a mechanism that can implement the Solidarity value.

In this paper, we will study the Shapley–Solidarity value from three aspects. First of all, we give axiomatic characterizations of the Shapley–Solidarity value mainly using intracoalitional quasi-balanced contributions, which is an extended version of quasi-balanced contributions [31]. Subsequently, we obtain an adjusted potential function with respect to the Shapley–Solidarity value. At last, we give a new implementation of the Solidarity value using bidding mechanism. Additionally, we apply this result to the games with coalition structure and implement the Shapley-Solidarity value.

The paper is organized, as follows. In Section 2, we introduce some basic definitions and notations. Axiomatic characterizations of the Shapley-Solidarity value will be presented in Section 3. Subsequently, we study the potential function for the Shapley–Solidarity value in Section 4. Finally, in Section 5, we provide a non-cooperative implementation of the Shapley–Solidarity value.

2. Preliminaries

A cooperative game with transferable utility or simply a game is a pair (N, v) where $N = \{1, 2, \cdots, n\}$ is a player set and $v : 2^n \to \mathbb{R}$ with $v(\emptyset) = 0$ is a characteristic function which assigns to each coalition $S \in 2^n$ the worth $v(S)$. Denote the family of all cooperative games by $\mathcal{G}$, and then denote the family of all cooperative games with player set N by $\mathcal{G}^N$. A game $(N, v) \in \mathcal{G}^N$ is zero-monotonic if $v(S) + v(\{i\}) \leq v(S \cup \{i\})$ for all $S \subseteq N$ with $i \notin S$, and is monotonic if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$.

A value on $\mathcal{G}^N$ is a function φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^n$ to every games $(N, v) \in \mathcal{G}^N$. The Shapley value [1], which is denoted by Sh, assigns to every player the expectation of his marginal contributions with respect to all coalitions. For any $(N, v) \in \mathcal{G}^N$ and $i \in N$,

$$Sh_i(N, v) = \sum_{i \in S, S \subseteq N} \frac{(|N| - |S|)!(|S| - 1)}{|N|!} [v(S) - v(S \backslash \{i\})],$$

where $|S|$ is the cardinality of player set S. The Solidarity value [2] is similar to the Shapley value, but it behaves more equalitarian, which assigns to every player the expectation of his average marginal contributions with respect to all coalitions. The Solidarity value $Sol(N,v) \in \mathbb{R}^n$ is defined as, for any $(N,v) \in \mathcal{G}^N$ and $i \in N$,

$$Sol_i(N,v) = \sum_{i \in S, S \subseteq N} \frac{(|N|-|S|)!(|S|-1)}{|N|!} \frac{1}{|S|} \sum_{i \in S} [v(S) - v(S \backslash \{i\})].$$

For a player set N, a coalition structure over N is a partition of N, i.e., $C = \{C_1, C_2, ..., C_m\}$, satisfying $\bigcup_{h \in M} C_h = N$ and $C_h \cap C_r = \emptyset$ when $h \neq r$, where $M = \{1, 2, ..., m\}$. The set $C_h \in C$ are called unions. Denote the set of all coalition structures over N by $\mathcal{C}(N)$. A game (N,v) with coalition structure $C \in \mathcal{C}(N)$ is denoted by (N,v,C). Denote the family of all the games with coalition structure with player set N by $\mathcal{CG}^N$.

For all games with coalition structure, the game defined between unions is called quotient game. Formally, for all game $(N,v,C) \in \mathcal{CG}^N$, with $C = \{C_1, C_2, ..., C_m\}$, the quotient game is denoted by $(M, v_C) \in \mathcal{G}^N$, where $M = \{1, 2, ..., m\}$ and $v_C(T) = v(\bigcup_{i \in T} C_i)$ for all $T \subseteq M$. For all $k \in M$ and all $S \subseteq C_k$, denote, by $C|_S$, the new coalition structure $(\bigcup_{j \neq k} C_j) \bigcup S$, which means that the union C_k is replaced by coalition S in the original coalition structure. The internal game (C_k, v_k) is defined in Owen [20] where $v_k(S) = Sh_k(M, v_{C|_S})$.

A coalitional value f on $\mathcal{CG}^N$ is a function assigning a vector $f(N,v,C) \in \mathbb{R}^N$ to each game with coalition structure $(N,v,C) \in \mathcal{CG}^N$. The Owen value [20] of the game (N,v,C) is the coalitional value, defined as

$$Ow_i(N,v,C) = Sh_i(C_h, v_h), \text{ for all } h \in M \text{ and all } i \in C_h.$$

Another interesting coalitional value, called the Shapley–Solidarity value [24], was introduced and characterized by Calvo and Gutiérrez. They stuck with the Shapley value among unions and applied the Solidarity value between players within the same union. Formally, for a game $(N,v,C) \in \mathcal{CG}^N$ the Shapley–Solidarity value is defined as

$$SS_i(N,v,C) = Sol_i(C_h, v_h), \text{ for all } h \in M \text{ and all } i \in C_h. \tag{1}$$

3. New Characterizations for the Shapley-Solidarity Value

Axiomatization is one of the main ways to characterize the reasonability of solutions with a set of properties in cooperative games. In this section, we characterize the Shapley–Solidarity value by introducing the intracoalitional quasi-balanced contributions property, inspired from the balanced contributions property.

The balanced contributions property, as introduced by Myerson [4], indicates that, for any pair of players, the influence of a player who leaves the grand coalition on the other player is the same as the impact of the other player's departure on it. Myerson [4] characterized the Shapley value by the balanced contributions property and efficiency.

Balanced contributions **(BC)**: a solution φ on $\mathcal{G}^N$ satisfies the balanced contributions property if for any $(N,v) \in \mathcal{G}^N$, $i, j \in N$ with $i \neq j$,

$$\varphi_i(N,v) - \varphi_i(N \backslash \{j\}, v) = \varphi_j(N,v) - \varphi_j(N \backslash \{i\}, v).$$

We first introduce two axioms, inspired by the balanced contributions property, and both of them are used in order to characterize the Solidarity value. In Xu et al. [31], the Solidarity value is characterized by the quasi-balanced contributions property with efficiency.

Quasi-balanced contributions **(QBC)**: a solution φ on $\mathcal{G}^N$ satisfies the quasi-balanced contributions property if for any $(N,v) \in \mathcal{G}^N$, $i, j \in N$ with $i \neq j$,

$$\varphi_i(N,v) - \varphi_i(N \backslash \{j\}, v) + \frac{1}{n} v(N \backslash \{j\}) = \varphi_j(N,v) - \varphi_j(N \backslash \{i\}, v) + \frac{1}{n} v(N \backslash \{i\}).$$

In Calvo and Gutiérrez [24], the Solidarity value is also characterized by the equal averaged gains property together with efficiency.

Equal averaged gains **(EAG)**: a solution φ on $\mathcal{G}^N$ satisfies the equal average gains property if for any $(N, v) \in \mathcal{G}^N$, $i, j \in N$ with $i \neq j$,

$$\frac{1}{n}\sum_{k\in N}[\varphi_i(N,v) - \varphi_i(N\backslash\{k\},v)] = \frac{1}{n}\sum_{k\in N}[\varphi_j(N,v) - \varphi_j(N\backslash\{k\},v)].$$

The previous two properties have been both used for characterizing the Solidarity value separately. However, one can easily check that these two properties are independent with each other.

Calvo et al. [41] introduced the property of intracoalitional balanced contributions to characterize the level structure value. It states that, for any two players in one union, the influence of a player who leaves the union on the other player is the same as the impact of the other player's departure on it.

Intracoalitional balanced contributions **(IBC)**: For all $(N, v, C) \in \mathcal{CG}^N$ and all $i, j \in C_h \in C$, $i \neq j$,

$$f_i(N,v,C) - f(N\backslash\{j\},v,C_{N\backslash\{j\}}) = f_j(N,v,C) - f(N\backslash\{i\},v,C_{N\backslash\{i\}}). \tag{2}$$

In Lorenzo-Freire [27], it is proved that, if a coalitional value satisfies the intracoalitional balanced contributions property, then it can be computed by means of the Shapley value. Next, we will define the intracoalitional quasi-balanced contributions property, and we will prove that if a coalitional value satisfies the intracoalitional quasi-balanced contributions property, it can be computed by means of the Solidarity value.

Intracoalitional quasi-balanced contributions **(IQBC)**: For all $(N, v, C) \in \mathcal{CG}^N$ and all $i, j \in C_h \in C$, $i \neq j$,

$$\begin{aligned} &f_i(N,v,C) - f_i(N\backslash\{j\},v,C_{N\backslash\{j\}}) + \frac{v^{f,C}(C_h\backslash\{j\})}{|C_h|} \\ =&f_j(N,v,C) - f_j(N\backslash\{i\},v,C_{N\backslash\{i\}}) + \frac{v^{f,C}(C_h\backslash\{i\})}{|C_h|}, \end{aligned} \tag{3}$$

where the game $(C_h, v^{f,C})$ [27] is defined as

$$v^{f,C}(T) = \sum_{i\in T} f_i((N\backslash\{C_h\})\cup T, v, (C\backslash\{C_h\})\cup T), \tag{4}$$

for all $T \subseteq C_h$, $T \neq \emptyset$. This game means that the payoff for each subset of players in the union is the sum of the payoffs of these players given by the coalitional value when the union is replaced by the subset. Obviously, this property is directly extended from the quasi-balanced contributions property.

Proposition 1. *A coalitional value f satisfies IQBC if and only if for all $(N, v, C) \in \mathcal{CG}^N$ and all $i \in C_h$ with $C_h \in C$,*

$$f_i(N,v,C) = \frac{1}{|C_h|}[v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\})] + \frac{1}{|C_h|}\sum_{j\in C_h\backslash\{i\}} f_i(N\backslash\{j\},v,C_{N\backslash\{j\}}). \tag{5}$$

Proof. Given $i \in C_h$, if a coalitional value f satisfies IQBC, then for all $j \in C_h\backslash\{i\}$,

$$\begin{aligned} &f_i(N,v,C) - f_i(N\backslash\{j\},v,C_{N\backslash\{j\}}) + \frac{v^{f,C}(C_h\backslash\{j\})}{|C_h|} \\ =&f_j(N,v,C) - f_j(N\backslash\{i\},v,C_{N\backslash\{i\}}) + \frac{v^{f,C}(C_h\backslash\{i\})}{|C_h|}. \end{aligned}$$

Summing over $j \in C_h\backslash\{i\}$, we obtain that

$$
\begin{aligned}
&(|C_h|-1)f_i(N,v,C) - \sum_{j\in C_h\setminus\{i\}} f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) + \frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} v^{f,C}(C_h\setminus\{j\}) \\
=& \sum_{j\in C_h\setminus\{i\}} f_j(N,v,C) - \sum_{j\in C_h\setminus\{i\}} f_j(N\setminus\{i\},v,C_{N\setminus\{i\}}) + \frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} v^{f,C}(C_h\setminus\{i\}).
\end{aligned}
$$

Consider the definition of the game $(C_h, v^{f,C})$, then we have

$$
\begin{aligned}
&(|C_h|-1)f_i(N,v,C) \\
=& \sum_{j\in C_h\setminus\{i\}} f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) - \frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} v^{f,C}(C_h\setminus\{j\}) \\
&- v^{f,C}(C_h\setminus\{i\}) + \frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} v^{f,C}(C_h\setminus\{i\}) + v^{f,C}(C_h) - f_i(N,v,C) \\
=& \sum_{j\in C_h\setminus\{i\}} f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) - \frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} v^{f,C}(C_h\setminus\{j\}) \\
&- \frac{1}{|C_h|}v^{f,C}(C_h\setminus\{i\}) + v^{f,C}(C_h) - f_i(N,v,C) \\
=& \sum_{j\in C_h\setminus\{i\}} f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\setminus\{j\}) + v^{f,C}(C_h) - f_i(N,v,C).
\end{aligned}
$$

Thus,

$$
f_i(N,v,C) = \frac{1}{|C_h|}[v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\setminus\{j\})] + \frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}).
$$

Now, it remains to proven the reverse part. The proof will be done by induction on $|C_h|$. Suppose that $C_h = \{i,j\}$, then we have

$$
\begin{aligned}
f_i(N,v,C) =& \frac{1}{2}[v^{f,C}(C_h) - \frac{1}{2}v^{f,C}(C_h\setminus\{j\}) - \frac{1}{2}v^{f,C}(C_h\setminus\{i\})] + \frac{1}{2}f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) \\
=& \frac{1}{2}[f_i(N,v,C) + f_j(N,v,C) - \frac{1}{2}v^{f,C}(C_h\setminus\{j\}) + \frac{1}{2}v^{f,C}(C_h\setminus\{i\})] \\
&- \frac{1}{2}f_j(N\setminus\{i\},v,C_{N\setminus\{i\}}) + \frac{1}{2}f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}).
\end{aligned}
$$

Accordingly, we have

$$
\begin{aligned}
&f_i(N,v,C) - f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) \\
=& f_j(N,v,C) - f_j(N\setminus\{i\},v,C_{N\setminus\{i\}}) + \frac{v^{f,C}(C_h\setminus\{i\})}{2} - \frac{v^{f,C}(C_h\setminus\{j\})}{2}.
\end{aligned}
$$

Suppose that $|C_h| > 2$. For any $i, j \in C_h$, we have

$$
\begin{aligned}
&|C_h|[f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) - f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})] \\
=& (|C_h|-1)[f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) - f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})] \\
&+ [f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) - f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})] \\
=& v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\}) - \frac{1}{|C_h|-1}\sum_{k\in C_h\setminus\{j\}} v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j,k\}) \\
&+ \sum_{k\in C_h\setminus\{i,j\}} f_i(N\setminus\{j,k\},v,C_{N\setminus\{j,k\}})
\end{aligned}
$$

$$
\begin{aligned}
&- [v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\}) - \frac{1}{|C_h|-1}\sum_{k\in C_h\setminus\{i\}} v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i,k\})\\
&+ \sum_{k\in C_h\setminus\{i,j\}} f_j(N\setminus\{i,k\}, v, C_{N\setminus\{i,k\}})]\\
&+ [f_i(N\setminus\{j\}, v, C_{N\setminus\{j\}}) - f_j(N\setminus\{i\}, v, C_{N\setminus\{i\}})]\\
=& v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\}) - \frac{1}{|C_h|-1}\sum_{k\in C_h\setminus\{i,j\}} [v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j,k\})\\
&- v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i,k\})] - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\})\\
&+ \sum_{k\in C_h\setminus\{i,j\}} [f_i(N\setminus\{j,k\}, v, C_{N\setminus\{j,k\}}) - f_j(N\setminus\{i,k\}, v, C_{N\setminus\{i,k\}})]\\
&+ [f_i(N\setminus\{j\}, v, C_{N\setminus\{j\}}) - f_j(N\setminus\{i\}, v, C_{N\setminus\{i\}})].
\end{aligned}
$$

By the induction hypothesis, the following result is true for all $k \in C_h\setminus\{i,j\}$,

$$
\begin{aligned}
&f_i(N\setminus\{j,k\}, v, C_{N\setminus\{j,k\}}) - f_j(N\setminus\{i,k\}, v, C_{N\setminus\{i,k\}})\\
=& f_i(N\setminus\{k\}, v, C_{N\setminus\{k\}}) - f_j(N\setminus\{k\}, v, C_{N\setminus\{k\}}) + \frac{1}{|C_h|-1} v^{f,C_{N\setminus\{k\}}}(C_h\setminus\{k,j\})\\
&- \frac{1}{|C_h|-1} v^{f,C_{N\setminus\{k\}}}(C_h\setminus\{k,i\}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|C_h|[f_i(N\setminus\{j\}, v, C_{N\setminus\{j\}}) - f_j(N\setminus\{i\}, v, C_{N\setminus\{i\}})]\\
=& v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\}) - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\})\\
&- \frac{1}{|C_h|-1}\sum_{k\in C_h\setminus\{i,j\}} [v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j,k\}) - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i,k\})]\\
&+ \sum_{k\in C_h\setminus\{i\}} f_i(N\setminus\{k\}, v, C_{N\setminus\{k\}}) - \sum_{k\in C_h\setminus\{j\}} f_j(N\setminus\{k\}, v, C_{N\setminus\{k\}})\\
&+ \frac{1}{|C_h|-1}\sum_{k\in C_h\setminus\{i,j\}} [v^{f,C_{N\setminus\{k\}}}(C_h\setminus\{k,j\}) - v^{f,C_{N\setminus\{k\}}}(C_h\setminus\{k,i\})]\\
=& v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\}) - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\})\\
&+ \sum_{k\in C_h\setminus\{i\}} f_i(N\setminus\{k\}, v, C_{N\setminus\{k\}}) - \sum_{k\in C_h\setminus\{j\}} f_j(N\setminus\{k\}, v, C_{N\setminus\{k\}})\\
=& v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\}) - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\})\\
&+ [v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{k\in C_h} v^{f,C}(C_h\setminus\{k\}) + \sum_{k\in C_h\setminus\{i\}} f_i(N\setminus\{k\}, v, C_{N\setminus\{k\}})]\\
&- [v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{k\in C_h} v^{f,C}(C_h\setminus\{k\}) + \sum_{k\in C_h\setminus\{j\}} f_j(N\setminus\{k\}, v, C_{N\setminus\{k\}})]\\
=& |C_h|[f_i(N, v, C) - f_j(N, v, C) + \frac{1}{|C_h|}(v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\}) - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\}))]. \quad \square
\end{aligned}
$$

Proposition 2. *A coalitional value f satisfies IQBC if and only if for all $(N, v, C) \in \mathcal{CG}^N$ and all $i \in C_h$ with $C_h \in C$, $f_i(N, v, C) = Sol_i(C_h, v^{f,C})$.*

Proof. Because the Solidarity value satisfies the quasi-balanced contributions, we have that for all i, $j \in C_h$,

$$
\begin{aligned}
& f_i(N,v,C) - f_i(N\backslash\{j\},v,C_h\backslash\{j\}) + \frac{v^{f,C}(C_h\backslash\{j\})}{|C_h|} \\
=& Sol_i(C_h,v^{f,C}) - Sol_i(C_h\backslash\{j\},v^{f,C_{N\backslash\{j\}}}) + \frac{v^{f,C}(C_h\backslash\{j\})}{|C_h|} \\
=& Sol_j(C_h,v^{f,C}) - Sol_j(C_h\backslash\{i\},v^{f,C_{N\backslash\{i\}}}) + \frac{v^{f,C}(C_h\backslash\{i\})}{|C_h|} \\
=& f_j(N,v,C) - f_j(N\backslash\{i\},v,C_h\backslash\{i\}) + \frac{v^{f,C}(C_h\backslash\{i\})}{|C_h|}.
\end{aligned}
$$

To prove the counterpart, we will use the induction on $|C_h|$.

For $C_h = \{i\}$, $Sol_i(C_h, v^{f,C}) = Sol_i(\{i\}, v^{f,C}) = v^{f,C}(\{i\}) = f_i(N,v,C)$. Suppose that the result is true for $|C_h| < m$ where $m \in \mathbb{R}$. For $|C_h| = m$, by previous proposition, we have

$$
\begin{aligned}
& |C_h| f_i(N,v,C) \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) + \sum_{j\in C_h\backslash\{i\}} f_i(N\backslash\{j\},v,C_{N\backslash\{j\}}) \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) + \sum_{j\in C_h\backslash\{i\}} Sol_i(C_h\backslash\{j\},v^{f,C_{N\backslash\{j\}}}) \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) \\
& + \sum_{j\in C_h\backslash\{i\}} \sum_{T\subseteq C_h\backslash\{i,j\}} \frac{t!(m-t-2)!}{(m-1)!} \sum_{k\in T\cup\{i\}} \frac{1}{t+1}[v^{f,C_{N\backslash\{j\}}}(T\cup\{k\}) - v^{f,C_{N\backslash\{j\}}}(T)] \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) \\
& + \sum_{j\in C_h\backslash\{i\}} \sum_{T\subseteq C_h\backslash\{i,j\}} \frac{t!(m-t-2)!}{(m-1)!} \sum_{k\in T\cup\{i\}} \frac{1}{t+1}[v^{f,C}(T\cup\{k\}) - v^{f,C}(T)] \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) \\
& + \sum_{T\subsetneq C_h\backslash\{i\}} \sum_{j\in C_h\backslash(T\cup\{i\})} \frac{t!(m-t-2)!}{(m-1)!} \sum_{k\in T\cup\{i\}} \frac{1}{t+1}[v^{f,C}(T\cup\{k\}) - v^{f,C}(T)] \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) \\
& + \sum_{T\subsetneq C_h\backslash\{i\}} \frac{t!(m-t-1)!}{(m-1)!} \sum_{k\in T\cup\{i\}} \frac{1}{t+1}[v^{f,C}(T\cup\{k\}) - v^{f,C}(T)],
\end{aligned}
$$

where t denotes the cardinality of the coalition T. Thus, we have

$$
f_i(N,v,C) = \sum_{T\subsetneq C_h\backslash\{i\}} \frac{t!(m-t-1)!}{c_h!} \sum_{k\in T\cup\{i\}} \frac{1}{t+1}[v^{f,C}(T\cup\{k\}) - v^{f,C}(T)] = Sol_i(C_h, v^{f,C}). \quad \square
$$

Calvo and Gutiérrez [24] extended the equal averaged gains property to games with coalition structure and they introduced the following property.

Intracoalitional equal averaged gains **(IEAG)**: For all $(N,v,C) \in \mathcal{CG}^N$, all $h \in M$ and all $i, j \in C_h$,

$$
\frac{1}{|C_h|}\sum_{k\in C_h}[f_i(N,v,C) - f_i(N\backslash\{k\},v,C_{N\backslash\{k\}})] = \frac{1}{|C_h|}\sum_{k\in C_h}[f_j(N,v,C) - f_j(N\backslash\{k\},v,C_{N\backslash\{k\}})].
$$

Proposition 3. *A coalitional value f satisfies IEAG if and only if for all $(N, v, C) \in \mathcal{CG}^N$ and all $i \in C_h$ with $C_h \in C$,*

$$f_i(N, v, C) = \frac{1}{|C_h|}[v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\})] + \frac{1}{|C_h|}\sum_{j\in C_h\backslash\{i\}} f_i(N\backslash\{j\}, v, C_{N\backslash\{j\}}). \tag{6}$$

Proof. If the coalitional value f satisfies IEAG then for all $(N, v, C) \in \mathcal{CG}^N$, all $h \in M$ and all $i, j \in C_h$,

$$\frac{1}{|C_h|}\sum_{k\in C_h}[f_i(N, v, C) - f_i(N\backslash\{k\}, v, C_{N\backslash\{k\}})] = \frac{1}{|C_h|}\sum_{k\in C_h}[f_j(N, v, C) - f_j(N\backslash\{k\}, v, C_{N\backslash\{k\}})].$$

Fixing i and summing over $j \in C_h$, we have

$$\begin{aligned}&|C_h|f_i(N, v, C) - \frac{1}{|C_h|}\sum_{j\in C_h}\sum_{k\in C_h} f_i(N\backslash\{k\}, v, C_{N\backslash\{k\}})\\ =&v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h}\sum_{k\in C_h} f_j(N\backslash\{k\}, v, C_{N\backslash\{k\}}).\end{aligned}$$

i.e.,

$$|C_h|f_i(N, v, C) - \sum_{k\in C_h\backslash\{i\}} f_i(N\backslash\{k\}, v, C_{N\backslash\{k\}}) = v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{k\in C_h} v^{f,C}(C_h\backslash\{k\}).$$

It remains to prove the counterpart,

$$\begin{aligned}&\frac{1}{|C_h|}\sum_{k\in C_h}[f_i(N, v, C) - f_i(N\backslash\{k\}, v, C_{N\backslash\{k\}})]\\ =&f_i(N, v, C) - \frac{1}{|C_h|}\sum_{k\in C_h\backslash\{i\}} f_i(N\backslash\{k\}, v, C_{N\backslash\{k\}})\\ =&\frac{1}{|C_h|}[v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{l\in C_h} v^{f,C}(C_h\backslash\{l\})] + \frac{1}{|C_h|}\sum_{l\in C_h\backslash\{i\}} f_i(N\backslash\{l\}, v, C_{N\backslash\{l\}})\\ &- \frac{1}{|C_h|}\sum_{k\in C_h\backslash\{i\}} f_i(N\backslash\{k\}, v, C_{N\backslash\{k\}})\\ =&\frac{1}{|C_h|}[v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{l\in C_h} v^{f,C}(C_h\backslash\{l\})]\\ =&\frac{1}{|C_h|}[v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{l\in C_h} v^{f,C}(C_h\backslash\{l\})] + \frac{1}{|C_h|}\sum_{l\in C_h\backslash\{j\}} f_j(N\backslash\{l\}, v, C_{N\backslash\{l\}})\\ &- \frac{1}{|C_h|}\sum_{k\in C_h\backslash\{j\}} f_j(N\backslash\{k\}, v, C_{N\backslash\{k\}})\\ =&f_j(N, v, C) - \frac{1}{|C_h|}\sum_{k\in C_h\backslash\{j\}} f_j(N\backslash\{k\}, v, C_{N\backslash\{k\}}). \quad \square\end{aligned}$$

Accordingly, we have following corollaries.

Corollary 1. *A coalitional value f satisfies IQBC if and only if it satisfies IEAG.*

Corollary 2. *A coalitional value f satisfies IEAG if and only if for all $(N, v, C) \in \mathcal{CG}^N$ and all $i \in C_h$ with $C_h \in C$, $f_i(N, v, C) = Sol_i(C_h, v^{f,C})$.*

Now, we introduce some properties that will be used in the characterizations of the Shapley–Solidarity value.

A coalitional value f on $\mathcal{CG}^N$ satisfies:

- *Efficiency* **(EFF)**: if for all $(N, v, C) \in \mathcal{CG}^N$, $\sum_{i \in N} f_i(N, v, C) = v(N)$.
- *Additivity* **(ADD)**: if for any (N, v_1, C), $(N, v_2, C) \in \mathcal{CG}^N$ and $i \in N$, $f_i(N, v_1 + v_2, C) = f_i(N, v_1, C) + f_i(N, v_2, C)$ where $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for any $S \subseteq N$.
- *Coalitional Symmetry* **(CSY)**: if for all $(N, v, C) \in \mathcal{CG}^N$ and all symmetry coalitions C_h, $C_r \in C$, $\sum_{i \in C_h} f_i(N, v, C) = \sum_{i \in C_r} f_i(N, v, C)$, where C_h and C_r are symmetry if $v(C_h \cup (\bigcup_{k \in T} C_k)) = v(C_r \cup (\bigcup_{k \in T} C_k))$ for all $T \subseteq M \backslash \{h, r\}$.
- *Coalitional Strong Marginality* **(CSM)**: if for all (N, v, C), $(N, v', C) \in \mathcal{CG}^N$, $v(S \cup C_h) - v(S) \geq v'(S \cup C_h) - v'(S)$ for all $S \in N \backslash C_h$, then $\sum_{i \in C_h} f_i(N, v, C) \geq \sum_{i \in C_h} f_i(N, v', C)$.
- *Null Union* **(NU)**: if for all $(N, v, C) \in \mathcal{CG}^N$ and $C_h \in C$, C_h is a null union, then $\sum_{i \in C_h} f_i(N, v, C) = 0$, where C_h is a null union if $v(C_h \cup (\bigcup_{k \in T} C_k)) = v(\bigcup_{k \in T} C_k)$ for all $T \subseteq M \backslash \{h\}$.

Lorenzo-Freire [26] introduced the following proposition.

Proposition 4. *For all $C_h \in C$, if a coalitional value f satisfies EFF, CSY and CSM, then $\sum_{i \in C_h} f_i(N, v, C) = Sh_h(M, v_C)$.*

Therefore, we have the following characterizations of the Shapley-Solidarity value.

Theorem 1.

(a) *The Shapley–Solidarity value is the only coalitional value that satisfies EFF, IQBC/IEAG, CSY, and CSM.*
(b) *The Shapley-Solidarity value is the only coalitional value that satisfies EFF, IQBC/IEAG, ADD, CSY, and NU.*

Proof. Because the property of IQBC and IEAG are equivalent, we only consider IEAG in the following proof.

(a) We first check that the Shapley–Solidarity value satisfies these properties. Calvo and Gutiérrez [24] have already proved the Shapley-Solidarity value satisfies EFF and IEAG. Because the Shapley-Solidarity value satisfies quotient game property i.e., $\sum_{i \in C_h} SS_i(N, v, C) = Sh_h(M, v_C)$, and the Shapley value satisfies symmetry and strong marginality, the Shapley–Solidarity value satisfies CSY and CSM.

Next, we will prove these properties can identify the Shapley-Solidarity value uniquely. Suppose the coalitional value f satisfies EFF, IEAG, CSY, and CSM. From former propositions, we have proved that if a coalitional value f satisfies IEAG, then for all $i \in C_h$ and $C_h \in C$ we have $f_i(N, v, C) = Sol_i(C_h, v^{f,C})$, where $v^{f,C}(T) = \sum_{i \in T} f_i((N \backslash \{C_h\}) \cup T, v, (C \backslash \{C_h\}) \cup T)$, $T \subseteq C_h$.

By Proposition 4, if f satisfies EFF, CSY, and CSM, then for all $C_h \in C$

$$\sum_{i \in C_h} f_i(N, v, C) = Sh_h(M, v_C).$$

Thus,

$$v^{f,C}(T) = \sum_{i \in T} f_i((N \backslash \{C_h\}) \cup T, v, (C \backslash \{C_h\}) \cup T) = Sh_h(M, v_{C_{(N \backslash C_h) \cup T}}).$$

From the definition of the Shapley–Solidarity value (1), we have $f(N, v, C) = SS_i(N, v, C)$.

(b) Because ADD and NU can imply CSM [26], the proof of this part is similar to the proof of **(a)**, and we omit it. □

Lemma 1. *Note that Calvo [24] characterized the Shapley–Solidarity value by EFF, IEAG and CBC, where CBC is coalitional balanced contributions property which is shown as follows.*

Coalitional balanced contributions ***(CBC)****: for all* $(N, v, C) \in \mathcal{CG}$ *and all* $C_i, C_j \in C$,

$$\begin{aligned} &\sum_{\alpha \in C_i} f_\alpha(N, v, C) - \sum_{\alpha \in C_i} f_\alpha(N \backslash \{C_j\}, v, C \backslash \{C_j\}) \\ =&\sum_{\alpha \in C_j} f_\alpha(N, v, C) - \sum_{\alpha \in C_j} f_\alpha(N \backslash \{C_i\}, v, C \backslash \{C_i\}). \end{aligned} \tag{7}$$

Obviously, the CBC property is extended from the balanced contributions in Myerson [4]. It states that, for all $C_i, C_j \in C$*, the influence of the members in* C_i *who leave the grand coalition on* C_j *is the same as the impact of the members's departure in* C_i *on* C_j*.*

4. The Potential Function for the Shapley-Solidarity Value

Hart and Mas-Colell [28] characterized the Shapley value by means of consistency property and standard for two-person games. An interesting concept, called potential function, is used for the characterization.

A function $P : \mathcal{G} \to \mathbb{R}$ with $P(\emptyset, v) = 0$ is called a potential function if it satisfies for any $(N, v) \in \mathcal{G}$,

$$\sum_{i \in N} D^i P(N, v) = v(N), \tag{8}$$

where $D^i P(N, v)$ represents the marginal contributions of a player $i \in N$ with respect to the potential function, which is defined by

$$D^i P(N, v) = P(N, v) - P(N \backslash \{i\}, v). \tag{9}$$

In a similar way, Xu et al. [31] defined $A^i P^*(N, v) = D^i P(N, v) + \frac{1}{n} v(N \backslash \{i\})$ as the adjusted marginal contribution for player i and they obtained an A-potential function. They also proved that the vector of the adjusted marginal contributions coincides with the Solidarity value.

For games with coalition structure, Winter [29] first extended the concept of potential function. Our potential function for the Shapley–Solidarity value is inspired by the potential function of the Owen value introduced in Winter [29] and use the similar way of adjustment in Xu et al. [31].

Now, we define a potential function for games with coalition structure.

Definition 1. *Let* P *be a function defined on* $\mathcal{CG}$ *s.t.* $P(N, v, C) \in \mathbb{R}^m$*, where* $C = (C_1, C_2, \cdots, C_m)$ *and* $P^j(N, v, C) = 0$ *when* $C_j \cap N = \emptyset$*. The marginal contribution of player* $i \in C_j$ *to* (N, v, C) *is*

$$D^i P(N, v, C) = P^j(N, v, C) - P^j(N \backslash \{i\}, v, C_{N \backslash \{i\}}). \tag{10}$$

The function P *is said to be an adjusted potential function for games with coalition structure if for all* $S \in C$,

$$\sum_{i \in S} [D^i P(N, v, C) + \frac{1}{|S|} D^{[S \backslash \{i\}]} P([C|_{S \backslash \{i\}}], v_C, [C|_{S \backslash \{i\}}])] = D^{[S]} P([C], v_C, [C]), \tag{11}$$

and

$$\sum_{i \in N} [D^i P(N, v, C) + \frac{1}{|S|} D^{[S \backslash \{i\}]} P([C|_{S \backslash \{i\}}], v_C, [C|_{S \backslash \{i\}}])] = v(N). \tag{12}$$

Denote, by $[S]$*, the coalition* $S \in C$ *when considered as a player, and the set of players in* v_C *by* $[C]$*.* $C|_{S \backslash \{i\}}$ *is the coalition structure that player* i *is deleted from* S*.*

For simplicity, we denote $D^iP(N,v,C) + \frac{1}{|S|}D^{[S\backslash\{i\}]}P([C|_{S\backslash\{i\}}], v_C, [C|_{S\backslash\{i\}}])$ the $A^iP(N,v,C)$ and call it the adjusted marginal contributions of player i in the rest of the paper.

Expression (11) says that the sum of the adjusted marginal contributions of players $i \in S \in C$ is the marginal contribution of the player $[S]$ to the $([C], v_C, [C])$, where the player set is $C = (C_1, C_2, \cdots, C_m)$, the game is v_C. So, combining with (12), $P(\emptyset, v, C) = 0$ and the definition of potential function for the Shapley value in Hart and Mas-Colell [28], we have

$$D^{[S]}P([C], v_C, [C]) = Sh^{[S]}([C], v_C), \text{ for all } S \in C. \tag{13}$$

Theorem 2. *There exists a unique adjusted potential function P for games with coalition structure. Moreover, for all $(N,v,C) \in \mathcal{CG}$ and $i \in N$, $A^iP(N,v,C) = SS_i(N,v,C)$.*

Proof. First, we show uniqueness. for any $(N,v,C) \in \mathcal{CG}$ and $S \in C$, from (10) and (11), we have following recursive form of the potential function P.

$$\begin{aligned} P(N,v,C) =& \frac{1}{|S|}[D^{[S]}P([C], v_C, [C]) - \frac{1}{|S|}\sum_{i\in S} D^{[S\backslash\{i\}]}P([C|_{S\backslash\{i\}}], v_C, [C|_{S\backslash\{i\}}]) \\ &+ \sum_{i \in S} P(N\backslash\{i\}, v, C_{N\backslash\{i\}})]. \end{aligned}$$

Together with $P(\emptyset, v, \emptyset) = 0$, $P(N,v,C)$ can be obtained by recursion and can be uniquely determined.

Now, we show that for all $(N,v,C) \in \mathcal{CG}$ and $i \in N$, $A^iP(N,v,C) = SS_i(N,v,C)$. We know that the Shapley–Solidarity value is characterized by EFF, IQBC, and CBC from the Corollary 1 and Calvo [24]. If we can prove that $A^iP(N,v,C)$ satisfies these three properties, we can claim that $A^iP(N,v,C)$ coincides with the Shapley–Solidarity value.

The EFF is obvious from the definition of the potential function. For any game with coalition structure $(N,v,C) \in \mathcal{CG}$, $h \in M$, and $i, j \in C_h$, if we want to prove that $A^iP(N,v,C)$ satisfies IQBC, we should prove the following equation.

$$\begin{aligned} &A^iP(N,v,C) - A^iP(N\backslash\{j\}, v, C_{N\backslash\{j\}}) + \frac{v^{AP,C}(C_h\backslash\{j\})}{|C_h|} \\ =&A^jP(N,v,C) - A^jP(N\backslash\{i\}, v, C_{N\backslash\{i\}}) + \frac{v^{AP,C}(C_h\backslash\{i\})}{|C_h|}, \end{aligned}$$

where AP denotes $AP(N,v,C)$. From the definition of the game $(C_h, v^{f,C})$ (4) and (11), we have $v^{AP,C}(C_h\backslash\{j\}) = D^{[C_h\backslash\{j\}]}P([C|_{C_h\backslash\{j\}}], v_C, [C|_{C_h\backslash\{j\}}])$. Thus,

$$\begin{aligned} &A^iP(N,v,C) - A^iP(N\backslash\{j\}, v, C_{N\backslash\{j\}}) + \frac{v^{AP,C}(C_h\backslash\{j\})}{|C_h|} \\ =&D^iP(N,v,C) + \frac{1}{|C_h|}D^{[C_h\backslash\{i\}]}P([C|_{C_h\backslash\{i\}}], v_C, [C|_{C_h\backslash\{i\}}]) \\ &- D^iP(N\backslash\{j\}, v, C_{N\backslash\{j\}}) - \frac{1}{|C_h|-1}D^{[C_h\backslash\{i,j\}]}P([C|_{C_h\backslash\{i,j\}}], v_C, [C|_{C_h\backslash\{i,j\}}]) \\ &+ \frac{1}{|C_h|}D^{[C_h\backslash\{j\}]}P([C|_{C_h\backslash\{j\}}], v_C, [C|_{C_h\backslash\{j\}}]) \\ =&P(N,v,C) - P(N\backslash\{i\}, v, C_{N\backslash\{i\}}) + \frac{1}{|C_h|}D^{[C_h\backslash\{i\}]}P([C|_{C_h\backslash\{i\}}], v_C, [C|_{C_h\backslash\{i\}}]) \\ &- P(N\backslash\{j\}, v, C_{N\backslash\{j\}}) + P(N\backslash\{i,j\}, v, C_{N\backslash\{i,j\}}) \\ &- \frac{1}{|C_h|-1}D^{[C_h\backslash\{i,j\}]}P([C|_{C_h\backslash\{i,j\}}], v_C, [C|_{C_h\backslash\{i,j\}}]) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|C_h|}D^{[C_h\backslash\{j\}]}P([C|_{C_h\backslash\{j\}}],v_C,[C|_{C_h\backslash\{j\}}])\\
=&D^jP(N,v,C)+\frac{1}{|C_h|}D^{[C_h\backslash\{i\}]}P([C|_{C_h\backslash\{i\}}],v_C,[C|_{C_h\backslash\{i\}}])\\
&-D^jP(N\backslash\{i\},v,C_{N\backslash\{i\}})-\frac{1}{|C_h|-1}D^{[C_h\backslash\{i,j\}]}P([C|_{C_h\backslash\{i,j\}}],v_C,[C|_{C_h\backslash\{i,j\}}])\\
&+\frac{1}{|C_h|}D^{[C_h\backslash\{j\}]}P([C|_{C_h\backslash\{j\}}],v_C,[C|_{C_h\backslash\{j\}}])\\
=&A^jP(N,v,C)-A^jP(N\backslash\{i\},v,C_{N\backslash\{i\}})+\frac{v^{AP,C}(C_h\backslash\{i\})}{|C_h|}.
\end{aligned}
$$

It verifies that $A^iP(N,v,C)$ satisfies IQBC.

It remains to prove that $A^iP(N,v,C)$ satisfies CBC. From (13), we know that $\sum_{i\in S}A^iP(N,v,C)=Sh^S([C],v_C)$. Because the Shapley value satisfies balanced contributions [4]. For any C_i, $C_j\in C$, we have

$$
Sh^{C_{k_1}}([C],v_C)-Sh^{C_{k_1}}([C]\backslash\{C_{k_2}\},v_C)=Sh^{C_{k_2}}([C],v_C)-Sh^{C_{k_2}}([C]\backslash\{C_{k_1}\},v_C).
$$

Subsequently, we have

$$
\begin{aligned}
&D^{C_{k_1}}P([C],v_C,[C])-D^{C_{k_1}}P([C]\backslash\{C_{k_2}\},v_C,[C])\\
=&D^{C_{k_2}}P([C],v_C,[C])-D^{C_{k_2}}P([C]\backslash\{C_{k_1}\},v_C,[C]).
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\sum_{\alpha\in C_{k_1}}A^{\alpha}P(N,v,C)-\sum_{\alpha\in C_{k_1}}A^{\alpha}P(N\backslash\{C_{k_2}\},v,C\backslash\{C_{k_2}\})\\
=&\sum_{\beta\in C_{k_2}}A^{\beta}P(N,v,C)-\sum_{\beta\in C_{k_2}}A^{\beta}P(N\backslash\{C_{k_1}\},v,C\backslash\{C_{k_1}\}).
\end{aligned}
$$

Thus, the Shapley-Solidarity value satisfies CBC. And we have that $A^iP(N,v,C)=SS_i(N,v,C)$. □

5. The Coalitional Bidding Mechanism for the Shapley-Solidarity Value

Mechanism design can be seen as the part of Nash program to bridge the gap between cooperative and non-cooperative game theory. It is a significant approach to characterize the rationality of solutions for cooperative games, and it has been widely studied in the field of cooperative games. Pérez-Castrillo and Wettstein [33] firstly proposed a bidding mechanism to give rise to the Shapley value as the result of equilibrium behavior. Our coalitional bidding mechanism is similar to the coalition bidding mechanism for the Owen value [34], which is extended from the bidding mechanism for the Shapley value [33]. Both of the smechanisms have two round and the second round is the same while the difference between our mechanism for the Shapley–Solidarity value and the one for the Owen value lies in the first round. We first give a new bidding mechanism for the Solidarity value, and we will restrict the underlying game to monotonic games.

5.1. A New Bidding Mechanism for the Solidarity Value

Our bidding mechanism is inspired from Pérez-Castrillo and Wettestein's for the Shapley value. The Solidarity value and Shapley value are the excepted values with respect to different considerations of the marginal contributions. The Shapley value only takes care of every individual himself, and it uses the pure marginal contributions, while the Solidarity value uses the average marginal contributions which behaves as the property of solidarity. When a player joins or leaves a coalition, he will share their gains or losses with other players in the coalition rather than himself only. In our bidding mechanism, we will consider the property of solidarity in the Solidarity value.

Now, we describe the bidding mechanism for the Solidarity value. When there is only one player i in the game, the player i receives $v(\{i\})$. Suppose that the rules of the bidding mechanism is known when there are at most $n-1$ players. The procedure of the bidding mechanism for a set of players $N=\{1,2,...,n\}$ is as follows.

- **Stage 1:** the players bid for each other for electing a proposer, i.e., each player $i \in N$ makes bid $b^i_j \in \mathbb{R}$ for every $j \in N\backslash\{i\}$. Let $B^i = \sum_{j\in N\backslash\{i\}} b^i_j - \sum_{j\in N\backslash\{i\}} b^j_i$ denote the net bid of player $i \in N$. Find a player α be the proposer whose net bid is max among all the players. If there is more than one maximizer, then randomly choose a player from the maximizers.
- **Stage 2:** the proposer α makes an offer $x^\alpha_j \in \mathbb{R}$ to every player $j \in N\backslash\{\alpha\}$.
- **Stage 3:** all other players, except proposer sequentially accept or reject the offer. If all other players agree the proposer, we say the offer is accepted otherwise the offer is rejected.
  If the offer is accepted, player $j \in N\backslash\{\alpha\}$ receives x^α_j, and the proposer receives $v(N) - \sum_{j\in N\backslash\{\alpha\}} x^\alpha_j$. Consider the bids in stage 1, the player $j \in N\backslash\{\alpha\}$ totally receives $x^\alpha_j + b^\alpha_j$ and the proposer eventually receives $v(N) - \sum_{j\in N\backslash\{\alpha\}}(b^\alpha_j + x^\alpha_j)$.
  If the offer is rejected, a player β randomly chosen from N with probability $\frac{1}{n}$ leaves the game and gets nothing. Note that the proposer and other players have same probability to be player β. Other players proceed with the game in same rules with player set $N\backslash\{\beta\}$. In this case, when we consider the bids in stage 1, every player j other than proposer α gets b^α_j and proposer α loses $\sum_{j\in N\backslash\{\alpha\}} b^\alpha_j$. The randomly chosen player β will be excluded in the rest of the games and his bid in stage 1 will not be changed.

Theorem 3. *For any monotonic game $(N,v) \in \mathcal{G}$, the above mechanism implements the Solidarity value in any subgame perfect equilibrium (SPE).*

Proof. We prove this result by induction on the number of players. Obviously, the theorem is true if there is only one player in the game. We assume that it holds for all player set number $m \le n-1$. Now, we show that it is true for $m = n$.

Firstly, we construct a subgame perfect equilibrium whose outcome is the Solidarity value of game (N,v). Consider the following strategy profiles.

At stage 1, each player $i \in N$ makes bids $b^i_j = Sol_j(N,v) - \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$ to every player $j \in N\backslash\{i\}$.

At stage 2, the proposer α offers $x^\alpha_j = \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$.

At stage 3, each player $j \in N\backslash\{\alpha\}$ will accept the offer if $x^\alpha_j \ge \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$, otherwise the offer is rejected.

It is easy to check that every player other than the proposer obtains their Solidarity value. Because the Solidarity value is an efficient solution, the proposer also gets his Solidarity value.

In these strategy profiles, the net bids B^i of every player $i \in N$ is equal to zero by the equal averaged gains property of the Solidarity value.

$$
\begin{aligned}
B^i &= \sum_{j\in N\backslash\{i\}} (b^i_j - b^j_i) \\
&= \sum_{j\in N\backslash\{i\}} [Sol_j(N,v) - \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v) - (Sol_i(N,v) - \frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\},v))] \\
&=0.
\end{aligned}
$$

Now, we prove that the above strategies constitute a SPE. At stage 3, in the case of rejection, one of the players, including the proposer, is randomly chosen and he/she will be excluded in the next round. The players who are left in the game will play the subgame according to the same rules. Subsequently, we can obtain the Solidarity value as the outcome of those games by the induction hypothesis. If any player rejects the proposal, then the expected payoff for any player $i \in N$ in the subgame is $\frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v)$. Thus, each player $i \in N\backslash\{\alpha\}$ accepts any offer $x_i^\alpha \geq \frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v)$, and rejects any offer $x_i^\alpha < \frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v)$. Therefore, the strategies are best responses at stage 3.

At stage 2, the strategies are also best responses. According to these strategies, the payoff of α starting from stage 2 is

$$\begin{aligned} v(N) - \sum_{j\in N\backslash\{\alpha\}} x_j^\alpha =& v(N) - \sum_{j\in N\backslash\{\alpha\}} \frac{1}{n} \sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v) \\ =& v(N) - \frac{1}{n}\sum_{k\in N} v(N\backslash\{k\}) + \frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\}, v) \\ \geq& \frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\}, v), \end{aligned} \tag{14}$$

where the last inequation holds from the fact that the game is a monotonic game. If he offers some player j the payoff x_j^α less than $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$, the proposal is rejected and the expected payoff he will obtain is $\frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\}, v)$. If he offers some player j the value x_j^α higher than $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$, the proposal is accepted, but his payoff is strictly worse off.

Now, consider the strategies at stage 1. Obviously, if a player increases his total bid, then he will be chosen as the proposer, but his payoff will decrease. If a player decreases his total bid, then his payoff is invariable, since other players will be chosen as the proposer. Thus, the strategy is the best response at stage 1. Hence, the above strategies constitute a SPE.

The proof of any subgame perfect equilibrium yields the Solidarity value needs five claims.

Claim a. In any SPE, at stage 3, each player $j \in N\backslash\{\alpha\}$ accepts any offer if $x_j^\alpha > \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$. The offer is rejected if $x_j^\alpha < \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$ for at least some $j \in N\backslash\{\alpha\}$.

If the proposal of player α is rejected, the excepted payoff of a player $j \neq \alpha$ is $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$ by the induction. Subsequently, the optimal strategy of the player j is that he accepts any offer higher than $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$ and rejects any offer lower than $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$.

Claim b. If there is a player $i \in N$, such that $v(N) > v(N\backslash\{i\})$, then the SPE strategies starting from stage 2 are as follows. At stage 2, the proposer α offers $x_j^\alpha = \frac{1}{n}\sum_{k\in N} Sol_j(N\backslash\{k\}, v)$ to all $j \in N\backslash\{\alpha\}$, and at stage 3, $j \in N\backslash\{\alpha\}$ accepts any offer $x_j^\alpha \geq \frac{1}{n}\sum_{k\in N} Sol_j(N\backslash\{k\}, v)$ and rejects the offer otherwise. If $v(N) = v(N\backslash\{k\})$ for all $k \in N$, there exist the SPE strategies besides the previous SPE strategies. At stage 2, the proposer α offers $x_j^\alpha \leq \frac{1}{n}\sum_{k\in N} Sol_j(N\backslash\{k\}, v)$ to all $j \in N\backslash\{\alpha\}$, and at stage 3, $j \in N\backslash\{\alpha\}$ rejects any offer $x_j^\alpha \leq \frac{1}{n}\sum_{k\in N} Sol_j(N\backslash\{k\}, v)$ and accepts the offer otherwise.

We prove that the previous strategies constitute a SPE. Suppose that there is a player $i \in N$, such that $v(N) > v(N\backslash\{i\})$. In the case of rejection, player α will expectantly receive $\frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\}, v)$. Let $0 < \varepsilon < v(N) - \sum_{j\in N\backslash\{\alpha\}} \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v) - \frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\}, v)$. The ε is definitely exists from Equation (14) and $v(N) > v(N\backslash\{i\})$. The player α can improve his payoff by offering any player $j \in N\backslash\{\alpha\}$ with $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v) + \frac{\varepsilon}{n-1}$. The offer will be accepted by claim a. Thus, in any SPE the offer of player α must be accepted. This implies that $x_j^\alpha \geq \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$ for all $j \in N\backslash\{\alpha\}$. Player α still can improve his payoff by offering $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v) + \frac{\varepsilon}{n-1}$ to player $j \in N\backslash\{\alpha\}$ with a smaller ε. These offers are accepted and α's payoff increases. Hence, $x_j^\alpha = \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$ for all $i \neq \alpha$ at any SPE and, at stage 3, every agent $i \neq \alpha$ accepts any offer if $x_j^\alpha \geq \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$.

If $v(N) = v(N\backslash\{k\})$ for all $k \in N$, the proposer α offers at least $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$ to every player $j \neq \alpha$, so that the offer can be accepted by the same argument in the previous case. The proposer expectantly receive $\frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\}, v)$ in the case of rejection, which is identical to the payoff in the case of acceptation. Therefore, any offer that leads to a rejection is also a SPE.

Claim c. In any SPE, $B^i = B^j = 0$ for all $i, j \in N$.

Claim d. In any SPE, the payoff of every player is invariable whoever is chosen as the proposer.

The proofs of the above two claims are similar to the proof of Theorem 1 in Pérez-Castrillo and Wettstein [33], and we omit them.

Claim e. In any SPE, the final payoff of every player coincides with the Solidarity value.

If a player i is the proposer, then the his payoff is $x_i^i = v(N) - \sum_{j\neq i}\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v) - \sum_{j\neq i} b_j^i$. If a player $j \neq i$ is the proposer, the payoff of player i is $x_i^j = \frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v) + b_i^j$. Thus, we have

$$\begin{aligned}
\sum_{j\in N} x_i^j =& [v(N) - \sum_{j\neq i}\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v) - \sum_{j\neq i} b_j^i] + \sum_{j\neq i}[\frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v) + b_i^j] \\
=& [v(N) - \sum_{j\neq i}\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)] + \sum_{j\neq i}[\frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v)] - B^i \\
=& v(N) - \sum_{j\in N}\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v) + \sum_{j\in N}[\frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v)] \\
=& v(N) - \frac{1}{n}\sum_{k\in N} v(N\backslash\{k\}) + \sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v) \\
=& nSol_i(N, v)
\end{aligned} \tag{15}$$

where the last equality holds by the recursive formula of the Solidarity value [42]. By claim (d), we have $x_i^j = x_i^k$ for all j, k. Thus $x_i^j = Sol_i(N, v)$ for all $i \in N$. $\square$

Remark 1. *Note that Xu et al. [31] also gave a bidding mechanism for the Solidarity value. When comparing our bidding mechanism with the work of Xu et al. [31], there is no difference between the two bidding mechanisms at Stage 1 and Stage 2, but an exceptional stage (Stage 0) is considered at the beginning of the mechanism in [31], where $a_i = \frac{1}{s}v(S\backslash\{i\})$ is distributed for player i that is in the active player set S. They call a_i as the compensation for player i from an ethical point of view. The main difference is at Stage 3. When the offer by proposer is rejected, a player β is randomly chosen from N with probability $\frac{1}{n}$ leaving the grand coalition in our bidding mechanism, while the rejected proposer leaves the grand coalition in the mechanism of Xu et al. [31].*

Remark 2. *When comparing our bidding mechanism with Pérez-Castrillo and Wettstein's mechanism [33] for the Shapley value, there is only difference at Stage 3. In Pérez-Castrillo and Wettstein's mechanism, if the offer is rejected, then the proposer α will leave from the grand coalition N and take away his individual worth $v(\{\alpha\})$. It means that the proposer himself takes the full responsibility when his proposal is rejected by others. In our mechanism for the Solidarity value, if the offer is rejected, a player β is randomly chosen from all players with probability $\frac{1}{n}$ leaving the grand coalition N and gets nothing. It indicates the preference of egalitarianism from the property of solidarity, which is, all players (not only the proposer or any rejector) need to take the responsibility for the breakdown of the coalition when the proposal is rejected in our mechanism.*

5.2. The Coalitional Bidding Mechanism for the Shapley-Solidarity Value

Vidal-Puga and Begantiños [34] extended the bidding mechanism for the Shapley value introduced in Pérez-Castrillo and Wettestein [33] to games with coalition structure. They used a two round coalitional bidding mechanism, where, at round 1, players in the same union play the bidding mechanism for obtaining the resources of the union and, at round 2, the players who have obtained the resources in round 1 play the bidding mechanism with the obtained resources.

In the former section, we have already proposed a new non-cooperative implementation of the Solidarity value. Next, we will use the coalitional bidding mechanism that was introduced in Vidal-Puga and Begantiños to implement the Shapley-Solidarity value. Considering the relationship between the Owen value and the Shapley-Solidarity value, we just need to change the first round of the coalitional bidding mechanism in the original implementation for the Owen value.

We will describe our bidding mechanism recursively. If there is only one player i, the player obtains $v(i)$. Suppose that the mechanism is played at most $n-1$ players. Then, for $N = \{1, \cdots, n\}$ and $C = \{C_1, \cdots, C_m\}$, the procedure of the bidding mechanism is as follows.

Round 1. At this round, the players in any union $C_h \in C$ play the bidding mechanism for obtaining the resources of C_h. If there is only one player i in C_h, this player has his resources. Assume the mechanism played by at most $|C_h| - 1$ players. For $|C_h|$ the process is as follows.

- **Stage 1.** Every player $i \in C_h$ makes bids $b_j^i \in \mathbb{R}$ for every $j \in C_h \backslash \{i\}$. The net bid $B^i = \sum_{j \in C_h \backslash \{i\}} b_j^i - \sum_{j \in C_h \backslash \{i\}} b_i^j$. Let $\alpha_h = argmax_i\{B^i\}$. If the maximizer is not unique, randomly choosing any player from them.
- **Stage 2.** The proposer α_h makes an offer $x_j^{\alpha_h}$ to every player $i \in C_h \backslash \{\alpha_h\}$.
- **Stage 3.** The players in $C_h \backslash \{\alpha_h\}$ sequentially decides whether or not to accept the offer. If all players accept the offer, then the offer is accepted, otherwise the offer is rejected.

Every union in C plays the bidding mechanism sequentially in the order $C_1, \cdots, C_m$ until we find C_{l_0} and α_{l_0} such that the offer of α_{l_0} is rejected or for any $C_l \in C$ the offer of α_l is accepted.

If the offer of α_{l_0} is rejected by some player in C_{l_0}, then randomly choose one player from coalition C_{l_0}. Suppose that the player β_{l_0} is chosen then he leaves the game with nothing. Consider the bid in the first stage, all he acquires from the game is $b_{\beta_{l_0}}^{\alpha_{l_0}}$ if he is not the proposer otherwise he will lose $\sum_{j \in C_{l_0} \backslash \{\alpha_{l_0}\}} b_j^{\alpha_{l_0}}$. The left players play the subgame (N', v', C') with same rules, where $N' = N \backslash \{\beta_{l_0}\}$, $v' = v_{-\beta_{l_0}}$ and $C' = C_{-\beta_{l_0}}$.

If for any $C_l \in C$ the offer of α_l is accepted, then the player α_l becomes the "representative" of the coalition C_l. The final payoff of the players other than the proposer in the coalition C_l will get the bid in stage 1 and the offer in stage 3 that is given from the proposer. Accordingly, the payoff of the proposer in this round is $p_{\alpha_l}^1 = -\sum_{i \in C_l \backslash \{\alpha_l\}} (b_i^{\alpha_l} + x_i^{\alpha_l})$. He will go to round 2 with the resources of C_l and the other players leave the game.

When the first round is finished, any coalition $C_l \in C$ have one representative r_l. We denote C_l^r the set of players whose resources are obtained by representative r_l. The set of the players $C \backslash C_l^r$ is the players that have been removed in C_l.

Round 2. All of representatives $N^r = \{r_1, \cdots, r_m\}$ play the bidding mechanism introduced by Pérez-Castrillo and Wettstein [33] according to the game (N^r, v^r), which is defined by for any $S \subset N^r$, $v^r(S) = v(\cup_{r_l \in S} C_l^r)$. For any $r_l \in N^r$, the payoff obtained by r_l at round 2 is denoted by $p_{r_l}^2$.

The bidding mechanism for the Shapley value in Pérez-Castrillo and Wettstein needs that the underlying game is zero monotonic game i.e., the game (N^r, v^r) should be a zero monotonic game. Thus, the game v should be a superadditive game. In our bidding mechanism for the Solidarity value, we restrict the underlying game to monotonic games. It is easy to check that, if a superadditive game is non-negative game, then the game is a monotonic game. Therefore, we will restrict the underlying game for implementing the Shapley–Solidarity value to the superadditive game where every coalition has non-negative worth.

Theorem 4. *Given any game (N, v, C), if (N, v, C) is superadditive and non-negative, then above coalitional bidding mechanism implements the Shapley–Solidarity value.*

Proof. The result holds when there is only one player. Suppose the result holds for at most $n-1$ players. Let $N = \{1, 2, ..., n\}$, and we consider the following strategies.

Round 1. For any coalition $C_h \in C$, we describe their strategies as follows.

- **Stage 1.** Every player $i \in C_h$ makes bids $b_j^i = SS_j(N, v, C) - \frac{1}{|C_h|}\sum_{k \in C_h} SS_j(N\backslash\{k\}, v, C_{N\backslash\{k\}})$ for every $j \in C_h\backslash\{i\}$.
- **Stage 2.** The proposer α_h offers $x_j^{\alpha_h} = \frac{1}{|C_h|}\sum_{k \in C_h} SS_j(N\backslash\{k\}, v, C_{N\backslash\{k\}})$ to every player $j \in C_h\backslash\{\alpha_h\}$.
- **Stage 3.** Any player $j \in C_h\backslash\{\alpha_h\}$ accepts the offer of α_h if and only if $x_j^{\alpha_h} \geq \frac{1}{|C_h|}\sum_{k \in C_h} SS_j(N\backslash\{k\}, v, C_{N\backslash\{k\}})$.

Round 2. The representatives play the bidding mechanism in terms of the strategies described in Pérez-Castrillo and Wettstein [33].

Following the similar proof procedure in Theorem 2 of Vidual-Puga and Bergantiños [34], we can prove the theorem and we will omit it. □

Remark 3. *Our non-cooperative implementation for the Shapley–Solidarity value is quit similar to the implementation for the Owen value. We just change the way to punish the players when the bargaining is broken in the first round.*

6. Conclusions

Coalition structure is adopted in order to describe the situation where the players form groups for bargaining payoffs in cooperative games. The Shapley-Solidarity value provides a way of dividing the total payoffs among the players in a game with coalition structure. It employs the Shapley value among the unions and the Solidarity value among the members inside each union, by considering that the players within a union were more willing to show their solidarity and each union was more inclined to protect its revenue. In this paper, we study the characterizations of the Shapley–Solidarity value from both cooperative and non-cooperative aspects.

Firstly, we present two axiomatic characterizations of the Shapley-Solidarity value. We characterized the Shapley–Solidarity value is the only coalitional value that satisfies efficiency, coalitional symmetry, coalitional strong marginality, intracoalitional equal averaged gains (intracoalitional quasi-balanced contributions). Additioally, the Shapley–Solidarity value is the only coalitional value that satisfies efficiency, additivity, coalitional symmetry, null union, intracoalitional equal averaged gains (intracoalitional quasi-balanced contributions). In the axiom system, the axiom of intracoalitional equal averaged gains can be replaced by the axiom of intracoalitional quasi-balanced contributions for these two properties are equivalent for a coalitional value.

Secondly, we combine the A-potential function for the Solidarity value and the potential function for the Owen value, in order to obtain the potential function for the Shapley–Solidarity value. We prove that there exists a unique adjusted potential function for games with coalition structure. Moreover, for each player of a game with with coalition structure, the player's marginal contribution of the unique adjusted potential function coincides with its Shapley–Solidarity value.

Finally, we propose a bidding mechanism in order to implement the Solidarity value. At beginning, we introduce a new bidding mechanism for the Solidarity value, inspiring from Pérez-Castrillo and Wettestein's bidding mechanism for the Shapley value. In our bidding mechanism for the Solidarity value, if the offer is rejected, then a player is randomly chosen from all players with probability $\frac{1}{n}$ leaving the grand coalition and gets nothing. It means that all players (not only the proposer or any rejector) need to take the responsibility for the breakdown of the coalition when the proposal is rejected, from the property of solidarity. This is the main difference with the Pérez-Castrillo and Wettstein's implementation of the Shapley value. Subsequengly, by changing the first round of the two round mechanism in the original implementation for the Owen value, we extend the coalitional bidding mechanism that was introduced by Vidal-Puga and Begantiños to implement the Shapley–Solidarity value.

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Rational Behavior in Dynamic Multicriteria Games

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Abstract: We consider a dynamic, discrete-time, game model where n players use a common resource and have different criteria to optimize. To construct a multicriteria Nash equilibrium the bargaining solution is adopted. To design a multicriteria cooperative equilibrium, a modified bargaining scheme that guarantees the fulfillment of rationality conditions is applied. The concept of dynamic stability is adopted for dynamic multicriteria games. To stabilize the multicriteria cooperative solution a time-consistent payoff distribution procedure is constructed. The conditions for rational behavior, namely irrational-behavior-proofness condition and each step rational behavior condition are defined for dynamic multicriteria games. To illustrate the presented approaches, a dynamic bi-criteria bioresource management problem with many players is investigated.

Keywords: dynamic games; multicriteria games; Nash bargaining solution; dynamic stability; rational behavior conditions

MSC: 22E46

1. Introduction

Mathematical models involving more than one objective [1] seem more adherent to real problems. Players often seek to achieve several goals simultaneously, which can be incomparable. These situations are typical for game-theoretic models in economics and ecology. For example, in bioresource management problems the players wish to maximize their exploitation rates and to minimize the harm to the environment. The multicriteria approach allows determining an optimal behavior in such situations.

In this paper, we consider a dynamic, discrete-time, game model where the players use a common resource and have different criteria to optimize. First, we construct a multicriteria Nash equilibrium applying the bargaining concept (via Nash products [2,3]). Then, we find a multicriteria cooperative equilibrium as a solution of modified bargaining scheme with the multicriteria Nash equilibrium payoffs playing the role of status quo points [4,5]. The presented approach guarantees that the cooperative payoffs of the players are greater than or equal to the multicriteria Nash payoffs.

As is well known, in ecological problems, cooperative behavior leads to a more sparing harvesting rate. The special importance of cooperative behavior for "common resource" exploitation was stressed by Nobel prize winer Ostrom E. [6]. The contract that satisfies the dynamic stability (time-consistency) condition [7,8] is concluded to maintain cooperative behavior. Haurie A. [9] raised the problem of instability of the Nash bargaining solution. The concept of time-consistency (dynamic stability) was introduced by Petrosyan L.A. [7]. Time consistency involves the property that, as the cooperation develops, participants are guided by the same optimality principle at each time moment and hence do not have incentives to deviate from cooperation. Petrosyan L.A. and Danilov N.N. [10] have developed the notion of time-consistent imputation distribution procedure.

An important problem arising in applications is to maintain cooperative behavior. Cooperative behavior and the dynamic stability of cooperative solutions were investigated in a number of papers; see [8,11–15]. The related works that study the cooperation processes in biological, economical and social sciences are [16,17]. An extensive form of multicriteria multistage game and the realization of the IDP-related concepts had been studied in [18,19]. Here, we adopt the concept of dynamic stability for dynamic multicriteria games and construct the payoff distribution procedure.

Still following the cooperative agreement there can be some irrational players who can break out the cooperation. To indemnify players against the loss of profits in this case Yeung D.W.K. [20] introduced the irrational-behavior-proofness condition. In the papers [21,22] the each step rational behavior condition, which is stronger than the Yeung's condition and is easier to verify, was presented. We adopt both conditions for rational behavior for dynamic multicriteria games.

To illustrate the presented approaches, a dynamic bi-criteria bioresource management problem with many players is investigated.

The remainder of the paper has the following structure. Section 2 describes the noncooperative and cooperative solution concepts for a finite horizon multicriteria dynamic game with many players in discrete time. The time-consistent imputation distribution procedure for dynamic multicriteria game is presented in Section 2.3. The conditions for rational behavior are constructed in Section 2.4. A bi-criteria discrete-time dynamic bioresource management model (harvesting problem) with a finite planning horizon is treated in Section 3. Finally, Section 4 provides the basic results and their discussion.

2. Dynamic Multicriteria Game with Finite Horizon

Consider a multicriteria dynamic game with finite horizon in discrete time. Let $N = \{1, \dots, n\}$ players exploit a common resource for k different goals. The state dynamics is in the form

$$x_{t+1} = f(x_t, u_{1t}, \dots, u_{nt}), \;\; x_0 = x, \tag{1}$$

where $x_t \geq 0$ denotes the quantity of resource at a time $t \geq 0$, $f(x_t, u_{1t}, \dots, u_{nt})$ is the natural growth function, and $u_{it} \in U_i = [0, \infty)$ specifies the strategy (resource exploitation rate) of player i at a time $t \geq 0$, $i \in N$.

Denote $u_t = (u_{1t}, \dots, u_{nt})$. Each player has k goals to optimize. The vector payoff functions of players on a finite planning horizon $[0, m]$ have the form

$$J_i = \begin{pmatrix} J_i^1 = \sum\limits_{t=0}^{m} \delta^t g_i^1(u_t) \\ \dots \\ J_i^k = \sum\limits_{t=0}^{m} \delta^t g_i^k(u_t) \end{pmatrix}, \; i \in N, \tag{2}$$

where $g_i^j(u_t) \geq 0$ are the instantaneous payoff functions, $j = 1, \dots, k$, $i \in N$, $\delta \in (0,1)$ denotes the discount factor.

2.1. Multicriteria Nash Equilibrium

We design the noncooperative behavior in dynamic multicriteria game applying the Nash bargaining products [2,3]. Therefore, we begin with the construction of guaranteed payoffs which play the role of status quo points.

The possible concepts to determine the guaranteed payoffs for the game with two players were presented in [2]. As it was demonstrated, the case in which the guaranteed payoffs are determined as the Nash equilibrium solutions is the best for the ecological system and also profitable for the players. Therefore, for the multicriteria game with n players we adopt this concept of guaranteed payoff points construction. Namely

$G_1^1, \dots, G_n^1$ — are the Nash equilibrium payoffs in the dynamic game $\langle x, N, \{U_i\}_{i=1}^n, \{J_i^1\}_{i=1}^n \rangle$,

...
$G_1^k, \ldots, G_n^k$ — are the Nash equilibrium payoffs in the dynamic game $\langle x, N, \{U_i\}_{i=1}^n, \{J_i^k\}_{i=1}^n \rangle$, where the state dynamics is described by (1). Please note that if the Nash equilibrium is not unique, one of the solutions is taken as guaranteed payoff points.

To construct multicriteria payoff functions we adopt the Nash products. The role of the status quo points belongs to the guaranteed payoffs of the players:

$$H_1(u_{1t}, \ldots, u_{nt}) = (J_1^1(u_{1t}, \ldots, u_{nt}) - G_1^1) \cdot \ldots \cdot (J_1^k(u_{1t}, \ldots, u_{nt}) - G_1^k),$$
$$\ldots$$
$$H_n(u_{1t}, \ldots, u_{nt}) = (J_n^1(u_{1t}, \ldots, u_{nt}) - G_n^1) \cdot \ldots \cdot (J_n^k(u_{1t}, \ldots, u_{nt}) - G_n^k).$$

Definition 1. *A strategy profile $u_t^N = (u_{1t}^N, \ldots, u_{nt}^N)$ is a multicriteria Nash equilibrium [2] of problem (1), (2) if*

$$H_i(u_t^N) \geq H_i(u_{1t}^N, \ldots, u_{i-1\,t}^N, u_{it}, u_{i+1\,t}^N, \ldots, u_{nt}^N) \ \ \forall u_{it} \in U_i\,, \ i \in N. \tag{3}$$

As it is demonstrated in Appendix A, the presented approach guarantees that the noncooperative payoffs of the players are greater than or equal to the guaranteed ones (for bi-criteria game for simplicity). Hence, the scheme for noncooperative behavior construction is meaningful since multicriteria payoff functions are nonnegative.

2.2. Multicriteria Cooperative Equilibrium

The cooperative equilibrium was obtained as a solution of the Nash bargaining scheme in [23,24]. For the multicriteria dynamic games, the Nash product with the sums of players' payoffs for the criteria in which the sums of their noncooperative payoffs act as the status quo points was applied in [3,4]. In [5], a new approach to determine cooperative behavior in dynamic multicriteria game with asymmetric players was presented. More specifically, the cooperative strategies and payoffs of players are determined from the modified bargaining solution for the entire game horizon. The status quo points are the noncooperative payoffs obtained by the players using the multicriteria Nash equilibrium strategies u_t^N:

$$J_1^N = \begin{pmatrix} J_1^{1N} = \sum\limits_{t=0}^{m} \delta^t g_1^1(u_t^N) \\ \ldots \\ J_1^{kN} = \sum\limits_{t=0}^{m} \delta^t g_1^k(u_t^N) \end{pmatrix}, \ldots, J_n^N = \begin{pmatrix} J_n^{1N} = \sum\limits_{t=0}^{m} \delta^t g_n^1(u_t^N) \\ \ldots \\ J_n^{kN} = \sum\limits_{t=0}^{m} \delta^t g_n^k(u_t^N) \end{pmatrix}. \tag{4}$$

The cooperative strategies and payoffs are constructed by solving the following problem:

$$\begin{aligned} &(V_1^{1c} - J_1^{1N}) \cdot \ldots \cdot (V_1^{kc} - J_1^{kN}) + \ldots + (V_n^{1c} - J_n^{1N}) \cdot \ldots \cdot (V_n^{kc} - J_n^{kN}) = \\ &= (\sum_{t=0}^{m} \delta^t g_1^1(u_{1t}^c, \ldots, u_{nt}^c) - J_1^{1N}) \cdot \ldots \cdot (\sum_{t=0}^{m} \delta^t g_1^k(u_{1t}^c, \ldots, u_{nt}^c) - J_1^{kN}) + \ldots \\ &+ (\sum_{t=0}^{m} \delta^t g_n^1(u_{1t}^c, \ldots, u_{nt}^c) - J_n^{1N}) \cdot \ldots \cdot (\sum_{t=0}^{m} \delta^t g_n^k(u_{1t}^c, \ldots, u_{nt}^c) - J_n^{kN}) \to \max_{u_{1t}^c, \ldots, u_{nt}^c}, \end{aligned} \tag{5}$$

where J_i^{jN} are the noncooperative payoffs given by (4), $i \in N$, $j = 1, \ldots, k$.

Definition 2. *A strategy profile $u_t^c = (u_{1t}^c, \ldots, u_{nt}^c)$ is a rational multicriteria cooperative equilibrium [5] of problem (1), (2) if it is the solution of problem (5).*

As was demonstrated in [5], with the presented approach, the cooperative payoffs of the players are greater than or equal to the multicriteria Nash payoffs. Hence, the conditions of individual rationality $V_i^{jc} \geq J_i^{jN}$, $i = 1, \ldots, n$, $j = 1, \ldots, k$ are fulfilled.

2.3. Dynamic Stability of Cooperative Solution

Classically, the solution optimality principle for a cooperative game includes: (1) an agreement on a set of cooperative controls, (2) a mechanism to distribute total payoff among the players. In cooperative setting players seek a set of strategies that yields a Pareto optimal solution, hence they maximize the sum of their individual payoffs. To determine the share of each player from the total payoff, that is called the imputation, some solution concepts, such as NM-solution, the core and the Shapley value are applied; see [25–27]. To construct the imputation of the cooperative game the characteristic function reflecting the payoff of any coalition of the players should be determined. There are some approaches how to define the characteristic function, for example α, β, γ—characteristic functions and others (see [8,25,28–30] for details).

In contrast to the classical one the cooperative behavior determination approach presented above needs no distribution of the total cooperative payoff among the players. As it is easily seen, the players seek jointly a set of strategies that optimize their individual payoffs presented as the Nash products. Hence, neither the characteristic function nor the imputation is required. Please note that the problems of construction and stability of the coalitions for multicriteria dynamic games have been also considered; see [31,32]. In the case of coalition games, naturally, the characteristic function and the imputation should be determined. However, in this paper we are not concerned with coalitions' formation processes and the players' cooperative payoffs for the whole game can be calculated without any imputations as

$$J_1^c(0) = \begin{pmatrix} J_1^{1c}(0) = \sum\limits_{t=0}^{m} \delta^t g_1^1(u_t^c) \\ \cdots \\ J_1^{kc}(0) = \sum\limits_{t=0}^{m} \delta^t g_1^k(u_t^c) \end{pmatrix}, \ldots, J_n^c(0) = \begin{pmatrix} J_n^{1c}(0) = \sum\limits_{t=0}^{m} \delta^t g_n^1(u_t^c) \\ \cdots \\ J_n^{kc}(0) = \sum\limits_{t=0}^{m} \delta^t g_n^k(u_t^c) \end{pmatrix},$$

where $u_t^c = (u_{1t}^c, \ldots, u_{nt}^c)$ are the cooperative strategies determined in (5).

Similarly we determine the cooperative payoffs $J_i^c(t)$, $i = 1, \ldots, n$, for every subgame started from the state x_t^c at a time t.

As is well known, the Nash bargaining scheme is not dynamically stable [9]. To stabilize the cooperative solution in multicriteria dynamic games we adopt the idea of imputation distribution procedure ([7,10,18,19]).

Definition 3. *A vector*

$$\beta(t) = (\beta_1(t), \ldots, \beta_n(t)),$$

where

$$\beta_1(t) = \begin{pmatrix} \beta_1^1(t) \\ \cdots \\ \beta_1^k(t) \end{pmatrix}, \ldots, \beta_n(t) = \begin{pmatrix} \beta_n^1(t) \\ \cdots \\ \beta_n^k(t) \end{pmatrix}$$

is a payoff distribution procedure (PDP) for the dynamic multicriteria game (1), (2), if

$$J_1^c(0) = \sum_{t=0}^{m} \delta^t \beta_1(t), \ldots, J_n^c(0) = \sum_{t=0}^{m} \delta^t \beta_n(t), \tag{6}$$

or in extended form,

$$\begin{cases} J_1^{1c}(0) = \sum\limits_{t=0}^{m} \delta^t \beta_1^1(t), \\ \dots \\ J_1^{kc}(0) = \sum\limits_{t=0}^{m} \delta^t \beta_1^k(t), \end{cases} , \dots , \begin{cases} J_n^{1c}(0) = \sum\limits_{t=0}^{m} \delta^t \beta_n^1(t), \\ \dots \\ J_n^{kc}(0) = \sum\limits_{t=0}^{m} \delta^t \beta_n^k(t). \end{cases}$$

The main idea of this scheme is to distribute the cooperative gain along the game path. Then β_i can be interpreted as the payment to player i in all criteria at a time t, $i = 1, \dots, n$.

Definition 4. *A vector $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$ is a time-consistent [7,10] PDP for dynamic multicriteria game (1), (2), if for every $t \geq 0$*

$$\begin{aligned} J_1^c(0) &= \sum_{\tau=0}^{t} \delta^\tau \beta_1(\tau) + \delta^{t+1} J_1^c(t+1), \\ &\dots \\ J_n^c(0) &= \sum_{\tau=0}^{t} \delta^\tau \beta_n(\tau) + \delta^{t+1} J_n^c(t+1), \end{aligned} \tag{7}$$

or in extended form,

$$\begin{cases} J_1^{1c}(0) = \sum\limits_{\tau=0}^{t} \delta^\tau \beta_1^1(\tau) + \delta^{t+1} J_1^{1c}(t+1), \\ \dots \\ J_1^{kc}(0) = \sum\limits_{\tau=0}^{t} \delta^\tau \beta_1^k(\tau) + \delta^{t+1} J_1^{kc}(t+1), \end{cases}$$

$$\dots$$

$$\begin{cases} J_n^{1c}(0) = \sum\limits_{\tau=0}^{t} \delta^\tau \beta_n^1(\tau) + \delta^{t+1} J_n^{1c}(t+1), \\ \dots \\ J_n^{kc}(0) = \sum\limits_{\tau=0}^{t} \delta^\tau \beta_n^k(\tau) + \delta^{t+1} J_n^{kc}(t+1). \end{cases}$$

Here the players following the cooperative trajectory are guided by the same optimal behavior determination approach (5) at each current time and hence do not have any reasonable motivation to deviate from the cooperation agreement.

Theorem 1. *A vector $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$, where*

$$\begin{aligned} \beta_1(t) &= J_1^c(t) - \delta J_1^c(t+1), \\ &\dots \\ \beta_n(t) &= J_n^c(t) - \delta J_n^c(t+1) \end{aligned} \tag{8}$$

is a time-consistent payoff distribution procedure for dynamic multicriteria game (1), (2).

Proof. The proof is given for the first player, for others it is similar. Conditions (6) of Definition 3 are satisfied:

$$\sum_{t=0}^{m} \delta^t \beta_1(t) = \sum_{t=0}^{m} \delta^t J_1^c(t) - \sum_{t=0}^{m} \delta^{t+1} J_1^c(t+1) =$$

$$= \begin{pmatrix} \sum\limits_{t=0}^{m} \delta^t J_1^{1c}(t) - \sum\limits_{t=0}^{m} \delta^{t+1} J_1^{1c}(t+1) \\ \dots \\ \sum\limits_{t=0}^{m} \delta^t J_1^{kc}(t) - \sum\limits_{t=0}^{m} \delta^{t+1} J_1^{kc}(t+1) \end{pmatrix} = \begin{pmatrix} J_1^{1c}(0) \\ \dots \\ J_1^{kc}(0) \end{pmatrix} = J_1^c(0)$$

as $J_1^{jc}(m+1) = 0$, $j = 1, \dots, k$. Similar considerations are true for $\beta_i(t)$, $i = 2, \dots, n$. Hence, $\beta(t)$ is a PDP.

Let us prove that this vector is a time-consistent payoff distribution procedure (7). It follows from the equalities

$$\sum_{\tau=0}^{t} \delta^\tau \beta_1(\tau) + \delta^{t+1} J_1^c(t+1) = \begin{pmatrix} \sum\limits_{\tau=0}^{t} \delta^\tau \beta_1^1(\tau) + \delta^{t+1} J_1^{1c}(t+1) \\ \dots \\ \sum\limits_{\tau=0}^{t} \delta^\tau \beta_1^k(\tau) + \delta^{t+1} J_1^{kc}(t+1) \end{pmatrix} =$$

$$= \begin{pmatrix} \sum\limits_{\tau=0}^{t} \delta^\tau J_1^{1c}(\tau) - \sum\limits_{\tau=0}^{t} \delta^{\tau+1} J_1^{1c}(\tau+1) + \delta^{t+1} J_1^{1c}(t+1) \\ \dots \\ \sum\limits_{\tau=0}^{t} \delta^\tau J_1^{kc}(\tau) - \sum\limits_{\tau=0}^{t} \delta^{\tau+1} J_1^{kc}(\tau+1) + \delta^{t+1} J_1^{kc}(t+1) \end{pmatrix} = \begin{pmatrix} J_1^{1c}(0) \\ \dots \\ J_1^{kc}(0) \end{pmatrix} = J_1^c(0),$$

and similarly for the other players. □

2.4. Conditions for Rational Behavior

The conditions to maintain the cooperative (rational) behavior in dynamic games are considered. Since there can be some irrational players who can break out the cooperation, Yeung D.W.K. [20] introduced the condition that protects players against the loss of profits in this case.

Definition 5. *The imputation $\xi = (\xi_1, \dots, \xi_n)$ satisfies irrational-behavior-proofness condition [20] if*

$$\sum_{\tau=0}^{t} \delta^\tau \beta_i(\tau) + \delta^{t+1} V(i, t+1) \geq V(i, 0) \tag{9}$$

for all $t \geq 0$, where $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$ – time-consistent imputation distribution procedure and $V(i,t)$ is the noncooperative payoff of player i, $i \in N$.

If this condition is satisfied, then each player is irrational-behavior-proof because irrational actions that break the cooperative agreement will not bring his payoff below the initial noncooperative payoff.

In the papers [21,22] for discrete-time problems, a new condition which is stronger than the Yeung's condition and is easier to verify was introduced.

Definition 6. *The imputation $\xi = (\xi_1, \dots, \xi_n)$ satisfies each step rational behavior condition if*

$$\beta_i(t) + \delta V(i, t+1) \geq V(i, t) \tag{10}$$

for all $t \geq 0$, where $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$—time-consistent imputation distribution procedure and $V(i,t)$ is the noncooperative payoff of player i, $i \in N$.

The proposed condition offers an incentive to each player to maintain cooperation because at every step she gains more from cooperation than from noncooperative behavior.

Here, we adopt rationality conditions for dynamic multicriteria games. Since no imputation procedure is required with the approach presented above, let us rewrite the definitions.

Definition 7. *The multicriteria cooperative solution $J^c(t) = (J_1^c(t), \dots, J_n^c(t))$ satisfies the irrational behavior proofness condition if*

$$\sum_{\tau=0}^{t} \delta^\tau \beta_i(\tau) + \delta^{t+1} J_i^N(t+1) \geq J_i^N(0) \tag{11}$$

for all $t \geq 0$, where $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$ – time-consistent payoff distribution procedure (8) and $J_i^N(t)$ is the noncooperative payoff (4) of player i, $i \in N$. Or in extended form,

$$\begin{cases} \sum\limits_{\tau=0}^{t} \delta^\tau \beta_1^1(\tau) + \delta^{t+1} J_1^{1N}(t+1) \geq J_1^{1N}(0), \\ \qquad \dots \\ \sum\limits_{\tau=0}^{t} \delta^\tau \beta_1^k(\tau) + \delta^{t+1} J_1^{kN}(t+1) \geq J_1^{kN}(0), \end{cases}$$

$$\dots$$

$$\begin{cases} \sum\limits_{\tau=0}^{t} \delta^\tau \beta_n^1(\tau) + \delta^{t+1} J_n^{1N}(t+1) \geq J_n^{1N}(0), \\ \qquad \dots \\ \sum\limits_{\tau=0}^{t} \delta^\tau \beta_n^k(\tau) + \delta^{t+1} J_n^{kN}(t+1) \geq J_n^{kN}(0). \end{cases}$$

Definition 8. *The multicriteria cooperative solution $J^c(t) = (J_1^c(t), \dots, J_n^c(t))$ satisfies each step rational behavior condition if*

$$\beta_i(t) + \delta J_i^N(t+1) \geq J_i^N(t) \tag{12}$$

for all $t \geq 0$, where $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$ – time-consistent payoff distribution procedure (8) and $J_i^N(t)$ is the noncooperative payoff (4) of player i, $i \in N$. Or in extended form,

$$\begin{cases} \beta_1^1(t) + \delta J_1^{1N}(t+1) \geq J_1^{1N}(t), \\ \qquad \dots \\ \beta_1^k(t) + \delta J_1^{kN}(t+1) \geq J_1^{kN}(t), \end{cases}$$

$$\dots$$

$$\begin{cases} \beta_n^1(t) + \delta J_n^{1N}(t+1) \geq J_n^{1N}(t), \\ \qquad \dots \\ \beta_n^k(t) + \delta J_n^{kN}(t+1) \geq J_n^{kN}(t). \end{cases}$$

For problem (1), (2) the conditions for rational behavior (11) and (12) can be rewritten as

$$(1 - \delta^{t+1})(J_i^c(0) - J_i^N(0)) + \delta^{t+1} \sum_{\tau=0}^{t} \delta^\tau (g_i(u_\tau^c) - g_i(u_\tau^N)) \geq 0 \ \forall t, \ i \in N, \tag{13}$$

$$(1 - \delta)(J_i^c(t) - J_i^N(t)) + \delta^{t+2}(g_i(u_t^c) - g_i(u_t^N)) \geq 0 \ \forall t, \ i \in N, \tag{14}$$

where

$$g_i(u) = \begin{pmatrix} g_i^1(u) \\ \dots \\ g_i^k(u) \end{pmatrix}.$$

Since with the presented cooperative behavior construction approach individual rationality conditions are satisfied, then the first parts of both inequalities are nonnegative. Hence, the each step rational behavior conditions is fulfilled if $g_i(u_t^c) - g_i(u_t^N)\ \forall t, i \in N$, and the irrational behavior proofness condition is true if $\sum_{\tau=0}^{t} \delta^\tau (g_i(u_\tau^c) - g_i(u_\tau^N))\ \forall t, i \in N$. As it easily seen, the each step rational behavior condition yields the Yeung's condition.

Next, we consider a dynamic bi-criteria model related with the bioresource management problem (harvesting) to illustrate the suggested concepts.

3. Dynamic Bi-Criteria Resource Management Problem

Consider a bi-criteria discrete-time dynamic bioresource management model with many players. Let n players (countries or firms) be exploiting a bioresource on a finite time horizon $[0, m]$. The population evolves according to the equation

$$x_{t+1} = \varepsilon x_t - u_{1t} - \ldots - u_{nt}\,, \quad x_0 = x, \tag{15}$$

where $x_t \geq 0$ is the population size at a time $t \geq 0$, $\varepsilon \geq 1$ denotes the natural birth rate, and $u_{it} \geq 0$ specifies the catch strategy of player i at a time $t \geq 0$, $i \in N = \{1, \ldots, n\}$.

Each player seeks to achieve two goals: to maximize the profit from resource sales and to minimize the catching costs. It will be assumed that the players have different market prices but the same costs that depend quadratically on the exploitation rate of each player. The vector payoffs of the players on the finite planning horizon take the form

$$J_1 = \begin{pmatrix} J_1^1 = \sum\limits_{t=0}^{m} \delta^t p_1 u_{1t} \\ J_1^2 = -\sum\limits_{t=0}^{m} \delta^t c u_{1t}^2 \end{pmatrix}, \ldots, J_n = \begin{pmatrix} J_n^1 = \sum\limits_{t=0}^{m} \delta^t p_n u_{nt} \\ J_n^2 = -\sum\limits_{t=0}^{m} \delta^t c u_{nt}^2 \end{pmatrix}, \tag{16}$$

where for $i \in N$, $p_i \geq 0$ is the market price of the resource for player i, $c \geq 0$ indicates the catching cost, and $\delta \in (0, 1)$ denotes the discount factor.

3.1. Multicriteria Nash Equilibrium

First, we construct the guaranteed payoffs using one of the modifications from [2]. The guaranteed payoff points $G_1^1, \ldots, G_n^1$ will be defined as the Nash equilibrium in the game $\langle N, \{U_i\}_{i=1}^n, \{J_i^1\}_{i=1}^n \rangle$. Applying the Bellman principle and assuming the linear form of the strategies and value functions, we obtain the Nash equilibrium strategies

$$u_{1t} = \ldots = u_{nt} = \frac{\varepsilon - 1}{n - 1} x_t,$$

and the dynamics becomes

$$x_t = \left(\frac{n - \varepsilon}{n - 1}\right)^t x_0.$$

Then the guaranteed payoff points take the form

$$G_1^1 = p_1 A x_0\,, \ldots, G_n^1 = p_n A x_0, \tag{17}$$

where

$$A = \frac{\varepsilon - 1}{n - 1} \frac{(\delta(n - \varepsilon))^{m+1} + (n - 1)^{m+1}}{(n - 1)^m (\delta(n - \varepsilon) - n + 1)}\,.$$

Similarly, determining the Nash equilibrium in the game with the second criteria of all players $\langle N, \{U_i\}_{i=1}^n, \{J_i^2\}_{i=1}^n \rangle$, yields n more guaranteed payoffs points

$$G_1^2 = \ldots = G_n^2 = -cGx_0^2, \tag{18}$$

where

$$G = \Big(\frac{2n - \varepsilon^2 + \varepsilon\sqrt{4n^2 + \varepsilon^2 - 4n}}{n(-\varepsilon + \sqrt{4n^2 + \varepsilon^2 - 4n})}\Big)^2$$
$$\times \frac{(2\delta n)^{m+1} - (\varepsilon - \sqrt{4n^2 + \varepsilon^2 - 4n})^{m+1}}{(\varepsilon - \sqrt{4n^2 + \varepsilon^2 - 4n})^m (2\delta n - \varepsilon + \sqrt{4n^2 + \varepsilon^2 - 4n})}.$$

In accordance with Definition 1, for designing the multicriteria Nash equilibrium of the game (15), (16) the following problem has to be solved:

$$p_1 c(\sum_{t=0}^{m} \delta^t u_{1t} - Ax)(-\sum_{t=0}^{m} \delta^t u_{1t}^2 + Gx^2) \to \max_{u_{1t}},$$
$$\ldots$$
$$p_n c(\sum_{t=0}^{m} \delta^t u_{nt} - Ax)(-\sum_{t=0}^{m} \delta^t u_{nt}^2 + Gx^2) \to \max_{u_{nt}}.$$

Considering the process starting from one-stage game to m-stage one and seeking the strategies in linear form, we obtain the multicriteria Nash equilibrium.

