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Economics of Conflict and Terrorism

Edited by
Joao Ricardo Faria, Daniel Arce

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About the Editors

Joao Ricardo Faria deals with dynamic models and differential games, having published in the areas of macroeconomics, terrorism and managerial economics. Faria is currently an Editorial Board member of EconomiA.

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Editorial

A Preface for the Special Issue "Economics of Conflict and Terrorism"

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The current Special Issue presents an interesting collection of seven articles that expand the existing literature on the subjects of terrorism and conflict. The papers present significant empirical, methodological, and theoretical contributions.

Two papers from the present collection use game-theoretic foundations in order to examine the empirical issues in relation to the notion of conflict. The first, by George and Sandler [1], uses two-step GMM estimates of the demands of E.U. members for defense spending based on alternative spatial-weight matrices. They found that the consistent and robust estimates of E.U. military spending during the post-Cold War differs from past non-spatial and spatial E.U. defense spending estimates. Most notably, free riding, indicative of strategic substitutes, characterizes E.U. members' military expenditure. In the second paper, Bang, Basuchoudhary, and Mitra [2] use machine learning to empirically shift between competing models of terrorism or nonlinear patterns. Machine-learning algorithms focus on predictive accuracy instead of tests of significance; in this sense, they can identify whether a variable is predictive or not, even if it is endogenous with the target variable, terrorism. Second, game-theoretic approaches often predict the nonlinear relationships between variables, where equilibria switch in comparative static scenarios. They found that models predicting economic opportunity, development assistance, and ethnic tensions may not be predictively salient. In contrast, those that predict a more formidable target would elicit more terrorist attacks and are predictively salient.

There are two papers that present methodological innovations. Balcaen, Du Bois and Buts [3] use prospect theory to study the uncertainty of conflicts between a State challenger and a defender. The article raises awareness with regard to cognitive bias associated with conflict choices. The article yields two specific recommendations. First, future research could confront test subjects (e.g., decision makers, such as politicians, or regular citizens) with hybrid threat scenarios that involve hypothetical policy responses and different outcomes. Second, as hybrid attacks occur frequently, we can conduct large-N statistical analyses. The article written by Ganzfried [4] studies a new algorithm for approximating Nash equilibrium strategies in continuous games, which are difficult to solve since the pure strategy space can be infinite. He implements the algorithm in the Blotto game. His algorithm converges quickly and is the first algorithm to solve the continuous case of the game.

Last but not least, three theoretical articles exist. Faria and Arce [5] studiy a dynamic game in discrete space and find a number of new results, namely, the fact that counter-terror is limited; defensive counter-terror limits the worst-case scenario, while proactive counter-terror reduces the capacity of terrorists; proactive counter-terror is the most effective of the two, however it is underprovided; and, finally, cyclical attacks are independent of counter-terror policy and depend on the terrorist's time preferences and tactic adjustment costs. Oliveira and Silva [6] study the incentives produced as a result of retaliation for the formation of an international counter-terror coalition. The benefits of joining such a coalition are the relatively lower spillover benefits as a result of the retaliation. The cost

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of joining a coalition is the anticipated backlash from retaliation. Boudreau, Matthews, Sanders, and Bagchi [7] examined the momentum in conflict, where victory in the initial stage can provide an advantage in the final stage. They discovered that the impact of elasticity of effort on levels of effort has no bearing on the value of momentum itself. Instead, momentum helps a player by enhancing the marginal chance for victory in the second-stage contest. This concept provides a theoretical foundation for Pyrrhic victories.

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Article

Conflicts with Momentum

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Abstract: *Take the fort, then take the city*. In a two-stage, two-party contest, victory in the initial stage can provide an advantage in the final stage. We examine such momentum in conflict scenarios and investigate how valuable it must be to avoid a *Pyrrhic* victory. Our main finding is that although the elasticity of effort—which we allow to vary between the two stages—does impact the contestants' effort levels, it has no bearing on the endogenously determined value of momentum itself. Further, rent dissipation in the two-stage conflict is equal across party whether or not an individual obtains first-stage momentum. Thus, momentum helps a player solely by enhancing marginal ability for victory in the second-stage contest. It does not, however, change the player's net calculus of second-stage contest spending. Such contestable advantage is also found to be more rent-dissipative than innate/uncontestable advantage. Therefore, *Pyrrhic* victories should be more common for contests with an intermediate stage or stages in which advantages can be earned, *ceteris paribus*. While intermediate targets appear as useful conflict benchmarks, they dissipate additional expected contest rents. This additional rent-dissipative toll exists even for backward-inductive equilibrium behavior in a complete information setting. Whereas the quagmire theory suggests parties can become involved in problematic conflicts due to incomplete information, the present paper finds that the setting of conflict—namely, contestable intermediate advantage—can alternatively generate rent-dissipative tolls. Similarly, contestable advantage can lead parties to optimally forego contest participation (i.e., if conflict parameters do not meet the participation constraint). This is in contrast to a one-stage simultaneous contest with second-stage parametric values of the present contest.

Keywords: conflict; contests; momentum

JEL Classification: C72; D74

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1. Introduction

The premise of our model is an intuitive one: *Take the fort, then take the city*. The concept is that many conflicts are not one-shot scenarios, but rather involve an initial stage in which one party can gain an advantage that improves its relative position in the ultimate stage.

More specifically, we set up a two-stage model of conflict in which the winner from the first-stage gains *momentum* in the sense that it has a reduced unit cost of effort in the second stage. One party gains an advantageous position or elevation through a first stage of conflict, so it is less costly in the second stage for that party to produce conflict inputs. One interpretation could be that one needs fewer soldiers to put forth an effective force when at an elevation gained earlier. The *Battle of the Alamo*, for example, had an ultimate 5-to-1 casualty ratio in favor of the advantaged side.

Our work is most similar to contest models that emphasize "head starts" in the sense of giving one party or another a cost advantage of some kind, as studied in [1–3]. These papers, however, focus in particular on the optimal setting of cost advantages (or disadvantages) in order to maximize effort expenditures by contestants, as if set by the contest designer.

Furthermore, refs. [1,3] uses contest success function formats that differ importantly from ours and one-stage settings, while [2] focuses on a multi-stage setting in which some participants are disqualified from later stages. Ref. [2] also uses a classic [4] lottery-form success function. (Ref. [5] also study the role of cost asymmetries in conflict outcome). The two-stage nested conflict approach has been used to study aspects of conflict other than momentum (see, e.g., [6]).

Here, we assume that two parties in conflict exert efforts at a cost in a two-stage game with a more generalized contest success function of the Tullock variety, but more akin to the version generalized by works such as [7,8]. (The results obtained may in some ways depend upon the assumed functional form of the contest success function. We feel that a ratio-form success function—for which each party has a strictly positive probability of victory so long as a positive amount of effort is exerted—is appropriate for the types of conflict that we have in mind as the primary motivators for the analysis (e.g., multi-stage armed conflict between combatants).) Our model assumes that there is some cost advantage to be achieved by winning the first stage, but that the first stage's sensitivity to effort spending may differ from that of the second stage. Ultimately, we find that while the overall size of the (exogenous) cost advantage in the second stage—which we term *momentum*—does matter for the ultimate probability of victory and spending by parties in the second stage, the sensitivity to effort in either stage does not matter for how valuable that advantage is in a crucial sense. Although the elasticity of effort—which we allow to differ between the two stages of conflict—does impact the contestants' effort levels, it has no bearing on the endogenously determined value of momentum itself. Further, rent dissipation in the two-stage conflict is equal across party whether or not an individual obtains momentum in the first stage. Thus, momentum helps a player solely by enhancing the player's marginal ability for victory in the second-stage contest. It does not, however, change the player's net calculus of second-stage contest spending. Contestable advantage in conflict is also found to be more rent-dissipative than innate or otherwise incontestable advantage. Similarly, contestable advantage can lead parties to optimally forego contest participation (i.e., if conflict parameters do not meet the participation constraint). This is in contrast to a one-stage simultaneous contest that takes on the second-stage parametric values of the present contest.

Therefore, we expect *Pyrrhic* victories to be more common, *ceteris paribus*, for contests that feature an intermediate stage or stages in which subsequent advantages can be earned. While intermediate targets may appear as useful benchmarks in conflict, they in fact dissipate additional expected contest rents to each party. This additional rent-dissipative toll exists even given a backward-inductive (equilibrium) behavior in a setting of complete information rather than one characterized by "fog of war" effects. The quagmire theory suggests that countries can become involved in problematic (i.e., rent-dissipative) conflicts due to incomplete information. The present paper finds that the setting of conflict—namely, the contestibility of intermediate, momentous advantage in a conflict—can effectively substitute for incomplete information in generating rent-dissipative tolls.

2. A Model of Conflicts with Momentum

2.1. Model Setup

Consider two parties, $i = \{1, 2\}$, in a two-stage conflict. The ultimate winner of the final conflict in the second stage of the game is awarded a prize commonly valued at V by both parties. But winning the first stage of the game provides a cost advantage of $0 < \alpha < 1$ to the first-stage winner in terms of competing in the second stage.

Solving backwards, in the second stage of the game, one party has already won the first stage, making their objective function for that stage

$$\frac{s_w^{r_2}}{s_w^{r_2} + s_\ell^{r_2}} V - \alpha s_w - s_1$$

where s_w is the expenditure by the first-stage winner in the second stage, s_ℓ is the expenditure by the first-stage loser in the second stage, and s_1 is the expenditure of each party in the first stage. The expenditure by both parties is equal in the first stage since we assume symmetric valuations of V.

The r_2 parameter represents the sensitivity of the second-stage contest success function (CSF) to the relative expenditures chosen by the conflicting parties in that stage (see [7,9] for axiomitizations of CSFs). We initially impose the restriction $0 < r_2 \leq 2$ as is standard in the literature (e.g., [10]), but will both verify this assumption and explore the more detailed joint restrictions on this parameter and α necessary for the participation of both parties in the next subsection.

The party that loses the first stage has a similar objective function in the second stage to that of the first-stage winner but without the cost advantage of α,

$$\frac{s_\ell^{r_2}}{s_w^{r_2} + s_\ell^{r_2}} V - s_\ell - s_1.$$

These objective functions lead to the first order conditions

$$\frac{r_2 s_w^{r_2-1} s_\ell^{r_2}}{(s_w^{r_2} + s_\ell^{r_2})^2} V = \alpha$$

and

$$\frac{r_2 s_\ell^{r_2-1} s_w^{r_2}}{(s_w^{r_2} + s_\ell^{r_2})^2} V = 1,$$

which imply $s_\ell = \alpha s_w$, allowing the conditions to be solved to determine the equilibrium efforts of

$$s_w^* = \frac{r_2 \alpha^{r_2-1} V}{(1+\alpha^{r_2})^2}$$

and

$$s_\ell^* = \frac{r_2 \alpha^{r_2} V}{(1+\alpha^{r_2})^2}.$$

These then lead to probabilities of victory (which are the same as they would be as in the case of the classic Tullock CSF)

$$P_w^* = \frac{1}{(1+\alpha^{r_2})}$$

and

$$P_\ell^* = \frac{\alpha^{r_2}}{(1+\alpha^{r_2})}$$

and the corresponding expected payoffs

$$\pi_w^* = \frac{V}{(1+\alpha^{r_2})} - \alpha s_w^* - s_1$$

and

$$\pi_\ell^* = \frac{\alpha^{r_2} V}{(1+\alpha^{r_2})} - s_\ell^* - s_1.$$

We then refer to the *value of the momentum* as the difference between the winner's and loser's expected payoffs,

$$\pi_w^* - \pi_\ell^* = \frac{(1-\alpha^{r_2})}{(1+\alpha^{r_2})} V.$$

The first stage of the contest is then a battle for this value of momentum.

Assuming a different degree of elasticity for the contest success function in this initial stage, denoted r_1, the objective functions in this first-stage contest between players I and II are

$$u_I = \frac{s_I^{r_1}}{s_I^{r_1} + s_{II}^{r_1}} \frac{(1-\alpha^{r_2})}{(1+\alpha^{r_2})} V - s_I$$

and

$$u_{II} = \frac{s_{II}^{r_1}}{s_I^{r_1} + s_{II}^{r_1}} \frac{(1-\alpha^{r_2})}{(1+\alpha^{r_2})} V - s_{II}$$

which lead to the standard equilibrium efforts of

$$s_I^* = s_{II}^* = s_1^* = \frac{r_1 V}{4} \frac{(1-\alpha^{r_2})}{(1+\alpha^{r_2})}$$

The equilibrium (second-stage) expected payoffs are then

$$\pi_w^* = \frac{1}{(1+\alpha^{r_2})} V - \frac{r_2 \alpha^{r_2} V}{(1+\alpha^{r_2})^2} - \frac{r_1 V}{4} \frac{(1-\alpha^{r_2})}{(1+\alpha^{r_2})}$$

(because of the α-reduction in cost in the winner's payoff), and

$$\pi_\ell^* = \frac{\alpha^{r_2}}{1+\alpha^{r_2}} V - \frac{r_2 \alpha^{r_2} V}{(1+\alpha^{r_2})^2} - \frac{r_1 V}{4} \frac{(1-\alpha^{r_2})}{(1+\alpha^{r_2})}.$$

Before moving on to considerations of rent dissipation, we first must consider whether or not the parties will find it in their own best interests to participate in the conflict at each stage, which is the topic of the next subsection.

2.2. Participation, Parameter Restrictions, and Pyrrhic Victories

Though we have seemingly solved for the model's equilibrium, we must still verify that the positive resource expenditure by each party is better than the option of sitting out the conflict and not spending at all. (We restrict our attention to pure-strategy Nash equilibria, since the mixed strategy equilibria that would result from one (or both) players not spending would simply involve parties mixing between the original game's equilibrium spending levels and zero, as per [11,12]).

To show why participation may be an issue for some parameter configurations, we present three-dimensional graphs of π_w^* and π_ℓ^* as r_1 and r_2 range from just over zero to two (our previously assumed ranges, and the ranges for unique interior solutions to exist in standard contest models). We provide three graphs for the equilibrium expected payoff of each party, one with $\alpha = 0.75$, one with $\alpha = 0.5$, and one with $\alpha = 0.25$, to show how the relationship with the CSFs' parameters changes with a lower reward to the first-stage winner (higher α) vs. a higher reward (lower α).

Figures 1–3 illustrate the $\partial \pi_w^* / \partial r_1 \leq 0$ relationship, and how the negative relationship gets stronger as r_2 increases for given α. They also show that overall, for given (r_1, r_2), a lower $alpha$ (a bigger second-stage advantage to the first-stage winner) means a higher expected payoff: $\partial \pi_w^* / \partial \alpha < 0$.

The $\partial \pi_w^* / \partial r_2$ relationship is more nuanced. For larger α (e.g., $\alpha = 0.75$), $\partial \pi_w^* / \partial r_2 < 0 \; \forall r_1$. For mid-range α (e.g., $\alpha = 0.5$), $\partial \pi_w^* / \partial r_2 < 0$ for large enough r_1, since the increased effort cost effect of r_2 dominates. But for small r_1, $\partial \pi_w^* / \partial r_2$ begins negative but eventually becomes positive for larger r_2 as the improved probability of victory from increased r_2 dominates. This goes to the extreme for small α (e.g., $\alpha = 0.25$), when the $\partial \pi_w^* / \partial r_2 > 0 \; \forall r_2$ at low levels of r_1 and is still "U-shaped" (negative at low r_2, then becoming positive as r_2 gets closer to 2) when r_1 is closer to 2.

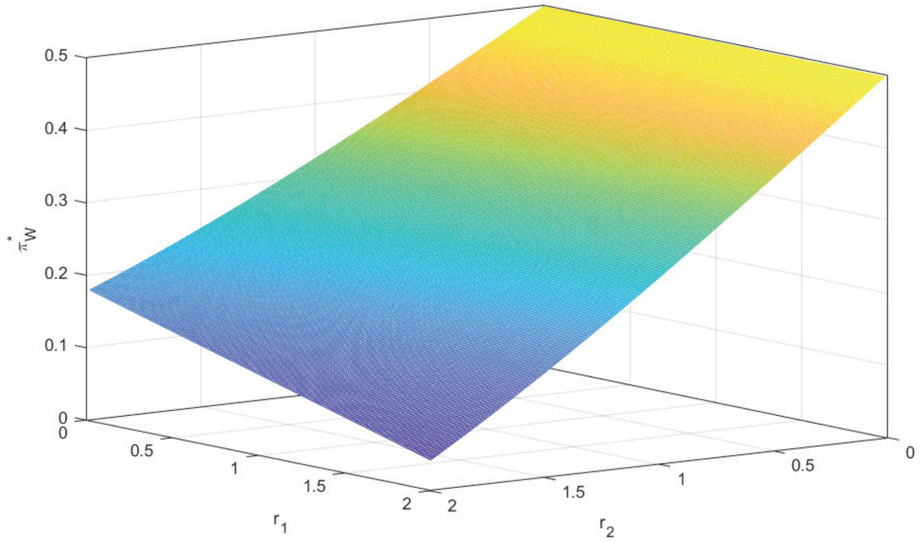

Figure 1. Equilibrium Payoff to the First−Stage Winner with $\alpha = 0.75$, $V = 1$, as r_1 and r_2 Vary.

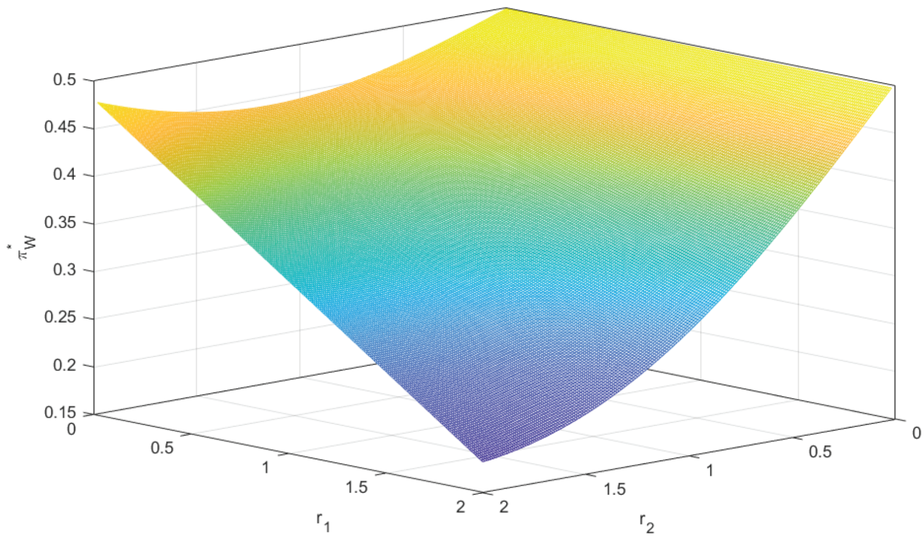

Figure 2. Equilibrium Payoff to the First−Stage Winner with $\alpha = 0.5$, $V = 1$, as r_1 and r_2 Vary.

Figures 4–6 illustrate the first-stage loser's equilibrium expected payoffs for the three example values of α ranging over r_1 and r_2. These relationships are similar to those for π_w^* but are easier to visualize, as $\partial \pi_\ell^* / \partial r_2 < 0$ for all $\alpha \in (0, 1)$ and $r_1 \in (0.2]$, and $\partial \pi_\ell^* / \partial r_1 < 0$ for all $\alpha \in (0, 1)$ and $r_2 \in (0, 2]$. The bigger issue revealed by these graphs is that π_ℓ can be negative for a variety of parameter combinations, which brings into question whether or not the parties will necessarily want to participate in the conflict.

We begin by considering second stage and assume that a party will not spend at all if their equilibrium expected payoff from that stage is lower than simply dropping out of the conflict and spending zero in the second stage. Since $\pi_w^* > \pi_\ell^*$, we know that if the losing party from the first stage is willing to expend effort, the winning party will be willing to as well, so we only need to check the necessary condition for the first-stage losing party.

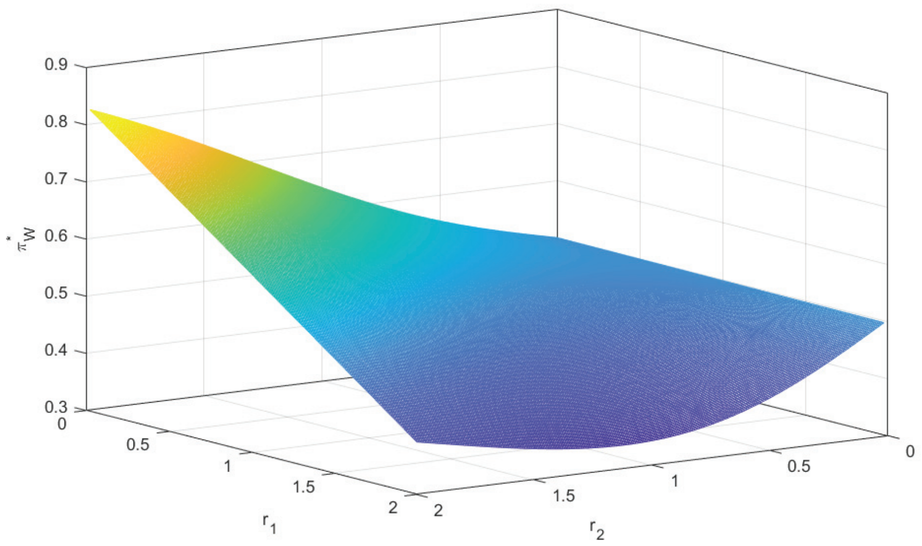

Figure 3. Equilibrium Payoff to the First−Stage Winner with $\alpha = 0.25$, $V = 1$, as r_1 and r_2 Vary.

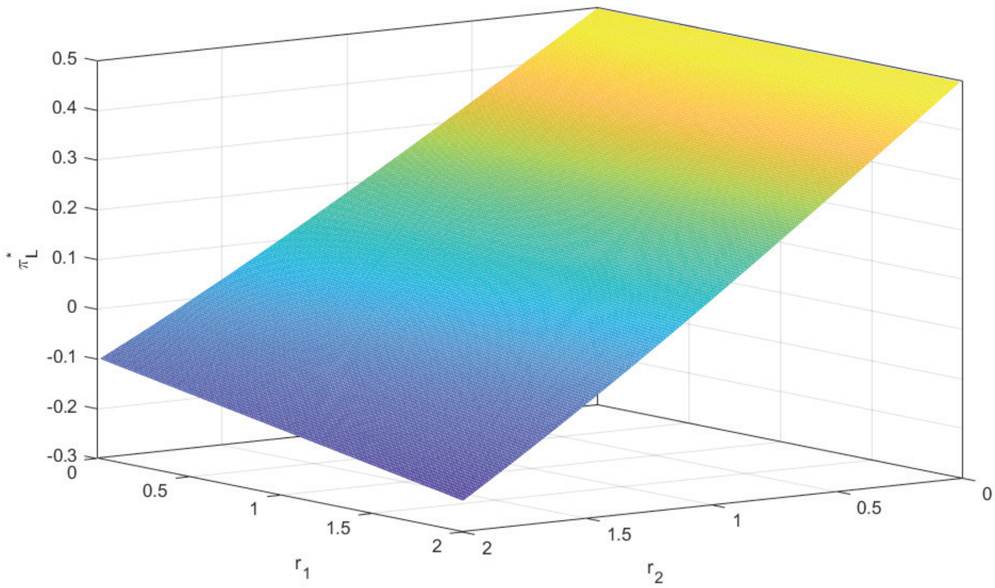

Figure 4. Equilibrium Payoff to the First−Stage Loser with $\alpha = 0.75$, $V = 1$, as r_1 and r_2 Vary.