Proposition 1. *The multicriteria Nash equilibrium strategies in problem (15), (16) have the form* $u_{it}^N = \gamma_{im-t}^N x_t$, $i \in N$,

$$\gamma_{1t}^N = \ldots = \gamma_{nt}^N = \gamma_t^N = \frac{\varepsilon^{t-1}\gamma_1^N}{1 + n\gamma_1^N \sum\limits_{j=0}^{t-2} \varepsilon^j}, \ t = 2, \ldots, m. \tag{19}$$

The players' strategy at the last stage γ_1^N *is determined from the following equation*

$$\Big[3\varepsilon^{2(m-1)} \sum_{j=0}^{m-1} \delta^j - 2\varepsilon^{m-1} n \sum_{j=0}^{m-2} \varepsilon^j A - n^2 (\sum_{j=0}^{m-2} \varepsilon^j)^2 G\Big] (\gamma_1^N)^2 -$$
$$-2(\varepsilon^{m-1} A + \sum_{j=0}^{m-2} \varepsilon^j G n)\gamma_1^N - G = 0.$$

3.2. Cooperative Equilibrium

To construct the cooperative payoffs and strategies the modified bargaining scheme will be applied [5]. First, we have to determine the noncooperative payoffs as the ones gained by the players using the multicriteria Nash strategies. Then, we construct the sum of the Nash products with the noncooperative payoffs of players acting as the status quo points.

In view of Proposition 1, the noncooperative payoffs have the form

$$J_i^{1N}(x) = \sum_{t=0}^{m-1} \delta^t p_i \gamma_{m-t}^N x, \ i \in N,$$
$$J_1^{2N}(x) = \ldots = J_n^{2N}(x) = -c \sum_{t=0}^{m-1} \delta^t (\gamma_{m-t}^N)^2 x^2.$$

In accordance with Definition 2, for designing the multicriteria cooperative equilibrium the following problem has to be solved:

$$p_1(\sum_{t=0}^{m}\delta^t u_{1t}^c - Px)(-\sum_{t=0}^{m}\delta^t (u_{1t}^c)^2 + Kx^2) + \ldots + p_n(\sum_{t=0}^{m}\delta^t u_{nt}^c - Px)(-\sum_{t=0}^{m}\delta^t (u_{nt}^c)^2 + Kx^2) \to \max_{u_{1t}^c,\ldots,u_{nt}^c},$$

where $P = \sum_{t=0}^{m-1}\delta^t \gamma_{m-t}^N$, $K = \sum_{t=0}^{m-1}\delta^t (\gamma_{m-t}^N)^2$.

Considering the process starting from one-stage game to m-stage one and seeking the strategies in linear form, we construct cooperative behavior.

Proposition 2. *The multicriteria cooperative equilibrium strategies in problem (15), (16) take the form* $u_{it}^c = \gamma_{im-t}^c x_t$, $i \in N$,

$$\gamma_{1t}^c = \ldots = \gamma_{nt}^c = \gamma_t^c = \frac{\varepsilon^{t-1}\gamma_1^c}{1 + n\gamma_1^c \sum_{j=0}^{t-2}\varepsilon^j}, \; t = 2, \ldots, m. \tag{20}$$

The players' strategy at the last stage γ_1^c *is determined from the following equation*

$$\Big[3\varepsilon^{2(m-1)}\sum_{j=0}^{m-1}\delta^j - 2\varepsilon^{m-1}n\sum_{j=0}^{m-2}\varepsilon^j P - n^2(\sum_{j=0}^{m-2}\varepsilon^j)^2 K\Big](\gamma_1^c)^2 -$$
$$-2(\varepsilon^{m-1}P + \sum_{j=0}^{m-2}\varepsilon^j K n)\gamma_1^c - K = 0.$$

3.3. Dynamic Stability and Conditions for Rational Behavior

Proposition 3. *The time-consistent payoff distribution procedure in the problem (15), (16) takes the form*

$$\beta_i(t) = \begin{pmatrix} \beta_i^1(t) \\ \beta_i^2(t) \end{pmatrix}, \; i = 1, \ldots, n, \; t = 0, \ldots, m-1,$$

where

$$\beta_i^1(t) = p_i \delta^t \gamma_{m-t}^c x_t + p_i(1-\delta)\sum_{\tau=t+1}^{m-1}\delta^\tau \gamma_{m-\tau}^c x_\tau,$$

$$\beta_i^2(t) = -c\delta^t(\gamma_{m-t}^c)^2 x_t^2 - c(1-\delta)\sum_{\tau=t+1}^{m-1}\delta^\tau(\gamma_{m-\tau}^c)^2 x_\tau^2, \; i = 1, \ldots, n.$$

Proof. follows from Theorem 1 and the form of cooperative strategies given in Proposition 2. □

Proposition 4. *The conditions for rational behavior in problem (15), (16) are fulfilled if* $\gamma_1^c \geq \gamma_1^N$.

Proof. The irrational-behavior-proofness condition (13) in problem (15), (16) takes the form

$$(1 - \delta^{t+1})(J_i^c(0) - J_i^N(0)) + \delta^{t+1}\begin{pmatrix} p_i \sum_{\tau=0}^{t}\delta^\tau(u_{i\tau}^c - u_{i\tau}^N) \\ -c\sum_{\tau=0}^{t}\delta^\tau((u_{i\tau}^c)^2 - (u_{i\tau}^N)^2) \end{pmatrix} \geq 0, \tag{21}$$

and each step rational behavior condition becomes

$$(1-\delta)(J_i^c(t) - J_i^N(t)) + \delta^{t+2}\begin{pmatrix} p_i(u_{it}^c - u_{it}^N) \\ -c((u_{it}^c)^2 - (u_{it}^N)^2) \end{pmatrix} \geq 0. \tag{22}$$

Let us consider each step rational behavior condition for the first criterium. Since the individual rationality conditions are fulfilled the first part of the inequality is positive. Hence, the sigh of the $u^c_{it} - u^N_{it}$ need to be checked. In accordance with Propositions 1 and 2

$$u^c_{it} - u^N_{it} = \frac{(\varepsilon-1)^2\varepsilon^{m-t-1}(\gamma^c_1 - \gamma^N_1)x_t}{(\varepsilon - 1 + n\gamma^c_1(\varepsilon^{m-t-1}-1))(\varepsilon - 1 + n\gamma^N_1(\varepsilon^{m-t-1}-1))}$$

that is nonnegative if $\gamma^c_1 \geq \gamma^N_1$.

For the second criterium the sigh of $-c(u^c_{it} - u^N_{it})(u^c_{it} + u^N_{it})$ needs to be checked. Since, the cooperative solution satisfy the individual rationality conditions for each stage t the first part of the Inequality (22) takes the form

$$-c\sum_{\tau=t}^{m-1}\delta^\tau(u^c_{i\tau} - u^N_{i\tau})(u^c_{i\tau} + u^N_{i\tau}) \geq 0$$

that yields

$$-c\delta^t(u^c_{it} - u^N_{it})(u^c_{it} + u^N_{it}) \geq c\sum_{\tau=t+1}^{m-1}\delta^\tau(u^c_{i\tau} - u^N_{i\tau})(u^c_{i\tau} + u^N_{i\tau})\,. \tag{23}$$

The right hand side of (23) is nonnegative if $u^c_{i\tau} - u^N_{i\tau} \geq 0 \; \forall \tau = t+1, \ldots, m$, that again is true if $\gamma^c_1 \geq \gamma^N_1$.

As the each step rational behavior condition is stronger than the Yeung's condition, this yields the fulfillment of irrational-behavior-proofness condition. □

3.4. Modelling

We have performed numerical simulation for symmetric case with the following parameters:

$$m = 15,\; n = 5,\; \varepsilon = 1.3,\; p_1 = \ldots = p_5 = 100,\; c = 50,\; \delta = 0.8.$$

These parameters are typical for the fish species in Karelian lake [33]. In the papers [22,34,35] the natural growth function of the population was estimated and its linear approximation with the appropriate parameter ε is applied in this paper. It should be stressed that the price and the cost parameters do not influence the form of the players' strategies, hence can be taken as any values.

The presented figures illustrate our theoretical results. Namely Figure 1 shows the dynamics of the population size, while Figure 2 presents the players' strategies for noncooperative and cooperative cases. As one can notice cooperative behavior improves the ecological situation as it limits bioresource exploitation. The population size increases in both settings but under cooperation much quicker (from x_0 = 50,000 to 110,000).

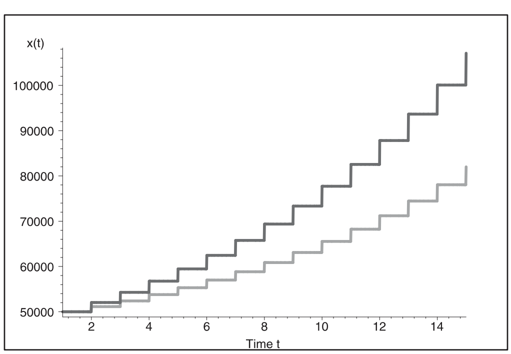

Figure 1. Population size: dark—cooperation, light—Nash equilibrium.

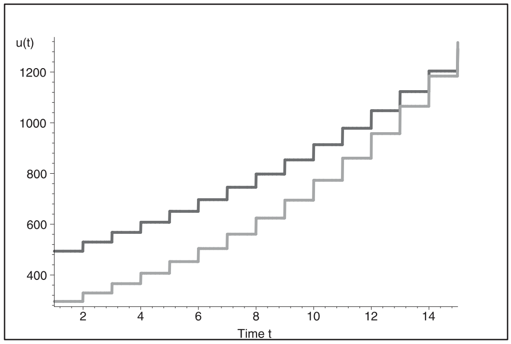

Figure 2. Players' strategies: dark—cooperation, light—Nash equilibrium.

Moreover, as Figure 2 shows the cooperative behavior is beneficial for the players. To emphasize the last conclusion the instantaneous payoffs ($\delta^t g_1^1(t)$) for both noncooperative and cooperative settings are presented in Figure 3. As it is easily seen the players' cooperative strategies (the catch) are larger than the noncooperative ones and some convergence can be noticed at the end of the planning horizon. It is related to the fact that the asymptotic values of the players' strategies in both cases (γ_t^N, γ_t^c) are $(\varepsilon - 1)/n$. The instantaneous payoffs decrease in both settings because of the discounting but under cooperation much slower (from 60,000 to 4000 monetary units).

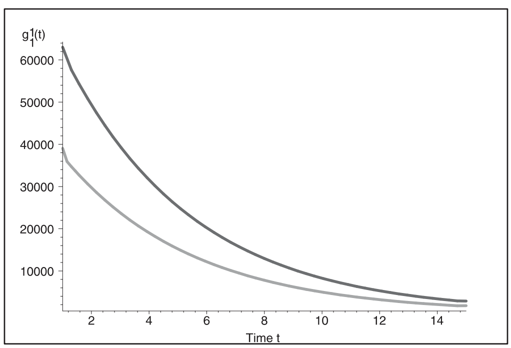

Figure 3. Instantaneous payoffs: dark—cooperation, light—Nash equilibrium.

Since the player's strategy at the last stage under cooperation is larger that noncooperative one the conditions for rational behavior are fulfilled. Figure 4 shows how to distribute the cooperative gain among the game path (PDP $\beta_1^1(t)$). It is quiet interesting that PDP differs from instantaneous payoffs very slightly. Please note that changing the number of players, time horizon and other parameters gives the similar pictures, hence are not presented.

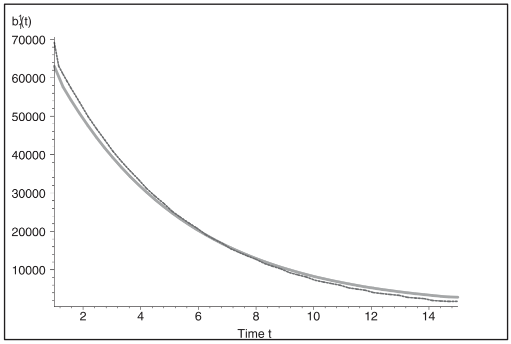

Figure 4. Instantaneous payoffs (light line) and PDP ($\beta_1^1(t)$).

4. Conclusions

The problem of dynamic stability in multicriteria dynamic games with finite horizon has been investigated. First, we have evaluated the multicriteria Nash equilibrium strategies. Second, we constructed the multicriteria cooperative strategies and payoffs via the modified bargaining scheme. We adopted the concept of dynamic stability for multicriteria dynamic games and have constructed the payoff distribution procedure. The conditions for rational behavior have been modified for dynamic multicriteria games.

The approaches presented in the paper give the possibility to find optimal solutions in various multicriteria dynamic games. To show one of the possible applications, we studied a bi-criteria discrete-time bioresource management problem, where the players differ in their aims. Multicriteria Nash and cooperative equilibria strategies have been derived analytically in linear forms. Hence, they can be directly applied to concrete populations with different values of parameters. As cooperative behavior improves the ecological situation, the dynamic stability concept has been applied to stabilize the cooperative agreement. The time-consistent payoff distribution procedure has been also derived analytically. The fulfillment of conditions for rational behavior has been proved.

The presented theoretical constructions can be applied for different management problems, where the decision maker often has several criteria to optimize. For example, to maximize the profit and to minimize the production cost or the labor involved in the manufacture. Moreover, the constructed payoff distribution procedure gives an incentive to maintain the cooperative agreement that is extremely important for management problems with common resources. Hence, the results presented in this paper can be applied in biological, economical and social game-theoretic models with vector payoffs.

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Appendix A. Nash Equilibrium Meaningful

Consider problem (3) with two criteria for simplicity with the constraints $J_i^j \geq G_i^j$, $i \in N$, $j = 1, 2$. Let deal with a problem for the first player, to construct noncooperative behavior we should maximize multicriteria payoff function or equally minimize

$$H_1(u_{1t}, u_{2t}^N, \ldots, u_{nt}^N) = (-J_1^1(u_{1t}, u_{2t}^N, \ldots, u_{nt}^N) + G_1^1)(J_1^2(u_{1t}, u_{2t}^N, \ldots, u_{nt}^N) - G_1^2) \to \min_{u_{1t}}$$

subject to (here and below $(u_{1t}, u_{2t}^N, \ldots, u_{nt}^N)$ is omitted)

$$G_1^1 - J_1^1 \leq 0,$$
$$G_1^2 - J_1^2 \leq 0,$$
$$u_{1t} \geq 0\,.$$

The Kuhn–Tucker (KT) conditions are applicable, and the Lagrangian for each time instant $t = 1, \ldots, m$ takes the form (t is omitted)

$$L = (-J_1^{1c} + G_1^1)(J_1^2 - G_1^2) + \lambda_1(G_1^1 - J_1^1) + \lambda_2(G_1^2 - J_1^2).$$

The KT conditions take the forms

$$-(J_1^1)'(J_1^2 - G_1^2 + \lambda_1) + (J_1^2)'(-J_1^1 + G_1^1 - \lambda_2) \geq 0,$$
$$u_1\Big[-(J_1^1)'(J_1^2 - G_1^2 + \lambda_1) + (J_1^2)'(-J_1^1 + G_1^1 - \lambda_2)\Big] = 0, \tag{A1}$$

$$J_1^1 - G_1^1 \geq 0, \tag{A2}$$

$$\lambda_1(J_1^1 - G_1^1) = 0, \tag{A3}$$

$$J_1^2 - G_1^2 \geq 0, \tag{A4}$$

$$\lambda_2(J_1^2 - G_1^2) = 0, \tag{A5}$$

$$u_1 \geq 0\,,\ \lambda_i \geq 0\,,\ i = 1, 2.$$

1. Consider the case $\lambda_1 > 0$, $\lambda_2 > 0$. From (A3) and (A5) it follows that

$$J_1^1 - G_1^1 = 0\,,\ J_1^2 - G_1^2 = 0.$$

If u_1 is equal to zero, then conditions (A2), (A4) are not satisfied (under assumptions that $J_1^j(0, u_{2t}^N, \ldots, u_{nt}^N) = 0$, $j = 1, 2$). Hence, $u_1 > 0$. In this case, the noncooperative behavior coincides with the guaranteed one, and the goal function H_1 is equal to zero.

2. Consider the case $\lambda_1 = 0$, $\lambda_2 > 0$. From (A5) it follows that

$$J_1^2 - G_1^2 = 0.$$

By analogy, $u_1 > 0$, and from (A1) it follows that

$$(J_1^2)'(-J_1^1 + G_1^1 - \lambda_2) = 0.$$

Consequently,

$$-J_1^1 + G_1^1 = \lambda_2 > 0\,,$$

which obviously contradicts condition (A2).

Similarly, in the case where $\lambda_1 > 0$, $\lambda_2 = 0$, we will naturally arrive in contradiction.

3. Finally, consider the case $\lambda_1 = 0$, $\lambda_2 = 0$. Similarly, it is easy to check that $u_1 > 0$, the minimum is achieved at an interior point and can be found via the first-order optimality condition:

$$-(J_1^1)'(J_1^2 - G_1^2) + (J_1^2)'(-J_1^1 + G_1^1) = 0\,.$$

Here, the goal function becomes

$$H_1 = (-J_1^1 + G_1^1)(J_1^2 - G_1^2)\,,$$

which is less than zero.

Similarly, for other players. Thus, the presented above scheme guarantees that the solution satisfies the conditions $J_i^j \geq G_i^j$, $i \in N$, $j = 1, 2$.

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On a Simplified Method of Defining Characteristic Function in Stochastic Games

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Abstract: In the paper, we propose a new method of constructing cooperative stochastic game in the form of characteristic function when initially non-cooperative stochastic game is given. The set of states and the set of actions for any player is finite. The construction of the characteristic function is based on a calculation of the maximin values of zero-sum games between a coalition and its anti-coalition for each state of the game. The proposed characteristic function has some advantages in comparison with previously defined characteristic functions for stochastic games. In particular, the advantages include computation simplicity and strong subgame consistency of the core calculated with the values of the new characteristic function.

Keywords: cooperative stochastic game; strong subgame consistency; characteristic function; core

1. Introduction

When a non-cooperative game is initially defined, the problem of construction of a cooperative version of the game is actual if players start acting as a unique coalition to maximize their joint payoff or minimize joint costs. The classical approach is to define cooperative game in a form of characteristic function that assigns the value for any coalition of players. Subsequently, based on this function one can calculate the imputation of the joint payoff allocating it among players. The component of the imputations may vary if we calculate them based on different characteristic functions. Therefore, the way of defining this function is important and it has influence on the players' payoffs in cooperative game. Moreover, some approaches to define characteristic function make it impossible to apply in dynamic or differential games because of computational difficulties. Additionally, the way of constructing characteristic function also influences on the consistency properties of cooperative solutions that are realized in dynamics.

The choice of the approach on how to define characteristic function also depends on the background of the considered problem if it arises from an applied area. The existence and uniqueness issues are also actual when one chooses the way of constructing characteristic function. There exist different approaches that can be applied to stochastic game. The so-called maxmin and minmax approaches define the value of the function for coalition S as maxmin and minmax payoff of coalition S in zero-sum game against coalition of all left-out players [1,2]. Another approach is proposed in [3,4] when the value of coalition S is defined as its payoff in the Nash equilibrium in the non-cooperative game between coalition S and left-out players acting individually. The calculation of characteristic function in two-step procedure is proposed in [5], in which the authors first find an n-player non-cooperative equilibrium and then allow coalition S to optimize its payoff, assuming that left-out players use their Nash equilibrium actions found at the first step. The properties of this function are examined in [6,7]. Another two-stage approach for defining characteristic function is proposed in [8], in which the strategies maximizing total payoff of the players are first found.

Subsequently, these strategies are used by the players from coalition S, while the out-coalition players use the strategies minimizing the total payoff of players from S. The joint payoff of players from the coalition equals the value of characteristic function for this coalition.

The new simplified method of constructing characteristic function in multistage games is introduced in [9]. They examine the properties of this function and proved that the corresponding core is strongly subgame-consistent in multistage game. This property cannot be proved in general case when the characteristic function is constructed with the classical approaches, like maxmin or minmax.

In the paper, we adopt the method of constructing the characteristic function proposed in [9] to stochastic games. Based on the values of the characteristic function, one can determine the core. Moreover, the core satisfies the strong subgame consistency property, which is a refinement of subgame consistency on the case of set-valued cooperative solutions. The problem of subgame consistency is originally examined for differential games in [10,11]. The construction of a special payment scheme, called imputation distribution procedure (see [11]), allows for coping with the problem of time inconsistency of cooperative solutions. This problem is described for stochastic games in [12–14] in the case of unique-valued cooperative solutions. The node-consistent core is constructed in dynamic games played over event trees in [15]. The strong subgame consistency of the set-valued cooperative solution, like the core, guarantees players to obtain, in total, the solution from initially defined core. It means that, in any intermediate time period, the solution is the sum of obtained payments up to the current period, and the core elements of subgame starting from the next time period. The strong subgame consistency condition is proposed in [16]. The subcore satisfying strong subgame property is constructed for multistage games in [17]. The problem of subgame consistency is actual for different classes of dynamic and differential games and it is examined in [18] for stochastic games with finite duration, in [19] for differential games with finite time horizon, in [20] for multistage games. In the paper, we construct characteristic function for stochastic game in a special way and calculate the core while using the values of this function. The core satisfies strong subgame consistency property. To prove this result, we define the imputation distribution procedure, which determines the payments to the players in any state realized in the game process.

The rest of the paper is organized, as follows. We describe the model of stochastic games in Section 2.1. In Section 2.2, we define the new approximated characteristic function for stage games, and then extend this approach to the case of stochastic game in Section 2.3. We formulate the definition of the imputation distribution procedure for stochastic games and describe the idea of strongly subgame consistency of the core in Section 3. We briefly conclude in Section 4.

2. Cooperative Stochastic Games

2.1. Model

Consider a non-cooperative stochastic game G given by

$$G = \left(N, \Omega, \{\Gamma(\omega)\}_{\omega\in\Omega}, \pi_0, \left\{ p(\omega''|\omega', a^{\omega'}) \right\}_{\substack{\omega',\omega''\in\Omega \\ a^{\omega'}\in\prod_{i\in N} A_i^{\omega'}}}, \delta \right), \tag{1}$$

where

- $N = \{1, \ldots, n\}$ is the set of players.
- $\Omega = \{\omega_1, \ldots, \omega_k\}$ is the finite set of states.
- $\Gamma(\omega)$ is the game in normal form associated with state ω. The set of players N is common for any state ω. Let A_i^ω be a finite set of actions of player $i \in N$ in state ω, $a_i^\omega \in A_i^\omega$ be an action of player $i \in N$ in this state; $K_i^\omega : \prod_{j\in N} A_j^\omega \to \mathbb{R}$ be a payoff function of player i in state ω.
- $p(\cdot|\omega, a^\omega) : \Omega \times A^\omega \to \Delta(\Omega)$ is a transition function from state ω when action profile $a^\omega \in \prod_{j\in N} A_j^\omega$ is realized, where $\Delta(\Omega)$ is a probability distribution over set Ω.
- $\pi_0 = (\pi_0^{\omega_1}, \ldots, \pi_0^{\omega_k})$ is an initial state distribution.

- $\delta \in (0,1)$ is a common discount factor.

Denote by G^{ω} the subgame of G starting from state ω defined by (1) with π_0, such that $\pi_0^{\omega} = 1$ and $\pi_0^{\omega'} = 0$ for any state $\omega' \neq \omega$.

We assume that, in stochastic game G, the set of any player's strategies H_i is stationary. The stationary strategy of player i is η_i assigning action (maybe mixed) $a_i \in \Delta(A_i^{\omega})$ to any state ω. The vector $(\eta_1, \dots, \eta_n) \in \prod_{j \in N} H_j$ is a stationary strategy profile in stochastic game G. It is obvious that a stationary strategy η_i of player $i \in N$ in game G is the stationary strategy of this player in any subgame G^{ω}.

By the payoff of player i, we assume the expected payoff in stochastic subgame G^{ω} given by

$$E_i^{\omega}(\eta) = K_i^{\omega}(a^{\omega}) + \delta \sum_{\omega' \in \Omega} p(\omega' | \omega, a^{\omega}) E_i^{\omega'}(\eta). \tag{2}$$

where $\eta \in H = \prod_{j \in N} H_j$ is a stationary strategy profile such that $\eta(\omega) = a^{\omega} \in \prod_{j \in N} A_j^{\omega}$. We rewrite Equation (2) in a vector form and obtain

$$E_i(\eta) = K_i(a) + \delta \Pi(\eta) E_i(\eta), \tag{3}$$

where $E_i(\eta) = (E_i^{\omega_1}(\eta), \dots, E_i^{\omega_k}(\eta))'$, $K_i(a) = (K_i^{\omega_1}(a^{\omega_1}), \dots, K_i^{\omega_k}(a^{\omega_k}))'$. A matrix of transition probabilities is formed in the following way

$$\Pi(\eta) = \begin{pmatrix} p(\omega_1|\omega_1, a^{\omega_1}) & \dots & p(\omega_k|\omega_1, a^{\omega_1}) \\ p(\omega_1|\omega_2, a^{\omega_2}) & \dots & p(\omega_k|\omega_2, a^{\omega_2}) \\ \dots & \dots & \dots \\ p(\omega_1|\omega_k, a^{\omega_k}) & \dots & p(\omega_k|\omega_k, a^{\omega_k}) \end{pmatrix} \tag{4}$$

in which each row contains transition probabilities from a corresponding state.

Equation (3) implies the explicit formula to calculate the expected payoff of player i when the stationary strategy profile η is realized:

$$E_i(\eta) = \left(\mathbb{I}_k - \delta \Pi(\eta)\right)^{-1} K_i(a),$$

where $\mathbb{I}_k$ is an identity matrix of size $k \times k$. Inverted matrix $\left(\mathbb{I}_k - \delta \Pi(\eta)\right)^{-1}$ always exists for discount factor $\delta \in (0,1)$.

Taking into account the probability distribution π_0, we calculate the expected payoff in game G, as

$$\bar{E}_i(\eta) = \pi_0 E_i(\eta) = \pi_0 \left(\mathbb{I}_k - \delta \Pi(\eta)\right)^{-1} K_i(a). \tag{5}$$

If players cooperate, they find the cooperative strategy profile η^* maximizing the total expected payoff, which is

$$\eta^* = \arg\max_{\eta \in H} \sum_{i \in N} \bar{E}_i(\eta).$$

We should notice that η^* is a pure stationary strategy profile. The profile η^* is such that $\eta_i^*(\omega) = a_i^{\omega *} \in A_i^{\omega}, \omega \in \Omega$. We also assume that the profile η^* is such that $\max_{\eta \in H} \sum_{i \in N} E_i^{\omega}(\eta) = \sum_{i \in N} E_i^{\omega}(\eta^*)$ for any state $\omega \in \Omega$, which means that the cooperative strategy profile maximizes the total payoff of the players independently of which state is initial. This assumption is usually satisfied for most stochastic games.

To define cooperative game when the non-cooperative stochastic game is given, we use the classical approach and define it in the form of characteristic function $v : 2^N \to \mathbb{R}^1$ whose values estimate the "power" of any coalition or the subset of players. In [21], the characteristic function value for coalition S in subgame starting at any state ω is defined in as maxmin value, which is

$$v^{\omega}(S) = val\, G_S^{\omega}, \tag{6}$$

where G_S^{ω} is a zero-sum stochastic subgame starting at state ω, in which coalition S is a maximizing player, coalition $N \backslash S$ is a minimizing player. Existence of the value of game G_S^{ω} for stochastic games is proved in [22].

2.2. Approximated Characteristic Function for State Games

Before we define a characteristic function in a new form, we need to make additional calculations. First, we consider state games and propose a scheme of calculation of the approximated characteristic function values for any state. Define characteristic function for a state $\omega \in \Omega$ or one-shot game $\Gamma(\omega)$ given in normal form while using the maxmin approach:

$$v(\omega, S) = \max_{a_S \in \prod\limits_{j \in S} A_j^{\omega}} \; \min_{a_{N\backslash S} \in \prod\limits_{j \in N\backslash S} A_j^{\omega}} \sum_{i \in S} K_i^{\omega}(a_S^{\omega}, a_{N\backslash S}^{\omega}), \tag{7}$$

where maxmin in (7) is found in pure strategies.

Let $C(\omega)$ be a non-empty core in the game defined in state ω using c.f. (7), which is

$$C(\omega) = \Big\{(\alpha_1(\omega), \ldots, \alpha_n(\omega)) : \sum_{i \in S} \alpha_i(\omega) \geqslant v(\omega, S), \forall S \subset N, \sum_{i \in N} \alpha_i(\omega) = v(\omega, N)\Big\} \tag{8}$$

Remark 1. *We assume that conditions under which the core $C(\omega)$ exists for any state ω are satisfied. The core $C(\omega)$ is non-empty if and only if for any function $\psi : 2^N \setminus \varnothing \to [0,1]$, where $\sum_{S \in 2^N : S \ni i} \psi(S) = 1$ for any $i \in N$, condition (see [23,24])*

$$\sum_{S \in 2^N \backslash \varnothing} \psi(S) v(\omega, S) \leq v(\omega, N) \tag{9}$$

holds. Characteristic function $v(\omega, S)$ is defined by (7). We refer to the book [25] for further discussion of non-emptiness of the core.

Second, for any coalition $S \subseteq N$ define maximal value of characteristic function (7) over set Ω:

$$\hat{w}(S) = \max_{\omega \in \Omega} v(\omega, S), \tag{10}$$

which is the maximal value that coalition S can obtain in state games.

The next step is to define the approximated value of the characteristic function for any state in the following way. Let for any state $\omega \in \Omega$ the approximated characteristic function $w(\omega, S)$ be given as

$$w(\omega, S) = \begin{cases} \sum\limits_{i \in S} K_i^{\omega}(a^{\omega *}), & \text{if } S = N, \\ \hat{w}(S), & \text{if } S \neq N. \end{cases} \tag{11}$$

In Equation (11), the summarized payoff of the players adopting cooperative action profile $a^{\omega *}$ is assigned to the grand coalition. The approximated (maximal possible value over all possible states) values of characteristic function $\hat{w}(S)$ given by (10) are assigned to any coalition S different from N. Denote the core constructed with the values of characteristic function (11) as $D(\omega)$ and assume that it is non-empty for any state ω,

$$D(\omega) = \Big\{(\alpha_1(\omega), \ldots, \alpha_n(\omega)) : \sum_{i \in S} \alpha_i(\omega) \geqslant w(\omega, S), \forall S \subset N, \sum_{i \in N} \alpha_i(\omega) = w(\omega, N)\Big\}. \tag{12}$$

Lemma 1. *Let for any coalition $S \subset N$, $S \neq N$, the inequality $\hat{w}(S) < \min_{\omega \in \Omega} v(\omega, N)$ hold. If condition*

$$\sum_{i \in N} K_i^{\omega}(a^{\omega *}) = \max_{a^{\omega} \in \prod_{j \in N} A_j^{\omega}} \sum_{i \in N} K_i^{\omega}(a^{\omega}), \tag{13}$$

is true, and the core $D(\omega)$ is non-empty for any ω, and then $D(\omega) \subset C(\omega)$.

Proof. If there exists coalition $S \subset N$, $S \neq N$, such that $\hat{w}(S) \geqslant \min_{\omega \in \Omega} v(\omega, N)$, then the core $D(\omega)$ is empty. Assuming the non-emptiness of the core $D(\omega)$, we consider any imputation $\alpha(\omega) \in D(\omega)$. If condition (13) is true, it means that $\sum_{i \in N} \alpha_i(\omega) = v(\omega, N) = w(\omega, N)$.

Subsequently, for any coalition $S \subset N$, we have $\sum_{i \in S} \alpha_i(\omega) \geqslant w(\omega, S) = \hat{w}(S) = \max_{\omega \in \Omega} v(\omega, S) \geqslant v(\omega, S)$, which proves that $\alpha(\omega) \in C(\omega)$. □

Remark 2. *Condition (13) states that the maximal total payoff of the players in state ω coincides with their payoff if players adopt actions prescribed by the cooperative strategy profile. It may not be satisfied in general case in dynamic games. If condition (13) is not true, the main result of the paper can be proved, but it requires a modification in the method of characteristic function definition. We leave this case for future research.*

Remark 3. *We assume that the approximated core $D(\omega)$ is non-empty for any ω. The conditions under which it is non-empty are similar to the ones given in Remark 1, but in Equation (9) characteristic function $w(\omega, S)$ given by (11) is used. If the conditions of Lemma 1 are satisfied, then $D(\omega) \subset C(\omega)$, and non-emptiness of approximated core $D(\omega)$ implies non-emptiness of core $C(\omega)$.*

Example 1. *Consider three-player stochastic game with two states (ω_1 and ω_2). The sets of actions of player 1, 2, and 3 in state ω_1 (ω_2) are $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$ ($\{\alpha_1, \alpha_2\}$, $\{\zeta_1, \zeta_2\}$, $\{\gamma_1, \gamma_2\}$), respectively. The payoff functions are given by the following matrices:*

- *in state ω_1:*

$$c_1: \quad \begin{array}{cc} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} (10,10,8) & (0,15,0) \\ (15,0,0) & (5,5,5) \end{pmatrix} \end{array} \qquad c_2: \quad \begin{array}{cc} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} (0,0,15) & (2,4,4) \\ (4,4,2) & (0,0,0) \end{pmatrix} \end{array}$$

- *in state ω_2:*

$$\gamma_1: \quad \begin{array}{cc} & \begin{array}{cc} \zeta_1 & \zeta_2 \end{array} \\ \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} & \begin{pmatrix} (2,1,1) & (4,0,2) \\ (0,4,2) & (7,5,3) \end{pmatrix} \end{array} \qquad \gamma_2: \quad \begin{array}{cc} & \begin{array}{cc} \zeta_1 & \zeta_2 \end{array} \\ \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} & \begin{pmatrix} (2,3,0) & (4,2,4) \\ (3,4,3) & (7,5,7) \end{pmatrix} \end{array}$$

Player 1 chooses a row, player 2 chooses a column and player 3 chooses a matrix.

The transition probabilities are written in the matrices:

- *for state ω_1:*

$$c_1: \quad \begin{array}{cc} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} (0.5,0.5) & (0,1) \\ (0,1) & (0,1) \end{pmatrix} \end{array} \qquad c_2: \quad \begin{array}{cc} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} (0,1) & (0.5,0.5) \\ (0.5,0.5) & (1,0) \end{pmatrix} \end{array}$$

- *for state ω_2:*

$$\gamma_1 : \quad \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{cc} \zeta_1 & \zeta_2 \\ \multicolumn{2}{c}{\begin{pmatrix} (0,1) & (1,0) \\ (1,0) & (1,0) \end{pmatrix}} \end{array} \qquad \gamma_2 : \quad \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{cc} \zeta_1 & \zeta_2 \\ \multicolumn{2}{c}{\begin{pmatrix} (0.2, 0.8) & (0,1) \\ (0,1) & (1,0) \end{pmatrix}} \end{array}$$

The first (second) element in any entry of the matrix is the probability of transition from the particular state and action profile to state ω_1 (state ω_2). One can easily notice that the probabilistic transitions are defined in state ω_1 when players choose action profiles (a_1, b_1, c_1), (a_2, b_1, c_2) and (a_1, b_2, c_2), and in state ω_2 when players choose action profiles $(\alpha_1, \zeta_1, \gamma_2)$. All other transitions are deterministic.

The discount factor equals 0.9.
Cooperative strategy profile $\eta^* = (\eta_1^*, \eta_2^*, \eta_3^*)$ is such that

$$\eta_1^* = (a_1, \alpha_2), \quad \eta_2^* = (b_1, \zeta_2), \quad \eta_2^* = (c_1, \gamma_2), \tag{14}$$

which prescribes any player to choose the first action in state ω_1 and the second action in state ω_2. The cooperative strategy profile defines a Markov chain with the structure that is depicted in Figure 1.

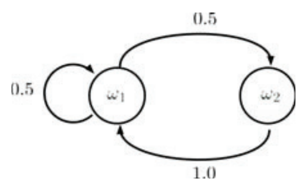

Figure 1. The transition probabilities defined by cooperative strategy profile η^*.

The players' payoffs are $(10, 10, 8)$ in state ω_1 and $(7, 5, 7)$ in state ω_2. We obtain that the maximal total payoff of the players in state games coincide with the payoff that players get in states implementing cooperative strategy profile η^*. However, Theorem 1 is also true for the case when this condition is not satisfied.

First, we calculate the characteristic function $v(\omega, S)$ by Equation (7) and its approximation $w(\omega, S)$ by (11) for state games. The values of these functions are represented in Table 1.

Table 1. Values of characteristic function v and approximated characteristic function w for states ω_1 and ω_2.

S	$\{1,2,3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1\}$	$\{2\}$	$\{3\}$
$v(\omega_1, S)$	28	8	10	10	0	0	0
$v(\omega_2, S)$	19	12	6	8	2	1	1
$w(\omega_1, S)$	28	12	10	10	2	1	1
$w(\omega_2, S)$	19	12	10	10	2	1	1

The cores of state games $C(\omega)$ and $D(\omega)$ calculated with values of functions $v(\omega, S)$ and $w(\omega, S)$ by Formulae (8) and (12) are non-empty for any ω and represented on Figures 2 and 3 for ω_1 and ω_2 respectively.

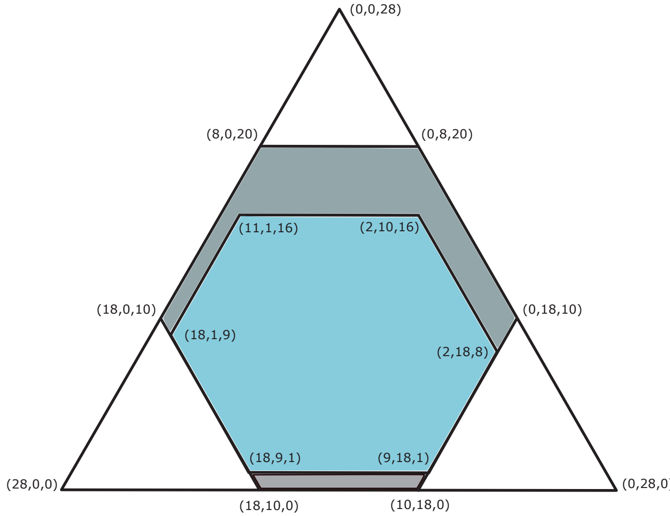

Figure 2. The core $C(\omega_1)$ (gray region) and approximated core $D(\omega_1)$ (blue region inside gray region) for ω_1 state game.

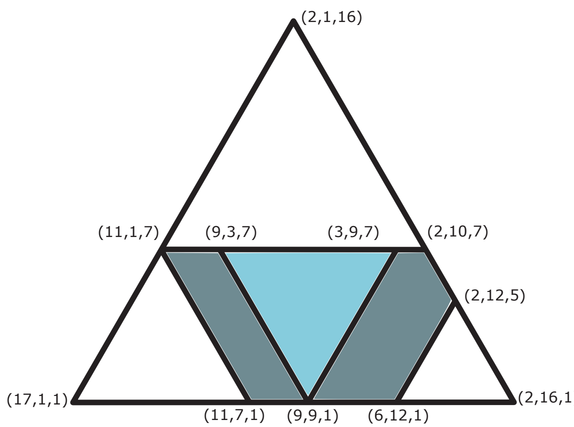

Figure 3. The core $C(\omega_2)$ (gray region) and approximated core $D(\omega_2)$ (blue region inside gray region) for ω_2 state game.

2.3. New Approximated Characteristic Function for Stochastic Games

We propose a new method of determining characteristic function for stochastic games based on the values of approximated characteristic function defined in states and given by Formula (11).

We assume that coalition S at any state of the game may obtain $\hat{w}(S)$ as maximum. Accordingly, this value is the maximal value that the coalition can get, regardless of the state that currently appears. If we summarize this value over infinite horizon with discount factor δ, we can calculate the approximation or the upper bound of the payoff that coalition S can get in stochastic subgame starting from state ω, which is

$$\bar{w}(\omega, S) = \begin{cases} \hat{w}(S) + \delta\hat{w}(S) + \ldots = \dfrac{1}{1-\delta}\hat{w}(S), & \text{if } S \subset N, S \neq N, \\ \sum\limits_{i \in N} E_i^{\omega}(\eta^*), & \text{if } S = N. \end{cases} \tag{15}$$

One should notice that, according to Equation (15), we save the value of characteristic function for grand coalition without approximation. The reason is that, when we define the allocation of a joint payoff, the players should redistribute the value that they obtain using cooperative strategy profile, but not the approximated one. The cooperative stochastic subgame is defined by the set of players N and function (15). In the following, we omit the set of players and refer the cooperative stochastic subgame as $\varpi(\omega, S)$ given by (15).

Let $\bar{D}(\omega)$ be the core calculated with the values of function (15), i.e.,

$$\bar{D}(\omega) = \big\{(\alpha_1(\omega), \ldots, \alpha_n(\omega)) : \sum_{i \in S} \alpha_i(\omega) \geqslant \varpi(\omega, S), \forall S \subset N, \ \sum_{i \in N} \alpha_i(\omega) = \varpi(\omega, N)\big\}. \tag{16}$$

Let $\bar{D}(\omega)$ be non-empty for any $\omega \in \Omega$. We can compare the core $\bar{D}(\omega)$ constructed with the values of approximated function (15) and the core defined with the values of characteristic function defined with the classical approach. For any subgame G^ω, we define characteristic function using the maxmin approach:

$$\bar{v}(\omega, S) = \max_{\eta_S \in \prod\limits_{j \in S} H_j} \ \min_{\eta_{N \setminus S} \in \prod\limits_{j \in N \setminus S} H_j} \sum_{i \in S} E_i^\omega(\eta_S, \eta_{N \setminus S}). \tag{17}$$

Let $\bar{C}(\omega)$ be a non-empty core of subgame G^ω constructed with the values of function (17).

Lemma 2. *Let for any coalition $S \subset N$, $S \neq N$ the inequality*

$$\hat{w}(S) < \min_{\omega \in \Omega} v(\omega, N) \tag{18}$$

hold, and $\bar{D}(\omega)$ is non-empty for any ω, then $\bar{D}(\omega) \subset \bar{C}(\omega)$.

Proof. If $\hat{w}(S) < \min\limits_{\omega \in \Omega} v(\omega, N)$ is not satisfied, then the core $\bar{D}(\omega)$ is empty by construction. Consider any imputation $\bar{\alpha}(\omega) \in \bar{D}(\omega)$ and prove that it belongs to the set $\bar{C}(\omega)$.

First, $\sum\limits_{i \in N} \bar{\alpha}_i(\omega) = \varpi(\omega, N) = \sum\limits_{i \in N} E_i^\omega(\eta^*) = \bar{v}(\omega, N)$.

Second, we prove that $\sum\limits_{i \in S} \bar{\alpha}_i(\omega) \geqslant \bar{v}(\omega, S)$ taking into account that $\sum\limits_{i \in S} \bar{\alpha}_i(\omega) \geqslant \varpi(\omega, S)$ for any $S \neq N$. We prove that $\varpi(\omega, S) \geqslant \bar{v}(\omega, S)$.

By definition, we have

$$\bar{v}(\omega, S) = \max_{\eta_S} \ \min_{\eta_{N \setminus S}} \sum_{i \in S} E_i^\omega(\eta_S, \eta_{N \setminus S}),$$

and we write the functional equation for the right-hand side of this equality and obtain the following

$$\max_{\eta_S} \ \min_{\eta_{N \setminus S}} \sum_{i \in S} E_i^\omega(\eta_S, \eta_{N \setminus S}) = \max_{\eta_S} \ \min_{\eta_{N \setminus S}} \Big\{ \sum_{i \in S} K_i^\omega(a_S^\omega, a_{N \setminus S}^\omega) + \delta p(\omega, a^\omega) \sum_{i \in S} E_i(\eta_S, \eta_{N \setminus S}) \Big\},$$

where $p(\omega, a^\omega)$ is a vector $\big(p(\omega' | \omega, a^\omega) : \omega' \in \Omega\big)$.

Let profile $(\eta_S, \eta_{N \setminus S})$ be such that maxmin is reached at this profile, we can write the functional equation, as follows:

$$\begin{aligned} \sum_{i \in S} E_i^\omega(\eta_S, \eta_{N \setminus S}) &= (\mathbb{I}_k - \delta \Pi(\eta_S, \eta_{N \setminus S}))^{-1} \sum_{i \in S} K_i(a_S, a_{N \setminus S}) \leqslant \frac{1}{1 - \delta} \max_{\omega \in \Omega} \max_{a_S} \ \min_{a_{N \setminus S}} \sum_{i \in S} K_i^\omega(a_S^\omega, a_{N \setminus S}^\omega) \\ &= \frac{1}{1 - \delta} \max_{\omega \in \Omega} v(\omega, S) = \varpi(\omega, S). \end{aligned}$$

In the last inequality, we use the property of stochastic matrices, i.e., the sum of the elements in any row of matrix $(\mathbb{I}_k - \delta \Pi(\eta_S, \eta_{N \setminus S}))^{-1}$ equal $1/(1 - \delta)$, because $\Pi(\eta_S, \eta_{N \setminus S})$ is a stochastic matrix. The lemma is proved. $\square$

Remark 4. *We assume non-emptiness of the approximated core $\bar{D}(\omega)$ in stochastic game with any initial state ω. If condition (18) in Lemma 2 is satisfied, the non-emptiness of the approximated cores $D(\omega)$ for any ω implies the non-emptiness of approximated core $\bar{D}(\omega)$. It follows from definition of characteristic function $\bar{w}(\omega, S)$ and formula (10). Moreover, the non-emptiness of approximated core $\bar{D}(\omega)$ implies non-emptiness of core $\bar{C}(\omega)$.*

Example 2. *(continuation of Example 1) We continue calculations for stochastic game described in Example 1. Define characteristic function $\bar{v}$ by (17) and approximated characteristic function $\bar{w}$ by (15). The values of these functions are given in Table 2.*

Table 2. Values of characteristic function $\bar{v}$ and approximated characteristic function $\bar{w}$ for stochastic game starting from states ω_1 and ω_2.

S	$\{1,2,3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1\}$	$\{2\}$	$\{3\}$
$\bar{v}(\omega_1, S)$	252.07	92.41	62.01	64.00	9.47	9.00	9.00
$\bar{v}(\omega_2, S)$	245.86	95.17	77.87	60.00	10.52	10.00	10.00
$\bar{w}(\omega_1, S)$	252.07	120.00	100.00	100.00	20.00	10.00	10.00
$\bar{w}(\omega_2, S)$	245.86	120.00	100.00	100.00	20.00	10.00	10.00

The cores $\bar{C}(\omega)$ and $\bar{D}(\omega)$ constructed with the values of functions $\bar{v}$ and $\bar{w}$, respectively, are non-empty and depicted on Figures 4 and 5 for initial states ω_1 and ω_2, respectively. One can notice that $\bar{D}(\omega) \subset \bar{C}(\omega)$ for any ω.

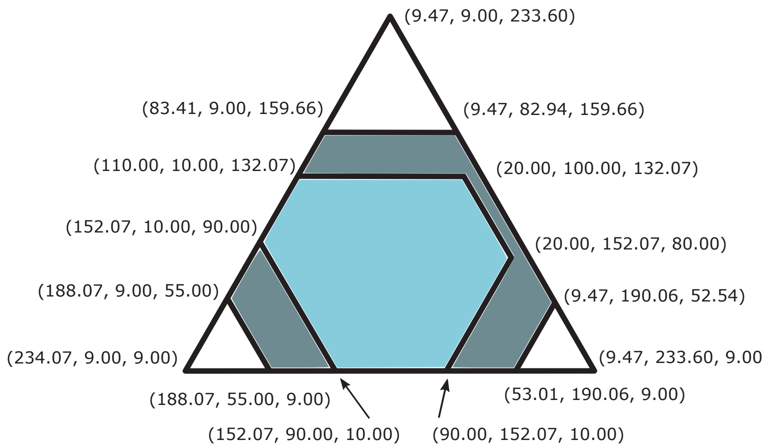

Figure 4. The core $\bar{C}(\omega)$ (gray region) and approximated core $\bar{D}(\omega)$ (blue region inside gray region) in stochastic game with ω_1 initial state.

The approximated core $\bar{D}(\omega_1)$ is defined as the set

$$\bar{D}(\omega_1) = \Big\{(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) : \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 = 252.07, \bar{\alpha}_1 + \bar{\alpha}_2 \geqslant 120.00, \bar{\alpha}_1 + \bar{\alpha}_3 \geqslant 100.00, \bar{\alpha}_2 + \bar{\alpha}_3 \geqslant 100.00, \bar{\alpha}_1 \geqslant 20.00, \bar{\alpha}_2 \geqslant 10.00, \bar{\alpha}_3 \geqslant 10.00\Big\}.$$

The approximated core $\bar{D}(\omega_2)$ is defined as the set

$$\bar{D}(\omega_2) = \Big\{(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) : \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 = 245.86, \bar{\alpha}_1 + \bar{\alpha}_2 \geqslant 120.00, \bar{\alpha}_1 + \bar{\alpha}_3 \geqslant 100.00, \bar{\alpha}_2 + \bar{\alpha}_3 \geqslant 100.00, \bar{\alpha}_1 \geqslant 20.00, \bar{\alpha}_2 \geqslant 10.00, \bar{\alpha}_3 \geqslant 10.00\Big\}.$$

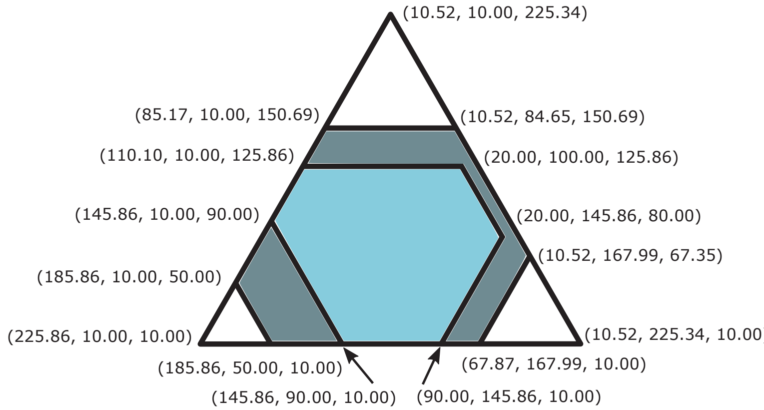

Figure 5. The core $\bar{C}(\omega_2)$ (gray region) and approximated core $\bar{D}(\omega_2)$ (blue region inside gray region) in stochastic game with ω_2 initial state.

3. Strongly Subgame-Consistent Core in Stochastic Games

3.1. Imputation Distribution Procedure

In cooperation, players follow the cooperative strategy profile η^* and then agree on the core as a cooperative solution of the game or the set of possible imputations of the joint payoff in the game. We assume that the core for any subgame G^ω is calculated based on function (15), which is $\bar{D}(\omega)$. Consider an imputation $\bar{\alpha}(\omega) \in \bar{D}(\omega)$. Obviously, if the players are paid step by step according to initially given payoff functions K_i^ω, $i \in N$, we cannot guarantee that they will get the components of imputation $\bar{\alpha}(\omega)$ as an expected payoff in subgame G^ω. Therefore, we define the scheme of state payments that, in total, will give the players to obtain the components of imputation $\bar{\alpha}(\omega)$.

Definition 1. *[10,11] We call the collection of vectors $(\beta_i : i \in N)$, where $\beta_i = (\beta_i(\omega_1), \ldots, \beta_i(\omega_k))$, $\beta_i(\omega)$ is a payment to player i in state ω in cooperative stochastic game, an imputation distribution procedure (IDP) of imputation $\bar{\alpha}(\omega) \in \bar{D}(\omega)$ if*

1. $\sum\limits_{i \in N} \beta_i(\omega) = \sum\limits_{i \in N} K_i^\omega(a^{\omega *})$ *for any $\omega \in \Omega$;*
2. $\bar{\alpha}_i(\omega) = B_i^\omega$, *where B_i^ω is the expected discounted sum of payments to player i in stochastic subgame starting from state ω, according to procedure β.*

The expected sum of payments to player i made according to IDP can be calculated by formula (see [14]):

$$B_i^\omega = \pi_0(\mathbb{I} - \delta\Pi(\eta^*))^{-1}\beta_i,$$

where π_0 is such that $\pi_0^\omega = 1$ and $\pi_0^{\omega'} = 0$ for any $\omega' \neq \omega$.

Remark 5. *The IDP determined in Definition 1 for an imputation $\bar{\alpha}(\omega) \in \bar{D}(\omega)$ may be non-unique.*

In the following section, we describe a property of the imputations from the core and corresponding IDP, which allows to narrow the set of IDP.

3.2. Strongly Subgame-Consistent Core

We formulate the property of strongly subgame consistency of the core and propose sufficient conditions of strongly subgame consistency of the core in stochastic games with characteristic function (15). We suppose that the cores of stochastic game G and any subgame G^ω, $\omega \in \Omega$, are non-empty.

In cooperation, players agree on the joint implementation of cooperative strategy profile η^* and expect to obtain the components of the imputation belonging to the core $\bar{D}(\omega)$ in the subgame stating from ω. Reaching an intermediate state $\omega \in \Omega$, Player i chooses action $a_i^{\omega*}$ prescribed by cooperative strategy profile η^* and gets payoff $K_i^{\omega}(a^{\omega*})$. If the players recalculate the solution in the current subgame and find solution of cooperative subgame G^{ω}, we would assume that the cooperative solution is chosen from the core $\bar{D}(\omega)$. It would be reasonable to require that the payoff received by a player in state ω summarized with the expected sum of any imputations from the cores $\bar{D}(\omega')$, $\omega' \in \Omega$, following state ω, would be an imputation from the core $\bar{D}(\omega)$. If this property holds for any intermediate state $\omega \in \Omega$, then the core of cooperative stochastic game with characteristic function (15) is strongly subgame-consistent.

To determine a strongly subgame-consistent core, we need to define the so-called expected core at state ω, i.e., we define the set of expected imputations belonging to the cores, which are cooperative solutions of the following subgames. We determine the expected core of state $\omega \in \Omega$, as follows:

$$E\bar{D}(\omega) = \Big\{\delta \sum_{\omega \in \Omega} p(\omega'|\omega, a^{\omega*})\bar{\alpha}(\omega'), \quad \bar{\alpha}(\omega') \in \bar{D}(\omega')\Big\}.$$

Definition 2. *We call the core $\bar{D}(\omega)$ strongly subgame consistent solution of cooperative stochastic game with approximated characteristic function $\bar{w}(\omega, S)$ starting from state ω if for any imputation $\bar{\alpha}(\omega) \in \bar{D}(\omega)$ there exists an IDP $\beta = (\beta_i : i \in N)$, where $\beta_i = (\beta_i(\omega) : \omega \in \Omega)$, satisfying condition:*

$$\beta \oplus E\bar{D} \subset \bar{D}, \tag{19}$$

where $E\bar{D}$ is the vector $(E\bar{D}(\omega_1), \ldots, E\bar{D}(\omega_k))'$ of expected cores for states $\omega_1, \ldots, \omega_k$ respectively, $\bar{D}$ is a vector with elements which are sets, i.e., $\bar{D} = (\bar{D}(\omega_1), \ldots, \bar{D}(\omega_k))'$.

Remark 6. *The inclusion (19) is written in a vector form. To explain it, we write the first row of vector inclusion (19):*

$$\beta(\omega_1) \oplus E\bar{D}(\omega_1) \subset \bar{D}(\omega_1)$$

where $\beta(\omega_1) \in \mathbb{R}^n$, $E\bar{D}(\omega_1) \subset \mathbb{R}^n$, $\bar{D}(\omega_1) \subset \mathbb{R}^n$. The operation $a \oplus C$, where $a \in \mathbb{R}^n$ and C is a set in $\mathbb{R}^n$, is defined as the set $\{a + c$, for all $c \in C\}$.

Theorem 1. *The core $\bar{D}(\omega)$, if it exists, is strongly subgame-consistent.*

Proof. Following Definition 2 we need to prove that there exists an IDP of the elements from the core $\bar{D}(\omega)$ defined in (16) satisfying two properties from Definition 1, such that inclusion (19) is true.

Let for any imputation $\bar{\alpha}_i(\omega) \in \bar{D}(\omega)$, the IDP is calculated as

$$\beta_i = (\mathbb{I}_k - \delta\Pi(\eta^*))\bar{\alpha}_i, \tag{20}$$

where $\beta_i = (\beta_i(\omega_1), \ldots, \beta_i(\omega_k))'$ and $\bar{\alpha}_i = (\bar{\alpha}_i(\omega_1), \ldots, \bar{\alpha}_i(\omega_k))'$.

First, we prove that β, defined in (20), satisfies properties 1 and 2 in Definition 1.

1. Find the sum of β_i over the set of players, we obtain

$$\sum_{i \in N} \beta_i = (\mathbb{I}_k - \delta\Pi(\eta^*)) \sum_{i \in N} \bar{\alpha}_i = (\mathbb{I}_k - \delta\Pi(\eta^*))(\bar{v}(\omega_1, N), \ldots, \bar{v}(\omega_k, N))'$$
$$= (\mathbb{I}_k - \delta\Pi(\eta^*))(\mathbb{I}_k - \delta\Pi(\eta^*))^{-1} \sum_{i \in N} K_i(a^*) = \sum_{i \in N} K_i(a^*),$$

or for any $\omega \in \Omega$ the equality $\sum_{i \in N} \beta_i(\omega) = \sum_{i \in N} K_i^{\omega}(a^{\omega*})$ is true.

2. We prove that $\bar{\alpha}_i(\omega) = B_i^{\omega}$ or in vector form $\bar{\alpha}_i = B_i$, where $B_i = (B_i(\omega_1), \ldots, B_i(\omega_k))'$. We have

$$B_i = (\mathbb{I}_k - \delta\Pi(\eta^*))^{-1}\beta_i = (\mathbb{I}_k - \delta\Pi(\eta^*))^{-1}(\mathbb{I}_k - \delta\Pi(\eta^*))\bar{\alpha}_i = \bar{\alpha}_i.$$

Therefore, the payment vector β_i, $i \in N$, is the distribution procedure of imputation $\bar{\alpha}_i$.

Now, we prove that inclusion (19) holds. Let β_i be given by Equation (20), there $\bar{\alpha}_i = (\bar{\alpha}_i(\omega_1), \ldots, \bar{\alpha}_i(\omega_k))'$ and $\bar{\alpha}_i(\omega_j) \in \bar{D}(\omega_j)$ for any $j = 1, \ldots, k$. Consider the sum $\beta(\omega) + \varepsilon(\omega)$, where $\varepsilon(\omega)$ is any vector from the expected core $E\bar{D}(\omega)$. Substituting expressions of β_i from Equation (20) and element of the expected core into the sum, we get

$$\beta + \varepsilon = (\mathbb{I}_k - \delta\Pi(\eta^*))\bar{\alpha} + \delta\Pi(\eta^*)\bar{\alpha} = \bar{\alpha} \in \bar{D},$$

which proves the theorem. □

Theorem 1 gives the method of construction of payment scheme of any element from the core $\bar{D}$ defined by (16) while using values of function (15).

Example 3. *(continuation of Example 1 and 2) We demonstrate how to define IDP using a method from the proof of Theorem 1. Let for ω_1 and ω_2 the core imputations $\bar{\alpha}(\omega_1) = (100.00, 100.00, 52.07) \in \bar{D}(\omega_1)$ and $\bar{\alpha}(\omega_2) = (50.00, 95.86, 100.00) \in \bar{D}(\omega_2)$ be chosen. To calculate IDP by Formula (20), we need to define matrix $\Pi(\eta^*)$, which is*

$$\Pi(\eta^*) = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 1 \end{pmatrix}$$

for cooperative strategy profile η^* determined by (14).

Using formula (20) with $\bar{\alpha}_1 = (100.00, 50.00)$, $\bar{\alpha}_2 = (100.00, 95.86)$, $\bar{\alpha}_3 = (52.07, 100.00)$, we obtain

$$\begin{aligned} \beta_1 &= (32.50, -40.00), \\ \beta_2 &= (11.86, 5.86), \\ \beta_3 &= (-16.36, 53.14), \end{aligned}$$

where the first component of vector β_i is the payment to player i in state ω_1 and the second component is the payment in state ω_2. We can easily check that collection of vectors $(\beta_i : i \in N)$ satisfies conditions from Definition 1 of IDP.

The approximated cores $\bar{D}(\omega_1)$ and $\bar{D}(\omega_2)$ are strongly subgame-consistent, which is proved in Theorem 1.

Remark 7. *The new method of construction of the characteristic function or the so-called approximated characteristic function proposed in the paper allows not only to find the strongly subgame-consistent subset of the core, but also simplifies calculations. In the example, each player has two actions in any state. Therefore, he has four pure stationary strategies in a stochastic game, and there are 64 strategy profiles in the game. The calculations of maxmin payoff of a coalition in such games is a complicated computational problem. The new approach allows for avoiding these calculations using the values of approximated characteristic function defined in state games to determine the function for a stochastic game.*

4. Conclusions

We have proposed a new method of constructing the characteristic function in stochastic games. The method simplifies calculations in comparison with the previously introduced approaches. An additional advantage of the method is that the core calculated with the values of this characteristic function satisfies strongly subgame consistency. This property positively characterizes the realization of the imputations from the core in a dynamic game process. The property of strongly subgame consistency is applied for set-valued cooperative solutions, like the core. We can briefly characterize the possible directions for future research in this area. We can also consider additional simplifications in characteristic function definitions, which allow not only to keep the strong subgame consistency properties of the core, but also to reduce the number of calculations defining cooperative stochastic game.

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Article

Maximizing the Minimal Satisfaction—Characterizations of Two Proportional Values

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Abstract: A class of solutions are introduced by lexicographically minimizing the complaint of coalitions for cooperative games with transferable utility. Among them, the nucleolus is an important representative. From the perspective of measuring the satisfaction of coalitions with respect to a payoff vector, we define a family of optimal satisfaction values in this paper. The proportional division value and the proportional allocation of non-separable contribution value are then obtained by lexicographically maximizing two types of satisfaction criteria, respectively, which are defined by the lower bound and the upper bound of the core from the viewpoint of optimism and pessimism respectively. Correspondingly, we characterize these two proportional values by introducing the equal minimal satisfaction property and the associated consistency property. Furthermore, we analyze the duality of these axioms and propose more approaches to characterize these two values on basis of the dual axioms.

Keywords: cooperative game; satisfaction criteria; proportional value; axiomatization

1. Introduction

In the process of economic globalization, multinational corporations usually reach a cooperative agreement and form a cooperative coalition in order to gain more benefits. It is a central problem of how to allocate the overall profits of cooperation among these multinational corporations. Cooperative game theory provides general mathematical methods to solve the allocation problems. The solution concepts, such as the Shapley value [1] and the nucleolus [2], offer concrete schemes of allocating the overall profits among players.

The nucleolus, introduced by Schmeidler [2], is a classical solution concept of cooperative games. The nucleolus is obtained by lexicographically minimizing the maximal excess of coalition over the non-empty imputation set. Here, the excess is an important criterion to describe the dissatisfaction with respect to the payoff vector. Thus, a positive excess of a coalition with respect to a payoff vector represents the loss that the coalition suffers from the payoff vector. Several central solutions of cooperative games are defined according to the idea of excess, for example, the nucleolus [2], the core, the kernel [3], and the τ value [4]. In particular, the core is the set of all payoff vectors with non-positive excesses for all coalitions. Besides the excess criterion, Hou et al. [5] proposed two other criteria to measure the dissatisfaction of coalition with respect to a payoff vector.

On the contrary, the satisfaction is a significant criterion to measure the preference degree of coalitions for a payoff vector. Thus, from the perspective of the satisfaction, we define a family of optimal satisfaction values in this paper. Two special optimal satisfaction values are given in terms of the optimistic satisfaction and the pessimistic satisfaction respectively. For a cooperative game with transferable utility (for short, TU-game), the individual worth vector is the lower bound of the core while the marginal contribution vector is the upper bound of the core. Thus, the individual worth vector and the marginal contribution vector can be viewed as the least potential payoff vector and the ideal payoff vector respectively. There are two representative biases in social comparisons [6], a comparative optimism bias (i.e., a tendency for people to evaluate themselves in a more positive light) and a comparative pessimism bias (i.e., a tendency for people to evaluate themselves in a more negative light). The optimistic satisfaction and the pessimistic satisfaction are defined by the individual worth vector and the marginal contribution vector from the viewpoints of optimism and pessimism respectively. On the optimistic side, players always take the individual worth of themselves into consideration and think of the ratio between the real payoff and the individual worth as the measure of satisfaction. The optimistic satisfaction of a coalition is defined by the ratio between the real payoff of the coalition and the sum of their individual worths with respect to a payoff vector. Conversely, pessimists always take the ideal payoff of themselves into consideration. The pessimistic satisfaction of a coalition is the ratio between the real payoff of the coalition and the sum of the marginal contributions of players in the coalition. Thus, the optimistic optimal satisfaction value and the pessimistic optimal satisfaction value are determined by maximizing the minimal optimistic satisfaction and the minimal pessimistic satisfaction in the lexicographic order over the non-empty pre-imputation set, respectively. Interestingly, the two values are coincident with the proportional division value (PD value) and the proportional allocation of non-separable contribution value (PANSC value), respectively.

The proportional principle is a relatively fair and reasonable allocation criterion in many economic situations. It is a norm of distributed justice rooted in law and custom [7]. Moulin's survey [8] of cost and surplus sharing opens by emphasizing the importance of the proportional principle. The PD value and the PANSC value are defined based on the idea of proportionality. The PD value, introduced by Banker [9], distributes the overall worth of the grand coalition in proportion to player's individual worth among all players. As the dual value of the PD value, the PANSC value distributes the overall worth in proportion to their marginal contributions with respect to the grand coalition. Moreover, some other proportional values have been studied in the literature, such as the proper Shapley value [10,11], the proportional value [12,13], and the proportional Shapley value [14,15]. In this paper, we mainly study the PD value and the PANSC value and propose several new axiomatizations of the PD value and the PANSC value.

Axiomatization is one of the main ways to characterize the reasonability of solutions in cooperative games. For the PD value, Zou et al. [16] proposed several characterizations on the basis of the equal treatment of equals, monotonicity and reduced game consistency. In this paper, we first propose the equal minimal optimistic satisfaction property and equal minimal pessimistic satisfaction property, which are inspired by the kernel concept [17]. The equal minimal optimistic satisfaction property states that for a pair of players $\{i, j\}$ and a payoff vector x, the minimal optimistic satisfaction of coalitions containing i and not j with respect to x should equal that of coalitions containing j and not i under the optimistic satisfaction criterion, while the equal minimal pessimistic satisfaction property describe this situation under the pessimistic satisfaction criterion. Then, the PD value and the PANSC value are characterized by these two properties with efficiency, respectively.

Associated consistency is also an important characteristic of solutions for TU-games. A solution satisfies associated consistency if it allocates the same payoff to players in the associated game as that in the initial game. The concept of associated consistency was firstly introduced by Hamiache [18] to characterize the Shapley value. Driessen [19] characterized the family of efficient, symmetric, and linear values by associated consistency on the basis of Hamiache's axiomatization system. Associated consistency is quite popular in the literature on the axiomatization of solutions for TU-games, for instance, the EANS value and the CIS value [20], linear and symmetric values [21] and the core [22]. We propose two associated consistency properties, optimistic associated consistency and pessimistic associated consistency, to characterize the PD value and the PANSC value in this paper. Furthermore, we also study the dual axioms of the two associated consistency properties and propose more approaches to characterize these two values.

This paper is organized as follows—in Section 2, some basic definitions and notation are introduced. We determine the PD value and the PANSC value by maximizing the minimal optimistic satisfaction and the minimal pessimistic satisfaction in the lexicographic order in Section 3. In Section 4, we propose two types of axioms, the equal minimal satisfaction property and the associated consistency property, to characterize the PD value and the PANSC value, and analyze the dual axioms of associated consistency. Finally, we give a brief conclusion in Section 5.

2. Preliminaries

Let $\mathcal{U} \subseteq \mathbb{N}$ be the set of potential players, where $\mathbb{N}$ is the set of natural numbers. A cooperative game with transferable utility or simply a TU-game is a pair $\langle N, v\rangle$, where $N \subseteq \mathcal{U}$ is a finite set of n players and $v : 2^N \to \mathbb{R}$ is a characteristic function that assigns to each coalition $S \in 2^N$, the worth $v(S)$ with $v(\emptyset) = 0$. Denote the set of all TU-games on player set N by $\mathcal{G}^N$. Hereafter, a TU-game $\langle N, v\rangle$ is simply denoted by v, the cardinality of a finite set S is denoted by s, and the set of all non-empty coalitions is denoted by Ω.

A TU-game $v \in \mathcal{G}^N$ is individually positive (or negative) if $v(\{i\}) > 0$ (or $v(\{i\}) < 0$) for all $i \in N$. Denote the set of all individually positive (or negative) TU-games on player set N by $\mathcal{G}^N_+$ (or $\mathcal{G}^N_-$). Without ambiguity, let $b^v(\{i\}) \equiv v(N) - v(N\backslash\{i\})$ be the marginal contribution of player i with respect to the grand coalition N. For all $v \in \mathcal{G}^N$ and $S \in \Omega$, let $b^v(S) \equiv \sum_{i\in S} b^v(\{i\})$. A TU-game $v \in \mathcal{G}^N$ is marginally positive (or negative) if $b^v(\{i\}) > 0$ (or $b^v(\{i\}) < 0$) for all $i \in N$. Denote the set of all marginally positive (or negative) TU-games on player set N by $\mathcal{G}^N_\oplus$ (or $\mathcal{G}^N_\ominus$). For convenience, we focus on the family of all individually positive TU-games $\mathcal{G}^N_+$ and the family of all marginally positive TU-games $\mathcal{G}^N_\oplus$ in the rest of this paper.

For any TU-game $v \in \mathcal{G}^N$, its dual game v^d is given as follows, for all $S \subseteq N$,

$$v^d(S) \equiv v(N) - v(N\backslash S), \tag{1}$$

where $v^d(S)$ represents the marginal worth of coalition S with respect to N. Obviously, the dual of a individually positive (or negative) TU-game is marginally positive (or negative). Thus, the duality operator is not closed on the class of individually positive (or negative) TU-games. Given any $\mathcal{A} \subseteq \mathcal{G}^N$, let $\mathcal{A}^d$ be the set of dual of TU-games in $\mathcal{A}$.

A payoff vector for a TU-game $v \in \mathcal{G}^N$ is an n-dimensional vector $x \in \mathbb{R}^n$ assigning a payoff $x_i \in \mathbb{R}$ to every player $i \in N$. Let $x(S) = \sum_{i\in S} x_i$ for all $S \in \Omega$. A payoff vector x satisfies efficiency if $x(N) = v(N)$ for all $v \in \mathcal{G}^N$, satisfies individual rationality if $x_i \geq v(\{i\})$ for all $v \in \mathcal{G}^N$ and $i \in N$, and satisfies group rationality if $x(S) \geq v(S)$ for all $v \in \mathcal{G}^N$ and $S \in \Omega$. According to these properties, the pre-imputation set $I^*(v)$ and the imputation set $I(v)$ are given by $I^*(v) = \{x \in \mathbb{R}^n | x(N) = v(N)\}$ and $I(v) = \{x \in I^*(v) | x_i \geq v(\{i\})$ for all $i \in N\}$.

A value on $\mathcal{G}^N$ is a function φ which assigns to every game $v \in \mathcal{G}^N$ a payoff vector $\varphi(v) \in \mathbb{R}^n$. Given any $\mathcal{A} \subseteq \mathcal{G}^N$ and a value φ on $\mathcal{A}$, its dual value φ^d is defined as, for all $v \in \mathcal{A}^d$, $\varphi^d(v) \equiv \varphi(v^d)$. The PD value, denoted by PD, assigns to every player the payoff in proportion to their singleton worths. For any $v \in \mathcal{G}^N_+$ and $i \in N$,

$$PD_i(v) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N).$$

The PANSC value, denoted by PANSC, assigns to every player the payoff in proportion to their marginal contributions with respect to the grand coalition. For any $v \in \mathcal{G}^N_\oplus$ and $i \in N$,

$$PANSC_i(v) = \frac{b^v(\{i\})}{b^v(N)} v(N).$$

Obviously, the PD value is the dual of the PANSC value.

3. The Optimal Satisfaction Value

The core is one of the most important set solutions for TU-games, which is defined through efficiency and group rationality. The core of a TU-game v is given by

$$C(v) = \{x \in I^*(v) | x(S) \geq v(S) \text{ for all } S \in \Omega\}.$$

Let $e(S, x, v) = v(S) - x(S)$ be the excess of a coalition $S \in \Omega$ with respect to x in a TU-game v. The excess $e(S, x, v)$ is usually used to measure the dissatisfaction degree of a coalition S with respect to x. Obviously, a payoff vector in the core only generates non-positive excesses for all coalitions. The larger the excess $e(S, x, v)$ is, the more unsatisfied the coalition S feel with respect to x. Conversely, the larger the minus excess $-e(S, x, v)$ is, the more satisfied the coalition S feel with respect to x.

From the perspective of measuring the satisfaction of coalitions for a payoff vector, we aim to introduce a family of optimal satisfaction values for TU-games. For a payoff vector $x \in \mathbb{R}^n$ and a TU-game $v \in \mathcal{G}^N$, let $\theta^v(x)$ be the $(2^n - 1)$-tuple vector whose components are the satisfactions of all coalitions $S \in \Omega$ with respect to x in non-decreasing order, that is, $\theta^v_t(x) \leq \theta^v_{t+1}(x)$ for all $t \in \{1, 2, \cdots, 2^n - 2\}$. For any $v \in \mathcal{G}^N$ and $x, y \in \mathbb{R}^n$, we call $\theta^v(x) \geq_L \theta^v(y)$ if and only if $\theta^v(x) = \theta^v(y)$, or there exists an $t \in \{1, 2, \cdots, 2^n - 2\}$ such that $\theta^v_l(x) = \theta^v_l(y)$ for all $l \in \{1, 2, \cdots, t-1\}$ and $\theta^v_t(x) > \theta^v_t(y)$.

Definition 1. *For any $v \in \mathcal{G}^N$, an optimal satisfaction value φ^{os} is a payoff vector y in the pre-imputation set satisfying $\theta^v(y) \geq_L \theta^v(x)$ for all $x \in I^*(v)$, that is,*

$$\varphi^{os}(v) = \{y \in I^*(v) | \theta^v(y) \geq_L \theta^v(x) \text{ for all } x \in I^*(v)\}.$$

The optimal satisfaction value can be viewed as a solution for an optimization problem aiming to maximize the minimal satisfaction with respect to the payoff vector over the pre-imputation set in the lexicographic order. It is easy to obtain that the optimal satisfaction value is consistent with the pre-nucleolus of a TU-game v under the satisfaction criterion of the minus excess $-e(S, x, v)$. Hou et al. [5] defined two linear complaint criteria which are given by $e^E(S, x, v) = b^v(S) - x(S)$ and $e^C(S, x, v) = x(N \backslash S) - \sum_{k \in N \backslash S} v(\{k\})$ for any $v \in \mathcal{G}^N$, $S \in \Omega$ and $x \in \mathbb{R}^n$. Conversely, $-e^E(S, x, v)$ and $-e^C(S, x, v)$ can be regarded as two different satisfaction criteria. Thus, two optimal satisfaction values are obtained by Definition 1, which coincide with the ENSC value and the CIS value according to Theorem 3.8 and Theorem 3.14 in Reference [5], respectively.

In this section, we define two special satisfaction criteria, the optimistic satisfaction and the pessimistic satisfaction, from the viewpoint of optimism and pessimism respectively. Given any $v \in \mathcal{G}^N$ and $x \in C(v)$, then it holds that $v(\{i\}) \leq x_i \leq b^v(\{i\})$ for all $i \in N$. Thus, the vector $(v(\{k\}))_{k \in N}$ can be regarded as the least potential payoff vector while $(b^v(\{k\}))_{k \in N}$ can be regarded as the ideal payoff vector of a TU-game v. On the optimistic side, the players prefer taking the least potential payoff of themselves into consideration and think of the ratio between the real payoff of coalition and their least potential payoff as the measure of satisfaction of the coalition. Conversely, pessimists prefer taking the ideal payoff of themselves into consideration. Formally, the optimistic satisfaction and the pessimistic satisfaction are defined as follows.

Definition 2. *For any payoff vector $x \in \mathbb{R}^n$ and $v \in \mathcal{G}^N_+$, $w \in \mathcal{G}^N_\oplus$, the optimistic satisfaction of a coalition $S \in \Omega$ with respect to x is given by*

$$e^o(S, x, v) = \frac{x(S)}{\sum_{k \in S} v(\{k\})}, \tag{2}$$

and the pessimistic satisfaction of a coalition $S \in \Omega$ with respect to x is given by

$$e^p(S, x, w) = \frac{x(S)}{b^w(S)}. \tag{3}$$

With respect to the two satisfaction criteria, we have two corresponding optimal satisfaction values, namely the optimistic optimal satisfaction value and the pessimistic optimal satisfaction value. In the following, we show that they are in coincidence with the PD value and the PANSC value, respectively.

3.1. The Optimistic Optimal Satisfaction Value and the PD Value

Formally, the optimistic optimal satisfaction value is given as follows.

Definition 3. *For any $v \in \mathcal{G}_{+}^{N}$, the optimistic optimal satisfaction value φ^o is the unique payoff vector y in the pre-imputation set satisfying $\theta^v(y) \geq_L \theta^v(x)$ for all $x \in I^*(v)$, that is,*

$$\varphi^o(v) = \{y \in I^*(v) | \theta^v(y) \geq_L \theta^v(x) \text{ for all } x \in I^*(v)\},$$

where θ^v is the satisfaction vector with respect to the optimistic satisfaction.

Next we show that the PD value is also obtained by lexicographically maximizing the minimal optimistic satisfaction, and coincides with the optimistic optimal satisfaction value.

Lemma 1. *Given any $v \in \mathcal{G}_{\oplus}^{N}$ and a payoff vector $x \in \mathbb{R}^n$, let $l = \arg\min_{l \in N}\{e^o(\{l\}, x, v)\}$. Then, we have $e^o(\{l\}, x, v) = \min_{S \in \Omega}\{e^o(S, x, v)\}$.*

Proof. Let $p = \min_{k \in N}\{e^o(\{k\}, x, v)\}$, then $p = e^o(\{l\}, x, v)$ and $x_k \geq p \cdot v(\{k\})$ for all $k \in N$. Then, we have

$$\begin{aligned} e^o(\{l\}, x, v) &\geq \min_{S \in \Omega}\{e^o(S, x, v)\} = \min_{S \in \Omega}\{\frac{x(S)}{\sum_{k \in S} v(\{k\})}\} \\ &\geq \min_{S \in \Omega}\{\frac{p \cdot \sum_{k \in S} v(\{k\})}{\sum_{k \in S} v(\{k\})}\} = p = e^o(\{l\}, x, v). \end{aligned}$$

Therefore, all inequalities are equalities and then $e^o(\{l\}, x, v) = \min_{S \in \Omega}\{e^o(S, x, v)\}$. □

Lemma 2. *Given any $v \in \mathcal{G}_{+}^{N}$ and a payoff vector $x \in \mathbb{R}^n$, let $l = \arg\min_{l \in N}\{e^o(\{l\}, x, v)\}$. If there exists a player $m \in N$ such that $e^o(\{m\}, x, v) > e^o(\{l\}, x, v)$, define a new payoff vector x^* given by*

$$x_k^* = \begin{cases} x_k, & \text{for } k \in N \backslash \{l, m\}; \\ x_l + \triangle, & \text{for } k = l; \\ x_m - \triangle, & \text{for } k = m, \end{cases}$$

where $\triangle = \frac{x_m \cdot v(\{l\}) - x_l \cdot v(\{m\})}{v(\{l\}) + v(\{m\})}$. Then the following five statements hold.

1. *$e^o(S, x^*, v) = e^o(S, x, v)$ for any $S \in \Omega$ and $S \not\ni l, m$.*
2. *$e^o(S, x^*, v) = e^o(S, x, v)$ for any $S \in \Omega$ and $S \ni l, m$.*
3. *$e^o(S, x^*, v) > e^o(S, x, v)$ for any $S \in \Omega$, $S \ni l$ and $S \not\ni m$.*
4. *$e^o(S, x^*, v) > e^o(\{l\}, x, v)$ for any $S \in \Omega$, $S \not\ni l$ and $S \ni m$.*
5. *$\theta^v(x^*) >_L \theta^v(x)$, where θ^v is the satisfaction vector with respect to the optimistic satisfaction.*

Proof.

1. It is obvious that $e^o(S,x^*,v) = e^o(S,x,v)$ for any $S \in \Omega$ and $S \not\ni l,m$ because $x^*(S) = x(S)$ for any $S \in \Omega$ and $S \not\ni l,m$.
2. It is trivial that $e^o(S,x^*,v) = e^o(S,x,v)$ for any $S \in \Omega$ and $S \ni l,m$.
3. It is easy to obtain that $\triangle > 0$ since $e^o(\{m\},x,v) > e^o(\{l\},x,v)$. Then for any $S \in \Omega$, $S \ni l$ and $S \not\ni m$,
$$e^o(S,x^*,v) = \frac{x^*(S)}{\sum_{k\in S} v(\{k\})} = \frac{x(S)+\triangle}{\sum_{k\in S} v(\{k\})} > \frac{x(S)}{\sum_{k\in S} v(\{k\})} = e^o(S,x,v).$$
4. Since $e^o(\{m\},x,v) > e^o(\{l\},x,v)$, we have
$$\begin{aligned} e^o(\{m\},x^*,v) =& \frac{x^*_m}{v(\{m\})} = \frac{x_m-\triangle}{v(\{m\})} = \frac{x_m+x_l}{v(\{m\})+v(\{l\})} \\ >& \frac{\frac{x_l}{v(\{l\})}v(\{m\})+x_l}{v(\{m\})+v(\{l\})} = \frac{x_l}{v(\{l\}))} = e^o(\{l\},x,v). \end{aligned}$$

   For any $S \in \Omega$, $S \not\ni l$ and $S \ni m$, we have
$$\begin{aligned} e^o(S,x^*,v) =& \frac{x(S\backslash\{m\})+x^*_m}{\sum_{k\in S\backslash\{m\}} v(\{k\})+v(\{m\})} \\ >& \frac{\frac{x_l}{v(\{l\})}\sum_{k\in S\backslash\{m\}} v(\{k\})+\frac{x_l}{v(\{l\})}v(\{m\})}{\sum_{k\in S\backslash\{m\}} v(\{k\})+v(\{m\})} \\ =& \frac{x_l}{v(\{l\}))} = e^o(\{l\},x,v), \end{aligned}$$

   where the second inequality holds because $e^o(\{m\},x^*,v) > e^o(\{l\},x,v)$ and $e^o(S\backslash\{m\},x,v) \geq e^o(\{l\},x,v)$ by Lemma 1.
5. It holds that $\theta^v(x^*) >_L \theta^v(x)$ by 1–4. □

Theorem 3. *For any $v \in \mathcal{G}^N_+$, the following two statements hold.*

1. $\frac{\varphi^o_i(v)}{v(\{i\})} = \frac{\varphi^o_j(v)}{v(\{j\})}$ *for all $i,j \in N$.*
2. $\varphi^o_i(v) = PD_i(v)$ *for all $i \in N$.*

Proof.

1. We will prove that $\frac{\varphi^o_i(v)}{v(\{i\})} = \frac{\varphi^o_j(v)}{v(\{j\})}$ for all $i,j \in N$ by reduction to absurdity. Given any $v \in \mathcal{G}^N_+$, suppose there exists $m,j \in N$ such that $\frac{\varphi^o_m(v)}{v(\{m\})} \neq \frac{\varphi^o_j(v)}{v(\{j\})}$. Without loss of generality, suppose that $\frac{\varphi^o_m(v)}{v(\{m\})} > \frac{\varphi^o_j(v)}{v(\{j\})}$. Let $y = \varphi^o(v)$ and $l = \arg\min_{l\in N}\{e^o(\{l\},y,v)\}$, then we have $e^o(\{m\},y,v) > e^o(\{l\},y,v)$. By Lemma 2, there exists $x \in I^*(v)$ such that $\theta^v(x) >_L \theta^v(y)$, where θ^v is the satisfaction vector with respect to the optimistic satisfaction, which contradicts with $\theta^v(y) \geq_L \theta^v(x)$ for all $x \in I^*(v)$. Therefore, $\frac{\varphi^o_i(v)}{v(\{i\})} = \frac{\varphi^o_j(v)}{v(\{j\})}$ for all $i,j \in N$.
2. It is immediate to deduce that $\varphi^o_i(v) = PD_i(v) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N)$ by the statement 1 and efficiency. □

Obviously, if $\sum_{k\in N} v(\{k\}) \leq v(N)$, then the optimistic optimal satisfaction value φ^o satisfies individual rationality, that is, $\varphi^o_k(v) \geq v(\{k\})$ for all $k \in N$. The following corollary is immediate for the reason that $C(v) \neq \emptyset$ implies $\sum_{k\in N} v(\{k\}) \leq v(N)$.

Corollary 4. *For any TU-game $v \in \mathcal{G}^N_+$ with $C(v) \neq \emptyset$, the optimistic optimal satisfaction value φ^o satisfies individual rationality, that is, $\varphi^o_k(v) \geq v(\{k\})$ for all $k \in N$.*

3.2. The Pessimistic Optimal Satisfaction Value and the PANSC Value

The pessimistic optimal satisfaction value is defined by lexicographically maximizing the minimal pessimistic satisfaction. We show that the PANSC value is also in coincidence with the pessimistic optimal satisfaction value in this subsection.

Definition 4. *For any $v \in \mathcal{G}^N_{\oplus}$, the pessimistic optimal satisfaction value φ^p is the unique payoff vector z in the pre-imputation set satisfying $\theta^v(z) \geq_L \theta^v(x)$ for all $x \in I^*(v)$, that is,*

$$\varphi^p(v) = \{z \in I^*(v) | \theta^v(z) \geq_L \theta^v(x) \text{ for all } x \in I^*(v)\},$$

where θ^v is the satisfaction vector with respect to the pessimistic satisfaction.

Next, we will verify that the PANSC value coincides with the pessimistic optimal satisfaction value. The proofs of Lemma 5, Lemma 6 and Theorem 7 are similar to those of Lemma 1, Lemma 2 and Theorem 3, and are omitted here.

Lemma 5. *Given any $v \in \mathcal{G}^N_{\oplus}$ and a payoff vector $x \in \mathbb{R}^n$, let $l = \arg\min_{l \in N}\{e^p(\{l\}, x, v)\}$. Then we have $e^p(\{l\}, x, v) = \min_{S \in \Omega}\{e^p(S, x, v)\}$.*

Lemma 6. *Given any $v \in \mathcal{G}^N_{\oplus}$ and a payoff vector $x \in \mathbb{R}^n$, let $l = \arg\min_{l \in N}\{e^p(\{l\}, x, v)\}$. If there is one player $m \in N$ such that $e^p(\{m\}, x, v) > e^p(\{l\}, x, v)$, define a new payoff vector x^* given by*

$$x^*_k = \begin{cases} x_k, & \text{for } k \in N \backslash \{l, m\}; \\ x_l + \triangle, & \text{for } k = l; \\ x_m - \triangle, & \text{for } k = m, \end{cases}$$

where $\triangle = \frac{x_m \cdot b^v_l - x_l \cdot b^v_m}{b^v_l + b^v_m}$. Then the following five statements hold.

1. *$e^p(S, x^*, v) = e^p(S, x, v)$ for any $S \in \Omega$ and $S \not\ni l, m$.*
2. *$e^p(S, x^*, v) = e^p(S, x, v)$ for any $S \in \Omega$ and $S \ni l, m$.*
3. *$e^p(S, x^*, v) > e^p(S, x, v)$ for any $S \in \Omega$, $S \ni l$ and $S \not\ni m$.*
4. *$e^p(S, x^*, v) > e^p(\{l\}, x, v)$ for any $S \in \Omega$, $S \not\ni l$ and $S \ni m$.*
5. *$\theta^v(x^*) >_L \theta^v(x)$, where θ^v is the satisfaction vector with respect to the pessimistic satisfaction.*

Theorem 7. *For any $v \in \mathcal{G}^N_{\oplus}$, the following two statements hold.*

1. *$\frac{\varphi^p_i(v)}{b^v(\{i\})} = \frac{\varphi^p_j(v)}{b^v(\{j\})}$ for all $i, j \in N$.*
2. *$\varphi^p_i(v) = PANSC_i(v)$ for all $i \in N$.*

By Theorem 7, it holds that $\varphi^p_k(v) \leq b^v(\{k\})$ for all $k \in N$ if $b^v(N) \geq v(N)$. Then the following corollary is immediate for the reason that $C(v) \neq \emptyset$ implies $b^v(N) \geq v(N)$.

Corollary 8. *For any $v \in \mathcal{G}^N_{\oplus}$ with $C(v) \neq \emptyset$, the pessimistic optimal satisfaction value φ^p is bounded by the ideal payoff vector $(b^v(\{k\}))_{k \in N}$, that is, $\varphi^p_k(v) \leq b^v(\{k\})$ for all $k \in N$.*

4. Axiomatizations of the PD Value and the PANSC Value

In this section, we propose two types of axioms, the equal minimal satisfaction property and the associated consistency property, to characterize the PD value and the PANSC value.

4.1. Equal Minimal Satisfaction Property

In this subsection, we introduce the equal minimal optimistic satisfaction property and the equal minimal pessimistic satisfaction property, inspired by the kernel concept [17]. The PD value and the PANSC value are characterized by these two properties with efficiency, respectively. Moreover, we study the dual relationship between the equal minimal optimistic satisfaction property and the equal minimal pessimistic satisfaction property.

Given any pair of axioms of TU-games, if whenever a value satisfies one of these axioms, the dual of the value satisfies the other, then these two axioms are dual to each other. More accurately, for any pair of axioms A and A^d, A and A^d are dual to each other, if for any value that satisfies A, its dual value satisfies A^d, and on the contrary, for any value that satisfies A^d, its dual value satisfies A. An axiom is called self-dual if the dual of the axiom is itself. Obviously, efficiency is a self-dual axiom.

For any payoff vector $x \in \mathbb{R}^n$ and any $v \in \mathcal{G}_+^N$, $w \in \mathcal{G}_\oplus^N$, the optimistic minimal satisfaction $m_{ij}^o(v,x)$ and the pessimistic minimal satisfaction $m_{ij}^p(w,x)$ of player $i \in N$ over player $j \in N\backslash\{i\}$ with respect to x are given as follows

$$m_{ij}^o(v,x) = \min\{e^o(S,x,v) | S \in \Omega, i \in S, j \notin S\},$$
$$m_{ij}^p(w,x) = \min\{e^p(S,x,w) | S \in \Omega, i \in S, j \notin S\}.$$

Definition 5. *Given any $v \in \mathcal{G}_+^N$, $w \in \mathcal{G}_\oplus^N$, a payoff vector x satisfies*

1. *equal minimal optimistic satisfaction property if for every $i,j \in N$, $m_{ij}^o(v,x) = m_{ji}^o(v,x)$.*
2. *equal minimal pessimistic satisfaction property if for every $i,j \in N$, $m_{ij}^p(w,x) = m_{ji}^p(w,x)$.*

The equal minimal optimistic satisfaction property states that for any $i,j \in N$, the minimal optimistic satisfaction of all coalitions containing i and not j should equal that of all coalitions containing j and not i with respect to a payoff vector under the optimistic satisfaction criterion. On the contrary, the equal minimal pessimistic satisfaction property describe this situation under the pessimistic satisfaction criterion.

Proposition 9. *The equal minimal optimistic satisfaction property and the equal minimal pessimistic satisfaction property are dual to each other.*

Proof. Given a value φ on $\mathcal{G}_+^N$, let φ^d be the dual of φ. It is sufficient to prove that φ satisfies the equal minimal optimistic satisfaction property if and only if φ^d satisfies the equal minimal pessimistic satisfaction property.

Suppose that φ satisfies the equal minimal optimistic satisfaction property. Given any $v \in \mathcal{G}_\oplus^N$ and its dual game $v^d \in \mathcal{G}_+^N$, by the equal minimal optimistic satisfaction property, we have $m_{ij}^o(v^d, \varphi(v^d)) = m_{ji}^o(v^d, \varphi(v^d))$ for all $i,j \in N$, and then it holds that

$$\min\{\frac{\sum_{k\in S}\varphi_k(v^d)}{\sum_{k\in S} v^d(\{k\})} | S \in \Omega, i \in S, j \notin S\} = \min\{\frac{\sum_{k\in S}\varphi_k(v^d)}{\sum_{k\in S} v^d(\{k\})} | S \in \Omega, j \in S, i \notin S\}.$$

By the duality theory, it holds that

$$\min\{\frac{\sum_{k\in S}\varphi_k^d(v)}{b^v(S)} | S \in \Omega, i \in S, j \notin S\} = \min\{\frac{\sum_{k\in S}\varphi_k^d(v)}{b^v(S)} | S \in \Omega, j \in S, i \notin S\}.$$

Then, we have $m_{ij}^p(v, \varphi^d(v)) = m_{ji}^p(v, \varphi^d(v))$ for all $i,j \in N$. Therefore, φ^d satisfies the equal minimal pessimistic satisfaction property.

Similarly, we can prove that φ satisfies the equal minimal optimistic satisfaction property if φ^d satisfies the equal minimal pessimistic satisfaction property, which is similar to the above proof.

Therefore, we can conclude that the equal minimal optimistic satisfaction property and the equal minimal pessimistic satisfaction property are dual to each other. □

Theorem 10.

1. *The PD value satisfies the equal minimal optimistic satisfaction property on $\mathcal{G}_{+}^{N}$.*
2. *The PANSC value satisfies the equal minimal pessimistic satisfaction property on $\mathcal{G}_{\oplus}^{N}$.*

Proof.

1. For any $v \in \mathcal{G}_{+}^{N}$, let $x = PD(v)$. Then for any $S \in \Omega$, $e^{o}(S, x, v) = \frac{v(N)}{\sum_{k\in N} v(\{k\})}$. Therefore, for every $i, j \in N$, we have
$$m_{ij}^{o}(v, x) = \frac{v(N)}{\sum_{k\in N} v(\{k\})} = m_{ji}^{o}(v, x).$$
2. For any $v \in \mathcal{G}_{\oplus}^{N}$, let $x = PANSC(v)$. Then for any $S \in \Omega$, $e^{p}(S, x, v) = \frac{v(N)}{b^{v}(N)}$. Therefore, we have
$$m_{ij}^{p}(v, x) = \frac{v(N)}{b^{v}(N)} = m_{ji}^{p}(v, x),$$
for every $i, j \in N$. □

Theorem 11. *The PD value is the unique value on $\mathcal{G}_{+}^{N}$ satisfying efficiency and the equal minimal optimistic satisfaction property.*

Proof. Firstly, it is easy to show that the PD value satisfies efficiency. Then the equal minimal optimistic satisfaction property follows from Theorem 10. It is left to show the uniqueness.

Suppose that x is a payoff vector of a TU-game $v \in \mathcal{G}_{+}^{N}$ which satisfies efficiency and the equal minimal optimistic satisfaction property. Now suppose that $x \neq PD(v)$, and then there must exist $i, j \in N$ such that $x_i > PD_i(v)$ and $x_j < PD_j(v)$ by the efficiency. Let $l = \arg\min_{l\in N}\{e^{o}(\{l\}, x, v)\}$, it holds that $e^{o}(\{l\}, x, v) = \min_{S\in\Omega}\{e^{o}(S, x, v)\}$ by Lemma 1. Then we have
$$e^{o}(\{l\}, x, v) \leq e^{o}(\{j\}, x, v) < \frac{v(N)}{\sum_{k\in N} v(\{k\})} < e^{o}(\{i\}, x, v).$$

Without loss of generality, let $S_0 \subseteq N\backslash\{l\}$ be a coalition containing i such that $m_{il}^{o}(v, x) = e^{o}(S_0, x, v)$. Thus, we have
$$m_{il}^{o}(v, x) = \frac{x(S_0\backslash\{i\}) + x_i}{\sum_{k\in S_0\backslash\{i\}} v(\{k\}) + v(\{i\})} > e^{o}(\{l\}, x, v) = m_{li}^{o}(v, x),$$
where the first inequality holds because $e^{o}(S_0\backslash\{i\}, x, v) \geq e^{o}(\{l\}, x, v)$ and $e^{o}(\{i\}, x, v) > e^{o}(\{l\}, x, v)$, and the last equality holds because $e^{o}(\{l\}, x, v) = \min_{S\in\Omega}\{e^{o}(S, x, v)\}$. But $m_{il}^{o}(v, x) > m_{li}^{o}(v, x)$ contradicts with the equal minimal optimistic satisfaction property. Therefore, the PD value is the unique value on $\mathcal{G}_{+}^{N}$ that satisfies efficiency and the equal minimal optimistic satisfaction property. □

In TU-games, the duality operator is a very useful tool to derive new axiomatizations of solutions. If there is an axiomatization of solution φ, then we can get one of axiomatization of its dual solution φ^{d} by determining the dual axioms of the axioms which are included in the axiomatization of φ. Oishi et al. [23] derived new axiomatizations of several classical solutions for TU-games by the duality theory. Since the equal minimal optimistic satisfaction property and the equal minimal pessimistic satisfaction property are dual to each other and efficiency is self-dual, we can obtain the following theorem.

Theorem 12. *The PANSC value is the unique value on $\mathcal{G}_{\oplus}^N$ satisfying efficiency and the equal minimal pessimistic satisfaction property.*

4.2. Associated Consistency Property

In the framework of the axiomatic system for TU-games, associated consistency is a significant characteristic of feasible and stable solutions. Associated consistency states that the solution should be invariant when the game changes into its associated game.

Throughout this subsection we deal with two types of associated games, the optimistic associated game and the pessimistic associated game. In these two associated games, every coalition reevaluates its own worth. Every coalition S just considers the players in $N \backslash S$ as individual elements and ignores the connection among players in $N \backslash S$. On the optimistic side, every coalition S always thinks that players in $N \backslash S$ should just receive their least potential payoff $(v\{k\})_{k \in N \backslash S}$. The amount $v(N) - v(S) - \sum_{k \in N \backslash S} v(\{k\})$ can be regarded as the optimistic surplus arising from mutual cooperation between S itself and all $j \in N \backslash S$. On the pessimistic side, every coalition S takes into consideration the ideal payoff vector and thinks that players in $N \backslash S$ can obtain their ideal payoff $(b^v(\{k\}))_{k \in N \backslash S}$. The amount $v(N) - v(S) - b^v(N \backslash S)$ is considered as the pessimistic surplus. Every coalition S believes that the appropriation of at least a part of the surpluses is within reach. Thus, every coalition S reevaluates its own worth $v_{\lambda,O}(S)$ in the optimistic associated game as the sum of its initial worth $v(S)$ and a percentage $\lambda \in (0,1)$ of a part $\frac{\sum_{k \in S} v(\{k\})}{\sum_{k \in N} v(\{k\})}$ of the optimistic surplus $v(N) - v(S) - \sum_{k \in N \backslash S} v(\{k\})$. Similarly, the pessimistic surplus is taken into account in the pessimistic associated game.

Definition 6. *Given any $v \in \mathcal{G}_{+}^N$ with $v(N) > 0$, and a real number $\lambda, 0 < \lambda < 1$, the optimistic associated game, denoted by $\langle N, v_{\lambda,O} \rangle$, is given by $v_{\lambda,O}(\emptyset) = 0$ and*

$$v_{\lambda,O}(S) = v(S) + \lambda \frac{\sum_{k \in S} v(\{k\})}{\sum_{k \in N} v(\{k\})} [v(N) - v(S) - \sum_{k \in N \backslash S} v(\{k\})], \textit{ for all } S \in \Omega. \tag{4}$$

The purpose of making $v(N) > 0$ is in order to ensure $v_{\lambda,O} \in \mathcal{G}_{+}^N$. For convenience, let $\mathcal{G}_{++}^N = \{v \in \mathcal{G}_{+}^N | v(N) > 0\}$. It is easy to obtain that $v_{\lambda,O} \in \mathcal{G}_{++}^N$ if $v \in \mathcal{G}_{++}^N$. Moveover, let $\mathcal{G}_{\oplus\oplus}^N = \{v \in \mathcal{G}_{\oplus}^N | v(N) > 0\}$. Obviously, $\mathcal{G}_{++}^N$ and $\mathcal{G}_{\oplus\oplus}^N$ are dual to each other.

Definition 7. *Given any $v \in \mathcal{G}_{\oplus}^N$ and a real number $\lambda, 0 < \lambda < 1$, the pessimistic associated game, denoted by $\langle N, v_{\lambda,P} \rangle$, is given by $v_{\lambda,P}(\emptyset) = 0$ and*

$$v_{\lambda,P}(S) = v(S) + \lambda \frac{b^v(S)}{b^v(N)} [v(N) - v(S) - b^v(N \backslash S)], \textit{ for all } S \in \Omega. \tag{5}$$

Obviously, $v_{\lambda,P} \in \mathcal{G}_{\oplus}^N$ if $v \in \mathcal{G}_{\oplus}^N$.

Definition 8.

1. *A value φ on $\mathcal{G}_{++}^N$ satisfies optimistic associated consistency if $\varphi(v) = \varphi(v_{\lambda,O})$ for any $v \in \mathcal{G}_{++}^N$.*
2. *A value φ on $\mathcal{G}_{\oplus}^N$ satisfies pessimistic associated consistency if $\varphi(v) = \varphi(v_{\lambda,P})$ for any $v \in \mathcal{G}_{\oplus}^N$.*

Next, let us consider the dual relation between these two associated consistency. Given any $v \in \mathcal{G}_{++}^N$ and its dual game $v^d \in \mathcal{G}_{\oplus\oplus}^N$, we only need to verify whether $(v^d)_{\lambda,O}$ is equal to $(v_{\lambda,P})^d$, to determine the dual relation between optimistic associated consistency and pessimistic associated consistency.

Remark 1. *Optimistic associated consistency and pessimistic associated consistency are not dual to each other.*

Theorem 13.

1. *The PD value satisfies optimistic associated consistency on* $\mathcal{G}^N_{++}$.
2. *The PANSC value satisfies pessimistic associated consistency on* $\mathcal{G}^N_{\oplus}$.

Proof.

1. By Definition 6, $v_{\lambda,O}(N) = v(N)$ and for all $i \in N$,

$$v_{\lambda,O}(\{i\}) = v(\{i\}) + \frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})}[v(N) - \sum_{k\in N} v(\{k\})]$$
$$= (1-\lambda)v(\{i\}) + \frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N) > 0.$$

Then we have, for all $i \in N$

$$PD_i(v_{\lambda,O}) = \frac{v_{\lambda,O}(\{i\})}{\sum_{k\in N} v_{\lambda,O}(\{k\})} v_{\lambda,O}(N) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N) = PD_i(v).$$

Therefore, the PD value satisfies optimistic associated consistency.

2. By Definition 7, $v_{\lambda,P}(N) = v(N)$ and $v_{\lambda,P}(N\backslash\{i\}) = v(N\backslash\{i\})$ for all $i \in N$. Then, for all $i \in N$, we have

$$PANSC_i(v_{\lambda,P}) = \frac{b^{v_{\lambda,P}}(\{i\})}{b^{v_{\lambda,P}}(N)} v_{\lambda,P}(N) = \frac{b^v(\{i\})}{b^v(N)} v(N) = PANSC_i(v).$$

Therefore, the PANSC value satisfies pessimistic associated consistency. □

Next we recall some classical properties of solutions for TU-games. A value φ satisfies

1. continuity, if for any convergent sequence of games $\{\langle N, v^k\rangle\}_{k=1}^{\infty}$ and its limit game $\langle N, \tilde{v}\rangle$ (i.e., for all $S \in \Omega$, $\lim_{k\to\infty} v^k(S) = \tilde{v}(S)$), the corresponding sequence of the values $\{\varphi(v^k)\}_{k=1}^{\infty}$ converges to the payoff vector $\varphi(\tilde{v})$.
2. inessential game property, if $\varphi_i(v) = v(\{i\})$ for any inessential game $v \in \mathcal{G}^N$ and $i \in N$. A game v is inessential if $v(S) = \sum_{k\in S} v(\{k\})$ for all $S \in \Omega$.
3. proportional constant additivity, if $\varphi_i(v+w) = \varphi(v) + \frac{b^v(\{i\})}{b^v(N)} w(N)$ for any $v \in \mathcal{G}^N_{\oplus}$, any constant game $w \in \mathcal{G}^N$ and $i \in N$. A game w is a constant game if $w(S) = \alpha$ for all $S \in \Omega$ and some $\alpha \in \mathbb{R}$.

The following lemma states the convergence of the sequence of repeated optimistic associated games, and its detailed proof is in Appendix A.

Lemma 14. *For any $v \in \mathcal{G}^N_{++}$, the sequence of repeated optimistic associated games $\{\langle N, v^t_{\lambda,O}\rangle\}_{t=1}^{\infty}$ converges, and its limit game $\langle N, \hat{v}\rangle$ is inessential, where $v^1_{\lambda,O} = v_{\lambda,O}$ and $v^{t+1}_{\lambda,O} = (v^t_{\lambda,O})_{\lambda,O}$, $t = 1, 2, \cdots$.*

Theorem 15. *The PD value is the unique value on $\mathcal{G}^N_{++}$ satisfying optimistic associated consistency, continuity and the inessential game property.*

Proof. It is easy to verify that the PD value satisfies continuity and the inessential game property. Optimistic associated consistency follows from Theorem 13. It is left to show the uniqueness.

Now suppose that a value φ on $\mathcal{G}^N_{++}$ satisfies these three axioms. For any $v \in \mathcal{G}^N_{++}$, by Lemma 14, the sequence of repeated optimistic associated games $\{\langle N, v^t_{\lambda,O}\rangle\}_{t=1}^{\infty}$ converges to an inessential game $\langle N, \hat{v}\rangle$. Then by optimistic associated consistency and continuity, it holds that

$$\varphi(v) = \varphi(v^1_{\lambda,O}) = \varphi(v^2_{\lambda,O}) = \cdots = \varphi(\hat{v}).$$

By the inessential game property, we have $\varphi_i(\hat{v}) = v(\{i\}) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N)$ for all $i \in N$. Thus, $\varphi(v) = PD(v)$. $\square$

Next we prove the convergence of the sequence of repeated pessimistic associated games. The detailed proof of the lemma is in Appendix A.

Lemma 16. *For any $v \in \mathcal{G}^N_{\oplus}$, the sequence of repeated pessimistic associated games $\{\langle N, v^t_{\lambda,P}\rangle\}_{t=1}^{\infty}$ converges and its limit game $\langle N, \check{v}\rangle$ is the sum of an inessential game $\langle N, u\rangle$ and a constant game $\langle N, w\rangle$. where $v^1_{\lambda,P} = v_{\lambda,P}$ and $v^{t+1}_{\lambda,P} = (v^t_{\lambda,P})_{\lambda,P}, t = 1, 2, \cdots$.*

Theorem 17. *The PANSC value is the unique value on $\mathcal{G}^N_{\oplus}$ satisfying pessimistic associated consistency, continuity, the inessential game property and proportional constant additivity.*

Proof. It is easy to verify that the PANSC value satisfies continuity, the inessential game property and proportional constant additivity. Pessimistic associated consistency follows from Theorem 13. It is left to show the uniqueness.

Suppose that a value φ on $\mathcal{G}^N_{\oplus}$ satisfies pessimistic associated consistency, continuity, the inessential game property and proportional constant additivity. For any $v \in \mathcal{G}^N_{\oplus}$, by Lemma 16, the sequence of repeated pessimistic associated games $\{\langle N, v^t_{\lambda,P}\rangle\}_{t=1}^{\infty}$ converges to a game $\langle N, \check{v}\rangle$ which is expressed as the sum of a constant game $\langle N, w\rangle$ and an inessential game $\langle N, u\rangle$, where $w(S) = v(N) - b^v(N)$ and $u(S) = b^v(S)$ for all $S \in \Omega$. By continuity and pessimistic associated consistency, we have

$$\varphi(v) = \varphi(v^1_{\lambda,P}) = \varphi(v^2_{\lambda,P}) = \cdots = \varphi(\check{v}).$$

Let $\alpha = v(N) - b^v(N)$. By the inessential game property and proportional constant additivity, for any $i \in N$

$$\begin{aligned} \varphi_i(\check{v}) =& \varphi_i(u + w) = \varphi(u) + \frac{b^u(\{i\})}{b^u(N)} w(N) \\ =& u(\{i\}) + \frac{b^v(\{i\})}{b^v(N)} [v(N) - b^v(N)] = \frac{b^v(\{i\})}{b^v(N)} v(N). \end{aligned}$$

Therefore, $\varphi(v) = \frac{b^v(\{i\})}{b^v(N)} v(N) = PANSC(v)$. $\square$

4.3. Dual Axioms of Associated Consistency

In Remark 1, we mentioned that optimistic associated consistency and pessimistic associated consistency are not dual to each other. Next let us consider the dual axioms of optimistic associated consistency and pessimistic associated consistency.

Definition 9. *Given any $v \in \mathcal{G}^N_{\oplus\oplus}$, and a real number $\lambda, 0 < \lambda < 1$, the dual optimistic associated game $\langle N, v^*_{\lambda,O}\rangle$ is given by*

$$v^*_{\lambda,O}(S) = \begin{cases} v(S) + \lambda \frac{b^v(N\setminus S)}{b^v(N)} [b^v(S) - v(S)], & \text{if } S \subset N, \\ v(N), & \text{if } S = N. \end{cases} \tag{6}$$

Definition 10. *Given any $v \in \mathcal{G}^N_{+}$, and a real number $\lambda, 0 < \lambda < 1$, the dual pessimistic associated game $\langle N, v^*_{\lambda,P}\rangle$ is given by*

$$v^*_{\lambda,P}(S) = \begin{cases} v(S) + \lambda \frac{\sum_{k\in N\setminus S} v(\{k\})}{\sum_{k\in N} v(\{k\})} [\sum_{k\in S} v(\{k\}) - v(S)], & \text{if } S \subset N, \\ v(N), & \text{if } S = N. \end{cases} \tag{7}$$

Obviously, $v^*_{\lambda,O} \in \mathcal{G}^N_{\oplus\oplus}$ if $v \in \mathcal{G}^N_{\oplus\oplus}$, and $v^*_{\lambda,P} \in \mathcal{G}^N_{+}$ if $v \in \mathcal{G}^N_{+}$.

Definition 11.

1. *A value φ on $\mathcal{G}^N_{\oplus\oplus}$ satisfies dual optimistic associated consistency if $\varphi(v) = \varphi(v^*_{\lambda,O})$ for any $v \in \mathcal{G}^N_{\oplus\oplus}$.*
2. *A value φ on $\mathcal{G}^N_{+}$ satisfies dual pessimistic associated consistency if $\varphi(v) = \varphi(v^*_{\lambda,P})$ for any $v \in \mathcal{G}^N_{+}$.*

Lemma 18. *For any $v \in \mathcal{G}^N_{++}$ and $w \in \mathcal{G}^N_{\oplus}$, the following two statements hold.*

1. $(v_{\lambda,O})^d = (v^d)^*_{\lambda,O}$.
2. $(w_{\lambda,P})^d = (w^d)^*_{\lambda,P}$.

Proof.

1. By Equations (4) and (6), for any $v \in \mathcal{G}^N_{++}$ and $S \subset N$, we have

$$\begin{aligned}(v_{\lambda,O})^d(S) =& v_{\lambda,O}(N) - v_{\lambda,O}(N\backslash S)\\ =& v(N) - v(N\backslash S) - \lambda \frac{\sum_{k\in N\backslash S} v(\{k\})}{\sum_{k\in N} v(\{k\})}[v(N) - v(N\backslash S) - \sum_{k\in S} v(\{k\})]\\ =& v^d(S) + \lambda \frac{b^{v^d}(N\backslash S)}{b^{v^d}(N)}[b^{v^d}(S) - v^d(S)]\\ =& (v^d)^*_{\lambda,O}(S).\end{aligned}$$

For $S = N$, we have $(v_{\lambda,O})^d(N) = v(N) = (v^d)^*_{\lambda,O}(N)$. Thus, $(v_{\lambda,O})^d = (v^d)^*_{\lambda,O}$.

2. By Equations (5) and (7), for any $w \in \mathcal{G}^N_{\oplus}$ and $S \subset N$, we have

$$\begin{aligned}(w_{\lambda,P})^d(S) =& w_{\lambda,P}(N) - w_{\lambda,P}(N\backslash S)\\ =& w(N) - w(N\backslash S) - \lambda \frac{b^w(N\backslash S)}{b^w(N)}[w(N) - w(N\backslash S) - b^w(S)]\\ =& w^d(S) + \lambda \frac{\sum_{k\in N\backslash S} w^d(\{k\})}{\sum_{k\in N} w^d(\{k\})}[\sum_{k\in S} w^d(\{k\}) - w^d(S)]\\ =& (w^d)^*_{\lambda,P}(S).\end{aligned}$$

For $S = N$, we have $(w_{\lambda,P})^d(N) = w(N) = (w^d)^*_{\lambda,P}(N)$. Thus, $(w_{\lambda,P})^d = (w^d)^*_{\lambda,P}$. □

Proposition 19. *Optimistic associated consistency and dual optimistic associated consistency are dual to each other.*

Proof. Given a value φ on $\mathcal{G}^N_{++}$, let φ^d be the dual of φ. We just prove that φ satisfies optimistic associated consistency if and only if φ^d satisfies dual optimistic associated consistency.

If φ satisfies optimistic associated consistency, for any $v \in \mathcal{G}^N_{\oplus\oplus}$ and its dual game $v^d \in \mathcal{G}^N_{++}$, we have

$$\varphi^d(v) = \varphi(v^d) = \varphi((v^d)_{\lambda,O}) = \varphi((v^*_{\lambda,O})^d) = \varphi^d(v^*_{\lambda,O}),$$

where the third equation holds by Lemma 18. Thus, φ^d satisfies dual optimistic associated consistency.

If φ^d satisfies dual optimistic associated consistency, for any $v \in \mathcal{G}^N_{++}$ and its dual game $v^d \in \mathcal{G}^N_{\oplus\oplus}$, we have

$$\varphi(v) = \varphi^d(v^d) = \varphi^d((v^d)^*_{\lambda,O}) = \varphi^d((v_{\lambda,O})^d) = \varphi(v_{\lambda,O}).$$

Then, φ satisfies optimistic associated consistency. □

The proof of Proposition 20 is similar to that of Proposition 19 and is left to readers.

Proposition 20. *Pessimistic associated consistency and dual pessimistic associated consistency are dual to each other.*

Next, let us identify the dual axioms of other axioms which are included in the axiomatizations of the PD value and the PANSC value appearing in Theorems 15 and 17. It is easy to verify that continuity and the inessential game property are self-dual. A value φ on $\mathcal{G}_+^N$ satisfies dual proportional constant additivity, if $\varphi_i(v+w) = \varphi(v) + \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} w(N)$ for any $v \in \mathcal{G}_+^N$, any constant game $w \in \mathcal{G}^N$ and $i \in N$. Obviously, proportional constant additivity and dual proportional constant additivity are dual to each other. Thus, it is straightforward to obtain the following two theorems by the duality theory.

Theorem 21. *The PANSC value is the unique value on $\mathcal{G}_{\oplus\oplus}^N$ satisfying dual optimistic associated consistency, continuity and the inessential game property.*

Theorem 22. *The PD value is the unique value on $\mathcal{G}_+^N$ satisfying dual pessimistic associated consistency, continuity, the inessential game property and dual proportional constant additivity.*

5. Conclusions

In this paper, we introduce the family of optimal satisfaction values from the perspective of the satisfaction criteria. According to the optimistic satisfaction criterion and the pessimistic satisfaction criterion, the PD value and the PANSC value are determined by lexicographically maximizing the corresponding minimal satisfaction. Then, we characterize these two proportional values by introducing the equal minimal satisfaction property, associated consistency and the dual axioms of associated consistency. As two representative values of the proportional principle, the PD value and the PANSC value are relatively fair and reasonable allocations applied in many economic situations. For instance, in China's bankruptcy law, the bankruptcy property shall be distributed on a proportional principle when it is insufficient to repay all the repayment needs within a single order of priority. The proportional principle is deeply rooted in law and custom as a norm of distributed justice.

In the future, we will study other characterizations of the PD value and the PANSC value relying on some existing characterizations of classical solutions for TU-games. Coordinating the optimistic satisfaction and the pessimistic satisfaction, we may elicit the combination of the PD value and the PANSC value by an underlying neutral satisfaction criterion, and apply to some real situations.

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Appendix A

Proof of Lemma 14. For any $v \in \mathcal{G}_{++}^N$, we have $v_{\lambda,O}^t(N) = v(N)$, $t = 1, 2, \cdots$. Next, we show the convergence of the sequence of repeated optimistic associated games in two cases.

Case 1 $|S| = 1$. We first show that $v_{\lambda,O}^t(\{i\}) = (1-\lambda)^t v(\{i\}) + [1-(1-\lambda)^t] \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N)$ for all $i \in N$ and $t \in \{1, 2, \cdots\}$ by induction on t. When $t = 1$, by Definition 6, we have

$v^1_{\lambda,O}(\{i\}) = (1-\lambda)v(\{i\}) + \frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N)$ for any $i \in N$. Suppose that $v^{t-1}_{\lambda,O}(\{i\}) = (1-\lambda)^{t-1}v(\{i\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N)$. Then we have

$$\begin{aligned}
v^t_{\lambda,O}(\{i\}) =&(1-\lambda)v^{t-1}_{\lambda,O}(\{i\}) + \frac{\lambda v^{t-1}_{\lambda,O}(\{i\})}{\sum_{k\in N} v^{t-1}_{\lambda,O}(\{k\})}v(N) \\
=&(1-\lambda)\left\{(1-\lambda)^{t-1}v(\{i\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N)\right\} \\
&+\frac{\lambda\left\{(1-\lambda)^{t-1}v(\{i\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N)\right\}}{\sum_{j\in N}\left\{(1-\lambda)^{t-1}v(\{j\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{j\})}{\sum_{k\in N} v(\{k\})}v(N)\right\}}v(N) \\
=&(1-\lambda)^t v(\{i\}) + (1-\lambda)[1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N) \\
&+\frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N) \\
=&(1-\lambda)^t v(\{i\}) + [1-(1-\lambda)^t]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N).
\end{aligned}$$

Thus, it holds that

$$v^t_{\lambda,O}(\{i\}) = (1-\lambda)^t v(\{i\}) + [1-(1-\lambda)^t]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N), \quad t = 1,2,\cdots \tag{A1}$$

Therefore, for any $0 < \lambda < 1$, we have

$$\hat{v}(\{i\}) = \lim_{t\to\infty} v^t_{\lambda,O}(\{i\}) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N).$$

Case 2 $|S| \geq 2$. For convenience, let $\rho_S = \frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})}$ and $\sigma = v(N) - \sum_{k\in N} v(\{k\})$. Next we will show that

$$\begin{aligned}
v^t_{\lambda,O}(S) =&(1-\lambda\rho_S)^t v(S) + [1-(1-\lambda\rho_S)^t]\rho_S v(N) \\
&+\lambda\rho_S\sigma(1-\rho_S)[\sum_{m=1}^{t}(1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-m}],
\end{aligned} \tag{A2}$$

for all $S \in \Omega$ and $t \in \{1,2,\cdots\}$ by induction on t. When $t = 1$, by Definition 6, it holds that $v^1_{\lambda,O}(S) = (1-\lambda\rho_S)v(S) + \lambda\rho_S\rho_S v(N) + \lambda\rho_S\sigma(1-\rho_S)$, and Equation (A2) holds. Without loss of generality, suppose that Equation (A2) holds at $t-1$. Then, by Definition 6 and Equation (A1), we have

$$\begin{aligned}
v^t_{\lambda,O}(S) =& v^{t-1}_{\lambda,O}(S) + \lambda \frac{\sum_{k\in S} v^{t-1}_{\lambda,O}(\{k\})}{\sum_{k\in N} v^{t-1}_{\lambda,O}(\{k\})}[v(N) - v^{t-1}_{\lambda,O}(S) - \sum_{k\in N\setminus S} v^{t-1}_{\lambda,O}(\{k\})] \\
=&(1-\lambda\rho_S)v^{t-1}_{\lambda,O}(S) + \lambda\rho_S[v(N) - (1-\rho_S)\sum_{k\in N} v^{t-1}_{\lambda,O}(\{k\})] \\
=&(1-\lambda\rho_S)v^{t-1}_{\lambda,O}(S) + \lambda\rho_S\rho_S v(N) + \lambda\rho_S\sigma(1-\rho_S)(1-\lambda)^{t-1} \\
=&(1-\lambda\rho_S)^t v(S) + (1-\lambda\rho_S)[1-(1-\lambda\rho_S)^{t-1}]\rho_S v(N) \\
&+ \lambda\rho_S\sigma(1-\rho_S)(1-\lambda\rho_S)[\sum_{m=1}^{t-1}(1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-1-m}] \\
&+ \lambda\rho_S\rho_S v(N) + \lambda\rho_S\sigma(1-\rho_S)(1-\lambda)^{t-1} \\
=&(1-\lambda\rho_S)^t v(S) + [1-(1-\lambda\rho_S)^t]\rho_S v(N) \\
&+ \lambda\rho_S\sigma(1-\rho_S)[\sum_{m=1}^{t}(1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-m}].
\end{aligned}$$

Thus, Equation (A2) holds for all $S \in \Omega$ and $t \in \{1,2,\cdots\}$.

Let $a_t = \sum_{m=1}^t (1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-m}$. Since $0<\lambda<1$ and $0<\rho_S<1$, we have $t(1-\lambda)^{t-1} \le a_t \le t(1-\lambda\rho_S)^{t-1}$. Since $\lim_{t\to\infty} t(1-\lambda)^{t-1} = 0$ and $\lim_{t\to\infty} t(1-\lambda\rho_S)^{t-1} = 0$, then $\lim_{t\to\infty} a_t = 0$. Thus, we have

$$\hat{v}(S) = \lim_{t\to\infty} v^t_{\lambda,O}(S) = \rho_S v(N) = \frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})} v(N).$$

Therefore, the sequence of repeated optimistic associated games $\{\langle N, v^t_{\lambda,O}\rangle\}_{t=1}^\infty$ converges and its limit game $\langle N,\hat{v}\rangle$ is given by $\hat{v}(S) = \frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})} v(N)$ for all $S\in\Omega$. $\square$

Proof of Lemma 16. For any $v \in \mathcal{G}^N_\oplus$, it holds that $v^t_{\lambda,P}(N) = v(N)$ and $v^t_{\lambda,P}(N\setminus\{i\}) = v(N\setminus\{i\})$ for all $i\in N$ and $t = 1,2,\cdots$. Then we can obtain that $b^{v^t_{\lambda,P}}(\{i\}) = b^v(\{i\})$ for all $i \in N$ and $t=1,2,\cdots$. For convenience, let $\tau_S = \frac{b^v(S)}{b^v(N)}$. Next we will prove that

$$v^t_{\lambda,P}(S) = (1-\lambda\tau_S)^t v(S) + [1-(1-\lambda\tau_S)^t][v(N) - b^v(N\setminus S)] \tag{A3}$$

for all $S \in \Omega$ and $t = 1,2,\cdots$, by induction on t. When $t=1$, by Definition 7, we have $v^1_{\lambda,P}(S) = (1-\lambda\tau_S)v(S) + \lambda\tau_S[v(N) - b^v(N\setminus S)]$, and Equation (A3) holds. Suppose that Equation (A3) holds at $t-1$. Then, by Definition 7, we have

$$\begin{aligned}
v^t_{\lambda,P}(S) =& v^{t-1}_{\lambda,P}(S) + \lambda\frac{b^{v^{t-1}_{\lambda,P}}(S)}{b^{v^{t-1}_{\lambda,P}}(N)}[v(N) - v^{t-1}_{\lambda,P}(S) - b^{v^{t-1}_{\lambda,P}}(N\setminus S)] \\
=&(1-\lambda\tau_S)v^{t-1}_{\lambda,P}(S) + \lambda\tau_S[v(N) - b^v(N\setminus S)] \\
=&(1-\lambda\tau_S)\left\{(1-\lambda\tau_S)^{t-1}v(S) + [1-(1-\lambda\tau_S)^{t-1}][v(N)-b^v(N\setminus S)]\right\} \\
&+ \lambda\tau_S[v(N) - b^v(N\setminus S)] \\
=&(1-\lambda\tau_S)^t v(S) + [1-(1-\lambda\tau_S)^t][v(N) - b^v(N\setminus S)]
\end{aligned}$$

Thus, Equation (A3) holds for all $S\in\Omega$ and $t\in\{1,2,\cdots\}$.

Due to $0 < \lambda < 1$ and $0 < \tau_S < 1$, for any $S \in \Omega$, we have

$$\check{v}(S) = \lim_{t \to \infty} v^t_{\lambda,P}(S) = v(N) - b^v(N \backslash S).$$

Let $u(S) = b^v(S)$ and $w(S) = v(N) - b^v(N)$ for all $S \in \Omega$. Obviously, $\langle N, u \rangle$ is an inessential game and $\langle N, w \rangle$ is a constant game. The limit game $\langle N, \check{v} \rangle$ is given by $\check{v}(S) = u(S) + w(S)$. $\square$

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Resource Exploitation in a Stochastic Horizon under Two Parametric Interpretations

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Abstract: This work presents a two-player extraction game where the random terminal times follow (different) heavy-tailed distributions which are not necessarily compactly supported. Besides, we delve into the implications of working with logarithmic utility/terminal payoff functions. To this end, we use standard actuarial results and notation, and state a connection between the so-called *actuarial equivalence principle*, and the feedback controllers found by means of the Dynamic Programming technique. Our conclusions include a conjecture on the form of the optimal premia for insuring the extraction tasks; and a comparison for the intensities of the extraction for each player under different phases of the lifetimes of their respective machineries.

Keywords: differential games; random time horizon; time until failure; discounted equilibrium; weibull distribution; chen distribution; equivalence principle

MSC: 91A10; 91A23; 49N90; 60E05

1. Introduction

In this work we study an extension of the extraction game presented in Reference [1] to the case where the random terminal times follow (different) heavy-tailed distributions which are not necessarily compactly supported. We use the framework of the problem of common non-renewable resource exploitation as was posed in Reference [2], from both—the game-theoretical (cf. Reference [3]) and the actuarial points of view (see References [4,5]).

The first reported works on the dynamic development of exhaustible resources by the members of an oligopoly are those by Hotelling (see References [6,7]). There, we can find the well-known principle of marginal revenue, as well as the standard Hypothesis on the equality between the growth rate and the market interest rate over time. The survey [8] constitutes an excellent introduction to the topic from an Economic point of view, see also Reference [9] for empirical investigation of common-pool resource users' dynamic and strategic behavior at the micro level using real-world data.

The area owes its main developments to a discussion that took place during the late '70s and the early '80s on the possibility of replacing the exploitation schemes with some cutting-edge technology to be attained in the near future. The relevance of the debate was the search of a path to move from extracting a non-renewable resource to extracting a renewable one. In this line, we can quote the works of Dasgupta et al. (e.g., References [10,11]; see also Reference [12] Chapter 10.2), and that of Reinganum and Stokey (see Reference [13]). In the former papers, two agents extract a resource that becomes extinct at a time instant which is not known a priori, and then, some technological breakthrough becomes a suitable replacement; while in the latter, the authors assume that the extraction costs equal zero to find an optimal extraction policy over time. From the point of view of our own research, one of the main features of these publications is a method for comparing aggregate extraction paths in terms of the impact of the commitment period of the players on how fast the resource becomes exhausted.

Harris and Vickers [14] revisited Dasgupta's model, enhanced his analyses on the state dynamics, prove the existence and uniqueness of a Nash equilibrium, and characterize it in terms of the slope of the extraction policies at equilibrium. Epstein [15] and Feliz [16] considered a degenerate game to study the policy of extraction from either one or two wells as the resource dynamics is affected by uncertainty. More recently, the empirical economic knowledge on the subject (along with Hartwick and Solow's developments—see References [17,18] respectively- on the transformation of an exhaustible resource into productive capital to sustain a steady level of consumption), allowed Van der Ploeg to model the relation between the risk of depletion of a resource with the government policy on debt and precautionary saving (see Reference [19]). In Reference [20], an evolutionary analysis of the renewable resources exploitation with differentiated technologies was undertaken. Comprehensive surveys of models of dynamic games for the development of (renewable and non-renewable) resources can be found in Reference [21] by Van Long, and Reference [12] (Chapter 10) by Dockner et al.

Almost all of the papers mentioned above include the notion of uncertainty. However, only Reference [1] uses random variables to model the terminal times of extraction of the competing firms. We propose a differential game for the extraction of exhaustible resources (see References [12,22] (Chapter 10; Chapter 7)), where we consider uncertainty and asymmetry in a cake-eating model (as shown in Reference [23]), and interpret it in actuarial terms to, for instance, insure the extraction tasks of the players. The uncertainty here is reflected by the fact that the game ends at a random time instant, while the asymmetry can be seen in the different distributions we use for each player involved in the game.

In the insurance literature on non-renewable resource extraction, the work of Stroebel and van Benthem (see Reference [4]) is relevant for our research because they prove that (should the economy is likely to expropriate) the insurance premium on the extraction tasks is increasingly expensive, and decreasingly expensive as the extractors become more expert. Our approach resembles the one used by Delacote (see Reference [24]), because what he sees as households near the extraction points can be, in our case, an agent subjected to a double risk: they will (surely) occupy the place of what we dub *Chen extractor* (which we prove to be the riskiest of the two players), and they will not be "able to get more than their subsistence requirement" from the wells (the press article [25] presents a narrative of one such situation in Mexico, and the essays in the book [26] delve with the problem in African and Asian countries). There are three more references, in the insurance literature on the extraction of renewable resources, that are important for our developments [27–29]. *Au lieu* of the control methods we adopt, these pieces use statistical methods to show that natural insurance is a normal economic good, and we all agree on the willingness of the agents to pay premia in exchange for a continued supply of the resource under consideration.

The main feature of our game model can be traced back to the works of Petrosyan and Murzov (cf. Reference [30]), and Yaari (cf. Reference [31]). The former used Dynamic Programming to solve a zero-sum pursuit differential game with random duration, while the latter studied the maximization of a utility function by the design of an optimal consumption plan. This topic was further investigated in Reference [32], where the process of technological innovation was supposed to have a random duration. To the best of our knowledge, Petrosyan and Shevkoplyas (Gromova) were the first to propose a general differential game model with random duration (see Reference [33]), including the derivation and solution in closed form under a logarithmic payoff structure, of the corresponding

Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations (see Reference [34], and the generalization brought about by Reference [35]).

Game theoretical tools have been applied to model phenomena of the interest of actuarial scientists since the time of the works of Borch and Lemaire, who modelled risk transfers between insurer and reinsurer by means of cooperative games tools (see References [36–38]). Among the most relevant and recent works in Actuarial literature related to our work, we can quote Schäl's paper on the application of stochastic Dynamic Programming for a specific form of the utility function (cf. Reference [5]), because in that research, as well as in ours, we intend to minimize risks in the insurance industry, we focus on a particular form of the reward functions, and we share the use of Dynamic Programming. Such technique is the cornerstone of our approach (the textbook [39] by Hinderer, Rieder and Stieglitz represents an excellent introduction to it, for it covers the deterministic and stochastic cases, and it presents some actuarial applications in insurance). This method has been widely used since its dawn in the decade of the 1960s for many applications, including warfare, resource extraction, and control of pollution in the environment, among others. Schmidli's textbook [40] presents a complete introduction to the subject with traditional Actuarial Sciences in view. Indeed, he starts with the presentation of stochastic control (by means of Bellman's principle) in discrete- and continuous-time, then goes to applications in life insurance, and finally presents the classic ruin theory and Merton's model (Sections 2.6 and 3.1 in the survey [41], and Reference [42] constitute two quick reference guides towards the techniques we use). The works of Dutang et al. (cf. References [43,44]), and that of Polborn (see Reference [45]) present a game theoretic model in discrete-time on a non-life insurance market; and despite the fact that they focus on the competition for the marketshare of the agents, our research can be thought of as a continuation of their model on the fair (optimal) premia to be charged to the agents so as to maximize their profits from the competitive process. Pliska and Ye's article on optimal life insurance strategies (see Reference [46]), and Mango's work (cf. Reference [47]) on catastrophe modelling resemble our own contribution, but the characterization of the probability laws that these two works use lies on another extreme of the spectrum of distributions. This is the reason for which the model of perishable inventories with fat-tailed distributions studied by Giri (cf. Reference [48]) is so appealing to us. As for our use of simple contingent functions in association with Markovian processes, the works of Mao and Ostaszewski, and Perry and Stadje on annuity theory (see References [49,50]) represent significant antecedents to our own ideas.

The problem with asymmetry in different random time instants has been studied for some differential games in References [1,51–53]. A similar cake-eating problem with different asymmetric discounting functions has been considered in Reference [54]. In our paper we also assume that the game stops at the moment when one of the players quits the extraction tasks. However, rather than a Verhulst-type dynamic for the stock of the resource (see Reference [55]), we use a model that resembles the analysis in Reference [12]. We have chosen to work with the classic two-parameter Weibull distribution and the Chen's law. Weibull's model has been widely used in life and non-life Actuarial Mathematics, while Chen's law owes its importance to the fact that its hazard rate function is –in a broad sense- remarkably stronger than that of the former model; this feature automatically turns it into a very suitable option for modelling extreme events (see Reference [56]). We focus our attention on three phases of the lifetime of the machineries of each player: early stage, normal operation stage, and aging stage. We do this by changing the *shape parameters* of each one of these laws, for lower value of these parameters ($\delta < 1$) results in a lower hazard rate for the distributions; while the choice of $\delta = 1$ yields a stable hazard rate function, and a greater value ($\delta > 1$) gives us strong failure rates.

Our main conclusions have to do with the implications of using stochastic Dynamic Programming with a particular form of the utility functions, as in Reference [5], however, all of our developments are presented in the continuous-time framework. To this end, we use standard actuarial results and notation, and state a connection between the so-called *actuarial equivalence principle* (see Chapter 6 in Reference [57], Section on Premium Calculation, Nonlife in References [46,58]), and the feedback controllers found by means of the Dynamic Programming technique. We will end-up showing that, as in References [4,27–29], the agents are willing to pay for the coverage of the insurance, in both: the one player context (see References [15,16]), and the game theoretic case with asymmetries (see References [1,51–53]); where the unbalance comes from the choice of different fat-tailed distributions for the terminal times of extraction of the agents (see References [50,56,59]).

The rest of the paper is organized as follows. In Section 2 we state the main hypotheses of our model and argue about the connection of the game theoretic framework and the actuarial perspective. Section 3.1 is a reduced version of our study for a degenerate game where only one player performs extraction tasks. This part of the work allows us to exemplify an actuarial interpretation of the verification result given by Theorem 1; and define what we call *ease of the extraction* of the player in terms of the intensity of the extraction rate. In Section 3.2 we state and explicitly solve the two-player extraction game where the random terminal times are distributed according to Weibull and Chen laws; compare the results in terms of the ease of extraction of each player for particular choices of the parameters; and interpret the verification Theorem 3 in actuarial terms. We give our conclusions in Section 4.

2. Problem Statement

In this section we present the problem we are interested in, introduce our hypotheses, and explain its connection with some ideas from the Actuarial Sciences.

2.1. Game Theoretical Framework

Let us consider the conflict-control process of the extraction of a non-renewable resource in which two participants are involved. We assume that this set of agents remains fixed for the whole duration of the process.

We describe the dynamics of the consumption of the resource by means of the model presented in Reference [12] (Chapter 10.3), according to which,

$$\dot{x}(t) = -u_1(t) - u_2(t), \text{with } \ x(t_0) = x_0, \tag{1}$$

where $x(t)$ is the amount of the resource available at time $t \geq 0$, $u_i(t)$ is the extraction rate of the i-th agent at time t, x_0 is the initial amount of the stock, and $i = 1, 2$.

Let $\Gamma(x_0)$ be a differential game whose system satisfies the following conditions.

Hypothesis 1. **(a)** *Both players act simultaneously and start the game at some initial time $t_0 = 0$ from the state x_0.*

(b) *The control variables of the players are their respective extraction rates at every moment in time, namely $u_1, u_2 : [0, \infty) \to \mathcal{U}$, where $\mathcal{U}$ is a compact subset of $[0, \infty)$.*

(c) *The dynamics of the system is given by (1).*

The system (1) mirrors the fact that the resource is non-renewable because, by Hypothesis 1(b), $x(\cdot)$ is non-increasing.

We assume that the extraction performed by each agent stops at some random moment of time T_i, for $i = 1, 2$. We impose the following hypothesis on the random terminal times T_i.

Hypothesis 2. *The cumulative distribution function of T_i satisfies the following:*

(a) *The random terminal times of the firms are distributed according to some (known) functions $F_i : [0, \infty) \to [0, 1]$, $i = 1, 2$, that satisfy the normalization condition*

$$\int_0^\infty \mathrm{d}F_i(t) = 1,$$

and that are absolutely continuous with respect to Lebesgue's measure.

(b) *Given the different characteristics of the firms, the terminal times of extraction of the same resource are mutually independent.*
(c) *As soon as one of the firms reaches its terminal time, it quits the game and the remaining one keeps extracting the resource until its terminal time is reached (which might happen when the resource becomes extinct).*

We define the *failure rate function associated with the i-th firm* (see References [57,60,61] (Chapter 3; Chapters 4 and 5; Chapter 8.5)) as

$$\lambda_i(t) := \frac{f_i(t)}{1 - F_i(t)}, \tag{2}$$

where $f_i := F_i'$ is a density function for the random terminal time of the i-th player. The existence of such density function is ensured by Hypothesis 2(a).

By virtue of Hypothesis 2(c), $\Gamma(x_0)$ collapses to a one-player game at a random terminal time that may be formed by means of the rule of correspondence

$$T := \min\{T_1, T_2\}. \tag{3}$$

Now, by Hypothesis 2(b), the results in Reference [61] (Chapter 16.3) and the well-known relation

$$1 - F_i(t) = \exp\left(- \int_0^t \lambda_i(s) \mathrm{d}s \right) \tag{4}$$

(see Reference [57] (Chapter 3)), we define the distribution function of the random variable T as

$$\begin{aligned} F(t) &:= 1 - (1 - F_1(t))(1 - F_2(t)) \\ &= 1 - \mathrm{e}^{-\int_{t_0}^t \lambda(s)\mathrm{d}s}, \end{aligned} \tag{5}$$

where $\lambda(t) := \lambda_1(t) + \lambda_2(t)$ stands for the hazard rate function of the random variable T (see, for instance Reference [57] (Chapter 9.3)).

We now introduce the payoff function and the performance indices for each player in the game $\Gamma(x_0)$. To do this, we assume the following conditions.

Hypothesis 3. *For $i = 1, 2$, the utility functions h_i and Φ_i are continuous in all of their arguments, and concave with respect to u_i. Additionally, the function h_i satisfies either:*

(a) *It is a nonnegative function, that is: $h_i : \mathbb{R} \times \mathcal{U}^n \to [0, \infty)$.*
(a') *The product $f_i \cdot h_i$ is such that*

$$\int_0^\infty \int_0^\infty |f_i(t) \cdot h_i(x(\tau), u_1(\tau), u_2(\tau))| \mathrm{d}t \mathrm{d}\tau < \infty.$$

At time T given by (3), if the i-th player is the only one remaining in the extraction game, she receives a terminal payoff $\Phi_i(x(T))$.

With Hypothesis 3, we can introduce the performance index we are interested in.

Define $u : [0, \infty) \to \mathcal{U}^2$ as the vector of functions (u_1, u_2). For $i = 1, 2$, the performance index for the i-th player is:

$$\begin{aligned} K_i(x_0, u_1, u_2) &= \mathbb{E}_{x_0}^{u_1,u_2}\left[\int_0^{T_i} h_i(x(\tau), u_1(\tau), u_2(\tau)) \mathrm{d}\tau \chi_{\{T_i \leq T_j\}} \right] && (6) \\ &+ \mathbb{E}_{x_0}^{u_1,u_2}\left[\int_0^{T_j} h_i(x(\tau), u_1(\tau), u_2(\tau)) \mathrm{d}\tau \chi_{\{T_i > T_j\}} \right] && (7) \\ &+ \mathbb{E}_{x_0}^{u_1,u_2}\left[\Phi_i(x(T)) \chi_{\{T_i > T_j\}} \right], && (8) \end{aligned}$$

where $\mathbb{E}_{x_0}^{u_1,u_2}[\cdot]$ is the conditional expectation of $\cdot$ given that the initial stock is x_0, and the agents use the strategies u_1 and u_2; $\chi_{\{\cdot\}}$ is an indicator function; T is as in (3); and $\Phi_i(\cdot)$ is the terminal payoff function referred to in the final part of Hypothesis 3.

Remark 1. *Note that the payoff of the game has two components: the integral payoff* (6) *and* (7)*; achieved while playing, and* (8)*, a final reward, which is assigned to the player that stayed longer in the system.*

Now we use the compactness mentioned in Hypothesis 1(b) to introduce the sort of optimality we are interested in.

Definition 1. *Let* $\Pi^i := \{u_i : [0,\infty) \times [0,x_0] \to [0,x_0] | u_i \text{ is Lebesgue-measurable}\}$*, for* $i = 1,2$*. We say that a pair of feedback strategies* $(u_1^*, u_2^*) \in \Pi^1 \times \Pi^2$ *is* optimal *if it is a so-called* Nash equilibrium*, that is, if*

$$\begin{aligned} K_1(x_0,u_1^*,u_2^*) &\geq K_1(x_0,u_1,u_2^*) \text{ for every } u_1 \in \Pi^1, \\ K_2(x_0,u_1^*,u_2^*) &\geq K_2(x_0,u_1^*,u_2) \text{ for every } u_2 \in \Pi^2. \end{aligned}$$

In this game, each firm intends to maximize its profit. Then, we designate the corresponding strategies of the players as u_1^*, u_2^*, and call them "optimal". We also write the trajectory under such strategies as x^*, and also call it "optimal trajectory". Let $h_i^*(t) := h_i(x^*, u_1^*, u_2^*)$, and rewrite the optimal expected payoff resulting from the maximization of (6)–(8) as

$$\begin{aligned} K_i(x_0,u_1^*,u_2^*) &= \mathbb{E}_{x_0}^{u_1^*,u_2^*}\left[\int_0^{T_i} h_i^*(\tau)\mathrm{d}\tau\chi_{\{T_i\leq T_j\}}\right] \\ &\quad +\mathbb{E}_{x_0}^{u_1^*,u_2^*}\left[\int_0^{T_j} h_i^*(\tau)\mathrm{d}\tau\chi_{\{T_i> T_j\}}\right] \\ &\quad +\mathbb{E}_{x_0}^{u_1^*,u_2^*}\left[\Phi_i(x^*(T))\chi_{\{T_i> T_j\}}\right]. \end{aligned}$$

The following result is an extension of Reference [1] (Corollary 3.1). We include its proof here for the sake of completeness.

Proposition 1. *Let Hypotheses 1–3 hold. If*

$$H_i(\theta) := \int_0^{\theta} h_i^*(t)\mathrm{d}t < \infty \tag{9}$$

for all $\theta > 0$*, then the optimal expected payoff for the problem starting at* $t = 0$ *is given by*

$$K_i(x_0,u_1^*,u_2^*) = \int_0^\infty h_i^*(\theta)(1-F(\theta)) + \Phi_i(x^*(\theta))f_j(\theta)(1-F_i(\theta))\mathrm{d}\theta.$$

Proof. It is easy to see that

$$\begin{aligned} K_i(x_0,u_1^*,u_2^*) &= \int_0^\infty\int_0^\infty\int_0^\theta h_i^*(t)\mathrm{d}t\chi_{\{\theta<\tau\}}f_j(\tau)\mathrm{d}\tau f_i(\theta)\mathrm{d}\theta \\ &\quad +\int_0^\infty\int_0^\infty\int_0^\tau h_i^*(t)\mathrm{d}t\chi_{\{\theta>\tau\}}f_i(\theta)\mathrm{d}\theta f_j(\tau)\mathrm{d}\tau \\ &\quad +\int_0^\infty\int_0^\infty \Phi_i(x^*(\tau))\chi_{\{\theta>\tau\}}f_j(\tau)\mathrm{d}\tau f_i(\theta)\mathrm{d}\theta. \end{aligned}$$

The use of (9) and Fubini's rule yield

$$K_i(x_0,u_1^*,u_2^*) = \int_0^\infty \int_0^\tau H_i(\theta)f_i(\theta)\mathrm{d}\theta f_j(\tau)\mathrm{d}\tau + \int_0^\infty \int_0^\theta H_i(\tau)f_j(\tau)\mathrm{d}\tau f_i(\theta)\mathrm{d}\theta \tag{10}$$

$$+ \int_0^\infty \int_0^\infty \Phi_i\left(x^*(\tau)\right)\chi_{\{\theta>\tau\}}f_j(\tau)\mathrm{d}\tau f_i(\theta)\mathrm{d}\theta. \tag{11}$$

An integration by parts yields

$$\int_0^\infty \int_0^\tau H_i(\theta)f_i(\theta)\mathrm{d}\theta f_j(\tau)\mathrm{d}\tau + \int_0^\infty \int_0^\theta H_i(\tau)f_j(\tau)\mathrm{d}\tau f_i(\theta)\mathrm{d}\theta \tag{12}$$

$$= \int_0^\infty H_i(\theta)f_i(\theta)\mathrm{d}\theta - \int_0^\infty H_i(\theta)f_i(\theta)F_j(\theta)\mathrm{d}\theta \tag{13}$$

$$+ \int_0^\infty H_i(\theta)f_j(\theta)\mathrm{d}\theta - \int_0^\infty H_i(\theta)f_j(\theta)F_i(\theta)\mathrm{d}\theta. \tag{14}$$

Working with $\int_0^\infty H_i(\theta)f_i(\theta)\mathrm{d}\theta$ and $\int_0^\infty H_i(\theta)f_i(\theta)F_j(\theta)\mathrm{d}\theta$ separately yields:

$$\int_0^\infty H_i(\theta)f_i(\theta)\mathrm{d}\theta = \lim_{\theta\to\infty} H_i(\theta) - \int_0^\infty h_i^*(\theta)F_i(\theta)\mathrm{d}\theta, \tag{15}$$

$$\int_0^\infty H_i(\theta)f_i(\theta)F_j(\theta)\mathrm{d}\theta = \lim_{\theta\to\infty} H_i(\theta) - \int_0^\infty H_i(\theta)f_j(\theta)F_i(\theta)\mathrm{d}\theta - \int_0^\infty h_i^*(\theta)F_i(\theta)F_j(\theta)\mathrm{d}\theta.$$

The last relations imply

$$\int_0^\infty H_i(\theta)f_i(\theta)F_j(\theta)\mathrm{d}\theta + \int_0^\infty H_i(\theta)f_j(\theta)F_i(\theta)\mathrm{d}\theta = \lim_{\theta\to\infty} H_i(\theta) - \int_0^\infty h_i^*(\theta)F_i(\theta)F_j(\theta)\mathrm{d}\theta. \tag{16}$$

Substituting (15) and (16) in (12)–(14) and collecting similar terms yield:

$$\int_0^\infty \int_0^\tau H_i(\theta)f_i(\theta)\mathrm{d}\theta f_j(\tau)\mathrm{d}\tau + \int_0^\infty \int_0^\theta H_i(\tau)f_j(\tau)\mathrm{d}\tau f_i(\theta)\mathrm{d}\theta$$

$$= \lim_{\theta\to\infty} H_i(\theta) - \int_0^\infty h_i^*(\theta)\left(F_i(\theta)+F_j(\theta)-F_i(\theta)F_j(\theta)\right)\mathrm{d}\theta$$

$$= \int_0^\infty h_i^*(\theta)\left(1-F(\theta)\right)\mathrm{d}\theta. \tag{17}$$

Note that the term $\int_0^\infty \int_0^\infty \Phi_i\left(x^*(\tau)\right)\chi_{\{\theta>\tau\}}f_j(\tau)\mathrm{d}\tau f_i(\theta)\mathrm{d}\theta$ in (11) equals

$$\int_0^\infty \Phi_i\left(x^*(\tau)\right)f_j(\tau)\mathrm{d}\tau f_i(\theta)\mathrm{d}\theta = \int_0^\infty \Phi_i\left(x^*(\theta)\right)f_j(\theta)\mathrm{d}\theta - \int_0^\infty F_i(\theta)\Phi_i\left(x^*(\theta)\right)f_j(\theta)\mathrm{d}\theta$$

$$= \int_0^\infty \Phi_i\left(x^*(\theta)\right)f_j(\theta)\left(1-F_i(\theta)\right)\mathrm{d}\theta, \tag{18}$$

where the first equality is due to an integration by parts. The substitution of (17) and (18) in (10) and (11) gives the result. □

2.2. Interconnection with the Actuarial Sciences

In the traditional Actuarial Sciences literature, there are two main principles under which it is possible to compute the amount (of money, time or effort) to be invested/reserved –know as "premium" (premia)- in exchange for a benefit (*e.g.*, the coverage of an insurance or the earnings of the extraction tasks). These principles are classified according to the following methodologies.

PI. *Optimization of the probability that the trajectory $x(\cdot)$ attains certain set.* This approach and its refinements yield what is called "percentile premium", "value-at-risk" (see References [60,62] (Chapters 3.5.3 and 3.5.4)) or "risk premium" (see Reference [61] (Example 6.5 and Chapter 17)).

PII. *Search of a premium such that the decision maker is indifferent between taking the risk or not.* This is typically done by setting the expectation of the utility function under uncertainty equal to the utility function without risk. A remarkable simplification arises when the utility function is linear, for in that case, the expectation of the prospective loss turns out to be null, and the computation of the corresponding surcharge is very straightforward (see Reference [57] (Example 6.1.1)). The result of this method is called "equivalence premium", "indifferent price", or "utility/benefit premium".

From the point of view of the actuarial scientist, we are using a third principle, which connects the traditional approaches. Thus, we state such third principle:

PIII. *Optimization of the conditional expectation* (6)–(8) (see References [40] and [63] (Chapter 6)) to find the premium which is to be exchanged for the benefits derived from the extraction tasks. Such a mixture resembles the approach of the tail-value-at-risk –also dubbed *conditional tail expectation* and *expected shortfall*- (cf. References [64–66] and [60] (Chapter 3.5.4)) from **PI**, in the sense that the objective function of the optimization program we work with is a conditional expectation. However, due to the logarithmic form of the reward functions we use; and the particular form of the HJBI equations that this yields, the premia we obtain are very much alike the indifferent prices one would get, should one have chosen to work with the equivalence principle from **PII** in the first place (see the Remarks 2 and 3 below). In the present context, we follow References [31,33] to present the development of the extraction tasks as a process with uncertain duration, and interpret the maximizers u_1^* and u_2^* as measures of the value's cost that each of the agents earns from such development, that is, as the premia that they should invest in order to ensure/insure their operation.

We consider games with a random terminal time T, to which the basic terminology of reliability theory can be applied directly. In fact, the failure rate function (2) can be thought of as a conditional density provided that the agent did not *default* (i.e., leave the game until the moment t). In our terminology, we would talk about the density of the terminal time of the game, provided that the game was not terminated before the moment t. The failure rate function $\lambda_i(t)$ that describes the life cycle for the player i has the form described by Figure 1. We follow Reference [67] (Chapter 1) to identify three phases of the hazard rate with respect to time.

- The first phase is called the pre-run phase. According to the theory of reliability, the failures in this phase arise due to undetectable latent defects. Specificity of this problem is understandable not only from the point of view of the application of elements to technical systems, in actuarial risk theory, for such a period the terms "newborn period", "infant mortality" and "early failures" can be used (see Reference [57] (Chapter 3)). From the point of view of game theory, early failures can be caused by inexperience, that is, inconsistency of the players just who just entered the game. The failure rate function $\lambda_i(\cdot)$ in this phase is a decreasing function of time.
- The next period of the life cycle of the system is the so-called period of normal system operation. The failure rate function $\lambda_i(\cdot)$ in this period is constant (or approximately constant), and the sudden failures themselves are caused by imperfection of the system itself, or are caused by some external factor. This is called the "adult" period of the process (see Reference [67] (Chapters 1, 3 and 5)). The game in the period under consideration can stop under the influence of some unforeseen circumstances of the external environment.
- In the last period, the system goes into the aging phase. The system failures in this period are associated with how the system ages, and that's why the failure rate function $\lambda(t)$ is an increasing function.

Moreover, we are interested in presenting a comparative analysis of the results when the duration of the extraction is distributed according to the laws of Weibull and Chen. The former has been widely used for modelling losses, time-until-failure of many non-renewable electronic devices (electronic lamps, semiconductor devices, some microwave devices) and lifetimes in general (see References [57,59–61,67]), while the latter is a fat-tailed distribution (see Reference [56]).

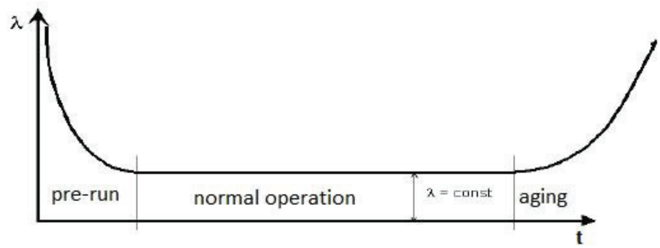


Figure 1. A *U*-shaped (bathtub) hazard rate function.

The cumulative distribution function of the Weibull distribution is:

$$F_1(t) = 1 - \exp\left(-\lambda_1 t^{\delta_1}\right), \tag{19}$$

where $\lambda_1 > 0$ is a scale parameter, and $\delta_1 > 0$ is a shape parameter that corresponds to one of the three phases in which the lifetime of the player can be located. Namely, the value $\delta_1 < 1$ corresponds to the pre-run period, here the failure rate function $\lambda_1(t)$ is a decreasing function of t. At $\delta_1 = 1$, the system is in the normal operation mode, and $\lambda_1(\cdot)$ equals a constant value of $\lambda_1 > 0$. We note that for $\delta_1 = 1$, the Weibull distribution corresponds to an exponential distribution. For $\delta_1 > 1$, the system is in an aging state, therefore $\lambda_1(\cdot)$ is an increasing function. A special case of the Weibull distribution for this instance is the so-called *Rayleigh* distribution (see Reference [60] (Appendix A.3)). We are in this case when $\delta_1 = 2$.

By (2), the corresponding failure rate function is

$$\lambda_1(t) = \lambda_1 \delta_1 t^{\delta_1 - 1}. \tag{20}$$

The cumulative distribution function of Chen distribution is

$$F_2(t) = 1 - \exp\left(\lambda_2\left(1 - e^{t^{\delta_2}}\right)\right), \tag{21}$$

where $\lambda_2 > 0$ is a scale parameter, and $\delta_2 > 0$ is a shape parameter. Now, for this case, it follows from (2) that the failure rate function takes the form

$$\lambda_2(t) = \lambda_2 \delta_2 t^{\delta_2 - 1} e^{t^{\delta_2}}. \tag{22}$$

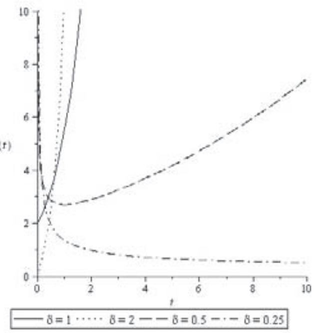
If $\delta_2 < 1$, we will be at the "newborn" phase. Here, the failure rate function $\lambda_2(\cdot)$ is bathtub-shaped. This corresponds to a realistic process of extracting natural resources. When $\delta_2 = 1$, the system is in the normal operation mode and the hazard rate function $\lambda_2(\cdot)$ is increasing. At $\delta_2 > 1$, the system is in aging state, and $\lambda_2(\cdot)$ is also an increasing function, but from

$$\lambda_2'(t) = \lambda_2 \delta_2 t^{\delta_2 - 1} e^{t^{\delta_2}} \text{ for } t > 0,$$

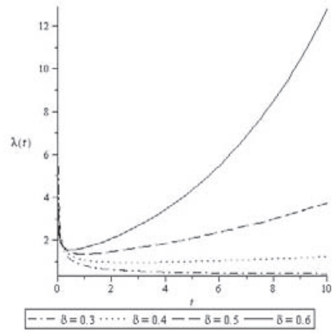
it is straightforward that the growth rate of $\lambda_2(\cdot)$ is noticeably larger than in the case of normal operation. This implies that there is a greater probability of failure at this stage of the extraction.

Graphical representations of the failure rate function of Chen distribution for two values of the scale parameter λ_2 are shown in Figure 2. In Figure 2b, note that, as $\delta \to 1$, the slope of the graph grows larger. This fact might be interpreted by arguing that Chen's distribution plausibly describes how the system goes from the pre-run state into the normal operation mode.

Graphical representations of the failure rate functions for the Weibull and Chen distributions for fixed $\lambda_1 = 2 = \lambda_2$, and the same values of the parameter δ_1 and δ_2 are shown in Figure 3. Note that we display these functions for each of the periods we have identified.

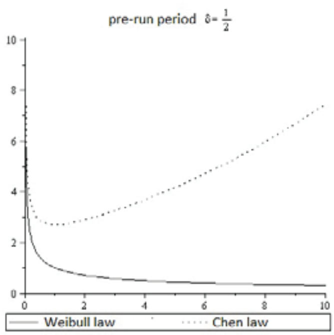


(**a**) $\lambda_2 = 2$, different modes of δ_2.

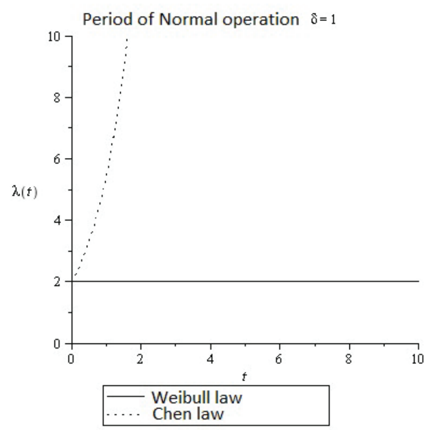


(**b**) $\lambda_2 = 1$, "newborn" mode.

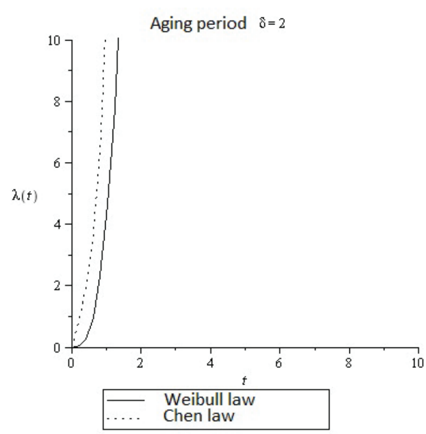
Figure 2. The failure rate function $\lambda_2(\cdot)$.



(**a**)



(**b**)



(**c**)

Figure 3. Failure rate functions for Weibull and Chen distributions with different shape parameters. (**a**) Comparison of the failure rate functions in the pre-run stage. (**b**) Comparison of the failure rate functions at the stage of normal operation. (**c**) Comparison of the failure rate functions during the system aging period.

3. Resource Exploitation under Two Parametric Interpretations

It is easy to see that Weibull and Chen distributions (19) and (21) verify the Hypothesis 2(a). We will consider the cases of these probability laws for the random terminal times of the players, and take $T = \min\{T_1, T_2\}$ for different parameters of these families. This will allow us to model failures of the equipments depending on their operation mode.

3.1. Dynamic Models for the Extraction of Natural Resources by One Agent

In this section, we follow Reference [32], and consider the situation where only one agent performs extraction tasks, that is, the degenerate game where $n = 1$ (in this case, we do not consider a terminal payoff function, because the *game* finishes as soon as the only player leaves the system). Let $x(t)$ be the stock of the non-renewable resource under consideration. Then, the dynamics (1) reduces to

$$\dot{x}(t) = -u(t),\ x(t_0) = x_0 > 0, \tag{23}$$

where $u(t) \geq 0$. An application of Proposition 1 yields that the expected payoff of the agent, provided that the terminal random time follows Weibull's law, that is

$$K(x_0, u) = \int_0^{\infty} h(x(t), u(t)) \mathrm{e}^{-\lambda t^{\delta}} \mathrm{d}t. \tag{24}$$

Note that adding a terminal cost does not make sense with a single agent, so we set $\Phi(x) \equiv 0$.
If we use Chen's law for the random terminal time, the winnings of the extractor are given by

$$K(x_0, u) = \int_0^{\infty} h(x(t), u(t)) \exp\left[-\lambda\left(1 - \mathrm{e}^{t^{\delta}}\right)\right] \mathrm{d}t. \tag{25}$$

The problem of obtaining feedback maximizers for the expected gains in Formulae (24) and (25) under the condition (23) can be solved using the following Bellman equation

$$\lambda(t) W(x,t) = \frac{\partial}{\partial t} W(x,t) + \max_u \left(h(t,u) - u \frac{\partial}{\partial x} W(x,t)\right), \tag{26}$$

with transversality condition

$$\lim_{t \to \infty} W(x,t) = 0. \tag{27}$$

In (26), $\lambda(\cdot)$ corresponds either to the failure rate of Weibull or Chen distributions. The details on the necessary calculations for achieving (26) and (27) can be seen in, for instance Reference [68] (Chapter I.5).

The following result assumes a specific form of the agent's utility function. There are, of course, many other forms, which need to be concave, continuous and non-decreasing. However, we have chosen this particular form because of the interest that the result has from the point of view of the Actuarial Scientist (see Remark 2 below).

Theorem 1. *If the utility function is of the form*

$$h(x,u) = \ln u, \tag{28}$$

then the optimal controller is given by the Lebesgue-measurable function

$$u^*(t,x) = \frac{x}{\bar{a}_t}, \tag{29}$$

where

$$\bar{a}_t := \int_0^{\infty} \frac{1 - F(t+s)}{1 - F(t)} \mathrm{d}s. \tag{30}$$

Proof. The substitution of the *ansatz* (see References [34,35]):

$$W(x,t) = A(t)\ln x + B(t), \tag{31}$$

into Bellman's Equation (26) yields

$$\begin{aligned} \frac{\partial}{\partial x} W(x,t) &= \frac{A(t)}{x}, && (32) \\ \frac{\partial}{\partial t} W(x,t) &= \dot{A}(t)\ln x + \dot{B}(t). && (33) \end{aligned}$$

Plugging (32) and (33) into (26) gives us that the maximized control is:

$$u^*(t,x) = \frac{x}{A(t)}, \tag{34}$$

and also the following system of differential equations:

$$\begin{aligned} \dot{A}(t) - \lambda(t)A(t) + 1 &= 0; && (35) \\ \dot{B}(t) - \lambda(t)B(t) - \ln(A(t)) - 1 &= 0. && (36) \end{aligned}$$

The transversality condition (27) takes the form

$$\begin{aligned} \lim_{t\to\infty} A(t) &= 0, && (37) \\ \lim_{t\to\infty} B(t) &= 0. && (38) \end{aligned}$$

Using (37) and (thus) the integrating factor (4) we can solve (35) and get

$$\begin{aligned} A(t) &= \frac{1}{1-F(t)} \int_t^\infty e^{-\int_0^s \lambda(\tau)d\tau} ds \\ &= \frac{\int_t^\infty 1 - F(s)ds}{1-F(t)}. \end{aligned}$$

The last equality follows from (4). Here, of course, $F(\cdot)$ is given by (19) or (21).

From (30), it is straightforward that

$$A(t) = \bar{a}_t. \tag{39}$$

The substitution of (39) in (34) gives (29).

The fact that the controller (29) is optimal for the degenerate game follows from Theorem I.7.1(a) in Reference [68]. The fact that that such controller is Lebesgue-measurable follows from Hypothesis 1(b), along with the so-named measurable selection theorems (see, for instance, References [69–71] (Theorem 12.1; Proposition D5(a); Theorem 3.4)). This proves the result. □

Remark 2. *An actuarial interpretation of Theorem 1 is the fact that the function $A(t)$ agrees with the expectation of the so-called* contingent life annuity with 0% interest rate for a life aged (t) *displayed in* (30) *(see Reference [57] (Chapter 5.2)) and write*

$$x - u^*(t,x)\cdot \bar{a}_t = 0. \tag{40}$$

*This expression is remarkably similar to the relations (6.2.3)–(6.2.4) in Reference [57] (see **PII** in Section 2.2, References [57,58] (Chapters 6.2 and 7.2; sections on Premium Calculation for Nonlife Insurance), which are used to establish the* equivalence premium $u^*(t,x)$ *that is to be* continuously paid *to obtain a benefit of x. In the Actuarial Mathematics literature, we use this expression to state the existence of a balance between an* expected income $u^*(t,x)\cdot\bar{a}_t$ *(that will be paid over a contingent horizon), and an* expected benefit x *(that* *will be received at a given moment of time). From this point of view, we might distinguish two parts within the Lebesgue-measurable optimal rate of extraction $u^*(t,x)$ from (29):*

(i) *the benefit that the agent will eventually get, x; and*
(ii) *the* intensity of the extraction *that the agent needs to apply to acquire $u^*(t,x)$, that is, $A(t) = \bar{a}_t$.*

Keeping this in mind, we can state that the resulting instantaneous utility of obtaining x by continuously extracting $u^*(t,x)$ with an intensity of $\bar{a}_t$ is given by $h(x,u)$ in (28).

Since the intensity of the extraction appears in the denominator of (29), we will dub $\frac{1}{\bar{a}_t}$ as ease of extraction*. Our goal is to emphasize the inverse proportionality of the optimal controller $u^*(t,x)$ with respect to the* intensity of extraction *$\bar{a}_t$.*

We obtain the following result as a by-product of Theorem 1.

Theorem 2. *If the utility function is of the form (28), then the value function for the optimal control problem of maximizing (6) within $\mathcal{U}$ subject to (1) is given by*

$$W(x,t) = \bar{a}_t \ln x + \int_0^\infty (1 + \ln \bar{a}_s) \frac{1 - F(s+t)}{1 - F(t)} \mathrm{d}s. \tag{41}$$

Proof. From (31), we readily know that

$$W(x,t) = \bar{a}_t \ln x + B(t).$$

To find the function $B(t)$, we note that the transversality condition (38) and again, the integrating factor (4) give that

$$\begin{aligned} B(t) &= \frac{\int_t^\infty \exp\left(-\int_0^s \lambda(\tau)\mathrm{d}\tau\right)(1 + \ln A(s))\,\mathrm{d}s}{\exp\left(-\int_0^t \lambda(s)\mathrm{d}s\right)} \\ &= \frac{\int_t^\infty (1 - F(s))(1 + \ln \bar{a}_s)\,\mathrm{d}s}{1 - F(t)}. \end{aligned} \tag{42}$$

The last equality follows from (4) and (39). Substituting (42) into (31) gives (41). The fact that this function is actually the value of the optimal control problem of maximizing (6) subject to (1) follows from the verification Theorem I.7.1(b) in Reference [68]. This proves the result. □

3.1.1. Normal Mode ($\delta_1 = 1 = \delta_2$)

As we already stated, for the case of Weibull distribution, when $\delta_1 = 1$, the random terminal time is exponentially distributed with mean λ_1^{-1}. Then, by Theorem 1, the optimal strategy of the agent is $u^*(t,x) = \lambda_1 x$. We solve (23) and get that the optimal trajectory is

$$x^*(\tau) = x_0 \mathrm{e}^{-\lambda_1 \tau}.$$

For the case of Chen's law, we let $\delta_2 = 1$ and substitute (22) into (29) to get

$$u^*(t,x) = \frac{x \exp\left(-\lambda_2 \mathrm{e}^t\right)}{\int_t^\infty \exp\left(-\lambda_2 \mathrm{e}^s\right) \mathrm{d}s}.$$

We substitute this controller into (23) and solve the differential equation to obtain the optimal trajectory when the random terminal time is distributed according to Chen's law and $\delta_2 = 1$. That is:

$$x^*(\tau) = x_0 \exp\left(-\int_0^\tau \frac{\exp\left(\mathrm{e}^{-\lambda_2 t}\right)}{\exp\left(-\int_\tau^\infty \mathrm{e}^{-\lambda_2 s}\mathrm{d}s\right)}\mathrm{d}t\right).$$

Figures 4–6 summarize these results when $\lambda_1 = 1 = \lambda_2$, which, since λ_1 and λ_2 are scale parameters of the distributions under study, we can have a sufficiently general idea of our developments. The reason is that, if we select other values for these parameters, we will end up having constant multiples of the random variables X_1 and X_2 analyzed in this study (see Reference [60] (Section 4.2.1)).

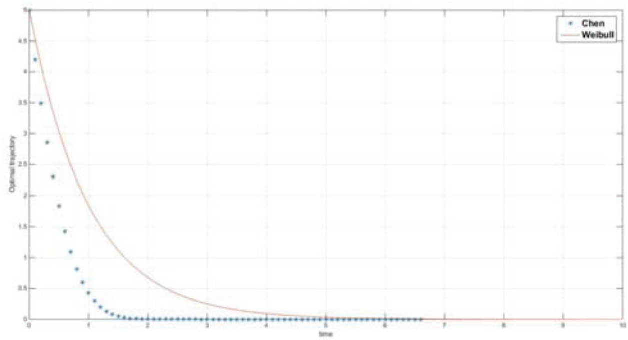


Figure 4. Optimal trajectories for the laws of Weibull (continuous-red) and Chen (dotted-blue) when $\delta_1 = 1 = \delta_2$ and $\lambda_1 = 1 = \lambda_2$.

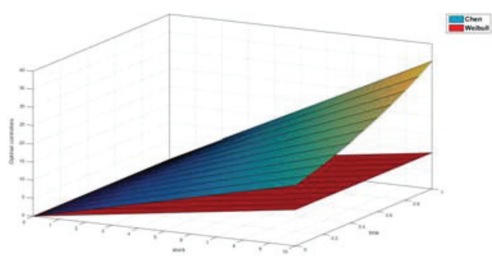


Figure 5. Optimal Weibull (red) and Chen (blue) controllers when $\delta_1 = 1 = \delta_2$ and $\lambda_1 = 1 = \lambda_2$ for $(t, x) \in [0, 1] \times [0, 10]$.

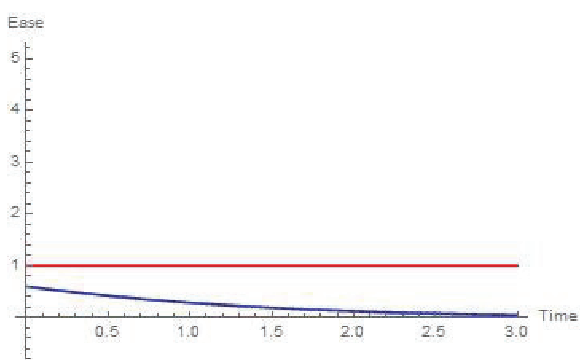


Figure 6. Ease of the extraction for the laws of Weibull (straight-red) and Chen (decreasing-blue) when $\delta_1 = 1 = \delta_2$ and $\lambda_1 = 1 = \lambda_2$.

We might give a plausible interpretation of Figures 5 and 6 in the direction of Remark 2. The optimal extraction rates are linear in the state variable and, as we established in Remark 2(ii), the largest the value of $A(t) = \bar{a}_t$ is, the easiest it is for the extractor to obtain $u^*(t, x)$. In this sense, it is straightforward that, for Chen's law, as time goes by, it becomes harder to extract at a rate of $u^*(t, x)$. On the other hand, it is very easy to see that, under Weibull's law, $A(t) \equiv 1$ (see References [12,57,61] (Example 5.2.1; p. 323; Chapter 8.10.1)). This implies that, in the normal mode, the agent whose random terminal time follows the Weibull distribution is indifferent to the moment of time when the extraction task takes place, and should only look at the remaining amount of the resource.

As for the optimal trajectories in Figure 4, observe that the system exploited by an agent affected by Chen's law becomes exhausted a lot faster than a system exploited by a Weibull extractor. This is consistent with the fact that, according to Figure 5, a Chen extractor would be more intense in his exploitation tasks.

3.1.2. Aging Mode ($\delta_1 = 2 = \delta_2$)

We stated before that for $\delta_1 = 2 = \delta_2$, Weibull's law coincides with Rayleigh distribution. In this case, (20) gives that $\lambda_1(t) = 2\lambda_1 t$; then from Theorem 1 we get

$$u^*(t, x) \quad = \quad \frac{x\mathrm{e}^{-\lambda_1 t^2}}{\int_t^\infty \mathrm{e}^{-\lambda_1 s^2} \mathrm{d}s}.$$

For Chen's distribution, when we have an aging system, by (22), the failure rate function is $\lambda_2(t) = 2\lambda_2 t \mathrm{e}^{t^2}$, and the Theorem 1 gives that the optimal control is

$$u^*(t, x) = \frac{x \exp\left(-\lambda_2 \mathrm{e}^{t^2}\right)}{\int_t^\infty \exp\left(-\lambda_2 \mathrm{e}^{s^2}\right) \mathrm{d}s}.$$

We solve (23) and get that the optimal trajectories under each law are:

$$x^*(\tau) = \begin{cases} x_0 \exp\left(-\int_0^\tau \frac{\mathrm{e}^{-\lambda_1 t^2}}{\int_t^\infty \mathrm{e}^{-\lambda_1 s^2} \mathrm{d}s} \mathrm{d}t\right) & \text{for Weibull's law with} \quad \delta_1 = 2 \\ x_0 \exp\left(-\int_0^\tau \frac{\exp\left(-\lambda_2 \mathrm{e}^{t^2}\right)}{\int_t^\infty \exp\left(-\lambda_2 \mathrm{e}^{s^2}\right) \mathrm{d}s} \mathrm{d}t\right) & \text{for Chen's law with} \quad \delta_2 = 2. \end{cases}$$

Figures 7–9 summarize these results when $\lambda_1 = 1 = \lambda_2$. There, we can see that, in the aging mode, as time passes by, the extraction tasks need to be more intense for both assumptions.

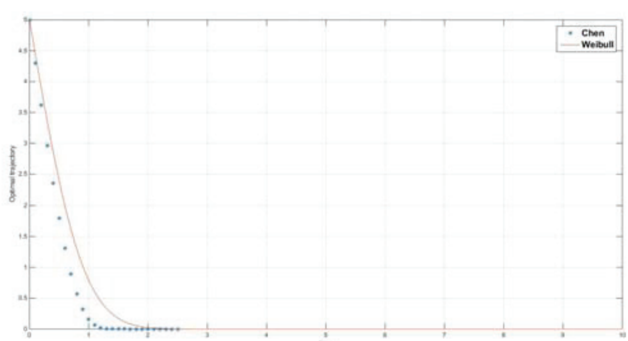


Figure 7. Optimal trajectories for laws of Weibull (continous-red) and Chen (dotted-blue) when $\delta_1 = 2 = \delta_2$ and $\lambda_1 = 1 = \lambda_2$.

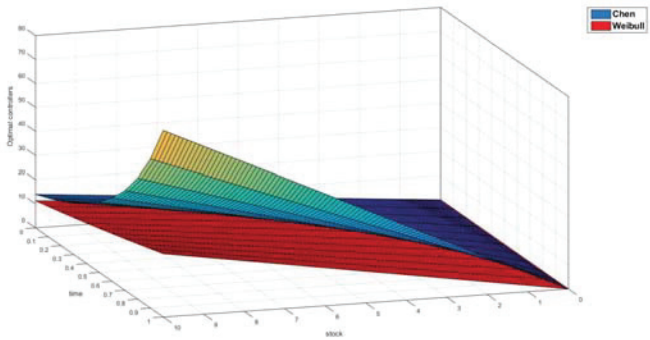


Figure 8. Optimal Weibull (red) and Chen (blue) controllers when $\delta_1 = 2 = \delta_2$ and $\lambda_1 = 1 = \lambda_2$ for $(t, x) \in [0, 1] \times [0, 10]$.

However, for the Chen extractor, the situation deteriorates very rapidly, and hence, it needs to speed up the pace of its tasks. It readily follows from (29) that both controllers are linear in the state. Again, Figure 7 shows how the stock becomes exhausted for each case.

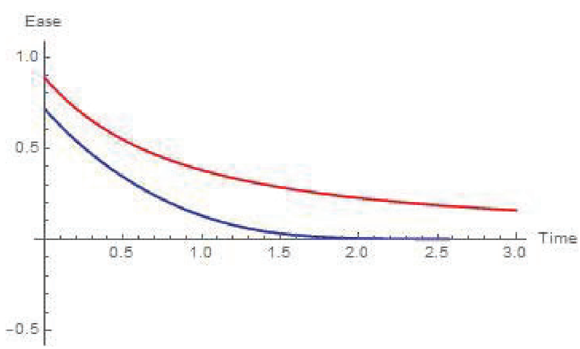


Figure 9. ease of the extraction for Weibull (red) and Chen (blue) laws when $\delta_1 = 2 = \delta_2$ and $\lambda_1 = 1 = \lambda_2$.