Since spending from the first stage is sunk, the necessary participation constraint for the second stage is $\pi_\ell + s_1^* \geq 0$. That is, if a party spends nothing in the second stage, they simply lose the battle with certainty and the sunk effort cost with it, so only the portion of the party's expected payoff that is relevant to the second stage must be positive.

$$\pi_\ell^* + s_1^* = \frac{\alpha^{r_2}}{1+\alpha^{r_2}}V - \frac{r_2\alpha^{r_2}V}{(1+\alpha^{r_2})^2} \geq 0$$

which simplifies to the following condition.

Participation Constraint (i) (*PC (i)*): $\alpha^{r_2} \geq (r_2 - 1)$.

This restriction could of course be simplified a step further to isolate α in terms of r_2, but we keep it as in $PC(i)$ for the purposes of illustration since the relationship is nonlinear.

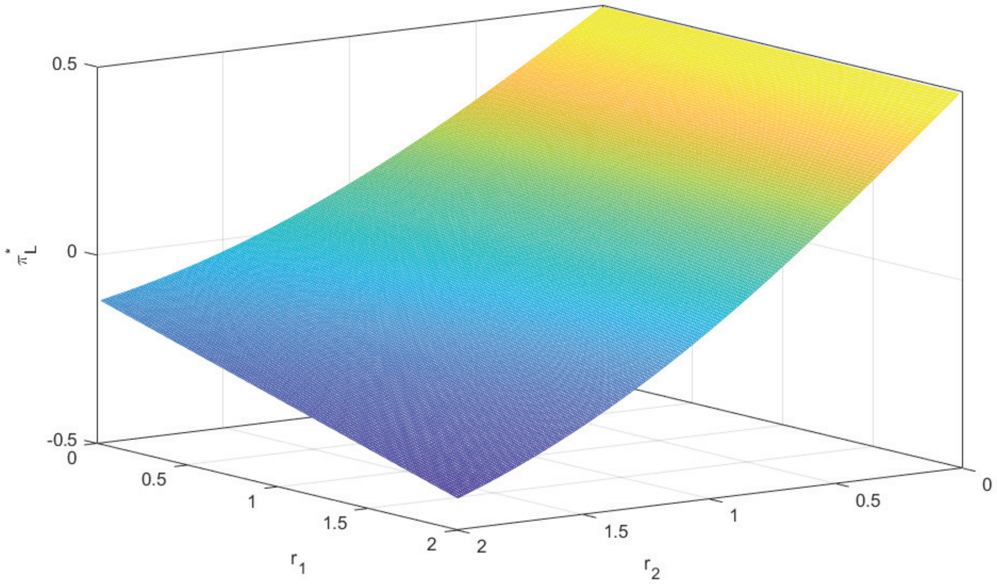

Figure 5. Equilibrium Payoff to the First−Stage Loser with $\alpha = 0.5$, $V = 1$, as r_1 and r_2 Vary.

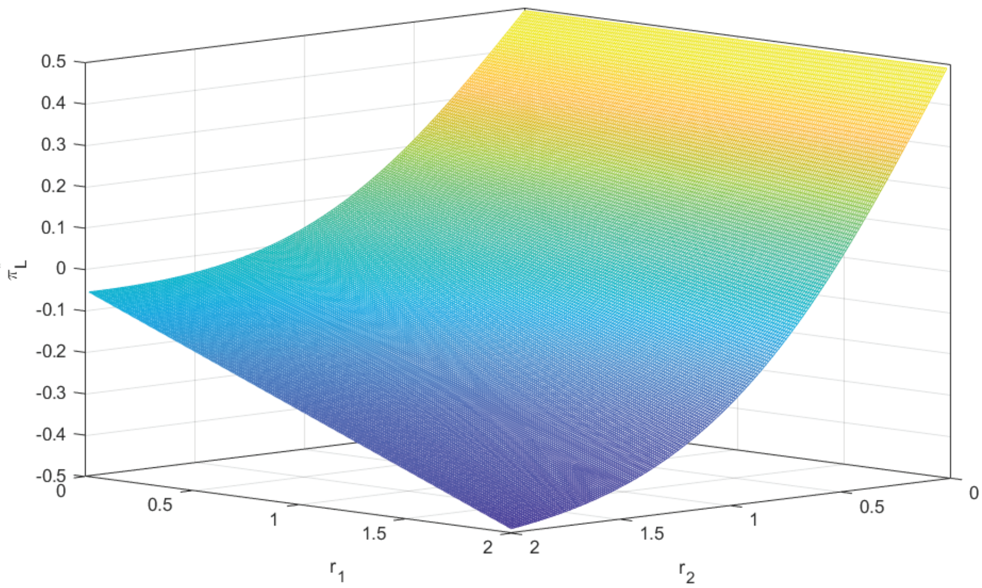

Figure 6. Equilibrium Payoff to the First−Stage Loser with $\alpha = 0.25$, $V = 1$, as r_1 and r_2 Vary.

Figure 7 graphs the left- and right-hand sides of $PC(i)$ in terms of α and r_2. All areas where $\alpha^{r_2} > (r_2 - 1)$ represent combinations of the two relevant parameters that result in second-stage conflict, with positive equilibrium expenditure by both parties as described in the previous section. All areas where $\alpha^{r_2} < (r_2 - 1)$ represent those combinations of α and r_2 that lead the losing party of the first stage to choose zero expenditure and abstain from conflict in the second stage. Intuitively, the participation constraint becomes more constrictive in terms of the allowable range of r_2 as the advantage to the first-stage winner increases (i.e., as α decreases) and vice versa. The larger the reward for winning the first stage, the less sensitive the second stage can be to effort without making it so much of an advantage that it completely deters the first-stage loser from continuing. The boundary of the maximum-allowable r_2 for given α can be traced along the curve of the intersection in Figure 7.

Our second participation constraint concerns the first stage of the conflict, when both parties have the option to either: expend effort seeking the advantage gained by the value of momentum or spend nothing and proceed to the second stage without that advantage with certainty (assuming the other party spends positively). In the latter case, of course, they also have zero sunk costs from the first stage. Thus, we compare the equilibrium expected payoff to a party—making positive equilibrium expenditures in each stage, since we assume $PC(i)$ is satisfied—to the expected payoff they would receive if they spend nothing in the first stage and competed only in the second stage at a disadvantage.

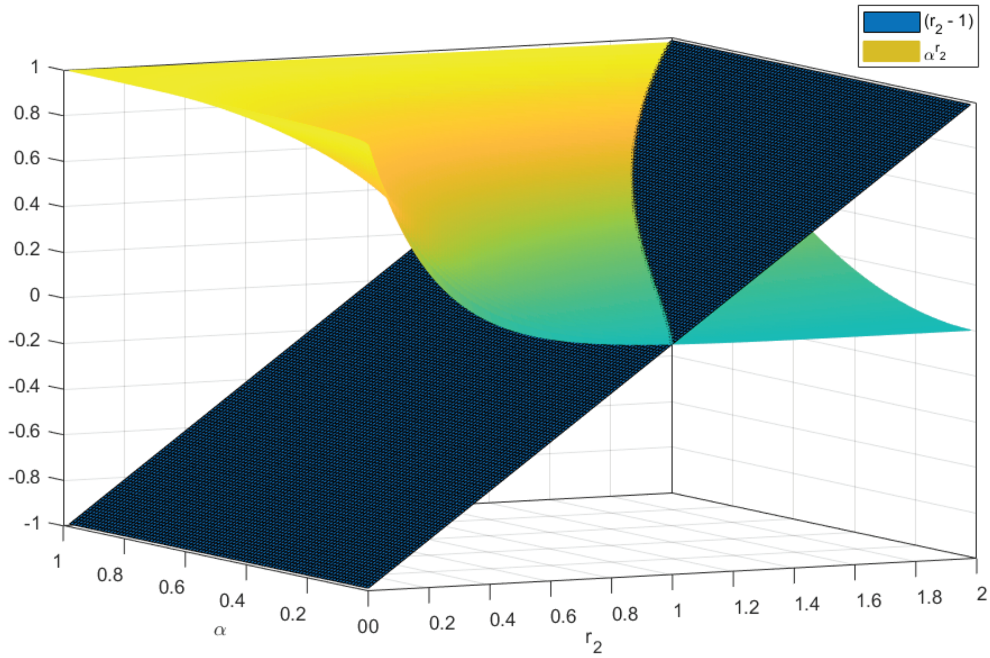

Figure 7. Participation Constraint (i): α^{r_2} vs. $(r_2 - 1)$.

Since the parties are equal in the first stage, the positive-effort equilibrium is symmetric, and each has an equal probability of victory or loss. Their expected payoff if they participate in the first-stage (again, assuming participation in the second stage) is $\frac{1}{2}\pi_w^* + \frac{1}{2}\pi_\ell^*$. If they choose to opt out of the first stage with certainty and sacrifice their chance at momentum, their equilibrium expected payoff is $P_\ell^* V - s_\ell^*$. The participation constraint is therefore

$$\frac{1}{2}\pi_w^* + \frac{1}{2}\pi_\ell^* \geq P_\ell^* V - s_\ell^*. \quad (1)$$

But since

$$\pi_w^* = P_w^* V - \alpha s_w^* - s_1^* \quad = \quad P_w^* V - s_\ell^* - s_1^*,$$

we have

$$\frac{1}{2}\pi_w^* + \frac{1}{2}\pi_\ell^* = \frac{1}{2}P_w^* V + \frac{1}{2}P_\ell^* V - s_\ell^* - s_1^*,$$

so (1) becomes

$$(P_w^* - P_\ell^*)V \geq 2s_1^*$$

or

$$\frac{(1-\alpha^{r_2})}{(1+\alpha^{r_2})}V \geq \frac{r_1 V}{2}\frac{(1-\alpha^{r_2})}{(1+\alpha^{r_2})},$$

which simplifies to the following condition.

Participation Constraint (ii) (*PC (ii)*): $2 \geq r_1$.

This makes sense given that the first stage is essentially a standard contest with the value of momentum as its prize, leading to the usual restriction for r_1 as our participation constraint for the first stage (assuming $PC(i)$ is satisfied).

The sensitivity of the first stage (or the initial "battle for momentum") to the expenditures of the conflicting parties is thus relatively unrestricted (so long as we are interested in pure-strategy equilibria), while the second stage must be permissible enough to vie for should a party lose the first stage. A second stage that is more sensitive to effort only enhances the advantage gained by momentum.

Even with these parameter constraints satisfied and both parties acting rationally according to backward induction, what is interesting is that Pyrrhic victories are still possible. In particular, note that in the game's first stage each party has an equal chance of victory and, so long as $PC(i)$ is satisfied, the loser of that stage will still compete in the second stage. But although $PC(i)$ ensures that positive spending is better than none at that point, it does not guarantee a net-positive overall expected payoff.