3.1.3. Early Period ($\delta_1 = \frac{1}{2} = \delta_2$)

For the Weibull case, (20) yields that the hazard rate function is $\lambda_1(t) = \frac{\lambda_1}{2\sqrt{t}}$. A substitution in (29) gives us

$$u^*(t, x) \quad = \quad \frac{x\mathrm{e}^{-\lambda_1\sqrt{t}}}{\int_t^\infty \mathrm{e}^{-\lambda_1\sqrt{s}}\mathrm{d}s}.$$

For Chen's law, the pre-run system has a hazard rate function of the form $\lambda_2(t) = \frac{\lambda_2}{2\sqrt{t}}\mathrm{e}^{\sqrt{t}}$. The corresponding optimal control has the form

$$u^*(t, x) \quad = \quad \frac{x\exp\left(-\lambda_2\mathrm{e}^{\sqrt{t}}\right)}{\int_t^\infty \exp\left(-\lambda_2\mathrm{e}^{\sqrt{s}}\right)\mathrm{d}s}.$$

We solve (23) and get that the optimal trajectories under each law are:

$$x^*(\tau) = \begin{cases} x_0 \exp\left(-\int_0^\tau \frac{\lambda_1^2}{2(\lambda_1\sqrt{\tau}+1)}\mathrm{d}t\right) & \text{for Weibull's law with} \quad \delta_1 = \frac{1}{2}, \\ x_0 \exp\left(-\int_0^\tau \frac{\exp\left(-\lambda_2 e^{\sqrt{t}}\right)}{\int_t^\infty \exp\left(-\lambda_2 e^{\sqrt{s}}\right)\mathrm{d}s}\mathrm{d}t\right) & \text{for Chen's law with} \quad \delta_2 = \frac{1}{2}. \end{cases}$$

Figures 10–12 summarize these results when $\lambda = 1$. It is of particular interest that, according to Figure 12, for a Chen extractor, we almost have the same situation displayed in Figure 6 with the Weibull extractor. However, in this case, the intensity function $A(t) = \bar{a}_t$ converges to a smaller value than the one displayed in that part of the illustration. This means that, during the early mode of the system, an agent whose random terminal time follows the Chen distribution should only mind about the remaining stock of the resource.

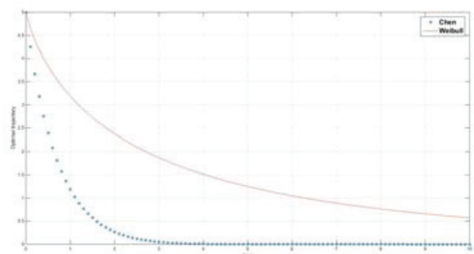


Figure 10. Optimal trajectories for laws of Weibull (continuous-red) and Chen (dotted-blue) when $\delta_1 = \frac{1}{2} = \delta_2$ and $\lambda_1 = 1 = \lambda_2$.

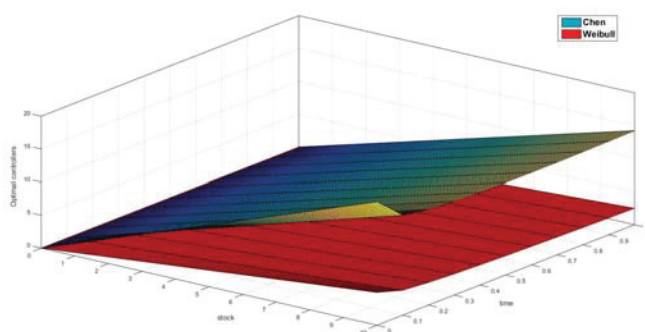


Figure 11. Optimal Weibull (red) and Chen (blue) controllers when $\delta_1 = \frac{1}{2} = \delta_2$ and $\lambda_1 = 1 = \lambda_2$ for $(t, x) \in [0, 1] \times [0, 10]$.

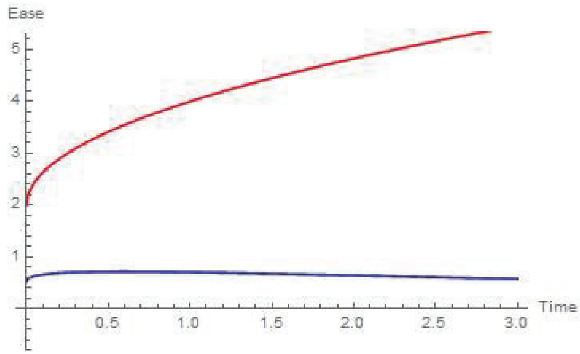


Figure 12. Ease of the extraction for Weibull (red) and Chen (blue) laws when $\delta_1 = \frac{1}{2} = \delta_2$ and $\lambda_1 = 1 = \lambda_2$.

For a Weibull extractor at the pre-run phase, as time goes by, the exploitation becomes less intense, and, in spite of the fact that the reserve will be consumed at an exponential rate, it will last for more time than under Chen assumption (look at Figure 10). An economic interpretation of this fact is that the optimal rules of behavior for the agents demand Chen's law to be more intense in its labors even from the early period, while they allow Weibull's to be less intense (look at Figure 11).

3.2. Game Theoretic Model for the Extraction of Natural Resources

Now we consider the non degenerate game model from Section 2. For that purpose, we will make an extensive use of the results presented in References [1,50] and [57] (Chapter 9.2). Note that the utility function of the agent will explicitly depend only on its own control, and there are no payoff transfers among the players, that is, on the extraction rate applied by the agent, and on the stock of the resource at time $t \geq 0$.

Theorem 3.1 in Reference [1] allows us to state the HJBI equations associated with the optimization problem for the i-th player. Namely,

$$\begin{aligned} & -\frac{\partial}{\partial t} W_i(x,t) + (\lambda_1(t) + \lambda_2(t)) W_i(x,t) \\ = \ & \max_{u_i \in \Pi^i} \left(h_i(t, u_1, u_2) + \Phi_i(x)\lambda_j(t) - \frac{\partial}{\partial x} W_i(x,t)(u_1 + u_2) \right), \qquad (43) \\ & W_i(x,t) \to 0 \text{ as } t \to \infty, \qquad (44) \end{aligned}$$

for $i = 1, 2$.

We will find explicit solutions for (43) when the functions h_i are analogous to (28) for $i = 1, 2$. That is, when

$$h_i(x, u_1(t,x), u_2(t,x)) = \ln(u_i(t,x)). \qquad (45)$$

We also suppose that

$$\Phi_i(x(t \wedge \tau)) = c_i \ln(x(t \wedge \tau)) = c_i \ln(x) \chi_{\tau \leq t}, \qquad (46)$$

for some positive constant values c_i and $i = 1, 2$. In this case, the HJBI Equation (43) turns out to be

$$\begin{aligned} & -\frac{\partial}{\partial t} W_i(x,t) + (\lambda_1(t) + \lambda_2(t)) W_i(x,t) \\ & = \max_{u_i \in \Pi^1} \left(\ln(u_i) + c_i \ln(x) \lambda_{-i}(t) - \frac{\partial}{\partial x} W_i(x,t)(u_1 + u_2) \right), \end{aligned}$$

for $i = 1, 2$. Here,

$$\lambda_{-i}(\cdot) := \begin{cases} \lambda_2(\cdot) & \text{if } i = 1, \\ \lambda_1(\cdot) & \text{if } i = 2. \end{cases}$$

In what follows, we find the optimal strategies and the value function for this problem by proceeding in the same way that led us to (29) and (41). The next result is similar to Reference [1] (Proposition 4.1).

Theorem 3. *If the utility functions are of the form* (28)*, and the terminal payoff functions are given by* (46)*, then the optimal strategies for the game* $\Gamma(x_0)$ *are Lebesgue-measurable, and are given by:*

$$u_i^*(t, x) = \frac{x}{\bar{a}_{[t]_1:[t]_2} + c_i \bar{A}^{1}_{[t]_i:[t]_{-i}}}, \tag{47}$$

for $i = 1, 2$, *where*

$$\bar{a}_{[t]_1:[t]_2} := \int_0^\infty \frac{1 - F(t+s)}{1 - F(t)} \mathrm{d}s, \tag{48}$$

and

$$\bar{A}^{1}_{[t]_i:[t]_{-i}} := \int_0^\infty \frac{1 - F(t+s)}{1 - F(t)} \lambda_{-i}(s+t) \mathrm{d}s. \tag{49}$$

Here, $F(t)$ *is as in* (5).

Proof. The substitution of the informed guesses

$$W_i(x, t) = A_i(t) \ln x + B_i(t),$$

for $i = 1, 2$; into (43) and (44) yields:

- that the maximizers of (43) are of the form

$$u_i^*(t, x) = \frac{x}{A_i(t)}. \tag{50}$$

  The fact that these controllers are Lebesgue-measurable follows from Hypothesis 1(b), along with the so-named measurable selection theorems (see, for instance, References [69–71] (Theorem 12.1; Proposition D5(a); Theorem 3.4)).
- the following Cauchy problem (which is analogous to (35)–(38)):

$$\begin{aligned} -\dot{A}_i(t) + A_i(t)(\lambda_i(t) + \lambda_{-i}(t)) &= 1 + c_i \lambda_{-i}(t), & (51) \\ \dot{B}_i(t) - B_i(t)(\lambda_i(t) + \lambda_{-i}(t)) &= \ln A_i(t) + 1 + \frac{A_i(t)}{A_{-i}(t)} & (52) \\ \lim_{t \to \infty} A_i(t) &= 0, & (53) \\ \lim_{t \to \infty} B_i(t) &= 0. & (54) \end{aligned}$$

We apply the technique of the integrating factor in (51); use the transversality condition (53), and get

$$\begin{aligned} A_i(t) &= \frac{\int_t^\infty (1 + c_i \lambda_{-i}(\tau)) \exp\left(-\int_0^\tau \lambda_i(s) + \lambda_{-i}(s) \mathrm{d}s\right) \mathrm{d}\tau}{\exp\left(-\int_0^t \lambda_i(\tau) + \lambda_{-i}(\tau) \mathrm{d}\tau\right)} \\ &= \frac{\int_t^\infty (1 + c_i \lambda_{-i}(\tau))(1 - F(\tau)) \mathrm{d}\tau}{1 - F(t)}. \end{aligned}$$

The last equality holds by virtue of (5).

We now use (48) and (49) and observe that

$$A_i(t) = \bar{a}_{[t]_1:[t]_2} + c_i \bar{A}_{[t]_i:[t]_{-i}^{\;1}}. \tag{55}$$

The substitution of this expression in (50) yields (47).

By Theorem I.7.1(a) in Reference [68] (see also Reference [52] (Theorem 2(i))), we know that (47) is optimal for the game with no explicit payoff transfer among the players. This proves the result. □

Remark 3. *Just as we interpreted* (39) *in Remark* 2 *as a continuous life annuity, we can do the same with* (55)*.*

- *In the Actuarial Mathematics literature, the* continuous annuity of the joint-life status of the times-until-failure T_1 and T_2 *is designed by the symbol* $\bar{a}_{[t]_1:[t]_2}$*, and represents a natural extension of* (30) *to the case where the payments stop when either player leaves the system (the use of the braces around* t *emphasizes the fact that each player has lived-up to the moment* $t > 0$*, and states that the* age-at-selection *of each player is* t*—see Reference [57] (Chapter 3.8)). Thus we define such annuity as in* (48) *(with null interest rate);*
- *we also design the mathematical expectation of the* present value of one monetary unit payable (to the i-th player) at the moment of failure of the $(-i)$-th player *(with null interest rate) by the symbol* $\bar{A}_{[t]_i:[t]_{-i}^{\;1}}$*, as in* (49) *(the superscript* 1 *means that the* $(-i)$*-th player is the first who fails—see Reference [57] (Chapter 9.7)). The reason is that the expression*

$$\frac{1 - F(t+s)}{1 - F(t)} \lambda_{-i}(s+t)$$

can be thought of as the probability that both *agents survive up to moment* t*, and the* $(-i)$*-th agent fails at moment* $t + s$*. (See References [46,57] (Chapter 9.9)).*

Keeping this in mind we can propose an extension of (40) *and write* (47) *as:*

$$x - u_i^*(t,x) \cdot \left(\bar{a}_{[t]_1:[t]_2} + c_i \bar{A}_{[t]_i:[t]_{-i}^{\;1}} \right) = 0. \tag{56}$$

From an actuarial perspective, this expression means that, for the i-th extractor, there is a balance between the eventual benefit x*, and the continuous rate of extraction* $u_i^*(t,x)$*, which includes a* final *payment (of size* $c_i u_i^*(t,x)$*) that covers the possibility that the other player fails and leaves the game.*

This means that the optimal rate of extraction of the i-th player in (47) *can be viewed as the composition of two parts:*

(i) *the benefit that the agent will eventually get,* x*; and*
(ii) *the* intensity *of the effort that the agent needs to apply to attain such benefit, that is,* (55)*. Observe that this function explicitly takes into account the fact that the i-th agent will receive a payment if the other one leaves the system before he/she does.*

The resulting utility of this exercise for the i-th player is given by (45)*.*

As a by-product of Theorem 3, we can state and prove the following result.

Theorem 4. *If the utility functions are of the form* (28)*, and the terminal payoff functions are given by* (46)*, then the value functions for game* $\Gamma(x_0)$ *are given by*

$$W_i(x,t) = \left(\bar{a}_{[t]_1:[t]_2} + c_i \bar{A}_{[t]_i:[t]_{-i}^{\;1}} \right) \ln x$$
$$+ \int_0^\infty \left(1 + \ln \left(\bar{a}_{[t]_1+s:[t]_2+s} + c_i \bar{A}_{[t]_i+s:[t]_{-i}^{\;1}+s} \right) + \frac{\bar{a}_{[t]_1+s:[t]_2+s} + c_i \bar{A}_{[t]_i+s:[t]_{-i}^{\;1}+s}}{\bar{a}_{[t]_1+s:[t]_2+s} + c_{-i} \bar{A}_{[t]_i+s:[t]_{-i}+s}^{\;1}} \right) \frac{1 - F(s+t)}{1 - F(t)} \mathrm{d}s,$$

for $i = 1, 2$.

Proof. From Theorem 3, we readily know that

$$W_i(x,t) = \left(\bar{a}_{[t]_1:[t]_2} + c_i \bar{A}^{\,1}_{[t]_i:[t]_{-i}}\right) \ln x + B_i(t),$$

for $i = 1, 2$. To find the functions $B_i(t)$, we apply the technique of the integrating factor to (52), and use the transversality condition (54). This gives:

$$\begin{aligned} B_i(t) &= \frac{\int_t^\infty \left(\ln A_i(s) + 1 + \frac{A_i(s)}{A_{-i}(s)}\right) \exp\left(-\int_0^s \lambda_i(\tau) + \lambda_{-i}(\tau) \mathrm{d}\tau\right) \mathrm{d}s}{\exp\left(-\int_0^t \lambda_i(s) + \lambda_{-i}(s) \mathrm{d}s\right)}, \\ &= \frac{\int_t^\infty \left(\ln A_i(s) + 1 + \frac{A_i(s)}{A_{-i}(s)}\right)(1 - F(s)) \mathrm{d}s}{1 - F(t)}, \end{aligned}$$

where $A_i(\cdot)$ is as in (55).

The optimality of the functions W_1 and W_2 follows from Reference [52] (Theorem 2(ii)). This completes the proof. $\square$

The optimal trajectory can be found by plugging (47) into (1) and solving. That is,

$$x^*(\tau) = x_0 \exp\left(-\sum_{i=1}^{2} \int_0^\tau \frac{\exp\left(-\int_0^t (\lambda_1(s) + \lambda_2(s)) \mathrm{d}s\right)}{\int_t^\infty (1 + c_i \lambda_{-i}(s)) \exp\left(-\int_0^s (\lambda_1(r) + \lambda_2(r)) \mathrm{d}r\right) \mathrm{d}s} \mathrm{d}t\right). \tag{57}$$

The relation (57) can also be compactly written as

$$x^*(\tau) = x_0 \exp\left(-\int_0^\tau \frac{1}{\bar{a}_{[t]_1:[t]_2} + c_1 \bar{A}^{\,1}_{[t]_1:[t]_2}} + \frac{1}{\bar{a}_{[t]_1:[t]_2} + c_2 \bar{A}^{1}_{[t]_1:[t]_2}} \mathrm{d}t\right),$$

thus showing the interaction between the players and their effect on the system.

3.3. An Illustration

We devote this Section to the analysis of the particular cases of our interest.

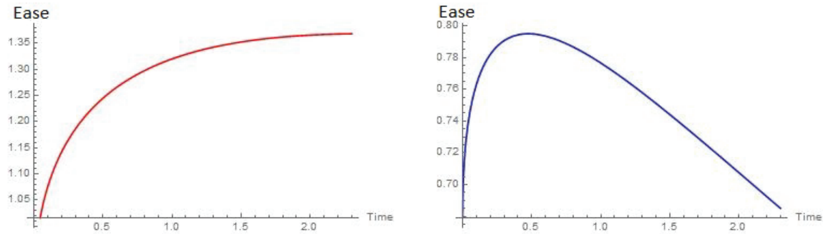
If the random terminal time of player 1 has a Weibull distribution, and that of player 2 follows Chen's law, the optimal extraction rates are:

$$u_1^*(t,x) = \frac{x \exp\left(-\int_0^t (\lambda_1 \delta_1 s^{\delta_1 - 1} + \lambda_2 \delta_2 s^{\delta_2 - 1} \mathrm{e}^{s^{\delta_2}}) \mathrm{d}s\right)}{\int_t^\infty (1 + c_1 \lambda_2 \delta_2 s^{\delta_2 - 1} \mathrm{e}^{s^{\delta_2}}) \exp\left(-\int_0^s (\lambda_1 \delta_1 r^{\delta_1 - 1} + \lambda_2 \delta_2 r^{\delta_2 - 1} \mathrm{e}^{r^{\delta_2}}) \mathrm{d}r\right) \mathrm{d}s},$$

and

$$u_2^*(t,x) = \frac{x \exp\left(-\int_0^t (\lambda_1 \delta_1 s^{\delta_1 - 1} + \lambda_2 \delta_2 s^{\delta_2 - 1} \mathrm{e}^{s^{\delta_2}}) \mathrm{d}s\right)}{\int_t^\infty (1 + c_2 \lambda_1 \delta_1 s^{\delta_1 - 1}) \exp\left(-\int_0^s (\lambda_1 \delta_1 r^{\delta_1 - 1} + \lambda_2 \delta_2 r^{\delta_2 - 1} \mathrm{e}^{r^{\delta_2}}) \mathrm{d}r\right) \mathrm{d}s}.$$

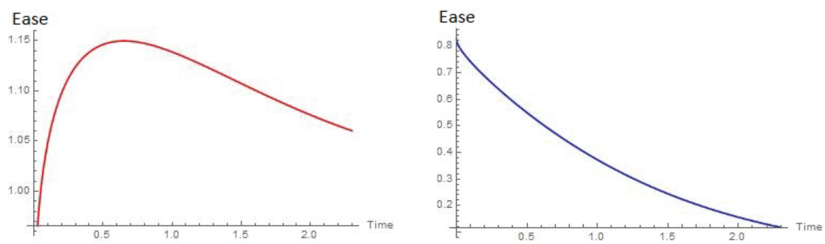
If we fix the initial stock at $x_0 = 5$, Figures 13–18 will show us a graphical depiction of the involved intensities in the process. From these images, we can notice that *almost all the time* the extraction rates of the player whose random terminal time follows the Chen distribution are higher than those of the other player. This fact is also consistent with the plots of Figures 4–6 and 10–12, which were exhibited in the illustration of Section 3.



(**a**) (**b**)

Figure 13. Ease of the extraction when the shape parameter of Weibull distribution is $\delta_1 = \frac{1}{2}$. (**a**) Ease of the extraction for Weibull's player when $\delta_1 = \frac{1}{2}$. (**b**) Ease of the extraction for Chen's player when $\delta_2 = \frac{1}{2}$.

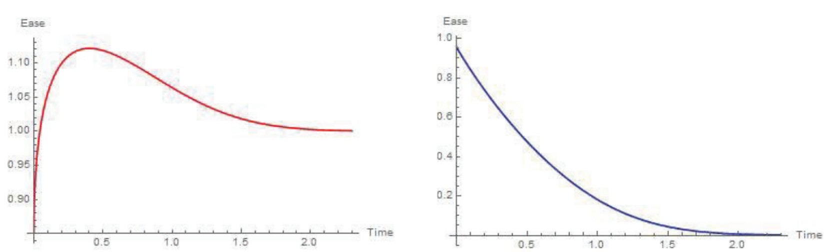
To have a proper interpretation of the results presented in Figure 13, recall Figures 10–12. Observe that the shapes of the plots in Figure 13a,b resemble those in Figure 12. It should be noted that the scales are much larger in the degenerate case. The reason is that, in this scenario, there are two actors making decisions and consuming the resource. Moreover, each of the agents needs to take into consideration the action of its counterpart (recall Theorem 3). As in the one-player case, it should be noticed that, in spite of the fact that the plots in Figure 13a,b have opposite trends, the optimal behavior of the Chen extractor is to obtain the resource at a much rapid pace than that of the Weibull extractor.



(**a**) (**b**)

Figure 14. Ease of the extraction when the shape parameters are $\delta_1 = \frac{1}{2}$ and $\delta_2 = 1$ for Weibull and Chen distributions, respectively. (**a**) Ease of the extraction for Weibull's player when $\delta_1 = \frac{1}{2}$. (**b**) Ease of the extraction for Chen's player when $\delta_2 = 1$.

In view of Figure 14, we must recall Figures 4–6 and 10– 12. Again, for the intensity function of the Chen extractor, $A_2(\cdot)$, we see the same general trend as that in Figure 6. However, for the Weibull case, the trend observed in Figure 12 becomes perturbed by the presence of the other agent in a more advanced mode of the extraction.



(**a**) (**b**)

Figure 15. Ease of the extraction when the shape parameters are $\delta_1 = \frac{1}{2}$ and $\delta_2 = 2$ for Weibull and Chen distributions, respectively. (**a**) Ease of the extraction for Weibull's player when $\delta_1 = \frac{1}{2}$. (**b**) Ease of the extraction for Chen's player when $\delta_2 = 2$.

Figure 15 can be interpreted by means of Figures 7–12. Note that from the comparison of Figure 15a with Figure 12 it is very clear how the presence of the Chen extractor affects the ease of the extraction of Weibull's player.

For the case $\delta_1 = 1$, we present the following result.

Proposition 2. *If $\delta_1 = 1$, then*

$$A_1(t) = c_1 + (1 - c_1\lambda_1)\frac{\mathrm{e}^t}{1 - F_2(t)}\left(\frac{1}{\lambda_1} - \mathcal{L}(F_2)\right),$$

where $\mathcal{L}(F_2) := \int_0^\infty \mathrm{e}^{-\lambda_1\tau}F_2(\tau)\mathrm{d}\tau$ stands for the Laplace transform of the distribution function F_2.

Proof. By (55), we readily know that

$$A_1(t) = \bar{a}_{[t]_1:[t]_2} + c_1\bar{A}_{[t]_1\overset{1}{:}[t]_2}. \tag{58}$$

Now, by (5) and (48) (with $v \equiv 1$) we can write

$$\begin{aligned}\bar{a}_{[t]_1:[t]_2} &= \int_0^\infty \frac{1 - F_1(t+\tau)}{1 - F_1(t)}\frac{1 - F_2(t+\tau)}{1 - F_2(t)}\mathrm{d}\tau \\ &= \int_0^\infty \mathrm{e}^{-\lambda_1\tau}\frac{1 - F_2(t+\tau)}{1 - F_2(t)}\mathrm{d}\tau.\end{aligned}$$

The last equality follows from plugging $\delta_1 = 1$ into Weibull's distribution. Define $s_2(\tau) := \frac{1-F_2(t+\tau)}{1-F_2(t)}$ as the conditional *survival* of the random variable $[T - t|T > t]$ and write

$$\bar{a}_{[t]_1:[t]_2} = \int_0^\infty \mathrm{e}^{-\lambda_1\tau}s_2(\tau)\mathrm{d}\tau. \tag{59}$$

We now use (5) and (49) (with $v \equiv 1$) to write

$$\begin{aligned}\bar{A}_{[t]_1\overset{1}{:}[t]_2} &= \int_0^\infty \frac{1 - F_1(t+\tau)}{1 - F_1(t)}\frac{1 - F_2(t+\tau)}{1 - F_2(t)}\lambda_2(t+\tau)\mathrm{d}\tau \\ &= \int_0^\infty \mathrm{e}^{-\lambda_1\tau}\frac{1 - F_2(t+\tau)}{1 - F_2(t)}\lambda_2(t+\tau)\mathrm{d}\tau.\end{aligned}$$

The last equality follows from the substitution of $\delta_1 = 1$. Now, by the results in Reference [57] (Chapter 3.2.4) we can argue that

$$\bar{A}_{[t]_1\overset{1}{:}[t]_2} = -\int_0^\infty \mathrm{e}^{-\lambda\tau}s_2'(\tau)\mathrm{d}\tau, \tag{60}$$

where s_2' stands for the derivative of s_2. Plugging (59) and (60) into (58) yields

$$\begin{aligned}A_1(t) &= \int_0^\infty \mathrm{e}^{-\lambda_1\tau}(s_2(\tau) - c_1s_2'(\tau))\mathrm{d}\tau \\ &= \mathcal{L}(s_2 - c_1s_2') \\ &= \mathcal{L}(s_2) - c_1\mathcal{L}(s_2') \\ &= \mathcal{L}(s_2) - c_1(\lambda_1\mathcal{L}(s_2) - s_2(0)).\end{aligned} \tag{61}$$

The last equality follows from the rule of Laplace Transform of Derivatives (see Reference [72] (Theorem 6.2.1)). Noting that $s_2(0) = 1$ and rearranging the terms in (61) gives us

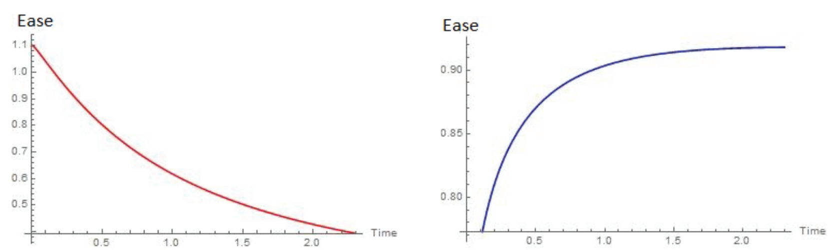
$$\begin{aligned}A_1(t) &= c_1 + (1 - c_1\lambda_1)\mathcal{L}(s_2) \\ &= c_1 + (1 - c_1\lambda_1)\int_0^\infty \mathrm{e}^{-\lambda_1\tau}\frac{1 - F_2(t+\tau)}{1 - F_2(t)}\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&= c_1 + (1 - c_1\lambda_1) \int_0^\infty e^{-\lambda_1\tau} \frac{1 - F_2(t+\tau)}{1 - F_2(t)} d\tau \\
&= c_1 + (1 - c_1\lambda_1) \frac{e^t}{1 - F_2(t)} \int_0^\infty e^{-\lambda_1\tau}(1 - F_2(\tau)) d\tau \\
&= c_1 + (1 - c_1\lambda_1) \frac{e^t}{1 - F_2(t)} \left(\frac{1}{\lambda_1} - \mathcal{L}(F_2) \right).
\end{aligned}
$$

This proves the result. □

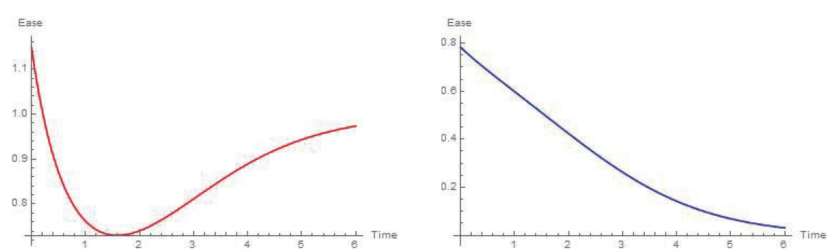
Proposition 2 gives us that, regardless of the distribution of T_2 (or of its shape parameter for the case of Chen's law), when T_1 follows the exponential distribution and $c_1\lambda_1 = 1$, the optimal rate of extraction will be invariant.

To interpret Figure 16, we recall Figures 7–12. This case corresponds to the situation where the Weibull extractor is already at the aging mode of the process, and Chen extractor is only starting its tasks. Here, the intensity of Weibull's extractor has to be higher as time passes, and the other player can take advantage of it because the intensity of its rate of extraction is decreasing. However, it is only at the beginning of the process when it is optimal for Chen extractor to have a stronger rate than that of Weibull (this is the only case in which this situation is observed). This is consistent with the behaviors shown by the intensity functions for the Weibull extractor in Figure 9, and for the Chen extractor in Figure 12.



(**a**) (**b**)

Figure 16. Ease of the extraction when the shape parameters of Weibull and Chen distributions are $\delta_1 = 2$ and $\delta_2 = \frac{1}{2}$, respectively. (**a**) Ease of the extraction for Weibull's player when $\delta_1 = 2$. (**b**) Ease of the extraction for Chen's player when $\delta_2 = \frac{1}{2}$.

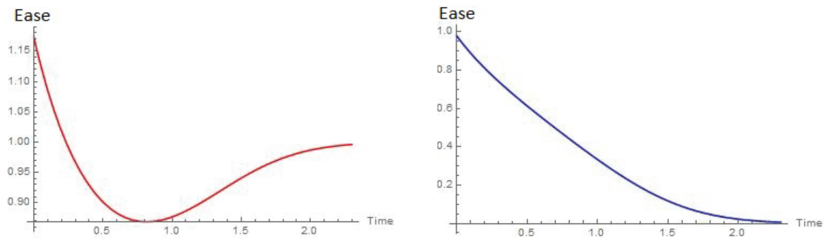


(**a**) (**b**)

Figure 17. Ease of the efforts when the shape parameters of Weibull and Chen distributions are $\delta_1 = 2$ and $\delta_2 = 1$, respectively. (**a**) Ease of the effort required Weibull extractor when $\delta_1 = 2$. (**b**) Ease of the effort required Chen extractor when $\delta_2 = 1$.

Figure 17a shows how the effort required by the Weibull extractor is a convex function that tends to stabilize in the long run. This feature is consistent with Figure 9. However, in the controlled case, we cannot observe an increasing trend. The behavior of the ease of Chen extractor mirrors that of the corresponding ease function in Figure 6.

Figure 18 shows a dramatized effect of the one shown in Figure 17. The reason is that both agents are in the aging mode of their respective processes.



(**a**) (**b**)

Figure 18. Ease of the efforts when the shape parameters of Weibull and Chen distributions are $\delta_1 = 2$ and $\delta_2 = 2$, respectively. (**a**) Ease of the effort required by Weibull extractor when $\delta_1 = 2$. (**b**) Ease of the effort required by Chen extractor when $\delta_2 = 2$.

If we fix the initial stock at $x_0 = 5$, and use Theorem 4, we can calculate the following table, where we have assumed, as in Section 3 that the scale parameters λ_1 and λ_2 equal the unity. Each of the entries represent the pair $(W_1(5,0), W_2(5,0))$.

Weibull/Chen	$\delta_2 = \frac{1}{2}$	$\delta_2 = 1$	$\delta_2 = 2$
$\delta_1 = \frac{1}{2}$	$(2.3257, 1.4487)$	$(2.4326, 1.6731)$	$(2.5882, 1.9514)$
$\delta_1 = 1$	$(2.5265, 1.4734)$	$(2.6221, 1.8187)$	$(2.7908, 2.2097)$
$\delta_1 = 2$	$(2.6941, 1.5161)$	$(2.9190, 1.9226)$	$(3.0567, 2.4373)$

It should be noted that the player whose random terminal time is distributed according to the Chen's law end's up earning less than the other player in all cases. This is consistent with the comparisons we made in Section 2, and in particular, in Figure 3c.

The optimal trajectory is given by the following exponential function:

$$x^*(\tau) = x_0 \exp\left(- \int_0^\tau \frac{\exp\left(-\int_0^t (\lambda_1\delta_1 s^{\delta_1-1} + \lambda_2\delta_2 s^{\delta_2-1}e^{s^{\delta_2}})ds\right)}{\int_t^\infty (1 + c_1\lambda_2\delta_2 s^{\delta_2-1}e^{s^{\delta_2}}) \exp\left(-\int_0^s (\lambda_1\delta_1 r^{\delta_1-1} + \lambda_2\delta_2 r^{\delta_2-1}e^{r^{\delta_2}})dr\right) ds} + \frac{\exp\left(-\int_0^t (\lambda_1\delta_1 s^{\delta_1-1} + \lambda_2\delta_2 s^{\delta_2-1}e^{s^{\delta_2}})ds\right)}{\int_t^\infty (1 + c_2\lambda_1\delta_1 s^{\delta_1-1}) \exp\left(-\int_0^s (\lambda_1\delta_1 r^{\delta_1-1} + \lambda_2\delta_2 r^{\delta_2-1}e^{r^{\delta_2}})dr\right) ds} dt \right).$$

4. Conclusions

In this paper we used standard Dynamic Programming techniques and classic Actuarial Mathematics tools for the analysis of a differential game with linear system and logarithmic reward function under the total payoff criteria with a random horizon. In Section 2.1 we presented the game model of our interest, and in Section 2.2 we presented a *third actuarial principle* to calculate premia under the total payoff criterion, which has been used to introduce the game with random terminal times. The distributions that we studied for these random variables are of great importance for risk analysts, and we devoted Section 3 to describe them and to present our main results and analyses. The first distribution that we considered is the classic two-parameter Weibull random variable, and the second is the heavy-tailed law of Chen. We compared the results of a resource extraction differential game model with each of these random variables and we identified the phases of the extraction with different values of the shape parameter of the distributions we used. In Section 3.1 we solved the degenerate case of the game (i.e., for only one player) and, in Section 3.2 we used some actuarial tools to study a two-player version of the game with two independent random terminal times for the extraction tasks.

For the purpose of illustrating our developments, we stated a degenerate game, and a two-person game with independent random terminal times for each player. It is important to emphasize the importance of the logarithmic utility of wealth function that we used, for it allowed us to find a connection between principles **PI** and **PII** from Section 2.2. This connection is clearly expressed by expressions (40) and (56) in Remarks 2 and 3, respectively. We believe that, regardless of the very particular form of the utility functions involved in our study, these expressions mean that the bond between the extraction game at hand (with random terminal times whose distributions are known), and the actuarial equivalence principle should be further studied, for it implies that, should there be an interest in insuring the operation of any of the agents, the optimal premium to be *continuously charged* to each player is given by $u^*(t,x)$. We think of such an extension as a plausible future line of work.

Additionally, we found short, explicit, numerical and graphical expressions in terms of actuarial nomenclature for the optimal rates of extraction of the agents in each case analyzed. We used this notation to characterize the optimal extraction rates in two parts, namely:

(i) the benefit that the agent will eventually get, x; and
(ii) the *intensity* of the effort that the agent needs to invest to acquire $u^*(t,x)$, that is, $A(t) = \bar{a}_t$ for the degenerate game in Section 3.1, and $A_i(t) = \bar{a}_{[t]_1:[t]_2} + c_i\bar{A}_{[t]_i:\overset{1}{[t]}_{-i}}$ for $i = 1,2$, for the game in Section 3.2. As this amount becomes larger, the extractor needs to become more intense in its extraction.

The resulting instantaneous utility of this exercise is given by (28) for the degenerate game, and (45) for the two-person differential game.

Moreover, with such representations of the premia available, it is possible to think of analyses of other kind; for instance, building confidence intervals and adding a loading factor to the premium to compute probabilities of ruin of the (hypothetical) insurance company, and thus extend our results to the realm of the actuarial risk theory. One of such extensions, that we look forward to work on, would be to consider the possibility of having an agent whose entry in the system happens at a random moment after the start of the extraction tasks (as is the case in Reference [52]) and (any of) the agents require(s) insurance to continue to develop the resource.

Another plausible extension is the one suggested by the fact that, in References [1,52], the authors found expressions for the optimal rates of extraction that can also be put in terms of simple contingent functions; which, on the one hand, is consistent with our findings in this paper, and on the other, can be used to model the presence of both, legal and illegal extracting agents. This means that in spite of the particular form of the utility functions we used for the agents, the *optimal* premia are given by expressions that follow the actuarial equivalence principle. With this in mind, we could try to perform an analysis of the optimality of equivalence premia with a broader class of reward functions, under the total payoff criterion, by using Dynamic Programming.

The speed of the extraction tasks differs for two different moments of the end of the game. And, as it was to be expected, Chen's distribution allows the most realistic description of the life cycle of the system. Our illustrations confirm that in normal operation, it is necessary to "dig" at a rate which needs to be gradually increased with time, but in the case where the end of the process is subject to Chen's law, the pace of resource extraction still needs a faster intensification. In the equipment aging mode, when the failure rate function increases, it does not matter what distribution law determines the completion of the development process, it is necessary to increase the rate of development of the fields.

Among our findings, we can also quote that the optimal behaviors of the agents differ for different game scenarios and moments of its completion. For example, during the pre-run phase (i.e., when the equipment is not yet established and the overall picture of the process is not fully clear), the development speed should be the smallest, which corresponds to the agent's caution. This is reflected in the entries of the row $\delta_1 = \frac{1}{2}$, and the column $\delta_2 = \frac{1}{2}$ of the table of Section 3.2. We can notice about this by observing that, for both players, the entries in the other rows and columns are greater than these ones. This feature is consistent with the conclusions drawn in Reference [4], in the sense that both our works prove that the price of insuring the extraction tasks is decreasing in the level of expertise of the extractor (which we might interpret as the stages where the agents are located at). This conclusion is coherent as well with the developments shown in References [27–29], on the willingness of the agents to pay for the coverage of an insurance strategy to guarantee the continued supply of the revenues from the extraction tasks.

To conclude, we mention that the results of this paper can be applied to a wide range of critical thecnologies related domains, including Internet of Things, energy management, and security to mention just a few. An extension of our results to the mentioned applications will be addressed in our subsequent work.

Author Contributions: The conceptualization of the model with random-time horizon, as well as its formal analysis for each of the players is original of E.V.G. The methodology for this particular problem is due to E.S.M., for it was the subject of her M.Sc. dissertation. The validation of the methods, and comparison with the results from the Actuarial Sciences field is original of J.D.L.-B. The financial resources came from Saint Petersburg State University and Universidad Anáhuac México. J.D.L.-B. was in charge of writing and preparing the original draft preparation. All authors contributed to the process of writing, reviewing and editing. All authors have read and agreed to the published version of the manuscript.

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Subgame Consistent Cooperative Behavior in an Extensive form Game with Chance Moves

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Abstract: We design a mechanism of the players' sustainable cooperation in multistage n-person game in the extensive form with chance moves. When the players agreed to cooperate in a dynamic game they have to ensure time consistency of the long-term cooperative agreement. We provide the players' rank based (PRB) algorithm for choosing a unique cooperative strategy profile and prove that corresponding optimal bundle of cooperative strategies satisfies time consistency, that is, at every subgame along the optimal game evolution a part of each original cooperative trajectory belongs to the subgame optimal bundle. We propose a refinement of the backwards induction procedure based on the players' attitude vectors to find a unique subgame perfect equilibrium and use this algorithm to calculate a characteristic function. Finally, to ensure the sustainability of the cooperative agreement in a multistage game we employ the imputation distribution procedure (IDP) based approach, that is, we design an appropriate payment schedule to redistribute each player's optimal payoff along the optimal bundle of cooperative trajectories. We extend the subgame consistency notion to extensive-form games with chance moves and prove that incremental IDP satisfies subgame consistency, subgame efficiency and balance condition. An example of a 3-person multistage game is provided to illustrate the proposed cooperation mechanism.

Keywords: time consistency; multistage game; chance moves; subgame perfect equilibria; cooperative trajectory; imputation distribution procedure

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1. Introduction

In a dynamic n-person game the players first choose their "optimal" strategies at the initial position x_0 (which form the optimal strategy profile for the whole game), and then have an option to change their strategies at any intermediate position x_t and switch to other strategies if these strategies constitute the locally optimal strategy profile for the subgame starting at x_t. The time consistency property (first introduced in References [1–3] for differential games) ensures that the players will not have an incentive to change their strategies at any subgame along the optimal game evolution, and hence plays an important role in the designing of the optimal players' behavior in non-cooperative and cooperative dynamic games (see, e.g., References [2–21], for details).

We consider an n-person finite multistage games in the extensive form (see, e.g., References [5,17,22,23]) with perfect information and with chance moves. Note that much research has been already done on time consistent solutions (or close concepts) in extensive-form games (see, e.g., References [4,6,13,17,21]). Time consistency concept was extended to dynamic games played over event trees in References [14,16,20] as well as to multicriteria extensive-form cooperative games (without chance moves) in References [7,8,10,11,15]. The property of "time consistency in the whole game" was extended to multicriteria extensive-form cooperative games with chance moves in Reference [9] (note that in these games an optimal pure strategy profile does not generate the unique optimal trajectory in the game tree but rather the whole optimal bundle of the trajectories).

In the paper, we mainly focus on the dynamic aspects of cooperation in a dynamic extensive-form game with chance moves, and propose to design a mechanism of the players' sustainable cooperation which satisfies three properties. First, a fragment of each cooperative trajectory from the optimal bundle for the original game Γ^{x_0} should "remain optimal" at each subgame Γ^{x_t} along the cooperative game evolution, that is, it should belong to the subgame optimal bundle of cooperative trajectories. Secondarily, a cooperative payoff-to-go at the subgame Γ^{x_t} is no less than the non-cooperative payoff-to-go for all players. Finally, when the players re-evaluate their expected cooperative payoff after each passed chance move, they have no incentive to change original cooperative agreement.

To this aim, we first need to provide a rule for choosing a unique cooperative strategy profile as well as the unique optimal bundle of cooperative trajectories. We introduce the *Players' Rank Based (PRB) algorithm* and prove that this algorithm generates the unique optimal bundle of cooperative trajectories which satisfies time consistency. Note that a rather close approach—the so-called Refined Leximin (RL) algorithm—was introduced recently in Reference [8]. Let us notice the main differences of these two algorithms. The RL algorithm is applicable for multicriteria game without chance moves and is based on the ranking of the criteria, while the PRB algorithm is designed for single-criterion extensive-form game with chance moves and employed the players' ranks. Further, the RL algorithm allows to choose a unique cooperative trajectory while the PRB algorithm generates the unique optimal bundle of the cooperative trajectories in the game tree. To the best of the authors' knowledge, other approaches to choosing an optimal bundle of the cooperative trajectories in extensive-form game with chance moves have not been considered yet.

Then, to construct a characteristic function (which describes the worth of each coalition in cooperative game) we use an equilibrium-based approach, namely the γ-characteristic function introduced in Reference [24]. Hence, the players have to accept a specific method for choosing a unique Subgame Perfect Equilibria (SPE) [25] in an extensive-form game with chance moves. To solve this problem we provide the novel refinement of the backwards induction procedure (see, e.g., References [5,17,23])—the so-called *Attitude SPE algorithm*. A similar approach to construct a unique SPE in extensive-form game with perfect information was explored in References [17,26,27] and was called the Type Equilibrium (TE) algorithm. Both algorithms are the refinements of the general backwards induction procedure that take into account the attitudes of each player towards other players. Let us point out the main differences of these algorithms. The TE algorithm is applicable for the game without chance moves and for the case when the payoffs are only determined in terminal nodes. In addition, the TE algorithm allows to construct SPE that is "unique" in the sense of payoffs (i.e., there may exist several optimal trajectories which generate the same equilibrium payoffs) while the Attitude SPE algorithm allows to choose unique SPE strategy profile as well as unique bundle of trajectories. Another rather close approach to find a unique SPE—the so-called Indifferent Equilibrium (IE) algorithm—was introduced in Reference [28]. Again, the IE algorithm is applicable only for the game without chance moves and for the partial case when the payoffs are determined in terminal nodes. Moreover, IE algorithm in general allows to construct a SPE in behavior strategies while the proposed Attitude SPE algorithm always generates a SPE in pure strategies.

It is worth noting, that other approaches to analyze an extensive-form game, except for the backwards induction procedure and its refinements mentioned above, imply that the researcher first needs to obtain a strategic representation of the original extensive game and then analyzes this strategic (or normal-form) game (see, e.g., References [29–31]). For instance, the software tool "Game Theory Explorer" [29] is based on the strategic-form representation and then applying the modified Lemke-Howson algorithm [32] to find all Nash equilibria. The majority of existing algorithms are developed to find Nash equilibria in mixed strategies for 2-person games and do not allow to construct SPE in pure strategies. Moreover, as it was noted in Reference [31], in general the strategic-form representation is exponential in the size of the original game tree. In contrast, the proposed Attitude

SPE algorithm is a rather simple recursive algorithm which deals with n-person extensive-form game (with perfect information) itself and allows to compute a unique SPE in pure strategies.

After computing the γ-characteristic function we suppose that the players adopt some single-valued cooperative solution φ (for instance, the Shapley value [33], the nucleolus [34], etc.) which satisfies the individual and collective rationality property. Finally, to guarantee the sustainability of the achieved long-term cooperative agreement we employ the *Imputation Distribution Procedure (IDP)* based approach (see, e.g., References [3,12,14,16–18,20,35]), that is, a payment schedule to redistribute the ith player's expected cooperative payoff along the optimal bundle of cooperative trajectories. In this paper, we mainly focus on the following good properties an IDP may satisfy: subgame efficiency, strict balance condition [10,15,17] and an appropriate refinement of the time consistency property, called *subgame consistency*. The point is that the "time consistency in the whole game" property [9,14,16,20] is based on an a priori assessment of the ith player's expected optimal payoff (before the game Γ^{x_0} starts). However, when the players make a decision in the subgame Γ^{x_t} after the chance move occurs, they need to re-estimate their expected optimal payoffs-to-go since the original optimal bundle of cooperative trajectories shrinks after each chance node. To deal with this interesting feature of the game with chance moves we adopt the notion of subgame consistency that was firstly proposed in Reference [36] for cooperative stochastic differential games and then extend it to stochastic dynamic games in References [37,38].

Since we derive a suitable definition of subgame consistency for other class of games, the proposed Definition 6 differs from ones provided in References [37,38] but captures the same idea. Let us point out the main differences with References [37,38]. While D. Yeung and L. Petrosyan do not consider the issue of multiple equilibria and study the stochastic games in which there exists a unique Nash equilibrium in each subgame, we focus on the problem of how to select a unique (subgame perfect) Nash equilibrium in extensive-form game with chance moves and derive the corresponding algorithm. Secondarily, the characteristic function has not been constructed in References [37,38] and, hence, the players are restricted to using the simplest cooperative solutions (for instance, they may share equally the excess of the total expected cooperative payoff over the expected sum of individual non-cooperative payoffs), whereas we provide a method for calculating the γ-characteristic function. Hence, the players may use different solution concepts based on the characteristic function approach. Finally, it turns out that the incremental IDP specified for extensive-form games with chance moves in Reference [9] satisfies not only the subgame consistency but also subgame efficiency and strict balance condition.

Therefore, the suggested PRB algorithm, the Attitude SPE algorithm combined with the γ-characteristic function, and the incremental payment schedule for any single-valued cooperative solution (meeting individual and collective rationality) together constitute a required mechanism of the players' sustainable cooperation that satisfies exactly three properties mentioned above for any extensive-form game with chance moves.

It is worth noting that the extensive-form games, as well as dynamic games played over event trees, differential games and multistage games with discrete dynamics are used to model various real-world situations where several decision makers (or players) with different objectives may cooperate (see, e.g., References [5,12,14,16,17,20,39–44]. Hence, a proposed approach to implement a long-term cooperative agreement may have a number of possible applications.

The rest of the paper is organized as follows: Section 2 recalls the main ingredients of the class of games of interest. In Section 3, we specify the attitude SPE algorithm that allows constructing a unique SPE in a extensive-form game with chance moves. In Section 4, we provide the PRB algorithm and prove that the optimal bundle of cooperative trajectories generated by this algorithm satisfy time consistency. Section 5 reveals a drawback of the IDP "time consistency in the whole game" property and presents a subgame consistency definition that is applicable for extensive-form games with chance moves. We prove that incremental IDP satisfies a number of good properties and consider an example of a 3-person multistage game with chance moves to illustrate the incremental IDP implementation. Section 6 provides a brief review of the results and discussion.

2. Extensive-Form Game with Chance Moves

We consider a finite multistage game in extensive form following References [6,13,17,22,23]. First we need to define the basic notations and briefly remind some properties of extensive-form game that will be used in the sequel:

- $N = \{1, \dots, n\}$ is the set of all players.
- K is the game tree with the root x_0 and the set of all nodes P.
- $S(x)$ is the set of all direct successors (descendants) of the node x, and $S^{-1}(y)$ is the unique predecessor (parent) of the node $y \neq x_0$ such that $y \in S(S^{-1}(y))$.
- P_i is the set of all decision nodes of the ith player (at these nodes the player i chooses the following node), $P_i \cap P_j = \varnothing$, for all $i, j \in N$, $i \neq j$.
- $P_{n+1} = \{z^j\}_{j=1}^{m}$ denotes the set of all terminal nodes (final positions), $S(z^j) = \varnothing \; \forall z^j \in P_{n+1}$.
- P_0 is the set of all nodes at which a chance moves, where $\pi(y|x) > 0$ denotes the probability of transition from node $x \in P_0$ to node $y \in S(x)$. We suppose that for each $x \in P_0$ it holds that $S(x) \cap P_0 = \varnothing$. Lastly, $\bigcup\limits_{i=0}^{n+1} P_i = P$.
- $\omega = (x_0, \dots, x_{t-1}, x_t, \dots, x_T)$ is the trajectory (or the path) in the game tree, $x_{t-1} = S^{-1}(x_t)$, $1 \leq t \leq T$, $x_T = z^j \in P_{n+1}$; where index t in x_t denotes the ordinal number of this node within the trajectory ω and can be interpreted as the "time index".
- $h_i(x) = (h_{i/1}(x), \dots, h_{i/r}(x))$ is the payoff of the ith player at the node $x \in P$. We assume that for all $i \in N$, $k = 1, \dots, r$, and $x \in P$ the payoffs are non-negative, that is, $h_{i/k}(x) \geqslant 0$.

In the following, we will use $G^{cm}(n)$ to denote the class of all finite multistage n-person games with chance moves in extensive form defined above, where $\Gamma^{x_0} \in G^{cm}(n)$ denotes a game with root x_0. Note that Γ^{x_0} is an extensive-form game with perfect information (see, e.g., References [17,22,23] for details).

Since all the solutions we are interested in throughout the paper are attainable when the players restrict themselves to the class of pure strategies we will focus on this class of strategies. The pure strategy $u_i(\cdot)$ of the ith player is a function with domain P_i that specifies for each node $x \in P_i$ the next node $u_i(x) \in S(x)$ which the player i has to choose at x. Let U_i denote the (finite) set of all ith player's pure strategies, $U = \prod_{i \in N} U_i$.

Denote by $p(y|x, u)$ the conditional probability that node $y \in S(x)$ is reached if node x has been already reached (the probability of transition from x to y) while the players use the strategies u_i, $i \in N$. Note that for all $x \in P_i$, $i = 1, \dots, n$, and for all $y \in S(x)$ $p(y|x, u) = 1$ if $u_i(x) = y$ and $p(y|x, u) = 0$ if $u_i(x) \neq y$. For chance moves, that is, if $x \in P_0$ $p(y|x, u) = \pi(y|x)$ for all $y \in S(x)$ for each $u \in U$.

Then one can calculate the probability $p(\omega, u)$ of realization of the trajectory $\omega = (x_0, \dots, x_\tau, x_{\tau+1}, \dots, x_T)$, $x_T \in P_{n+1}$, $x_{\tau+1} \in S(x_\tau)$, $\tau = 0, \dots, T-1$, when the players use the strategies u_i from the strategy profile $u = (u_1, \dots, u_n)$.

$$p(\omega, u) = p(x_1|x_0, u) \cdot p(x_2|x_1, u) \cdot \ldots \cdot p(x_T|x_{T-1}, u) = \prod_{\tau=0}^{T-1} p(x_{\tau+1}|x_\tau, u). \tag{1}$$

Denote by $\Omega(u) = \{\omega_k(u) | p(\omega_k, u) > 0\}$ the finite set (or the bundle) of the trajectories ω_k which are generated by strategy profile $u \in U$. Note that for all $\omega_k(u) \in \Omega(u)$, $u_j(x_\tau) = x_{\tau+1}$ for all $x_\tau \in \omega_k(u) \cap P_j$, $j \in N$, $0 \leq \tau \leq T-1$.

Let $\tilde{h}_i(\omega) = \sum\limits_{\tau=0}^{T} h_i(x_\tau)$ denote the ith player's vector payoff corresponding to the trajectory $\omega = (x_0, \dots, x_t, x_{t+1}, \dots, x_T)$.

Denote by

$$H_i(u) = \sum_{\omega_k \in \Omega(u)} p(\omega_k, u) \cdot \tilde{h}_i(\omega_k) = \sum_{\omega_k \in \Omega(u)} p(\omega_k, u) \cdot \sum_{\tau=0}^{T(k)} h_i(x_\tau) \quad (2)$$

the (expected) value of the ith player's payoff function which corresponds to the strategy profile $u = (u_1, \dots, u_n)$. Let $\Omega_{n+1}(u) = \{\Omega(u) \cap P_{n+1}\}$ denote the set of all terminal nodes of the trajectories $\omega_k(u) \in \Omega(u)$.

Remark 1 ([9]). *If the pure strategy profiles u and v generate different bundles $\Omega(u)$ and $\Omega(v)$ of the trajectories, that is, $\Omega(u) \neq \Omega(v)$, then $\Omega_{n+1}(u) \cap \Omega_{n+1}(v) = \emptyset$.*

According to References [17,22,23] each intermediate node $x_t \in P \setminus P_{n+1}$ generates a subgame Γ^{x_t} with the subgame tree K^{x_t} and the subgame root x_t as well as a factor-game Γ^D with the factor-game tree $K^D = (K \setminus K^{x_t}) \cup \{x_t\}$. Decomposition of the original extensive game Γ^{x_t} at node x_t onto the subgame Γ^{x_t} and the factor-game Γ^D generates the corresponding decomposition of the pure (and mixed) strategies (see References [17,22] for details).

Let $P_i^{x_t}(P_i^D)$, $i \in N$, denote the restriction of P_i on the subgame tree $K^{x_t}(K^D)$, and $u_i^{x_t}(u_i^D)$, $i \in N$, denote the restriction of the ith player's pure strategy $u_i(\cdot)$ in Γ^{x_0} on $P_i^{x_t}(P_i^D)$. The pure strategy profile $u^{x_t} = (u_1^{x_t}, \dots, u_n^{x_t})$ generates the bundle of the subgame trajectories $\Omega^{x_t}(u^{x_t}) = \{\omega_k^{x_t}(u^{x_t}) | p(\omega_k^{x_t}, u^{x_t}) > 0\}$. Similarly to (2), let us denote by

$$H_i^{x_t}(u^{x_t}) = \sum_{\omega_k^{x_t} \in \Omega^{x_t}(u^{x_t})} p(\omega_k^{x_t}, u^{x_t}) \cdot \sum_{\tau=t}^{T(k)} h_i(x_\tau) = \sum_{\omega_k^{x_t} \in \Omega^{x_t}(u^{x_t})} p(\omega_k^{x_t}, u^{x_t}) \cdot \tilde{h}_i(\omega_k^{x_t}) \quad (3)$$

the expected value of the ith player's payoff in Γ^{x_t}, and by $U_i^{x_t}$ the set of all possible ith player's pure strategies in the subgame Γ^{x_t}, $U^{x_t} = \prod_{i \in N} U_i^{x_t}$. Note that for each trajectory $\omega = (x_0, \dots, x_t, x_{t+1}, \dots, x_T)$, $1 \leqslant t \leqslant T-1$, $x_T \in P_{n+1}$,

$$\begin{aligned} p(\omega, u) &= \prod_{\tau=0}^{t-1} p(x_{\tau+1} | x_\tau, u) \cdot \prod_{\tau=t}^{T-1} p(x_{\tau+1} | x_\tau, u) = \\ &= p(\underline{\omega}^{x_t}, u) \cdot p(\omega^{x_t}, u) = p(\underline{\omega}^{x_t}, u^D) \cdot p(\omega^{x_t}, u^{x_t}), \end{aligned} \quad (4)$$

where $\underline{\omega}^{x_t} = (x_0, x_1, \dots, x_{t-1}, x_t)$ denotes a fragment of trajectory ω implemented before the subgame Γ^{x_t} starts, and $p(\underline{\omega}^{x_t}, u) = p(x_t, u)$ denotes the probability that node x_t is reached when the players employ the strategies u_i, $i \in N$. It is worth noting that factor-game $\Gamma^D = \Gamma^D(u^{x_t})$ is usually defined for given strategy profile u^{x_t} in the subgame Γ^{x_t} since we assume that

$$h_i^D(x_0, x_1, \dots, x_{t-1}, x_t) = \sum_{\tau=0}^{t-1} h_i(x_\tau) + H_i^{x_t}(u^{x_t}) = \tilde{h}_i(\underline{\omega}^{x_t} \setminus \{x_t\}) + H_i^{x_t}(u^{x_t}) \quad (5)$$

(see, e.g., References [17,22] for details). Moreover, given intermediate node x_t, the bundle $\Omega(u) = \{\omega_k(u) | p(\omega_k, u) > 0\}$ can be divided in two subsets, that is, $\Omega(u) = \{\Psi_m\} \cup \{\chi_l\}$, where $x_t \in \Psi_m$, and $x_t \notin \chi_l$, $\{\Psi_m\} \cap \{\chi_l\} = \emptyset$. Then, taking (1), (3), (4) and (5) into account, we get

$$
\begin{aligned}
H_i(u) &= \sum_m p(\Psi_m, u) \cdot \tilde{h}_i(\Psi_m) + \sum_l p(\chi_l, u) \cdot \tilde{h}_i(\chi_l) = \\
&= \sum_m p(x_t, u) \cdot p(\Psi_m^{x_t}, u^{x_t}) \cdot \left[\tilde{h}_i(\underline{\Psi}_m^{x_t} \setminus \{x_t\}) + \tilde{h}_i(\Psi_m^{x_t})\right] + \\
&+ \sum_l p(\chi_l, u) \cdot \tilde{h}_i(\chi_l) = p(x_t, u^D) \cdot \tilde{h}_i(x_0, \ldots, x_{t-1}) \cdot \sum_m p(\Psi_m^{x_t}, u^{x_t}) + \\
&+ p(x_t, u^D) \cdot \sum_m p(\Psi_m^{x_t}, u^{x_t}) \cdot \tilde{h}_i(\Psi_m^{x_t}) + \sum_l p(\chi_l, u) \cdot \tilde{h}_i(\chi_l) = \\
&= p(x_t, u^D) \cdot \tilde{h}_i(x_0, \ldots, x_{t-1}) + p(x_t, u^D) \cdot H_i^{x_t}(u^{x_t}) + \\
&+ \sum_l p(\chi_l, u) \cdot \tilde{h}_i(\chi_l) = p(x_t, u^D) \cdot h_i^D(x_0, \ldots, x_t) + \sum_l p(\chi_l, u) \cdot \tilde{h}_i(\chi_l).
\end{aligned} \tag{6}
$$

Note that, since $P_i = P_i^{x_t} \cup P_i^D$, one can compose the ith player's pure strategy $W_i = (u_i^D, v_i^{x_t}) \in U_i$ in the original game Γ^{x_0} from her strategies $v_i^{x_t} \in U_i^{x_t}$ in the subgame Γ^{x_t} and $u_i^D \in U_i^D$ in the factor-game Γ^D [17,22].

3. Refined Backwards Induction Procedure to Construct a Unique SPE

Definition 1 ([45])**.** *A strategy profile $u = (u_1, u_2, \ldots, u_n)$ is a Nash Equilibrium (NE) in $\Gamma^{x_0} \in G^{cm}(n)$, if*

$$H_i(v_i, u_{-i}) \leqslant H_i(u_i, u_{-i}), \ \forall v_i \in U_i, \ \forall i \in N.$$

Let $NE(\Gamma^{x_0})$ denote the set of all pure strategy Nash equilibria in Γ^{x_0}.

Definition 2 ([25])**.** *A strategy profile u is a subgame perfect (Nash) equilibrium (SPE) in $\Gamma^{x_0} \in G^{cm}(n)$, if $\forall x \in P \setminus P_{n+1}$ it holds that $u^x \in NE(\Gamma^x)$, i. e. the restriction of u on each subgame Γ^x forms a NE in this subgame.*

To construct SPE in an extensive-form game with perfect information one may employ a so-called backwards induction procedure (see, e.g., References [12,17,22,23,46,47]).

However, the backwards induction procedure may generate multiple subgame perfect equilibriums in an extensive form game with different payoffs to the players (see, e.g., References [5,12,17,23]). To choose a unique SPE and unique corresponding bundle of trajectories we use an approach based on the players' attitude vectors. Namely, let the ith player's attitude vector $F_i = \{f_i(1), \ldots, f_i(n)\}$ be a permutation of numbers $\{1, \ldots, n\}$ meeting the condition $f_i(i) = 1$. If $f_i(j) = k$ one may interpret the player j as an "ith player's associate of level k".

In the paper we will use these attitude vectors when constructing SPE via backwards induction procedure in the following way. Let $x \in P_i$, $H_i^y(\underline{u}^y)$ denote the ith player's expected payoff in the subgame Γ^y, $y \in S(x)$ while $\underline{u}^y$ be a SPE in this subgame. Assume that there exist multiple nodes $y_1, \ldots, y_q$ such that $h_i(y_1) + H_i^{y_1}(\underline{u}^{y_1}) = \ldots = h_i(y_q) + H_i^{y_q}(\underline{u}^{y_q})$, that is, player i is indifferent to the choice of particular node $\overline{y}$ from $\{y_1, \ldots, y_q\}$ while the ith player's choice may affect the other players' payoffs. If $f_i(j) = 2$, suppose that the ith player aims to maximize firstly the jth player's expected payoff $H_j^y(\underline{u}^y)$ when choosing a unique node y from $y_1, \ldots, y_q$. If again there are several nodes y with the same value $H_j^y(\underline{u}^y)$ the ith player purposes to maximize secondarily the expected payoff $H_l^y(\underline{u}^y)$ of such player l that $f_i(l) = 3$, and so on. Note that similar approach to construct a unique SPE in extensive-form game with perfect information but without chance moves was explored in References [17,26,27] for the case when the payoffs are only determined in terminal nodes.

Now let us provide a rigorous specification of this backwards induction procedure refinement which we will refer to as the *Attitude SPE or A-SPE algorithm*.

Attitude SPE algorithm. Suppose that the players attitude vectors $F_1, F_2, \ldots, F_n$ are of common knowledge, i. e. each player knows these vectors, and all the players are aware of it. Let the length of the trajectory $\omega = (x_0, \dots, x_t, x_{t+1}, \dots, x_T)$ equals to $T-1$, and the multistage game Γ^{x_0} length equals to the maximal length of the trajectory ω in Γ^{x_0}. We'll construct the unique subgame perfect equilibrium $\underline{u} = (\underline{u}_1, \dots, \underline{u}_n)$ in Γ^{x_0} by induction in the length L of the subgame Γ^x.

Step $L = 1$: Consider a subgame Γ^x of the length $L = 1$. If $x \in P_i$, $i = 1, \dots, n$, we have two cases.

Case 1: there exists a unique $z^k \in S(x) = P^x_{n+1}$ such that $h_i(z^k) = \max_{z \in S(x)} h_i(z)$. Then suppose that $\underline{u}_i(x) = z^k$, $p(z^k|x, \underline{u}) = 1$, $p(z|x, \underline{u}) = 0 \ \forall z \in S(x) \setminus \{z^k\}$.

Case 2: there exist $q > 1$ nodes $z^{k_q} \in S(x) = P^x_{n+1}$ such that $h_i(z^{k_1}) = h_i(z^{k_2}) = \dots = h_i(z^{k_q}) = \max_{z \in S(x)} h_i(z)$. Then suppose that the ith player chooses the terminal position $z^k \in \{z^{k_1}, \dots, z^{k_q}\} = S^{i,1}(x)$ such that

$$h_j(z^k) = \max_{z \in S^{i,1}(x)} h_j(z), \text{ where } f_i(j) = 2. \tag{7}$$

Let $S^{i,2}(x)$ denote the set of all nodes $z^k \in S^{i,1}(x)$ meeting (7). If $S^{i,2}(x)$ consists of unique node z^k then $\underline{u}_i(x) = z^k$, $p(z^k|x, \underline{u}) = 1$, $p(z|x, \underline{u}) = 0 \ \forall z \in S(x) \setminus \{z^k\}$. Otherwise, suppose that the ith player chooses terminal node $z^k \in S^{i,2}(x)$ such that

$$h_l(z^k) = \max_{z \in S^{i,2}(x)} h_l(z), \text{ where } f_i(l) = 3. \tag{8}$$

Let $S^{i,3}(x)$ denote the set of all final nodes $z^k \in S^{i,2}(x)$ satisfying (8), and so on.

$\vdots$

Finally, if $S^{i,n}(x)$ contains unique node z^k, then $\underline{u}_i(x) = z^k$, $p(z^k|x, \underline{u}) = 1$, $p(z|x, \underline{u}) = 0$ $\forall z \in S(x) \setminus \{z^k\}$. Otherwise, suppose that player i chooses the final node z^k from $S^{i,n}(x)$ with minimal ordinal number k.

Note that for all cases $H_j(\underline{u}) = h_j(z^k)$, $j \in N$.

If $x \in P_0$ then $S(x) = P^x_{n+1}$ and we do not need to define a strategy of any player at x, while $H_j(\underline{u}) = \sum_{z^k \in S(x)} \pi(z^k|x) \cdot h_j(z^k)$. Hence, the players' behavior $\underline{u}^x = (\underline{u}^x_1, \dots, \underline{u}^x_n) \in NE(\Gamma^x)$ and the expected payoffs $H^x_j(\underline{u}^x)$, $j \in N$ are defined for all subgames Γ^x of the length 1. In addition, for games Γ^y, $y \in P_{n+1}$ of length $L = 0$ we assume that $H^y_i(\underline{u}^y) = h_i(y)$, $i \in N$.

Step $2, \dots, L-1$: Suppose that at each subgame Γ^y of the length $(L-1)$ or less the unique SPE $\underline{u}^y = (\underline{u}^y_1, \dots, \underline{u}^y_n)$ has been already constructed ("inductive assumption"), and $H^y_i(\underline{u}^y)$, $i \in N$, is the corresponding vector of all the players' payoffs.

Step L: Consider the game Γ^{x_0} of the length $L \geqslant 1$. Note that for all $y \in S(x_0)$ the length of the subgame Γ^y is less than L. If $x_0 \in P_0$ then

$$H_j(\underline{u}) = \sum_{y \in S(x_0)} \pi(y|x_0) \cdot (h_j(y) + H^y_j(\underline{u}^y)) \geqslant \sum_{y \in S(x_0)} \pi(y|x_0) \cdot (h_j(y) + H^y_j(u^y_j, \underline{u}^y_{-j})) = H_j(u^y_j, \underline{u}^y_{-j}) \tag{9}$$

for all $u_j = u^y_j \in U^y_j = U_j$ since $\underline{u}^y \in NE(\Gamma^y)$ due to induction assumption, and each player $j \in N$ can deviate from $\underline{u}_j$ only in the subgames Γ^y, $y \in S(x_0)$.

If $x_0 \in P_i$ for some $i \in N$, we have two cases.

Case 1: there exists a unique $\overline{y} \in S(x_0)$ such that

$$h_i(\overline{y}) + H^{\overline{y}}_i(\underline{u}^{\overline{y}}) = \max_{y \in S(x_0)} \left(h_i(y) + H^y_i(\underline{u}^y) \right). \tag{10}$$

Then we suppose that $\underline{u}_i(x_0) = \overline{y}$; $\underline{u}_j(x) = \underline{u}^y_j(x)$ if $x \in P_j \cap K^y$, $y \in S(x_0)$, $j = 1, \dots, n$.

Case 2: there exist $q > 1$ nodes $\overline{y}_1, \dots, \overline{y}_q \in S(x_0)$ such that

$$h_i(\overline{y}_1) + H_i^{\overline{y}_1}(\underline{u}^{\overline{y}_1}) = \dots = h_i(\overline{y}_q) + H_i^{\overline{y}_q}(\underline{u}^{\overline{y}_q}) = \max_{y \in S(x_0)} \left(h_i(y) + H_i^y(\underline{u}^y) \right). \tag{11}$$

Then we suppose that the ith player chooses $\overline{y} \in \{\overline{y}_1, \dots, \overline{y}_q\} = S^{i,1}(x_0)$ such that

$$h_j(\overline{y}) + H_j^{\overline{y}}(\overline{u}^{\overline{y}}) = \max_{y \in S^{i,1}(x_0)} \left(h_j(y) + H_j^y(\underline{u}^y) \right), \text{ where } f_i(j) = 2. \tag{12}$$

Let $S^{i,2}(x_0)$ denote the set of all nodes $\overline{y} \in S^{i,1}(x_0)$ satisfying (12). If $S^{i,2}(x_0)$ consists of unique node $\overline{y}$ then we suppose that $\underline{u}_i(x_0) = \overline{y}$; $\underline{u}_j(x) = \underline{u}_j^y(x)$ if $x \in P_j \cap K^y$, $y \in S(x_0)$, $j = 1, \dots, n$. Otherwise, suppose that the ith player chooses node $\overline{y} \in S^{i,2}(x_0)$ such that

$$h_l(\overline{y}) + H_l^{\overline{y}}(\overline{u}^{\overline{y}}) = \max_{y \in S^{i,2}(x_0)} \left(h_l(y) + H_l^y(\underline{u}^y) \right), \text{ where } f_i(l) = 3. \tag{13}$$

Let $S^{i,3}(x_0)$ denote the set of all nodes $\overline{y} \in S^{i,2}(x_0)$ meeting (13), and so on.

$\vdots$

Finally, if $S^{i,n}(x_0)$ contains several nodes $\overline{y}_m$, denote by $\underline{l} = \min_{\overline{y}_m \in S^{i,n}(x_0)} \{l \mid z^l \in P_{n+1}^{\overline{y}_m} \cap \Omega(\underline{u}^{\overline{y}_m})\}$ the minimal number of terminal nodes of the trajectories generated by subgame perfect equilibriums $\underline{u}^{\overline{y}_m}$ in the subgames $\Gamma^{\overline{y}_m}$, $\overline{y}_m \in S^{i,n}(x_0)$ (see Remark 1). Note that there exists unique trajectory $\omega = (x_0, \dots, z^{\underline{l}})$ from x_0 to $z^{\underline{l}}$ in the game Γ^{x_0}, and let $\overline{y} = \omega \cap S^{i,n}(x_0)$. Again, we suppose that $\underline{u}_i(x_0) = \overline{y}$; $\underline{u}_j(x) = \underline{u}_j^y(x)$ if $x \in P_j \cap K^y$, $y \in S(x_0)$, $j = 1, \dots, n$.

Now we prove that for both cases no player has profitable deviation in Γ^{x_0} from the strategy profile $\underline{u} = (\underline{u}_1, \dots, \underline{u}_n)$ constructed above.

$$H_i(\underline{u}) = h_i(\overline{y}) + H_i^{\overline{y}}(\underline{u}^{\overline{y}}) \geqslant h_i(y) + H_i^y(\underline{u}^y) \geqslant h_i(y) + H_i^y(u_i^y, \underline{u}^y_{-i}) \tag{14}$$

for all $y \in S(x_0)$, $u_i^y \in U_i^y$ due to (10), (11) and the induction assumption that $\underline{u}^y \in NE(\Gamma^y)$, $y \in S(x_0)$.

For other players $j \in N$, $j \neq i$, we have

$$H_j(\underline{u}) = h_j(\overline{y}) + H_j^{\overline{y}}(\underline{u}^{\overline{y}}) \geqslant h_j(\overline{y}) + H_j^{\overline{y}}(u_j^{\overline{y}}, \underline{u}^{\overline{y}}_{-j}) = H_j(u_j, \underline{u}_{-j}) \tag{15}$$

for all $u_j \in U_j$ since $\underline{u}^{\overline{y}} \in NE(\Gamma^{\overline{y}})$, and the only deviation of player $j \in N$, $j \neq i$ from $\underline{u}_j$ in the subgame $\Gamma^{\overline{y}}$ may affect the players' payoffs.

Hence, taking (9), (14) and (15) into account we obtain by induction that the strategy profile $\underline{u} = (\underline{u}_1, \dots, \underline{u}_n)$ constructed above forms unique subgame perfect equilibria in Γ^{x_0}.

Proposition 1. *If the players attitude vectors $F_1, F_2, ..., F_n$ are of common knowledge, the Attitude SPE algorithm allows to construct a unique subgame perfect equilibrium $\underline{u} = (\underline{u}_1, \dots, \underline{u}_n)$ in pure strategies for any extensive-form game $\Gamma^{x_0} \in G^{cm}(n)$ with chance moves as well as a unique bundle of trajectories $\Omega(\underline{u})$.*

It is worth noting than the existence of (subgame perfect) pure strategy equilibrium in extensive form game with perfect information and chance moves was first proved in References [46,47] for the partial case when the payoffs are only defined in terminal nodes. Hence, Proposition 1 could be considered as a corollary of these results. However, we provide a rigorous algorithm how to construct a unique SPE in extensive-form game with chance moves as well as a (unique) corresponding bundle of trajectories. We will use this algorithm, in particular, to calculate the characteristic function of the cooperative extensive-form game in Section 4.

Let us use the following example to demonstrate how the Attitude SPE algorithm works.

Example 1. *(A 3-player multistage game with chance moves).*

Let $P_0 = \{\overline{x}_1, \overline{x}_3\}$, $P_1 = \{\overline{x}_0, \overline{x}_4^2\}$, $P_2 = \{\overline{x}_2^1, \overline{x}_5\}$, $P_3 = \{x_2^2, x_4^3\}$, $P_{n+1} = \{z_1, \ldots, z_{10}\}$. *The players' payoffs and probabilities* $\pi(y|x)$, $x \in P_0$ *are written in the game tree.*

Suppose that the players' attitude vectors are $F_1 = (f_1(1), f_1(2), f_1(3)) = (1,3,2)$, $F_2 = (2,3,1)$ *and* $F_3 = (3,1,2)$.

When using the Attitude SPE algorithm, at each node $x \in P_i$, $i = 1,2,3$, *the ith player has to choose the alternative marked in bold violet in Figure 1. Note that*

$$H^{x_3}(\underline{u}^{x_3}) = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 \\ 24 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 24 \\ 0 \\ 24 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 \\ 18 \\ 12 \end{pmatrix} = \begin{pmatrix} 24 \\ 10 \\ 16 \end{pmatrix} \text{ and } H^{z_2} = h(z_2) = \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}.$$

Hence, $S^{2,1}(x_2^1) = \{z_2, x_3\}$, *and* $\underline{u}_2(x_2^1) = z_2$ *due to the player's 2 attitude vector* F_2.

The A-SPE algorithm generates unique SPE $\underline{u} = (\underline{u}_1, \underline{u}_2, \underline{u}_3)$, *where* $\underline{u}_1(x_0) = x_1$, $\underline{u}_1(x_4^2) = z_8$; $\underline{u}_2(x_2^1) = z_2$, $\underline{u}_2(x_5) = z_9$; $\underline{u}_3(x_2^2) = z_4$, $\underline{u}_3(x_4^3) = z_6$, *while* $H(\underline{u}) = (11, 22, 18)$. *We will use this SPE later in Section 4 when calculating the* γ*-characteristic function.*

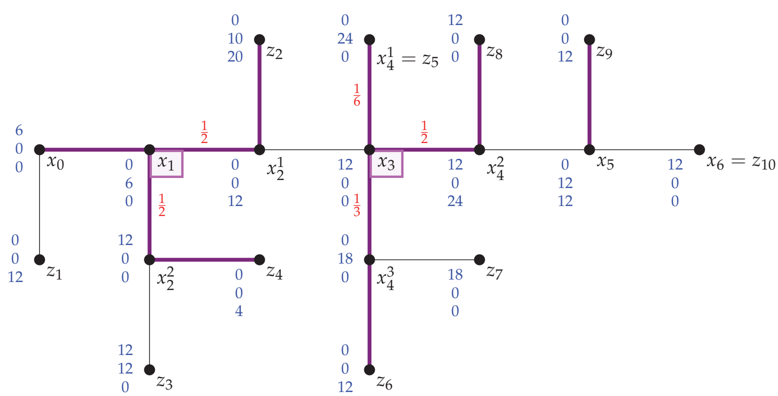

Figure 1. 3-person extensive-form game: A-Subgame Perfect Equilibria (SPE) algorithm implementation.

4. Cooperative Strategies and Trajectories

If the players agree to cooperate in multicriteria game Γ^{x_0}, first they are expected to maximize the total payoff $\sum_{i=1}^{n} H_i(u)$ of the grand coalition. Let $\overline{U}(\Gamma^{x_0})$ denote the set of all pure strategy profiles u, such that

$$\sum_{i \in N} H_i(u) = \max_{v \in U} \sum_{i \in N} H_i(v) = \overline{H}. \tag{16}$$

The set $\overline{U}(\Gamma^{x_0})$ is known to be nonempty and it may contain multiple strategy profiles (see, e.g., Reference [17]). Hence, the players need to agree on a specific approach they are going to use to choose a unique optimal cooperative strategy profile $\overline{u} \in \overline{U}(\Gamma^{x_0})$ as well as the corresponding optimal bundle of cooperative trajectories in the game tree. To this aim we introduce the so-calle *Players' Rank Based (PRB)* algorithm. Note that rather close approach—using the ranking of the criteria to choose a unique cooperative trajectory—was proposed recently in Reference [8] for multicriteria extensive-form games without chance moves. Namely, suppose that the players have agreed on the so-called "rank" of each player within the grand coalition N, and $r(k) = i$ means that the rank of player i equals k, $k = 1, ..., n$.

Players' rank based (PRB) algorithm.

Step 0. Consider the set $\overline{U}(\Gamma^{x_0})$. If all strategy profiles $u \in \overline{U}(\Gamma^{x_0})$ generate the same bundle of trajectories $\Omega(u)$ (see, e.g., References [17,22,23] for discussion on a certain redundancy of the pure strategy definition in extensive game), let the players choose any strategy profile $\overline{u} \in \overline{U}(\Gamma^{x_0})$ as the cooperative strategy profile and $\Omega(\overline{u})$ denote the corresponding bundle of cooperative trajectories.

Step $k = 1$. Otherwise, that is, if the strategy profiles from $\overline{U}(\Gamma^{x_0})$ generate different (and hence, disjoint—see Remark 1) bundles of the trajectories, calculate

$$\max_{v \in \overline{U}(\Gamma^{x_0})} H_{r(1)}(v) = \overline{H}_{r(1)}.$$

Let $\overline{U}_{r(1)}(\Gamma^{x_0})$ denote the set of all strategy profiles u such that $H_{r(1)}(u) = \overline{H}_{r(1)}$. If all strategy profiles $u \in \overline{U}_{r(1)}(\Gamma^{x_0})$ generate the same bundle of trajectories $\Omega(u)$, the players may choose any strategy profile $\overline{u} \in \overline{U}_{r(1)}(\Gamma^{x_0})$ as the cooperative strategy profile. Otherwise proceed to the next step.

Step $k = 2$. Consider the set $\overline{U}_{r(1)}(\Gamma^{x_0})$. If all strategy profiles $u \in \overline{U}_{r(1)}(\Gamma^{x_0})$ generate the same bundle of trajectories $\Omega(u)$, the players may choose any strategy profile $\overline{u} \in \overline{U}_{r(1)}(\Gamma^{x_0})$ as the cooperative strategy profile. Otherwise, proceed to the next step.

Step k ($k = 2, \ldots, n$).

$\vdots$

Step $k + 1$. Finally, if the strategy profiles from $u \in \overline{U}_{r(n)(\Gamma^{x_0})}$ generate different bundles of the trajectories, we suppose that the players choose such $\overline{u} \in \overline{U}_{r(n)(\Gamma^{x_0})}$ that $\Omega(\overline{u}) = \{\omega_m(\overline{u}) = (x_0, \ldots, x_{T(m)} = z_l) \mid p(\omega_m, \overline{u}) > 0\}$ contains the trajectory $\omega(\overline{u})$ with minimal number l of the terminal node z_l (see Remark 1).

Henceforth, we will refer to the strategy profile $\overline{u} \in \overline{U}(\Gamma^{x_0})$ and the bundle of the trajectories $\Omega(\overline{u})$ as the *optimal cooperative strategy profile* and the *optimal bundle of cooperative trajectories* respectively.

In the dynamic setting it is significant that a specific method which the players agreed to accept in order to choose a unique optimal cooperative strategy profile $\overline{u} \in \overline{U}(\Gamma^{x_0})$ as well as the corresponding optimal bundle of cooperative trajectories satisfies time consistency (see, e.g., References [1,2,6,13,17]), that is, a fragment of the optimal bundle of the cooperative trajectories in the subgame should remain optimal in this subgame. Suppose that at each subgame $\Gamma^{\overline{x}_t}$ along the cooperative trajectories, that is $\overline{x}_t \in \omega(\overline{u})$, $\omega(\overline{u}) \in \Omega(\overline{u})$, the players choose the strategy profile $u^{\overline{x}_t} \in U^{\overline{x}_t}$ such that

$$u^{\overline{x}_t} \in \arg\max_{v^{\overline{x}_t} \in U^{\overline{x}_t}} \sum_{i \in N} H_i^{\overline{x}_t}(v^{\overline{x}_t}). \tag{17}$$

Let $\overline{U}(\Gamma^{\overline{x}_t})$ denote the set of all pure strategy profiles $u^{\overline{x}_t} \in U^{\overline{x}_t}$ which satisfy (17) and the players use the same approach to choose a unique optimal cooperative strategy profile $\overline{u}^{\overline{x}_t} \in \overline{U}(\Gamma^{\overline{x}_t})$ in the subgame as for the original game Γ^{x_0} (namely, the PRB algorithm).

Proposition 2. *A cooperative strategy profile for $\Gamma^{x_0} \in G^{cm}(n)$ based on the PRB algorithm satisfies time consistency. Namely, let $\overline{u} \in U$ satisfies (16), and $\Omega(\overline{u})$ be the optimal bundle of cooperative trajectories. Then for each subgame $\Gamma^{\overline{x}_t}$, $\overline{x}_t \in \omega(\overline{u}) = (\overline{x}_0, \ldots, \overline{x}_t, \overline{x}_{t+1}, \ldots, \overline{x}_T), 1 \leqslant t < T$, with $\overline{x}_0 = x_0$, $\omega(\overline{u}) \in \Omega(\overline{u})$, it holds that*

$$\sum_{i \in N} H_i^{\overline{x}_t}(\overline{u}^{\overline{x}_t}) = \max_{v^{\overline{x}_t} \in U^{\overline{x}_t}} \sum_{i \in N} H_i^{\overline{x}_t}(v^{\overline{x}_t}), \tag{18}$$

while $\omega^{\overline{x}_t} = (\bar{x}_t, \bar{x}_{t+1}, \ldots, \bar{x}_T) \in \Omega(\overline{u}^{\overline{x}_t})$, that is, $\omega^{\overline{x}_t}$ belongs to the optimal bundle of cooperative trajectories in the subgame $\Gamma^{\overline{x}_t}$.

Proof. The optimal bundle of cooperative trajectories $\Omega(\overline{u})$ generated by $\overline{u} \in \overline{PO}(\Gamma^{x_0})$ can be divided onto two subsets $\{\Psi_m\} = \{\omega \in \Omega(\overline{u}) \mid \overline{x}_t \in \omega\}$ and $\{\chi_l\} = \{\omega \in \Omega(\overline{u}) \mid \overline{x}_t \notin \omega\}$ while $\{\Psi_m\} \cap \{\chi_l\} = \emptyset$, $\{\Psi_m\} \cup \{\chi_l\} = \Omega(\overline{u})$. Then, taking (5) and (6) into account we get

$$\begin{aligned} H_i(\overline{u}) &= \sum_m p(\Psi_m, \overline{u}) \cdot \tilde{h}_i(\Psi_m) + \sum_l p(\chi_l, \overline{u}) \cdot \tilde{h}_i(\chi_l) = \\ &= p(\overline{x}_t, \overline{u}) \cdot \left[\tilde{h}_i(\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{t-1}) + H_i^{\overline{x}_t}(\overline{u}^{\overline{x}_t})\right] + \sum_l p(\chi_l, \overline{u}^D) \cdot \tilde{h}_i(\chi_l), \end{aligned} \tag{19}$$

and (16) for $\overline{u}$ takes the form

$$\begin{aligned} &\sum_{i \in N} p(\overline{x}_t, \overline{u}) \cdot \left(\tilde{h}_i(\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{t-1}) + H_i^{\overline{x}_t}(\overline{u}^{\overline{x}_t})\right) + \\ &+ \sum_{i \in N} \sum_l p(\chi_l, \overline{u}^D) \cdot \tilde{h}_i(\chi_l) = \max_{v \in U} H_i(v). \end{aligned} \tag{20}$$

Suppose that $\overline{u}^{\overline{x}_t}$ does not satisfy (18), that is, there exists $v^{\overline{x}_t} \in U^{\overline{x}_t}$ such that

$$\sum_{i \in N} H_i^{\overline{x}_t}(\overline{u}^{\overline{x}_t}) < \sum_{i \in N} H_i^{\overline{x}_t}(v^{\overline{x}_t}). \tag{21}$$

Denote by $\Omega(v^{\overline{x}_t}) = \{\lambda_m^{\overline{x}_t} = (\overline{x}_t, \ldots, \overline{x}_{T(m)}) \mid p(\lambda_m^{\overline{x}_t}, v^{\overline{x}_t}) > 0\}$ the bundle of all trajectories in the subgame $\Gamma^{\overline{x}_t}$ generated by $v^{\overline{x}_t}$. Then (21) takes the form

$$\sum_{i \in N} \sum_m p(\Psi_m^{\overline{x}_t}, \overline{u}^{\overline{x}_t}) \cdot \tilde{h}_i^{\overline{x}_t}(\Psi_m^{\overline{x}_t}) < \sum_{i \in N} \sum_m p(\lambda_m^{\overline{x}_t}, v^{\overline{x}_t}) \cdot \tilde{h}_i^{\overline{x}_t}(\lambda_m^{\overline{x}_t}). \tag{22}$$

Denote by $W_i = (\overline{u}_i^D, v_i^{\overline{x}_t})$, $i \in N$, the ith player's compound pure strategy in Γ^{x_0}. The strategy profile $W = (W_1, \ldots, W_n)$ generates the strategy bundle $\Omega(W)$ that can be divided onto two disjoint subsets $\{\lambda_m\} = \{\omega \in \Omega(W) \mid \overline{x}_t \in \omega\}$ and $\{\chi_l\} = \{\omega \in \Omega(W) \mid \overline{x}_t \notin \omega\}$, where the second subset for $\Omega(W)$ coincides with the second subset for $\Omega(\overline{u})$ since $W^D = u^D$, and $\lambda_m = (\overline{x}_0, \ldots, \overline{x}_t) \cup (\overline{x}_t, \ldots, \overline{x}_{T(m)}) = (\overline{x}_0, \ldots, \overline{x}_t) \cup \lambda_m^{\overline{x}_t}$.

Adding $\sum_{i \in N} \tilde{h}_i(\overline{x}_0, \overline{x}_1 \ldots, \overline{x}_{t-1})$ to both sides of (22) we get

$$\sum_{i \in N} \left(\tilde{h}_i(\overline{x}_0, \ldots, \overline{x}_{t-1}) + H_i^{\overline{x}_t}(\overline{u}^{\overline{x}_t})\right) < \sum_{i \in N} \left(\tilde{h}_i(\overline{x}_0, \ldots, \overline{x}_{t-1}) + H_i^{\overline{x}_t}(v^{\overline{x}_t})\right). \tag{23}$$

Then we can multiply both sides of (23) on $p(\overline{x}_t, \overline{u}) = p(\overline{x}_t, \overline{u}^D) = p(\overline{x}_t, W^D) = p(\overline{x}_t, W) > 0$ and then add $\sum_{i \in N} \sum_l p(\chi_l, \overline{u}^D) \cdot \tilde{h}_i(\chi_l)$ to both sides of the last inequality. Taking into account (4)–(6) and (20) we obtain

$$\sum_{i \in N} H_i(\overline{u}) < \sum_{i \in N} H_i(W)$$

for the constructed strategy profile $W \in U$. The last inequality contradicts the fact that $\overline{u} \in \overline{U}(\Gamma^{x_0})$, hence (18) is valid.

Arguing in a similar way (for the case when different strategy profiles from $\overline{U}(\Gamma^{\overline{x}_t})$ generate different bundles of the trajectories) we can verify that $\omega^{\overline{x}_t} = (\overline{x}_t, \ldots, \overline{x}_T)$ — a fragment of the cooperative trajectory $\omega \in \Omega(\overline{u})$, starting at $\overline{x}_t$ — belongs to the optimal bundle of cooperative trajectories in the subgame $\Gamma^{\overline{x}_t}$, that is, $\omega^{\overline{x}_t} \in \Omega(\overline{u}^{\overline{x}_t})$. $\square$

We will assume in this paper that all the players have agreed to apply the PRB algorithm in order to choose the cooperative strategy profile $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_n)$ that generates the optimal bundle $\Omega(\overline{u})$ of cooperative trajectories in $\Gamma^{x_0} \in G^{cm}(n)$. The next step of cooperation is to define a characteristic function $V^{x_0}(S)$. There are different notions of characteristic functions (see, e.g., References [23,24,48]), in this paper we adopt the so-called γ-characteristic function introduced in Reference [24]. Namely, we assume that $V^{x_0}(S)$ is given by the SPE (based on the Attitude SPE algorithm) outcome of S in the noncooperative game between members of S maximizing their joint payoff, and non members playing individually.