Consider $\pi_\ell^* = P_\ell^* V - s_\ell^* - s_1^*$. As long as ℓ competes (i.e., spends a positive amount) in the second-stage conflict, $P_\ell^* > 0$, meaning they have a positive probability of victory. But their expected payoff (including their effort expenditure from the first stage, s_1^*) may be negative. The condition $\pi_\ell^* < 0$ simplifies to $P_\ell^* V - s_\ell^* < s_1^*$ or

Pyrrhic Victory (PV): $\alpha^{r_2}(1+\alpha^{r_2}) - r_2\alpha^{r_2} < \frac{r_1}{4}(1-\alpha^{2r_2})$.

For example, consider $r_1 = r_2 = 1$, for which $PC(i)$ and $PC(ii)$ are each satisfied for all $0 < \alpha < 1$ (so that both parties compete in both stages of the conflict). Condition PV is satisfied, for all $\alpha < \sqrt{\frac{1}{5}}$. So, for $\alpha < \sqrt{\frac{1}{5}}$, the combatant who ends up losing the first-stage battle for momentum is set up to realize an expected Pyrrhic victory, in that his overall expected payoff is negative.

The condition under which such an expected Pyrrhic victory arises depends upon the values of all three parameters (i.e., r_1, r_2, and α). Figure 8 provides a plot of the left- and right-hand sides of condition PV as functions of r_1 and r_2 for $\alpha = \frac{1}{3}$. Condition PV is satisfied—so that the first-stage loser realizes an expected Pyrrhic victory—when the black surface lies above the multi-colored surface.

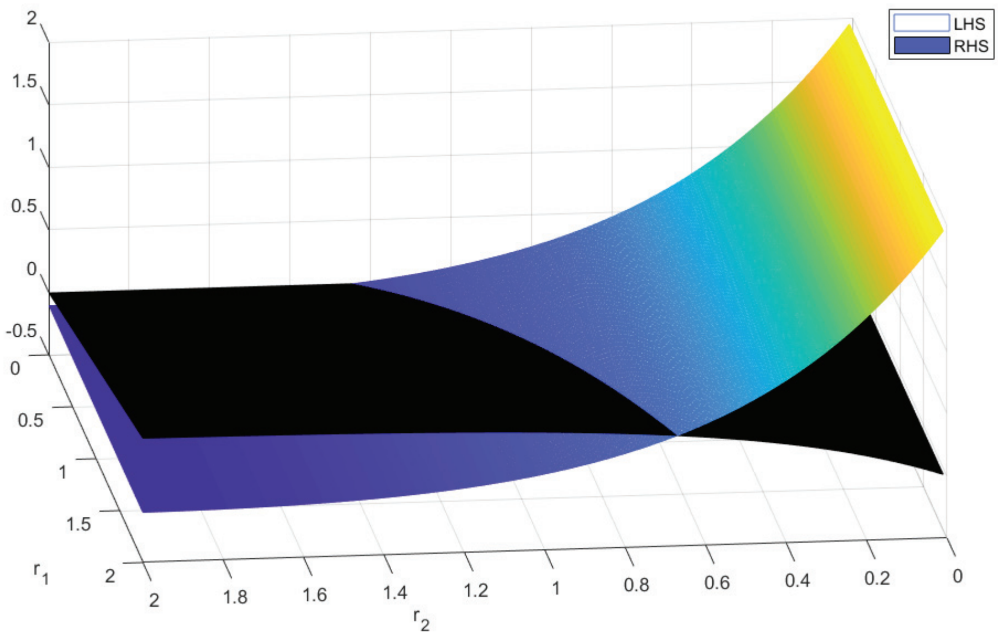

Figure 8. Pyrrhic Victory Conditions, $\alpha = 0.33$.

2.3. Rent Dissipation

Total rent dissipated across the two stages is

$$\frac{2r_2\alpha^{r_2}V}{(1+\alpha^{r_2})^2} + \frac{r_1 V}{2}\frac{1-\alpha^{r_2}}{1+\alpha^{r_2}}$$

such that total post-conflict rents are

$$V - \frac{2r_2\alpha^{r_2}V}{(1+\alpha^{r_2})^2} - \frac{r_1 V}{2}\frac{1-\alpha^{r_2}}{1+\alpha^{r_2}}.$$

Hence, the rent dissipation across the two players is equal, though the winner of the first stage does benefit from its win in terms of a higher likelihood of victory. What is perhaps interesting is that the elasticity of the contest success function does not determine that likelihood, but rather only the effort expenditure of the two contestants. Thus, momentum helps a player solely by enhancing the player's ability for victory in the second-stage contest. It does not, however, change the player's calculus of second-stage contest nor create any welfare changes therefrom.

From the second-stage objective functions, it is straightforward to see that the present contest is more rent-dissipative than a one-stage contest that features the same level of cost asymmetry, α, and noise parameter, r_2, as observed in the second round of this two-stage contest. Specifically, such an alternative contest would be less rent-dissipative by $2 \cdot s_1^*$ units of input expenditure. We know this because objective functions for each party in this alternative one-shot contest would be the same as the second-stage objective functions observed herein, but would exclude the $(-s_1)$ term from each function, where this term is not marginal to the decision calculus of that stage. Therefore, this term exactly measures additional rent-dissipation for each party in the two-stage game with contestable momentum. To check this reasoning, we can reconsider total rent-dissipation in the present contest:

$$\frac{2r_2\alpha^{r_2}V}{(1+\alpha^{r_2})^2} + \frac{r_1 V}{2}\frac{1-\alpha^{r_2}}{1+\alpha^{r_2}}$$

The second term in the sum above is simply $2 \cdot s_1^*$. Then, we expect the first term in the sum above to represent total rent dissipation for the alternative one-shot contest discussed previously. It is straightforward to verify that this is the case. That is, we find that $2 \cdot s_1^* = \frac{r_1 V}{2}\frac{1-\alpha^{r_2}}{1+\alpha^{r_2}}$. From this result, we conclude that, *ceteris paribus*, contestable advantage in conflict is more rent-dissipative than innate or otherwise incontestable advantage. Therefore, we expect *Pyrrhic* victories to be more common for contests that feature an intermediate stage or stages in which subsequent advantages can be earned, *ceteris paribus*. This additional rent-dissipative toll exists even given a backward-inductive (equilibrium) behavior in a setting of complete information rather than one characterized by "fog of war" effects. The quagmire theory suggests that countries can become involved in problematic (i.e., rent-dissipative) conflicts due to incomplete information. The present paper finds that the setting of conflict—namely, the contestibility of intermediate, momentous advantage in a conflict—can effectively substitute for incomplete information in generating rent-dissipative tolls.

3. Discussion and Conclusions

In this study, we have examined the role of momentum in conflict outcome. We model momentum as an "intermediate target". For example, one might take the fort before taking the city. By taking the fort, one then faces a lower unit input cost of contesting for the city. We model this as a two-stage contest in which the first stage is a conflict for the (value of) momentum, and the second stage is a battle for the ultimate conflict prize. The intermediate target is simply an instrument by which to gain an advantage toward the ultimate prize.

Our main finding is that although the elasticity of effort—which we allow to vary between the two stages of conflict—does impact the contestants' effort levels, it has no bearing on the endogenously determined value of momentum itself. Further, rent dissipation in the two-stage conflict is equal across party whether or not an individual obtains momentum in the first stage. Thus, momentum helps a player solely by enhancing the player's marginal ability for victory in the second-stage contest. It does not, however, change the player's net calculus of second-stage contest spending. Contestable advantage in conflict is also found to be more rent-dissipative than innate or otherwise incontestable advantage. Therefore, we expect *Pyrrhic* victories to be more common for contests that feature an intermediate stage or stages in which subsequent advantages can be earned, *ceteris paribus*.

An alternative version or extension of the model could incorporate additional parameters, for example giving the winner of the first stage a different elasticity of effort as compared to the first-stage loser in the second stage. In other words, an extra impact of momentum. Letting r_w and r_ℓ denote the second-stage elasticities, the results are qualitatively similar to the original model, but efforts would then be modified by those elasticities. Rather than $s_\ell = \alpha s_w$ as in the model analyzed, we would instead have $\frac{r_w}{r_\ell} s_\ell = \alpha s_w$. We chose to focus on the simpler model in this paper, with momentum as just a cost advantage, for clarity of presentation.

While intermediate targets may appear as useful benchmarks in conflict, they in fact dissipate additional expected contest rents to each party. This additional rent-dissipative toll exists even given a backward-inductive (equilibrium) behavior in a setting of complete information rather than one characterized by "fog of war" effects. That is, rather than countries becoming involved in problematic (i.e., rent-dissipative) conflicts due to incomplete information, the present paper finds that the setting of conflict—namely, the contestability of intermediate, momentous advantage in a conflict—can effectively substitute for incomplete information in generating rent-dissipative tolls. Similarly, we find that contestable advantage can lead parties to optimally forego contest participation (i.e., if conflict parameters do not meet the participation constraint). This is in contrast to a one-stage simultaneous contest that takes on the second-stage parametric values of the present contest.

An alternative application for our model could be a conflict between not military parties but business organizations—particularly a union organization versus management. Businesses may be very willing to invest early on to prevent workers' organizations from gaining any advantage going forward. And this may be true regardless of how hard the first-stage struggle is to prevent that advantage, or the degree of difficulty going forward. An additional example may be in the area of attack and defense of information networks, in which players engage in a first-stage battle over network access and alteration (e.g., undetected installation of a "backdoor" access point), sometimes followed by a second-stage battle over control of the network. In this case, undetected access can help an attacker gain knowledge about the architecture of the network. In turn, this knowledge will raise the attacker's effectiveness in stage 2 of the contest.

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games

Article

Self-Enforcing Collective Counterteror Retaliation

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Abstract: Motivated by recent examples of collective effort on the war on terror, we examine the incentives that retaliation may produce for the endogenous formation of an international counterterror coalition. We show that there are quite reasonable circumstances under which any nation that is a target of a terrorist attack finds it desirable to be a member of the international counterterror coalition, holding the choices of all other nations as given. The incentives to join the coalition are the group-specific benefits from retaliation enjoyed by each coalition member, the relatively lower spillover benefit from retaliation enjoyed by each stand-alone nation, and the inability of pre-emptive measures to avert terrorist attacks. The disincentive to join is the anticipated backlash from retaliation, which targets coalition members only.

Keywords: retaliation; counterterror; coalition; backlash

JEL Classification: C72; D74; F53; H87

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1. Introduction

Governments often retaliate after some citizens they represent become victims of terrorist attacks (see, e.g., Crenshaw [1], Merari [2], Lee [3], Lee and Sandler [4], Kydd and Walter [5], Benmelech et al. [6], Carter [7], and Gaibulloev and Sandler [8]). The U.S. bombed targets in Tripoli and Benghazi, Libya, on 15 April 1986 as retaliation for the Libyan sponsored terrorist attack in Berlin that killed two and injured sixty-two U.S. citizens on 4 April 1986 (Lee and Sandler [4]). Since 1967, Israel has demolished houses in areas occupied by Palestinians as retaliation for Palestinian terrorist attacks (Benmelech et al. [6]). In response to the 9/11 attacks, the United States and Britain conducted airstrikes on October 2001 and, later, together with many other allies in Operation Enduring Freedom, engaged in other military operations, as retaliation against the Taliban and al-Qaeda in Afghanistan. In coordination with the United States, France bombed ISIS targets in Raqqa, Syria, on 15 November 2015, following a number of ISIS terrorist attacks in Paris on 13 November 2015.