The γ-characteristic function $V^{\bar{x}_t}$ for the subgame $\Gamma^{\bar{x}_t}$, $\bar{x}_t \in \omega_m(\bar{u}) = (\bar{x}_0, \ldots, \bar{x}_t, \ldots, \bar{x}_{T(m)})$, $\omega_m(\bar{u}) \in \Omega(\bar{u})$ along the optimal bundle of cooperative trajectories can be constructed using the same approach. Note that

$$V^{\bar{x}_t}(N) = \sum_{\omega_m^{\bar{x}_t} \in \Omega(\bar{u}^{\bar{x}_t})} p(\omega_m^{\bar{x}_t}, \bar{u}^{\bar{x}_t}) \cdot \sum_{\tau=t}^{T(m)} \sum_{i \in N} h_i(\bar{x}_\tau), \; t = 0, 1, \ldots, T(m). \tag{24}$$

Let $\Gamma^{x_0}(N, V^{x_0})$ denote extensive-form cooperative game $\Gamma^{x_0} \in G^{cm}(n)$ with γ-characteristic function, and $\Gamma^{\bar{x}_t}(N, V^{\bar{x}_t})$ denote the corresponding subgame.

We assume that the players adopt a single-valued cooperative solution φ^{x_0} (for instance, the Shapley value [33], the nucleolus [34], etc.) for the cooperative game $\Gamma^{x_0}(N, V^{x_0})$ which satisfies the collective rationality property

$$\sum_{i=1}^{n} \varphi_i^{x_0} = V^{x_0}(N) = \sum_{\omega_m \in \Omega(\bar{u})} p(\omega_m, \bar{u}) \cdot \sum_{\tau=0}^{T(m)} \sum_{i \in N} h_i(\bar{x}_\tau) \tag{25}$$

and the individual rationality property

$$\varphi_i^{x_0} \geqq V^{x_0}(\{i\}), \; i = 1, \ldots, n. \tag{26}$$

In addition, we assume that the same properties (25) and (26) are valid for the cooperative solutions $\varphi^{\bar{x}_t}$ at each subgame $\Gamma^{\bar{x}_t}(N, V^{\bar{x}_t})$, $t = 0, \ldots, T-1$.

It is worth noting that the last assumption as well as the choice of γ-characteristic function ensure that every player has an incentive to cooperate at each subgame along the optimal game evolution since the ith player's cooperative payoff-to-go at $\Gamma^{\bar{x}_t}(N, V^{\bar{x}_t})$, $t = 0, \ldots, T-1$, is at least equal to her non-cooperative counterpart: $\varphi_i^{\bar{x}_t} \geqslant H_i^{\bar{x}_t}(\underline{u}^{\bar{x}_t})$.

5. Subgame Consistency and Incremental IDP

Let $\beta = \{\beta_i(\bar{x}_\tau)\}$, $i = 1, \ldots, n$; $\tau = 1, \ldots, T(l)$, $\bar{x}(\tau) \in \omega_l(\bar{u})$, $\omega_l(\bar{u}) \in \Omega(\bar{u})$ denote the Imputation Distribution Procedure (IDP) for the cooperative solution $\left(\varphi_i^{\bar{x}_0}\right)_{i \in N}$ or the payment schedule (see, e.g., References [3,8–12,14–18,20] for details). The IDP approach means that all the players have agreed to allocate the total cooperative payoff $V^{x_0}(N)$ between the players along the optimal bundle $\Omega(\bar{u})$ of cooperative trajectories $\omega_l(\bar{u})$ according to some specific rule which is called IDP. Namely, $\beta_i(\bar{x}_\tau)$ denotes the actual current payment which the player i receives at position $\bar{x}_\tau$ (instead of $h_i(\bar{x}_\tau)$) if the players employ the IDP β. Moreover, one can design such an IDP β that all the players will be interested in cooperation in any subgame $\Gamma^{\bar{x}_\tau}$, $\bar{x}(\tau) \in \omega_l(\bar{u})$, $\omega_l(\bar{u}) \in \Omega(\bar{u})$, that is, at any intermediate time instant.

Definition 3. *The IDP $\beta = \{\beta_i(\bar{x}_\tau)\}$ satisfies subgame efficiency, if at any intermediate node $\bar{x}_t \in \omega(\bar{u})$, $\omega(\bar{u}) \in \Omega(\bar{u})$, $0 \leqslant t < T$, it holds that:*

$$\sum_{\omega_m^{\bar{x}_t} \in \Omega(\bar{u}^{\bar{x}_t})} p(\omega_m^{\bar{x}_t}, \bar{u}^{\bar{x}_t}) \cdot \sum_{\tau=t}^{T(m)} \beta_i(\bar{x}_\tau) = \varphi_i^{\bar{x}_t}, \; i \in N. \tag{27}$$

Equation (27) means that the expected sum of the payments to player i along the optimal subgame $\Gamma^{\bar{x}_t}$ evolution equals to what she is entitled to in this subgame. Then the IDP for each player can be reasonably implemented as a rule for step-by-step allocation of the ith player's current expected optimal payoff. Note that for $t = 0$ the subgame efficiency definition coincides with the efficiency at initial node x_0 or the efficiency in the whole game Γ^{x_0} condition (see References [9,14,16,20]).

Definition 4 ([10]). *The IDP $\beta = \{\beta_i(\bar{x}_\tau)\}$ satisfies the strict balance condition if for each node $\bar{x}_\tau \in \omega_m(\overline{u})$, $\omega_m(\overline{u}) \in \Omega(\overline{u})$ $\forall t = 0, \ldots, T(m)$*

$$\sum_{i \in N} \beta_i(\bar{x}_\tau) = \sum_{i \in N} h_i(\bar{x}_\tau). \tag{28}$$

Equation (28) ensures the "admissibility" of the IDP, that is, the sum of payments to the players in any node $\bar{x}_\tau$ is equal to the sum of payoffs that they can collect in this node.

The next advantageous dynamic property of an IDP—the time consistency, introduced in Reference [3]—was extended to dynamic games played over event trees in References [14,16,20] as well as to multicriteria extensive-form cooperative games (with chance moves) in Reference [9].

To write down properly the time consistency condition for some intermediate node $\bar{x}_t \in \omega(\overline{u}) = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{t-1}, \bar{x}_t, \bar{x}_{t+1}, \ldots, \bar{x}_T)$, $\omega(\overline{u}) \in \Omega(\overline{u})$, $1 \leqslant t < T$, in multistage game Γ^{x_0} with chance moves we need to pay attention to all chance nodes on the path $(\bar{x}_0, \ldots, \bar{x}_{t-1}) = \underline{\omega}^{x_t} \setminus \{\overline{x}_t\}$.
Namely, let us numerate the chance nodes from $P_0 \cap (\underline{\omega}^{x_t} \setminus \{\overline{x}_t\})$ in order of their occurrence on the path $(\bar{x}_0, \ldots, \bar{x}_{t-1})$, that is, $y_1 = \overline{x}_{t(1)}, y_2 = \overline{x}_{t(2)}, \ldots, y_\theta = \overline{x}_{t(\theta)}$, $0 \leqslant t(1) < t(2) < \ldots < t(\theta) < t$.

Definition 5 ([9]). *The IDP $\beta = \{\beta_{i/k}(\overline{x}_\tau)\}$ for the cooperative solution φ^{x_0} is called time consistent in the whole game $\Gamma^{x_0}(N, V^{x_0}) \in G^{cm}(n)$ if at any intermediate node $\overline{x}_t \in \omega(\overline{u})$, $\omega(\overline{u}) \in \Omega(\overline{u})$, $1 \leqslant t < T$, for all $i \in N$, it holds that*

<u>case $\theta = 0$</u> (no chance nodes on the path $(\bar{x}_0, \ldots, \bar{x}_{t-1})$):

$$\sum_{\tau=0}^{t-1} \beta_i(\overline{x}_\tau) + \varphi_i^{\overline{x}_t} = \varphi_i^{x_0}, \tag{29}$$

<u>case $\theta = 1$</u> (only one chance node $y_1 = \overline{x}_{t(1)}$ before $\bar{x}_t$):

$$\begin{aligned} &\sum_{\tau=0}^{t(1)} \beta_i(\overline{x}_\tau) + p(\overline{x}_{t(1)+1}, \overline{u}) \cdot \left\{ \sum_{\tau=t(1)+1}^{t-1} \beta_i(\overline{x}_\tau) + \varphi_i^{\overline{x}_t} \right\} + \\ &\quad + \sum_{x^k \in S(\overline{x}_{t(1)}) \setminus \{\overline{x}_{t(1)+1}\}} p(x^k, \overline{u}) \cdot \varphi_i^{x^k} = \varphi_i^{x_0}, \end{aligned} \tag{30}$$

<u>case $\theta = 2$</u> (two chance nodes $y_1 = \overline{x}_{t(1)}, y_2 = \overline{x}_{t(2)}$ before $\bar{x}_t$):

$$\begin{aligned} &\sum_{\tau=0}^{t(1)} \beta_i(\overline{x}_\tau) + p(\overline{x}_{t(1)+1}, \overline{u}) \cdot \left\{ \sum_{\tau=t(1)+1}^{t(2)} \beta_i(\overline{x}_\tau) + p(\overline{x}_{t(2)+1} \mid \overline{x}_{t(2)}, \overline{u}) \times \right. \\ &\times \left[\sum_{\tau=t(2)+1}^{t-1} \beta_i(\overline{x}_\tau) + \varphi_i^{\overline{x}_t} \right] + \left. \sum_{x^m \in S(\overline{x}_{t(2)}) \setminus \{\overline{x}_{t(2)+1}\}} p(x^m \mid \overline{x}_{t(2)}, \overline{u}) \cdot \varphi_i^{x^m} \right\} + \\ &+ \sum_{x^k \in S(\overline{x}_{t(1)}) \setminus \{\overline{x}_{t(1)+1}\}} p(x^k, \overline{u}) \cdot \varphi_i^{x^k} = \varphi_i^{x_0}, \end{aligned} \tag{31}$$

...

Note that for partial case when $\overline{x}_t \in S(\overline{x}_{t(1)})$, that is, if $\overline{x}_t$ follows the chance node $\overline{x}_{t(1)}$ Equation (30) takes the simpler form

$$\sum_{\tau=0}^{t(1)} \beta_i(\overline{x}_\tau) + \sum_{x^k \in S(\overline{x}_{t(1)})} p(x^k, \overline{u}) \cdot \varphi_i^{x^k} = \varphi_i^{x_0}.$$

A similar note is valid for equation (31), and so forth.

Roughly speaking, Definition 5 implies that the payments collected by the ith player (according to the payment schedule β) before reaching some intermediate node $\overline{x}_t$ plus the expected ith player's component of the Shapley value in the subgame $\Gamma^{\overline{x}_t}$ starting at $\overline{x}_t$ plus this player's expected Shapley value components in other subgames along the cooperative trajectories which do not contain $\overline{x}_t$ corresponds to what the player i is entitled to in the original game $\Gamma^{x_0}(N, V^{x_0})$.

It is worth noting that Definition 5 indeed provides a reasonable consistency requirements which a good payment schedule β should satisfy when the player evaluates IDP β at the initial node x_0, that is, before the game $\Gamma^{x_0}(N, V^{x_0})$ starts (and the words "in the whole game" in Definition 5 properly reflect this feature). However, when the player purposes to evaluate IDP β in the subgame $\Gamma^{\overline{x}_t}$, that is, after reaching some intermediate node $\overline{x}_t$ (in case when $\theta \geqslant 1$) this player will unlikely take into account the expected future payoffs in all the subgames which are unattainable if the node $\overline{x}_t$ has been already reached, that is, the last addends in the LHS of (30) and (31). To overcome this problem we suggest the players to use a notion of subgame consistency—a refinement of time consistency that was firstly proposed in Reference [36] for cooperative stochastic differential games and then extend it to stochastic dynamic games in References [37,38]. Let us provide a rigorous definition of the IDP subgame consistency for extensive-form games with chance moves that is applicable in all the subgames along the optimal bundle of cooperative trajectories.

Definition 6. *The IDP $\beta = \{\beta_i(\overline{x}_\tau)\}$ is called subgame consistent if at any intermediate node $\overline{x}_t \in \omega(\overline{u})$, $\omega(\overline{u}) \in \Omega(\overline{u})$, $1 \leqslant t \leqslant T$, for all $i \in N$, it holds that*

case $1 \leqslant t \leqslant t(1)$ (no chance nodes before the subgame $\Gamma^{\overline{x}_t}$ root $\overline{x}_t$):

$$\sum_{\tau=0}^{t-1} \beta_i(\overline{x}_\tau) + \varphi_i^{\overline{x}_t} = \varphi_i^{x_0}, \tag{32}$$

case $t(1)+1 < t \leqslant t(2)$ (only one chance node $y_1 = \overline{x}_{t(1)}$ before $\overline{x}_t$):

$$\sum_{\tau=t(1)+1}^{t-1} \beta_i(\overline{x}_\tau) + \varphi_i^{\overline{x}_t} = \varphi_i^{\overline{x}_{t(1)+1}}, \tag{33}$$

case $t(2)+1 < t \leqslant t(3)$ (two chance nodes before $\overline{x}_t$):

$$\sum_{\tau=t(2)+1}^{t-1} \beta_i(\overline{x}_\tau) + \varphi_i^{\overline{x}_t} = \varphi_i^{\overline{x}_{t(2)+1}}, \tag{34}$$

$\vdots$

case $t(\theta)+1 < t \leqslant T$ (no chance nodes after $\overline{x}_t$):

$$\sum_{\tau=t(\theta)+1}^{t-1} \beta_i(\overline{x}_\tau) + \varphi_i^{\overline{x}_t} = \varphi_i^{\overline{x}_{t(\theta)+1}}. \tag{35}$$

The subgame consistency definition differs from the "time consistency in the whole game" property (see References [9,14,16,20]) which is based on an a priori assessment of the ith player's expected optimal payoff (before the game starts). However, when the players make a decision in the subgame after the chance move occurs they need to recalculate the expected optimal payoff since the original optimal bundle of cooperative trajectories shrinks after each chance node. Note that we can not write out the subgame consistency condition for $t = t(1) + 1, t(2) + 1, ..., t(\theta) + 1$, that is, for the nodes $\overline{x}_t$ that immediately follow the chance nodes.

One can suggest different imputation distribution procedures that may or may not satisfy the useful properties listed above. The review of different IDP for multistage games (without chance moves) as well as the analysis of their properties can be found in References [10,12,15,17]. Below we consider the refinement of the so-called incremental IDP (see, e.g., References [10,14,16,17,20,21]) that was recently introduced for multistage games with chance moves [9].

Definition 7 ([9])**.** *The incremental IDP for the cooperative solution φ^{x_0} in multistage game with chance moves Γ^{x_0} is defined as follows:*

$$\beta_i(x_t) = \varphi_i^{x_t} - \sum_{x_{t+1}^k \in S(x_t)} p(x_{t+1}^k | x_t, \overline{u}) \cdot \varphi_i^{x_{t+1}^k} \tag{36}$$

for $x_t \in \omega_l(\overline{u}) = (x_0, \ldots, x_t, \ldots, x_{T(l)})$, $\omega_l(\overline{u}) \in \Omega(\overline{u})$, $t = 0, \ldots, T(l) - 1$;

$$\beta_i(x_{T(l)}) = \varphi_i^{x_{T(l)}} \tag{37}$$

for $x_{T(l)} \in \Omega(\overline{u}) \cap P_{n+1}$.

Remark 2. *Formulas (36), (37) are similar to the imputation distribution procedures suggested in References [14,16,20] for (single-criterion) stochastic discrete-time dynamic games played over event trees. If $x_t \in P_i$, $i = 1, \ldots, n$ Equation (36) takes the simpler form $\beta_i(x_t) = \varphi_i^{x_t} - \varphi_i^{x_{t+1}}$, where $\overline{u}_i(x_t) = x_{t+1}$, that coincides with the "classical" incremental IDP.*

Let us use again 3-person extensive-form game from Example 1 to demonstrate a proposed scheme of cooperation.

Example 2. *(Cooperative behavior in 3-player game from Ex. 1).*

Suppose that the players have agreed on the following ranks: $r(1) = 1$, $r(2) = 2$ and $r(3) = 3$. When implementing the PRB algorithm we get the optimal bundle $\Omega(\overline{u})$ which contains four cooperative trajectories (marked in bold, deep blue in Figure 2): $\omega_1 = (\overline{x}_0, \overline{x}_1, \overline{x}_2^1, \overline{x}_3, \overline{x}_4^2, \overline{x}_5, \overline{x}_6)$, $\omega_2 = (\overline{x}_0, \overline{x}_1, x_2^2, z_3)$, $\omega_3 = (\overline{x}_0, \overline{x}_1, \overline{x}_2^1, \overline{x}_3, x_4^1)$ and $\omega_4 = (\overline{x}_0, \overline{x}_1, \overline{x}_2^1, \overline{x}_3, x_4^3, z_7)$. Note that players use the ranks when making decision at node $\overline{x}_5$.

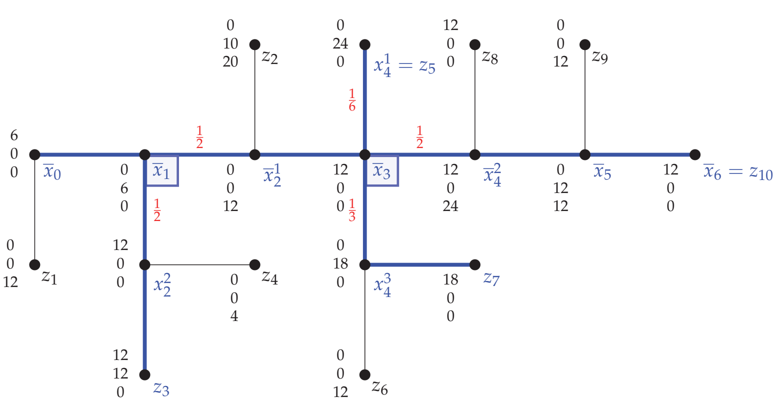

Figure 2. 3-player extensive-form game: cooperative behavior.

To demonstrate the implementation of the incremental IDP and its properties we will adopt the Shapley value as a single valued cooperative solution. The values of the γ-characteristic function V^{x_0} for the original game $\Gamma^{x_0}(N, V^{x_0})$ and the Shapley value φ^{x_0} are

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$V^{x_0}(S)$	12	11	18	38	48	33	68

$$\varphi^{x_0} = \begin{pmatrix} 25\frac{1}{6} \\ 17\frac{1}{6} \\ 25\frac{2}{3} \end{pmatrix}.$$

Consider, for instance, the incremental IDP along the longest cooperative trajectory $\omega_2 = (\overline{x}_0, \ldots, \overline{x}_6)$ from $\Omega(\overline{u})$. If we calculate γ-characteristic functions using Attitude SPE algorithm for the subgames, we get the following results.

Subgame $\Gamma^{\overline{x}_1}(N, V^{\overline{x}_1})$:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$V^{\overline{x}_1}(S)$	6	11	18	32	42	33	62

$$\varphi^{\overline{x}_1} = \begin{pmatrix} 19\frac{1}{6} \\ 17\frac{1}{6} \\ 25\frac{2}{3} \end{pmatrix}.$$

Subgame $\Gamma^{\overline{x}_2^1}(N, V^{\overline{x}_2^1})$:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$V^{\overline{x}_2^1}(S)$	0	10	32	40	60	42	76

$$\varphi^{\overline{x}_2^1} = \begin{pmatrix} 21 \\ 17 \\ 38 \end{pmatrix}.$$

Subgame $\Gamma^{x_2^2}(N, V^{x_2^2})$:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$V^{x_2^2}(S)$	12	0	4	12	24	12	36

$$\varphi^{x_2^2} = \begin{pmatrix} 17\frac{1}{3} \\ 5\frac{1}{3} \\ 13\frac{1}{3} \end{pmatrix}.$$

Subgame $\Gamma^{\overline{x}_3}(N, V^{\overline{x}_3})$:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$V^{\overline{x}_3}(S)$	24	10	16	40	48	26	64

$$\varphi^{\overline{x}_3} = \begin{pmatrix} 31 \\ 13 \\ 20 \end{pmatrix}.$$

Subgame $\Gamma^{\overline{x}_4^2}(N, V^{\overline{x}_4^2})$:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$V^{\overline{x}_4^2}(S)$	24	0	24	36	60	24	72

$$\varphi^{\overline{x}_4^2} = \begin{pmatrix} 36 \\ 6 \\ 30 \end{pmatrix}.$$

Subgame $\Gamma^{x_4^3}(N, V^{x_4^3})$:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$V^{x_4^3}(S)$	0	18	12	18	18	30	36

$$\varphi^{x_4^3} = \begin{pmatrix} 3 \\ 18 \\ 15 \end{pmatrix}.$$

Subgame $\Gamma^{\overline{x}_5}(N, V^{\overline{x}_5})$:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$V^{\overline{x}_5}(S)$	0	12	24	12	24	36	36

$$\varphi^{\overline{x}_5} = \begin{pmatrix} 0 \\ 12 \\ 24 \end{pmatrix}.$$

Finally, $\varphi^{\overline{x}_6} = h^{\overline{x}_6} = (12, 0, 0)$.

One can calculate the incremental IDP $\{\beta_i(\overline{x}_\tau),\ \overline{x}_\tau \in \omega_2\}$ using (36) and (37):

	$\overline{x}_0$	$\overline{x}_1$	$\overline{x}_2^1$	$\overline{x}_3$	$\overline{x}_4^2$	$\overline{x}_5$	$\overline{x}_6$
$\beta_1(\overline{x}_\tau)$	6	0	-10	12	36	-12	12
$\beta_2(\overline{x}_\tau)$	0	6	4	0	-6	12	0
$\beta_3(\overline{x}_\tau)$	0	0	18	0	6	24	0

Note that the subgame consistency conditions at nodes $\overline{x}_1$, $\overline{x}_3$ and $\overline{x}_5$ according to (32)–(34) respectively take the form:

$$\beta_i(\overline{x}_0) + \varphi_i^{\overline{x}_1} = \varphi_i^{x_0}, i \in N, \text{ or } \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 19\frac{1}{6} \\ 17\frac{1}{6} \\ 25\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 25\frac{1}{6} \\ 17\frac{1}{6} \\ 25\frac{2}{3} \end{pmatrix},$$

$$\beta_i(\overline{x}_2^1) + \varphi_i^{\overline{x}_3} = \varphi_i^{x_2^1}, i \in N, \text{ or } \begin{pmatrix} -10 \\ 4 \\ 18 \end{pmatrix} + \begin{pmatrix} 31 \\ 13 \\ 20 \end{pmatrix} = \begin{pmatrix} 21 \\ 17 \\ 38 \end{pmatrix},$$

$$\beta_i(\overline{x}_4^2) + \varphi_i^{\overline{x}_5} = \varphi_i^{x_4^2}, i \in N, \text{ or } \begin{pmatrix} 36 \\ -6 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 \\ 12 \\ 24 \end{pmatrix} = \begin{pmatrix} 36 \\ 6 \\ 30 \end{pmatrix}.$$

It is known that the classical incremental IDP for multistage (and differential) games may imply negative current payments to some players at some positions (see References [4,10,17,38] for details). As one can observe in Ex. 2, this drawback of the incremental IDP may appear in the extensive-form game with chance moves as well. Two approaches how to overcome this possible disadvantage were suggested in References [4,10]. Unfortunately, as it was firstly proved in Reference [10], in general it is impossible to design a time consistent IDP which satisfies both the balance condition and non-negativity constraint.

Proposition 3. *The incremental IDP (36), (37) satisfies strict balance condition (28), the subgame efficiency condition (27), and the subgame consistency conditions (32)–(35).*

Proof. Incremental IDP β was proved to satisfiy strict balance condition (28) in Reference [9]. The proof of subgame consistency can be carried out by direct verification. For instance, consider the case when $t(1) + 1 < t \leqslant t(2)$. Then, using Remark 2 we get

$$\sum_{\tau=t(1)+1}^{t-1} \beta_i(\overline{x}_\tau) = \left(\varphi_i^{\overline{x}_{t(1)+1}} - \varphi_i^{\overline{x}_{t(1)+2}}\right) + \ldots + \left(\varphi_i^{\overline{x}_{t-1}} - \varphi_i^{\overline{x}_t}\right) = \varphi_i^{\overline{x}_{t(1)+1}} - \varphi_i^{\overline{x}_t}.$$

Obviously, (33) is satisfied.

The proof that IDP (36), (37) satisfies subgame efficiency (27) is based on direct calculations but rather cumbersome in general case (i.e., for arbitrary game Γ^{x_0}). Let us demonstrate how it works for the game in Example 2. For instance we verify that the incremental IDP meets the subgame efficiency condition at node $\overline{x}_3$.

Note that $\Omega(\overline{u}^{\overline{x}_3}) = \left\{\omega_1^{\overline{x}_3} = (\overline{x}_3, x_4^1); \omega_2^{\overline{x}_3} = (\overline{x}_3, \overline{x}_4^2, \overline{x}_5, \overline{x}_6); \omega_3^{\overline{x}_3} = (\overline{x}_3, x_4^3, z_7)\right\}$ while $p(\omega_1^{\overline{x}_3}, \overline{u}^{\overline{x}_3}) = \pi(x_4^1|\overline{x}_3)$, $p(\omega_2^{\overline{x}_3}, \overline{u}^{\overline{x}_3}) = \pi(\overline{x}_4^2|\overline{x}_3)$ and $p(\omega_3^{\overline{x}_3}, \overline{u}^{\overline{x}_3}) = \pi(x_4^3|\overline{x}_3)$. Then, using (32), (33), Remark 2, equality $\sum\limits_{x_4^k \in S(\overline{x}_3)} \pi(x_4^k|\overline{x}_3) = 1$ and the notation

$$\Phi_i^4 = \sum_{k=1}^{3} \pi(x_4^k|\overline{x}_3) \cdot \varphi_i^{x_4^k},$$

we obtain

$$\begin{aligned}
\sum_{\omega_k^{\overline{x}_3} \in \Omega(\overline{u}^{\overline{x}_3})} p(\omega_k^{\overline{x}_3}, \overline{u}^{\overline{x}_3}) \cdot \sum_{\tau=3}^{T(k)} \beta_i(\overline{x}_\tau) = \pi(x_4^1|\overline{x}_3) \cdot \left[\left(\varphi_i^{\overline{x}_3} - \Phi_i^4\right) + \varphi_i^{x_4^1}\right] + \\
+\pi(\overline{x}_4^2|\overline{x}_3) \cdot \left[\left(\varphi_i^{\overline{x}_3} - \Phi_i^4\right) + \left(\varphi_i^{\overline{x}_4^2} - \varphi_i^{\overline{x}_5}\right) + \left(\varphi_i^{\overline{x}_5} - \varphi_i^{\overline{x}_6}\right) + \varphi_i^{\overline{x}_6}\right] + \\
+\pi(\overline{x}_4^3|\overline{x}_3) \cdot \left[\left(\varphi_i^{\overline{x}_3} - \Phi_i^4\right) + \left(\varphi_i^{x_4^3} - \varphi_i^{z_7}\right) + \varphi_i^{z_7}\right] = \varphi_i^{\overline{x}_3} \cdot \sum_{k=1}^{3} \pi(x_4^k|\overline{x}_3) + \\
+\pi(x_4^1|\overline{x}_3) \cdot \left[-\Phi_i^4 + \varphi_i^{x_4^1}\right] + \pi(\overline{x}_4^2|\overline{x}_3) \cdot \left[-\Phi_i^4 + \varphi_i^{\overline{x}_4^2}\right] + \pi(x_4^3|\overline{x}_3) \cdot \left[-\Phi_i^4 + \varphi_i^{x_4^3}\right] = \\
= \varphi_i^{\overline{x}_3} - \Phi_i^4 \cdot \sum_{k=1}^{3} \pi(x_4^k|\overline{x}_3) + \sum_{k=1}^{3} \pi(x_4^k|\overline{x}_3) \cdot \varphi_i^{x_4^k} = \varphi_i^{\overline{x}_3}.
\end{aligned}$$

□

According to Proposition 3, the incremental payment schedule (36), (37) can be used to implement a long-term cooperative agreement in an extensive-form game with chance moves.

6. Conclusions

In the paper we purposes to design a mechanism of the players' sustainable long-term cooperation that satisfies a number of good properties. To this aim we formalised the players' rank based algorithm for selecting a unique optimal bundle of cooperative trajectories, and proved that corresponding cooperative strategy profile satisfies time consistency. To calculate γ-characteristic function one need to have a specific method for constructing a unique (subgame perfect) equilibrium at any extensive-form game with chance moves. Hence, we formalised a backwards induction procedure refinement based on the players' attitude vectors—the so-called attitude SPE algorithm.

As a result of reexamination of the "IDP time consistency in the whole game" concept, we suggest to adopt the concept of subgame consistency, introduced in Reference [36] for differential stochastic games and then extend it to dynamic stochastic games in References [37,38]. The definition of subgame consistency for extensive-form game with chance moves is provided. This property takes into account such an interesting feature of the games under consideration that when the players make a decision in the subgame Γ^{x_t} after the chance move occurs, they need to recalculate their expected optimal payoffs-to-go since the original optimal bundle of cooperative trajectories shrinks after each chance node. It is worth noting that a similar approach, based on the IDP subgame consistency notion could be applied to dynamic games played over event trees ([14,16,20]). We proved that the incremental IDP specified for multistage games with chance moves in Reference [9] satisfy subgame consistency and subgame efficiency as well as the strict balance condition.

It follows from Propositions 1–3 that two specified algorithms combined with the γ-characteristic function, and the incremental payment schedule together constitute a mechanism of the players' sustainable cooperation that satisfies a number of good properties and could be used in extensive-form games with chance moves. Note that the main result of the paper—Proposition 3—does not depend on the specific method which the players employ to calculate the characteristic function as well as on the specific single-valued cooperative solution meeting (25) and (26).

Since this is the first time that subgame consistent solutions are examined for extensive-form games with chance moves, further research along this line is expected. It is surely of interest to develop appropriate software application to implement proposed algorithms in arbitrary extensive-form game with chance moves. Possibly, one can use the so-called Game Theory Explorer [30] when developing such software tools for 2-person extensive games. Further, it might be interesting to run experiments with large-scale datasets, after the software application that allows to construct unique SPE, the optimal bundle of cooperative trajectories, γ-characteristic function, and so forth, will be developed.

Let us notice some preliminary suggestions on how one can use such software application to run simulations. First, one can vary the main parameter—the length of the game tree, and the additional parameters such as the game structure, the players' payoffs, probabilities of transitions, and so forth, to obtain practical estimations of the proposed algorithms complexity and scalability. Secondarily, one can generate external disturbances of the stage payoffs and probabilities and vary the players' attitude vectors to carry out the sensitivity analysis of the proposed non-cooperative and cooperative solutions. Further, it is of interest to get experimental estimations of the price of anarchy and the price of stability for the class of games under consideration. Finally, one can use such software application to check whether the additional properties (non-negativity, irrational-behavior-proof conditions, etc.) of the proposed incremental IDP and other payment schedules (see, e.g., Reference [15]) are satisfied for given extensive-form game with chance moves.

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The Relationship between the Core and the Modified Cores of a Dynamic Game

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Abstract: The core as a solution to a cooperative game has the advantage that any imputation from it is undominated. In cooperative dynamic games, there is a known transformation that demonstrates another advantage of the core—time consistency—keeping players adhering to it during the course of the game. Such a transformation may change the solution, so it is essential that the new core share common imputations with the original one. In this paper, we will establish the relationship between the original core of a dynamic game and the core after the transformation, and demonstrate that the latter can be a subset of the former.

Keywords: discrete-time games; cooperation; the core; linear transformation

MSC: 91A12; 91A50

1. Introduction

The theory of cooperative dynamic games is useful for modeling and analyzing real world problems. Examples include advertising, public goods provision, resource extraction, environmental management, and others which are extensively discussed in [1,2]. The core as a solution to a cooperative game has the advantage that any imputation from it is undominated. This solution is quite popular in the literature on the application of dynamic game theory, not only because of the aforementioned property, but also because of its flexibility, allowing allocating the cooperative outcome in several ways, for instance, in lot sizing [3–5], pollution control [6–8], or non-renewable resource extraction [9]. In cooperative dynamic games, there is a known transformation of a characteristic function, which is a key component of any cooperative game measuring the claims of any group of players [10,11]. The core determined by the modified characteristic function possesses another advantage—time consistency—keeping players adhering to it and being non-negotiable during the course of the game [1,12]. Such a transformation, however, may change the core, so it is essential that the modified core share common imputations with the original one. This allows players to expect if not all of the imputations from the original one, but a part of them. For this reason, we will establish the relationship between the original core of a dynamic game and the core after the transformation, and demonstrate that in some instances the latter can be a subset of the former. It was proven in [11] that the proposed transformation rule applied an infinite number of times converges when the total players' payoffs along the agreed upon behavior are positive. Here we will relax this assumption and refine the conditions that ensure the convergence of the transformation rule.

The structure of the paper is as follows. Section 2 introduces necessary definitions and concepts. The main results of the paper are formulated in Section 3. We then study the relationship between the original core of a dynamic game and the modified ones and establish conditions for the limiting core to be a subset of the original core also when considering classes of linear symmetric games, linear-state games, and two-stage network games. Section 4 concludes.

2. Background

We consider a standard formulation of a dynamic game with complete information. Let $N = \{1, \ldots, n\}$, $|N| = n \geqslant 2$, be a finite set of players. The set of game stages (periods) is described by a finite set $\mathcal{T} = \{1, \ldots, T\}$. We denote a state variable at stage $t \in \mathcal{T}$ by $x(t)$ which belongs to a state space X. Let $x(1) = x_1$. Next, we denote an action of player $i \in N$ at stage $t \in \mathcal{T}$ by $u_i(t)$ which belongs to her action space $U_i(t)$. (Since the sets of actions $U_1(t), \ldots, U_n(t)$, $t \in \mathcal{T}$, and the state space X have not been precisely defined, we suppose that they are not empty, and the values of all optimization problems below exist and are finite). We suppose that the state dynamics is governed by the difference equation

$$x(t+1) = f_t(x(t), u(t)) \in X \tag{1}$$

for any $t \in \mathcal{T}$ from the initial state x_1. It is supposed that $x(t+1)$ is uniquely defined. At each of T game stages, players simultaneously choose actions and thus form an action profile $u = (u_1, \ldots, u_n)$ with $u_i = (u_i(1), \ldots, u_i(T))$ for $i \in N$. The payoff to player $i \in N$ in the game is defined by the real-valued function

$$J_i(x_1, u) = \sum_{t=1}^{T} h_{it}(x(t), u(t)), \quad i \in N,$$

and amounts to the sum of her stage payoffs. (As with [11], players are not rewarded at a terminal state $x(T+1)$, yet the setting can easily be generalized to this case as well).

A player chooses an action according to her *strategy* which accounts for the current information about the game available to her at the time of decision: this can be the information about the game stage, the value of the state variable, the actions that players have taken at previous stages, etc. We denote the information available to player $i \in N$ at stage $t \in \mathcal{T}$ by $\eta_i(t)$. A strategy u_i of player i is a rule that maps the player's information space to her action space, i.e., at stage t player i with information $\eta_i(t)$ chooses the action $u_i(t) = \mathrm{u}_i(\eta_i(t)) \in U_i(t)$. (See [13] for more details). A collection $\mathrm{u} = (\mathrm{u}_1, \ldots, \mathrm{u}_n)$ is a strategy profile. Each strategy profile generates a *trajectory* which is a profile $x = (x(1), \ldots, x(T))$ whose entries are determined by (1). One can introduce the *payoff function* J_i of player $i \in N$ defined on the set of players' strategy profiles as follows: $\mathrm{J}_i(\mathrm{u}) = J_i(x_1, u)$ where u is an action profile corresponding to u.

In the cooperative formulation of the game, players choose their strategies jointly to maximize the payoff they generate, that is to maximize the sum $\sum_{i \in N} \mathrm{J}_i(\mathrm{u})$. Let a strategy profile $\mathrm{u}^* = (\mathrm{u}_1^*, \ldots, \mathrm{u}_n^*)$ maximize the latter sum. This profile is called a cooperative strategy profile and the associated trajectory $x^* = (x^*(1), \ldots, x^*(T))$ with $x^*(1) = x_1$ is called a cooperative trajectory.

We now define a cooperative transferable utility game, or a TU game, (N, v) which is determined by the same player set N and a characteristic function v. This function is defined on 2^N, that is the set of all subsets of set N, and for a subset $S \subseteq N$, called a coalition, its value (a real number) $v(S)$ measures the worth, or claims, of this coalition in the game. Additionally, $v(\varnothing) = 0$. We will not specify how this function is determined as it is not relevant to the analysis we will perform; we only note that $v(N) = \sum_{i \in N} \mathrm{J}_i(\mathrm{u}^*)$, i.e., the *grand coalition* claims the maximum payoff it generates. (See different concepts for determining the characteristic function in dynamic games in [12]). Once the value of $v(N)$ is obtained, players allocate it among them as an *imputation* which is a profile $\xi(v) = (\xi_1(v), \ldots, \xi_n(v))$ satisfying efficiency, i.e., $\sum_{i \in N} \xi_i(v) = v(N)$, and individual rationality, i.e., $\xi_i(v) \geqslant v(\{i\})$, $i \in N$. The set of all imputations, or the imputation set, will be denoted by $\mathcal{I}(v)$. A *cooperative solution*, or simply a *solution*, to the cooperative dynamic game (N, v) is a rule that assigns a subset $\mathcal{M}(v) \subseteq \mathcal{I}(v)$ to this game. In this paper, we suppose that the solution is the core, that is the subset of the imputation set given by $\mathcal{C}(v) = \{\xi(v) \in \mathcal{I}(v) : \sum_{i \in S} \xi_i(v) \geqslant v(S),\ S \subset N\}$. Having chosen the agreed upon cooperative solution $\mathcal{C}(v)$, players jointly implement cooperative strategy profile u^* moving along cooperative trajectory x^*, and after obtaining the value $v(N)$ as their payoff, the players allocate it among them as an imputation from the chosen solution $\mathcal{C}(v)$.

In cooperative dynamic games, it is important that players adhere to the same solution chosen at the initial stage as the game develops along the agreed upon cooperative trajectory x^*. A *time-consistent* solution is stable to its revision during the course of the game, and implementing certain mechanisms one can make cooperation sustainable. When the solution is time inconsistent, there are effective mechanisms of game *regularization*, that is a change in players' stage payoffs along the cooperative trajectory, so that the solution becomes time consistent in the regularized game. (See [1,12] for a comprehensive analysis of sustainable cooperation and the associated time consistency property of a cooperative solution). In the vast majority of cases, such mechanisms are designed on a special redistribution of players' stage payoffs determined by an imputation distribution procedure and they require consideration of proper subgames of the original game along the cooperative trajectory. Each subgame is a dynamic game of $T - t + 1$ stages starting from the initial state $x^*(t)$, $t \in \mathcal{T} \setminus \{1\}$. In a similar way, one can define a cooperative subgame (N, v_t), the imputation set $\mathcal{I}(v_t)$, and the cooperative solution (the core $\mathcal{C}(v_t)$) to each subgame, $t \in \mathcal{T} \setminus \{1\}$. (From now on, the original cooperative game (N, v) will be denoted by (N, v_1) for consistency in notation). We suppose that the solution is not empty along the cooperative trajectory x^*. In other words, for each state $x^*(t)$, $t \in \mathcal{T}$, the core $\mathcal{C}(v_t)$ is not empty. If it is not the case, then from the first game stage when this assumption is violated, the players are unable to follow the agreed upon solution.

Petrosyan et al. [11] examine the time consistency of the core based on a *transformation* of the characteristic functions and reveal that the core of the transformed game becomes strong time consistent. (Strong time consistency is a stricter property of a cooperative solution to a dynamic game; it is applicable to set solutions and it coincides with the property of time consistency for point solutions (see [10,14–16] for details)). The strong time consistency of the core was established with the use of a *modified* characteristic function $\hat{v}_t$, $t \in \mathcal{T}$, which for each coalition $S \subseteq N$ accounts for values $v_\tau(S)$ and $v_\tau(N)$, $\tau \geqslant t$, along the cooperative trajectory x^* and is given by:

$$\hat{v}_t(S) = \sum_{\tau=t}^{T} \frac{v_\tau(S) \sum_{i \in N} h_{i\tau}(x^*(\tau), u^*(\tau))}{v_\tau(N)}, \quad S \subseteq N. \tag{2}$$

We call the sets $\mathcal{I}(\hat{v}_t)$ and $\mathcal{C}(\hat{v}_t)$ the modified imputation set and the modified core: these sets are the imputation set and the core in the modified game $(N, \hat{v}_t)$, $t \in \mathcal{T}$.

Since the transformation rule changes the solution, a player or a group of players may want to apply the rule again (or several times subsequently) to change the characteristic function of the game and therefore their payoffs prescribed by the solution which is based on the characteristic function. For a given cooperative trajectory x^* and a coalition $S \subseteq N$, let $\mathbf{v}(S) = (v_1(S), \ldots, v_T(S))'$ and $\hat{\mathbf{v}}(S) = (\hat{v}_1(S), \ldots, \hat{v}_T(S))'$. Using this notation, transformation rule (2) can be written in matrix form:

$$\hat{\mathbf{v}}(S) = \Theta\, \mathbf{v}(S), \tag{3}$$

where Θ is the upper-triangular matrix

$$\Theta = \begin{pmatrix} \theta_1 & \theta_2 & \cdots & \theta_{T-1} & \theta_T \\ 0 & \theta_2 & \cdots & \theta_{T-1} & \theta_T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \theta_{T-1} & \theta_T \\ 0 & 0 & \cdots & 0 & \theta_T \end{pmatrix},$$

whose entries are given by

$$\theta_t = \frac{\sum_{i \in N} h_{it}(x^*(t), u^*(t))}{\sum_{\tau=t}^{T} \sum_{i \in N} h_{i\tau}(x^*(\tau), u^*(\tau))}, \quad t \in \mathcal{T}.$$

Since $\theta_T = 1$, the last column of Θ consists of ones. Using relation (3), for each coalition $S \subseteq N$ we construct an iterative process $\mathbf{v}^{(m)}(S) = \Theta\, \mathbf{v}^{(m-1)}(S)$, $m = 1, 2, \ldots$ with the initial condition $\mathbf{v}^{(0)}(S) = \mathbf{v}(S)$ where $\mathbf{v}^{(m)}(S) = (v_1^{(m)}(S), \ldots, v_T^{(m)}(S))'$ and $\mathbf{v}^{(1)}(S) = \hat{\mathbf{v}}(S)$. The iterative process can be rewritten as:

$$\mathbf{v}^{(m)}(S) = \Theta^m\, \mathbf{v}(S), \quad m = 1, 2, \ldots \tag{4}$$

It was established in [11] that under the assumption of the non-negativity of players' stage payoffs along the cooperative trajectory, the sequence of modified characteristic functions defined by (4) converges. A *limiting* characteristic function $\bar{v}_t$, $t \in \mathcal{T}$, is given by:

$$\bar{v}_t(S) = \frac{v_t(N)}{v_T(N)} \cdot v_T(S), \quad S \subseteq N. \tag{5}$$

We call the sets $\mathcal{I}(\bar{v}_t)$ and $\mathcal{C}(\bar{v}_t)$ the limiting imputation set and the limiting core: these sets are the imputation set and the core in the limiting game $(N, \bar{v}_t)$, $t \in \mathcal{T}$. In [11], it was shown that when the iterative process (4) converges and (i) when the core $\mathcal{C}(v_t) \neq \varnothing$ for any $t \in \mathcal{T}$, then the modified core $\mathcal{C}(\hat{v}_t) \neq \varnothing$ for any $t \in \mathcal{T}$, (ii) when the core at the terminal stage $\mathcal{C}(v_T) \neq \varnothing$, the limiting core $\mathcal{C}(\bar{v}_t) \neq \varnothing$ for any $t \in \mathcal{T}$. Since we suppose that the original cores are non-empty along the cooperative trajectory, all modified and limiting cores $\mathcal{C}(v_t^{(m)})$ and $\mathcal{C}(\bar{v}_t)$ will be non-empty as well for all $t \in \mathcal{T}$, $m = 1, 2, \ldots$, provided that (4) converges.

3. The Results

3.1. General Results

The convergence of transformation rule (4) was only established for non-negative payoffs along the cooperative trajectory. We now relax the non-negativity condition, yet still assume that the total payoff $\sum_{i \in N} h_{it}(x^*(t), u^*(t))$ is non-zero at each game stage, i.e., at least one player contributes into the grand coalition's payoff.

Proposition 1. *The limiting characteristic function $\bar{v}_1$ exists if and only if $v_t(N)v_{t+1}(N) > 0$ and $|v_{t+1}(N)| \leqslant 2|v_t(N)|$ for $t \in \mathcal{T} \setminus \{T\}$.*

Proof. We suppose that matrix Θ can be decomposed as $\Theta = P\Lambda P^{-1}$ where Λ is a diagonal matrix whose diagonal entries are the eigenvalues of Θ, and P is a matrix whose columns are the corresponding eigenvectors. When the limiting characteristic function exists, it holds that

$$\bar{\mathbf{v}}(S) = \lim_{m \to \infty} \mathbf{v}^{(m)}(S) = \lim_{m \to \infty} \Theta^m\, \mathbf{v}(S) = \lim_{m \to \infty} P\Lambda^m P^{-1}\, \mathbf{v}(S). \tag{6}$$

Since the transformation matrix Θ is upper triangular, we have that $\Lambda = \mathrm{diag}\{\theta_1, \ldots, \theta_T\}$ and $\Lambda^m = \mathrm{diag}\{\theta_1^m, \ldots, \theta_T^m\}$ for $m = 1, 2, \ldots$ According to (6), the limiting characteristic function exists if and only if the limit $\lim_{m \to \infty} \Lambda^m$ exists. This is the case when the absolute values of the eigenvalues of matrix Θ do not exceed 1: $|\theta_t| \in [0, 1]$ for $t \in \mathcal{T}$. Recall that $\theta_T = 1$.

If $\theta_t = (v_t(N) - v_{t+1}(N))\,/\,v_t(N) = 1$ for some $t \in \mathcal{T} \setminus \{T\}$, then $v_{t+1}(N) = 0$. However, the linear transformation requires that $v_t(N) \neq 0$ for all $t \in \mathcal{T}$. Therefore, it must hold that $\theta_t \in [-1, 1)$ for $t \in \mathcal{T} \setminus \{T\}$ which is equivalent to

$$-1 \leqslant \frac{v_t(N) - v_{t+1}(N)}{v_t(N)} < 1, \quad t \in \mathcal{T} \setminus \{T\}. \tag{7}$$

For $t \in \mathcal{T} \setminus \{T\}$ if $v_t(N) > 0$, then (7) is equivalent to $0 < v_{t+1}(N) \leqslant 2v_t(N)$, whereas if $v_t(N) < 0$, (7) is equivalent to $2v_t(N) \leqslant v_{t+1}(N) < 0$. Summarizing the above, (7) is equivalent to the conditions mentioned in the statement of the proposition for every $t \in \mathcal{T} \setminus \{T\}$. $\square$

Remark 1. *When the limiting characteristic function $\bar{v}_1$ exists, then the limiting characteristic functions $\bar{v}_t$, $t \in \mathcal{T} \setminus \{1\}$, exist as well for any subgame along the cooperative trajectory.*

The conditions that ensure the convergence of the iterative process (4) and, therefore, the existence of the limiting characteristic function, have the following meaning. First, the grand coalition's payoff in the original game and its proper subgames along the cooperative trajectory are of same sign. Second, the grand coalition's payoff in any subgame must be at most twice its payoff in the preceding subgame in absolute values.

Now we study the relationship between the core of the cooperative dynamic game and the modified (limiting) core. As we noted, the transformation rule (4) changes the solution. Therefore, players having agreed on the core $\mathcal{C}(v_1)$ as a solution to game (N, v_1) have to be sure that they will be able to realize an imputation from it even after game transformation. Since for the grand coalition N it holds that $v_t(N) = v_t^{(1)}(N) = \cdots = v_t^{(m)}(N) = \cdots = \bar{v}_t(N)$ for every $t \in \mathcal{T}$, the value to be allocated is invariant to the transformation rule. Our main goal is to establish the relationship between the original core $\mathcal{C}(v_1)$, modified cores $\mathcal{C}(v_1^{(m)})$, $m = 1, 2, \ldots$, and the limiting one $\mathcal{C}(\bar{v}_1)$. When the latter cores intersect with $\mathcal{C}(v_1)$, players are able to realize an imputation from the original core after one-time or even multifaceted transformation of the characteristic function. We will need the following definitions. A set-function $v : 2^N \mapsto \mathbb{R}$ is *monotone* if for every $R \subset S \subseteq N$ we have that either $v(R) \leqslant v(S)$ or $v(R) \geqslant v(S)$. A set-function v is called *supermodular* if for any subsets (coalitions) $S, R \subseteq N$ the following holds: $v(S \cup R) + v(S \cap R) \geqslant v(S) + v(R)$. When the opposite inequality holds for every pair of coalitions, the function v is called *submodular*. It is well known that the core of a *convex* cooperative game, i.e., the game whose characteristic function is supermodular, is not empty [17]. Therefore, when the characteristic functions $\max\{v_1, v_1^{(m)}\}$, $m = 1, 2 \ldots$, and $\max\{v_1, \bar{v}_1\}$ are supermodular, we will have non-empty intersections $\mathcal{C}(v_1) \cap \mathcal{C}(v_1^{(m)}) \neq \varnothing$ and $\mathcal{C}(v_1) \cap \mathcal{C}(\bar{v}_1) \neq \varnothing$ respectively. As it is pointed out in [18], neither the minimum nor the maximum of two submodular set-functions is in general submodular. However, the following result is useful:

Proposition 2 ([18])**.** *Let v and w be real-valued submodular set-functions on 2^N such that $v - w$ is either monotone increasing or decreasing. Then $\min\{v, w\}$ is also submodular.*

The case when the modified cores $\mathcal{C}(v_1^{(m)})$, $m = 1, 2, \ldots$, and the limiting core $\mathcal{C}(\bar{v}_1)$ are the subsets of $\mathcal{C}(v_1)$ is even more desirable. It ensures that players can realize an imputation from the original core after the transformation(s) of the characteristic function. We would like to establish the conditions providing a nested structure for the cores. Obviously, the inclusions $\mathcal{C}(v_t) \subseteq \mathcal{C}(v_t^{(m)})$, $m = 1, 2, \ldots$, and $\mathcal{C}(v_t) \subseteq \mathcal{C}(\bar{v}_t)$ with $t \in \mathcal{T}$ hold if and only if $v_t(S) \geqslant v_t^{(m)}(S)$ and $v_t(S) \geqslant \bar{v}_t(S)$ for every coalition $S \subset N$. Recall that in the subgame which starts at the terminal stage, the original, the modified, and the limiting cores coincide. Similarly, $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(m)})$ and $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ if and only if $v_t(S) \leqslant v_t^{(m)}(S)$ and $v_t(S) \leqslant \bar{v}_t(S)$ for every $S \subset N$. As the above inequalities require the comparison of the original and the modified characteristic functions, we would prefer to establish relationship that require only the definition of the original characteristic function. The following proposition addresses this issue.

Proposition 3. *Let $v_t(N)$ be non-increasing in t and positive. It holds that*

1. *If $\frac{v_1(S)}{v_1(N)} \leqslant \cdots \leqslant \frac{v_T(S)}{v_T(N)}$ for any coalition $S \subseteq N$, then $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(1)}) \supseteq \mathcal{C}(v_t^{(2)}) \supseteq \cdots \supseteq \mathcal{C}(\bar{v}_t)$ for every game stage $t \in \mathcal{T}$.*

2. *If* $\frac{v_1(S)}{v_1(N)} \geqslant \cdots \geqslant \frac{v_T(S)}{v_T(N)}$ *for any coalition* $S \subseteq N$*, then* $\mathcal{C}(v_t) \subseteq \mathcal{C}(v_t^{(1)}) \subseteq \mathcal{C}(v_t^{(2)}) \subseteq \cdots \subseteq \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.
3. *If* $\frac{v_1(S)}{v_1(N)} = \cdots = \frac{v_T(S)}{v_T(N)}$ *for any coalition* $S \subseteq N$*, then* $\mathcal{C}(v_t) = \mathcal{C}(v_t^{(1)}) = \mathcal{C}(v_t^{(2)}) = \cdots = \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.

Proof. Prove the first claim. We suppose that $\frac{v_1(S)}{v_1(N)} \leqslant \frac{v_2(S)}{v_2(N)} \leqslant \cdots \leqslant \frac{v_T(S)}{v_T(N)}$ for any coalition $S \subseteq N$. Then for any $t \in \mathcal{T}$ and S it holds that the modified characteristic function

$$\begin{aligned} v_t^{(1)}(S) &= \sum_{\tau=t}^{T-1} \frac{v_\tau(S)}{v_\tau(N)}[v_\tau(N) - v_{\tau+1}(N)] + v_T(S) \\ &\geqslant \frac{v_t(S)}{v_t(N)} \left(\sum_{\tau=t}^{T-1} [v_\tau(N) - v_{\tau+1}(N)] + v_T(N) \right) = v_t(S). \end{aligned} \tag{8}$$

For the modified characteristic function, we next prove that $\frac{v_1^{(1)}(S)}{v_1^{(1)}(N)} \leqslant \frac{v_2^{(1)}(S)}{v_2^{(1)}(N)} \leqslant \cdots \leqslant \frac{v_T^{(1)}(S)}{v_T^{(1)}(N)}$ for every coalition $S \subseteq N$. Given a coalition S and a game stage $t \in \mathcal{T} \setminus \{T\}$, we obtain

$$\begin{aligned} \frac{v_{t+1}^{(1)}(S)}{v_{t+1}^{(1)}(N)} - \frac{v_t^{(1)}(S)}{v_t^{(1)}(N)} &= \frac{v_{t+1}^{(1)}(S)}{v_{t+1}(N)} - \frac{v_{t+1}^{(1)}(S) + \frac{v_t(S)}{v_t(N)}(v_t(N) - v_{t+1}(N))}{v_t(N)} \\ &= \frac{v_t(N)v_{t+1}^{(1)}(S) - v_{t+1}(N)\left(v_{t+1}^{(1)}(S) + \frac{v_t(S)}{v_t(N)}(v_t(N) - v_{t+1}(N))\right)}{v_t(N)v_{t+1}(N)} \\ &= \frac{(v_t(N) - v_{t+1}(N))\left(v_{t+1}^{(1)}(S) - v_{t+1}(N)\frac{v_t(S)}{v_t(N)}\right)}{v_t(N)v_{t+1}(N)} \\ &\geqslant \frac{(v_t(N) - v_{t+1}(N))}{v_t(N)v_{t+1}(N)}\left(v_{t+1}(S) - v_{t+1}(N)\frac{v_t(S)}{v_t(N)}\right) \\ &= \frac{(v_t(N) - v_{t+1}(N))}{v_t(N)}\left(\frac{v_{t+1}(S)}{v_{t+1}(N)} - \frac{v_t(S)}{v_t(N)}\right) \geqslant 0. \end{aligned}$$

The latter inequality holds true because $\frac{v_t(S)}{v_t(N)} \leqslant \frac{v_{t+1}(S)}{v_{t+1}(N)}$ for any stage $t \in \mathcal{T} \setminus \{T\}$ and $v_t(N)$ is non-increasing in t and positive. Therefore, $\frac{v_1^{(1)}(S)}{v_1^{(1)}(N)} \leqslant \frac{v_2^{(1)}(S)}{v_2^{(1)}(N)} \leqslant \cdots \leqslant \frac{v_T^{(1)}(S)}{v_T^{(1)}(N)}$ for every coalition $S \subseteq N$. Similar to (8), we conclude with $v_t^{(2)}(S) \geqslant v_t^{(1)}(S)$ for every S and $t \in \mathcal{T}$.

By induction, we get the following relation $v_t(S) \leqslant v_t^{(1)}(S) \leqslant \cdots \leqslant v_t^{(m)}(S) \leqslant v_t^{(m+1)}(S) \leqslant \ldots$ for all $S \subseteq N$ and $t \in \mathcal{T}$. It immediately implies that $\mathcal{C}(v_t^{(m)}) \supseteq \mathcal{C}(v_t^{(m+1)})$ for $m = 0, 1, \ldots$ with understanding $v_t^{(0)}(S) = v_t(S)$. Since $v_t(N)$ is positive and non-increasing in t, then by Proposition 1, the limiting characteristic function exists. Thus, $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(1)}) \supseteq \mathcal{C}(v_t^{(2)}) \supseteq \cdots \supseteq \mathcal{C}(\bar{v}_t)$ for $t \in \mathcal{T}$.

The second claim is proved in a similar way with the third one being a special case. □

We note that the conditions in the above proposition require the monotonicity of the *relative* worth of all coalitions along the cooperative trajectory. This proposition can be extended for the case when $v_t(N)$ is non-decreasing in t and negative. We formulate additional instances in the next corollary. As we already showed in Proposition 1, the case when $v_t(N)$ changes its sign in t does not lead to the convergence of the iterative process and, as a result, to the existence of the limiting core.

Corollary 1. *Let* $v_t(N)$ *be non-decreasing in t and negative. It holds that*

1. *If* $\frac{v_1(S)}{v_1(N)} \leqslant \cdots \leqslant \frac{v_T(S)}{v_T(N)}$ *for any coalition* $S \subseteq N$*, then* $\mathcal{C}(v_t) \subseteq \mathcal{C}(v_t^{(1)}) \subseteq \mathcal{C}(v_t^{(2)}) \subseteq \cdots \subseteq \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.
2. *If* $\frac{v_1(S)}{v_1(N)} \geqslant \cdots \geqslant \frac{v_T(S)}{v_T(N)}$ *for any coalition* $S \subseteq N$*, then* $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(1)}) \supseteq \mathcal{C}(v_t^{(2)}) \supseteq \cdots \supseteq \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.
3. *If* $\frac{v_1(S)}{v_1(N)} = \cdots = \frac{v_T(S)}{v_T(N)}$ *for any coalition* $S \subseteq N$*, then* $\mathcal{C}(v_t) = \mathcal{C}(v_t^{(1)}) = \mathcal{C}(v_t^{(2)}) = \cdots = \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.

3.2. Linear Symmetric Games

As a special class of cooperative dynamic games, we consider a class of linear symmetric games with the characteristic function depending only upon the number of players in a coalition, that is, $v_t(S) = A_t|S| + B_t$ for all coalitions $S \subseteq N$ and game stages $t \in \mathcal{T}$. Following [19], cooperative game (N, v_t) has a non-empty core $\mathcal{C}(v_t)$ if and only if $\frac{v_t(S)}{|S|} \leqslant \frac{v_t(N)}{|N|}$ for any non-empty coalition $S \subseteq N$. For the characteristic function under consideration, the latter inequality transforms into $\frac{B_t}{|S|} \leqslant \frac{B_t}{|N|}$, $S \subseteq N$, $t \in \mathcal{T}$, which holds true for non-positive B_t. Since players consider the core to be the solution to the cooperative dynamic game, the solution must prescribe a non-empty subset of the imputation set. For this reason, we introduce the assumption $B_t \leqslant 0$ for each $t \in \mathcal{T}$. In practical situations, it is reasonable to assume that the worth of grand coalition $v_1(N)$ is positive, i.e., players generate a positive gain in the game under cooperation. At the same time in view of Proposition 1, the iterative process (4) converges when the grand coalition's payoff does not change its sign along the cooperative trajectory. This implies that $v_t(N) \geqslant 0$ and, therefore, $A_t \geqslant 0$ as well for all $t \in \mathcal{T}$. The next results summarize the relationship between the cores for the class of games under consideration. We let $s = |S|$.

Corollary 2. *Let* $v_t(S) = A_t s + B_t$, $A_t \geqslant 0$, $B_t \leqslant 0$, $t \in \mathcal{T}$. *If the limiting characteristic function* $\bar{v}_t$ *exists, then* $\mathcal{C}(v_t) \bigcap \mathcal{C}(\bar{v}_t) \neq \varnothing$ *for every game stage* $t \in \mathcal{T}$.

Proof. By the definition of the limiting characteristic function (5), we note that v_t and $\bar{v}_t$ are monotone. Taking into account their difference, it holds that $v_t - \bar{v}_t$ is monotone as well. Using Proposition 2, we prove the result. □

Corollary 3. *Let* $v_t(N) = A_t n + B_t$ *be non-increasing in* t *and positive with* $A_t \geqslant 0$, $B_t < 0$ *for all* $t \in \mathcal{T}$. *It holds that*

1. *If* $\frac{A_1}{B_1} \geqslant \cdots \geqslant \frac{A_T}{B_T}$*, then* $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(1)}) \supseteq \mathcal{C}(v_t^{(2)}) \supseteq \cdots \supseteq \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.
2. *If* $\frac{A_1}{B_1} \leqslant \cdots \leqslant \frac{A_T}{B_T}$*, then* $\mathcal{C}(v_t) \subseteq \mathcal{C}(v_t^{(1)}) \subseteq \mathcal{C}(v_t^{(2)}) \subseteq \cdots \subseteq \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.
3. *If* $\frac{A_1}{B_1} = \cdots = \frac{A_T}{B_T}$*, then* $\mathcal{C}(v_t) = \mathcal{C}(v_t^{(1)}) = \mathcal{C}(v_t^{(2)}) = \cdots = \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.

Proof. Prove the first claim. We suppose that $\frac{A_t}{B_t} \geqslant \frac{A_{t+1}}{B_{t+1}}$ for any $t \in \mathcal{T} \setminus \{T\}$. Then the following sequence of equivalent relations holds:

$$\begin{aligned}
\frac{A_t}{B_t} \geqslant \frac{A_{t+1}}{B_{t+1}} &\Leftrightarrow A_t B_{t+1} \geqslant A_{t+1} B_t \\
&\Leftrightarrow A_t B_{t+1}(s-n) \leqslant A_{t+1} B_t (s-n) \\
&\Leftrightarrow A_t B_{t+1} s + A_{t+1} B_t n \leqslant A_t B_{t+1} n + A_{t+1} B_t s \\
&\Leftrightarrow A_t A_{t+1} sn + A_t B_{t+1} s + A_{t+1} B_t n + B_t B_{t+1} \leqslant A_t A_{t+1} sn + A_t B_{t+1} n + A_{t+1} B_t s + B_t B_{t+1} \\
&\Leftrightarrow (A_t s + B_t)(A_{t+1} n + B_{t+1}) \leqslant (A_{t+1} s + B_{t+1})(A_t n + B_t) \\
&\Leftrightarrow v_t(S) v_{t+1}(N) \leqslant v_{t+1}(S) v_t(N).
\end{aligned}$$

Since $v_t(N)$ is positive for all $t \in \mathcal{T} \setminus \{T\}$, the latter inequality is equivalent to $\frac{v_t(S)}{v_t(N)} \leqslant \frac{v_{t+1}(S)}{v_{t+1}(N)}$, $t \in \mathcal{T} \setminus \{T\}$. By Proposition 3, we get the inclusions $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(1)}) \supseteq \mathcal{C}(v_t^{(2)}) \supseteq \cdots \supseteq \mathcal{C}(\bar{v}_t)$, $t \in \mathcal{T}$.

The second claim is proved in a similar way with the third one being a special case. □

To establish the relationship between the core $\mathcal{C}(v_t)$ and the limiting core $\mathcal{C}(\bar{v}_t)$, we can relax the monotonicity of the ratio A_t/B_t.

Corollary 4. *Let $v_t(N) = A_t n + B_t$ be positive with $A_t \geqslant 0$, $B_t < 0$ for all $t \in \mathcal{T}$. If the limiting characteristic function $\bar{v}_t$ exists, then for any game stage it holds that*

1. *If $\frac{A_t}{B_t} \geqslant \frac{A_T}{B_T}$, then $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$.*
2. *If $\frac{A_t}{B_t} \leqslant \frac{A_T}{B_T}$, then $\mathcal{C}(v_t) \subseteq \mathcal{C}(\bar{v}_t)$.*
3. *If $\frac{A_t}{B_t} = \frac{A_T}{B_T}$, then $\mathcal{C}(v_t) = \mathcal{C}(\bar{v}_t)$.*

Proof. We prove the first statement. As with the proof of Corollary 3, it is easy to verify that $\frac{v_t(S)}{v_t(N)} \leqslant \frac{v_T(S)}{v_T(N)}$, $t \in \mathcal{T}$, $S \subseteq N$, and provided that the limiting characteristic function $\bar{v}_t$ exists, we obtain the inclusion $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$.

The second and the third statements are proved similarly. □

Remark 2. *It is worth noting that Corollaries 3 and 4 can be extended to the non-decreasing in game stage and negative values of the grand coalition's payoffs along the cooperative trajectory (recall that by Proposition 1 for convergence, these values must be of same sign). If it is the case, then relaxing the assumption $A_t \geqslant 0$ for all $t \in \mathcal{T}$, one can easily show that the core inclusions become opposite.*

3.3. Two-Stage Network Games

In this section, we establish the relationship between the cores for a class of cooperative two-stage network games studied in [20,21] for a general model and in [22] for their applications in public goods provision and market competition. We will define the characteristic functions in the two-stage cooperative network game according to transformation rule (2) when implementing the cooperative agreement. Taking into account that players receive their payoffs only at the second stage of the game, that is $v_1(N) = v_2(N)$, then it holds that $\hat{v}_1(S) = \hat{v}_2(S) = v_2(S)$ for any coalition $S \subseteq N$. Next, the transformation matrix Θ takes the form

$$\Theta = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Although the players' payoffs at the first game stage are zero, the iterative process (4) converges. From (5), we conclude that $\bar{v}_1(S) = \bar{v}_2(S) = v_2(S)$ for any coalition $S \subseteq N$ as well. Since characteristic functions $\hat{v}_1$, $\bar{v}_1$ in the cooperative two-stage network game and characteristic functions $\hat{v}_2$, $\bar{v}_2$ in its cooperative one-stage subgame coincide, we get the equality $\mathcal{C}(\hat{v}_1) = \mathcal{C}(\hat{v}_2) = \mathcal{C}(\bar{v}_1) = \mathcal{C}(\bar{v}_2) = \mathcal{C}(v_2)$ for the cores.

3.4. A Class of Linear-State Games

Now we examine a class of linear-state games. For the model under consideration, we take one studied in [23] with the purpose to establish the relationship between the cores in this class of games. For convenience, we change the set of game stages $\mathcal{T} = \{0, 1, \ldots, T\}$ and start indexing stages from zero. In the model, the state dynamics is governed by the state equation

$$x(t+1) = b_0 x(t) + b_1 \sum_{i \in N} u_i(t) \in X, \quad t \in \mathcal{T} \setminus \{T\},$$

with the initial condition $x(0) = x_0 \in X$. Here $u_i(t) \in \operatorname{comp} U_i \subset \mathbb{R}_+$ for each player $i \in N$ and $X = \mathbb{R}_+$. The player i's stage payoffs are defined by the functions

$$h_{it}(x(t), u(t)) = a_{i0}u_i(t) + \frac{a_{i1}}{2}u_i^2(t) + a_{i2}x(t), \quad t \in \mathcal{T} \setminus \{T\},$$
$$h_{iT}(x(T)) = a_{i2}x(T).$$

Additionally, we assume that $a_{i1} < 0$, and $a_{i2} \neq 0$ are of same sign for each $i \in N$, and $b_0, b_1 \neq 0$.

When the game is played cooperatively, players jointly maximize the sum $\sum_{i \in N} J_i(x_0, u) = \sum_{i \in N}(\sum_{t=0}^{T-1} h_{it}(x(t), u(t)) + h_{iT}(x(T)))$.

First, we introduce the following functions of stage number $t \in \mathcal{T}$:

$$\chi_1(t) = \sum_{\tau=1}^{T-t} b_0^\tau, \qquad \chi_2(t) = \sum_{\tau=1}^{T-t}\sum_{m=0}^{\tau-1} b_0^{\tau-m}, \qquad \chi_3(t) = \sum_{\tau=0}^{T-t-1}\left(\sum_{m=1}^{T-t-\tau} b_0^m\right)^2, \qquad t \in \mathcal{T} \setminus \{T\},$$

with $\chi_1(T) = \chi_2(T) = \chi_3(T) = 0$. In [23], it was established that the cooperative trajectory is given by

$$x^*(t) = \begin{cases} x_0, & t = 0, \\ b_0^t x_0 - \dfrac{b_1}{b_0}\displaystyle\sum_{i \in N}\frac{1}{a_{i1}}\sum_{\tau=0}^{t-1} b_0^{t-\tau}\left(a_{i0} + \frac{b_1}{b_0}\chi_1(\tau)\sum_{j \in N} a_{j2}\right), & t \in \mathcal{T} \setminus \{0\}, \end{cases}$$

and the characteristic functions in the game and its cooperative proper subgames along this trajectory equal

$$\begin{aligned} v_t(N) = &\sum_{i \in N}\left(-\frac{a_{i0}^2}{2a_{i1}}(T-t) + a_{i2}x^*(t)(1+\chi_1(t)) - \frac{a_{i2}b_1}{b_0}\chi_2(t)\sum_{j \in N}\frac{a_{j0}}{a_{j1}}\right) \\ &- \left(\frac{b_1}{b_0}\sum_{j \in N} a_{j2}\right)^2 \chi_3(t)\sum_{j \in N}\frac{1}{2a_{j1}}, \\ v_t(S) = &\sum_{i \in S}\left(-\frac{a_{i0}^2}{2a_{i1}}(T-t) + a_{i2}x^*(t)(1+\chi_1(t)) - \frac{a_{i2}b_1}{b_0}\chi_2(t)\sum_{j \in N}\frac{a_{j0}}{a_{j1}}\right) \\ &- \frac{b_1^2}{b_0^2}\left(\left(\sum_{j \in S} a_{j2}\right)^2\sum_{j \in S}\frac{1}{2a_{j1}} + \sum_{j \in S} a_{j2}\sum_{j \in N \setminus S}\frac{a_{j2}}{a_{j1}}\right)\chi_3(t), \quad S \subset N, \end{aligned}$$

while for $t = T$ and any $S \subseteq N$, we have $v_T(S) = \sum_{i \in S} a_{i2}x^*(T)$. Please note that characteristic function v_T is additive, therefore, the core $\mathcal{C}(v_T)$ is non-empty and consists of a single imputation $\xi(v_T) = (a_{12}x^*(T), \ldots, a_{n2}x^*(T))$. Moreover, if there exists the core $\mathcal{C}(\bar{v}_T)$, it consists of the same imputation as $\mathcal{C}(v_T) = \mathcal{C}(\bar{v}_T)$.

Before studying the relationship between the cores, we consider the following example. It demonstrates that for the class of games under consideration (i) the modified core and the limiting core can be subsets of the original one, (ii) they can share no common imputation with the original core, and (iii) the original core can intersect with the modified core, but does not intersect with the limiting one.

Example 1. *We consider a 3-person game with $T = 3$ and perform simulation with the following parameters: $x_0 = 15$, $b_0 = 1$, $b_1 = -1$, $a_{11} = a_{21} = a_{31} = -2$, $a_{12} = a_{22} = a_{32} = 0.05$ whereas parameters a_{10}, a_{20}, a_{30} vary. Figure 1 demonstrates the situation when the original core $\mathcal{C}(v_0)$ intersects with the modified core $\mathcal{C}(v_0^{(1)})$, but does not intersect with the limiting core $\mathcal{C}(\bar{v}_0)$. Next, the instance when the modified core and the limiting core are subsets of the core $\mathcal{C}(v_0)$ is depicted in Figure 2. Finally, in Figure 3, the original core intersects neither with the modified core nor with the limiting one.*

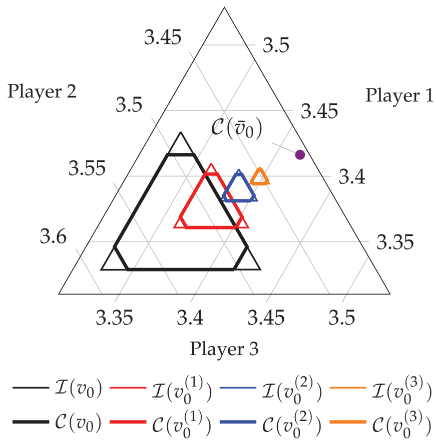

Figure 1. A non-empty intersection of the original core and the modified core ($a_{10} = 1$, $a_{20} = 1.1$, $a_{30} = 1$).

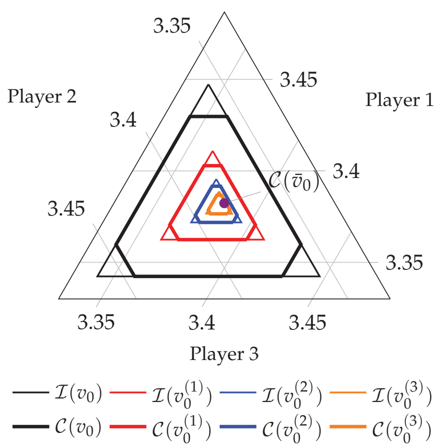

Figure 2. A nested cores pattern ($a_{10} = 1$, $a_{20} = 1.01$, $a_{30} = 1$).

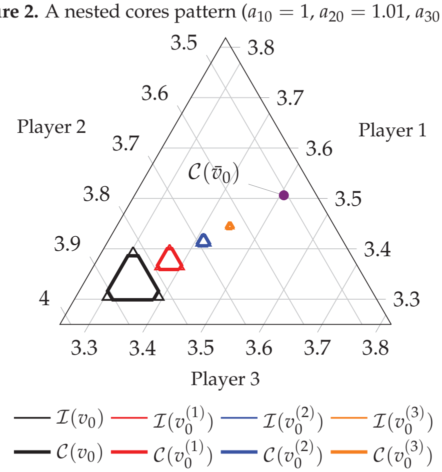

Figure 3. All cores are pairwise disjoint ($a_{10} = 1$, $a_{20} = 1.3$, $a_{30} = 1$).

The next proposition provides conditions under which the limiting core is a subset of the original core.

Proposition 4. *Let the limiting characteristic function $\bar{v}_0$ exist. If the inequality $(\sum_{j\in S}\frac{a_{j0}^2}{a_{j1}})/(\sum_{j\in N}\frac{a_{j0}^2}{a_{j1}}) \leqslant (\sum_{j\in S} a_{j2})/(\sum_{j\in N} a_{j2})$ holds for every coalition $S \subseteq N$ and $\frac{a_{i2}}{a_{j2}} + \frac{a_{j1}}{a_{i1}} \geqslant 1$ for any $i, j \in N$, then $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ for every $t \in \mathcal{T}$. Moreover, when players are symmetric, $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ for every $t \in \mathcal{T}$.*

Proof. First, we prove the statement assuming that players are asymmetric. Having the required conditions satisfied, for any $a_{i2}, i \in N$, of same sign we have:

$$\begin{aligned}
\bar{v}_t(S) - v_t(S) &= \frac{v_t(N)}{v_T(N)} v_T(S) - v_t(S) \\
&\geqslant -\frac{b_1^2}{b_0^2}\chi_3(t)\sum_{i\in S} a_{i2}\left(\sum_{i\in N} a_{i2}\sum_{j\in N}\frac{1}{2a_{j1}} - \sum_{i\in S} a_{i2}\sum_{j\in S}\frac{1}{2a_{j1}} - \sum_{j\in N\setminus S}\frac{a_{j2}}{a_{j1}}\right) \\
&= -\frac{b_1^2}{2b_0^2}\chi_3(t)\sum_{i\in S} a_{i2}\left(\sum_{i\in S} a_{i2}\sum_{j\in N\setminus S}\frac{1}{a_{j1}} + \sum_{i\in N\setminus S} a_{i2}\sum_{j\in S}\frac{1}{a_{j1}} + \sum_{i\in N\setminus S} a_{i2}\sum_{j\in N\setminus S}\frac{1}{a_{j1}} - 2\sum_{j\in N\setminus S}\frac{a_{j2}}{a_{j1}}\right) \\
&\geqslant -\frac{b_1^2}{2b_0^2}\chi_3(t)\sum_{i\in S} a_{i2}\left(\sum_{i\in S} a_{i2}\sum_{j\in N\setminus S}\frac{1}{a_{j1}} + \sum_{i\in N\setminus S} a_{i2}\sum_{j\in S}\frac{1}{a_{j1}} - \sum_{j\in N\setminus S}\frac{a_{j2}}{a_{j1}}\right).
\end{aligned}$$

When a_{i2} is positive for all $i \in N$, the following sequence of relations holds true:

$$\begin{aligned}
\frac{a_{i2}}{a_{j2}} + \frac{a_{j1}}{a_{i1}} \geq 1,\ \forall i, j \in N &\Rightarrow \sum_{i\in S}\left(\frac{a_{i2}}{a_{j2}} + \frac{a_{j1}}{a_{i1}}\right) \geqslant 1,\ \forall S \subseteq N, \forall j \in N \\
&\Leftrightarrow \frac{1}{a_{j2}}\sum_{i\in S} a_{i2} + a_{j1}\sum_{i\in S}\frac{1}{a_{i1}} \geqslant 1,\ \forall S \subseteq N, \forall j \in N \\
&\Leftrightarrow \frac{1}{a_{j1}}\sum_{i\in S} a_{i2} + a_{j2}\sum_{i\in S}\frac{1}{a_{i1}} \leqslant \frac{a_{j2}}{a_{j1}},\ \forall S \subseteq N, \forall j \in N \\
&\Rightarrow \sum_{j\in N\setminus S}\left(\frac{1}{a_{j1}}\sum_{i\in S} a_{i2} + a_{j2}\sum_{i\in S}\frac{1}{a_{i1}} - \frac{a_{j2}}{a_{j1}}\right) \leqslant 0,\ \forall S \subseteq N \\
&\Leftrightarrow \left(\sum_{i\in S} a_{i2}\sum_{j\in N\setminus S}\frac{1}{a_{j1}} + \sum_{i\in N\setminus S} a_{i2}\sum_{j\in S}\frac{1}{a_{j1}} - \sum_{j\in N\setminus S}\frac{a_{j2}}{a_{j1}}\right) \leqslant 0,\ \forall S \subseteq N.
\end{aligned}$$

Thus, $\bar{v}_t(S) - v_t(S) \geqslant 0$ and $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ for $t \in \mathcal{T}$.

When a_{i2} is negative for all $i \in N$, we obtain:

$$\left(\sum_{i\in S} a_{i2}\sum_{j\in N\setminus S}\frac{1}{a_{j1}} + \sum_{i\in N\setminus S} a_{i2}\sum_{j\in S}\frac{1}{a_{j1}} - \sum_{j\in N\setminus S}\frac{a_{j2}}{a_{j1}}\right) \geqslant 0,\ \forall S \subseteq N.$$

Then $\bar{v}_t(S) - v_t(S) \geqslant 0$ and $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ for $t \in \mathcal{T}$. Therefore, when $a_{i2} \neq 0, i \in N$, are of same sign, $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ for $t \in \mathcal{T}$.

Now suppose that players are symmetric. We note that in this case, the required conditions from the first part are always met. Therefore, the inclusion $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ holds as well for $t \in \mathcal{T}$. □

4. Conclusions

In this paper, we studied the relationship between the core of the original game and the cores of modified games determined by a transformation rule because these cores may not intersect for a dynamic game of a general structure. First, we extended the conditions known in the literature which lead to the convergence of an iterative process based on this transformation rule. Second, we found conditions under which one core is a subset of the other: these conditions require the monotonicity of the relative worth of coalitions along the cooperative trajectory. Finally, for several classes of dynamic games, we characterized the relationship between the cores.

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Article

On the Effect of the Absorption Coefficient in a Differential Game of Pollution Control

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Abstract: We consider various approaches for a characteristic function construction on the example of an n players differential game of pollution control with a prescribed duration. We explore the effect of the presence of an absorption coefficient in the game on characteristic functions. As an illustration, we consider a game in which the parameters are calculated based on the real ecological situation of the Irkutsk region. For this game, we compute a number of characteristic functions and compare their properties.

Keywords: differential games; prescribed duration; characteristic function; environmental resource management; pollution control

1. Introduction

Differential games are used for describing continuous processes of decision making in conflict situations that happen in industry, ecology, biology, political science, and so on. Models of differential games are often utilized for solving problems in the field of environmental protection policy and the optimal exploitation of natural resources [1–7].

To solve the problems of environmental management, it is effective to use games with negative externalities [4–7]. In that class of games, the increasing of the controls of some players, which are the volumes of environmental pollution over time, leads to the decreasing value of the payoff functions for others.

In this paper, we consider a cooperative differential game of pollution control with negative externalities modeling the behavior of several enterprises. They have an agreement to limit environmental pollution. This limitation has a negative effect on the total profit of each enterprise. Moreover, the high level of environmental pollution leads to profit loss associated with the higher environmental costs and taxes. We focus our attention on two cases when there is an absorption coefficient and when there is not.

In the theory of cooperative differential games, the concept of a characteristic function is one of the basic ones. The characteristic function shows the worth of a coalition and affects the formation of coalitions [8,9]. Therefore, players or coalitions have the motivation to cooperate with each other if the characteristic function of the joint coalition is greater than the sum of the original characteristic functions. Additionally, the significance of each player in the coalition can be determined by its marginal contribution to the characteristic function. For example, this approach is used in constructing the Shapley value and Banzhaf power index [10–12].

We present different techniques for characteristic function construction [13–15]. For the above-mentioned cooperative differential game, we compute a number of characteristic functions and compare their properties. Further, we analyze the effect of the absorption coefficient representing the natural environment purification on player's payoffs.

As an illustration, we explore a game of pollution control based on the acute ecological situation of the Irkutsk region. We use a cooperative differential game of three players, which are the largest enterprises of Bratsk [16]. Furthermore, we add the absorption coefficient to this model and focus on the effect this coefficient has on the payoffs.

2. Cooperative Differential Game in the form of the Characteristic Function

2.1. Differential Game in Normal Form

Let $N = \{1, 2, ..., n\}$ be a set of players participating in a classical cooperative differential game $\Gamma(x_0, t_0, T)$ with a prescribed duration [17]. The game starts from the initial state x_0 at time t_0 and evolves over the interval $t \in [t_0, T]$.

The dynamics of the game are described by the system of differential equations:

$$\dot{x}(t) = f(x, u_1, ..., u_n), \quad x(t_0) = x_0, \tag{1}$$

where $x \in R^n,\ u_i \in U_i \subset \text{comp}R^k$.

We assume that all standard restrictions [18] on the parameters, controls, and trajectory function are satisfied.

The payoff function of the i^{th} player is:

$$K_i(t_0, x_0, u) = \int\limits_{t_0}^{T} h_i(x(\tau), u(\tau))d\tau, \quad i = \overline{1, n},$$

where $h_i(x, u)$ are continuous functions and $x(t)$ is a solution of the Cauchy problem for System (1) under controls $u(t) = (u_1(t), \dots, u_n(t))$.

2.2. Different Methods of Characteristic Function Construction

To define the cooperative game, we have to construct a characteristic function $V(S)$ for every coalition $S \in N$ in the game.

A characteristic function is a mapping from the set of all possible coalitions:

$$V(S) : 2^N \to R, \quad V(\emptyset) = 0.$$

The value $V(S)$ is typically interpreted as the worth or the power of the coalition S. One of the most important properties of the characteristic function is superadditivity:

$$V(S_1 \cup S_2) \geq V(S_1) + V(S_2), \qquad \forall S_1, S_2 \subseteq N,\ S_1 \cap S_2 = \emptyset. \tag{2}$$

Superadditive characteristic functions provide some useful advantages in solving various problems in the field of cooperative game theory in static and dynamic settings. More information about this can be found in [19].

Currently, there are different approaches to the calculation of the characteristic function (see [7,19–22]). A systematic overview of different characteristic functions and their properties was presented in [23]. This paper provides an analysis of α-, δ-, ζ-, and η-characteristic functions.

2.2.1. α-Characteristic Function

A classical approach to the construction the characteristic function is called the α-characteristic function. It was introduced in [20] and was the only way to construct a cooperative game for a long time. The main idea of this method is using the lower value of the zero-sum game $\Gamma_{S,N\setminus S}$ between the coalition S as the first player and coalition $N \setminus S$ as the second player.

$$V^{\alpha}(S) = \begin{cases} 0, & S = \{\emptyset\}, \\ \max\limits_{\substack{u_i,\\ i\in S}} \min\limits_{\substack{u_j,\\ j\in N\setminus S}} \sum\limits_{i\in S} K_i(t_0, x_0, u), & S \subseteq N. \end{cases} \tag{3}$$

We assume that the maximum and minimum is achieved on (3). The value $V^{\alpha}(S)$ is interpreted as the maximum value that coalition S can get when $N \setminus S$ acts against S.

It was proved in [24] that the α-characteristic function is superadditive.

This approach of defining the characteristic function has some issues. It is necessary to solve $2^n - 1$ complicated optimization problems. It is hard to find (3) in analytical form in differential games due to computational problems. Finally, from an economic standpoint, it is unlikely that players of $N \setminus S$ form an anti-coalition [7].

2.2.2. δ-Characteristic Function

The technique of the construction the δ-characteristic function was proposed in [7]. The process of the calculation of this function consists of two steps. Firstly, one has to calculate the Nash equilibrium strategies for all players. Secondly, players from S maximize their total payoff $\sum\limits_{i\in S} K_i$ while players from $N \setminus S$ use strategies from the Nash equilibrium.

$$V^{\delta}(S) = \begin{cases} 0, & S = \{\emptyset\}, \\ \max\limits_{\substack{u_i,\ i\in S\\ u_j = u_j^{NE},\ j\in N\setminus S}} \sum\limits_{i\in S} K_i(t_0, x_0, u_S, u^{NE}_{N\setminus S}), & S \subseteq N. \end{cases} \tag{4}$$

This form of the characteristic function requires fewer computational operations compared the with α-characteristic function. Additionally, the previously constructed Nash equilibrium simplifies the computation of $V^{\delta}(S)$. Moreover, (4) has a practical economic interpretation. Players not from the coalition S do not tend to form anti-coalition $N \setminus S$ in real models (see [25–27]).

Nevertheless there are some problems of this approach. In general, the δ-characteristic function is a non-superadditive function (see examples in [28]). Besides, one has to consider the problem of the existence and uniqueness of the Nash equilibrium solution.

2.2.3. ζ-Characteristic Function

The ζ-characteristic function was introduced in [19]. The first step of calculation of this characteristic function for coalition S is finding optimal controls maximizing the total payoff of the players. In the second step, players from coalition S use the cooperative optimal strategies, while the left-out players from $N \setminus S$ use the strategies minimizing the total payoff of the players from S.

$$V^{\zeta}(S) = \begin{cases} 0, & S = \{\varnothing\}, \\ \min\limits_{\substack{u_j \in U_j,\, j \in N\setminus S, \\ u_i = u_i^*,\, i \in S}} \sum\limits_{i \in S} K_i(t_0, x_0, u_S^*, u_{N\setminus S}), & S \subseteq N. \end{cases} \tag{5}$$

We assume that the maximum and minimum are attained in (5).

The constructed $V^{\zeta}(S)$ is superadditive in general [19]. Additionally, already computed optimal controls are used for the ζ-characteristic function, which simplifies the computation process compared with the α-characteristic function. Besides, these controls exist and could be found for a wide class of games under rather weak constraints. Lastly, the ζ-characteristic function is applicable to games with fixed coalition structures [29].

2.2.4. η-Characteristic Function

The idea of the η-characteristic function was presented in [22]. This characteristic function is based on strategies from the optimal profile u^* and strategies from the Nash equilibrium u^{NE}. We use u_S^* for players from S (as in the ζ-characteristic function) and $u_{N\setminus S}^{NE}$ for players from $N \setminus S$ (as in the δ-characteristic function).

$$V^{\eta}(S) = \begin{cases} 0, & S = \{\varnothing\}, \\ \sum\limits_{i \in S} K_i(t_0, x_0, u_S^*, u_{N\setminus S}^{NE}), & S \subseteq N. \end{cases} \tag{6}$$

This function models the case when players from $N \setminus S$ decide instead of optimal strategies to use strategies from Nash equilibrium u^{NE}.

The construction of the η-characteristic function has some technical advantages. It is much simpler in terms of calculation compared with the α-characteristic function. As mentioned above, optimal controls exist and could be found for a wide class of games. The drawback of this function is the problem of the existence and uniqueness of the Nash equilibrium solution. Furthermore, $V^{\eta}(S)$ is not superadditive in the general case [23].

3. Problem of Optimal Pollution Control

3.1. Problem Statement

We assume there is an enterprise having a production site in its territory. The volume of production is directly proportional to harmful emissions to the atmosphere $u(t) \in [0, b]$, which the enterprise controls.

The dynamics of the total amount of pollution $x(t)$ is described by the following differential equation:

$$\dot{x}(t) = u(t) - \delta x(t), \quad x(t_0) = x_0,$$

where $\delta \geq 0$ is the absorption coefficient, $[\delta] = 1/[t]$ (we use square brackets to denote the dimension of the respective variable). Note that δ does not need to belong to the interval $[0, 1]$ as the value of δ is determined by the dimension of the time unit. For instance, 1 [1/day]=30 [1/month] .

The instantaneous profit of the enterprise is defined as:

$$R(u(t)) = \left(b - \frac{1}{2}u(t)\right) u(t), \quad t \in [t_0, T].$$

There is also an ecotax that is proportional to the amount of pollution. Hence, the net instantaneous payoff is obtained as the difference between the profit and the tax:

$$R(u(t)) - dx(t),$$

where $d > 0$ is the tax coefficient. Thus, the total payoff is:

$$K(t_0, x_0, u) = \int_{t_0}^{T} \left(\left(b - \frac{1}{2} u(t) \right) u(t) - dx(t) \right) dt.$$

It is straightforward to show that the optimal control maximizing the payoff $K(t_0, x_0, u)$ is:

$$u^*(t) = b + d\frac{e^{-\delta(T-t)} - 1}{\delta}. \quad (7)$$

We start by analyzing what values the optimal control can achieve. Before proceeding to the main result, we define:

$$\bar{\delta} = \frac{d}{b} + \frac{1}{T - t_0} W_0 \left(-\frac{d(T-t_0)}{b} e^{-\frac{d(T-t_0)}{b}} \right),$$

where $W_0(z)$ is the principal branch of the Lambert function, defined as the solution to the equation $we^w = z$ [30].

Proposition 1. *The optimal control Function (7) is bounded by b. The optimal control $u^*(t) \geq 0$ for all $t \in [t_0, T]$ if $(T - t_0) \leq \frac{b}{d}$ or $(T - t_0) > \frac{b}{d}$ and $\delta \geq \bar{\delta}$. If $(T - t_0) > \frac{b}{d}$ and $\delta < \bar{\delta}$, then the optimal control function changes sign from minus to plus at the point:*

$$\bar{t} = T + \frac{1}{\delta} \ln \left(1 - \frac{b}{d}\delta \right) \in (t_0, T).$$

The proof is given in Appendix A.

Proposition 1 gives the conditions guaranteeing that the optimal control is defined by (7). Then, the corresponding trajectory is given by:

$$x^*(t) = \frac{e^{-\delta t}}{2\delta^2} \left(e^{-\delta(T-t)} \left(de^{\delta t} + 2(b\delta - d)e^{\delta T} \right) - e^{-\delta(T-t_0)} \left(de^{\delta t_0} + 2(b\delta - d - \delta^2 x_0)e^{\delta T} \right) \right)$$

and the payoff is:

$$K(t_0, x_0, u^*) = x_0 \frac{d(e^{-\delta\Delta} - 1)}{\delta} - \frac{d^2}{4\delta^3} e^{-2\delta\Delta} + \left(\frac{d^2}{\delta^3} - \frac{bd}{\delta^2} \right) e^{-\delta\Delta} + \left(\frac{d^2}{2\delta^2} - \frac{bd}{\delta} + \frac{b^2}{2} \right) \Delta - \frac{3d^2}{4\delta^3} + \frac{bd}{\delta^2}, \quad (8)$$

where $\Delta = T - t_0$.

If $(T - t_0) > \frac{b}{d}$ and $\delta < \bar{\delta}$, the function $u^*(t)$ is defined as:

$$u^*(t) = \begin{cases} 0, & t \in [t_0, \bar{t}], \\ b + d\frac{e^{-\delta(T-t)} - 1}{\delta}, & t \in (\bar{t}, T]. \end{cases}$$

and the corresponding trajectory is:

$$x^*(t) = \begin{cases} e^{-\delta(t-t_0)} x_0, & t \in [t_0, \bar{t}], \\ \frac{e^{-\delta t}}{2\delta^2} \left(e^{-\delta(T-t)} \left(de^{\delta t} + 2(b\delta - d)e^{\delta T} \right) + 2\delta^2 x_0 e^{\delta t_0} + \frac{(b\delta - d)^2}{d} e^{\delta T} \right), & t \in (\bar{t}, T]. \end{cases}$$

The corresponding value of the payoff is:

$$K(t_0, x_0, u^*) = x_0 \frac{d(e^{-\delta\Delta} - 1)}{\delta} - \frac{d^2}{4\delta^3} e^{-2\delta\bar{\Delta}} + \left(\frac{d^2}{\delta^3} - \frac{bd}{\delta^2}\right) e^{-\delta\bar{\Delta}} + \left(\frac{d^2}{2\delta^2} - \frac{bd}{\delta} + \frac{b^2}{2}\right) \bar{\Delta} - \frac{3d^2}{4\delta^3} + \frac{bd}{\delta^2},$$

where $\bar{\Delta} = T - \bar{t}$. Note that the latter case differs from the former one in that the optimal control is equal to zero on the interval $[t_0, \bar{t}]$.

3.2. Influence of the Absorption Coefficient on the Payoff

Proposition 2.

$$\lim_{\delta \to +\infty} K(t_0, x_0, u^*) = \frac{b^2}{2} \Delta.$$

Proposition 2 is proven by applying L'Hospital's rule to (8) three times.

Figures 1–5 demonstrate the dependence of the payoff from the absorption coefficient in special cases when the control has a switch point. Here and later on, we follow the convention that the overall plot is shown on the left, while the right plot shows a zoomed-in part of the graph.

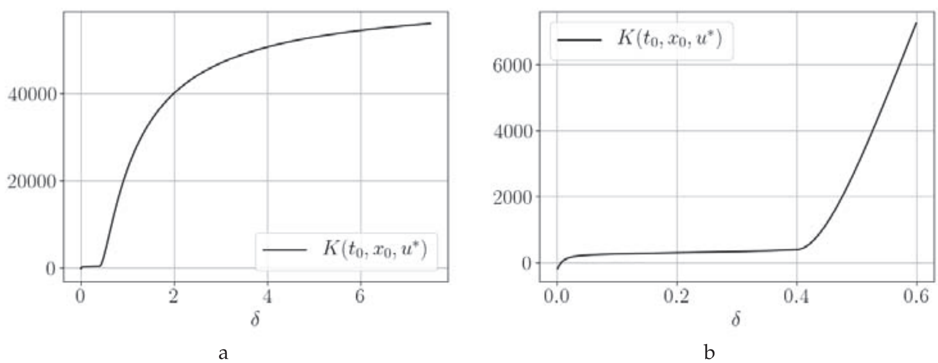

Figure 1. $d = 10$, $b = 25$, $t_0 = 0$, $T = 200$, $x_0 = 0.25$. (**a**) $\delta \in [0, 8]$ and (**b**) $\delta \in [0, 0.6]$.

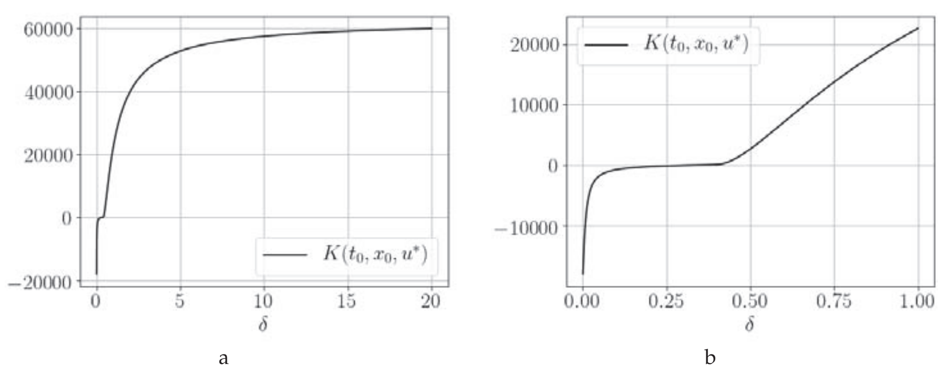

Figure 2. $d = 10$, $b = 25$, $t_0 = 0$, $T = 200$, $x_0 = 10$. (**a**) $\delta \in [0, 20]$ and (**b**) $\delta \in [0, 1]$.

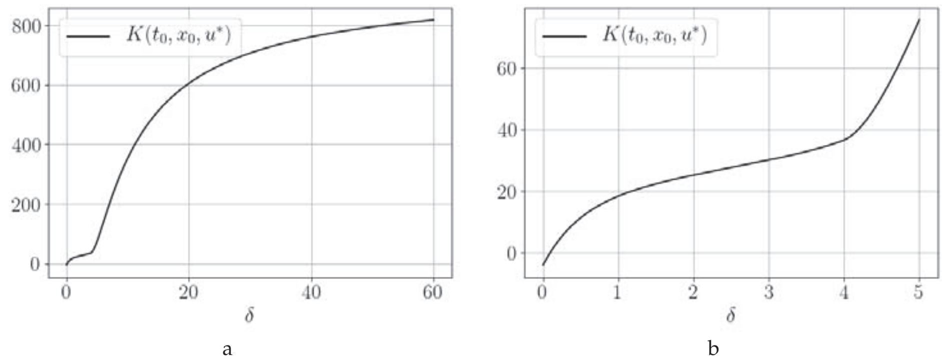

Figure 3. $d = 100$, $b = 25$, $t_0 = 0$, $T = 3$, $x_0 = 0.1$. (**a**) $\delta \in [0, 60]$ and (**b**) $\delta \in [0, 5]$.

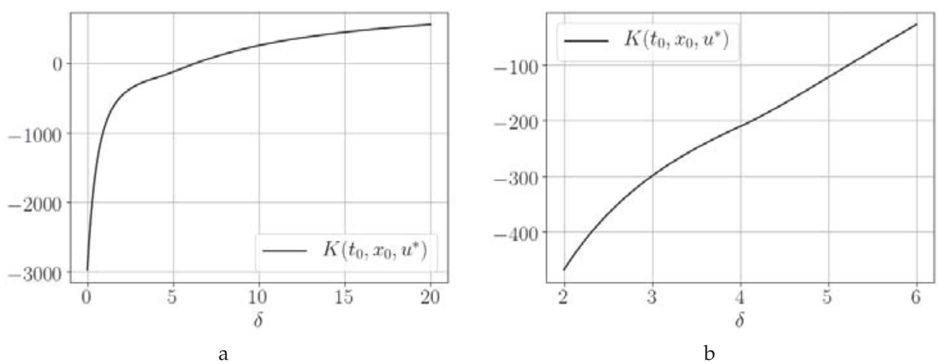

Figure 4. $d = 100$, $b = 25$, $t_0 = 0$, $T = 3$, $x_0 = 10$. (**a**) $\delta \in [0, 20]$ and (**b**) $\delta \in [2, 6]$.

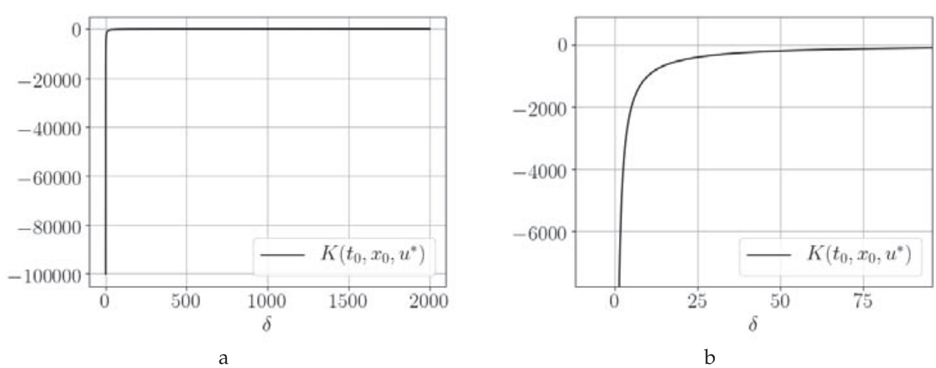

Figure 5. $d = 1000$, $b = 1$, $t_0 = 8$, $T = 100$, $x_0 = 10$. (**a**) $\delta \in [0, 2000]$ and (**b**) $\delta \in [0, 90]$.

4. Game-Theoretical Model of Pollution Control

4.1. No Absorption Coefficient Model

In this section, we consider a differential game of pollution control with a prescribed duration based on the game-theoretical models issued in [4]. The game involves n players (companies or countries), and each of them has an industrial production site in its territory. A three player game was considered in [31].

Let $N = \{1, 2, ..., n\}$ with $n \geq 2$ be a set of players. The strategy of player i is the amount of pollution emitted to the atmosphere over time $u_i \in [0; b_i]$. We will look for the solution in the class of open-loop strategies $u_i(t)$.

The dynamics of the total amount of pollution $x(t)$ is described by the following differential equation:

$$\dot{x}(t) = \sum_{i=1}^{n} u_i(t), \quad x(t_0) = x_0 \geq 0. \tag{9}$$

The total payoff of the i^{th} player is defined along the same lines as in Section 3.1. Thus, we write the payoff of the i^{th} player as:

$$K_i(t_0, x_0, u) = \int_{t_0}^{T} \left(\left(b_i - \frac{1}{2} u_i \right) u_i - d_i x \right) dt, \tag{10}$$

where b_i and d_i are the respective parameters that can be different for each player.

To simplify notation, we introduce the following parameters:

$$D_N = \sum_{i=1}^{n} d_i, \quad D_S = \sum_{i \in S} d_i, \quad D_{N \setminus S} = \sum_{i \in N \setminus S} d_i, \quad B_N = \sum_{i=1}^{n} b_i, \quad \tilde{B}_S = \sum_{i \in S} b_i^2, \text{ and } s = |S|.$$

We also assume that the regularity constraints $(T - t_0) \leq \frac{b_i}{D_N}$ hold for all $i \in N$. These constraints guarantee that $u_i(t) \geq 0, \forall t \in [t_0, T]$.

We assume the players have made a cooperative agreement on maximizing the total payoff of the players. To compute the cooperative solution (Shapley value, core, Harsanyi dividend, Banzhaf power index, etc.), one has to compute a characteristic function that plays a key role in cooperative game theory. In [23], we constructed the following characteristic functions for every coalition $S \subset N$.

$$V^{\alpha}(S, \Delta, x_0) = -D_S x_0 \Delta + \frac{1}{2}\tilde{B}_S \Delta - \frac{1}{2} B_N D_S \Delta^2 + \frac{1}{6} s D_S^2 \Delta^3. \tag{11}$$

$$V^{\delta}(S, \Delta, x_0) = -D_S x_0 \Delta + \frac{1}{2}\tilde{B}_S \Delta - \frac{1}{2} B_N D_S \Delta^2 + \frac{1}{6}(2 D_{N \setminus S} D_S + s D_S^2)\Delta^3. \tag{12}$$

$$V^{\zeta}(S, \Delta, x_0) = -D_S x_0 \Delta + \frac{1}{2}\tilde{B}_S \Delta - \frac{1}{2} B_N D_S \Delta^2 - \frac{1}{6} s D_N (D_N - 2 D_S)\Delta^3. \tag{13}$$

$$V^{\eta}(S, \Delta, x_0) = -D_S x_0 \Delta + \frac{1}{2}\tilde{B}_S \Delta - \frac{1}{2} B_N D_S \Delta^2 + \frac{1}{6}(-s D_N^2 + 2 s D_N D_S + 2 D_{N \setminus S} D_S)\Delta^3. \tag{14}$$

Functions (11)–(14) can be shown to be superadditive (see [23]). Moreover, the relations between characteristic functions were found in [23].

$$\begin{aligned} V^{\delta}(S,\Delta,x_0) &\geq V^{\alpha}(S,\Delta,x_0), \\ V^{\eta}(S,\Delta,x_0) &\geq V^{\zeta}(S,\Delta,x_0), \\ V^{\alpha}(S,\Delta,x_0) &\geq V^{\zeta}(S,\Delta,x_0), \\ V^{\delta}(S,\Delta,x_0) &\geq V^{\eta}(S,\Delta,x_0) \end{aligned} \tag{15}$$

4.2. Absorption Coefficient Model

We consider a modification of the differential game of pollution control. A special case of this game where $n = 3$ was considered in detail in [32]. In the paper [33], the construction of the η-characteristic function was presented for the example based on real data.

The dynamics of the total amount of pollution $x(t)$ is described by an extended equation:

$$\dot{x}(t) = \sum_{i=1}^{n} u_i(t) - \delta x(t), \quad x(t_0) = x_0 \geq 0, \tag{16}$$

where $\delta > 0$ is an absorption coefficient introduced earlier. The payoff function is defined by (10).

In the following, we assume that all additional regularity constraints defined above are satisfied.

Using the definitions (3)–(6), we get the following characteristic functions:

$$\begin{aligned} V^{\alpha}(S,\Delta,x_0) = x_0 \frac{D_S\left(e^{-\delta\Delta}-1\right)}{\delta} - \frac{sD_S^2}{4\delta^3}e^{-2\delta\Delta} + \left(\frac{sD_S^2}{\delta^3} - \frac{B_N D_S}{\delta^2}\right)e^{-\delta\Delta} \\ + \left(\frac{sD_S^2}{2\delta^2} - \frac{B_N D_S}{\delta} + \frac{\tilde{B}_S}{2}\right)\Delta - \frac{3sD_S^2}{4\delta^3} + \frac{B_N D_S}{\delta^2}. \end{aligned} \tag{17}$$

$$\begin{aligned} V^{\delta}(S,\Delta,x_0) = x_0 \frac{D_S\left(e^{-\delta\Delta}-1\right)}{\delta} - \frac{sD_S^2 + 2D_S D_{N\setminus S}}{4\delta^3}e^{-2\delta\Delta} + \left(\frac{sD_S^2 + 2D_S D_{N\setminus S}}{\delta^3} - \frac{B_N D_S}{\delta^2}\right)e^{-\delta\Delta} \\ + \left(\frac{sD_S^2 + 2D_S D_{N\setminus S}}{2\delta^2} - \frac{B_N D_S}{\delta} + \frac{\tilde{B}_S}{2}\right)\Delta - \frac{3(sD_S^2 + 2D_S D_{N\setminus S})}{4\delta^3} + \frac{B_N D_S}{\delta^2}. \end{aligned} \tag{18}$$

$$\begin{aligned} V^{\zeta}(S,\Delta,x_0) = x_0 \frac{D_S\left(e^{-\delta\Delta}-1\right)}{\delta} - \frac{-sD_N^2 + 2sD_S D_N}{4\delta^3}e^{-2\delta\Delta} + \left(\frac{-sD_N^2 + 2sD_S D_N}{\delta^3} - \frac{B_N D_S}{\delta^2}\right)e^{-\delta\Delta} \\ + \left(\frac{-sD_N^2 + 2sD_S D_N}{2\delta^2} - \frac{B_N D_S}{\delta} + \frac{\tilde{B}_S}{2}\right)\Delta - \frac{3(-sD_N^2 + 2sD_S D_N)}{4\delta^3} + \frac{B_N D_S}{\delta^2}. \end{aligned} \tag{19}$$

$$V^{\eta}(S,\Delta,x_0) = x_0\frac{D_S\left(e^{-\delta\Delta}-1\right)}{\delta} - \frac{s(2D_S-D_N)D_N+2D_SD_{N\setminus S}}{4\delta^3}e^{-2\delta\Delta}$$
$$+\left(\frac{s(2D_S-D_N)D_N+2D_SD_{N\setminus S}}{\delta^3}-\frac{B_ND_S}{\delta^2}\right)e^{-\delta\Delta}+\left(\frac{s(2D_S-D_N)D_N+2D_SD_{N\setminus S}}{2\delta^2}-\frac{B_ND_S}{\delta}+\frac{\tilde{B}_S}{2}\right)\Delta$$
$$-\frac{3\left(s(2D_S-D_N)D_N+2D_SD_{N\setminus S}\right)}{4\delta^3}+\frac{B_ND_S}{\delta^2}. \quad (20)$$

Functions (17)–(20) are superadditive. The construction of characteristic functions and the proof of their superadditivity are given in Appendix B and Appendix C. It can also be checked that the relations between characteristic Functions (15) are satisfied. In addition, we obtain the result:

$$V^{\delta}(S,\Delta,x_0)+V^{\zeta}(S,\Delta,x_0)=V^{\eta}(S,\Delta,x_0)+V^{\alpha}(S,\Delta,x_0).$$

Theorem 1. *The limits of the α-, δ-, ζ-, and η-characteristic functions exist and are equal as δ tends to infinity.*

$$\lim_{\delta\to+\infty}V^{\alpha}(S,\Delta,x_0)=\lim_{\delta\to+\infty}V^{\delta}(S,\Delta,x_0)=\lim_{\delta\to+\infty}V^{\zeta}(S,\Delta,x_0)=\lim_{\delta\to+\infty}V^{\eta}(S,\Delta,x_0)=\frac{\tilde{B}_S}{2}\Delta.$$

Theorem 1 is proven by computing limits with the use of the L'Hospital rule.

5. Optimal Control of Pollution Emissions for the Irkutsk Region

As an illustration, we considered a differential game of pollution control based on real data for enterprises of the Irkutsk region [16,33]. The ecological situation in Bratsk is one of the most acute in the region. We observed that the three largest enterprises of Bratsk pollute the environment: OJSC «RUSAL Bratsk», OJSC «ILIM Group», and units of OJSC «Irkutskenergo».

We used the coefficients of the differential game that were found in the paper [16].

We considered two cases. In the first case, the dynamics of the total amount of pollution $x(t)$ is described by (9). This means that there is no absorption coefficient in the model.

Using the provided numerical values of the parameters, we used (11)–(14) to compute α-, δ-, ζ-, and η-characteristic functions as functions of x_0 and Δ.

Figure 6 shows the influence of the initial time t_0 on the characteristic functions of the individual players. Similar results can be shown for the case of coalitions. In the following figures, we present the respective dependence on the left side and the zoomed-in fragments of the respective plots on the right side.

We now proceed to the case with non-zero pollution absorption. The dynamics of $x(t)$ is described by (16). Using the numerical values from Table 1 and the expressions for the characteristic Functions (17)–(20), we calculated the α-, δ-, ζ-, and η-characteristic functions.

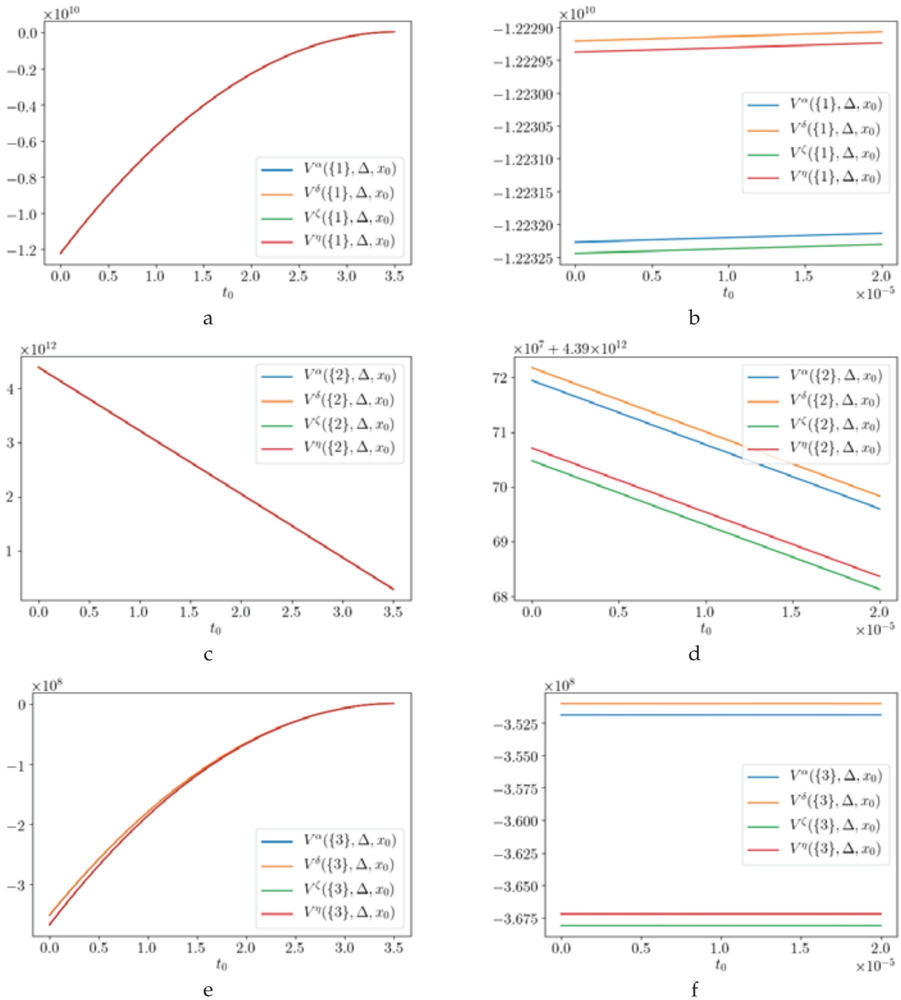

Figure 6. (**a**) $V(\{1\}), t_0 \in [0, 3.5]$, (**b**) $V(\{1\}), t_0 \in [0, 2 \times 10^{-5}]$, (**c**) $V(\{2\}), t_0 \in [0, 3.5]$, (**d**) $V(\{2\}), t_0 \in [0, 2 \times 10^{-5}]$, (**e**) $V(\{3\}), t_0 \in [0, 3.5]$, and (**f**) $V(\{3\}), t_0 \in [0, 2 \times 10^{-5}]$.

Table 1. Coefficient values.

Enterprise	b_i	d_i
OJSC «RUSAL Bratsk»	28,838.01	1254.97
OJSC «ILIM Group»	1,530,463	102.27
Units of OJSC «Irkutskenergo»	5228.4	36.65

Below, we present the plots showing the effect of the initial time t_0 and the absorption coefficient δ on the values of the respective characteristic functions. In particular, Figure 7 shows the influence of the initial time t_0 on the characteristic functions of the individual players, while Figure 8 illustrates the dependence of characteristic functions on the parameter δ. All presented results can be demonstrated for the coalitions as well.

Finally, in Figure 9, we illustrate the difference between the characteristic functions for the cases when the absorption coefficient is equal to zero (blue line) and when the absorption coefficient is taken from Table 1. For the illustration, we chose the η-characteristic function.

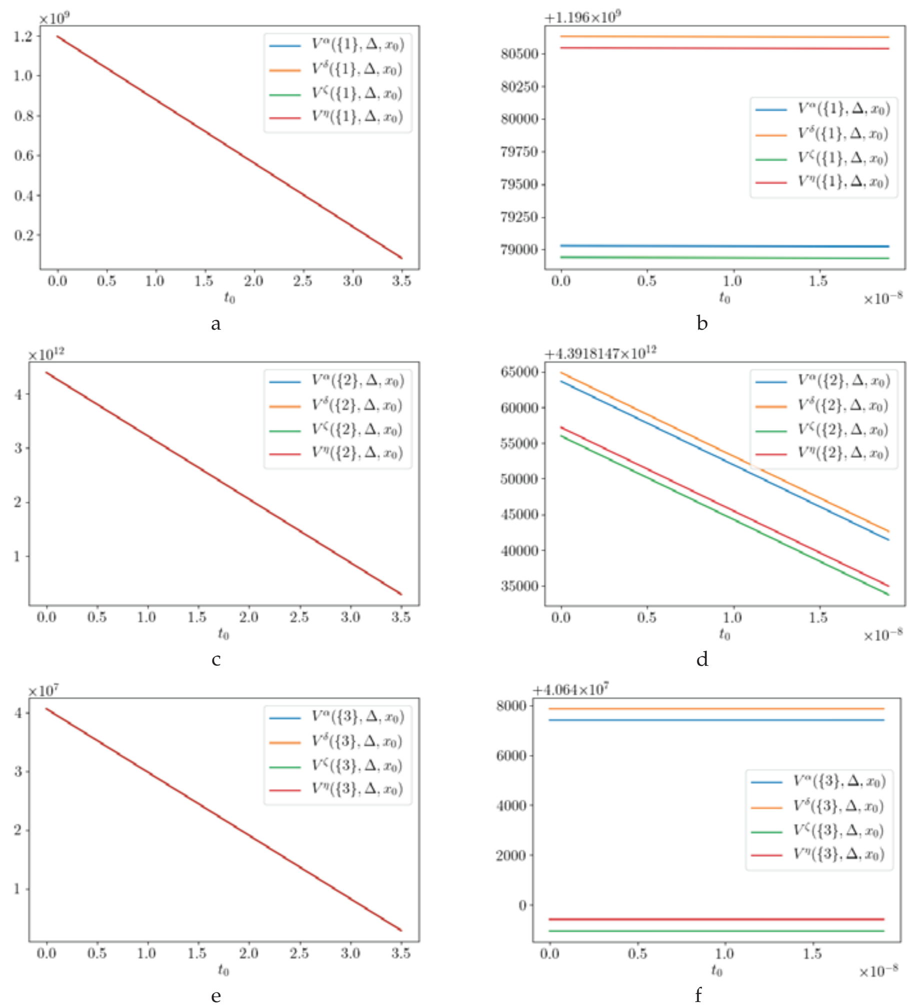

Figure 7. (**a**) $V(\{1\}), t_0 \in [0, 3.5]$, (**b**) $V(\{1\}), t_0 \in [0, 2 \times 10^{-8}]$, (**c**) $V(\{2\}), t_0 \in [0, 3.5]$, (**d**) $V(\{2\}), t_0 \in [0, 2 \times 10^{-8}]$, (**e**) $V(\{3\}), t_0 \in [0, 3.5]$, and (**f**) $V(\{3\}), t_0 \in [0, 2 \times 10^{-8}]$.

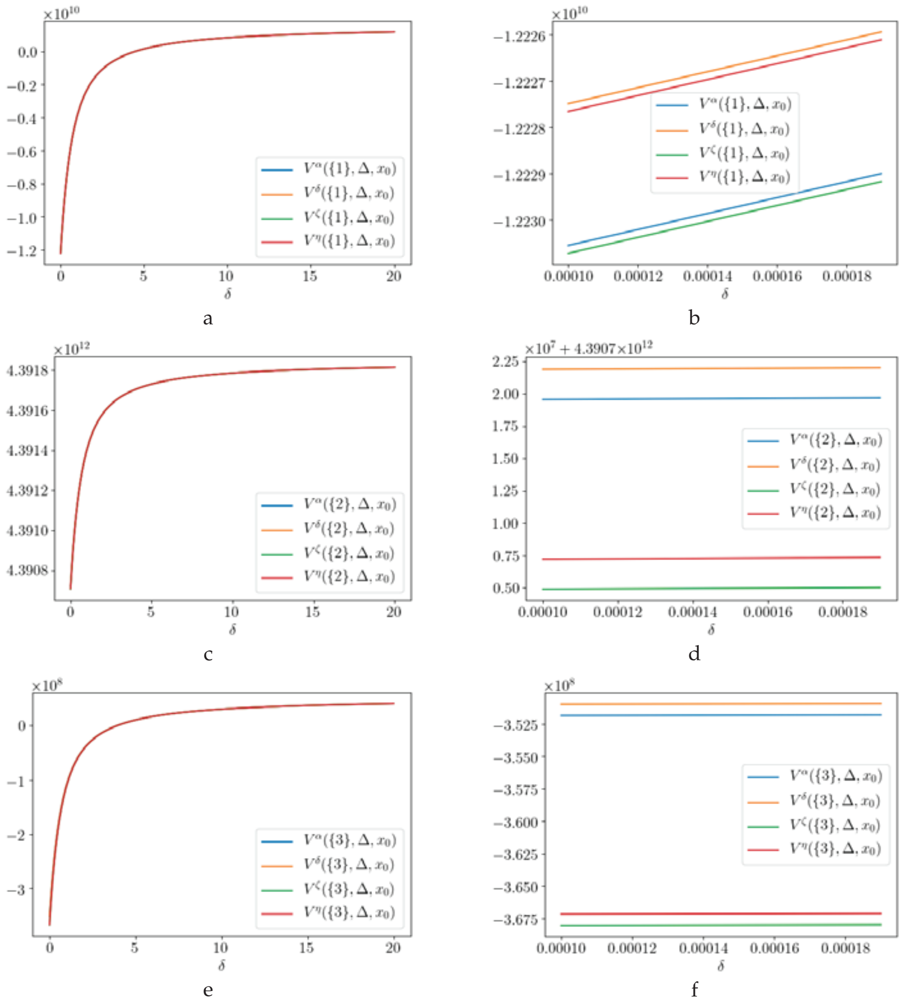

Figure 8. (**a**) $V(\{1\}), \delta \in [0.1, 20]$, (**b**) $V(\{1\}), \delta \in [0.0001, 0.00018]$, (**c**) $V(\{2\}), \delta \in [0.1, 20]$, (**d**) $V(\{2\}), \delta \in [0.0001, 0.00018]$, (**e**) $V(\{3\}), \delta \in [0.1, 20]$, and (**f**) $V(\{3\}), \delta \in [0.0001, 0.00018]$.

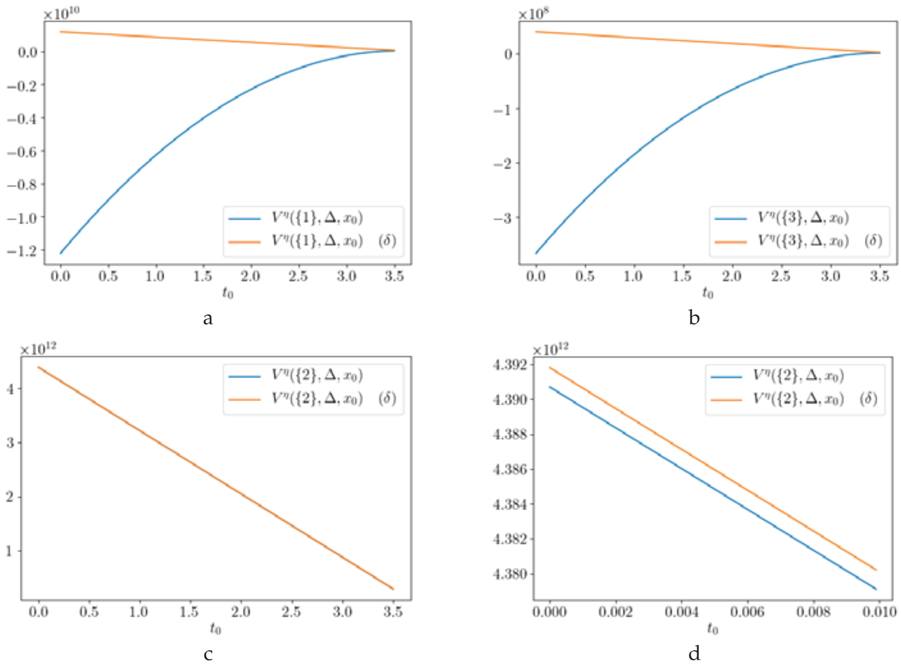

Figure 9. $\delta = 20$ (**a**) $V^\eta(\{1\})$, (**b**) $V^\eta(\{3\})$, (**c**) $V^\eta(\{2\}), t_0 \in [0, 3.5]$, and (**d**) $V^\eta(\{2\}), t_0 \in [0, 0.01]$.

Conclusions

In this paper, we considered a cooperative differential game of pollution control with a prescribed duration. We analyzed two cases when there was an absorption coefficient δ and when there was not. For these games, α-, δ-, ζ-, and η-characteristic functions were constructed for the general case of n players.

It was shown that the parameter δ had a significant impact on the characteristic functions. We also obtained analytical formulas for the limiting values of the characteristic functions with increasing absorption coefficient.

All results were illustrated with the differential game of pollution control based on real data for enterprises of the Irkutsk region.

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Appendix A. Construction of Optimal Pollution Control

Proof of Proposition 1

Proof. The proof of $u^*(t) \leq b$ is trivial because the second term is not positive.

We differentiate $u^*(t)$ with respect to t:

$$\frac{\mathrm{d}u^*(t)}{\mathrm{d}t} = de^{-\delta(T-t)} > 0, \quad \forall t \in [t_0, T].$$

Therefore, $u^*(t)$ is an increasing function of t. This means that the function takes a minimum value at a point $t = t_0$:

$$u^*(t_0) = b + d\frac{e^{-\delta(T-t_0)} - 1}{\delta}.$$

We differentiate $u^*(t)$ with respect to δ:

$$\frac{\mathrm{d}u^*(t)}{\mathrm{d}\delta} = \frac{de^{-\delta(T-t)}(-\delta(T-t) + e^{\delta(T-t)} - 1)}{\delta^2}.$$

Obviously, $de^{-\delta(T-t)} > 0$ and $\delta^2 > 0$. To define the sign of $(-\delta(T-t) + e^{\delta(T-t)} - 1)$:

$$\frac{\mathrm{d}(-\delta(T-t) + e^{\delta(T-t)} - 1)}{\mathrm{d}\delta} = (T-t)(e^{\delta(T-t)} - 1) > 0.$$

Hence, $(-\delta(T-t) + e^{\delta(T-t)} - 1)$ is an increasing function of δ. It follows that the function is positive for $\delta > 0$. This completes the proof that $\frac{\mathrm{d}u^*(t)}{\mathrm{d}\delta} > 0$.

Thus, $u^*(t)$ is an increasing function of δ, then:

$$u^*(t_0) \geq \lim_{\delta \to 0} \left(b + d\frac{e^{-\delta(T-t_0)} - 1}{\delta} \right) = b - d(T - t_0).$$

If $b - d(T-t_0) \geq 0$, which is the same as $(T-t_0) \leq \frac{b}{d}$, then $u^*(t) \geq 0, \forall \delta$ and $t \in [t_0, T]$. Next, we consider the case $(T-t_0) > \frac{b}{d}$. We find δ that provides $u^*(t) \geq 0$.

Note that $u^*(t_0) \geq 0$ for sufficiently large δ. If δ tends to zero, then $u^*(t) < 0$ at $t = t_0$ at least. Define the delta at which $u^*(t_0) = 0$ under the condition $u^*(t)$ is an increasing function of t and δ.

$$b + d\frac{e^{-\delta(T-t_0)} - 1}{\delta} = 0,$$

whence, upon some algebraic manipulation, we get:

$$\delta(T-t_0) - \frac{d(T-t_0)}{b} + \frac{d(T-t_0)}{b}e^{-\delta(T-t_0)} = 0.$$

We define:

$$\rho = \frac{d(T-t_0)}{b}, \quad \varepsilon = \delta(T-t_0).$$

Since $(T-t_0) > \frac{b}{d}$, then $\rho > 1$. Hence, we obtain the equation $\varepsilon - \rho + \rho e^{-\varepsilon} = 0$. Solving this equation with respect to ε, we get:

$$\varepsilon = \rho + W(-\rho e^{-\rho}).$$

Finally, we arrive at the following expression for the threshold value of δ:

$$\delta = \frac{1}{T-t_0}\left(\frac{d(T-t_0)}{b} + W\left(-\frac{d(T-t_0)}{b}\exp\left(-\frac{d(T-t_0)}{b}\right)\right)\right)$$
$$= \frac{d}{b} + \frac{1}{T-t_0}W\left(-\frac{d(T-t_0)}{b}\exp\left(-\frac{d(T-t_0)}{b}\right)\right) = \bar{\delta}.$$

If $\delta \geq \bar{\delta}$, then $u^*(t_0) \geq 0$ and $u^*(t) \geq 0, \forall t \in [t_0, T]$. Otherwise, if $\delta < \bar{\delta}$, then $u^*(t_0) < 0$. We determine the moment $\bar{t} > t_0$ at which the optimal control turns to zero: $u^*(\bar{t}) = 0$. Solving:

$$b + d\frac{e^{-\delta(T-t)} - 1}{\delta} = 0,$$

we get:

$$\bar{t} = T + \frac{1}{\delta}\ln\left(1 - \frac{b\delta}{d}\right).$$

Thus, $\bar{t}$ is the point where the control changes sign. □

Appendix B. Computation of the Characteristic Functions

Appendix B.1. Nash Equilibrium

The computation of the Nash equilibrium strategies (NE) is fairly obvious, so we skip most details. Using the Pontryagin maximum principle, we find the Nash equilibrium strategies:

$$u^{NE}(t) = \begin{pmatrix} b_1 + d_1\frac{e^{-\delta(T-t)}-1}{\delta} \\ \ldots \\ b_n + d_n\frac{e^{-\delta(T-t)}-1}{\delta} \end{pmatrix} \tag{A1}$$

and the corresponding trajectory:

$$x^{NE}(t) = \frac{e^{-\delta t}}{2\delta^2}\left(e^{-\delta(T-t)}\left(D_N e^{\delta t} + 2(B_N\delta - D_N)e^{\delta T}\right) - e^{-\delta(T-t_0)}\left(D_N e^{\delta t_0} + 2(B_N\delta - D_N - \delta^2 x_0)e^{\delta T}\right)\right).$$

Following the same scheme as that shown in Appendix A, we obtain that if $(T - t_0) \leq \frac{b_i}{d_i}$, then $u_i^{NE}(t) \geq 0, \forall\delta > 0$ and $\forall t \in [t_0, T]$.

Appendix B.2. Cooperative Agreement

The optimal cooperative strategies are:

$$u^*(t) = \begin{pmatrix} b_1 + D_N\frac{e^{-\delta(T-t)}-1}{\delta} \\ \ldots \\ b_n + D_N\frac{e^{-\delta(T-t)}-1}{\delta} \end{pmatrix} \tag{A2}$$

and the optimal trajectory is:

$$x^*(t) = \frac{e^{-\delta t}}{2\delta^2}\left(e^{-\delta(T-t)}\left(nD_N e^{\delta t} + 2(B_N\delta - nD_N)e^{\delta T}\right) - e^{-\delta(T-t_0)}\left(nD_N e^{\delta t_0} + 2(B_N\delta - nD_N - \delta^2 x_0)e^{\delta T}\right)\right).$$

For the optimal control, we have that if $b_i - D_N(T - t_0) \geq 0$, which is the same as $(T - t_0) \leq \frac{b_i}{D_N}$, then $u^*(t) \geq 0, \forall\delta$ and $t \in [t_0, T]$.

Appendix B.3. Construction of the α-Characteristic Function

We consider a minimization problem:

$$\min_{\substack{u_j \\ j\in N\setminus S}} \sum_{i\in S}\int_{t_0}^{T} \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i - d_i x\Big)dt, \quad S \in N.$$

For this problem, the optimal strategies are:

$$u_j = b_j. \tag{A3}$$

Next, we consider a maximization problem:

$$\max_{\substack{u_i,\, i\in S, \\ u_j=b_j, j\in N\setminus S}} \sum_{i\in S}\int_{t_0}^{T} \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i - d_i x\Big)dt, \quad S \in N.$$

For this problem, we obtain the controls:

$$u_i^S(t) = b_i + D_S\frac{e^{-\delta(T-t)} - 1}{\delta}, \quad i \in S. \tag{A4}$$

The corresponding trajectory is:

$$x^S(t) = \frac{e^{-\delta t}}{2\delta^2}\Big(e^{-\delta(T-t)}\Big(sD_S e^{\delta t} + 2(B_N\delta - sD_S)e^{\delta T}\Big) - e^{-\delta(T-t_0)}\Big(sD_S e^{\delta t_0} + 2(B_N\delta - sD_S - \delta^2 x_0)e^{\delta T}\Big)\Big).$$

For the controls (A4), we have that if $(T - t_0) \leq \frac{b_i}{D_S}$, then $u_i^S(t) \geq 0, \forall\delta > 0$ and $t \in [t_0, T]$. Combining (3), (A3), and (A4), we construct the α-characteristic function:

$$V^\alpha(S, t_0, x_0) = x_0\frac{D_S\left(e^{-\delta(T-t_0)} - 1\right)}{\delta} - \frac{sD_S^2}{4\delta^3}e^{-2\delta(T-t_0)} + \left(\frac{sD_S^2}{\delta^3} - \frac{B_N D_S}{\delta^2}\right)e^{-\delta(T-t_0)}$$
$$+ \left(\frac{sD_S^2}{2\delta^2} - \frac{B_N D_S}{\delta} + \frac{\tilde{B}_S}{2}\right)(T - t_0) - \frac{3sD_S^2}{4\delta^3} + \frac{B_N D_S}{\delta^2}.$$

Appendix B.4. Construction of the δ-Characteristic Function

We consider a maximization problem:

$$\max_{\substack{u_i,\, i\in S, \\ u_j=u_j^{NE}, j\in N\setminus S}} \sum_{i\in S}\int_{t_0}^{T} \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i - d_i x\Big)dt, \quad S \in N.$$

For this problem, the optimal controls are:

$$u_i^S(t) = b_i + D_S \frac{e^{-\delta(T-t)} - 1}{\delta} \tag{A5}$$

and the corresponding trajectory is:

$$\begin{aligned} x^S(t) =& \frac{e^{-\delta t}}{2\delta^2}\Big(e^{-\delta(T-t)} \left((sD_S + D_{N\setminus S})e^{\delta t} + 2(B_N\delta - sD_S - D_{N\setminus S})e^{\delta T} \right) \\ & - e^{-\delta(T-t_0)} \left((sD_S + D_{N\setminus S})e^{\delta t_0} + 2(B_N\delta - sD_S - D_{N\setminus S} - \delta^2 x_0)e^{\delta T} \right) \Big). \end{aligned}$$

Similar to the previous cases, we have that if $(T - t_0) \le \frac{b_i}{D_S}$, then $u_i^S(t) \ge 0, \forall \delta > 0$ and $t \in [t_0, T]$. Combining (4), (A1), and (A5), we construct the δ-characteristic function:

$$\begin{aligned} V^\delta(S, t_0, x_0) =& x_0 \frac{D_S\left(e^{-\delta(T-t_0)} - 1\right)}{\delta} - \frac{sD_S^2 + 2D_S D_{N\setminus S}}{4\delta^3} e^{-2\delta(T-t_0)} \\ &+ \left(\frac{sD_S^2 + 2D_S D_{N\setminus S}}{\delta^3} - \frac{B_N D_S}{\delta^2} \right) e^{-\delta(T-t_0)} \\ &+ \left(\frac{sD_S^2 + 2D_S D_{N\setminus S}}{2\delta^2} - \frac{B_N D_S}{\delta} + \frac{\tilde{B}_S}{2} \right)(T - t_0) - \frac{3(sD_S^2 + 2D_S D_{N\setminus S})}{4\delta^3} + \frac{B_N D_S}{\delta^2}. \end{aligned}$$

Appendix B.5. Construction of the ζ-Characteristic Function

According to (5), players from coalition $S \in N$ use optimal controls (A2).
We consider the minimization problem:

$$\min_{u_j, j \in N\setminus S} \sum_{i\in S} \int_{t_0}^{T} \left(\left(b_i - \frac{1}{2}u_i^* \right) u_i^* - d_i x \right) dt, \quad S \in N.$$

In this case, the optimal strategies are:

$$u_j = b_j. \tag{A6}$$

Combining (5), (A2), and (A6), we construct the ζ-characteristic function:

$$\begin{aligned} V^\zeta(S, t_0, x_0) =& x_0 \frac{D_S\left(e^{-\delta(T-t_0)} - 1\right)}{\delta} - \frac{-sD_N^2 + 2sD_S D_N}{4\delta^3} e^{-2\delta(T-t_0)} \\ &+ \left(\frac{-sD_N^2 + 2sD_S D_N}{\delta^3} - \frac{B_N D_S}{\delta^2} \right) e^{-\delta(T-t_0)} \\ &+ \left(\frac{-sD_N^2 + 2sD_S D_N}{2\delta^2} - \frac{B_N D_S}{\delta} + \frac{\tilde{B}_S}{2} \right)(T - t_0) - \frac{3(-sD_N^2 + 2sD_S D_N)}{4\delta^3} + \frac{B_N D_S}{\delta^2}. \end{aligned}$$

Appendix B.6. Construction of the η-Characteristic Function

According to (6), players from $S \in N$ use (A2) when players from $N \setminus S$ use (A1). We construct the η-characteristic function:

$$
\begin{aligned}
V^{\eta}(S,t_0,x_0) &= x_0\frac{D_S\left(e^{-\delta(T-t_0)}-1\right)}{\delta} - \frac{s(2D_S-D_N)D_N+2D_SD_{N\setminus S}}{4\delta^3}e^{-2\delta(T-t_0)} \\
&+\left(\frac{s(2D_S-D_N)D_N+2D_SD_{N\setminus S}}{\delta^3}-\frac{B_ND_S}{\delta^2}\right)e^{-\delta(T-t_0)} \\
&+\left(\frac{s(2D_S-D_N)D_N+2D_SD_{N\setminus S}}{2\delta^2}-\frac{B_ND_S}{\delta}+\frac{\tilde{B}_S}{2}\right)(T-t_0) \\
&-\frac{3\left(s(2D_S-D_N)D_N+2D_SD_{N\setminus S}\right)}{4\delta^3}+\frac{B_ND_S}{\delta^2}.
\end{aligned}
$$

Appendix C. Proofs of Superadditivity Characteristic Functions

Appendix C.1. Additional Statement

Proposition A1.

$$-e^{-2\delta(T-t_0)}+4e^{-\delta(T-t_0)}+2\delta(T-t_0)-3>0.$$

Proof. Let $x=e^{-\delta(T-t_0)}$, then we need to prove:

$$-x^2+4x+2\delta(T-t_0)-3>0.$$

We solve the quadratic equation:

$$-x^2+4x+2\delta(T-t_0)-3=0.$$

The roots of the equation are:

$$x_1=2-\sqrt{1+2\delta(T-t_0)},\quad x_2=2+\sqrt{1+2\delta(T-t_0)}.$$

The branches of the parabola $y=-x^2+4x+2\delta(T-t_0)-3$ are directed downwards. This means that:

$$-x^2+4x+2\delta(T-t_0)-3>0$$

when:

$$x\in\left(2-\sqrt{1+2\delta(T-t_0)},\quad 2+\sqrt{1+2\delta(T-t_0)}\right).$$

We change x to $e^{-\delta(T-t_0)}$. Let us prove that:

$$e^{-\delta(T-t_0)}\in\left(2-\sqrt{1+2\delta(T-t_0)},\quad 2+\sqrt{1+2\delta(T-t_0)}\right). \tag{A7}$$

We make a replacement $m=\delta(T-t_0)>0$. Therefore, we prove:

$$e^{-m}\in\left(2-\sqrt{1+2m},\quad 2+\sqrt{1+2m}\right). \tag{A8}$$

First, let us show that:

$$2+\sqrt{1+2m}-e^{-m}>0. \tag{A9}$$

We have:

$$\sqrt{1+2m} \geq 1,$$

$$e^{-m} \leq 1.$$

Hence:

$$2 + \sqrt{1+2m} - e^{-m} \geq 2.$$

This completes the proof (A9).

Secondly, let us show that:

$$e^{-m} - 2 + \sqrt{1+2m} > 0. \tag{A10}$$

We transform the expression:

$$e^{-m} - 2 + \sqrt{1+2m} = e^{-m}(e^m\sqrt{1+2m} - 2e^m + 1).$$

We clearly have $e^{-m} > 0$. To prove $e^m\sqrt{1+2m} - 2e^m + 1 > 0$, we need to differentiate $(e^m\sqrt{1+2m} - 2e^m + 1)$ with respect to m:

$$\frac{d(e^m\sqrt{1+2m} - 2e^m + 1)}{dm} = \frac{2e^m(m - \sqrt{1+2m} + 1)}{\sqrt{1+2m}} \tag{A11}$$

Obviously, $e^m > 0$, $\sqrt{1+2m} > 0$ for $m > 0$. Let us show:

$$m - \sqrt{1+2m} + 1 > 0. \tag{A12}$$

Assume the converse. Then:

$$m - \sqrt{1+2m} + 1 \leq 0.$$

Hence:

$$m + 1 \leq \sqrt{1+2m}$$

$$(m+1)^2 \leq 1 + 2m$$

because $m > 0$. Therefore:

$$m^2 \leq 0,$$

$$m = 0.$$

This contradiction proves (A12). This means that in (A11):

$$\frac{d(e^m\sqrt{1+2m} - 2e^m + 1)}{dm} > 0$$

That is, the function $y = e^m\sqrt{1+2m} - 2e^m + 1$ is increasing for $m > 0$ and:

$$e^m\sqrt{1+2m} - 2e^m + 1 > 0.$$

This implies the inequality (A10). Finally, using (A9) and (A10), we get (A8). This yields

$$e^{-\delta(T-t_0)} \in \left(2 - \sqrt{1+2\delta(T-t_0)}, \quad 2 + \sqrt{1+2\delta(T-t_0)}\right).$$

and hence, $-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 > 0$. $\square$

Appendix C.2. Superadditivity of the α-Characteristic Function

We prove that the α-characteristic function is superadditive using (2).

$$\begin{aligned}
&V^{\alpha}(S_1 \cup S_2, t_0, x_0) - V^{\alpha}(S_1, t_0, x_0) - V^{\alpha}(S_2, t_0, x_0) \\
&= -\frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2}\left(s_1+s_2\right)}{4\delta^3} e^{-2\delta(T-t_0)} \\
&+ \left(\frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2}\left(s_1+s_2\right)}{\delta^3}\right) e^{-\delta(T-t_0)} \\
&+ \left(\frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2}\left(s_1+s_2\right)}{2\delta^2}\right) (T-t_0) \\
&- \frac{3\left(s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2}\left(s_1+s_2\right)\right)}{4\delta^3} \\
&= \frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2}\left(s_1+s_2\right)}{4\delta^3}\left(-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3\right).
\end{aligned}$$

Clearly,

$$\frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2}\left(s_1+s_2\right)}{4\delta^3} > 0.$$

According to Proposition A1:

$$-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 > 0.$$

Hence, $V^{\alpha}(S_1 \cup S_2, t_0, x_0) - V^{\alpha}(S_1, t_0, x_0) - V^{\alpha}(S_2, t_0, x_0) > 0$.

Appendix C.3. Superadditivity of the δ-Characteristic Function

We verify the superadditivity of the δ-characteristic function using the definition (2).

$$\begin{aligned}
&V^{\delta}(S_1 \cup S_2, t_0, x_0) - V^{\delta}(S_1, t_0, x_0) - V^{\delta}(S_2, t_0, x_0) \\
&= -\frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2}\left(s_1+s_2-2\right)}{4\delta^3} e^{-2\delta(T-t_0)} \\
&+ \left(\frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2}\left(s_1+s_2-2\right)}{\delta^3}\right) e^{-\delta(T-t_0)} \\
&+ \left(\frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2}\left(s_1+s_2-2\right)}{2\delta^2}\right) (T-t_0) \\
&- \frac{3\left(s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2}\left(s_1+s_2-2\right)\right)}{4\delta^3} \\
&= \frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2}\left(s_1+s_2-2\right)}{4\delta^3}\left(-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3\right).
\end{aligned}$$

Using $s_1 = |S_1| \geq 1, s_2 = |S_2| \geq 1$, we get $s_1 + s_2 - 2 \geq 0$. It is obvious that:

$$\frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2D_{S_1} D_{S_2} (s_1 + s_2 - 2)}{4\delta^3} > 0.$$

According to Proposition A1:

$$-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 > 0.$$

Therefore, $V^\delta(S_1 \cup S_2, t_0, x_0) - V^\delta(S_1, t_0, x_0) - V^\delta(S_2, t_0, x_0) > 0$.

Appendix C.4. Superadditivity of the ζ-Characteristic Function

Let us show that the ζ-characteristic function is superadditive using (2).

$$\begin{aligned}
& V^\zeta(S_1 \cup S_2, t_0, x_0) - V^\zeta(S_1, t_0, x_0) - V^\zeta(S_2, t_0, x_0) \\
&= -\frac{D_N(s_2 D_{S_1} + s_1 D_{S_2})}{2\delta^3} e^{-2\delta(T-t_0)} + \frac{2D_N(s_2 D_{S_1} + s_1 D_{S_2})}{\delta^3} e^{-\delta(T-t_0)} \\
&+ \frac{D_N(s_2 D_{S_1} + s_1 D_{S_2})}{\delta^2} - \frac{3D_N(s_2 D_{S_1} + s_1 D_{S_2})}{2\delta^3}(T-t_0) \\
&= \frac{D_N(s_2 D_{S_1} + s_1 D_{S_2})}{2\delta^3} \left(-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 \right).
\end{aligned}$$

Trivially,

$$\frac{D_N(s_2 D_{S_1} + s_1 D_{S_2})}{2\delta^3} > 0.$$

According to Proposition A1:

$$-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 > 0.$$

It follows that $V^\zeta(S_1 \cup S_2, t_0, x_0) - V^\zeta(S_1, t_0, x_0) - V^\zeta(S_2, t_0, x_0) > 0$.

Appendix C.5. Superadditivity of the η-Characteristic Function

Let us check that the η-characteristic function is superadditive using (2).

$$\begin{aligned}
& V^\eta(S_1 \cup S_2, t_0, x_0) - V^\eta(S_1, t_0, x_0) - V^\eta(S_2, t_0, x_0) \\
&= -\frac{D_{S_1}(s_2 D_N - D_{S_2}) + D_{S_2}(s_1 D_N - D_{S_1})}{2\delta^3} e^{-2\delta(T-t_0)} \\
&+ \frac{2D_{S_1}(s_2 D_N - D_{S_2}) + 2D_{S_2}(s_1 D_N - D_{S_1})}{\delta^3} e^{-\delta(T-t_0)} \\
&+ \frac{D_{S_1}(s_2 D_N - D_{S_2}) + D_{S_2}(s_1 D_N - D_{S_1})}{\delta^2}(T-t_0) \\
&- \frac{3\left(D_{S_1}(s_2 D_N - D_{S_2}) + D_{S_2}(s_1 D_N - D_{S_1})\right)}{2\delta^3} \\
&= \frac{D_{S_1}(s_2 D_N - D_{S_2}) + D_{S_2}(s_1 D_N - D_{S_1})}{2\delta^3} \left(-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 \right).
\end{aligned}$$

We obtain:

$$\frac{D_{S_1}(s_2 D_N - D_{S_2}) + D_{S_2}(s_1 D_N - D_{S_1})}{2\delta^3} \geq \frac{D_{S_1}(D_N - D_{S_2}) + D_{S_2}(D_N - D_{S_1})}{2\delta^3}$$
$$= \frac{D_{S_1} D_{N\setminus S_2} + D_{S_2} D_{N\setminus S_1}}{2\delta^3} > 0.$$

According to Proposition A1:

$$-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 > 0.$$

Therefore, $V^\eta(S_1 \cup S_2, t_0, x_0) - V^\eta(S_1, t_0, x_0) - V^\eta(S_2, t_0, x_0) > 0$.

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Hotelling's Duopoly in a Two-Sided Platform Market on the Plane

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Abstract: Equilibrium in a two-sided market represented by network platforms on the plane and heterogeneous agents is investigated. The advocated approach is based on the duopoly model which implies a continuum of agents of limited size on each side of the market and examines the agents' heterogeneous utility with the Hotelling specification. The exact values were found for the equilibrium in the case of duopoly in a two-sided market with two platforms on the plane. The dependence of the platforms' benefits on network externalities was investigated. The problem of the optimal location of platforms in the market was considered.

Keywords: two-sided platform market; pricing; Hotelling's duopoly on the plane; Nash equilibrium; optimal location of platforms

1. Introduction

Digital economy has formed a paradigm of accelerated economic development. A central position in it is occupied by network technologies, which have led to the establishment of network markets. Network markets are set up on network platforms, which are a novel business element in the modern economy, providing benefits for both the platform's operator and the community.

The two-sided market involving a platform is a network type of market, where users belonging to two different groups interact. Consumption by any one of the groups generates external effects on the other group. Assuming a network effect exists between the two sides of the market (network externalities), the problem of optimizing the platform's profit can be formulated. Reference [1] suggested a model of a monopoly platform in a two-sided market with the total number of agents on each side equaling one. The agents have different transport costs to access the platform via Hotelling specification. As an illustration, imagine that agents in the model are sellers and customers in an online store, job-seekers and employers in a job centre, men and women in a dance floor, and so forth. Following the Hotelling specification [2], agents compare their costs before taking the decision about visiting the platform. Our study assumes agents to be heterogeneous and demand to be endogenous, and investigates the platform's behavior and revenues under these assumptions.

A duopoly in a two-sided in-plane platform market is considered, equilibrium in such a market and the dependence of platform revenues on network externalities are determined. For the case of identical agents and identical platforms, the conditions were detected enabling services in such platforms to exist under competition, and the exact expressions for the platforms' optimal strategies were found.

Also, the social optimum problem was solved and the dependence of the optimal value on the structure of the two-sided market's costs and network externalities was determined.

A number of papers have been published on network platforms with externalities. Since the analysis of monopolistic markets [3–7], studies in this sphere have been focused on the pricing platforms for markets with different structures, including the problem of the social optimum for private platforms and associations. The aspects addressed were the models of private platforms' competition in different scenarios for homogeneous and differentiated products, bottlenecks, exclusive contracts, as well as the effects of various important factors such as tariffs, market price elasticity, transaction size, and buyers and sellers net surplus.

Rochet and Tirole study the pricing strategy of payment-card type of platforms in two-sided markets, considering both the monopoly case and the duopoly case, with different objective functions of the platforms: either to maximize the profit or to maximize the social welfare [8]. Later on, Rochet and Tirole consider this type of platform model that integrates both usage and membership externalities and study obtain new results on the mix of usage and membership charges [8]. Later on, Rochet and Tirole placed the model of a platform of this type into the context of externalities with membership fees [6].

In Reference [1], which is the closest to our study, Armstrong looked into the pricing strategy in a two-sided market for platforms of the *shopping-mall* type for the monopoly scenario and a duopoly. In the monopoly case, there were no constraints on the size of the market and the demand function depended on exogenous reasons.

Later on, Armstrong and Wright explored the duopoly competition between two platforms in a linear market, included the Hotelling specification in the model, and considered three cases: (1) product differentiation on both sides; (2) products differentiation on one side only; (3) no product differentiation on either side [9]. For models (1) and (2), the agents' utilities were large enough for using the services of at least one of the platforms. For model (3), the agents were assumed to be homogeneous, and only the network effect was taken into account.

Where in References [1,9,10] each agent is supposed to choose one of the platforms for the service, agents in the model from Reference [11] are not allowed to join any platform if they do not benefit from it. It was demonstrated in the latter paper that assuming a limited market size, heterogeneous agents and endogenous demand, the optimal strategy always results in a corner solution, whether the goal is to maximize the platform's profit or social welfare. Also, the conditions under which the monopoly is socially optimal were found.

Another important issue associated with platform economy is the optimal platform location. This problem was investigated for a one-sided market by Hotelling in Reference [2], and then in some models for a linear market [12–14], and in Reference [15] for a market on the plane. Research on the optimal location of platforms is now in an early stage.

Some papers on two-sided platforms investigated pricing in a market depending on the seller's behavior for special platforms [16–21]. Reference [16] dealt with sequential competition between sellers in a two-sided software market, while Reference [20] focused on competition between virtual operators (sellers) in a two-sided telecommunication market.

Research on these issues is a new area in platform economy. In this study we analyze the performance of platforms under the new assumption that the market is in the plane space. This approach results in new intriguing effects, the market boundary is a second-order curve, which breaks the linearity of solutions and, hence, alters the form of the conditions for the feasibility of competitive service and the form of the solutions as such. Furthermore, the problem of platform location in the market is also meaningful. This is demonstrated below for the case of two platforms.

The article is structured as follows—Section 2 describes the model and its distinctions from previous models; Section 3 finds the equilibrium in the model with identical platforms and identical agents, while Section 4 finds the solution for the model with identical groups of agents and platforms occupying different positions in the market; Section 5 investigates the optimal location of platforms; Section 6 finds the equilibrium in the general case of the pricing problem; the article is completed with the conclusions and visions for further research in the area.

2. Description of the Model

Suppose there are two non-intersecting groups of agents in the market: group 1 (also known as group of buyers), and group 2 (also known as group of sellers). Agents in both groups, whose size equals a unit, are distributed uniformly over a plane in the square $S = [-1, 1] \times [-1, 1]$. The location of the members of groups 1 and 2 in the square S is defined by the points (x_1, y_1) and (x_2, y_2) respectively, where $-1 \leq x_i \leq 1$ and $-1 \leq y_i \leq 1$, $i = 1, 2$. Agents from both groups meet in platforms I and II, located at the boundary of the square S in diametrically opposite points, for example, $I(-1, 0)$ and $II(1, 0)$.

The following notations are introduced: $n_i^{(j)}$ is the size of the group i in the platform j ($i = 1, 2; j = I, II$); $p_i^{(j)}$ is the price of visiting the platform j by an agent of the group i ($i = 1, 2; j = I, II$); α, β are the degrees of the effect that the number of second (first) group agents in the platform has on the payoff of the first (second) group; t_i is the effect of the transport costs for visiting the ith platform where $t_i > 0$ ($i = 1, 2$).

In a two-sided market with two platforms, agents in both groups choose between them based on the utility they can derive from visiting the respective platform. For group 1 agents, the utility of visiting the platform I, situated in the point $(-1, 0)$ has the form

$$u_1^{(I)} = \alpha \cdot n_2^{(I)} - p_1^{(I)} - \sqrt{(x_1 + 1)^2 + y_1^2} \cdot t_1,$$

and the utility of visiting the platform II, in the point $(1, 0)$ is

$$u_1^{(II)} = \alpha \cdot n_2^{(II)} - p_1^{(II)} - \sqrt{(x_1 - 1)^2 + y_1^2} \cdot t_1.$$

In the above expressions, the first term captures the utility from network externality of the other side of the market, the second term is the cost of price the agent has to pay to access the platform, and the third term is essentially the agent's heterogeneous transport cost of accessing the platform, via Hotelling specification.

The boundary between regions of the market for group 1 is found from the equation $u_1^{(I)} = u_1^{(II)}$, which is solved to get

$$\frac{x_1^2}{s_1^2} - \frac{y_1^2}{1 - s_1^2} = 1,$$

where $s_1 = \dfrac{\alpha - 2\alpha n_2^{(I)} + p_1^{(I)} - p_1^{(II)}}{2t_1}$, given that $n_2^{(I)} + n_2^{(II)} = 1$ (see Figure 1). Parameter s_1 is the coordinate value at which the hyperbola crosses the abscissa axis. In the symmetric case, when the market is divided equally, $s_1 = 0$. If $s_1 > 0$ ($s_1 < 0$), then platform I (II) is more attractive for agents.

The market being divided into two regions, the respective numbers of group 1 agents visiting platforms I and II are

$$n_1^{(I)} = \frac{1}{2} - \frac{1}{2} \int_0^1 s_1 \sqrt{1 + \frac{y^2}{1 - s_1^2}}\, dy, \tag{1}$$

$$n_1^{(II)} = \frac{1}{2} + \frac{1}{2}\int_0^1 s_1\sqrt{1+\frac{y^2}{1-s_1^2}}\,dy. \tag{2}$$

The utilities of group 2 agents from visiting platforms I and II are determined similarly, and are equal, respectively

$$u_2^{(I)} = \beta \cdot n_1^{(I)} - p_2^{(I)} - \sqrt{(x_2+1)^2+y_2^2}\cdot t_2,$$

$$u_2^{(II)} = \beta \cdot n_1^{(II)} - p_2^{(II)} - \sqrt{(x_2-1)^2+y_2^2}\cdot t_2,$$

and the boundary between market regions for group 2 is

$$\frac{x_2^2}{s_2^2} - \frac{y_2^2}{1-s_2^2} = 1,$$

where $s_2 = \dfrac{\beta - 2\beta n_1^{(I)} + p_2^{(I)} - p_2^{(II)}}{2t_2}$, given that $n_1^{(I)} + n_1^{(II)} = 1$. The number of group 2 agents visiting the respective platforms is

$$n_2^{(I)} = \frac{1}{2} - \frac{1}{2}\int_0^1 s_2\sqrt{1+\frac{y^2}{1-s_2^2}}\,dy, \tag{3}$$

$$n_2^{(II)} = \frac{1}{2} + \frac{1}{2}\int_0^1 s_2\sqrt{1+\frac{y^2}{1-s_2^2}}\,dy. \tag{4}$$

We thus arrive at the pricing game for the two platforms I and II, which respectively serve $n_1^{(I)} + n_2^{(I)}$ and $n_1^{(II)} + n_2^{(II)}$ agents of both groups. The payoffs of platforms I and II in the pricing game are determined as follows:

$$H^{(I)}\left(p_1^{(I)}, p_2^{(I)}\right) = n_1^{(I)}\left(p_1^{(I)} - g_1\right) + n_2^{(I)}\left(p_2^{(I)} - g_2\right),$$

$$H^{(II)}\left(p_1^{(II)}, p_2^{(II)}\right) = n_1^{(II)}\left(p_1^{(II)} - g_1\right) + n_2^{(II)}\left(p_2^{(II)} - g_2\right),$$

where g_1 and g_2 are the platforms' costs of serving users from the respective groups. Note that service to any group $i = 1, 2$ in a platform can be provided only if $p_i^{(I,II)} \geq g_i$.

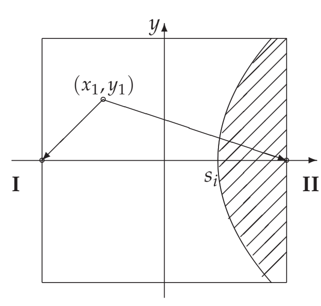

Figure 1. Two-sided platform market on a plane.

3. Identical Platforms and Identical Groups of Agents

Consider the case of identical platforms and identical agents, where the effect of the other group's size on the payoff and the transport costs of agents of the two groups are equal, that is, $\alpha = \beta$ and $t_1 = t_2 = t$, and the service costs of the platforms are equal, $g_1 = g_2 = g$. In this case, it suffices to find the optimal solution for one of the platforms, for example, for the platform I. The price equilibrium is found from the first-order optimality condition $\frac{\partial H^{(I)}}{\partial p_1^{(I)}} = 0$, which has the following form:

$$\frac{\partial H^{(I)}}{\partial p_1^{(I)}} = \frac{\partial n_1^{(I)}}{\partial p_1^{(I)}}\left(p_1^{(I)} - g\right) + n_1^{(I)} + \frac{\partial n_2^{(I)}}{\partial p_1^{(I)}}\left(p_2^{(I)} - g\right). \tag{5}$$

From Equations (1) and (2) we get that

$$\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} = \frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial s_1}{\partial p_1^{(I)}} = \frac{\partial n_1^{(I)}}{\partial s_1}\frac{1}{2t}\left(1 - 2\alpha\frac{\partial n_2^{(I)}}{\partial p_1^{(I)}}\right)$$

$$\frac{\partial n_2^{(I)}}{\partial p_1^{(I)}} = \frac{\partial n_2^{(I)}}{\partial s_2}\frac{\partial s_2}{\partial p_1^{(I)}} = \frac{\partial n_2^{(I)}}{\partial s_2}\frac{1}{2t}\left(-2\alpha\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}}\right),$$

from where we find that

$$\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} = \frac{1}{2t}\frac{\partial n_1^{(I)}}{\partial s_1}\left(1 - \frac{\alpha^2}{t^2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\right)^{-1}, \tag{6}$$

$$\frac{\partial n_2^{(I)}}{\partial p_1^{(I)}} = -\frac{\alpha}{2t^2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\left(1 - \frac{\alpha^2}{t^2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\right)^{-1}. \tag{7}$$

It follows from the symmetry of the problems that in the equilibrium the prices set by both platforms for agents of groups 1 and 2 have to be equal, that is, $p_1^{(I)} = p_1^{(II)} = p_2^{(I)} = p_2^{(II)} = p$. Furthermore, the size of both groups in the platforms should be equal, that is, $n_1^{(I)} = n_1^{(II)} = \frac{1}{2}$ and $n_2^{(I)} = n_2^{(II)} = \frac{1}{2}$. It follows from the equality of the prices and numbers of agents in the platforms that $s_1 = s_2 = 0$ and that

$$\frac{\partial n_1^{(I)}}{\partial s_1} = \frac{\partial n_2^{(I)}}{\partial s_2} = -\frac{1}{2}\int_0^1 \sqrt{1+y^2}\,dy.$$

Denote for brevity $I = \int_0^1 \sqrt{1+y^2}\,dy \approx 1.147$.

Thus, Equations (6) and (7) in the form

$$\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} = -\frac{1}{4t}I\left(1 - \frac{\alpha^2}{4t^2}I^2\right)^{-1},$$

$$\frac{\partial n_2^{(I)}}{\partial p_1^{(I)}} = -\frac{\alpha}{8t^2}I^2\left(1 - \frac{\alpha^2}{4t^2}I^2\right)^{-1},$$

are substituted into (5) to find

$$\frac{\partial H^{(I)}}{\partial p_1^{(I)}} = \left(\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} + \frac{\partial n_2^{(I)}}{\partial p_1^{(I)}} \right)(p-g) + \frac{1}{2} = -\frac{1}{4t} I \left(1 - \frac{\alpha^2}{4t^2} I^2\right)^{-1} \left(1 + \frac{\alpha I}{2t}\right)(p-g) + \frac{1}{2} = 0.$$

Derive from here the price in the equilibrium, which is equal to

$$p = g + \frac{4t^2\left(1 - \frac{\alpha^2}{4t^2} I^2\right)}{I(2t+\alpha I)} = g + \frac{2t}{I} - \alpha = g + \frac{2t}{\int\limits_0^1 \sqrt{1+y^2}\,dy} - \alpha. \tag{8}$$

Observe that competition exists only if $p \geq g$, or if

$$\frac{\alpha}{t} \leq \frac{2}{I} \approx 1.742.$$

Test the sufficient conditions for the maximum of the function $H^I\left(p_1^I, p_2^{II}\right)$. To this end, find the following expressions $A = \frac{\partial^2 H^{(I)}}{\partial^2 p_1^{(I)}}$, $B = \frac{\partial^2 H^{(I)}}{\partial p_1^{(I)} \partial p_2^{(I)}}$ and $C = \frac{\partial^2 H^{(I)}}{\partial^2 p_2^{(I)}}$. Considering the symmetry of the problem we have

$$A = \frac{\partial^2 H^{(I)}}{\partial^2 p_1^{(I)}} = \frac{\partial^2 n_1^{(I)}}{\partial^2 p_1^{(I)}}(p_1^{(I)} - g_1) + \frac{\partial^2 n_2^{(I)}}{\partial^2 p_1^{(I)}}(p_2^{(I)} - g_2) + 2\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} = \left(\frac{\partial^2 n_1^{(I)}}{\partial^2 p_1^{(I)}} + \frac{\partial^2 n_2^{(I)}}{\partial^2 p_1^{(I)}} \right)(p-g) + 2\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}},$$

$$B = \frac{\partial^2 H^{(I)}}{\partial p_1^{(I)} \partial p_2^{(I)}} = \frac{\partial^2 n_1^{(I)}}{\partial p_1^{(I)} \partial p_2^{(I)}}(p_1^{(I)} - g_1) + \frac{\partial^2 n_2^{(I)}}{\partial p_1^{(I)} \partial p_2^{(I)}}(p_2^{(I)} - g_2) + \frac{\partial n_1^{(I)}}{\partial p_2^{(I)}} + \frac{\partial n_2^{(I)}}{\partial p_1^{(I)}} =$$

$$= \left(\frac{\partial^2 n_1^{(I)}}{\partial p_1^{(I)} \partial p_2^{(I)}} + \frac{\partial^2 n_2^{(I)}}{\partial p_1^{(I)} \partial p_2^{(I)}} \right)(p-g) + \frac{\partial n_1^{(I)}}{\partial p_2^{(I)}} + \frac{\partial n_2^{(I)}}{\partial p_1^{(I)}},$$

$$C = \frac{\partial^2 H^{(I)}}{\partial^2 p_2^{(I)}} = \frac{\partial^2 n_1^{(I)}}{\partial^2 p_2^{(I)}}(p_1^{(I)} - g_1) + \frac{\partial^2 n_2^{(I)}}{\partial^2 p_2^{(I)}}(p_2^{(I)} - g_2) + 2\frac{\partial n_2^{(I)}}{\partial p_2^{(I)}} = \left(\frac{\partial^2 n_1^{(I)}}{\partial^2 p_2^{(I)}} + \frac{\partial^2 n_2^{(I)}}{\partial^2 p_2^{(I)}} \right)(p-g) + 2\frac{\partial n_2^{(I)}}{\partial p_2^{(I)}}.$$

It follows from the symmetry of the problem that $\frac{\partial n_1^{(I)}}{\partial p_2^{(I)}} = \frac{\partial n_2^{(I)}}{\partial p_1^{(I)}}$ and $\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} = \frac{\partial n_2^{(I)}}{\partial p_2^{(I)}}$, and also that $\frac{\partial^2 n_1^{(I)}}{\partial p_1^{(I)} \partial p_2^{(I)}} = \frac{\partial^2 n_2^{(I)}}{\partial p_1^{(I)} \partial p_2^{(I)}} = 0$ and $\frac{\partial^2 n_1^{(I)}}{\partial^2 p_1^{(I)}} = \frac{\partial^2 n_1^{(I)}}{\partial^2 p_2^{(I)}} = \frac{\partial^2 n_2^{(I)}}{\partial^2 p_2^{(I)}} = 0$. Therefore

$$A = C = 2\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} = -\frac{1}{2t} I \left(1 - \frac{\alpha^2}{4t^2} I^2\right)^{-1}, \qquad B = 2\frac{\partial n_1^{(I)}}{\partial p_2^{(I)}} = -\frac{\alpha}{4t^2} I^2 \left(1 - \frac{\alpha^2}{4t^2} I^2\right)^{-1}.$$

Since the expression $AC - B^2 = \frac{I^2}{4t^2}\left(1 - \frac{\alpha^2 I^2}{4t^2}\right)^{-2}\left[1 + \frac{\alpha^2 I^2}{4t^2}\right] > 0$ when $\frac{\alpha}{t} < \frac{2}{I}$ and since $A < 0$, the function $H^{(I)}\left(p_1^{(I)}, p_2^{(II)}\right)$ has a maximum at (p, p), where p is found from Formula (8).

Hence, the following theorem is valid for the case of identical platforms and identical groups of agents.

Theorem 1. *In the Hotelling model for a two-sided platform market on a plane with identical groups of agents, competitive service will take place given that*

$$\frac{\alpha}{t} < \frac{2}{\int\limits_0^1 \sqrt{1+y^2}\,dy},$$

and the price in the equilibrium in this case will have the form (8).

4. Identical Platforms and Different Groups of Agents

Suppose agents in the two groups differ in their parameters. Remark that owing to their symmetric location in the market the platforms are identical. In this case, the price equilibrium is derived from the first-order optimality condition for each group of agents in one of the platforms, for example, platform I, which has the form $\frac{\partial H^{(I)}}{\partial p_i^{(I)}} = 0, i = 1,2.$

The equation $\frac{\partial H^{(I)}}{\partial p_1^{(I)}} = 0$ has the form

$$\frac{\partial H^{(I)}}{\partial p_1^{(I)}} = \frac{\partial n_1^{(I)}}{\partial p_1^{(I)}}\left(p_1^{(I)} - g_1\right) + n_1^{(I)} + \frac{\partial n_2^{(I)}}{\partial p_1^{(I)}}\left(p_2^{(I)} - g_2\right). \tag{9}$$

From the Equations (1)–(4), we find that

$$\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} = \frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial s_1}{\partial p_1^{(I)}} = \frac{\partial n_1^{(I)}}{\partial s_1}\frac{1}{2t_1}\left(1 - 2\alpha\frac{\partial n_2^{(I)}}{\partial p_1^{(I)}}\right),$$

$$\frac{\partial n_2^{(I)}}{\partial p_1^{(I)}} = \frac{\partial n_2^{(I)}}{\partial s_2}\frac{\partial s_2}{\partial p_1^{(I)}} = \frac{\partial n_2^{(I)}}{\partial s_2}\frac{1}{2t_2}\left(-2\beta\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}}\right),$$

from where it follows that

$$\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} = \frac{1}{2t_1}\frac{\partial n_1^{(I)}}{\partial s_1}\left(1 - \frac{\alpha\beta}{t_1 t_2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\right)^{-1},$$

$$\frac{\partial n_2^{(I)}}{\partial p_1^{(I)}} = -\frac{\beta}{2t_1 t_2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\left(1 - \frac{\alpha\beta}{t_1 t_2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\right)^{-1}.$$

It follows from the symmetry of the problem that in the equilibrium the prices set by the platforms for each group of customers should be equal, although they may differ for different groups, that is, $p_1^{(I)} = p_1^{(II)} = p_1, p_2^{(I)} = p_2^{(II)} = p_2$. Furthermore, the size of both groups on both platforms should be equal, that is, $n_1^{(I)} = n_1^{(II)} = \frac{1}{2}$ and $n_2^{(I)} = n_2^{(II)} = \frac{1}{2}$. Hence $s_1 = s_2 = 0$ and

$$\frac{\partial n_1^{(I)}}{\partial s_1} = \frac{\partial n_2^{(I)}}{\partial s_2} = -\frac{1}{2}I.$$

It follows that the expression (9) has the form

$$\frac{\partial H^{(I)}}{\partial p_1^{(I)}} = \frac{\partial n_1^{(I)}}{\partial p_1^{(I)}}(p_1 - g_1) + \frac{\partial n_2^{(I)}}{\partial p_1^{(I)}}(p_2 - g_2) + \frac{1}{2}$$

$$= -\frac{1}{8t_1t_2} I \left(1 - \frac{\alpha\beta}{4t_1t_2} I^2\right)^{-1} (2t_2(p_1 - g_1) + \beta I(p_2 - g_2)) + \frac{1}{2} = 0. \tag{10}$$

Similar reasoning for equation $\frac{\partial H^{(I)}}{\partial p_2^{(I)}} = 0$, which has the form

$$\frac{\partial H^{(I)}}{\partial p_2^{(I)}} = \frac{\partial n_1^{(I)}}{\partial p_2^{(I)}} \left(p_1^{(I)} - g_1\right) + \frac{\partial n_2^{(I)}}{\partial p_2^{(I)}} \left(p_2^{(I)} - g_2\right) + n_2^{(I)},$$

leads to the equation

$$\frac{\partial H^{(I)}}{\partial p_2^{(I)}} = \frac{\partial n_1^{(I)}}{\partial p_2^{(I)}} \left(p_1^{(I)} - g_1\right) + \frac{\partial n_2^{(I)}}{\partial p_2^{(I)}} \left(p_2^{(I)} - g_2\right) + \frac{1}{2}$$

$$= -\frac{1}{8t_1t_2} I \left(1 - \frac{\alpha\beta}{4t_1t_2} I^2\right)^{-1} (\alpha I(p_1 - g_1) + 2t_1(p_2 - g_2)) + \frac{1}{2} = 0. \tag{11}$$

From (10) and (11), we find the prices for the groups in the equilibrium

$$p_1 = g_1 + \left(\frac{2}{I} t_1 - \beta\right), \qquad p_2 = g_2 + \left(\frac{2}{I} t_2 - \alpha\right). \tag{12}$$

Hence, the following theorem is valid for the case with identical platforms and different groups of agents.

Theorem 2. *In the Hotelling model for a two-sided platform market on the plane with identical platforms, competitive service will take place given that*

$$\max\left\{\frac{\alpha}{t_2}, \frac{\beta}{t_1}\right\} \leq \frac{2}{I} \approx 1.742,$$

and the service prices in the equilibrium in this case will have the form (12).

5. Different Platforms and Identical Groups of Agents

Consider the model of a two-sided market with different platforms located in points $(a, 0)$ and $(b, 0)$ of the square S, and with identical groups of agents, which have the same parameters of the power of the group size effect and the transport costs. Assume for definiteness that $a \leq b$. In this case, the utility for group 1 agents from visiting the platform I in the point $(a, 0)$ has the form

$$u_1^{(I)} = \alpha \cdot n_2^{(I)} - p_1^{(I)} - \sqrt{(x_1 - a)^2 + y_1^2} \cdot t,$$

while from visiting the platform II in the point $(b, 0)$ it is

$$u_1^{(II)} = \alpha \cdot n_2^{(II)} - p_1^{(II)} - \sqrt{(x_1 - b)^2 + y_1^2} \cdot t.$$

The boundary between regions of the market for group 1 is determined from the equation $u_1^{(I)} = u_1^{(II)}$, which is solved to get that

$$\frac{\left(x_1 - \dfrac{a+b}{2}\right)^2}{s_1^2} - \frac{y_1^2}{\left(\dfrac{b-a}{2}\right)^2 - s_1^2} = 1,$$

where $s_1 = \dfrac{\alpha(n_2^{(II)} - n_2^{(I)}) + p_1^{(I)} - p_1^{(II)}}{2t}$ and $n_1^{(I)} + n_1^{(II)} = 1$. The number of group 1 agents visiting the platforms I and II, respectively, is found from the formulas

$$n_1^{(I)} = \frac{1}{2} + \frac{a+b}{4} - \frac{1}{2}\int_0^1 s_1 \sqrt{1 + \frac{y^2}{\left(\dfrac{b-a}{2}\right)^2 - s_1^2}}\, dy,$$

$$n_1^{(II)} = \frac{1}{2} - \frac{a+b}{4} + \frac{1}{2}\int_0^1 s_1 \sqrt{1 + \frac{y^2}{\left(\dfrac{b-a}{2}\right)^2 - s_1^2}}\, dy.$$

The utilities of group 2 agents from visiting platforms I and II are determined similarly, and equal, respectively

$$u_2^{(I)} = \alpha \cdot n_1^{(I)} - p_2^{(I)} - \sqrt{(x_2 - a)^2 + y_2^2} \cdot t,$$
$$u_2^{(II)} = \alpha \cdot n_1^{(II)} - p_2^{(II)} - \sqrt{(x_2 - b)^2 + y_2^2} \cdot t,$$

and the boundary between market regions for group 2 is found from the equation of the form

$$\frac{\left(x_2 - \dfrac{a+b}{2}\right)^2}{s_2^2} - \frac{y_2^2}{\left(\dfrac{b-a}{2}\right)^2 - s_2^2} = 1,$$

where $s_2 = \dfrac{\alpha(n_1^{(II)} - n_1^{(I)}) + p_2^{(I)} - p_2^{(II)}}{2t}$ and $n_2^{(I)} + n_2^{(II)} = 1$. The number of group 2 agents visiting the respective platforms is

$$n_2^{(I)} = \frac{1}{2} + \frac{a+b}{4} - \frac{1}{2}\int_0^1 s_2 \sqrt{1 + \frac{y^2}{\left(\dfrac{b-a}{2}\right)^2 - s_2^2}}\, dy,$$

$$n_2^{(II)} = \frac{1}{2} - \frac{a+b}{4} + \frac{1}{2}\int_0^1 s_2 \sqrt{1 + \frac{y^2}{\left(\dfrac{b-a}{2}\right)^2 - s_2^2}}\, dy.$$

Since the groups of agents in this model are identical, the price equilibrium can be found from the first-order optimality condition for any one of the groups of agents (e.g., the first one) in both platforms:

$$\frac{\partial H^{(I)}}{\partial p_1^{(I)}} = 0, \qquad \frac{\partial H^{(II)}}{\partial p_1^{(II)}} = 0.$$

Therefore,

$$\begin{cases} \dfrac{\partial H^{(I)}}{\partial p_1^{(I)}} = \dfrac{\partial n_1^{(I)}}{\partial p_1^{(I)}} \left(p_1^{(I)} - g\right) + n_1^{(I)} + \dfrac{\partial n_2^{(I)}}{\partial p_1^{(I)}} \left(p_2^{(I)} - g\right) = 0, \\ \dfrac{\partial H^{(II)}}{\partial p_1^{(II)}} = \dfrac{\partial n_1^{(II)}}{\partial p_1^{(II)}} \left(p_1^{(II)} - g\right) + n_1^{(II)} + \dfrac{\partial n_2^{(II)}}{\partial p_1^{(II)}} \left(p_2^{(II)} - g\right) = 0. \end{cases} \tag{13}$$

The derivatives $\dfrac{\partial n_1^{(I)}}{\partial p_1^{(I)}}$ and $\dfrac{\partial n_2^{(I)}}{\partial p_1^{(I)}}$ are taken from the Formulas (6) and (7). The expressions for the derivatives $\dfrac{\partial n_1^{(II)}}{\partial p_1^{(II)}}$ and $\dfrac{\partial n_2^{(II)}}{\partial p_1^{(II)}}$ are found in a similar way

$$\frac{\partial n_1^{(II)}}{\partial p_1^{(II)}} = -\frac{1}{2t}\frac{\partial n_1^{(II)}}{\partial s_1}\left(1 - \frac{\alpha^2}{t^2}\frac{\partial n_1^{(II)}}{\partial s_1}\frac{\partial n_2^{(II)}}{\partial s_2}\right)^{-1}, \tag{14}$$

$$\frac{\partial n_2^{(II)}}{\partial p_1^{(II)}} = -\frac{\alpha}{2t^2}\frac{\partial n_1^{(II)}}{\partial s_1}\frac{\partial n_2^{(II)}}{\partial s_2}\left(1 - \frac{\alpha^2}{t^2}\frac{\partial n_1^{(II)}}{\partial s_1}\frac{\partial n_2^{(II)}}{\partial s_2}\right)^{-1}. \tag{15}$$

It follows from the symmetry of the problem that in the equilibrium the prices in the platforms should be equal for each group of customers, although they may differ for different groups, that is, $p_1^{(I)} = p_2^{(I)} = p^{(I)}$, $p_1^{(II)} = p_2^{(II)} = p^{(II)}$, and the size of the groups in each platform is equal, that is, $n_1^{(I)} = n_2^{(I)} = n^{(I)}$ and $n_1^{(II)} = n_2^{(II)} = 1 - n^{(I)}$, wherefore

$$s = s_1 = s_2 = \frac{\alpha - 2\alpha n^{(I)} + p^{(I)} - p^{(II)}}{2t}, \tag{16}$$

and

$$\frac{\partial n_1^{(I)}}{\partial s_1} = \frac{\partial n_2^{(I)}}{\partial s_2} = -\frac{\partial n_1^{(II)}}{\partial s_1} = -\frac{\partial n_2^{(II)}}{\partial s_2} = -\frac{1}{2}\int_0^1 \left(\sqrt{1 + \frac{y^2}{(\frac{b-a}{2})^2 - s^2}} + \frac{s^2 y^2}{((\frac{b-a}{2})^2 - s^2)^2\sqrt{1 + \frac{y^2}{(\frac{b-a}{2})^2 - s^2}}}\right) dy. \tag{17}$$

Denote the expression (17) as the function $D(s)$. Then, it follows from (6)–(7) and (14)–(15) that

$$\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} = \frac{\partial n_1^{(II)}}{\partial p_1^{(II)}} = \frac{1}{2t}D(s)\left(1 - \frac{\alpha^2}{t^2}D^2(s)\right)^{-1},$$

$$\frac{\partial n_2^{(I)}}{\partial p_1^{(I)}} = \frac{\partial n_2^{(II)}}{\partial p_1^{(II)}} = -\frac{\alpha}{2t^2}D^2(s)\left(1 - \frac{\alpha^2}{t^2}D^2(s)\right)^{-1}.$$

The resultant expressions are substituted into system (13):

$$\begin{cases} \frac{1}{2t}D(s)\left(1 - \frac{\alpha^2}{t^2}D^2(s)\right)^{-1}(p^{(I)} - g) - \frac{\alpha}{2t^2}D^2(s)\left(1 - \frac{\alpha^2}{t^2}D^2(s)\right)^{-1}(p^{(I)} - g) + n^{(I)} = 0, \\ \frac{1}{2t}D(s)\left(1 - \frac{\alpha^2}{t^2}D^2(s)\right)^{-1}(p^{(II)} - g) - \frac{\alpha}{2t^2}D^2(s)\left(1 - \frac{\alpha^2}{t^2}D^2(s)\right)^{-1}(p^{(II)} - g) + 1 - n^{(I)} = 0, \end{cases} \tag{18}$$

where

$$n^{(I)} = \frac{1}{2} + \frac{a+b}{4} - \frac{1}{2}\int_0^1 s\sqrt{1 + \frac{y^2}{\left(\frac{b-a}{2}\right)^2 - s^2}}\,dy. \tag{19}$$

From system (18) we find that

$$\begin{cases} tD(s)\left(p^{(I)} - g\right) - \alpha D^2(s)\left(p^{(I)} - g\right) + 2n^{(I)}(t^2 - \alpha^2 D^2(s)) = 0, \\ tD(s)\left(p^{(II)} - g\right) - \alpha D^2(s)\left(p^{(II)} - g\right) + 2(1 - n^{(I)})(t^2 - \alpha^2 D^2(s)) = 0, \end{cases}$$

and subsequently the optimal prices are found using the formulas

$$\begin{cases} p^{(I)} = g - 2n^{(I)} \cdot \dfrac{t + \alpha D(s)}{D(s)}, \\ p^{(II)} = g - 2(1 - n^{(I)}) \cdot \dfrac{t + \alpha D(s)}{D(s)}. \end{cases} \tag{20}$$

Theorem 3. *In the Hotelling model for a two-sided platform market on the plane with identical agents and different platforms, service prices in the equilibrium satisfy Equations (16), (19) and (20).*

From Equations (16) and (20), we find that

$$p^{(I)} - p^{(II)} = 2(1 - 2n^{(I)}) \cdot \frac{t + \alpha D(s)}{D(s)} = 2ts - \alpha + 2\alpha n^{(I)},$$

wherefore

$$n^{(I)} = \frac{1}{2} - \frac{tsD(s)}{3\alpha D(s) + 2t}. \tag{21}$$

Together with (19), this leads to the equation for finding the parameter s in the equilibrium:

$$\frac{tsD(s)}{3\alpha D(s) + 2t} = \frac{1}{2}\int_0^1 s\sqrt{1 + \frac{y^2}{\left(\frac{b-a}{2}\right)^2 - s^2}}\,dy - \frac{a+b}{4}. \tag{22}$$

Thus, optimal prices in the pricing game can be found by finding the parameter s from Equation (22), then finding the size of the group of agents in the first platform using Formula (21), and substituting the resultant parameters into Formula (20). The results of the numerical calculations are shown in Table 1.