Terrorists, however, may respond to retaliatory actions with further attacks (see e.g., Lee and Sandler [4], Jacobson and Kaplan [9], Argomaniz and Vidal-Diez [10], Benmelech et al. [6], Gaibulloev and Sandler [8], Matthews et al. [11], and Kattelman [12]). The U.S. retaliatory strikes in Libya in 1986, which received partial support from Britain, produced several terrorist attacks against U.S. and British interests soon after the airstrikes (Lee and Sandler, [4]). The Israeli policy of demolishing houses as retaliation against Palestinian terrorists generated an increase in terrorist attacks after precautionary house demolitions in 2004 and 2005, because properties of some non-terrorists (i.e., neutrals) were demolished (Benmelech et al. [6]). Both examples reveal that terrorist attacks following retaliation are likely if retaliation generates large or nondiscriminatory collateral damages. Airplane and drone strikes, for example, are prone to cause collateral damages owing to inaccuracy of information about precise location or signatures of terrorist targets (see, e.g., Gaibulloev and Sandler [8] and Allen et al. [13]). Owing to negative publicity, moral outrage, and the desire for vengeance, counterterror proactive policies, of which retaliation is an example, may facilitate a terrorist group's acquisition of resources as well as induce neutrals to become leaderless jihadists (see, e.g., Enders, Sandler and Cauley [14], Enders and Sandler [15], Pape [16], Kaplan et al. [17], Faria and Arce [18,19], and Sageman [20]). With a larger resource endowment, the terrorist organization may supply a greater amount of terrorist attacks in response to retaliatory actions.

In this paper, we consider the pros and cons of collective retaliation effort against a terrorist organization. As the paragraphs above reveal, retaliation for terror attacks is common even though there is evidence that it causes backlash—the phenomenon that counterterror policies expand the resources available to terrorists (Faria and Arce [19]). Motivated by Lee [3], Lee and Sandler [4], Cárceles-Poveda and Tauman [21], de Oliveira et al. [22], Kattelman [12], and the recent examples of collective effort on the war on terror, we examine the incentives that retaliation may produce for the endogenous formation of an international counterterror coalition. Lee [3] notes that retaliation against transnational terrorists yields country-specific and international benefits. An example of a country-specific benefit is the increased security level enjoyed by citizens of a retaliating nation whenever retaliation reduces the incidence of terrorist attacks. In addition, a nation's retaliation effort generates international benefits whenever it leads to a subsequent overall reduction in terror attacks produced by the targeted terrorist organization. Lee and Sandler [4] characterize retaliation against transnational terrorists as an action that produces country-specific and global, purely public, benefits. Unlike Lee [3], they argue that retaliation yields global consumption benefits that are both nonrival and nonexcludable. This purely public characteristic motivates free-riding behavior, which makes voluntary cooperation in the provision of retaliation effort difficult, if not impossible. More recently, Cárceles-Poveda and Tauman [21] point out that proactive counter-terror measures generate group benefits from cooperation to members of an international counter-terror coalition, which are not enjoyed by non-coalition members. Examples of group benefits from cooperation are international recognition and trade benefits enjoyed by trading agreements among members of the coalition only. Another important contribution to the study of the effectiveness of collective counterterror effort is provided by de Oliveira et al. [22]. They show that a coalition containing three nations is stable if the nations are symmetric and utilize defensive measures to prevent terrorist attacks promoted by a common terrorist organization.

In our analysis, the coalition engages in defensive and proactive measures. The latter include pre-emptive actions, which occur prior to terrorist attacks and are observed by the terrorist organization, and retaliatory actions, which occur after the counterterror coalition observes terrorist attacks. Coordinated retaliatory actions are desirable because pre-emptive actions are unable to completely deter attacks from a terrorist organization. Retaliation affects the terrorist organization as a pecuniary externality, yielding a monetary increase in its resources. As retaliation is known to cause substantial backlash, which subsequently may lead to an increase in the terrorist organization's available resources, we include this effect in the model. For the perpetrators of retaliatory actions, we postulate that retaliation, per se, may yield group-specific benefits originating from at least three sources: (1) an internationally coordinated tough position on terrorist attacks in order to deter future terror from the attacker or other terrorist organizations (i.e., reputation for counterterror leadership); (2) the sense of increased safety, being avenged (i.e., retribution), well represented by their elected officials (i.e., politics), or globally empowered (i.e., global prestige) felt by citizens of coalition members (as in Lee [3]); and (3) as in Cárceles-Poveda and Tauman [21], the possibility of exclusive trade deals among coalition members. As retaliation effort carried out by the coalition should produce future global benefits in terms of reduced terrorist activity, it yields a positive spillover to non-coalition nations. This is in line with the view advanced by Lee and Sandler [4] that retaliation generates purely public global benefits.

All nations and the terrorist organization play a sequential game of complete, but imperfect information as follows. In stage 0, each nation makes a choice to join or not to join an international counterterror coalition, taking the choices of all other nations as given. The choices are observed by all nations and the terrorist organization prior to the subsequent stage of the game. After being formed, the coalition represents its members and makes choices to maximize the sum of its members' payoffs. In stage 1, the coalition and the stand-alone nations choose their pre-emptive activities, taking each other's actions as given. In stage 2, the coalition and the stand-alone nations choose their defensive measures, taking each other's choices as given. In stage 3, the terrorist organization makes its choices concerning terrorist attacks. In stage 4, the coalition decides on the level of retaliation. The equilibrium concept is subgame perfect equilibrium.

We show that there are quite reasonable circumstances under which each nation in the globe, holding the choices of all other nations as given, finds it desirable to be a member of the international counterterror coalition. The incentives to join the coalition are the group-specific benefits from retaliation enjoyed by each coalition member, the relatively lower spillover benefit from retaliation enjoyed by each stand-alone nation, and the inability of pre-emptive measures to avert terrorist attacks. The disincentive to join is the anticipated backlash from retaliation, which targets coalition members only.

To the best of our knowledge, this is the first paper in the game-theoretic terrorism literature that explicitly separates retaliatory actions from other proactive actions and examines the incentives associated with retaliation to the endogenous formation of a counterterror coalition.

From this point on, the paper is organized as follows. Section 2 presents the simple model. Section 3 examines the solution to the game played from stages 1 to 4. Section 4 considers the choice made by each nation of whether to join the coalition. Section 5 offers concluding remarks.

2. Model

We consider a complete information coalition formation game with five stages, one terrorist organization, and I nations. In the coalitional stage (stage 0), nations decide whether or not to join a coalition. A generic coalition is denoted by S and has cardinality (# of members) $|S| \equiv s$. In stage 1, the coalition chooses the preemptive measures of its members and non-coalition members decide their pre-emptive measures independently. Stage 2, where each nation decides the level of its defensive measures, is followed by the terrorist organization's selection of a set of countries to attack as well as the magnitudes of the attacks, in stage 3. Finally, in stage 4, the coalition chooses whether to carry out retaliatory measures. We call this game a retaliation game.

Pre-emptive measures are actions that increase the costs of or reduce the resources available to a terrorist organization, reducing the terrorist threat for all potential targets. In our model, decisions about pre-emptive actions precede those about defensive actions. The latter can be thought of as measures that improve the nation's homeland security.

The terrorist organization derives benefits from its attacks according to the function $b(\boldsymbol{t}) = \sum_{i=1}^{I} b_i t_i$, where $\boldsymbol{t} = (t_1, t_2, \ldots, t_I)$ is the vector of attacks (damage inflicted) on countries $i = 1, \ldots, I$ and b_i is the marginal effect of an attack on nation i. It incurs a specific cost $c_i(t_i, d_i) = \frac{1}{2}(d_i + t_i)^2$ when it carries out an attack of magnitude t_i on country i, whose defensive effort is d_i. It also sustains a (common) cost $c_c(\boldsymbol{p}) = \left(\sum_{i=1}^{I} t_i\right)\xi(u)$, where $\xi(u) = \alpha u$, $\alpha \in (0, 1]$ and $u = \sum_{i=1}^{I} p_i$ (In addition to reducing the burden of notation, the advantage of specifying the common cost this way is that it makes clear how our findings would change with the specification of $\xi(u)$), which is imposed on the organization by the pre-emptive measures $\boldsymbol{p} = (p_1, p_2, \ldots, p_I)$ chosen in stage 1. The parameter α can be interpreted as the sensitivity rate of the terrorist organization's common cost to pre-emptive actions.

The objective of the terrorist organization is to maximize

$$\pi^T(\boldsymbol{t},\boldsymbol{d},\boldsymbol{p}) = \sum_{i=1}^{I} b_i t_i - \sum_{i=1}^{I} \frac{1}{2}(d_i + t_i)^2 - \left(\sum_{i=1}^{I} t_i\right)\xi(u) \tag{1}$$

where $\boldsymbol{d} = (d_1, d_2, \ldots, d_I)$. The impact of retaliatory actions by the coalition on the terrorist organization happens through the marginal benefit parameters. We define the marginal benefit of attacking nation i as $b_i = \begin{cases} b + \omega R, \text{ if } i \in S \\ b, \text{ if } i \notin S \end{cases}$, where ωR is the pecuniary externality caused by retaliation. The parameters $\omega \in [0,1]$ and R are the marginal external gain and the magnitude of the retaliation carried out by the coalition, respectively. Retaliation generates a gain (positive ω) for the terrorist organization when it attacks a coalition member because backlash leads to an increase in the terrorist organization's available resources[1].

The terrorist organization knows the size and composition of the counterterror coalition when it makes its choices, as well as the identities of stand-alone nations and the pre-emptive and defensive actions $\boldsymbol{p}$ and $\boldsymbol{d}$ undertaken by all nations. In addition, it knows how the pecuniary externality associated with retaliation affects its resources and fully anticipates the amount of retaliation that it will face if it attacks the counterterror coalition.

The payoff of nation i is given by

$$\begin{aligned} &\pi(t_i, r, d_i, p_i) = \beta R - \tfrac{1}{2} r_i^2 - \theta t_i - \tfrac{1}{2}\left(d_i^2 + p_i^2\right), \text{ if } i \in S \\ &\pi(t_i, r, d_i, p_i) = \tfrac{\gamma}{2}\beta R - \theta t_i - \tfrac{1}{2}\left(d_i^2 + p_i^2\right), \text{ if } i \notin S \end{aligned} \tag{2}$$

where $R = \sum_{i=1}^{I} r_i$, r_i is the retaliatory effort of nation i, $\beta > 0$ is marginal benefit of retaliation, $\gamma \in [0,2]$ is a scale parameter that controls how the benefit of retaliation to stand-alone nations compares to that of coalition members, and $\theta > 0$ is the marginal damage from a terrorist attack suffered by each nation. The first two terms in a member nation's payoff comprise the benefit it gets from retaliatory actions and a variable cost that includes monetary expenditures. For simplicity, we assume that stand-alone nations do not find it desirable to carry out retaliatory actions—thus, only the coalition retaliates, implying $r_i = 0$ for $i \notin S$ and $R = \sum_{i \in S} r_i$.

The coalition faces different scenarios in stages 1 and 4 of the game. Its objective function in the first stage is to maximize the sum of its members' payoffs, that is,

$$\Pi_C(\boldsymbol{r},\boldsymbol{d},\boldsymbol{p},\boldsymbol{t}) = \sum_{k \in S} \pi(t_k, r, d_k, p_k) = \sum_{k \in S}\left[\beta R - \frac{1}{2} r_i^2 - \theta t_k - \frac{1}{2}\left(d_k^2 + p_k^2\right)\right] \tag{3}$$

given the optimal values of $\boldsymbol{r} = \left(r_{i_1}, r_{i_2}, \ldots, r_{i_s}\right)$, $\boldsymbol{p}$ and $\boldsymbol{t}$, where $i_1, i_2, \ldots, i_s$ are the members of coalition S. In the last stage, the coalition chooses r_i, $i \in S$, which maximizes $\sum_{k \in S}\left[\beta R - \frac{1}{2} r_i^2\right] = s\beta R - \frac{1}{2}\sum_{k \in S} r_i^2$ if $\sum_{k \in S} t_k > 0$, and $R = 0$ otherwise.