Table 1. Numerical results for $g = 0$, $\alpha = 1$, $t = 1$.

a	b	s	n_1	n_2	p^I	p^{II}	H^I	H^{II}
−1	1	0	0.5	0.5	0.742	0.742	0.742	0.742
−1	0.9	−0.009	0.480	0.520	0.692	0.749	0.692	0.749
−1	0.8	−0.016	0.459	0.541	0.640	0.754	0.640	0.754
−1	0.75	−0.019	0.449	0.551	0.614	0.754	0.614	0.754
−1	0.7	−0.021	0.438	0.562	0.586	0.753	0.586	0.753
−1	0.6	−0.021	0.424	0.576	0.542	0.736	0.542	0.736

The table shows that when the second platform is shifted to the center, s shifts to the left, which reduces the market for platform I. This means that customers in the center are getting closer to platform II.

6. Optimal Location of Platforms

Observe the results of numerical calculations in Table 1. When the location of the first platform I is fixed in the position $a = -1$ and the position of the second platform b changes from 1 to 0.6 the second player's payoff $H^{(II)}$ first grows and then declines. The maximum is reached in the position $b = 0.8$. This means that if the position of the first platform is fixed, the second player should put his/her platform in the position $b = 0.8$.

If, however, the second player puts his/her platform in that position, the first player may also relocate. Table 2 shows the calculations simulating this situation. Let the players switch roles. The first player is now in the fixed position $a = -0.8$, while the second player's position is variable. It follows from the numerical calculations shown in Table 2 that his/her optimal position in this situation is $b = 0.9$. Continuing this sequence of the best responses, we arrive at the equilibrium location of platforms in the market $a = -0.85, b = 0.85$.

Table 2. Numerical results for $g = 0, \alpha = 1, t = 1$.

a	b	s	n_1	n_2	p^I	p^{II}	H^I	H^{II}
−0.8	1	0.016	0.541	0.459	0.754	0.640	0.754	0.640
−0.8	0.9	0.007	0.521	0.479	0.698	0.642	0.698	0.642
−0.8	0.8	0	0.5	0.5	0.640	0.640	0.640	0.640
−0.8	0.7	−0.005	0.478	0.522	0.579	0.632	0.579	0.632
−0.8	0.6	−0.006	0.454	0.546	0.515	0.620	0.615	0.620

7. Different Platforms and Different Groups of Agents

Consider the case where different platforms serve different groups of agents, which are described by the parameters $\alpha \neq \beta$ and $t_1 \neq t_2$. In this case, the payoffs of both platforms have the form

$$\begin{cases} H^{(I)}\left(p_1^{(I)}, p_2^{(I)}\right) = n_1^{(I)}\left(p_1^{(I)} - g_1\right) + n_2^{(I)}\left(p_2^{(I)} - g_2\right), \\ H^{(II)}\left(p_1^{(II)}, p_2^{(II)}\right) = \left(1 - n_1^{(I)}\right)\left(p_1^{(II)} - g_1\right) + \left(1 - n_2^{(I)}\right)\left(p_2^{(I)} - g_2\right). \end{cases}$$

The price equilibrium for the platforms is found from the first-order optimality condition

$$\frac{\partial H^{(I)}}{\partial p_i^{(I)}} = 0, \qquad \frac{\partial H^{(II)}}{\partial p_i^{(II)}} = 0, \qquad i = 1, 2.$$

Therefore,

$$\begin{cases} \dfrac{\partial H^{(I)}}{\partial p_1^{(I)}} = \dfrac{\partial n_1^{(I)}}{\partial p_1^{(I)}}\left(p_1^{(I)} - g_1\right) + n_1^{(I)} + \dfrac{\partial n_2^{(I)}}{\partial p_1^{(I)}}\left(p_2^{(I)} - g_2\right) = 0, \\ \dfrac{\partial H^{(I)}}{\partial p_2^{(I)}} = \dfrac{\partial n_1^{(I)}}{\partial p_2^{(I)}}\left(p_1^{(I)} - g_1\right) + n_2^{(I)} + \dfrac{\partial n_2^{(I)}}{\partial p_2^{(I)}}\left(p_2^{(I)} - g_2\right) = 0, \\ \dfrac{\partial H^{(II)}}{\partial p_1^{(II)}} = -\dfrac{\partial n_1^{(I)}}{\partial p_1^{(II)}}\left(p_1^{(II)} - g_1\right) + 1 - n_1^{(II)} - \dfrac{\partial n_2^{(I)}}{\partial p_1^{(II)}}\left(p_2^{(II)} - g_2\right) = 0, \\ \dfrac{\partial H^{(II)}}{\partial p_2^{(II)}} = -\dfrac{\partial n_1^{(I)}}{\partial p_2^{(II)}}\left(p_1^{(II)} - g_1\right) + 1 - n_2^{(II)} - \dfrac{\partial n_2^{(I)}}{\partial p_2^{(II)}}\left(p_2^{(II)} - g_2\right) = 0, \end{cases} \tag{23}$$

where the respective derivatives satisfy the following equations:

$$\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} = -\frac{\partial n_1^{(I)}}{\partial p_1^{(II)}} = \frac{1}{2t_1}\frac{\partial n_1^{(I)}}{\partial s_1}\left(1 - \frac{\alpha\beta}{t_1 t_2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\right)^{-1}, \tag{24}$$

$$\frac{\partial n_2^{(I)}}{\partial p_1^{(I)}} = \frac{\partial n_2^{(I)}}{\partial p_1^{(II)}} = -\frac{\beta}{2t_1t_2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\left(1-\frac{\alpha\beta}{t_1t_2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\right)^{-1}, \tag{25}$$

$$\frac{\partial n_1^{(I)}}{\partial p_2^{(I)}} = -\frac{\partial n_1^{(I)}}{\partial p_2^{(II)}} = -\frac{\alpha}{2t_1t_2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\left(1-\frac{\alpha\beta}{t_1t_2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\right)^{-1}, \tag{26}$$

$$\frac{\partial n_2^{(I)}}{\partial p_2^{(I)}} = -\frac{\partial n_2^{(I)}}{\partial p_2^{(II)}} = \frac{1}{2t_2}\frac{\partial n_2^{(I)}}{\partial s_2}\left(1-\frac{\alpha\beta}{t_1t_2}\frac{\partial n_1^{(I)}}{\partial s_1}\frac{\partial n_2^{(I)}}{\partial s_2}\right)^{-1}. \tag{27}$$

The following notations are introduced:

$$D(s_1) = \frac{\partial n_1^{(I)}}{\partial s_1} = -\frac{1}{2}\int_0^1 \left(\sqrt{1+\frac{y^2}{(\frac{b-a}{2})^2 - s_1^2}} + \frac{s_1^2 y^2}{((\frac{b-a}{2})^2 - s_1^2)^2\sqrt{1+\frac{y^2}{(\frac{b-a}{2})^2 - s_1^2}}}\right) dy, \tag{28}$$

$$D(s_2) = \frac{\partial n_2^{(I)}}{\partial s_2} = -\frac{1}{2}\int_0^1 \left(\sqrt{1+\frac{y^2}{(\frac{b-a}{2})^2 - s_2^2}} + \frac{s_2^2 y^2}{((\frac{b-a}{2})^2 - s_2^2)^2\sqrt{1+\frac{y^2}{(\frac{b-a}{2})^2 - s_2^2}}}\right) dy. \tag{29}$$

It follows from Formulas (24)–(29) that

$$\frac{\partial n_1^{(I)}}{\partial p_1^{(I)}} = \frac{1}{2t_1}D(s_1)\left(1-\frac{\alpha\beta}{t_1t_2}D(s_1)D(s_2)\right)^{-1}, \qquad \frac{\partial n_2^{(I)}}{\partial p_1^{(I)}} = -\frac{\beta}{2t_1t_2}D(s_1)D(s_2)\left(1-\frac{\alpha\beta}{t_1t_2}D(s_1)D(s_2)\right)^{-1},$$

$$\frac{\partial n_1^{(I)}}{\partial p_2^{(I)}} = -\frac{\alpha}{2t_1t_2}D(s_1)D(s_2)\left(1-\frac{\alpha\beta}{t_1t_2}D(s_1)D(s_2)\right)^{-1}, \qquad \frac{\partial n_2^{(I)}}{\partial p_2^{(I)}} = \frac{1}{2t_2}D(s_2)\left(1-\frac{\alpha\beta}{t_1t_2}D(s_1)D(s_2)\right)^{-1},$$

and the system of Equation (23) is transformed into the following:

$$\begin{cases} t_2D(s_1)\left(p_1^{(I)}-g_1\right) + 2t_1t_2n_1^{(I)}\left(1-\frac{\alpha\beta}{t_1t_2}D(s_1)D(s_2)\right) - \beta D(s_1)D(s_2)\left(p_2^{(I)}-g_2\right) = 0, \\ -\alpha D(s_1)D(s_2)\left(p_1^{(I)}-g_1\right) + 2t_1t_2n_2^{(I)}\left(1-\frac{\alpha\beta}{t_1t_2}D(s_1)D(s_2)\right) + t_1D(s_2)\left(p_2^{(I)}-g_2\right) = 0, \\ t_2D(s_1)\left(p_1^{(II)}-g_1\right) + 2t_1t_2\left(1-n_1^{(I)}\right)\left(1-\frac{\alpha\beta}{t_1t_2}D(s_1)D(s_2)\right) - \beta D(s_1)D(s_2)\left(p_2^{(II)}-g_2\right) = 0, \\ -\alpha D(s_1)D(s_2)\left(p_1^{(II)}-g_1\right) + 2t_1t_2\left(1-n_2^{(I)}\right)\left(1-\frac{\alpha\beta}{t_1t_2}D(s_1)D(s_2)\right) + t_1D(s_2)\left(p_2^{(II)}-g_2\right) = 0, \end{cases} \tag{30}$$

where

$$s_1 = \frac{\alpha - 2\alpha n_2^{(I)} + p_1^{(I)} - p_1^{(II)}}{2t_1}, \tag{31}$$

$$s_2 = \frac{\beta - 2\beta n_1^{(I)} + p_2^{(I)} - p_2^{(II)}}{2t_2}, \tag{32}$$

$$n_1^{(I)} = \frac{1}{2} + \frac{a+b}{4} - \frac{1}{2}\int_0^1 s_1 \sqrt{1 + \frac{y^2}{\left(\frac{b-a}{2}\right)^2 - s_1^2}}\, dy, \tag{33}$$

$$n_2^{(I)} = \frac{1}{2} + \frac{a+b}{4} - \frac{1}{2}\int_0^1 s_2 \sqrt{1 + \frac{y^2}{\left(\frac{b-a}{2}\right)^2 - s_2^2}}\, dy. \tag{34}$$

The system of Equation (30) yields formulas for finding the optimal prices in the equilibrium:

$$\begin{cases} p_1^{(I)} = g_1 - \dfrac{2\left(\beta D(s_1) n_2^{(I)} + t_1 n_1^{(I)}\right)}{D(s_1)}, \\ p_2^{(I)} = g_2 - \dfrac{2\left(\alpha D(s_2) n_1^I + t_2 n_2^I\right)}{D(s_2)}, \\ p_1^{(II)} = g_1 - \dfrac{2\left(\beta D(s_1)(1 - n_2^I) + t_1(1 - n_1^I)\right)}{D(s_1)}, \\ p_2^{(II)} = g_2 - \dfrac{2\left(\alpha D(s_2)(1 - n_1^I) + t_2(1 - n_2^I)\right)}{D(s_2)}. \end{cases} \tag{35}$$

Thus, the following theorem is valid for the case of different platforms and different agents.

Theorem 4. *In the Hotelling model for a two-sided platform market on the plane with different agents and different platforms, service prices in the equilibrium satisfy Equations (30)–(35).*

From Equations (31), (32) and (35), we have that

$$p_1^{(I)} - p_1^{(II)} = \frac{2\left(\beta D(s_1)\left(1 - 2n_2^I\right) + t_1\left(1 - 2n_1^I\right)\right)}{D(s_1)} = 2s_1 t_1 - \alpha + 2\alpha n_2^{(I)},$$

$$p_2^{(I)} - p_2^{(II)} = \frac{2\left(\alpha D(s_2)\left(1 - 2n_1^I\right) + t_2\left(1 - 2n_2^I\right)\right)}{D(s_2)} = 2s_2 t_2 - \beta + 2\beta n_2^{(I)},$$

from where the number of agents in the first platform is determined using the formulas

$$n_1^{(I)} = \frac{1}{2} - \frac{t_2 D(s_1)\left(s_2 D(s_2)(2\beta + \alpha) - 2s_1 t_1\right)}{D(s_1) D(s_2)(2\beta + \alpha)(2\alpha + \beta) - 4t_1 t_2}, \tag{36}$$

$$n_2^I = \frac{1}{2} - \frac{t_1 D(s_2)\left(s_1 D(s_1)(2\alpha + \beta) - 2s_2 t_2\right)}{D(s_1) D(s_2)(2\beta + \alpha)(2\alpha + \beta) - 4t_1 t_2}. \tag{37}$$

The Equations (36) and (37) together with (33) and (34) lead to the system of equations for finding the parameters s_1 and s_2 in the equilibrium:

$$\begin{cases} \dfrac{t_2 D(s_1)\left(s_2 D(s_2)(2\beta + \alpha) - 2s_1 t_1\right)}{D(s_1) D(s_2)(2\beta + \alpha)(2\alpha + \beta) - 4t_1 t_2} = \dfrac{a+b}{4} - \dfrac{1}{2}\int_0^1 s_1 \sqrt{1 + \dfrac{y^2}{\left(\frac{b-a}{2}\right)^2 - s_1^2}}\, dy, \\ \dfrac{t_1 D(s_2)\left(s_1 D(s_1)(2\alpha + \beta) - 2s_2 t_2\right)}{D(s_1) D(s_2)(2\beta + \alpha)(2\alpha + \beta) - 4t_1 t_2} = \dfrac{a+b}{4} - \dfrac{1}{2}\int_0^1 s_2 \sqrt{1 + \dfrac{y^2}{\left(\frac{b-a}{2}\right)^2 - s_2^2}}\, dy. \end{cases} \tag{38}$$

Thus, the values of the parameters s_1 and s_2 are found from the system of Equation (38), then the sizes of the first and second groups on the first platform are found using the Formulas (36) and (37), and the optimal prices the platforms charge different groups are determined by the Formula (35). The results of numerical calculations are shown in Table 3 for different values of the parameters α and β.

Table 3. Numerical results for $\alpha = 1$ and $\beta = 0.7$ when $t_1 = t_2 = 1$ and $g_1 = g_2 = 0$.

a	b	s_1	s_2	n_1^I	n_1^{II}	n_2^I	n_2^{II}	p_1^I	p_1^{II}	p_2^I	p_2^{II}	H^I	H^{II}
−1	1	0	0	0.5	0.5	0.5	0.5	1.04	1.04	0.742	0.742	0.892	0.892
−1	0.9	0.057	−0.064	0.508	0.492	0.488	0.512	1.064	0.973	0.660	0.776	0.863	0.875
−1	0.8	0.113	−0.126	0.518	0.482	0.476	0.524	1.076	0.897	0.571	0.797	0.828	0.851
−1	0.7	0.167	−0.182	0.525	0.475	0.465	0.535	1.073	0.809	0.471	0.800	0.782	0.812
−1	0.6	0.217	−0.235	0.534	0.466	0.454	0.546	1.051	0.707	0.356	0.779	0.723	0.754
−1	0.5	0.265	−0.289	0.544	0.456	0.444	0.556	1.000	0.583	0.220	0.725	0.642	0.669

8. Conclusions

This paper investigated the structure of prices in the equilibrium in a two-sided market on the plane. Service is provided in two platforms for heterogeneous agents and endogenous demand. Agents are uniformly distributed in a square. The target functions are the platform's maximum profit. The exact expressions were found for the prices which depend on the cost structure and network externalities. Analytical expressions were derived for equilibrium prices. The numerical experiments for finding the equilibrium prices in the market for different parameters of the problem and different locations of platforms in the market were described. The study was concerned with the case where agents of both groups were uniformly distributed over the market. We plan to study this problem in application to different distributions of agents in space.

The paper demonstrated the validity of the problem of the optimal platform location in the market. In the future, the optimal platform location problem will be considered in more detail for various distributions of agents over the market and for a greater number of competing platforms.

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IDP-Core: Novel Cooperative Solution for Differential Games

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Abstract: IDP-core is a new cooperative solution for dynamic and differential games. A novel approach of constructing solutions for dynamic and differential games was employed in which the time consistency property was used as the main axiom property for the cooperative solution. Another new and important approach used for constructing IDP-core is the IDP dominance, which allows to select undominated imputation distribution procedures and construct the cooperative solution or imputation set. This approach shows the potential of using the time consistency property as the main axiom for solutions in various fields such as Social Choice and Mechanism Design. The overall procedure for defining the cooperative solution is also new since IDP-core was constructed using imputation distribution procedures but not by using imputations directly.

Keywords: differential games; cooperative differential games; time consistency; IDP-core; IDP dominance

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1. Introduction

The theory of cooperative games examines how optimal parameters of cooperative and strategic agreements are to be determined. The main problem in the theory of cooperative games with transferable utilities is to determine the allocation procedure for total payoff in cases when all players cooperate. The rule of how to allocate cooperative payoff among the players is called the imputation. In the theory of cooperative games with non-transferable utilities, the main problem is to define agreement on strategies or a game outcome favorable to all players.

Within the framework of classical cooperative game theory with transferable utilities, numerous cooperative solutions or allocation rules were studied. One of them is the Core. The concept of Core was proposed by D. Gillis [1], which is a generalization of the contractual Edgeworth curve [2]. Edgeworth described a market with two products and two participantsp; here, the Core is defined as a part of the Pareto front. The Core is the set of undominated imputations, each of which can be used as a solution in the game.

It is important to study the non-emptiness property of a cooperative solution, which is to determine the conditions under which the cooperative solution is not empty since its applicability depends on the wideness of the class of games to which this solution can be applied. G. Scarf [3] showed that the Core is not empty for the class of convex games in characteristic function form. Characteristic function is a function of coalition or subset of players in the game, which shows the profit of coalition. Generalization of Scharf results can be found in the paper of L. Biller [4] and Shapley [5]. Necessary and sufficient conditions for the non-emptiness of Core were formulated by Bondareva [6] and Shapley [7], where the main role of proof is the concept of a balanced game. Unfortunately, based on this concept it is impossible to apply a constructive method for choosing the specific imputations from the Core.

V. Zakharov in [8] proposed the necessary and sufficient conditions for the non-emptiness of Core, which simplify the test for a single-point solution (imputation), such as the Shapley value, Banzhaf power index and others, whenever they belong to the Core. In [9,10] based on of this approach, geometric properties for several cooperative solutions were investigated. This approach implies that the non-emptiness property of Core can be formulated by a linear programming problem constructed using the values of the characteristic function.

It is also important to construct cooperative solutions for a class of dynamic and differential games. Solutions for such models can be used for modeling cooperative and strategic agreements where conditions are defined over a long time interval [11,12]. The theory of differential games was developed as a separate class of applied mathematics in the 1950s. One of the first works in the field of differential games is the work of R. Isaacs [13] in which the notions of state, controls, and the problem of aircraft interception by a guided missile were formulated, and a fundamental equation for defining the solution derived. A comprehensive description of dynamic cooperative games is presented in [14].

A natural approach for researching cooperative differential games is an attempt at transferring the results of classical static cooperative theory [15] to the theory of differential games. However, in order to use the results of classical theory, it is additionally necessary to study the time consistency and strong time consistency properties of cooperative solutions. Time consistency of cooperative solution is the property that shows that for the players it is not beneficial to deviate from the chosen cooperative solution during the game. The use of time-inconsistent cooperative solutions in the field of economics, ecology, and management makes these solutions unfeasible because players might find it profitable to reconsider the cooperative solution. The notion of the cooperative solution's time consistency was first formulated mathematically by L.A. Petrosyan in 1977 [16]. In [17] a method was proposed to construct time-consistent cooperative solutions using a special payment scheme, called the imputation distribution procedure (IDP). The notion of strong time consistency was formulated in [18]. Recent papers [19–22] are devoted to the study of the time consistency property of cooperative solutions.

In order to solve the time inconsistency problem in a classical cooperative solution, an imputation distribution procedure should be used. However, there exists another, rather new approach that allows for constructing time-consistent cooperative solutions. This approach uses the time consistency property as a basic axiomatic property for defining the cooperative solution. This approach is the subject of this paper and carries an innovative character. It is important to notice that the further use of time consistency property for dynamic cooperative games, Social Choice, and Mechanisms Design, as an axiom, is promising. Another important property considered in this paper is the IDP dominance property. According to this property, the corresponding cooperative solution is constructed using the imputation distribution procedures, which are undominated. We say that the IDP is undominated by coalition S if there does not exist another IDP, coalition S, and time instant such that the instant payments corresponding to IDP are higher for players from coalition S at a given time instant than in the current IDP.

In the paper [23] the notion of a strong time-consistent subset of the Core was introduced. Their authors constructed a new cooperative solution using the geometric approach and proved that it was a subset of Core and possesses a strong time consistency property. Later on, this solution was called the IDP-core and it can be constructed using a system of linear constraints for imputation distribution procedures. These conditions are defined for each time instant of a differential game. From the non-emptiness of a set described by these constraints, the non-emptiness of the corresponding set of IDPs at each time instant, it follows that the IDP-core is not empty. In the paper [24] we apply the technique proposed in [8] to study the non-emptiness of IDP-core for each time instant, and if it is non-empty, we conclude that IDP-core is non-empty. Obtained results can be used for the construction of IDP-core and verification of its non-emptiness as a numerical example. Moreover, a special case of this approach is presented for 3-player differential games. It is possible to analytically construct conditions for non-emptiness of IDP-core depending on the characteristic function. Furthermore, it is possible to define an analytical formula for selectors of IDP-core, in particular, the formula for imputation distribution procedures of IDP-core selectors.

The paper is structured as follows. Section 2 contains preliminary information, including the definition of a cooperative solution and time consistency property. Section 3 is devoted to the description of IDP-dominance, to the definition of IDP-core and corresponding necessary and sufficient conditions. Section 4 is devoted to studying the non-emptiness of IDP-core using linear programming methods. Section 5 presents the differential game model of resource extraction, IDP-core for this model is constructed using the corresponding necessary and sufficient conditions, non-emptiness conditions are studied and conclusions are drawn.

2. Problem Statement and Preliminary Information

2.1. Differential Game Model

In this section, the general description of the differential game model is given. The main concepts of this model are the type of model, payoff functions of players, motion equations, and solution concept. Type of the game model reflects what we intend to do with the model, in this paper we consider the cooperative game model. Here we need to define how to allocate joint cooperative payoff among the players. Payoff functions of players define the objectives of players depending on the state of the game, strategies and are calculated on some specific time interval (in our case closed time interval). Motion equations define of how the state of the game changes according to the strategies of players. In the case of the cooperative game model solution concept defines the exact type of imputation set that will be used to allocate joint payoff among the players.

Consider an n-player differential game $\Gamma(x_0, T - t_0)$ with prescribed duration $T - t_0$ and initial condition x_0. Game dynamics are defined by the system of differential equations:

$$\begin{aligned} &\dot{x} = f(x, u_1, \ldots, u_n),\ x \in R^n,\ u_i \in U_i \subset \mathrm{comp}R^k,\ t \in [t_0, T],\ i = \overline{1,n}, \\ &x(t_0) = x_0, \end{aligned} \tag{1}$$

for which the conditions of existence, uniqueness and continuity of solution $x(t)$ for any admissible measurable controls $u_1(\cdot), \ldots, u_n(\cdot)$ are satisfied. Open-loop control $u_i(t)$ satisfying the system (1) is a strategy of player i and $\mathrm{comp}R^k$ is the compact set in k-dimension real number space (k is integer).

Let $N = \{1, \ldots, n\}$ be the set of players. Payoff of player i is defined in the following way:

$$K_i(x_0, T - t_0; u_1, \ldots, u_n) = \int\limits_{t_0}^{T} h_i(x(\tau), u_1(\tau), \ldots, u_n(\tau))d\tau,\ i = \overline{1,n}, \tag{2}$$

where $h_i(x, u_1, \ldots, u_n) \geq 0$, $i = \overline{1,n}$ and $f(x, u_1, \ldots, u_n)$ are integrable functions, $x(t)$ is the solution of system (1) with controls $u(t) = (u_1(t), \ldots, u_n(t))$ involved.

2.2. Cooperative Differential Game Model

In the cooperative differential game model with transferable utility there are two problems:

1. Determination of a strategy set for players which maximizes the sum of their payoffs or determination of strategies corresponding to cooperative behavior. These strategies $u^* = (u_1^*, \ldots, u_n^*)$ are called optimal, the corresponding trajectory is called the cooperative trajectory and denoted by $x^*(t)$.
2. Determination of the allocation rule for the maximum joint payoff of players corresponding to the optimal strategies $u^*(t)$ and determination of optimal trajectory $x^*(t)$. Namely, the determination of a cooperative solution as a subset of the imputation set.

Let $u^* = (u_1^*, \ldots, u_n^*)$ be the vector of optimal strategies (open-loop controls) for players; i.e., a set of controls that maximizes the joint payoff of players:

$$u^* = (u_1^*, \ldots, u_n^*) = \arg\max_{u_1,\ldots,u_n} \sum_{i=1}^{n} K_i(x_0, T - t_0; u_1, \ldots, u_n). \tag{3}$$

Suppose that the maximum in (3) is achieved on the set of admissible strategies.

In order to determine how to allocate the maximum total payoff among players, it is necessary to define the notion of the characteristic function of coalition $S \subseteq N$. The characteristic function shows the strength of a coalition and thus allows the contribution of players to each coalition to be taken into account.

Suppose that in the game $\Gamma(x_0, T - t_0)$ characteristic function $V(S; x_0, T - t_0)$, $S \subseteq N$ is constructed in any relevant way (for example, as in [25]). We assume that the superadditivity conditions are satisfied:

$$\begin{aligned} &V(S_1 \cup S_2; x_0, T - t_0) \geq V(S_1; x_0, T - t_0) + V(S_2; x_0, T - t_0), \\ &\forall\, S_1, S_2 \subseteq N,\ S_1 \cap S_2 = \emptyset. \end{aligned}$$

Denote by $L(x_0, T - t_0)$ the set of imputations [26] in the game $\Gamma(x_0, T - t_0)$:

$$\begin{aligned} L(x_0, T - t_0) = \Big\{ \xi(x_0, T - t_0) = (\xi_1(x_0, T - t_0), \ldots, \xi_n(x_0, T - t_0)) : \\ \sum_{i=1}^{n} \xi_i(x_0, T - t_0) = V(N; x_0, T - t_0), \\ \xi_i(x_0, T - t_0) \geq V(\{i\}; x_0, T - t_0),\ i \in N \Big\}, \end{aligned}$$

where $V(\{i\}; x_0, T - t_0)$ is a value of characteristic function $V(S; x_0, T - t_0)$ for coalition $S = \{i\}$.

By $M(x_0, T - t_0)$ denote an arbitrary cooperative solution or subset of imputation set $L(x_0, T - t_0)$:

$$M(x_0, T - t_0) \subseteq L(x_0, T - t_0).$$

Suppose that at the beginning of game $\Gamma(x_0, T - t_0)$ at the instant t_0, players agreed to select a subset of $L(x_0, T - t_0)$ or some cooperative solution. However, suppose that at some instant $\bar{t}$ players decided to reconsider the chosen cooperative solution, or decided to reconsider the allocation rule for a cooperative payoff. In order to model their behavior, it is necessary to define the notion of subgame $\Gamma(x^*(t), T - t)$ along the cooperative trajectory $x^*(t)$ starting at the instant $t \in [t_0, T]$.

For each subgame $\Gamma(x^*(t), T - t)$, $t \in [t_0, T]$ along the trajectory $x^*(t)$, we define the superadditive characteristic function $V(S; x^*(t), T - t)$, $S \subseteq N$ in the same way as it was done for the initial game $\Gamma(x_0, T - t_0)$:

$$\begin{aligned} &\forall\, S, A \subseteq N,\ S \cap A = \emptyset : \\ &\qquad V(S \cup A; x^*(t), T - t) \geq V(S; x^*(t), T - t) + V(A; x^*(t), T - t). \end{aligned} \tag{4}$$

It is also possible to define the notion of imputation $\xi(x^*(t), T-t)$ for a subgame $\Gamma(x^*(t), T-t)$ along the cooperative trajectory $x^*(t)$, $t \in [t_0, T]$. The set of all possible imputations in the subgame $\Gamma(x^*(t), T-t)$ is denoted by $L(x^*(t), T-t)$, $t \in [t_0, T]$:

$$L(x^*(t), T-t) = \Big\{ \xi(x^*(t), T-t) = (\xi_1(x^*(t), T-t), \ldots, \xi_n(x^*(t), T-t)) : \\ \sum_{i=1}^{n} \xi_i(x^*(t), T-t) = V(N; x^*(t), T-t), \\ \xi_i(x^*(t), T-t) \geq V(\{i\}; x^*(t), T-t),\ i \in N \Big\}. \tag{5}$$

The superaditivity property (4) for characteristic function $V(S; x^*(t), T-t)$ guarantees the non-emptiness of imputation set $L(x^*(t), T-t)$, $t \in [t_0, T]$. The cooperative solution of subgame $\Gamma(x^*(t), T-t)$ is denoted correspondingly by $M(x^*(t), T-t)$.

2.3. Core

In cooperative game theory, the main problem is "fair" allocation of the maximum joint payoff $V(N; x_0, T-t_0)$ among the players from grand coalition $N = \{1, \ldots, n\}$.

Suppose that players in the cooperative differential game $\Gamma(x_0, T-t_0)$ (subgame $\Gamma(x^*(t), T-t)$, $t \in [t_0, T]$ along the cooperative trajectory $x^*(t)$) made an agreement on the allocation rule $\xi(x_0, T-t_0)$ (imputation $\xi(x^*(t), T-t)$), where none of imputations dominates $\xi(x_0, T-t_0)$ ($\xi(x^*(t), T-t)$) [26]. Such an allocation rule is stable in the sense that there not exists imputation that would be better for each coalition at every time instant $t \in [t_0, T]$.

Definition 1. *We call the set of undominated imputations of cooperative differential game $\Gamma(x^*(t), T-t)$ by the Core and denote it by $C(x^*(t), T-t)$, $t \in [t_0, T]$.*

The following theorem holds:

Theorem 1. *Imputation $\xi(x^*(t), T-t)$ belongs to the Core $C(x^*(t), T-t)$, if and only if for all $S \subseteq N$ the following inequalities are satisfied:*

$$V(S; x^*(t), T-t) \leq \sum_{i \in S} \xi_i(x^*(t), T-t),\ t \in [t_0, T].$$

2.4. Non-Emptiness of Core in Static Games

These are the main results concerning the nonemptiness conditions of Core in static games. Necessary and sufficient conditions for non-emptiness of Core were formulated by O. Bondareva [6] and by L. Shapley [7]. These conditions are based on the concept of a balanced game, but the application of this approach for a specific game model is difficult.

In the paper [27] G. Owen showed that in the game (N, v) exists a non-empty Core, if and only if the optimal value of the linear programming problem

$$\sum_{i \in N} \xi_i \longrightarrow min \\ \sum_{i \in S} \xi_i \geq v(S),\ \forall S \subseteq N,\ S \neq \emptyset$$

is equal to $v(N)$.

The papers [8–10] also make use of linear programming problem for Core's non-emptiness. Consider the following linear programming problem:

$$\sum_{i\in N}\xi_i \longrightarrow min$$
$$\sum_{i\in S}\xi_i \geq v(S),\ \forall S\subseteq N,\ S\neq N,\ \varnothing. \tag{6}$$

Suppose that $\xi^0 = (\xi^0_1,\ldots,\xi^0_n)$ is some arbitrary optimal solution of the linear programming problem (6). The set of all optimal solutions of the optimization problem (6) is denoted by $X^0(v)$. In [8] it is shown that the necessary and sufficient conditions of non-emptiness of Core can be formalized in the following way:

Theorem 2. *The Core in cooperative game with transferable utility (N, v) is nonempty, if and only if the following inequality is satisfied:*

$$\sum_{i\in N}\xi^0_i \leq v(N), \tag{7}$$

where $\xi^0 \in X^0(v)$ is a solution of the linear programming problem (6).

2.5. Time-Consistency of Cooperative Solution and Imputation Distribution Procedure

Transferring the results of static cooperative game theory to the field of cooperative differential games brings about the problem of defining the time-consistent cooperative solution. The problem of defining the solution of the differential game with prescribed duration was studied in the papers of L.A. Petrosyan [16,17]. Time consistency of cooperative solution is the property that shows that for the players it is not beneficial to deviate from the chosen cooperative solution during the game.

The main approach for solving the problem of time inconsistency of cooperative solution in the differential game is the imputation distribution procedure, proposed in [17]. In this paper, imputation distribution procedure was defined as a vector function for a fixed imputation. In this paper, we consider another approach that generalizes the notion of IDP.

Assume IDP's in cooperative differential game $\Gamma(x_0, T-t_0)$ are integrable vector functions that constitute some imputation from the imputation set:

$$\beta(t):\ \int_{t_0}^{T}\beta(\tau)d\tau \in L(x_0, T-t_0) \tag{8}$$

or

$$\int_{t_0}^{T}\beta_i(\tau)d\tau \geq V(\{i\}; x_0, T-t_0),\ i\in N,$$
$$\sum_{i\in N}\int_{t_0}^{T}\beta_i(\tau)d\tau = V(N; x_0, T-t_0).$$

Therefore, in the above definition, IDP is not based on the imputation itself but generates it. We define also the so-called corresponding IDP, the concept of which is close to the initial definition of IDP in the paper [17].

Definition 2. *The integrable function $\beta(t) = (\beta_1(t),\ldots,\beta_n(t))$, $t\in[t_0, T]$ is called a corresponding imputation distribution procedure (IDP) for $\xi(x_0, T-t_0)\in L(x_0, T-t_0)$, if the following equalities hold:*

$$\xi_i(x_0, T-t_0) = \int_{t_0}^{T}\beta_i(\tau)d\tau,\ i\in N. \tag{9}$$

Actually, the corresponding IDP $\beta(t)$ depends on $\xi(x_0, T-t_0)$ and is not unique for this imputation. We can represent it in the form

$$\beta(t) = \beta(t, \xi(x_0, T-t_0))$$

or

$$\beta_i(t) = \beta_i(t, \xi_i(x_0, T-t_0)),\ i \in N.$$

From (9) we have for $t \in [t_0, T]$, $i \in N$:

$$\xi_i(x_0, T-t_0) = \int_{t_0}^{t} \beta_i(\tau)d\tau + \int_{t}^{T} \beta_i(\tau)d\tau$$

or

$$\int_{t}^{T} \beta_i(\tau)d\tau = \xi_i(x_0, T-t_0) - \int_{t_0}^{t} \beta_i(\tau)d\tau.$$

That is IDP shares at instant t imputations in two parts: payoffs to player i, which are received in interval $[t_0, t]$ and in interval $(t, T]$.

Definition 3. *The cooperative solution $M(x_0, T-t_0)$ in the game $\Gamma(x_0, T-t_0)$ is called time-consistent, if for each imputation $\xi(x_0, T-t_0) \in M(x_0, T-t_0)$ there exists a corresponding IDP $\beta(t) = \beta(t, \xi(x_0, T-t_0))$ such that:*

$$\int_{t}^{T} \beta(\tau)d\tau \in M(x^*(t), T-t),\ t \in [t_0, T] \tag{10}$$

or equivalently

$$\xi(x_0, T-t_0) - \int_{t_0}^{t} \beta(\tau)d\tau \in M(x^*(t), T-t),\ t \in [t_0, T]. \tag{11}$$

Notice that from condition (10) we have the following equality

$$\sum_{i \in N} \int_{t}^{T} \beta_i(\tau)d\tau = V(N; x^*(t), T-t),\ t \in [t_0, T]. \tag{12}$$

It is obvious that if $M(x^*(t), T-t) \neq \emptyset$ for $\forall t \in [t_0, T]$, then for any differentiable by t function $\xi(x^*(t), T-t) \in M(x^*(t), T-t)$ $(\xi(x^*(t_0), T-t_0) = \xi(x_0, T-t_0))$ IDP $\beta(t)$ can be defined using the formula:

$$\begin{aligned} &\beta(t) = -\frac{d}{dt}\xi(x^*(t), T-t),\ t \in [t_0, T],\ i \in N, \\ &\xi(x^*(t_0), T-t_0) = \xi(x_0, T-t_0). \end{aligned} \tag{13}$$

Then imputation $\xi(x_0, T-t_0)$ is defined by the formula:

$$\xi(x_0, T-t_0) = \int_{t_0}^{t} \beta(\tau)d\tau + \xi(x^*(t), T-t),\ t \in [t_0, T].$$

Define an imputation in the current cooperative game $\Gamma(x^*(t), T-t)$ with characteristic function $V(S; x^*(t), T-t)$ which corresponds to a given IDP $\beta(t) = \beta(t, \xi(x_0, T-t_0))$ as

$$\xi(x^*(t), T-t) = \int_{t}^{T} \beta(\tau)d\tau. \tag{14}$$

From Definition 3 we have

$$\xi(x^*(t), T-t) \in M(x^*(t), T-t). \tag{15}$$

We will call the imputation (14) the dynamic imputation generated by the corresponding IDP $\beta(t) = \beta(t, \xi(x_0, T - t_0))$.

3. IDP-Core and Dominance of Imputation Distribution Procedures

Consider the development of game at instant $t \in (t_0, T)$. Suppose that at instant t_0 players agreed to realize imputation $\xi(x_0, T - t_0) = (\xi_1(x_0, T - t_0), \ldots, \xi_n(x_0, T - t_0))$. Then, according to the corresponding IDP $\beta(t)$, until the instant t, player $i \in N$ receives the payoff:

$$\int_{t_0}^{t} \beta_i(\tau) d\tau.$$

However, for some players IDP $\beta(t, \xi(x_0, T - t_0))$ would not be beneficial if there exists another imputation distrubution procedure $\overline{\beta}(t, \xi(x_0, T - t_0))$, according to which player i at interval $[t_0, t]$ receives more payoff:

$$\int_{t_0}^{t} \overline{\beta}_i(\tau) d\tau > \int_{t_0}^{t} \beta_i(\tau) d\tau. \tag{16}$$

In such a case IDP $\beta(t)$ may be considered as less beneficial for the player i at least in interval $[t_0, t]$. It is important to notice that the notion of IDP-dominance can be applied to imputation distribution procedures, which are not necessarily defined for a unique imputation. As IDP defines how dynamic imputation is to be constructed then it also makes sense to consider the notion of IDP-dominance not only for a fixed imputation.

3.1. Dominance of Imputation Distribution Procedures

In this section we consider the IDP $\beta(t)$ defined by the formula (8). Suppose that the function $V(S; x^*(t), T - t)$, $S \subseteq N$ is continuously differentiable by $t \in [t_0, T]$. Define the function $U(S; x^*(t), T - t)$ in the following way:

$$U(S; x^*(t), T - t) = -\frac{d}{dt} V(S; x^*(t), T - t),\ t \in [t_0, T],\ S \subseteq N. \tag{17}$$

Definition 4. *IDP $\beta(t)$ dominates IDP $\overline{\beta}(t)$ by coalition $S \subseteq N$ and at the instant $\bar{t} \in [t_0, T]$ (denote by $\beta(t) \overset{S,\,\bar{t}}{\succ} \overline{\beta}(t)$), if the following inequalities hold:*

$$\begin{aligned} &\beta_i(\bar{t}) > \overline{\beta}_i(\bar{t}),\ i \in S, \\ &\sum_{i \in S} \beta_i(\bar{t}) \leq U(S; x^*(\bar{t}), T - \bar{t}). \end{aligned} \tag{18}$$

Definition 5. *IDP $\beta(t)$ is undominated if at any $\bar{t} \in [t_0, T]$ there does not exist $\overline{\beta}(t)$, which dominates $\beta(t)$ by coalition $S \subseteq N$:*

$$\overline{\beta}(t) \overset{S,\,\bar{t}}{\not\succ} \beta(t), \quad \forall \overline{\beta}(t), S. \tag{19}$$

3.2. IDP-Core

In the paper [23] the authors first introduced and treated a subset of the imputation set in a cooperative differential game which was named subcore. This subset was designed using a set of imputation distribution procedures satisfying the system of inequalities and equalities. This approach is not classical for the theory of differential games since it uses IDP's for imputations, not vice versa. Based on this subcore notion, we in the paper [24] redefined this notion for the dynamic case, named it IDP-core, and formulated necessary and sufficient conditions of the existence of IDP-core along the cooperative trajectory of the game. In the current paper, we define a solution concept for IDP-core by introducing the notion of IDP dominance and using the time consistency properties or axioms defined above. It is proved that IDP-core has the necessary and sufficient conditions for a dynamic imputation when defined by the system of inequalities introduced in the paper [23,28].

Suppose that players in the game $\Gamma(x_0, T-t_0)$ agreed on the allocation rule for total payoff of grand coalition N (imputation $\xi(x_0, T-t_0)$) using the cooperative solution of IDP-core:

Definition 6. *By the dynamic $IDP-core(x^*(t), T-t)$ along the cooperative trajectory $x^*(t)$, $t \in [t_0, T]$ ($IDP-core(x_0, T-t_0)$), we call the solution in cooperative differential game $\Gamma(x^*(t), T-t)$ ($\Gamma(x_0, T-t_0)$), which includes all time-consistent imputations generated by undominated IDPs $\beta(\tau)$, $\tau \in [t, T]$ (8), $t \in [t_0, T]$ ($\beta(t)$, $t \in [t_0, T]$):*

$$\begin{aligned} IDP-core(x^*(t), T-t) = \Big\{ & \xi(x^*(t), T-t) = \int_t^T \beta(\tau) d\tau : \\ & \xi(x^*(t), T-t) \text{ and corresponding } \beta(\tau) \text{ satisfies } (10), \\ & \nexists \overline{\beta}(t),\ S,\ \overline{t} :\ \overline{\beta}(t) \overset{S,\,\overline{t}}{\succ} \beta(t) \Big\}. \end{aligned} \tag{20}$$

We note that $IDP-core(x^*(t), T-t)$ includes such imputations from the Core $C(x^*(t), T-t)$ of cooperative game $\Gamma(x_0, T-t_0)$ for which there exists corresponding undominated IDP and this IDP generates a dynamic imputation belonging $C(x^*(t), T-t)$ for each $t \in [t_0, T]$.

Theorem 3. *Let $C(x^*(t), T-t)$ be not empty for any $t \in [t_0, T)$. Dynamic imputation $\xi(x^*(t), T-t)$ in cooperative differential game $\Gamma(x_0, T-t_0)$ belongs to the dynamic $IDP-core(x^*(t), T-t)$, if and only if for corresponding $\beta(t) = \beta(t, \xi(x_0, T-t_0))$ the following conditions are satisfied $\forall t \in [t_0, T]$:*

$$\sum_{i \in S} \beta_i(t) \geq U(S; x^*(t), T-t),\ \forall S \subset N, \tag{21}$$

$$\sum_{i \in N} \beta_i(t) = U(N; x^*(t), T-t). \tag{22}$$

Proof. Sufficiency. Let for corresponding IDP $\beta(t) = \beta(t, \xi(x_0, T-t_0))$ conditions (21) and (22) hold at any $t \in [t_0, T]$. By integrating (21) and (22) in interval we obtain

$$\begin{aligned} & \sum_{i \in S} \xi(x^*(t), T-t) \geq V(S; x^*(t), T-t),\ \forall S \subset N, \\ & \sum_{i \in N} \xi(x^*(t), T-t) = V(N; x^*(t), T-t). \end{aligned} \tag{23}$$

This means the imputation

$$\xi(x_0, T-t_0) = \int_{t_0}^T \beta(t) dt \in C(x^*(t_0), T-t_0) = C(x_0, T-t_0).$$

Let us show that $\beta(t) = \beta(t, \xi(x_0, T-t_0))$ is undominated. It is proven by contradiction. Suppose for some $\overline{S} \subset N$ there exists $\overline{t} \in [t_0, T]$ and $\overline{\beta}(t)$ such that

$$\begin{aligned} & \xi(x_0, T-t_0) = \int_{t_0}^T \overline{\beta}(t) dt \in L(x_0, T-t_0), \\ & \overline{\beta}_i(\overline{t}) > \beta_i(\overline{t}),\ i \in \overline{S}, \\ & \sum_{i \in \overline{S}} \overline{\beta}_i(\overline{t}) \leq U(\overline{S}; x^*(\overline{t}), T-\overline{t}). \end{aligned} \tag{24}$$

Therefore
$$\sum_{i \in \overline{S}} \beta_i(\bar{t}) < U(\overline{S}; x^*(\bar{t}), T - \bar{t}).$$

This inequality contradicts to (21). Thus $\beta(t) = \beta(t, \xi(x_0, T - t_0))$ is undominated on the interval $[t_0, T]$. Notice that $\beta(t)$ is undominated on any subinterval $[\tau, T]$, $\tau \in [t_0, T]$, in subgame $\Gamma(x^*(\tau), T - \tau)$.

For $S = \{i\}$ condition (21) is represented in the form
$$\beta_i(t) \geq U(\{i\}; x^*(t), T - t),\ i \in N.$$

Integrating these inequalities and equality (22) and taking into account (9) and (17) we obtain
$$\xi_i(x^*(t), T - t) = \int_t^T \beta_i(\tau) d\tau \geq V(\{i\}; x^*(t), T - t),\ i \in N,$$
$$\sum_{i \in N} \xi_i(x^*(t), T - t) = \sum_{i \in N} \int_t^T \beta_i(\tau) d\tau = V(N; x^*(t), T - t). \tag{25}$$

Thus dynamic payoff
$$\xi(x^*(t), T - t) = \int_t^T \beta(\tau) d\tau$$
is the imputation in current game $\Gamma(x^*(t), T - t)$ generated by corresponding IDP $\beta(t) = \beta(t, \xi(x_0, T - t_0))$.

Due to the non-emptiness of $C(x^*(t), T - t)$ for any $t \in [t_0, T]$ the IDP $\beta(t) = \beta(t, \xi(x_0, T - t_0))$ which satisfies (21) and (22) generates the payoff vector
$$\xi(x^*(t), T - t) = \int_t^T \beta(\tau) d\tau,$$
which belongs to $C(x^*(t), T - t)$. Thus $\xi(x^*(t), T - t)$ satisfies (10) and therefore imputation $\xi(x_0, T - t_0)$ is time-consistent and lies in $IDP - core(x_0, T - t_0)$.

Necessity. Let imputation $\xi(x_0, T - t_0)$ in the cooperative differential game $\Gamma(x_0, T - t_0)$ belong to $IDP - core(x_0, T - t_0)$. Therefore by Definition 6 the imputation $\xi(x_0, T - t_0)$ generated by undominated IDP $\beta(t) = \beta(t, \xi(x_0, T - t_0))$ belongs to $C(x_0, T - t_0)$ and is time-consistent. Due to Definition 6 the time-consistency of imputation $\xi(x_0, T - t_0)$ means that there exists IDP $\beta(t)$ such that
$$\xi(x^*(t), T - t) = \int_t^T \beta(\tau) d\tau \in C(x^*(t), T - t),\ t \in [t_0, T].$$

Let us show that this inclusion takes place for the undominated $\beta(t) = \beta(t, \xi(x_0, T - t_0))$ corresponding to imputation $\xi(x_0, T - t_0)$. Suppose this is not the fact. Then there exists coalition $\overline{S} \subset N$ and some instant $\bar{t} \in [t_0, T]$ such that the following inequality holds
$$\sum_{i \in \overline{S}} \int_{\bar{t}}^T \beta(t) dt < V(\overline{S}; x^*(\bar{t}), T - \bar{t}). \tag{26}$$

Notice that $\beta(\bar{t}) = \beta(\bar{t}, \xi(x_0, T - t_0))$ belongs to the set of undominated imputations in a cooperative game with the characteristic function $U(S; x^*(\bar{t}), T - \bar{t})$, $S \subseteq N$. Therefore as follows from Theorem 1 for $\overline{S} \subset N$ the following has to be fulfilled
$$\sum_{i \in \overline{S}} \beta_i(\bar{t}) dt \geq U(\overline{S}; x^*(\bar{t}), T - \bar{t}). \tag{27}$$

Integrating this inequality in interval $[\bar{t}, T]$ we receive

$$\sum_{i \in \bar{S}} \int_{\bar{t}}^{T} \beta_i(t)dt \geq V(\bar{S}; x^*(\bar{t}), T - \bar{t}), \tag{28}$$

which contradicts (27). Thus the Theorem is proved. □

Proposition 1. *If* $C(x^*(t), T - t)$ *is empty for some* $t = \bar{t} \in [t_0, T]$, *then* $IDP - core(x^*(t_0), T - t_0)$ *is empty.*

Proof. Suppose we can find the time-consistent imputation $\xi(x_0, T - t_0) \in IDP - core(x^*(t_0), T - t_0)$ and corresponding undominated IDP $\beta(t)$, which satisfies (21) and (22). Integrating inequalities (21) and (22) we obtain

$$\sum_{i \in S} \int_{\bar{t}}^{T} \beta_i(\tau)d\tau \geq V(S; x^*(\bar{t}), T - \bar{t}), \; S \subset N,$$
$$\sum_{i \in N} \int_{\bar{t}}^{T} \beta_i(\tau)d\tau = V(N; x^*(\bar{t}), T - \bar{t}). \tag{29}$$

Thus we receive

$$\xi(x^*(\bar{t}), T - \bar{t}) = \int_{\bar{t}}^{T} \beta(\tau)d\tau \in C(x^*(\bar{t}), T - \bar{t}). \tag{30}$$

It contradicts with emptiness of $C(x^*(\bar{t}), T - \bar{t})$. Therefore, $IDP - core(x^*(t_0), T - t_0)$ is empty. The Proposition is proved. □

Proposition 2. *If* $C(x^*(t), T - t)$ *is not empty for any* $t \in [t_0, T]$ *then* $IDP - core(x^*(t_0), T - t_0) = C(x^*(t_0), T - t_0)$.

Proof. Consider an imputation $\xi(x_0, T - t_0) \in C(x^*(t_0), T - t_0)$ that does not belong to $IDP - core(x^*(t_0), T - t_0)$. That is $\xi(x_0, T - t_0)$ belongs to $C(x^*(t_0), T - t_0)$, but is time inconsistent. According to the time consistency imputation definition $\xi(x_0, T - t_0)$ is time inconsistent if there does not exist IDP $\beta(t) = \beta(t, \xi(x_0, T - t_0))$ such that at any $t \in [t_0, T]$ dynamic imputation $\xi(x^*(t), T - t)$ generated by this IDP belongs to core $C(x^*(t), T - t)$.

But as follows from Theorem 3, if the corresponding IDP $\beta(t) = \beta(t, \xi(x_0, T - t_0))$ satisfies conditions (21) and (22), then $\xi(x_0, T - t_0)$ belongs to $IDP - core(x_0, T - t_0)$. That is $\xi(x_0, T - t_0)$ is time-consistent by Definition 6, that is the following inclusion holds for any $t \in [t_0, T]$

$$\xi(x^*(t), T - t) = \int_{t}^{T} \beta(\tau)d\tau \in C(x^*(t), T - t).$$

The Proposition is proved. □

Remark 1. *Proposition 2 states that if the Core is not empty, then IDP-core and Core coincide in the current game or equivalently imputations from the Core coincide with the imputations from the IDP-core. The system of inequalities (21) and (22) allows extracting the subset from the set of imputation distribution procedures which provides time-consistency and IDP-nondominance of imputations from the Core. For other subsets of the IDP set, this kind of result generally speaking is not true. Note also that the set of imputation distribution procedures (21) and (22) can be empty. In the next section consider the approach to check nonemptiness and construction of IDP's from the IDP-core.*

Suppose that the characteristic function $V(S; x^*(t), T - t)$, $t \in [t_0, T]$ *is defined in some relevant way (for example, as in [25]). Suppose that it is a strictly monotonically decreasing function for any* $t \in [t_0, T]$, *decreasing faster than the linear law. Construct the Core* $C(x_0, T - t_0)$ *for the initial instant* $t = t_0$. *Afterwards choose imputation from the Core* $\xi(x_0, T - t_0) \in C(x_0, T - t_0)$. *According to the Definition 2 we choose the* *corresponding imputation distribution procedure $\beta(t) = \beta(t, \xi(x_0, T - t_0))$ (IDP) for $\xi(x_0, T - t_0)$ in some relevant way. As it follows from Proposition 2 for the imputations $\xi(x_0, T - t_0) \in IDP - core(x^*(t_0), T - t_0) = C(x^*(t_0), T - t_0)$ we can always find IDP $\beta'(t)$ satisfying the conditions (21) and (22) such that:*

$$\int_{t_0}^{T} \beta'(t)dt = \xi(x_0, T - t_0).$$

On the other hand $\beta(t) = \beta(t, \xi(x_0, T - t_0))$ could not satisfy conditions (21) and (22). It could be the case if $\beta(t) = \beta(t, \xi(x_0, T - t_0))$ is chosen in the following way

$$\beta(t) = c = const,\ t \in [t_0, t'),\ \left(t' \in (t_0, T)\right),$$
$$\beta(t) = \frac{\xi(x_0, T - t_0) - c(t' - t_0)}{T - t'},\ t \in [t', T]$$

and the derivative of $V(S; x^*(t), T - t)$

$$U(S; x^*(t), T - t) > 0,\ t \in [t_0, T].$$

This case will be demonstrated on the model example in Section 5.

Proposition 3. *If $C(x^*(t), T - t)$ is not empty for any $t \in [t_0, T]$, then all imputations of $C(x^*(t_0), T - t_0)$ are time-consistent.*

Proof. Consider am imputation $\xi(x_0, T - t_0)$ from Core $C(x^*(t_0), T - t_0)$. According to the Proposition 2 $IDP - core(x^*(t_0), T - t_0) = C(x^*(t_0), T - t_0)$ and therefore $\xi(x_0, T - t_0) \in IDP - core(x^*(t_0), T - t_0)$. From the definition of the IDP-core it follows that $\xi(x_0, T - t_0)$ is time-consistent. The Proposition is proved. □

4. Application of Linear Programming Methods for Nonemptiness Properties

In this section, we consider the linear programming problem described in Section 2.4, for the non-emptiness properties of Core. IDP-core can be constructed using a system of linear constraints for the imputation distribution procedures. These constraints are defined for each instant in the game. From the nonemptiness of the set described by these constraints, it follows that the IDP-core is not empty.

Consider the following linear programming problem for a fixed t:

$$\begin{aligned} &\sum_{i \in N} \beta_i \longrightarrow min \\ &\sum_{i \in S} \beta_i \geq U(S; x^*(t), T - t),\ \forall S \subseteq N, S \neq N,\ S \neq \varnothing. \end{aligned} \tag{31}$$

Suppose that $\beta_i^0 = (\beta_1^0, \ldots, \beta_n^0)$ is an optimal solution of linear programming problem (31) with fixed t. The set of optimal solutions of problem (31) we denote by Y^0.

Then the following theorem is true:

Theorem 4. *The set of IDPs satisfying the conditions (25), $t \in [t_0, T)$ is not empty, if and only if $\forall t \in [t_0, T)$ the following condition is satisfied:*

$$\sum_{i \in N} \beta_i^0 \leq U(N; x^*(t), T - t), \tag{32}$$

where $\beta^0 \in Y^0$ is any solution of the linear programming problem (31).

Proof. Start the proof with the sufficient condition. Suppose that the condition (32) is satisfied, then according to (31) for any $t \in [t_0, T]$ there exists $\hat{\beta}^0$ such that for

$$\beta_i = \beta_i^0 + \frac{U(N; x^*(t), T-t)}{n} - \frac{\sum_{i \in N} \beta_i^0}{n},\ i \in N \tag{33}$$

conditions (21) and (22) are satisfied for any fixed $t \in [t_0, T]$. If it is true, then we can compose the integrable function $\hat{\beta}^0(t)$ as a function of time, for which the conditions (21) and (22) will be satisfied.

Proof of the necessity condition. If the IDP-core is not empty, then there exists at least one integrable function $\beta(t)$ satisfying the conditions (21) and (22). As a result for the solution of (31) condition (32) should be satisfied. □

5. Differential Game Model of Resource Extraction

Consider a game-theoretical model of non-renewable resource extraction with asymmetric players [29,30]. The amount of resource depends on the rates of extraction which are chosen by the players. The game involves n asymmetric players, with utility functions depending on the current amount of resource and rates of extraction.

Denote by $x(t) \in R^1$ the amount of resource at instant t and by $u_i(t, x)$ resource extraction rate chosen by player i at instant t. As a class of strategies we will consider a class of feedback strategies, where the strategies are the functions of time t and state x. We assume that $\forall t$, $u_i(t, x) \geq 0$, and $x(t) = 0$ implies $u_i(t, x) = 0$. The amount of the resource $x(t)$ as a function of t depends in the following way on $u_i(t, x)$:

$$\begin{gathered} \dot{x} = -\sum_{i=1}^{n} a_i u_i(t, x), \quad a_i > 0, i = 1, \ldots, n. \\ x(t_0) = x_0. \end{gathered} \tag{34}$$

Payoff function representing the income of player i:

$$K_i(x_0, T - t_0) = \int_{t_0}^{T} \log(u_i(\tau, x)) d\tau,\ i = 1, \ldots, n. \tag{35}$$

5.1. Cooperative Strategies and Cooperative Trajectory

Consider the cooperative version of a non-renewable resource extraction game [30]. Here, players unite in a grand coalition and maximize total utility, acting as one player. The corresponding optimal control problem is formalized in the following way:

$$\sum_{i=1}^{n} K_i(x_0, T - t_0) = \sum_{i=1}^{n} \int_{t_0}^{T} \log(u_i(\tau, x)) d\tau \to \max_{u_i, i=\overline{1,n}} \tag{36}$$

subject to

$$\begin{aligned} &\dot{x} = -\sum_{i=1}^{n} a_i u_i(t, x), \\ &x(t_0) = x_0 > 0, \\ &u(t, x) \geq 0. \end{aligned} \tag{37}$$

To solve the optimization problem (36) and (37), we use the dynamic programming principle proposed by Bellman. To do this we define the Bellman function as the maximum value of the total payoff of players (35) in the subgame $\Gamma(x, T-t)$ starting at the instant t in the position x:

$$W(t,x) = \max_{u_i, i=\overline{1,n}} \left\{ \sum_{i=1}^{n} K_i(x, T-t) \right\} = \max_{u_i, i=\overline{1,n}} \left\{ \sum_{i=1}^{n} \int_{t}^{T} \log u(\tau, x) d\tau \right\} \tag{38}$$

subject to Equation (37), when $x_0 = x$ and $t_0 = t$.

It is proved that if there exists a continuously differentiable function $W(t,x)$ that satisfies the Hamilton-Jacobi-Bellman equation

$$-W_t(t,x) = \max_{u_i, i=\overline{1,n}} \left\{ \sum_{i=1}^{n} \log u_i(t,x) - W_x(t,x) \left(\sum_{i=1}^{n} a_i u_i(t,x) \right) \right\},$$
$$\lim_{t \to T-0} W(t,x) = 0, \tag{39}$$

then strategies $u_i^*(t,x)$ defined by maximizing the right hand side (39) deliver the maximum to the functional in the optimization problem (36) and (37).

From the first order extremum condition of (39), we obtain:

$$u_i^* = \frac{1}{a_i W_x(t,x)},$$

then substituting to (39):

$$W_t(t,x) = n \log W_x(t,x) + \log A^{[N]} + n, \quad A^{[N]} = \prod_{i=1}^{n} a_i$$
$$\lim_{t \to T-0} W(t,x) = 0. \tag{40}$$

We will consider a Bellman function as a function of the form:

$$W(t,x) = A(t) \log x + B(t),$$

then, by substituting in (40), we obtain:

$$\dot{A} \log x + \dot{B} = n \log A - n \log x + \log A^{[N]} + n, \tag{41}$$
$$\lim_{t \to T-0} A(t) = \lim_{t \to T-0} B(t) = 0.$$

The solution of (41) are the functions:

$$\begin{aligned} A(t) &= n(T-t), \\ B(t) &= -(T-t) \left(\log A^{[N]} + n \log n(T-t) \right). \end{aligned} \tag{42}$$

By substituting $A(t)$ and $B(t)$ into the Bellman function we obtain:

$$W(t,x) = n(T-t) \log \frac{x}{n(T-t)} - (T-t) \log A^{[N]}, \ t \in [t_0, T). \tag{43}$$

The corresponding form of optimal control or cooperative strategy:

$$u_i^*(t,x) = \frac{1}{a_i W_x(t,x)} = \frac{x}{a_i n(T-t)}, \ t \in [t_0, T). \tag{44}$$

Substituting the optimal control into the motion Equation (37), we obtain the differential equation for the trajectory corresponding to the optimal control:

$$\begin{aligned} \dot{x} &= -\frac{x}{T-t}, \\ x(t_0) &= x_0. \end{aligned} \tag{45}$$

The solution has the form:

$$x^*(t) = x_0 \frac{T-t}{T-t_0}, \ t \in [t_0, T). \tag{46}$$

Trajectory $x^*(t)$ and strategy (control) $u^*(t,x)$ we will call cooperative.

In order to determine the value of players' maximum total payoff that corresponds to the optimization problem (36) and (37) in the subgame along the cooperative trajectory $x^*(t)$ (46), it is necessary to substitute the expression for the cooperative trajectory by the expression for the Bellman function (43):

$$W(t, x^*(t)) = n(T-t) \log \frac{x_0}{n(T-t_0)} - (T-t) \log A^{[N]}, \ t \in [t_0, T). \tag{47}$$

5.2. Characteristic Function

To construct the rule for allocating the maximum joint payoff among players, it is necessary to define the characteristic function for each coalition $S \subseteq N$:

$$V(S; x, T-t) = \begin{cases} \sum_{i=1}^{n} K_i(x, T-t), & S = N \\ W_S(t,x), & S \subset N, \\ 0, & S = \emptyset, \end{cases} \tag{48}$$

where $W_S(t,x)$ is defined as the maximum joint payoff of coalition S given that the players from coalition $N \backslash S$ use strategies from a fixed Nash equilibrium $u^{NE} = (u_1^{NE}, \dots, u_n^{NE})$ in the initial game.

It can be shown that in the case of a non-cooperative game, Nash equilibrium strategies are

$$u_i^{NE}(t,x) = \frac{x}{a_i(T-t)}, \ i \in N. \tag{49}$$

Consider a case of coalition $S \subset N$. We introduce the Bellman function $W_S(t,x)$, as the maximum total payoff of players from coalition S in the subgame $\Gamma(x, T-t)$ starting at the instant t in the position x:

$$W_S(t,x) = \max_{u_i, i \in S} \sum_{i \in S} \left\{ \int_t^T \log u_i d\tau \right\} \tag{50}$$

$$\text{subject to } \dot{x}(\tau) = -\sum_{i \in N} a_i u_i \tag{51}$$

$$u_i = u_i^{NE}, \quad i \in N \backslash S. \tag{52}$$

The Hamilton-Jacobi-Bellman equation for this problem has the form:

$$\begin{aligned} -\frac{\partial W_S(t,x)}{\partial t} &= \max_{u_i, i \in S} \left\{ \sum_{i \in S} \log u_i(t,x) - \frac{\partial W_S(t,x)}{\partial x} \left(\sum_{j=1}^{n} a_j u_j(t,x) \right) \right\}, \\ &\lim_{t \to T-0} W_S(t,x) = 0. \end{aligned} \tag{53}$$

From the first order extremum condition for (53) we obtain

$$u_i^* = \frac{1}{a_i \frac{\partial W_S(t,x)}{\partial x}}, \tag{54}$$

substitute in (53):

$$\begin{gathered} \frac{\partial W_S}{\partial t} = k \log \frac{\partial W_S}{\partial x} + \log A^{[S]} + k + \frac{\partial W_S}{\partial x} \sum_{j \in N \setminus S} \frac{x}{T-t}, \quad A^{[S]} = \prod_{i=1}^{n} a_i \\ \lim_{t \to T-0} W_S(t,x) = 0, \end{gathered} \tag{55}$$

where $k = |S|$, $n = |N|$. Consider the following form of the Bellman function:

$$W_S(t,x) = A(t) \log x + B(t),$$

then by substituting in (55) we obtain:

$$\begin{gathered} \dot{A} \log x + \dot{B} = k \log A - k \log x + \log A^{[S]} + k + (n-k) \frac{A}{T-t}, \\ \lim_{t \to T-0} A(t) = \lim_{t \to T-0} B(t) = 0. \end{gathered} \tag{56}$$

The solution of (56) are the functions:

$$\begin{aligned} A(t) &= k(T-t), \\ B(t) &= -k(T-t) \left(\frac{\log A^{[S]}}{k} + \log k(T-t) + n - k \right). \end{aligned} \tag{57}$$

Solution of the optimization problem (50):

$$W_S(t,x) = k(T-t) \left[\log \frac{x}{T-t} - \log k - \frac{\log A^{[S]}}{k} - n + k \right]. \tag{58}$$

According to the definition, we obtain the characteristic function for the coalition $S \neq N$:

$$V(S, x, T-t) = W_S(t,x).$$

In order to determine the way to allocate the maximum joint payoff of players (47) among them along the cooperative trajectory $x^*(t)$ (46), namely, for the subgame starting at the instant t on the cooperative trajectory $x^*(t)$ (46) it is necessary to define the characteristic function along the cooperative trajectory. Let us substitute the expression for $x^*(t)$ (46) into the expression for characteristic function $V(S, T-t, x)$, $S \subset N$ (58).

$$W_S(t, x^*(t)) = k(T-t) \left[\log \frac{x_0}{T-t_0} - \log k - \frac{\log A^{[S]}}{k} - n + k \right]. \tag{59}$$

For the case when $S = N$, the characteristic function is calculated in accordance with (47).

5.3. IDP-Core

Suppose that all players unite in grand coalition N, then they can guarantee themselves joint payoff equal to $V(N; x^*(t), T-t)$. In order to determine how to allocate the maximum joint payoff among players, we use the notion of imputations $\xi(x, T-t)$. In particular, we will use IDP-core as a cooperative solution in the game. According to Theorem 3, IDP-core can be constructed using the conditions for IDP's $\beta_i(t), i \in N$:

$$\sum_{i \in S} \beta_i(t) \geq -kt \left[\log \frac{x_0}{T-t_0} - \log k - \frac{\log A^{[S]}}{k} - n + k \right], \ \forall S \subset N,$$
$$\sum_{i \in N} \beta_i(t) = -nt \log \frac{x_0}{n(T-t_0)} - (T-t) \log A^{[N]}, \ \forall t \in [t_0, T]. \tag{60}$$

5.4. Non-Emptiness of IDP-Core

In order to study non-emptiness conditions we solve the linear programming problem, as presented in the paper [24], for $t \in [t_0, T]$ with a fixed step Δt. As a result, the vector function $\beta^0 = (\beta_1^0, \ldots, \beta_n^0)$ is obtained using the numerical methods and corresponding conditions are to be verified in order for the IDP-core to be non-empty:

$$\sum_{i \in N} \beta_i^0 \leq U(N; x^*(t), T-t). \tag{61}$$

We construct IDP $\hat{\beta}^0(t)$ using $\beta^0(t)$ and show that it satisfies the conditions (25):

$$\hat{\beta}_i^0(t) = \beta_i^0(t) + \frac{U(N; x^*(t), T-t) - \sum_{i \in N} \beta_i^0(t)}{n}. \tag{62}$$

5.5. Core and IDP-Core

According to the Theorem 3, the imputation that corresponds to the IDP $\hat{\beta}^0(t)$

$$\xi(x_0, T-t_0) = \int_{t_0}^{T} \hat{\beta}^0(t) dt \tag{63}$$

belongs to the Core $C(x_0, T-t_0)$ because, for given parameters IDP $\hat{\beta}^0(t)$, $t \in [t_0, T]$ satisfies conditions (21) and (22) or $\xi(x_0, T-t_0)$ belongs to $IDP-core(x_0, T-t_0)$. But if we use the Core $C(x_0, T-t_0)$ instead of $IDP-core(x_0, T-t_0)$ as a cooperative solution in the game, then we can use any IDP for the imputation (63), such as

$$\beta(t) = c = const, \ t \in [t_0, t'), \ \left(t' \in (t_0, T)\right),$$
$$\beta(t) = \frac{\xi(x_0, T-t_0) - c(t'-t_0)}{T-t'}, \ t \in [t', T], \tag{64}$$

but this does not necessarily satisfy conditions (21) and (22) at some instant and therefore it appears not to be undominated and corresponding to this IDP imputation $\xi(x_0, T-t_0)$ is time-inconsistent.

On Figure 1 the set defined by the system of constrains (25) shown, the solid line is the solution $\beta^0(t)$ of corresponding linear programming problem (31) as a function of time, the dashed line is IDP $\hat{\beta}^0(t)$ and IDP $\beta(t)$ (64) corresponding to the imputation (63).

Function $\hat{\beta}_i^0(t)$ satisfies the constrains (25). It can be seen that the IDP-core in this game model is not empty and conditions (32) of Theorem 4 are satisfied.

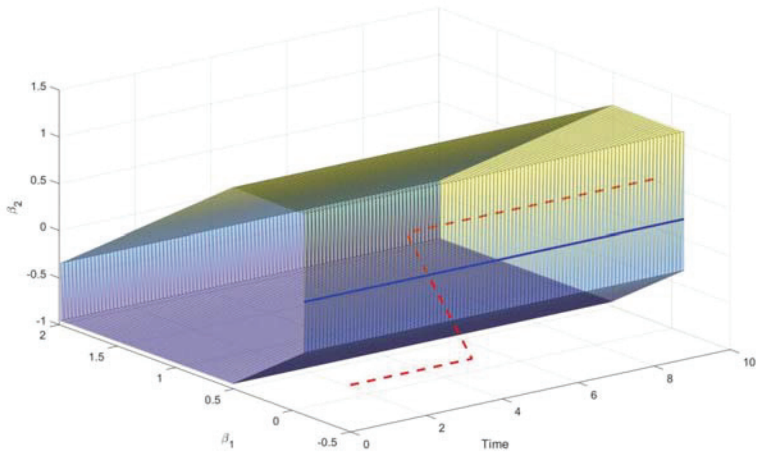

Figure 1. Axes: β_1, β_3, t. β_2 can be found using the equality in (25).

Using Figure 2 it is possible to verify the non-emptiness conditions (32) of Theorem 6, the solid line shows the sum of values $\beta_i^0(t)$, $i = \overline{1,3}$:

$$S_{\beta^0}(t) = \beta_1^0(t) + \beta_2^0(t) + \beta_3^0(t), \tag{65}$$

the dashed line in the Figure 2 shows the value of characteristic function for a grand coalition

$$U(N; x^*(t), T-t) = U(\{1,2,3\}; x^*(t), T-t),$$

where $U(\{1,2,3\}; x^*(t), T-t)$ is defined in (17). In the Figure 2 it can be seen that

$$S_{\beta^0}(t) \leq U(\{1,2,3\}; x^*(t), T-t) \; \forall t \in [t_0, T].$$

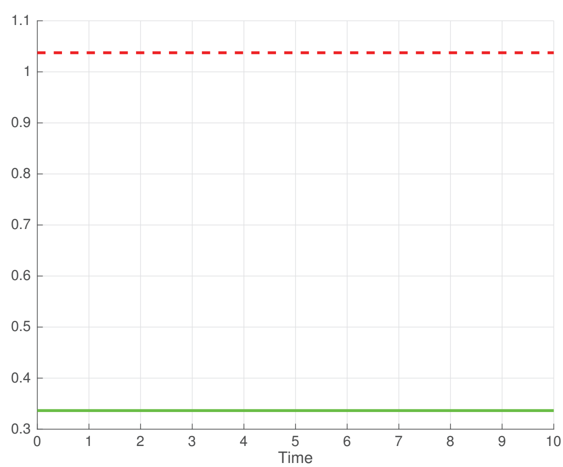

Figure 2. $U(\{1,2,3\}; x^*(t), T-t)$ (17) is a dashed line, $S_{\beta^0}(t)$ (65) is a solid line.

6. Conclusions

This paper examines a new approach for defining a cooperative solution for differential games. Our approach uses the time consistency property as a basic axiom for constructing the cooperative solution. It is important to notice that the further use of the time consistency property as the axiom for the theory of dynamic cooperative games, the theory of social choice, and mechanism design is promising. The approach also defines the notion of IDP-dominance, which allows for selecting undominated imputation distribution procedures. Properties of time consistency and IDP-dominance are the key properties for constructing a new cooperative solution, namely IDP-core. The necessary and sufficient conditions for the IDP-core defining geometric properties of this solution are presented. It is also proved that the set of imputations that corresponds to the Core and to IDP-core coincides, but, as the simulation demonstrates, the IDPs that would be naturally proposed for use sometimes might not appear to be undominated and therefore lead to the time inconsistency of the corresponding imputations they generate.

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An Optimal Pursuit Differential Game Problem with One Evader and Many Pursuers

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Abstract: The objective of this paper is to study a pursuit differential game with finite or countably number of pursuers and one evader. The game is described by differential equations in l_2-space, and integral constraints are imposed on the control function of the players. The duration of the game is fixed and the payoff functional is the greatest lower bound of distances between the pursuers and evader when the game is terminated. However, we discuss the condition for finding the value of the game and construct the optimal strategies of the players which ensure the completion of the game. An important fact to note is that we relaxed the usual conditions on the energy resources of the players. Finally, some examples are provided to illustrate our result.

Keywords: pursuit; control functions; integral constraints; strategies; value of the game

1. Introduction

The differential game has been an area of great interest to many applied mathematicians due to its application in solving real life problems in knowledge areas such as economics, engineering, missile guidance, behavioral biology. The first to study differential game was Rufus Isaacs [1], and one of the games analyzed was "the homicidal chauffeur game". For the fundamental concepts of differential games, see [1–8].

There are many different types of differential game problems, and one type is called pursuit-evasion differential game. Pursuit-evasion differential game is a game involving two players, called pursuer and evader, with conflicting goals. The aim of the pursuer is to complete the game in a finite time, whereas that of the evader is contrary. Strategies of pursuit and evasion play a role in many areas of life, such as missile launched at enemy aircraft, coastguard saving shipwrecked sailors, etc. The problem of constructing optimal strategies and finding the value of the game in a pursuit-evasion differential game motivated a lot of researchers to study this class of differential game problems, and fundamental results have been obtained, see [9–13].

Differential games of many pursuers with the integral and geometry constrained were studied by [9,14–19]. The case where the state variables are constraints was studied in [20], they considered a nonempty closed convex set in a plane, with the pursuers and evader movement restricted within the set during the game. Conditions under which pursuit could be completed were obtained and the strategies for the pursuers were constructed.

The evasion differential game of two dimensions, which involves one evader and several pursuers, was studied in [11]. The control functions of the players were subject with integral constraints. The game is solved by presenting explicit strategy for the evader, which guarantees evasion under the condition that there is no relation between the energy resource of the players.

In [21] Levchenko and Pashkov considered differential games described by simple differential equations, where the controls obeyed integral constraints. However, they showed that irrespective of the resources for controls of an individual, the completion of the game remains doable.

Ibragimov and Satimov [22] obtained sufficient condition for the completion of pursuit in a differential game problem of several pursuers and evaders in the space $\mathbb{R}^n$, with integral constraints on the control functions of the players. The results were obtained under the condition that the energy resource of the pursuers is greater than that of the evaders.

Ibragimov [9] studied a differential game of a countable number of pursuers pursuing one evader in Hilbert space l_2, with geometric constraints on the control functions of the pursuers and evader. Optimal strategies of the players were constructed and optimal pursuit time was found, under the assumption that the energy resource of the pursuers is greater than that of the evader.

Ibragimov and Kuchkarov [10] considered the same problem in [9], with integral constraints imposed on the control functions of the players. In this case, optimal strategies were constructed and value of the game was found under the assumption that energy resource of the evader is greater than that of any pursuers.

Salimi and Ferrara [12] studied a simple motion differential game with finite number of pursuers and one evader with integral constraints imposed on control of the players in Hilbert space l_2. The equations of motion are described by

$$\begin{aligned} P_i &: \dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \\ E &: \dot{y}(t) = v(t), \quad y(0) = y_0, \end{aligned} \tag{1}$$

where u_i is the control function of the i^{th} pursuers and v is that of the evader. The authors solved the problem and found the value of the game under the assumption that the energy resource of each pursuer is not necessarily greater than that of the evader, and optimal strategies of the pursuers were also constructed.

Inspired by the results in [9–12,22] and some known results on optimal pursuit problem in a Hilbert space l_2, the objective of this paper is to construct the optimal strategies and finding value of the game such that there is no relation between the energy resource of the players.

This paper is sectioned as follows. The second section present statement of the problems and some useful definitions which will be required for the later sections. In the third section, attainability domain of the players, optimal strategies and value of the game and some examples are given to show the application of the obtained results. In the last section, we give the concluding part of the paper.

2. Statement of the Problem

In this section, we present the statement of the problem and some useful definitions that will be used to prove our main theorem.

Here, we will consider an optimal pursuit problem with finite or countably many pursuers and one evader in a Hilbert space l_2, in such away that there is no relation between the their energy resources. In the space l_2, with elements

$$\alpha = (\alpha_1, ..., \alpha_k, ...), \quad \sum_{k=1}^{\infty} \alpha_k^2 < \infty,$$

the inner product and norm are defined as

$$(\alpha, \beta) = \sum_{k=1}^{\infty} \alpha_k \beta_k, \quad \|\alpha\| = \left(\sum_{k=1}^{\infty} \alpha_k^2 \right)^{1/2}.$$

Let P_i and E denote the motions of the pursuers and the evader, whose equations are described by

$$\begin{aligned} P_i : \dot{x}_i(t) &= \eta(t) u_i(t), \quad x_i(0) = x_{i0}, \\ E : \dot{y}(t) &= \eta(t) v(t), \quad y(0) = y_0, \end{aligned} \tag{2}$$

where $x_i(t), x_{i0}, u_i(t), y(t), y_0, v(t) \in l_2$, $u_i = (u_{i1}, \ldots, u_{ik}, \ldots)$ and $v = (v_1, \ldots, v_k, \ldots)$ are the control parameters of pursuer P_i and evader E, respectively. Throughout this paper, $i \in I = \{1, 2, 3, ...m\}$.

Let θ be a fixed time, and the function $\eta(t)$ be nonzero on any open interval—scalar measurable and square integrable over the interval $[0, \tau]$, $\tau > 0$. It is also assumed to satisfy the following conditions:

$$a(\tau) = \left(\int_0^{\tau} \eta^2(t) dt \right)^{1/2} < \infty. \tag{3}$$

$$H(x_0, r) = \{x \in l_2 : \|x - x_0\| \leq r\}, \quad S(x_0, r) = \{x \in l_2 : \|x - x_0\| = r\},$$

denote the ball and sphere, respectively, in the space l_2 with center x_0 and radius r. We now give some useful definitions.

Definition 1. *The admissible control of the ith pursuer is a function $u(\cdot)$, $u_i : [0, \theta] \to l_2$ defined as*

$$\|u_i(\cdot)\|_2 = \left(\int_0^{\theta} \|u_i(s)\|^2 ds \right)^{1/2} \leq \rho_i, \quad \|u_i\| = \left(\sum_{k=1}^{\infty} u_{ik}^2 \right)^{1/2},$$

provided that $u_{ik} : [0, \theta] \to \mathbb{R}^1$, $k = 1, 2, \ldots$, are Borel measurable functions and ρ_i is a fixed positive number $\forall i$. Let $B(\rho_i)$ denote the set of all admissible controls of the pursuer x_i.

Definition 2. *The admissible control of the evader is a function $v(\cdot)$, $v : [0, \theta] \to l_2$ defined as*

$$\|v(\cdot)\|_2 = \left(\int_0^{\theta} \|v(s)\|^2 ds \right)^{1/2} \leq \sigma,$$

where $v_k : [0, \theta] \to \mathbb{R}^1$, $k = 1, 2, \ldots$, are Borel measurable functions and σ is a fixed positive number. Let $B(\sigma)$. denote the set of all admissible controls of the evader y.

Once the players admissible controls $u_i(\cdot) \in B(\rho_i)$ and $v(\cdot) \in B(\sigma)$ are chosen, the corresponding motion $x_i(\cdot)$ and $y(\cdot)$ of the players are defined as

$$x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{ik}(t), \ldots), \quad y(t) = (y_1(t), y_2(t), \ldots, y_k(t), \ldots),$$

$$x_{ik}(t) = x_{ik0} + \int_0^{\theta} \eta(s) u_{ik}(s) ds, \quad y_k(t) = y_{k0} + \int_0^{\theta} \eta(s) v_k(s) ds, \quad i \in I, \quad k \in N.$$

It is not difficult to see that $x_i(\cdot), y(\cdot) \in C(0,\theta;l_2)$, where $C(0,\theta;l_2)$ is the space of functions

$$f(t) = (f_1(t), f_2(t), \ldots,) \in l_2, \quad t \geq 0,$$

such that

(i) $f_k(t), 0 \leq t \leq \theta, k \in N$, are absolutely continuous functions;
(ii) $f(t), 0 \leq t \leq \theta$, is continuous with respect to the norm on l_2 space.

Definition 3. *The strategy of the pursuer P_i is a function $U_i(t, x_i, y, v).U_i : [0,\infty) \times l_2 \times l_2 \times l_2 \to l_2$, such that the system of equations*

$$\begin{aligned}
&\dot{x}_i(t) = \eta(t)U_i(t, x_i, y, v(t)), \quad x_i(0) = x_{i0}, \\
&\dot{y}(t) = \eta(t)v(t), \quad y(0) = y_0,
\end{aligned}$$

has a unique solution $(x_i(\cdot), y(\cdot))$ for any $x_i(\cdot), y(\cdot) \in C(0,\theta;l_2)$ and admissible control $v = v(t)$, $0 \leq t \leq \theta$, of the evader E. We said that the strategy U_i is admissible if each control formed by strategy U_i is admissible.

Definition 4. *The optimal strategies U_{i0} of the pursuers P_i are defined as*

$$\inf_{U_1,\ldots,U_m,\ldots} \Gamma_1(U_1,\ldots,U_m,\ldots) = \Gamma_1(U_{10},\ldots,U_{m0},\ldots),$$

such that

$$\Gamma_1(U_1,\ldots,U_m,\ldots) = \sup_{v(\cdot)} \inf_{i\in I} ||x_{i0} - y(\theta)||,$$

where U_i and $v(\cdot)$ are admissible strategies of the pursuers P_i and evader E, respectively.

Definition 5. *The strategy of evader E is a function $V(t, x_1, \ldots, x_m, \ldots, y)$, $V : [0,\infty) \times l_2 \times \cdots \times l_2 \times \cdots \times l_2 \to l_2$, such that the system of equations*

$$\begin{aligned}
&\dot{x}_i(t) = \eta(t)u_i(t, x_i, y, v(t)), \quad x_i(0) = x_{i0}, \\
&\dot{y}(t) = \eta(t)V(t, x_1, \ldots, x_m, \ldots, y,), \quad y(0) = y_0,
\end{aligned}$$

has a unique solution $(x_i(\cdot), \ldots, x_m(\cdot), \ldots y(\cdot))$ for any $x_i(\cdot), y(\cdot) \in C(0,\theta;l_2)$, and admissible controls $u_i = u_i(t)$, $0 \leq t \leq \theta$ of the pursuers P_i. We said that strategy V is admissible if each control formed by strategy V is admissible.

Definition 6. *The optimal strategy V_0 of the evader E is defined as*

$$\sup_V \Gamma_2(V) = \Gamma_2(V_0),$$

provided that

$$\Gamma_2(V) = \inf_{u_1(\cdot),\ldots,u_m(\cdot),\ldots} \inf_{i\in I} ||x_{i0} - y(\theta)||,$$

u_i are admissible control of the pursuers P_i and V is that of the evader E.

If $\Gamma_1(U_{10},\ldots,U_{m0},\ldots) = \Gamma_2(V_0) = \lambda$, then, problem (2) has a value λ.

Our aim is to find the optimal strategies U_{i0}, V_0 of the players and value of the game, respectively.

3. Auxiliary Game

It is easily to see that the attainability domain of the pursuer P_i from the initial position x_{i0} at time θ is the closed ball $H(x_{i0}, a(\theta)\rho_i)$. Indeed

$$\begin{aligned}\|x(\theta) - x_{i0}\| &= \left\| \int_0^\theta \eta(s)u_i(s)ds \right\| \leq \int_0^\theta \eta(s)\|u_i(s)\|ds \\ &\leq \left(\int_0^\theta \eta^2(s)ds \int_0^\theta ||u_i(s)||^2 ds \right)^{1/2} \leq a(\theta)\rho_i.\end{aligned}$$

Moreover, if $\bar{x} \in H(x_{i0}, a(\theta)\rho_i)$, that is, $||\bar{x} - x_{i0}|| \leq a(\theta)\rho_i$, then, for the control

$$u_i(s) = \frac{\eta(s)}{a^2(\theta)}(\bar{x} - x_{i0}), \quad 0 \leq s \leq \theta$$

of the pursuer, we get

$$\begin{aligned}x_i(\theta) &= x_{i0} + \int_0^\theta \eta(s)u_i(s)ds \\ &= x_{i0} + \int_0^\theta \eta(s)\left(\frac{\eta(s)}{a^2(\theta)}(\bar{x} - x_{i0})\right)ds \\ &= x_{i0} + \frac{\bar{x} - x_{i0}}{a^2(\theta)} \int_0^\theta \eta^2(s)ds \\ &= \bar{x}.\end{aligned}$$

Therefore, the admissibility of this control follows from the relation

$$\int_0^\theta \|u_i(s)\|^2 ds = \int_0^\theta \left\| \frac{\eta(s)}{a^2(\theta)}(\bar{x} - x_{i0}) \right\|^2 ds \leq \frac{\|\bar{x} - x_{i0}\|^2}{a^4(\theta)} a^2(\theta) \leq \rho_i^2.$$

Moreover, applying the same procedure one can see that the attainability domain of the evader E at time θ from the initial state y_0 is the ball $H(y_0, a(\theta)\sigma)$.