To finalize the description of our model, we need to take a closer look at the coalition formation stage 0. Nations simultaneously choose whether they want to join a coalition S, $|S| \equiv s \leq I$, or play the game independently. To test which coalitions are stable, we follow D'Aspremont et al. [23] and apply the internal and external stability criteria:

$$\begin{aligned} &\text{Internal stability}: \pi_m^*(S) \geq \pi_m^*(S \backslash \{m\}),\ \forall m \in S \\ &\text{External stability}: \pi_n^*(S) \geq \pi_n^*(S \cup \{n\}),\ \forall n \notin S \end{aligned} \tag{4}$$

3. Equilibrium Analysis

Our equilibrium concept is subgame perfection. Utilizing backward induction, we start the analysis with an examination of the last stage of the game. In the last stage, the coalition chooses $r_i,\ i \in S$, in order to maximize $s\beta R - \frac{1}{2}\sum_{k\in S} r_i^2$ if $\sum_{k\in S} t_k > 0$, and $R = 0$ otherwise. If $\sum_{k\in S} t_k > 0$, which is the case under our assumptions, the first-order condition yields

$$s\beta - r_i = 0 \Rightarrow r_i = s\beta, \tag{5}$$

which implies $R = \sum_{i\in S} r_i = s^2\beta$. The optimal retaliation level for each coalition member is equal to the sum of the marginal benefits of retaliation enjoyed by the entire coalition. As the objective function of the coalition is strictly concave, the unique solution is a maximum.

In stage 3, the terrorist organization chooses the attack levels that maximize its payoff under the constraint that the retaliation of the coalition follows the formula above. It will thus solve the maximization problem below:

$$\max \pi(\boldsymbol{t}, \boldsymbol{d}, \boldsymbol{p}) = \sum_{i\in S}\left(b + s^2\omega\beta\right)t_i + \sum_{i\notin S} bt_i - \sum_{i=1}^{I}\frac{1}{2}(d_i + t_i)^2 - \left(\sum_{i=1}^{I} t_i\right)\xi(u) \text{ s.t. } t_i \geq 0,\ \forall i \tag{6}$$

The Lagrangean function is

$$L(\boldsymbol{t}, \boldsymbol{\mu}) = \sum_{i\in S}\left(b + s^2\omega\beta\right)t_i + \sum_{i\notin S} bt_i - \sum_{i=1}^{I}\frac{1}{2}(d_i + t_i)^2 - \left(\sum_{i=1}^{I} t_i\right)\xi(u) + \sum_{i=1}^{I}\mu_i t_i, \tag{7}$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_I)$ is the vector of Lagrange multipliers. Before we look at the first-order conditions, we show that this Lagrangean is concave. All the terms of $L(\boldsymbol{t}, \boldsymbol{\mu})$ that depend on the t_i's are linear, with the exception of $-\sum_{i=1}^{I}\frac{1}{2}(d_i + t_i)^2$, so it suffices to show that the latter is a concave function of t. It is easy to see that this is the case, as its Hessian matrix is negative definite:

$$\begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix} \tag{8}$$

The necessary and sufficient first-order conditions are as follows:

$$\begin{aligned} &(i)\ \tfrac{\partial L}{\partial t_i} = b + s^2\omega\beta - d_i - t_i - \xi(u) + \mu_i = 0, \text{ if } i \in S \\ &(ii)\ \tfrac{\partial L}{\partial t_i} = b - d_i - t_i - \xi(u) + \mu_i = 0, \text{ if } i \notin S \\ &(iii)\ t_i \geq 0, \mu_i \geq 0 \text{ and } \mu_i t_i = 0\ \forall \text{i} \end{aligned} \tag{9}$$

We assume that the terrorist organization chooses a positive level of attack for every nation (and later check if this assumption is satisfied in equilibrium). Then, $\mu_i = 0\ \forall i$ and

$$\begin{aligned} &b + s^2\omega\beta - d_i - t_i - \xi(u) = 0 \Rightarrow t_i = b + s^2\omega\beta - d_i - \xi(u),\ i \in S \\ &b - d_i - t_i - \xi(u) = 0 \Rightarrow t_i = b - d_i - \xi(u),\ i \notin S \end{aligned} \tag{10}$$

Note that the sole difference in attack levels is a function of the pecuniary externality that retaliation produces. As the externality is non-negative, the amount of terrorist activity in a nation that is a member of the counterterror coalition is at least as large as the activity in a stand-alone nation.

In stage 2, nation i maximizes its payoff with respect to d_i. It knows that the terrorist organization will behave according to the reaction functions derived above. The other arguments in its payoff functions are given. Therefore, we can rewrite their payoff functions as follows:

$$\begin{aligned} &\pi(d_i) = \tfrac{s^2\beta^2}{2} - \theta\left[b + s^2\omega\beta - d_i - \xi(u)\right] - \tfrac{1}{2}\left(d_i^2 + p_i^2\right), \text{ if } i \in S \\ &\pi(d_i) = \tfrac{\gamma s^2\beta^2}{2} - \theta[b - d_i - \xi(u)] - \tfrac{1}{2}\left(d_i^2 + p_i^2\right), \text{ if } i \notin S \end{aligned} \tag{11}$$

As the payoff functions are strictly concave, the necessary and sufficient conditions for a unique maximum are as follows:

$$\frac{\partial \pi}{\partial d_i} = \theta - d_i = 0 \Rightarrow d_i = \theta,\ \forall i \tag{12}$$

Each nation finds it optimal to set its level of defensive effort equal to its marginal damage from terrorism.

In stage 1, the coalition and stand-alone nations choose their pre-emptive measures. Let us start with a non-member nation. It chooses its pre-emptive measure p_i to maximize

$$\pi(p_i) = \frac{\gamma}{2}s^2\beta^2 - \theta[b - \theta - \xi(u)] - \frac{1}{2}\left(\theta^2 + p_i^2\right) = \frac{\gamma}{2}s^2\beta^2 + \frac{\theta^2}{2} - \theta[b - \xi(u)] - \frac{p_i^2}{2} \tag{13}$$

Clearly, this is a concave function of p_i for any $\xi(u)$, such that $\partial^2\xi(u)/\partial p_i^2 = 0$ (which is the case under the functional form of $\xi(u)$ in our model). The first-order conditions give us, for all $i \notin S$,

$$\frac{\partial \pi}{\partial p_i} = \theta\frac{d\xi(u)}{du} - p_i = 0 \Rightarrow p_i = \alpha\theta, \tag{14}$$

where we used the facts that $\partial\xi(u)/\partial p_i = d\xi(u)/du$, because $\partial u/\partial p_i = 1$ and $d\xi(u)/du = \alpha$. Each stand-alone nation sets its amount of pre-emptive effort equal to its marginal effective damage from terrorism avoided with pre-emptive action. The latter is proportional to the terrorist organization's sensitivity rate to pre-emptive actions.

The coalition wishes to maximize

$$\Pi_C\left(r, \boldsymbol{t}_{-i\notin S}, \boldsymbol{d}_{-i\notin S}, \boldsymbol{p}_{-i\notin S}\right) = \sum_{k\in S}\left[\beta R - \frac{1}{2}r_i^2 - \theta t_k - \frac{1}{2}\left(d_k^2 + p_k^2\right)\right] \tag{15}$$

subject to the expressions for the optimal levels of $\boldsymbol{t}_{-i\notin S}$, $\boldsymbol{d}_{-i\notin S}$, and r. After some algebra, the objective function becomes

$$\Pi_C(\boldsymbol{p}) = \sum_{k\in S}\left[\frac{s^2\beta^2}{2} - \theta\left[b + s^2\omega\beta - \theta - \xi(u)\right] - \frac{1}{2}\left(\theta^2 + p_k^2\right)\right] \tag{16}$$

The coalition wants to maximize the expression above with respect to p_k, $k \in S$. Notice that the objective function is concave. The first-order conditions are

$$\frac{\partial \Pi_C(\boldsymbol{p})}{\partial p_k} = s\theta\frac{d\xi(u)}{du} - p_k = 0 \ \Rightarrow \ p_k = s\alpha\theta,\ k \in S \tag{17}$$

As externalities within the coalition are internalized, each coalition member provides pre-emptive effort equal to the sum of effective marginal damages. As pre-emptive efforts are a global public good, stand-alone nations "easy ride" on the higher provision levels of coalition members. This is a disincentive to join the counterterror coalition.

The proposition below summarizes the equilibrium of the retaliation game for $\xi(u) = \alpha u$.

Proposition 1: The unique pure strategy subgame-perfect Nash equilibrium of the retaliation game is given by the following:

i. Retaliation: $r_i^* = s\beta$.
ii. Defensive measures: $d_i^* = \theta,\ \forall i$.
iii. Preemptive measures: $p_i^* = \alpha\theta,\ i \notin S; p_i^* = s\alpha\theta,\ i \in S$
iv. Terrorism activities:

$$t_i^* = b + s^2\omega\beta - \theta - \alpha^2\theta\left(I - s + s^2\right) = b + s^2\omega\beta - \theta\left[1 + \alpha^2\left(I - s + s^2\right)\right],\ i \in S$$
$$t_i^* = b - \theta - \alpha^2\theta\left(I - s + s^2\right) = b - \theta\left[1 + \alpha^2\left(I - s + s^2\right)\right],\ i \notin S$$

Before we proceed, we need to check under what conditions our assumption that the terrorist organization chooses a positive level of attack for every nation is valid. It is easy to see from part (iv) of Proposition 1 that $t_i^* > 0\ \forall i$ if $b > \theta + \alpha^2\theta\left(I - s + s^2\right)$. As $I - s + s^2$ reaches a maximum at $s = I$, this condition is satisfied if $b > \theta + \alpha^2\theta I^2$, which depends only on model parameters. In words, the benefit the terrorist organization enjoys when it attacks a non-member nation needs to be sufficiently high. This condition, which we assume holds true, does not affect the coalition stability results of the next section, where the parameter b, as it turns out, plays no role.

We will now highlight some important features of the equilibrium allocation. First, note that an increase in ω, the marginal external transfer from retaliation, increases terrorist attacks on nations that are members of the counterterror coalition, but does not affect attacks on stand-alone nations. The impact of ω is augmented by the size of the coalition because the larger the number of coalition members attacked, the larger the effects of retaliation on the terrorist organization. An increase in b, the marginal benefit of a terrorist attack, increases the terrorist organization's attacks on both coalition members and stand-alone nations at the same rate. As one expects, terror attacks decrease with the effectiveness of pre-emptive measures (i.e., α) and with the marginal damage caused by terror (higher θ) owing to defensive and pre-emptive measures. As pre-emptive and retaliatory measures are members of a family of proactive measures, they are naturally competing measures to achieve the same goal—namely, to reduce the terrorist organization's available resources. The key difference is the timing at which they occur. Pre-emptive actions occur before attacks and retaliatory actions occur afterwards. A necessary condition for retaliation is the failure of pre-emptive actions to completely deter terrorist attacks, because retaliation occurs only if coalition members are attacked. The incentive to retaliate and thus to join the counterterror coalition is higher the lower the effectiveness of pre-emptive actions. We will clearly demonstrate this connection below in our analysis of coalitional stability and size.

An increase in β, the marginal benefit to a member nation of retaliatory actions, increases the amount of retaliation and increases terrorist attacks on a member nation. Terrorist attacks on stand-alone nations are not affected by changes in β.

4. Stable Coalitions

We start this section with a proposition providing the conditions for internal and external stability.