In this section, we study a differential game of one pursuer x and one evader y. For simplicity, we use the notation $\rho_i = \rho$, $x_{i0} = x_0$, and $x_i = x$. Then, dynamics of x and y are described by

$$\begin{aligned}P &: \dot{x} = \eta(t)u, \quad x(0) = x_0, \\ E &: \dot{y} = \eta(t)v, \quad y(0) = y_0.\end{aligned} \tag{4}$$

The target of the pursuer P is to perceive the equality $x(\tau) = y(\tau)$ at some τ, $0 \leq \tau \leq \theta$; and that of the evader E is contrary.

Let

$$X = \{z : 2(y_0 - x_0, z) \leq a^2(\theta)(\rho^2 - \sigma^2) + \|y_0\|^2 - \|x_0\|^2\}.$$

Lemma 1. *If $y(\theta) \in X$, then, there exists an admissible strategy of the pursuer P which ensures $x(\theta) = y(\theta)$ in the game (4).*

Proof. Suppose the assumption of Lemma 1 holds, construct the strategy of the pursuer as follows:

$$u(t) = \frac{\eta(t)}{a^2(\theta)}(y_0 - x_0) + v(t), \quad 0 \leq t \leq \theta. \tag{5}$$

Then, admissibility of this strategy can be proved as follows. Since

$$y(\theta) = y_0 + \int_0^\theta \eta(s)v(s)ds \in X,$$

then,

$$\begin{aligned} 2(y_0 - x_0, y(\theta)) &\le a^2(\theta)(\rho^2 - \sigma^2) + \|y_0\|^2 - \|x_0\|^2, \\ 2\left(y_0 - x_0, y_0 + \int_0^\theta \eta(t)v(t)dt\right) &\le a^2(\theta)(\rho^2 - \sigma^2\theta) + \|y_0\|^2 - \|x_0\|^2, \\ 2(y_0 - x_0, \int_0^\theta \eta(t)v(t)) &\le a^2(\theta)(\rho^2 - \sigma^2) - (\|y_0\|^2 - 2(x_0, y_0) + \|x_0\|^2). \end{aligned} \tag{6}$$

Hence from the strategy (5) and inequality (6), we have

$$\begin{aligned} \int_0^\theta \|u(t)\|^2 dt &= \int_0^\theta \left\| \left(\frac{\eta(t)}{a^2(\theta)}(y_0 - x_0) + v(t) \right) \right\|^2 dt \\ &= \int_0^\theta \frac{\eta^2(t)}{a^4(\theta)} \|y_0 - x_0\|^2 dt + \frac{2}{a^2(\theta)} \int_0^\theta \eta(t)(y_0 - x_0, v(t))dt + \int_0^\theta \|v(t)\|^2 dt \\ &\le \frac{\|y_0 - x_0\|^2}{a^4(\theta)} a^2(\theta) + \frac{1}{a^2(\theta)} \left(a^2(\theta)(\rho^2 - \sigma^2) - \|y_0 - x_0\|^2 \right) + \sigma^2 \\ &= \rho^2. \end{aligned} \tag{7}$$

This shows that the strategy (5) is admissible. Therefore,

$$\begin{aligned} x(\theta) &= x_0 + \int_0^\theta \eta(t) \left(\frac{\eta(t)}{a^2(\theta)}(y_0 - x_0) + v(t) \right) dt \\ &= x_0 + \frac{(y_0 - x_0)}{a^2(\theta)} \int_0^\theta \eta^2(t)dt + \int_0^\theta \eta(t)v(t)dt \\ &= x_0 + y_0 - x_0 + \int_0^\theta \eta(t)v(t)dt \\ &= y(\theta). \end{aligned}$$

This proves the lemma. □

Remark 1. *It should be noted that in the construction of the pursuer's strategy we do not require the inequality* $\rho \ge \sigma$.

4. Main Result

We recall the following lemmas in order to prove our main theorem.

Lemma 2. *(see Ibragimov et al. 2005. Lemma 9). Suppose* r *and* R_i, $i \in I$, *are fixed positive real numbers and* $H(x_{i0}, R_i)$, $i \in I$, *and* $H(y_0, r)$ *are collections of finitely or a countable number of closed balls. Let*

$$\begin{aligned} I_0 &= \{i \in I : S(y_0, r) \cap H(x_{i0}, R_i) \neq \emptyset\}, \\ X_i &= \{z \in l_2 : 2(y_0 - x_{i0}, z) \le R_i^2 - r^2 + \|y_0\|^2 - \|x_{i0}\|^2\}, \quad i \in I_0. \end{aligned}$$

If $(y_0 - x_{i0}, p_0) \ge 0, \quad i \in I$, *for a nonzero vector* p_0, *and*

$$H(y_0, r) \subset \bigcup_{i \in I} H(x_{i0}, R_i),$$

then,

$$H(y_0, r) \subset \bigcup_{i \in I_0} X_i.$$

Lemma 3. *(See Ibragimov 2005. Assertion 5). Let* $\inf_{i\in I} R_i = R_0 > 0$ *and* $(y_0 - x_{i0}, p_0) \geq 0$, $i \in I$, *for a nonzero vector* p_0, *if for any* $0 < \delta < R_0$ *the set* $\bigcup_{i\in I} H(x_{i0}, R_i - \delta)$ *does not contain the ball* $H(y_0, r)$, *then, there exists a point* $\bar{y} \in H(y_0, r)$ *such that* $\|\bar{y} - x_{i0}\| \geq R_i$, *for all* $i \in I$.

Theorem 1. *Let* $(y_0 - x_{i0}, p_0) \geq 0$ *for all* $i \in I$, *for a nonzero vector* $p_0 \in l_2$, *then, the number*

$$\lambda = \inf\{l \geq 0 : H(y_0, a(\theta)\sigma) \subset \bigcup_{i\in I}^{\infty} H(x_{i0}, a(\theta)\rho_i + l)\} \tag{8}$$

is the value of the game (2).

Proof. The proof of the theorem in divided into three parts:

1. Constructing the strategies of the pursuers:

   We introduce the fictitious pursuers (FPs) z_i, whose equations of motions are described by

$$\dot{z_i} = \eta(t)w_i(t), \quad z_i(0) = x_{i0}, \quad \left(\int_0^\theta ||w_i(s)||^2 ds\right)^{1/2} \leq \bar{\rho}_i = \rho_i + \frac{\lambda}{a(\theta)}. \tag{9}$$

It can be shown easily that the attainability domain of the fictitious pursuers z_i at time θ from the initial position x_{i0} is the ball

$$H(x_{i0}, \bar{\rho}_i a(\theta)) = H(x_{i0}, a(\theta)\rho_i + \lambda).$$

Next, we define the strategy of the fictitious pursuers $z_i, i \in I$, as follows:

$$w_i(t, v) = \begin{cases} \frac{\eta(t)}{a^2(\theta)}(y_0 - x_{i0}) + v(t), & if \quad 0 \leq t \leq \theta_{i0}, \\ 0, & if \quad \theta_{i0} < t \leq \theta, \end{cases} \tag{10}$$

where $\theta_{i0} \in [0, \theta]$ is the time for which

$$\int_0^{\theta_{i0}} \left\| \frac{\eta(t)}{a^2(\theta)}(y_0 - x_{i0}) + v(t) \right\|^2 dt = \bar{\rho}_i^2. \tag{11}$$

Note that such time θ_{i0} may not exist.
We now define the strategies of the real pursuers $x_i, i \in I$, by

$$u_i(t, v) = \frac{\rho}{\bar{\rho}_i} w_i(t, v), \quad 0 \leq t \leq \theta. \tag{12}$$

where $\bar{\rho}_i = \bar{\rho}_i(0) = \rho_i + \frac{\lambda}{a(\theta)}$.

2. The value λ is guaranteed for the pursuers.

   We now show that the strategies (12) satisfy the inequality

$$\sup_{v(\cdot)} \inf_{i\in I} \|y(\theta) - x_i(\theta)\| \leq \lambda. \tag{13}$$

Thus, it follows from definition of λ that

$$H(y_0, a(\theta)\sigma) \subset \bigcup_{i\in I} H(x_{i0}, a(\theta)\rho_i + \lambda).$$

Then, it follows from Lemma 2 where $R_i = a(\theta)\rho_i + \lambda$ and $r = a(\theta)\sigma$, that

$$H(y_0, a(\theta)\sigma) \subset \bigcup_{i\in I} X_i,$$

where

$$\begin{aligned} I_0 &= \{i \in I : S(y_0, a(\theta)\sigma) \cap H(x_{i0}, a(\theta)\rho_i + \lambda) \neq \emptyset\} \\ &\textit{and} \\ X_i &= \{z \in l_2 : 2(y_0 - x_{i0}, z) \leq (a(\theta)\rho_i + \lambda)^2 - a^2(\theta)\sigma^2 + \|y_0\|^2 - \|x_0\|^2\}. \end{aligned} \tag{14}$$

Accordingly, the point $y(\theta) \in H(y_0, a(\theta)\sigma)$ belongs to some half-space X_s, with $s \in I_0$ and hence,

$$2(y_0 - x_{is}, y(\theta)) \leq (a(\theta)\rho_s + \lambda)^2 - a(\theta)\sigma^2 + \|y_0\|^2 - \|x_{s0}\|^2. \tag{15}$$

By Lemma 1, for the strategies of the pursuers $z_i, i \in I$, we obtain $z_s(\theta) = y(\theta)$ and

$$\int_0^\theta \left\| \frac{\eta(t)}{a^2(\theta)}(y_0 - x_{s0}) + v(t) \right\|^2 dt \leq \bar{\rho}^2. \tag{16}$$

By strategy (12), we have

$$\begin{aligned} \|x_s(\theta) - y(\theta)\| &= \|x_s(\theta) - z(\theta)\| \\ &= \left\| \int_0^\theta \eta(t)u_s(t)dt - \int_0^\theta \eta(t)w_s(t)dt \right\| \\ &= \left\| \int_0^\theta \eta(t)(\frac{\rho_s}{\bar{\rho}_s} - 1)w_s(t)dt \right\| \\ &\leq \frac{\lambda}{a(\theta)\bar{\rho}_s} \int_0^\theta \eta(t)\|w_s(t)\|dt. \end{aligned} \tag{17}$$

Using Cauchy–Schwatz inequality, we obtain

$$\int_0^\theta \eta(t)\|w_s(t)\|dt \leq \left(\int_0^\theta \eta^2(t)dt \right)^{1/2} \left(\int_0^\theta \|w_s(t)\|^2 dt \right)^{1/2} \leq a(\theta)\bar{\rho}_s. \tag{18}$$

Therefore, from inequalities (17) and (18) we obtain

$$\|x_s(\theta) - y(\theta)\| \leq \lambda. \tag{19}$$

Hence, the value λ is guaranteed by the actual pursuers.

3. The value λ is guaranteed for the evader.

   Define the evader's strategy that satisfies

$$\inf_{u_1(\cdot),\ldots,u_m(\cdot),\ldots} \inf_{i \in I} \|y(\theta) - x_i(\theta)\| \geq \lambda, \tag{20}$$

where $u_1(\cdot), \ldots, u_m(\cdot), \ldots$ are the admissible control of the pursuers. If $\lambda = 0$, then, the result follows from (20). Suppose that $\lambda > 0$, then, by definition of λ, for any $0 < \delta < \lambda$, the set

$$\bigcup_{i \in I} H(x_{i0}, a(\theta)\rho_i + \lambda - \delta)$$

does not include the ball $H(y_0, a(\theta)\sigma)$. According to Lemma 3, there exists a point $\bar{y} \in S(y, a(\theta)\sigma)$ such that $\|\bar{y} - x_{i0}\| \geq a(\theta)\rho_i + \lambda, i \in I$. Therefore, by the inequality

$$\|x_i(\theta) - x_{i0}\| \leq a(\theta)\rho_i, \quad i \in I,$$

We obtain

$$\|\bar{y} - x_i(\theta)\| \geq \|\bar{y} - x_{i0}\| - \|x_i(\theta) - x_{i0}\| \geq a(\theta)\rho_i + \lambda - a(\theta)\rho_i = \lambda.$$

Hence, if the evader comes to the point $\bar{y}$ at the time θ, the inequality (20) is guaranteed. This is achievable using the control function

$$v(t) = \frac{\sigma}{a(\theta)}\eta(t)e, \quad 0 \leq t \leq \theta, \quad e = \frac{\bar{y} - y_0}{||\bar{y} - y_0||}, \tag{21}$$

that is,

$$y(\theta) = y_0 + \int_0^\theta \eta(t)v(s)ds = \bar{y}. \tag{22}$$

Hence, the value of the game is not less than λ and the inequality is satisfied. This complete the proof of Theorem 1. □

We now present some examples to demonstrate our result.

Example 1. *Consider the differential game problem (2) with $\rho_i = 1, \quad \theta = 9, \quad \sigma = 5$, and denoting $\eta(t) = 1$. Consider the following initial positions of the players:*

$$x_{i0} = (0, \cdots, 0, 8, 0, \cdots), \quad y_0 = (0, 0, \cdots).$$

We can easily see that by simple computation we obtain that $a(\theta) = 3, a(\theta)\rho_i = 3$, and $a(\theta)\sigma = 15$. Next, we show that $\lambda = 14.$ is value of the game. Firstly, it is enough to show that

i) *Given any $\epsilon > 0$, the insertion*

$$H(O, 15) \subset \bigcup_{i=1}^{\infty} H(x_{i0}, 17 + \epsilon)$$

holds, where O is the origin.

ii) *Since the ball $H(O, 15)$ is not contained in the set $\bigcup_{i=1}^{\infty} H(x_{i0}, 13)$, that is, let $z = (z_1, z_2, \dots)$ be an arbitrary point of the ball $H(O, 10)$. This implies that $\sum_{i=1}^{\infty} z_i^2 \leq 225$. The following two cases are possible, either z has a non negative coordinate or all coordinates of z are negative. Suppose that there exists a non-negative coordinate z_k of the vector z, then,*

$$\begin{aligned} \|z - x_{k0}\| &= \left(z_i^2 + \dots + z_{k-1}^2 + (8 - z_k)^2 + z_{k+1}^2 + \dots\right)^{1/2} \\ &= \left(\sum_{i=1}^{\infty} z_i^2 + 64 - 16z_k\right)^{1/2} \\ &\leq (289 - 16z_k)^{1/2} \leq 17 \leq 17 + \epsilon, \end{aligned} \tag{23}$$

hence, $z \in H(x_{i0}, 17 + \epsilon)$.

Now, suppose that all the coordinates of z are negative. This implies that the inequality

$$\|z - x_{k0}\| = \left(\sum_{i=1}^{\infty} z_i^2 + 64 - 16z_k\right)^{1/2} \leq (289 - 16z_k)^{1/2} \leq 17 + \epsilon$$

is satisfied for sufficiently large k, since the series $\sum_{i=1}^{\infty} z_k^2$, $z_k \to 0$ *as* $k \to \infty$ *is convergent. Additionally, any point* $z \in S(O, 15)$ *with negative coordinates does not belong to the set* $\bigcup_{i=1}^{\infty} H(x_{i0}, 13)$, *since for any number i,* $\|z - x_{i0}\| = (289 - 16z_i)^{1/2} > 13$.

Therefore, the number

$$\begin{aligned} \lambda &= \inf\left\{ l \geq 0 : \quad H(y_0, a(\theta)\sigma) \subset \bigcup_{i=1}^{\infty} H(x_{i0}, a(\theta)\rho_i) + l \right\} \\ &= \inf\left\{ l \geq 0 : \quad H(O, 15) \subset \bigcup_{i=1}^{\infty} H(x_{i0}, 3) + l \right\} = 14 \end{aligned} \tag{24}$$

is the value of the game by the theorem.

5. Conclusions

We have studied a fixed duration pursuit–evasion differential game of a countable number of pursuers pursuing one evader with integral constraints on control functions of the players. We obtained a sufficient condition for completion of pursuit and obtained the value of the game. It worth noting the following points:

- There is no relationship between the players energy resources;
- The case $\eta(t) = 1$ was studied in [12].

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mathematics

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Review

Applications of Game Theory in Project Management: A Structured Review and Analysis

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Abstract: This paper provides a structured literature review and analysis of using game theory to model project management scenarios. We select and review thirty-two papers from Scopus, present a complex three-dimensional classification of the selected papers, and analyse the resultant citation network. According to the industry-based classification, the surveyed literature can be classified in terms of construction industry, ICT industry or unspecified industry. Based on the types of players, the literature can be classified into papers that use government-contractor games, contractor–contractor games, contractor-subcontractor games, subcontractor–subcontractor games or games involving other types of players. Based on the type of games used, papers using normal-form non-cooperative games, normal-form cooperative games, extensive-form non-cooperative games or extensive-form cooperative games are present. Also, we show that each of the above classifications plays a role in influencing which papers are likely to cite a particular paper, though the strongest influence is exerted by the type-of-game classification. Overall, the citation network in this field is sparse, implying that the awareness of authors in this field about studies by other academics is suboptimal. Our review suggests that game theory is a very useful tool for modelling project management scenarios, and that more work needs to be done focusing on project management in ICT domain, as well as by using extensive-form cooperative games where relevant.

Keywords: decision-making; game theory; project management

1. Introduction

A project is a time-bound exercise undertaken by on organisation to obtain a product, service or result. "Project management" is defined by the PMBoK (Project Management Body of Knowledge) as "a temporary endeavour to create a unique product, service or result" [1–3]. The concept of project management has evolved through several schools of thought, and uses a range of theories originating from mathematics, computer science, economics, and other related fields in its modelling and analysis.

Sensible decision-making is critical for the success of projects [1–3]. Every project begins with a decision: the decision to invest. Often, an investor has several investment options, and each option will result in a different project, and thus one of the investment options has to be chosen before the project charter can be produced. Similarly, any large project involving subcontractors, for instance, a construction project, has a complex interplay between the main contractor (the project manager) and subcontractors, or among the subcontractors themselves, which typically has several decision points. For example, if there is an ambiguity in the contract between the contractor and subcontractor, each must decide how hard to push their case without jeopardising the whole project, and thus their own stake in it. Similarly, when projects from competing organisations are launched, the marketing personnel have to decide what is the best timing and strategy to market the project, or its resultant product or service, so that it can gain maximum traction in the face of competition. In each of these scenarios described above, the required decisions depend on the decisions of others who, in some way, have competing interests to the interests of the decision-maker.

Project management uses a range of concepts and tools in decision-making. These include investment analysis methods such as force field analysis, the life cycle cost method, internal rate of return etc., and other tools such as utility theory, prospect theory, Net Present Value (NPV) method, Monte Carlo analysis, linear programming, queueing theory and so on [4,5]. Recently, game theory [6–11] has been gaining prominence as a tool useful in decision-making in project management scenarios [12,13]. Compared to the other tools mentioned above, game theory is particularly useful in scenarios where a number of entities are trying to achieve the same outcome (either in competition with each other, or in cooperation with each other), but have independent and rational decision-making abilities. Thus, it is especially useful in decision-making in the face of competition, particularly in scenarios such as those described above. Game theory also offers a rigorous mathematical framework, and has been successfully used in fields such as economics, social science, biology and computer science, presenting many precedents and examples for project management researchers to follow. Thus, papers analysing project management scenarios have arisen in the past decade, to use game theory in their modelling. It is, however, a nascent field, and the purpose of this review is to advocate for the widespread use of game theory as a modelling and analysis tool in project management, by summarising the state-of-the-art in this niche, classifying the existing works and highlighting gaps in the literature and opportunities for future research.

There is a considerable (but not overwhelming) body of work which has attempted to use game theory in modelling project management scenarios. There is even a hypothesis (the "Bilton and Cummings" hypothesis) that states, "the use of game theory makes it possible to understand the needs and interests of the involved persons in a better way and to finalize the project successfully" [14,15]. Papers which use game theory in project management relate to construction projects, information and communications technology (ICT) projects and projects from other domains. They use various types of games, and model a diverse range of players such as governments, project managers/contractors, subcontractors and clients. Therefore, a well-structured review of the field becomes necessary to comprehend the state-of-the-art, classify existing studies and identify gaps in the literature. To our knowledge, such a review spanning several application domains yet focusing exclusively on project management and games does not exist. Indeed, there are works such as Kaplinski and Tamosaitiene [16], which focus on reviewing the work of individual authors or research groups or reviewing aspects of the use of game theory in operations research, which has some relevance to project management; still, a structured review focusing on the use of game theory in project management in all its domains and applications is lacking. Therefore, we present such a review here.

We select papers that use game theory in project management scenarios from the Scopus database using a rigorous selection process, and review and analyse these papers in great detail. We also analyse the relative impact of each paper to the particular niche of "game theory in project management", and to project management research is general. We introduce and use a complex multidimensional, yet principled, classification scheme that helps to highlight the areas where most effort has been exerted to date and areas where there are gaps in the literature. We also consider citation networks of the papers in the niche, and show how these networks can be related to and explained by the classification scheme that we present. Our review and analysis highlight why game theory is a very useful tool to model project management scenarios, and in what further ways it could be applied in project management contexts in the future.

In summary, the significance of this review is attested by the following aspects.

- We follow a comprehensive and principled method for searching and filtering relevant papers.
- We review papers across several disciplines, such as construction, ICT and education, and highlight the similarities and differences between them in their application of game theory in project management.
- We present a detailed multidimensional classification of the papers that we have reviewed.
- We present and analyse the citation network of the papers we have reviewed, highlighting their interdependency and relative impact.

- We identify gaps in the literature that point to potentially fruitful future research directions.

Overall, our review not only presents descriptions of the papers that we have considered, but also an in-depth analysis of the field.

The rest of this paper is organised as follows. In Section 2, we review relevant concepts in project management and game theory, which are alluded to throughout this paper. In Section 3, we describe the process by which papers were selected from Scopus to be included in this review. In Section 4 we describe how the reviewed papers can be classified and the justification for our classifications. In Section 5, we describe the papers that we review, summarise their findings, and compare their findings. In Section 6, we analyse the relative importance of the papers that we have reviewed by constructing citation networks among the papers reviewed, and comment on how the citation networks are influenced by the classification(s) of the papers. Finally, in Section 7, we summarise our findings, identify gaps in literature and directions of high potential for future research and provide our conclusions. In the rest of the paper, we use the terms "the niche" or "the field" to refer to the particular area of research that we are reviewing: the application of game theory in project management.

2. Background

2.1. Project Management

The concept of project management is well known, but for the sake of completeness, we give a brief overview here. Projects are the results of organisations responding to threats, such as market forces, regulatory requirements, financial constraints or opportunities such as competitive advantage, compliance or operational efficiency [1–3]. Essentially, when an organisation undertakes a project, it deviates from its "business as usual" mode and, through a time-bound transition, moves to a state which becomes the new "business as usual" at project completion. Projects are usually classified as mandatory, business critical, discretionary or stewardship projects [1,2,17]. The theory of project management has undergone an evolutionary process in the last several decades, with several schools of thought gaining prominence from time to time, such as the optimisation school of thought, the governance school of thought, the modelling school of thought, the decision school of thought, etc. [1,18,19].

2.2. Game Theory

Game theory, which is the study of strategic decision making, was first developed as a branch of microeconomics [6,7,9,20]. However, later, it has been adopted in diverse fields of study, such as evolutionary biology, sociology, psychology, political science and computer science [21–24]. Game theory is used to study many phenomena and behavioural patterns in human societies and socio-economical systems, such as the emergence and sustaining of cooperation in communities and organisations [22,25–28], modelling of unethical or criminal behaviour [29,30] or the decision-making processes involved in vaccination against epidemics [31–33]. Game theory has gained such wide applicability due to the prevalence of strategic decision-making scenarios across different disciplines. A typical game defined in game theory has two or more players, a set of strategies available to these players, and a corresponding set of pay-off values (sometimes called utility values) for each player (which are, in the case of two-player games, often presented as a pay-off-matrix). Game theory can be classified into two broad domains: non-cooperative game theory and cooperative game theory.

2.2.1. Non-Cooperative Games and Cooperative Games

Typically, games are played for the self-interest of the players, even when the players cooperate; cooperation is the best strategy under the circumstances to maximise the individual pay-offs of the players. In such games, the cooperative behaviour, if it emerges, is driven by selfish goals and is transient. These games can be termed "non-cooperative games". Non-cooperative game theory is the branch of game theory that analyses such games. On the other hand, in a cooperative game, or coalitional game, players form coalitions, or groups, often due to external enforcement of cooperative behaviour, and competition is between these coalitions [10,34,35]. Cooperative games are analysed using cooperative game theory, which predicts the coalitions that will form and the pay-offs of these coalitions. Cooperative game theory focuses on surplus or profit sharing among the coalition members, where the coalition is guaranteed a certain amount of pay-off by virtue of the coalition being formed. Often, the outcome of a cooperative game played in a system is equivalent to the result of a constrained optimisation process [36], and therefore many of the papers we review use a linear programming framework to solve the cooperative games they model.

2.2.2. Nash Equilibrium

Nash equilibrium is one of the core concepts in (non-cooperative) game theory. It is a state (a set of strategies) in a strategic game from which no player has an incentive to deviate unilaterally, in terms of pay-offs. Both pure strategy and mixed strategy Nash equilibria can be defined. A strategic game can have more than one Nash equilibrium [10,37]. It is proven that every game with a finite number of players in which each player can choose from finitely many pure strategies has at least one Nash equilibrium [37].

The formal definition of Nash equilibrium is as follows. Let (S, f) be a game with n players, where S_i is the strategy set of a given player i. Thus, the strategy profile S consisting of the strategy sets of all players would be $S = S_1 \times S_2 \times S_3 .... \times S_n$. Let $f(x) = (f_1(x), ..., f_n(x))$ be the pay-off function for strategy set $x \in S$. Suppose x_i is the strategy of player i and x_{-i} is the strategy set of all players except player i; thus, when each player $i \in 1, ..., n$ chooses strategy x_i that would result in the strategy set $x = (x_1, ..., x_n)$, giving a pay-off of $f_i(x)$ to that particular player, which depends on both the strategy chosen by that player (x_i) and the strategies chosen by other players (x_{-i}). A strategy set $x^* \in S$ is in Nash equilibrium if no unilateral deviation in strategy by any single player would return a higher utility for that particular player [38]. Formally put, x^* is in Nash equilibrium if and only if

$$\forall i, x_i \in S_i : f_i(x_i^*, x_{-i}^*) \geq f_i(x_i, x_{-i}^*) \tag{1}$$

2.2.3. Zero-Sum Games

Zero-sum games are a class of non-cooperative games where the total of the pay-offs of all players is zero. In two player games, this implies that one player's loss in pay-off is equal to another player's gain in pay-off. A two-player zero-sum game can therefore be represented by a pay-off matrix, which shows only the pay-offs of one player. Zero-sum games can be solved with the mini–max theorem [39], which states that in a zero-sum game there is a set of strategies which minimises the maximum losses (or maximises the minimum pay-off) of each player. This solution is sometimes referred to as a "pure saddle point". It can be argued that the stock market is a zero-sum game. In contrast, most valid economic transactions are non-zero-sum, since each party considers that what it receives is more valuable (to itself) than what it parts with.

2.2.4. Common Interest Games

Common interest games are another class of non-cooperative games in which there is an action profile that all players strictly prefer over all other profiles [40]. In other words, in common interest games, the interests of players are perfectly aligned. It can be argued that common interest games are the antithesis of zero-sum games in which the interests of the players are perfectly opposed, so that any increase in fortune for one player must necessarily result in the collective decrease in fortune for others. Common interest games were first studied in the context of cold war politics, to understand and prescribe strategies for handling international relations [41–43].

It makes sense to classify non-cooperative games into common interest games and non-common interest games, just as much as it makes sense to classify them into zero-sum games and non-zero sum games, since these two concepts (zero-sum games and common interest games) represent extreme cases of non-cooperative games. However, the papers we reviewed do not use common interest games to model project management scenarios, and it would be rare to find scenarios in project management where the interests of players are perfectly aligned. Therefore, we do not use the common interest games-based classification in our classification process, as it would add another dimension and increase the complexity of classification needlessly.

2.2.5. Normal-Form Games and Extensive-Form Games

In a normal-form game, only a single round of decision-making takes place, where all players make decisions simultaneously. An extensive-form game is, on the other hand, an iterative game where there are several rounds of decision-making [44,45]. On each round, players could make decisions simultaneously or in some predefined order. An extensive-form game is often represented by a game tree, where each node (except terminal nodes) is a decision point, and each link corresponds to a decision or a set of decisions that could be made by the relevant player/players at that point. The terminal nodes represent an end to the extensive-form game, corresponding to pay-offs for each player involved.

2.2.6. Simultaneous Games and Sequential Games

A simultaneous game is either a normal-form game or an extensive-form game where on each iteration, all players make decisions simultaneously. Therefore, each player is forced to make the decision without knowing about the decisions made by other players (on that iteration). On the contrary, a sequential game is a type of extensive-form game where players make their decisions (or choose their strategies) in some predefined order [44,45]. For example, a negotiation process can be modelled as a sequential game if one party always has the privilege of making the first offer; the other parties make their offers or counteroffers after that. In a sequential game, at least some players can observe at least some of the actions of other players before making their own decisions (otherwise, the game becomes a simultaneous game, even if the moves of players do not happen simultaneously in time). However, it is not required that every move of each previous player should be observable to a given player. If a player can observe every move of every previous player, such a sequential game is known to have "perfect information"; otherwise, the game is known to have "imperfect information". Sequential games are often used by papers that we are reviewing here, to model bargaining or negotiation processes.

2.2.7. Subgames

A subgame is a subset of a sequential game, such that at its beginning, every player knows all the actions of the players that preceded it [6]. That is, a subgame is a section of the game tree of a sequential game where the first decision node that belongs to this section has perfect information.

2.2.8. Subgame Perfect Nash Equilibrium

In a sequential game, a subgame perfect Nash equilibrium is a set of strategies representing each player such that they constitute a Nash equilibrium for every subgame of that sequential game [6]. Thus, when a game tree of the sequential game is considered, if a set of strategies could be identified so that they represent a Nash equilibrium for every branch of game tree originating from a node representing a point where every player knows all preceding actions of all players, such a set of strategies represent a subgame perfect Nash equilibrium for that sequential game. For example, when two players are bargaining, they are in a subgame perfect Nash equilibrium if they are presently employing a set of strategies, which will represent a Nash equilibrium between them at any future stage of the bargaining process given that they are aware of the full history of the bargaining process up to that point.

2.2.9. Stackelberg Games

A Stackelberg game is a particular type of two player sequential game commonly used in economics [46]. In a Stackelberg game, there is a leader and a follower, which are typically companies operating in a market. The leader's firm has some sort of market advantage that enables it to move first and make the first decision, and the follower's optimal decision depends on the leader's decision. If a follower chose a non-optimal action given the action of the leader, it will adversely affect not only the pay-off of the follower, but the pay-off of the leader as well. Therefore, the optimal decision of the leader is made on the assumption that the follower is able to see the leader's action and that it will act to maximise its own pay-off given the leader's action. Several papers we review here have used Stackelberg games to model project management scenarios.

2.2.10. Nash Bargaining

In a Nash bargaining game [47], sometimes referred to as a bargaining problem or bargaining game, two players could choose from an identical set of alternatives, however each alternative has varying pay-offs for the players. Typically, some alternatives have better pay-off for one player, whereas other alternatives have better pay-off for the other player. If both players choose the same alternative, then each get the pay-off corresponding to that alternative. If they choose differing alternatives, then there is no agreement, and they each get a fixed pay-off which corresponds to the cost of non-agreement, and therefore typically negative. Thus, there is incentive to choose an alternative which may not necessarily be the best for a player. If there is perfect information, that is, the full set of alternatives and pay-offs is known to both players, then there is an equilibrium solution to the Nash bargaining game [48].

2.2.11. Evolutionary Game Theory

Evolutionary game theory is an outcome of the adoptation of game theory into the field of evolutionary biology [49,50]. Some of the critical questions asked in evolutionary game theory follow. Which populations/strategies are stable? which strategies can 'invade' (become popular) in populations where other strategies are prevalent? How do players respond to other players receiving or perceived to be receiving better pay-offs in an iterated game setting? Evolutionary games are often modelled as iterative games where a population of players play the same game iteratively in a well-mixed or a spatially distributed environment [38,51].

A strategy can be identified as an evolutionary stable strategy (ESS) if, when prevalent, it has the potential to prevent any mutant strategy from percolating its environment [38]. Alternatively, an ESS is the strategy which, if adopted by a population in a given environment, cannot be invaded by any alternative strategy. Hence, there is no benefit for a player to switch from an ESS to another strategy. Therefore, essentially, an ESS ensures an extended Nash equilibrium. For strategy, S_1, to be ESS against another "invading" strategy, S_2, one of the two conditions mentioned below needs to be met, in terms of expected pay-off, E.

1. $E(S_1, S_1) > E(S_2, S_1)$: By unilaterally changing strategy to S_2, the player will lose out against another player who sticks with the ESS S_1.
2. $E(S_1, S_1) = E(S_2, S_1)$ & $E(S_1, S_2) > E(S_2, S_2)$: a player, by converting to S_2, neither gains nor loses against another player who sticks with the ESS S_1, but playing against a player who has already "converted" to S_2, a player is better off playing the ESS S_1.

If either of these conditions are met, the new strategy S_2 is incapable of invading the existing strategy S_1, and thus S_1 is an ESS against S_2. Evolutionary games are typically modelled as iterative games, whereby players in a population play the same game iteratively.

3. Selection Methodology

In this section, we describe how we have selected the papers to review and classify. The papers were selected from the Scopus database [52], with any paper which was cited by one of the papers selected from Scopus but not in Scopus itself, and which satisfied all the selection criteria as described below, also being "manually" included. The selection of papers from the Scopus database included a four-step process.

1. Scopus database was searched for a number of key phrases, as elaborated below, and all papers deemed relevant in this search were downloaded. A total of 776 papers were downloaded and considered in this manner.
2. A brief manual screening was conducted considering the title and abstract of the papers, selecting some papers for detailed inspection. A total of 72 papers remained at the end of this stage.
3. After detailed reading of each paper, some papers were excluded from our review and classification, as described below. A total of 25 papers remained at the end of this stage.
4. The reference list of each remaining paper was considered, and checked against the list of papers already excluded. If a paper was not already excluded or not in Scopus, then it went through steps 2 and 3 of the screening process, and included in our review and classification if selected. At the end of this step, 32 papers were selected for inclusion in this review.

Clearly, it must be acknowledged that such a selection process has some limitations: it only considers papers from Scopus or papers having some citation relationship with the papers in Scopus, and it is also, implicitly, focused on papers which are published in international journals and internationally recognised conferences, which are more likely to be included in Scopus, as opposed to regionally focused papers or papers which are published in regional outlets. Therefore, the search is not exhaustive, and there may be other papers of merit. Nevertheless, the selection process is principled and represents at least a very significant section of the field, including almost all papers which have had international research impact.

Now we describe each step of this selection process in more detail.

3.1. Step 1

In step one, the following combinations of key phrases were used in Scopus to search the database and select the relevant papers. Only the titles of the papers were searched.

1. ("game theo*") and ("project management" or "construction management" or "*contract")
2. ("decision") and ("project management" or "construction management" or "*contract")
3. ("games") and ("project management" or "construction management" or "*contract")

These key phrases were selected so that papers having "construction management" or '1contract management" instead of "project management" in the title could also be considered. After deleting the duplicates, a total of 776 papers were found and downloaded.

3.2. Step 2

In step two, the title and abstract of each of the 776 papers are selected in the previous step were manually considered, and papers were selected according to the following criteria.

1. Title: those papers having titles where the keywords have been used in a different context were omitted. For example, if the word "game" was used in the sense of video game, computer game or other simulated game, those papers were omitted. Similarly, papers with titles where the word "contract" was used in a context not related to project management were omitted.
2. Abstract: Those papers which had abstracts that made it clear that the paper deals with ongoing operational issues, and not project management issues, were omitted.

3. English language: only records written in English were further considered. Records nominally written in English where the quality of the narrative was so poor as to not make sense to a reasonable native English speaker, were also excluded. Papers which had their title and/or abstract in English, but the body of the paper in another language, were also excluded.
4. Availability: Papers which were not publicly available (either freely, or for a fee) were excluded. Note, papers that were publicly available for a fee were not excluded. Only those papers which did not have the full content publicly available freely or for a fee were excluded.
5. In cases where two very similar papers by the same authors were found, where one paper is an extension of the other but included all content of the previous paper, only the later (and thus the more 'matured') paper was considered. For example, if a conference publication was later developed into a journal paper by the inclusion of additional material, the conference paper was excluded and only the journal paper was considered.

At the end of step 2, 72 papers were selected for complete and careful scrutiny.

3.3. Step 3

In this step, each of the remaining 72 papers was carefully and completely read and considered; papers were excluded if they use game theory in operational management scenarios, but not necessarily in project management particularly. To this end, the definition of project management as a time-bound activity was considered important [1–3], and thus papers describing the use of game theory to model day-to-day operational issues, or the 'business-as-usual' operation of the company, were excluded. For example, a paper describing the negotiation process between an airline and a freight forwarder using game theory was excluded [53]; this negotiation reflects the routine business operations of both parties, which they do not necessarily need to implement a project to undertake. Papers were also excluded if it was possible that a project management application may exist for their modelling, but they did not explicitly describe such an application. In Section 5.3, we describe some of the papers which passed stage 2 and were excluded at stage 3, so that the interested reader may appreciate why such papers were excluded, and how they may still be of some relevance. However, we do not classify these papers, as per the reasons described for each individual paper later, we consider them out of scope for this review.

It should also be pointed out that a considerable body of management-related literature exists in fields such as physics and sociology, which uses game theory. For example, the incentives for cooperation in society as well as organisations have been studied from a rigorous statistical mechanics point of view, in papers such as the works by the authors of [28,54], or from a systemic point of view, in papers such as the work by the authors of [55–58]. However, such papers also would be out of the scope of this review due to the definition of project management as a time-bound exercise, which we strictly adhere to in selecting papers to review. In other words, these papers do not clearly distinguish project management from organisational management, which is concerned with the day-to-day "business as usual" of the organisation, and as such, these papers are not within scope of this review. Nevertheless, the interested reader is referred to the above-mentioned papers which highlight some interesting overlaps between game theory and the concept of management as perceived in other fields.

3.4. Step 4

In this step, we compiled a combined list of references from the 25 papers in step 3, using a simple computer script. From this list, any paper selected in step 1 (selected from Scopus based on the key phrases) was excluded. The remaining papers were either not in Scopus or were in Scopus but not selected in step 1. Then, these papers were put through steps 2 and 3 (above), respectively (that is, title and abstract scan, and selected papers to detailed reading), and seven papers were selected after step 3. These seven papers were added to the pool of twenty-five papers already selected at stage 4. This step is used to ensure that any relevant paper cited by other papers in the niche, but is not present in Scopus or selected in keyword search, is considered.

Therefore, after step 4, a total of 25 + 7 = 32 papers remained, which are reviewed and classified in this study, as described below.

4. Classification of the Selected Papers

We propose that the selected papers shall be classified as follows.

- Classification based on the application domain
- Classification based on the players of the game
- Classification based on the type of game played

4.1. Classification Based on the Application Domain

Here, we propose that, based on the domain of the industry in which it is applied, papers undertaking game-theoretic analysis of project management can be classified into three classes.

- Papers focusing on project management in construction projects
- Papers focusing on project management in ICT (Information and Communications Technology) projects
- Papers focusing on project management in other fields or generic project management

4.2. Classification Based on Players of the Games

In any game-theoretic modelling of real world, the "players" of the game usually represent some real world being or entity. When game theory is used in project management, the players of the game usually represent investors, project managers, contractors, subcontractors, the government or other individuals or organisations involved in running the project. Since a game needs multiple players, several combinations of above-mentioned individuals can be represented in a game, and a complex classification of games is possible based on these combinations. Fortunately, most of the papers we reviewed use two-player games, and even when the number of players is more than two, the types of players present in a game is usually restricted to two. Therefore, the games present in the literature can be classified based on the following combinations of players. In our terminology, and for classification purposes, a "contractor" is the governing entity of a project which makes decisions on behalf of the project, variously called project manager, main contractor, private company or contractor in the literature. We distinguish a contractor from a "subcontractor" in the sense that the contractor is responsible for the overall delivery of the project, represents the project and makes decisions for the overall benefit of the project, whereas a subcontractor is responsible for one particular task or aspect of the project. Thus we propose the following classification.

- Papers focusing on contractor—contractor games (including investment decision games)
- Papers focusing on contractor—subcontractor games
- Papers focusing on contractor—government games (sometimes called Public–Private Partnerships, where the phrase 'public' represents the government and 'private' represents the private company which is the contractor)
- Papers focusing on subcontractor—subcontractor games
- Papers focusing on games with other player combinations

4.3. Classification Based on the type of Game Played

Here, we propose a complex matrix classification. On one-hand, games employed in game theory can be classified into non-cooperative games and cooperative games. One subclassification of non-cooperative games is zero-sum games and non zero-sum games. Zero-sum games cannot by definition exist in a cooperative game, as if the total pay-off for the community is by definition zero, it cannot be a fixed (non-zero) value and it cannot be shared. Another possible subclassification of non-cooperative games is common interest games and non-common interest games, but since the papers we review do not use common interest games, this subclassification is not necessary here. Both non-cooperative games and cooperative games are used by papers reviewed here. On the other hand, games can also be classified into normal-form games and extended-form games. Both normal-form games and extended-form games are used in the papers we review here.

Therefore, a matrix classification with four categories is possible: normal-form non-cooperative games, normal-form cooperative games, extensive-form non-cooperative games and extensive-form cooperative games. Normal-form non-cooperative games have two subclasses: normal-form non-cooperative zero-sum games and normal-form non-cooperative non-zero-sum games. Similarly, extensive-form non-cooperative games have two subclasses: extensive-form non-cooperative zero-sum games, and extensive-form non-cooperative non-zero-sum games. The resultant complex matrix can be used to classify the papers according to the type of game used.

These classifications, containing the relevant papers, are shown (Note that, in order to conserve space, a naming convention is used in the figures, which is different from the standard nomenclature and that used in the text of the paper. In the text of the paper, a paper by a single author is denoted by their surname, a paper by two authors is denoted by both their surnames, whereas a paper by three or more authors is denoted as "surname-of-first author et al.", as is the normal convention. To add further clarification, the publication year is also mentioned—for example, Peldschus et al., 2010. In figures however, to conserve space, the papers are denoted simply by the surname of first author and the publication year, regardless of the number of authors. For example, Peldschus et al., 2010 is simply denoted as Peldschus 2010. In either case, an alphabet is added if the same set of authors published more than one paper in the same year. For example, 'Bergantinos and Sanchez, 2002 a' is one of two papers published by these two authors in 2002, which is simply denoted as 'Bergantinos 2002 a' in the figures.) in Figures 1–3. Since these classifications are overlapping, each paper that we reviewed is present in some class of each classification. Therefore, overall, it could be argued that the reviewed literature is best captured in a multidimensional complex classification.

Construction sector	ICT sector	Other sectors or generic project management
Kaplinski 2010	Angelou 2009	Lagesse 2006
Sacks 2006 a	Butterfield 2001	Bockova 2015
Asgari 2013	Bakshi 2012	Kwon 2010
Barough 2012		Lippman 2013
Medda 2007		Bockova 2016
Peldschus 2008		Branzei 2002
Shen 2007		Bergantinos 2002 a
Shen 2002		Bergantinos 2002 b
Javed 2014		Estevez 2007
Li 2016		Estevez 2012
Asgari 2014		Peldschus 2006
Sacks 2006 b		
Zou 2009		
Peldschus 2010		
Kemblowski 2017		
Cristoba 2014		
Cristobal 2015 a		
Cristobal 2015 b		

Figure 1. The classification of papers reviewed based on the application domain.

Government sector – Private sector game (Public - Private partnership)	Contractor - contractor game	Contractor - subcontractor game	Subcontractor - subcontractor game	Other players
Medda 2007	Kaplinski 2010	Sacks 2006 a	Asgari 2013	Lagesse 2006
Shen 2002	Angelou 2009	Barough 2012	Cristoba 2014	Peldschus 2008
Shen 2007	Bockova 2015	Kwon 2010	Cristobal 2015 a	Peldschus 2010
Javed 2014	Butterfield 2001	Lippman 2013	Asgari 2014	
Li 2016	Bakshi 2012	Sacks 2006 b	Cristobal 2015 b	
Zou 2009	Kemblowski 2017		Branzei 2002	
Bockova 2016	Peldschus 2006		Bergantinos 2002 a	
			Bergantinos 2002 b	
			Estevez 2007	
			Estevez 2012	

Figure 2. The classification of papers reviewed based on the players of the games.

	Non Co-operative game		Co-operative game
	Zero-sum	Non zero-sum	
Normal form game	Kaplinski 2010 Peldschus 2008 Peldschus 2010	Barough 2012 Medda 2007 Bockova 2015 Bockova 2016 Kemblowski 2017	Lagesse 2006 Cristoba 2014 Cristobal 2015 a Cristobal 2015 b Zou 2009 Bakshi 2012 Branzei 2002 Bergantinos 2002 a Bergantinos 2002 b Estevez 2007 Estevez 2012 Peldschus 2006 Asgari 2013 Asgari 2014
Extensive form game		Sacks 2006 a Kwon 2010 Angelou 2009 Shen 2002 Shen 2007 Butterfield 2001 Javed 2014 Li 2016 Lippman 2013	Sacks 2006 b

Figure 3. The classification of papers reviewed based on the type of game played.

Overall, the rest of the paper is structured based on the game-based classification. Nevertheless, some papers use more than one type of game. In this case, the paper is described in the subsection which is most relevant, as it is pointless to describe the same paper in multiple sections. Similarly, some papers also use more than one type of player combination or may belong to more than one application domain, though these classifications are not used to structure this review. It should be noted that the Figures 1–3 highlight each paper according to its most relevant classification.

5. Description of the Papers Reviewed

In this section, we review the 32 papers selected and classified above, following the game-based classification to structure the review.

5.1. Papers Using Normal-Form Games

As shown in Figure 3, there are in total 22 papers using normal-form games.

5.1.1. Papers Using Normal-Form Non-Cooperative Games

As shown in Figure 3, there are eight papers using normal-form non-cooperative games.

Papers using normal-form non-cooperative zero-sum games: In this section, we describe papers which exclusively use zero-sum games, or their primary modelling uses zero-sum games, within the normal-form non-cooperative game category. As shown in Figure 3, there are three such papers.

Peldschus et al. (2010) [59] apply game theory in construction management, for the specific task of selecting a suitable construction site. It can be argued that construction management is a branch of project management, as most construction works are time-bound, and thus can be considered as projects. Peldschus et al. (2010) define a number of selection criteria, and construct a zero-sum game where the choices for potential construction sites are modelled as 'strategies' of one player, while the choices for selection criteria (such as price per square metre, total area, reputation/appeal of area, soak density, etc.) are modelled as the 'strategies' of another player. Then they present the mini–max solution for this game. The mini–max solution could be justified by the following argument (though such an argument is not explicitly presented in the paper). In the face of uncertainty about what sites the other player will choose, each player will take a minimal risk approach and accept the mini–max solution; that is, each player will take the solution that maximises the minimal value of the site (where the minimal value is obtained by comparing values according to a number of criteria), given uncertainty about other players' (builders') choices. The mini–max solution is modelled within the context of a zero-sum game, based on the assumption that summation of the intrinsic value of all construction sites available is constant, therefore all criteria must result in the same total value across all sites (therefore, if compared to criteria C_1, criteria C_2 increases the value of alternative (site) A_1 by u dollars, then all other alternatives (sites) assessed according to criteria C_2 must in total face a value reduction of u compared to criteria C_1. In this sense, this game could be called a constant-sum game rather than a zero-sum game). However, the above line of justification holds only when the competing investors are represented as players, whereas Peldschus et al. (2010) models a two player zero-sum game where the players have no corresponding real world roles.

Peldschus (2008) [12] presents a mix of theoretical frameworks and practical examples to demonstrate the use of game theory in construction project management. Partially a review of his previous works, his study focuses in particular on zero-sum games applied on investment decision scenarios, where the strategies of one "player" represent the available investment options (alternatives), whereas the strategies of the other player represents different investment evaluation methods (criteria). This representation of players is similar to his representation [59], described above. In this sense, the two players do not represent two human actors but rather two dimensions of the problem analysed, and thus the approach differs from most other studies in this field, or indeed in most other fields where game theory is applied. He also demonstrates the use of fuzzy games, and provides several real world examples where he demonstrates the application of the methods introduced.

Kaplinski and Tamosaitiene (2010) [16] reviewed the research of one particular academic (Professor Peldschus mentioned above) in the area of the application of game theory in construction engineering and management. Nevertheless, even though the particular academic has apparently done seminal work in the field, this article, which reviews one academic's work, obviously cannot be considered a comprehensive review of the field. The individual contributions of the particular academic mentioned in this review are reviewed individually in our own review, whenever such contributions are in English and satisfy our selection criteria for inclusion in this review as mentioned in Section 3. This paper discusses both zero-sum and non-zero-sum games in the construction management context, but mainly focuses on zero-sum games, as it discusses papers such as the work by the authors of [12,59] described above.

Papers using normal-form non-cooperative non-zero-sum games: There are five papers in our review, which exclusively or primarily use normal-form non-cooperative non-zero-sum games to model scenarios in project management, as shown in Figure 3.

Medda (2007) [60] uses game theory to model the allocation of risks between governments and private companies in public-private partnership projects as a bargaining process. Specifically, this process is modelled as a final offer arbitration game. Their particular focus is on Public–Private Partnership (PPP) projects in transport infrastructure. The "final offer arbitration" is an arbitration process designed to avoid court litigation, wherein the parties or arbitrator have no authority to further negotiate, but make a decision based on the final offers made by all parties concerned, making it ideal for game-theoretic modelling. Unlike many similar papers [61–64] that model bargaining processes using extensive-form games; Medda (2007) is able to use a normal-form non-cooperative game specifically because they focus on final offer arbitration. Medda (2007) presents a detailed analytical model and pay-off model, and calculates the equilibrium solution of the game-based on these.

Kemblowski et al. (2017) [65] use game theory, specifically normal-form non-cooperative non-zero-sum games, to model the bidding process for a highway construction project. The bidding process is further complicated by the fact that there are hidden costs in the project (such as the cost of purchasing a gravel pit). They show that the solution of this game depends on the probability of the hidden costs being non-zero, and for a certain range of this probability there are pure strategy equilibria, and beyond which there are mixed strategy equilibria. This brief paper is very specific in addressing only a well-defined and hypothetically formulated problem, which might be more extensive and complex in real world settings.

Bockova et al. (2015) [15] analyses the utility of game theory as a tool to manage educational projects in the Czech republic, during a specific period in time which they term "the post-conflict period". The paper is not a methodical literature review, but a combination of examples, results from other papers and results of interviews and surveys conducted by the authors. It essentially attempts to present and justify the case for game theory to be used as a tool in project management, using diverse methods. The paper claims to validate the hypothesis presented in another work by Bilton and Cummings [14]: "The use of game theory makes it possible to understand the needs and interests of the involved persons in a better way and to finalize the project successfully." This paper can be included in the category of papers using normal-form non-cooperative non-zero-sum games, on the basis that many examples the paper describes seem to lend themselves to modelling by such games. The study by Bilton and Cummings [14] itself does not figure in our review, due to it being included in a book, which, overall, does not satisfy our criteria for inclusion, and is concerned rather with operations research.

Bockova et al. (2016) [66] also claim to verify the so-called Bilton and Cumming's hypothesis. They claim to do so by a mixture of reviewing a select set of papers as well as conducting interviews with and running surveys among industry experts. Unlike [15], however, this paper focuses on public–private partnerships in post-conflict Czech republic. This paper has aspects of a review, however it does not provide justification for how the papers it comments on were selected from the literature. The paper also details several scenarios where game theory might be applicable based on media stories and the authors' knowledge of the country in question: the Czech republic. Thus, this study is qualitative and illustrative in nature. It is also included in the category of papers using normal-form non-cooperative non-zero-sum games, on the basis that many examples the paper describes seem to use such games.

5.1.2. Papers Using Normal-form Cooperative Games

There are 14 papers using normal-form cooperative games in our review, as shown in Figure 3. In some of these papers, cooperative games are used to model a scenario where optimisation of some quantity (such as the project cost) under certain constraints is required. Such papers are most often not domain-specific, but nevertheless have several examples inspired from the construction industry.

In a construction project, each subcontractor working on it have periods where they can work more efficiently and thus be more cost effective, and sometimes it is possible to 'trade' time-periods between subcontractors so that overall cost effectiveness or time efficiency of projects increases. Asgari and Afshar (2008) [67] present a cooperative game, where a group of subcontractors working on a number of simultaneous projects can trade their time commitment to each project for the benefit of all players. The cooperative game they present can also be thought of as a constrained optimisation problem, where the project cost or time for completion is minimised by trading time between subcontractors. They conclude the study with an illustrative example.

Similarly, Asgari et al. (2013) [68] present a game-theoretic framework for resource management in construction projects. This paper analyses how subcontractors could cooperate among themselves and share resources for maximum resource utility and profit. Thus they use a normal-form cooperative game to model the scenario.

Lagesse (2006) [69] employs game theory for task assignment within a project. Specifically, they show how game theory can be used to develop an algorithm for task assignments to each employee within a project based on employee skills, time available to each employee, as well as employee and management preference. The algorithm they present also includes in-built feedback mechanisms. Interestingly, though the authors state this is a game-theoretic model, the algorithm they present is essentially a matching algorithm that produces a stable matching solution between the set of employees and the set of tasks at any point in time, and is game-theoretic only in the sense that the "stable matching solution" is essentially an equilibrium solution.

Cristoba (2014) [70] (It appears that the author of this paper has spelt their surname as Cristoba in some papers and Cristobal in others.) proposes a game-theoretic framework to identify the particular activities that were responsible for delays in a project, and accordingly split the cost of the project delay among the respective subcontractors of these activities. They present a multiplayer game and use a linear programming approach to solve it. In this sense, this paper presents an optimisation solution rather than an equilibrium solution. This is a feature of several papers which use cooperative games to model project management scenarios, both in normal-form and extended-form, as we will see below.

Cristobal (2015) [71] also proposes a game-theoretic model to identify activities that are responsible for delays in (primarily construction) projects, and apportion the costs associated with delays among these activities accordingly. Again, the game they propose is a cooperative game, which they solve by modelling it as a constrained linear optimisation problem: an approach very similar to [72–74]. The overall benefit to the project is considered as the optimisation criteria, and individual activities (or the subcontractors responsible for them) are assumed to give priority to the cooperative over their self-interest.

Cristobal (2015) [13] is a very broad review of management science methods and methodologies used in project management, and one of the techniques it discusses is the use of game theory. However, its discussion of the use of game theory in project management is very brief and without any examples, since the paper overall is fairly abstract. This paper however is part of the body of work contributed by Cristobal and thus mentioned and classified here along with his other works.

Branzei et al. (2002) [75] focuses on cost sharing in delayed projects: specifically, how the costs associated with a delay can be "fairly" shared between the players who are individually and collectively responsible for a delay. Therefore, it could be argued that, rather than modelling the decision-making processes of the players involved, this study presents a cooperative game: an algorithm to share costs fairly, so that a ready justification for the cost allocation can be provided against the potential "protest" of selfish players. Thus their approach is broadly equivalent to solving a linear optimisation problem. Their approach, they say, is inspired by literature in bankruptcy and taxation, but their work itself is explicitly focused on project management. They present two cooperative games (which they term "coalitional" games), namely, the pessimistic delay game and the optimistic delay game. Both games are based on the activity graph of the project, and the cost sharing process is modelled as a game between activities of the project in the first game, and a game between paths in the activity graph in the second game. However, with their stated assumption that different agents (players) are responsible for different activities, and given that a player who is responsible for a specific activity in a project is typically a subcontractor, ultimately it could be argued that they model subcontractor–subcontractor games. This also holds true for several other papers which we review here, which adopt a similar approach, such as the works by the authors of [72–74]. Therefore, in Figure 2, these are all classified as papers that model subcontractor–subcontractor games.

Bergantinos and Sanchez (2002) [72] also consider how to share the additional cost of a project delay among the firms (subcontractors) responsible for it, using game theory. Again, following an approach similar to [70,71,75], they formulate a "cost game" and use constrained optimisation techniques, utilising the "Shapley value" of the cost game. Therefore, this is an algorithm for fair allocation of costs that each subcontractor then needs to be convinced about, rather than an equilibrium solution reached by selfish behaviour of each subcontractor whereby the subcontractor will have no incentive to deviate from it. Therefore, the practical value of such an approach is as a fair algorithm rather than a model of the behaviour of (typically selfish) subcontractors who accept their part of the cost in delaying the project.

Bergantinos and Sanchez (2002) [76] use non-transferable utility (NTU) games to divide slack time into different activities of the project, within the context of Project Evaluation and Review Technique (PERT). In this sense, this work has similarities to works by the authors of [71–75], which use game-theoretic modelling to assign costs related to project delay among participating activities (or subcontractors responsible for those activities) in the project.

The work of Estevez-Fernandez et al. (2007) [73] is essentially a generalisation on work done in papers such as the works by the authors of [71,72,75,76]. It presents a cost sharing model in situations where some activities in a project do not run according to schedule. It presents three categories: (a) situations where some activities are delayed, (b) situations where some activities are expedited and (c) situations where some activities are delayed and others are expedited. It models each of these scenarios using "project games". They show that delayed project games have a nonempty core. The expedited project games are shown to be convex. In the third scenario where some activities may be delayed and some activities may be expedited, related project games are shown to have a nonempty core. Again, it could be argued that this study presents an algorithm for cost allocation rather than a behavioural game mimicking the behaviour of subcontractors and an equilibrium solution.

Estevez-Fernandez (2012) [74] is a further generalisation of Estevez-Fernandez et al. (2007) [73], which models not only models the costs associated with delays, but also the rewards associated with expediting some activities. Again, a set of project games are defined and "coreness" of these games is analysed. The cost and reward assignment is again undertaken by treating the project game as a constrained linear optimisation problem.

It should be noted here, therefore, that the above-mentioned six papers [71–76] have similar approaches, and, as we will show later, they cite each other. This group of papers form an important "core" in the literature of applications of game theory in project management, which shares the cost of delays and all use cooperative games, while presenting slight variations and extensions from each other. As the citation networks we present later show, these papers form the most densely connected subnetwork of the citation network in this field.

Zou and Kumarasawamy (2010) [77] use game theory to model aspects of Public–Private Partnerships (PPPs), including risk sharing, financial negotiation and operation. The first two are concerned with project management, while the third aspect is not directly a project management aspect, since it takes place after the project has been completed. Yet, due to the presence of the first two aspects, this paper falls within the ambit of our review. They assume that the behaviour of the public sector in their interactions with private sector is predictable, whereas the private sector can be unpredictable in its interactions with public sector, and that the utility for public sector is derived from socio-economic benefits, whereas the utility for private sector is derived solely from economic benefits. Because pay-offs are not modelled mathematically, this paper does not present a rigorous mathematical equilibrium solution for the games it proposes, and is thus can be more appropriately considered as a conceptual paper. It also claims to have used interviews with industry experts to guide its modelling.

Bakshi et al. (2012) [78] focuses on software engineering project management. They primarily focus on investment decisions; that is, which software project among a number of viable choices is the best to invest in. They convert the game-theoretic framework into a constrained linear optimisation problem which is equivalent to a cooperative game, and they solve it using linear programming. They use four well-known investment decision techniques in project management as "strategies" of the players in the game: Net Present Value (NPV) method, Rate Of Return (ROR) method, Payback Period (which they denote as PB) method, and Project Risk (which they denote as PR) method. They demonstrate an equilibrium solution, by obtaining pay-off values from what they call 'expert decision' and from literature which is outside the scope of this review, being in the domain of operations research rather than project management.

Peldschus (2006) [79] is an invited editorial to the proceedings of a conference, among the many topics which were discussed therein is the use of game theory in project management. This contribution gives a general overview of the use of game theory in project-management related scenarios, without making a specific research contribution or providing a structured analysis. However, it seems to refer primarily to scenarios similar to those studied in the above-mentioned papers, utilising cooperative games, and therefore this paper could be categorised with the above-mentioned papers in terms of the type of game discussed.

5.2. Papers Using Extensive-Form Games

Ten papers among the papers covered by this review use extensive-form games to analyse project management related scenarios, as shown in Figure 3. Many of these papers focus on multiround bargaining within the project management context.

5.2.1. Papers Using eXtensive-Form Non-Cooperative Games

Nine out of the ten papers which use extensive-form games use non-cooperative extensive-form games to model the project management scenarios they study.

Papers using extensive-form non-cooperative zero-sum games: We did not find any papers eligible for our review, as specified in Section 3, which use extensive-form non-cooperative zero-sum games.

Papers using extensive-form non-cooperative non-zero-sum games: Nine papers covered in this review use extensive-form non-cooperative non-zero sum games to model project management scenarios. Several of these papers model bargaining between competing entities (such as contractor and subcontractor, or private contractor and government), even though, assuming some solutions are better for all players on average compared to others, these games are not zero-sum.

Sacks and Harel (2006) [80] model the behaviour of subcontractors in terms of resource allocation by using game theory. This paper models the main contractor (or the project manager representing the main contractor) and the subcontractors undertaking various components of a construction project, as the protagonists playing a game with each other. The subcontractor often works in several projects at the same time, and the project manager engages several subcontractors to complete the construction work. The project manager tries to get the subcontractor to give priority to their project in resource allocation, so that they can meet the schedule of the project, while the subcontractor tries to allocate their resources in a way that brings them maximum utility and profit. The paper describes several equilibrium solutions, based on the level of information either party (main contractor or subcontractor) possesses.

Angelou and Economides (2009) [81] present a game-theoretic model for irreversible investment decisions in the ICT field, where each potential investor is a player. They show that either a leader–follower equilibrium or a simultaneous investment equilibrium may result in investments being made. They apply their method to a real-life case study where a particular Greek company (Egnatia Odos) considers its investment options in the ICT industry under intense competition from other companies.

Shen et al. (2002) and Shen et al. (2007) [61,62] apply game theory to a Build-Operate-Transfer (BOT) Construction model, which is typically governed by an agreement between a government and a private company. In the BOT model, the private company which builds the infrastructure recovers its costs and makes a profit by running whatever it just built for a certain period of time (known as the concession period), before handing the facility over to the government. Therefore, the length of concession period is always a subject of negotiation between the private firm and the host government, and Shen et al. [61,62] use game theory to determine this period. They model the bargaining process as an iterative game whereby the pay-off of the government decreases with the concession time, while the pay-off of the private company increases: however, there is another factor—the time taken to strike the deal itself—which both parties want to make shorter, because it is in both parties' interest to strike a deal as quickly as possible. This time pressure increases with each iteration, so that both parties know that it is better to strike a deal early and make some concession rather than adopt a hard line and risk many iterations. Using this model, Shen et al. [61,62] find the solution of the game, which calculates the concession period agreed upon as the "equilibrium" solution.

Butterfield and Pendegraft (2001) [82] use game theory and an extension of it, the theory of moves, to identify best investment decisions in the IT sector, in the face of competition from other investors. In particular, they model the fact that investors may take actions sequentially after observing others' actions by using the theory of moves, which differs from simultaneous games in that players are able to decide on their strategy after observing the other's strategy; thus, they play sequential games (Note well here that an extensive-form game can be either simultaneous game or sequential game. We have not considered this difference in our classification of the literature reviewed in this study, since that will add a further dimension and make the game-based classification too complex.). Butterfield and Pendegraft particularly model the scenario where a number of IT companies are considering investing in a new technology (so that the strategies available are to invest or not to invest), one of these strategies (to invest) is irreversible, and the pay-off for each strategy depends on what the other players decide to do, thus lending itself to a classical scenario for the theory of moves. The paper offers a number of valuable examples involving real-world IT companies or IT-related services where their modelling can be applied.

Javed et al. [63] use game theory to model change negotiations in Public-Private partnerships (PPP) during the concession period—the period during which the private partner raises revenue from the project, before handing it over to the government involved. They use a software called "Z-Tree" to simulate the changes that would trigger the change negotiations. They used actual people to make decisions in a game-theoretic scenario, where each game is an iterated ultimatum game played by two human subjects using a computer interface, one representing the government and the other representing the private firm, under multiple output specifications. The multiple output specifications for each change scenario described the same change in varying levels of detail. For each output specification, each party made multiple offers to foot a certain proportion of the cost of the change, and the equilibrium solution was reached when the parties agreed on the proportions of cost each party would accept, or when the change was abandoned. Data was collected and aggregated from these experiments. They concluded that the level of detail described in the output influences the strategies adopted by players in the iterative bargaining games.

Li et al. (2016) [64] illustrated the use of bargaining game theory for risk allocation in Public–Private Partnerships, by using iterative bargaining games. They use Harsanyi transformation theory to model incomplete information games, whereby each player cannot entirely predict the other's strategy but can estimate it using a subjective probability distribution. They highlight that the cost of bargaining increases with each iteration, at some point subsuming costs associated with the risks being negotiated about, so that an equilibrium solution will always be reached. They analyse the difference in outcome when the first round offer was made by the public sector and when the first round offer was made by the private sector. They show that being the first mover, and having better information about the likely strategy of the other player, are advantages that result in a smaller share of risk being allocated to that party.

Lippman et al. (2013) [83] present the cost sharing equilibrium solution between a risk-neutral project manager (main contractor) and a risk-averse sub contractor, when their negotiation process can be modelled by Nash bargaining. They consider a '1fixed fee plus fraction of cost" model, of which both fixed price contracts and "cost + fixed fee" contracts are subclasses, and show that the fixed fee and fraction of cost borne by the main contractor can both be derived as the solution of a Nash bargaining model. They assume that the pay-off for each party is zero if the project is not launched, so that a Nash bargaining equilibrium where the pay-offs are not zero will always be reached. They further assume that the solution of Nash bargaining maximizes the product (not summation) of the pay-offs for each party, which they term the "Nash product". They conclude that the (full) cost plus fixed fee type of contract dominates all other types of (fixed fee plus fraction of cost) contracts, in arriving at highest Nash product at equilibrium solution.

5.2.2. Papers Using Extensive-Form Cooperative Games

A single paper within the scope of our review, Sacks and Harel (2006) [84], uses an extensive-form cooperative game in modelling a project management scenario. Given that in construction projects, subcontractors will be reluctant to allocate their labour resources to a particular project if they perceive that the project plan or schedule is unreliable, Sacks and Harel explore the relationship between project plan reliability and subcontractor resource allocation behaviour using a game-theoretic framework. In particular, they use extended-form games to explore the trust between project managers (main contractors) and subcontractors, and demonstrate what level of trust in project plan is needed before this relationship changes from competitive to collaborative. In their model, the project manager estimates the work that the subcontractor needs to do during a particular phase, and then, without divulging this information, can demand a certain quantum of work from the subcontractor, which could in fact be lower, equal, or higher than the estimate of the project manager. The subcontractor could, in turn, allocate resources which are less than, equal to, or more than what is demanded (the last action will be in the hope that the project manager has either underestimated work or under-demanded resources, and more resources will be needed than what the project manager has demanded). The pay-offs for both parties will depend on the strategy adopted by both parties, even though they do not divulge to each other what strategy they have adopted. For example, if the project manager has over-demanded resources and the subcontractor oversupplied resources, the utility for the project manager will be high, since the work will be finished quickly and smoothly, but the utility for the subcontractor is low, because s/he will over-allocate and waste resources. The win-win scenario arises when the project manager demands just enough resources, and the contractor allocates what is demanded, but for this to happen, the level of trust between the parties should be high. Sacks and Harel quantify this relationship between trust and pay-off, and further show that through the transparent 'last planner' system they introduce, where the subcontractors are given active roles in project planning, the overall pay-off for all parties could be increased, and the game turns from a competitive (non-cooperative) one into a cooperative game. In this sense, the paper deals with an extensive-form cooperative game.

5.3. Papers Relevant to Project Management but not Explicitly Set in Project Management Context

In our paper selection process, several papers were excluded after stage 3, because they did not provide specific project management-related examples or they were applied in scenarios which were not time-bound, and thus related more to operations research than project management. Nevertheless, several of these papers could be of interest to project management professionals, or describe modelling techniques which could be reused in project management related scenarios. We briefly describe here, some of the papers which were excluded at stage 3, so that the reader may appreciate why such papers were excluded, and how they may still be of some relevance. Note well though that this is merely a set of examples and not an exhaustive list of such papers which were excluded at stage 3. The examples we consider are listed in Figure 4.

Outside project Management Scope
Amaruchkul 2018
Barth 2012
Mobarakeh 2013
Liu 2017
Watson 2013
Burato 2007
Pinto 2015
Magalhaes 2015
Kim 2007
Elfakir 2015
Wei 2018
Li 2014
Peldschus 2005

Figure 4. Some examples of papers which were not classified, as they were not project management-specific.

Amaruchkul (2008) [53] uses game theory to model contract negotiations between air cargo carriers and freight forwarders, which determine the size of allotments made by the carrier to each freight forwarder. The pay-off of the freight forwarded is decided by the rate it receives from the carrier for their cargo, and they hope to receive a discount rate compared to the spot rate by securing an allotment beforehand. The pay-off for the carrier is decided by maximum utility of space, and by giving allotments beforehand, the carrier seeks to ensure efficient utility of space. The equilibrium solution (for each game where a particular carrier and particular freighter is involved) indicates the size of the allotment made to that particular freight forwarder by the carrier. Amaruchkul demonstrates that this solution offers higher pay-offs to both parties compared to spot purchasing of space. The Stackleberg game they formulate can also be formulated as a constrained optimisation problem. The authors provide examples of analytical as well as numerical solutions.

Magalhaes et al. (2015) [85] produced a game-theoretic framework to assess the competency and integrity of academics in universities. They explain that given the academics are often expected to perform multiple activities, such as teaching, research, and administration, within a certain period of time, their competency in each of these areas of work must necessarily be assessed against the time they spend in each of these. They point out that given individual academics have preferences about each of these branches of work, if they were to be required to provide a timesheet about the time spent in each of these activities, it is necessary to verify the claims made in these time sheets against the incentive to do each of these activities, and they use game theory as a tool to achieve this aim. Given that academia is competitive, they postulate that the decision by an academic to spend a certain percentage of their time in each of these activities is a function of how they perceive their colleagues are allocating their time, more than what their contracts actually demand of them. They ran a survey among academics, to model their average response to time-allocation patterns among their colleagues, and based on this, they model pay-off functions for academics to devote certain percentage of their time in each of the above-mentioned activities. However, they do not derive equilibrium solutions which would be optimal for academics given the work allocating patterns of their colleagues; instead, they derive results about the level of integrity prevalent among the academics that they surveyed, based on their input in time management systems. Even though academics may at whiles work in particular projects, and the title of this paper explicitly mentions project management, the relevance of this work to the field of project management would have to be considered limited, since ultimately it models the working patterns of academics which are not necessarily time-bound.

Wei et al. (2018) [86] discuss a psychological contract that exists in people's minds, rather than a physical contract that exists on paper, in settings where collaboration occurs between individuals who have specialist knowledge (such as academic researchers). They use game theory to model such contracts. They use an extensive set of parameters which are attributes of the individuals concerned (such as their age, seniority, professional status etc.), as well as attributes of the collaboration itself (such as collaboration risk, length of collaboration etc.) to model pay-off functions for the individuals, and based on this, provide a game-theoretic solution which determines the 'terms' of the contract (such as how much time and effort each individual will commit into the collaboration etc.). The concept of "psychological contract" is novel in our context, and clearly of relevance to project management. However, the paper does not explicitly mention project management and neither does it highlight the relevance to project management.

There are several other papers which address ongoing contractual and/or managerial issues, and thus do not qualify to be considered papers that directly address project management problems and thus be included in our classification. For example, Barth et al. (2012) [87] discusses a (continuous) resource management problem, and its solution using game theory, in an interdomain routing network. Since the management seems to be an ongoing issue and not time bound, this is outside the ambit of project management. Mobarakeh et al. (2013) [88] presented a game-theoretic framework to determine the price of energy based on distributed generation. Again, this paper develops an activity which is ongoing, and cannot be defined as a project: as such, this paper is outside the scope of our review. Similarly, Pinto et al. (2015) [89] introduced a game-theoretic framework to model the decision-making behaviour of electricity market players in bilateral negotiations. Contracts made by such players among themselves have varying timespans, but do not reflect the time bound nature of projects, and may last for any length of time. Thus, this paper is not necessarily about project management and does not fall under the scope of this review. Elfakir and Tkiouat (2015) [90] discuss a profit sharing contract based on religious scriptures (called the Musharakah contract). They use game-theoretical modelling to compare the pay-offs for players under effort-based and output-based Musharakah contracts. The context explained by them makes clear that the Musharakah contract is intended as a model for long term profit sharing among business partners, and not intended to be applied in a project management context. Thus, this paper is outside the scope of our review. Kim and Kwak (2007) [91] deal with using game theory to model the bargaining process of long-term replenishment contracts between supplier and buyer; the contracts are explicitly stated to be negotiated on a long term basis and not for particular projects, and thus this paper is not within scope of the present review (We have reviewed other papers which deal with very similar problems specifically within a project management perspective, such as Lippman et al. [83] and Li et al. [64]). Li (2014) [92] offers a game-theoretic perspective on information technology outsourcing, by focusing on particular the information that must necessarily be shared as part of an outsourcing deal, and thus can be exploited by firms. They show that exploitation of sensitive information depends on the believe held by the firms about the future potential of the contracts, and service providers have higher incentive to exploit sensitive information about the client when they believe that future potential of the outsourcing contract is not great. Even though outsourcing can happen in the context of a project as well as in the business-as-usual (day to day operation) context of a firm, this paper does not specifically show any project management related examples, and thus is outside the scope of our review. Liu et al. (2017) [93] deals with contract negotiation within the context of electricity generation: therefore, game theory is used in an ongoing issue and not a project management issue here. Thus, this paper also is out of scope for our review.

There are also several papers which appear to have relevance to project management, but do not directly provide examples in the project management context (being under no constraint to do so), and thus could not be included in our classification process. For example, Watson (2013) [94] proposes a (rather general and abstract) modelling framework for contracts involving multi-period settings, with both self enforcement and external enforcement. A notion of contractual equilibrium, which combines a bargaining solution and individual incentive constraints, is proposed and analysed. The paper may have relevance to project management scenarios, but this is not very clear at the outset, and the paper does not delve into practical examples which are necessarily time bound, since the author is under no constraint to specify whether the scenarios are relevant to project management or not. Burato and Christani (2007) [95] present a zero-sum bargaining game to model the meaning negotiation problem in formal contract negotiation. Again, this could have applications to project management but a project management example per se is not presented, and the authors, under being no constraint to do so, do not clarify whether their modelling is relevant to project management scenarios.

Finally, Peldschus (2005) [96] also could be mentioned as an example here. This paper does not directly address game theory, but discusses decision-making in engineering, particularly civil engineering. Their method addresses multicriteria decision-making in construction projects using fuzzy sets and matrix game theory in the context of incomplete information. Therefore, strictly speaking, this study is outside the scope of our review.

5.4. Gaps in Literature

Our classifications illustrate potential areas of research where sufficient effort has not yet been extended. For example, considering Figure 1, it can be observed that game-theoretic concepts have not been applied sufficiently in ICT project management. This is somewhat surprising because game theory has been applied in wider ICT or computer scientific contexts quite often. For instance, several studies in the field of networked computing systems have modelled the dynamics of these systems as games between self-interested players. Examples include TCP congestion control, computer security level allocation, peer-to-peer routing, peer-to-peer overlay network formation and peer-to-peer file sharing patterns [97–101]. Nevertheless, there has not been sufficient research in using game theory for modelling ICT project management specifically. Similarly, considering Figure 3, note that certain types of games are not frequently used to model project management scenarios. These include extensive-form zero-sum non-cooperative games, as well as extensive-form cooperative games. Although a researcher typically chooses the problem they are interested in studying and then chooses a form of game appropriate to model that problem, and not vice versa, it is illuminating that project management scenarios do not seem to lend themselves often to be modelled as either extensive-form zero-sum non-cooperative games or extensive-form cooperative games. Nevertheless, it should be noted that the relative complexity of extensive-form games might be a reason why some types of extensive-form games are not often used. We have not found any paper in the niche that we reviewed that uses extensive-form zero-sum games. There was a single paper which used extensive-form cooperative games (Sacks and Harel, 2006 [84]), however even this paper was modelling a scenario where in an extended interaction, competition turns into cooperation through trust. Here lies an important hint: extensive-form cooperative games could be particularly useful in scenarios where players are initially competitive, but develop trust and are induced to cooperate after several rounds of interactions. Such scenarios are certainly not uncommon in project management, so researchers that model such scenarios could perhaps consider using extensive-form cooperative games more often.

6. Citation Network of Reviewed Papers

In this section, we present a "citation network" of the 32 papers which were described and classified in this review. The purpose of this exercise is two-fold: (a) Understanding the relative importance of each paper in the field, and how it has inspired or guided other studies in the field. (b) Understanding how papers are grouped together in terms of citation patterns, with the view of highlighting which classifications are the most significant in terms of authors being aware of other papers falling under the same classification. The citation network is presented in Figure 5. While Figure 5 shows the un-annotated network, Figure 6, Figure 7, and Figure 8 respectively show the network annotated according to the three classifications—domain-based, player-based and game-based classifications, respectively. These in turn correspond to Figure 1, Figure 2 and Figure 3, respectively.

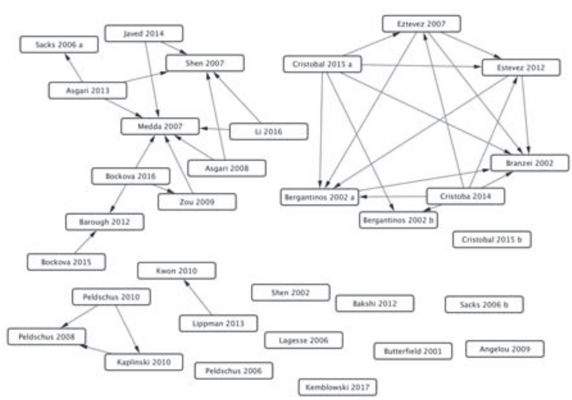

Figure 5. The citation network of papers reviewed—no classification shown.

In a citation network, the nodes are papers, and the links are citations from one paper to another, which are directed. We constructed the citation network of the 32 papers we have analysed, by manually checking which other papers within our review each of these papers have cited. This was possible, because the number of citations within the field was relatively sparse, there being only 34 instances where one paper from our review is cited by another. Therefore, the average out-degree of the citation network (the average number of times when a paper within the field, as we defined it, cited another within the field), was only 1.0625, indicating that, in general, studies about the use of game theory in project management were conducted without much awareness, or the explicit expression of such awareness, of other studies in the field. Quite often, we found that the authors cited only their own work in the field, or the work of close collaborators. Of course, each paper made a number of citations to papers outside the field, due to their relevance to project management or game theory but not both, which is justifiable. In fact, the average number of papers cited by papers we reviewed (the average length of the reference list of the papers we reviewed) is 35.0, which means that most of the citations were made to papers which were outside the niche that we review.

The citation network we constructed is shown in Figure 5. It could be immediately observed that there are two large clusters, two small clusters, and nine singleton papers. Medda (2007) is at the centre of one large cluster (henceforth called the "Medda cluster"), and the other large cluster seems to be composed of papers written by Cristobal each citing several others not written by Cristobal (henceforth called the "Cristobal cluster"). The smaller clusters consist of citations among papers of the same author or his collaborators (Peldschus in the first case, Lippman in the second, henceforth called "Peldschus cluster" and "Lippman cluster", respectively).

In order to understand how the large clusters were formed, we now proceed to highlight the "type" of papers, according to our classification scheme, in Figures 6–8 below. Accordingly, Figure 6 highlights the domain-based classification, Figure 7 highlights the classification based on the type of players, and Figure 8 highlights the classification based on the type of game used. Since the third classification is a complex matrix classification, we consider only the four basic types of games in this figure: normal-form non-cooperative games, normal-form cooperative games, extensive-form non-cooperative games and extensive-form cooperative games. The node colour indicates the class each node belongs to in each figure, as indicated in the caption of each figure.

From Figure 6, we can see that the large clusters are composed of construction industry papers or generic papers (many of which are nevertheless inspired by the construction industry). The Medda cluster and Peldschus cluster are predominantly composed of construction industry papers, while the Cristobal cluster is composed of author Cristobal, focusing on construction industry, citing a number of generic project management papers in each of their works. The few ICT domain papers present are singletons, apparently not being aware of each other. It should be noted particularly that the one Feldschus paper that we have classified as 'generic' lies outside of the Peldschus cluster, not citing or being cited by any of his other works.

However, the clustering patterns become clearer when we consider the player-based, and game-based classifications, as shown in Figures 7 and 8. It could be observed that the "Cristobal cluster" is composed of papers employing normal-form cooperative games, and modelling subcontractor—subcontractor interactions. Hence they are tightly clustered in terms of citations. The Medda cluster is more diverse, though it seems to focus primarily on public-private partnerships (government-contractor games). The Peldschus cluster focuses on normal-form non-cooperative games. The Lippman cluster focuses on extensive-form non-cooperative games which are between a contractor and a subcontractor. Therefore, we may see that all three classifications we presented above influence the citation patterns in the field.

In terms of relative importance within the field (niche) that we considered, it is obvious from Figure 5 that Medda (2007), Branzei et al. (2002), Bergantinos and Sanchez (2002), Estevez-Fernandez (2012) and Shen et al. (2007), in that order, are cited the most times from within the field. Thus, these papers occupy the most prominent positions within the citation network of the field, and have presumably had the most influence on other papers within the field. For comparison, we present in Figure 9 the Google Scholar citation counts of each paper that we have reviewed and classified (as they stood on the 4 April 2019). Here, obviously the citations are from papers in any field, not just in the particular niche that we are reviewing. The number of references in the reference list of each paper is also presented for completeness. Comparing this with Figure 5, we may note unsurprisingly that the Medda, 2007 paper, which is at the centre of the Medda cluster, is also the most cited paper overall. The papers by Shen et al. are the next most cited, one of which is also in the Medda cluster, and cited by many other papers in the field. A paper by Peldschus which belongs to the Peldschus cluster is also among the most cited. Therefore, there is close correlation between the reception and impact of a paper within the niche, and its overall reception and impact in the wider scientific community. However, since the citation network within the niche is sparse, the number of times papers are cited within the niche is a lot smaller than the number of times they are cited overall.

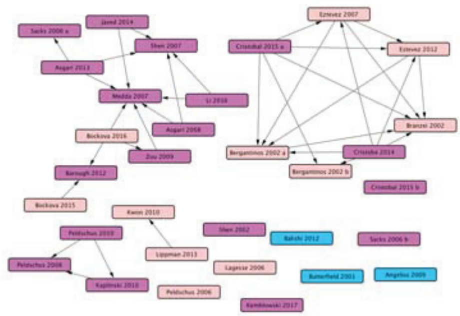

Figure 6. The citation network of papers reviewed—domain-based classification shown. Bright pink: construction domain; blue: ICT domain; brownish-pink: other domains or generic project management.

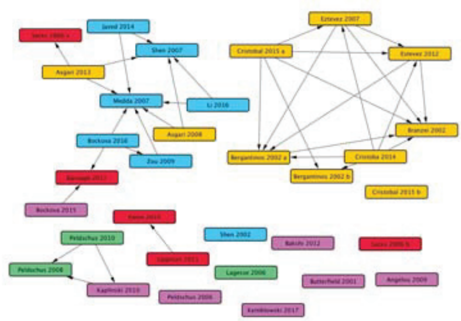

Figure 7. The citation network of papers reviewed—player-based classification shown. Blue: government sector—private sector game; pink: contractor—contractor game; red: contractor—subcontractor game; yellow: subcontractor—subcontractor game; green: other types of players.

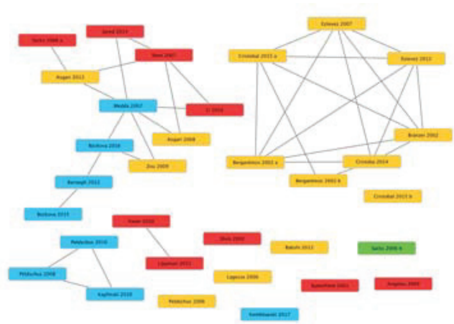

Figure 8. The citation network of papers reviewed—game-based classification shown. Blue: normal-form non-cooperative game; yellow: normal-form cooperative game; red: extensive-form non-cooperative game; green: extensive-form cooperative game.

Paper ID	Paper name (First author name + publication year)	Google scholar citations (incoming citations)	Number of references (outgoing citations)	Rank based on incoming citations
1	Angelou 2009	46	48	10
2	Asgari 2013	33	60	13
3	Asgari 2008	14	12	22
4	Bakshi 2012	3	15	29
5	Barough 2012	43	32	11
6	Bergantinos 2002 a	35	9	12
7	Bergantinos 2002 b	18	10	20
8	Bockova 2015	14	22	22
9	Bockova 2016	1	30	30
10	Branzei 2002	51	19	6
11	Butterfield 2001	22	13	18
12	Cristoba 2014	9	57	26
13	Cristobal 2015 a	12	80	25
14	Cristobal 2015 b	1	18	30
15	Estevez 2012	49	24	9
16	Eztevez 2007	28	5	16
17	Javed 2014	30	127	15
18	Kaplinski 2010	51	131	6
19	Kemblowski 2017	1	8	30
20	Kwon 2010	33	26	13
21	Lagesse 2006	19	6	19
22	Li 2016	13	63	24
23	Lippman 2013	26	33	17
24	Medda 2007	308	24	1
25	Peldschus 2008	63	33	5
26	Peldschus 2006	4	0	28
27	Peldschus 2010	50	87	8
28	Sacks 2006 a	78	44	4
29	Sacks 2006 b	17	18	21
30	Shen 2002	194	9	2
31	Shen 2007	161	29	3
32	Zou 2009	7	28	27

Figure 9. Citation counts of the papers classified according to Google scholar, as accessed on the 4 April 2019. The number of references in each paper is also shown; that is, the incoming citations and outgoing citations of each paper we reviewed are shown. The publication year of the paper is mentioned as part of the paper name.

7. Conclusions

Several project management scenarios where decision-making takes place lend themselves to be modelled using game theory. A considerable number of studies have applied game-theoretic analysis in project management, however a concise review of such efforts was lacking. In this review paper, we undertook the exercise of systematically searching, selecting and reviewing papers which have used game-theoretic analysis in project management. We also proposed a set of classifications which structure and define this niche, and demonstrated the importance of such classifications. We analysed the relative importance and impact of the papers reviewed, and identified gaps in the literature which represent future research opportunities.

Our analysis showed that papers in this field can be classified based on the domain of application, the way the players are modelled, or the type of game utilised. In terms of the domain, papers focusing on construction domain are the most prominent, though there are also papers that focus on the ICT domain, as well as generic project management focused papers. In terms of modelling the players, we showed that five classes of papers exist: papers that model government-private sector games, papers that model contractor–contractor games, papers that model contractor-subcontractor games, papers that model subcontractor–subcontractor games and papers that model games involving other players. In terms of the type of games used, we showed that a complex matrix-based classification exists, though the four basic classes present are papers which use normal-form non-cooperative games, papers which use normal-form cooperative games, papers which use extensive-form non-cooperative games and papers which use extensive-form cooperative games. Based on these classifications, we showed that papers which focus on ICT domain, as well as papers which use extensive-form cooperative games, are relatively few in number, representing gaps in literature.

We also showed that within the niche, a paper is more likely to cite another paper if they are both focusing on the same application domain, or use the same types of games. In particular, we showed that two large "citation clusters" exist: papers which use non-cooperative games mainly in the construction domain, and papers that use normal-form cooperative games which are generic project management focused and not specific to any application domain. Overall, we made a very strong case for the widespread use of game theory to model and analyse project management-related scenarios, by highlighting a range of scenarios where it could be used, and the types of games which could be used in each such scenario.

Even though there have been some efforts in the past to summarise efforts made in applying game theory to project management, they have been focused on specific authors, countries or application domains. This review, on the other hand, employed a principled and methodical selection process which was not centred on any author, country or application domain. This review also put emphasis on structuring, classification and citation-analysis of the literature that it covered. It also focused explicitly and methodically on citation relationships between the papers that it reviewed, highlighting how the works are interrelated and which papers acted as catalysts for further research in the field.

This review highlighted that game theory is a very useful tool to analyse project management scenarios, and efforts to apply game-theoretic analysis in project management have great potential, but at present this is a nascent field. Studies in this niche are often not aware of each other and the citation density within the niche is relatively low. Therefore, it is important that more collaboration efforts take place among researchers which apply game theory in project management, spanning domains and choice of games. It is expected that this review will be a catalyst for increased interest in applying game theory in project management, and will encourage cross-domain collaboration and sharing of expertise to realise the full potential of game theory in analysing project management problems.

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Conflicts of Interest: The author declares no conflicts of interest.

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