Proposition 2: The internal and external stability conditions are as follows, where $|S| \leq I$:

Internal stability:

$\left(\beta[\beta(1-\gamma)-2\theta\omega]-\alpha^2\theta^2\right)s^2+2\left(\beta^2\gamma+2\alpha^2\theta^2\right)s-\left(\beta^2\gamma+3\alpha^2\theta^2\right)\geq 0$ for all $i \in S$.

External stability:

$\left(\beta[\beta(1-\gamma)-2\theta\omega]-\alpha^2\theta^2\right)s^2+2\left[\beta(\beta-2\theta\omega)+\alpha^2\theta^2\right]s+\beta(\beta-2\theta\omega)\leq 0$ for all $i \notin S$.

Proof. See Appendix A. □

This proposition has important implications for the size of stable coalitions. For instance, it implies that full cooperation in the form of a grand coalition is possible under certain conditions. We will establish these conditions momentarily, but first, we introduce new notation.

Define the internal and external stability functions, respectively, as

$$\begin{aligned}
&\psi(s|\beta,\theta,\omega,\alpha)=\left(\beta[\beta(1-\gamma)-2\theta\omega]-\alpha^2\theta^2\right)s^2+2\left(\beta^2\gamma+2\alpha^2\theta^2\right)s-\left(\beta^2\gamma+3\alpha^2\theta^2\right)\\
&\text{and}\\
&\varphi(s|\beta,\theta,\omega,\alpha)=\left(\beta[\beta(1-\gamma)-2\theta\omega]-\alpha^2\theta^2\right)s^2+2\left[\beta(\beta-2\theta\omega)+\alpha^2\theta^2\right]s+\beta(\beta-2\theta\omega)
\end{aligned} \tag{18}$$

Both functions are quadratic in s, and thus can be written in the generic form ax^2+bx+c. We define

$$\begin{aligned}
a&=a_{\text{int}}=a_{ext}=\beta[\beta(1-\gamma)-2\theta\omega]-\alpha^2\theta^2\\
b_{\text{int}}&=2\left(\beta^2\gamma+2\alpha^2\theta^2\right)\\
c_{\text{int}}&=-\left(\beta^2\gamma+3\alpha^2\theta^2\right)\\
b_{ext}&=2\left[\beta(\beta-2\theta\omega)+\alpha^2\theta^2\right]\\
c_{ext}&=\beta(\beta-2\theta\omega)\\
\Delta_{\text{int}}&=b_{\text{int}}^2-4ac_{\text{int}}\\
\Delta_{ext}&=b_{ext}^2-4ac_{ext}\\
s_{\text{int}}^-&=\text{smallest root of }\psi(s|\beta,\theta,\omega,\alpha)\\
s_{\text{int}}^+&=\text{largest root of }\psi(s|\beta,\theta,\omega,\alpha)\\
s_{ext}^-&=\text{smallest root of }\varphi(s|\beta,\theta,\omega,\alpha)\\
s_{ext}^+&=\text{largest root of }\varphi(s|\beta,\theta,\omega,\alpha)
\end{aligned} \tag{19}$$

To better understand how $\psi(s|\beta,\theta,\omega,\alpha)$ and $\varphi(s|\beta,\theta,\omega,\alpha)$ behave, we generate a few pictures for different values of the parameters, shown in Figure 1. In the first two, $\alpha=\theta=1$ and $\gamma=\omega=0.5$, with $\beta = 3$ in Figure 1A and $\beta = 1$ in Figure 1B. The values of the parameters in Figure 1C are $\alpha = \theta = 1$, $\omega = 0.5$, $\gamma = 1.5$, and $\beta = 10$. The graphs of the internal and external stability functions are shown in red and blue, respectively. For a coalition to be stable, the red curve needs to be on or above the x axis, and the blue curve needs to be on or below the x axis.

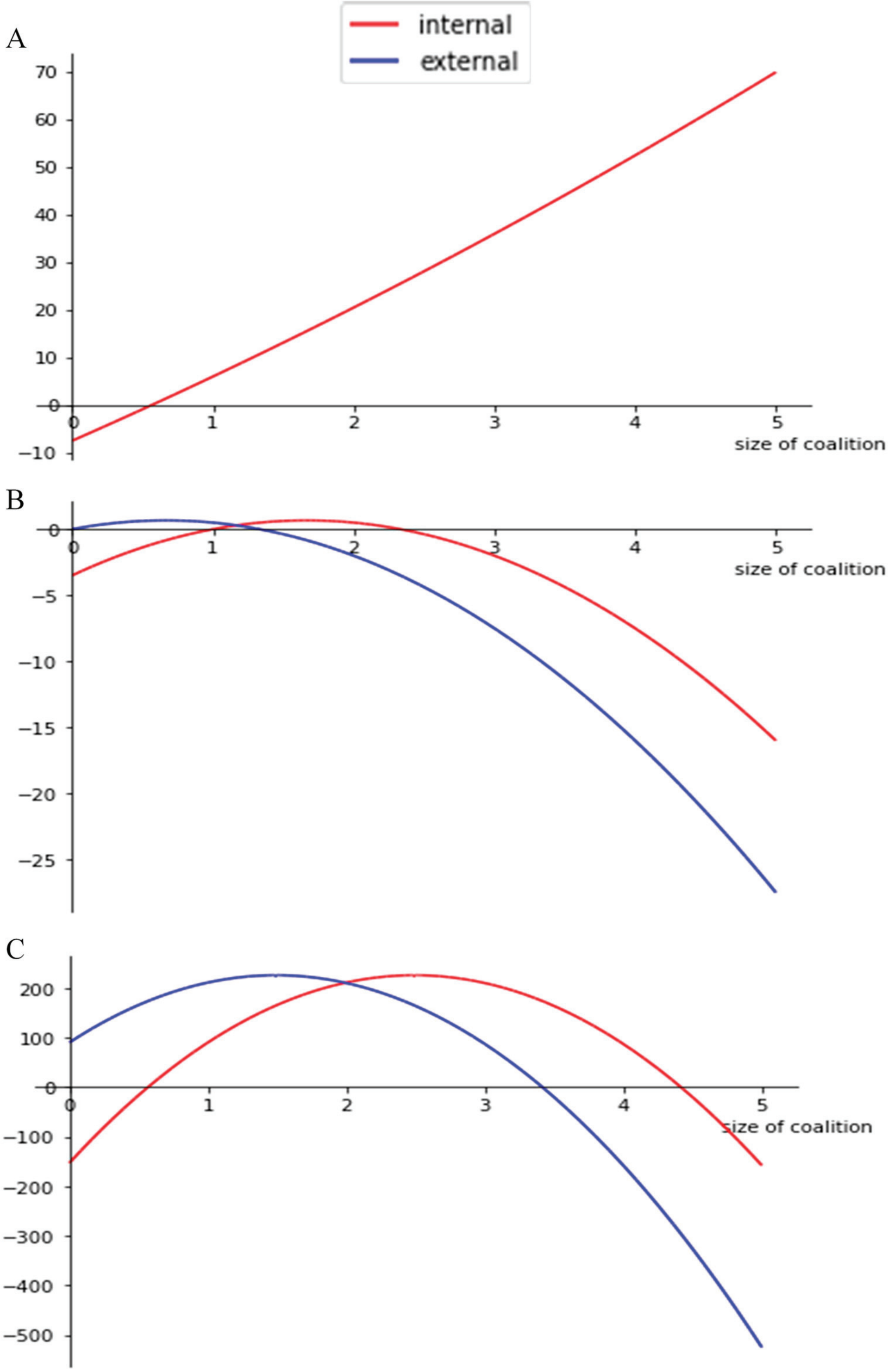

Figure 1. Graphs of stability conditions. (**A**) Grand coalition is stable, (**B**) Coalition of size 2 is stable, (**C**) Coalition of size 4 is stable.

In Figure 1A, the grand coalition is stable (The blue, external stability curve is not shown because the grand coalition satisfies external stability by default). In Figure 1B,C, coalitions of sizes 2 and 4, respectively, are the only stable coalitions. This shows that there is a variety of possible scenarios as far as coalition stability is concerned. The size of a stable coalition can be quite small or as large as the total number of nations, depending on specific combinations of values of the parameters. It is also possible for parameter values to be such that no coalition of any positive size is stable (these cases will be identified and discussed at the end of this section).

The corollaries below systematize our findings in this regard, starting with the grand coalition.

Corollary 1: The grand coalition is stable under the following conditions:

(i) $a \geq 0$.
(ii) $a < 0$, $\Delta_{\text{int}} \geq 0$, and $s^-_{\text{int}} \leq I \leq s^+_{\text{int}}$.

Proof. See Appendix A. □

Before we explore scenarios where conditions (i) and (ii) hold, it is important to stress that the findings in the corollary depart from the results frequently obtained in the literature on internal and external stability of coalitions, which Barrett [24] refers to as the "paradox of cooperation": Stable coalitions are either small or, if they are large, the full cooperation aggregate payoff is not much larger than the no cooperation aggregate payoff. The reason for this phenomenon is the positive externality generated by players' actions. As coalitions become larger, the payoffs of outsiders increase, making it more difficult to sustain stability.

Barrett [24] studies a pollution abatement game with identical countries with independent cost functions and shows, through simulations, that large coalitions are only stable when the cost of abatement is relatively small compared with its benefit. However, when this is true, coalitions with many countries do not increase net benefits by very much compared with the non-cooperative outcome. Cooperation would increase net benefits considerably when the cost and benefit of abatement are both large, but in this case, large coalitions are not stable. Yi [25] also analyses a game with identical countries, but considers a more general framework where several coalitions of different sizes can be formed. He considers a variety of endogenous coalition formation rules and shows that the grand coalition is usually not an equilibrium outcome in the presence of positive externalities. More recently, Finus and McGinty [26] show analytically that the largest stable coalition in a pure public good game with no transfers and where coalition members have identical individual benefit and cost functions is comprised of three nations.

Our findings show that the grand coalition is stable under a variety of conditions. Condition (i) in the corollary requires $\gamma < 1$, which means that the marginal benefits generated by the coalition's retaliatory actions are substantially larger for coalition members. That is not sufficient for $a \geq 0$ though, which can be written as $\beta[\beta(1-\gamma) - 2\theta\omega] \geq \alpha^2\theta^2$. According to this inequality, the marginal benefit of retaliation β has to be high enough with respect to factors that measure sensitivity to terrorist activities (θ), backlash (ω), and the impact of pre-emptive measures on terrorist costs (α).

The fact that the grand coalition is stable for high enough β (modulated by γ) is surprising. A high β is associated with strong positive externalities, in which case nations have a strong incentive to free ride, typically leading to a violation of internal stability. What is happening here is that retaliation also has a private good component, measured by γ. When the private benefit to coalition members is high enough (γ is low enough), it pays to stay in the coalition.

Put differently, part (i) of Corollary 1 states that, when the benefit stand-alone nations enjoy from retaliation is relatively small compared with that of coalition members ($\gamma < 1$), the internal stability condition will be satisfied for any coalition size if β is large enough. The rationale is that, if a coalition member stays in the coalition, it stands to benefit substantially from the retaliatory actions of the coalition, whereas as a stand-alone nation, it still benefits from retaliation, but to a considerably smaller extent[2].

The condition $a \geq 0$ can also be satisfied when θ, ω, and α are sufficiently low. The parameter θ measures the marginal damage suffered by a nation when it is attacked. When θ is low enough, coalition members do not care much about the fact that retaliation increases the likelihood they will suffer a terrorist attack, making internal stability easier to satisfy. A similar reasoning applies to the parameter ω, which captures the marginal external gain (due to backlash) from retaliation enjoyed by the terrorist organization[3]. A smaller ω translates into fewer attacks (or attacks of smaller magnitude) on member nations, which makes them less sensitive to the negative effects of retaliation. Finally, the impact of the parameter α on the possibility of full cooperation (and thus maximal coordinated retaliation effort) is also reasonable. As this parameter measures the sensitivity rate of the terrorist organization's common cost to preemptive actions, a decrease in α means that pre-emptive actions become less effective as deterrence instruments, generating weaker positive externalities enjoyed by free riders.

Full cooperation is also feasible when condition (ii) in Corollary 1 is satisfied. Let us assume that $\Delta_{\text{int}} \geq 0$, which can be shown to be true, with a little bit of algebra, when $\beta > 2\theta\omega$[4]. One scenario where condition (ii) may hold is $\gamma < 1$ and yet $a < 0$. In this case, it can be shown that there is a $\beta > 0$ such that $s^-_{\text{int}} \leq I \leq s^+_{\text{int}}$ (see proof in the Appendix A). The interpretation is similar to that of part (i) of Corollary 1; that is, if the positive spillover of retaliation on stand-alone nations is limited ($\gamma < 1$), full cooperation is possible when the marginal benefit of retaliation (β) is high enough in relation to the following: (a) factors that channel potential negative effects of retaliation on coalition members, namely, the marginal damage caused by terrorist attacks (θ) and backlash (ω), which increase terrorist attacks in member nations; and (b) the effectiveness rate of pre-emptive measures in making the terrorist activities costly (α).

Figure 2 below illustrates this scenario. We set $\alpha = \theta = 1$ and $\gamma = \omega = 0.5$, and let β vary from 0.5 to 2.732[5] in increments of 0.05. The graph shows how the maximum number of stable coalitions depends on β.

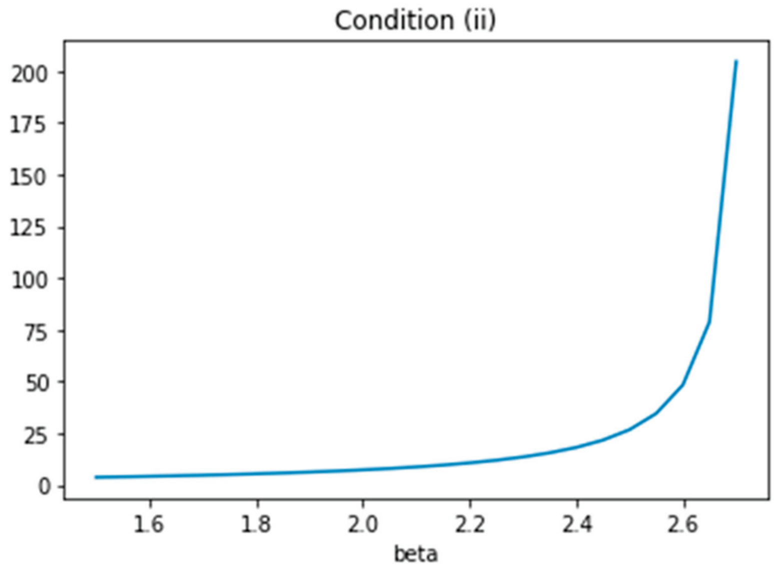

Figure 2. Maximum size of stable coalition as a function of β—condition (ii).

The maximum number of stable coalitions is an increasing function of β, and it is always possible to find a β such that the grand coalition is stable.

Another scenario under which condition (ii) of Corollary 1 may be satisfied is $\gamma \geq 1$, for it implies $a < 0$. However, in this case, the number of nations I cannot be too high. In fact, as s^+_{int} has a limit as β increases without bound (see Appendix A), it is possible for $I > s^+_{\text{int}}$, and then the grand coalition is not stable. In sum, even if the marginal benefit of retaliation becomes larger and larger compared with the other parameters, there is a limit to the size of stable coalitions. If the total number of nations is higher than that limit, full cooperation will not be possible. It is important to point out that this result is driven by the fact that the spillover effect of retaliation on stand-alone nations is relatively high in this case ($\gamma \geq 1$), making it harder to satisfy internal stability.

A possible combination of parameters in this scenario is shown in Figure 3 below. We again set $\alpha = \theta = 1$ and $\omega = 0.5$, but now $\gamma = 1.5$. β varies from 1.5 to 10 in increments of 0.1. The graph shows how the maximum number of stable coalitions depends on β.

Now that we are done discussing the stability of the grand coalition, we shift gears and focus on situations where only smaller coalitions or even no coalitions are stable.

Corollary 2: When $a < 0$ and $\Delta_{\text{int}} \geq 0$, a coalition of size $|S| = s < I$ is stable if $s^+_{ext} \leq s \leq s^+_{\text{int}} < I$.

Proof. See Appendix A. □

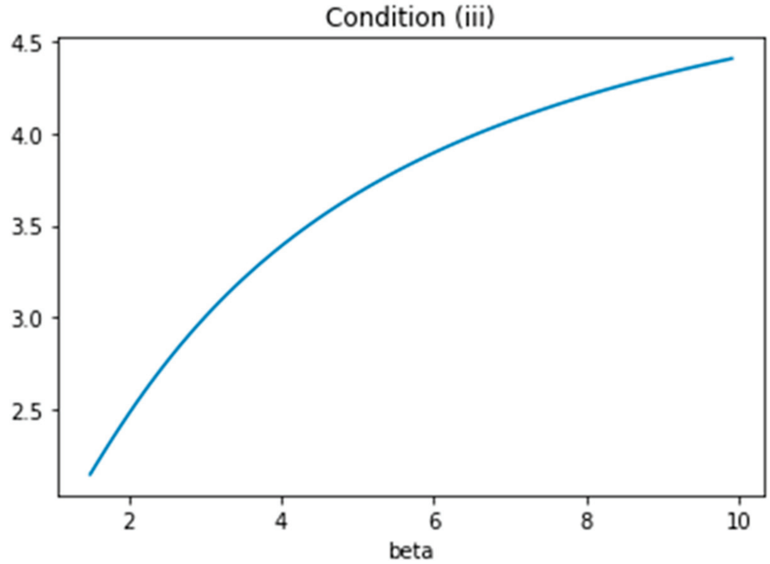

Figure 3. Maximum size of stable coalition as a function of β—condition (iii).

The importance of Corollary 2 is that it shows it is possible to obtain different degrees of cooperation between nations depending on how large the marginal benefit of retaliation β is in comparison with the potential negative effects of retaliation on coalition members, measured by θ and ω, and the effect of pre-emptive measures on the terrorist's common cost, measured by α[6]. As mentioned in our discussion about stability under $\gamma < 1$ and $a < 0$, larger values of β, all else the same, increase the maximum size of a stable coalition. We can always ensure full cooperation for large enough β, but, given β, only coalitions smaller than the grand coalition will be stable if the number of nations is such that $s^+_{\text{int}} < I$.

A similar reasoning applies to the case $\gamma \geq 1$, but now there is a limit to the maximum size of a stable coalition. It is still possible for the grand coalition to be stable, but only if the total number of nations is relatively small.

Our last corollary establishes conditions under which there are no stable coalitions with more than one nation. In this case, only the non-cooperative solution is viable.

Corollary 3: There is no stable coalition with size $|S| = s > 1$ if one of the following conditions is satisfied: (i) $a < 0$ and $\Delta_{\text{int}} < 0$; (ii) $a < 0$, $\Delta_{\text{int}} \geq 0$ and $s^+_{\text{int}} < 2$.

Proof. See Appendix A. □

Once again, stability hinges on the relationship between the benefit of retaliation parameter β, the "cost" of retaliation (from the perspective of coalition members) parameters θ and ω, and the effectiveness of pre-emption parameter α. When β is not large enough with respect to the other parameters, no coalition is stable. Figure 4 illustrates this scenario.

In Figure 4, the parameters were set at $\alpha = \theta = 1$, $\gamma = 1.5$, $\omega = 0.5$, and $\beta = 0.5$. Notice how internal stability is not satisfied for any s greater than approximately 1.7.

Table 1 collects all the results of this section.

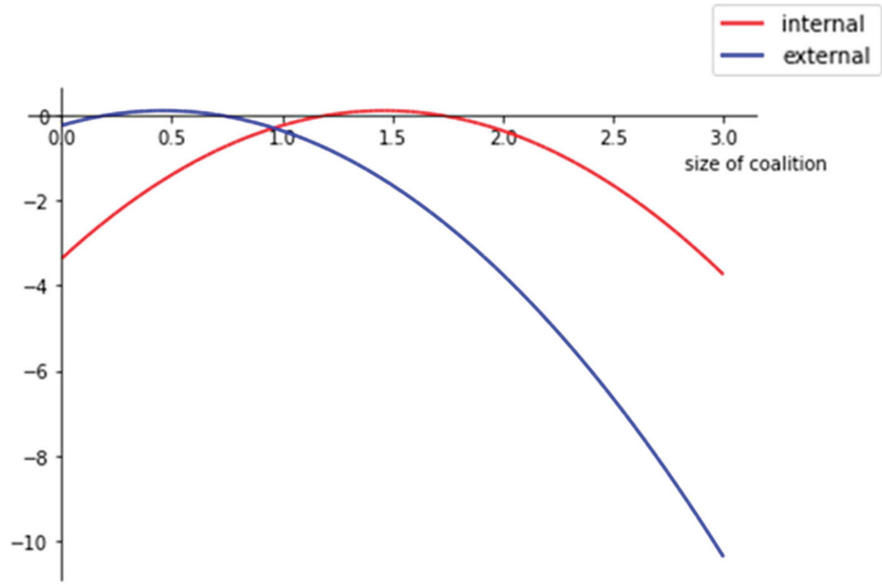

Figure 4. Only the grand coalition is stable—Corollary 3.

Table 1. Summary of stability conditions.

Condition	Internally Stable	Externally Stable	Stable
$a \geq 0$	All coalitions	Only the grand coalition (by default). No other coalition is externally stable because $s^+_{ext} < s^+_{\text{int}} < 1$.	Only the grand coalition
$a < 0,\ \Delta_{\text{int}} < 0$	No coalition	This case was not investigated.	No coalition
$a < 0,\ \Delta_{\text{int}} \geq 0,\ s^+_{\text{int}} < 2$	No coalition	Grand coalition (by default) and coalition of size $s < I$ if $s \geq s^+_{ext}$.	No coalition
$a < 0,\ \Delta_{\text{int}} \geq 0,\ s^+_{\text{int}} \geq 2$	Coalition of size $s^-_{\text{int}} \leq s \leq s^+_{\text{int}}$.	Grand coalition (by default) and coalition of size $s < I$ if $s \geq s^+_{ext}$.	Grand coalition, if $s^-_{\text{int}} \leq I \leq s^+_{\text{int}}$, or a coalition of size $s < I$ if $s^+_{ext} \leq s \leq s^+_{\text{int}}$.

If we take a closer look at the scenarios where stable coalitions are possible, we realize there are two possibilities: (1) only the grand coalition is stable (when $a \geq 0$); (2) either the grand coalition or a coalition of size $s < I$ is stable, but not both (when $a < 0$, $\Delta_{\text{int}} \geq 0$, and $s^+_{\text{int}} \geq 2$). If the stable coalition in this scenario is of size $s < I$, there are no stable coalitions of other sizes $s' < I$ (this follows from the fact that $s^+_{\text{int}} - s^+_{ext} = 1$).

To summarize, we have shown how coalition formation in the retaliation game depends on the intricate relationship between the overall and private marginal benefits of retaliation, the marginal damage caused by a terrorist attack, the backlash after retaliation, and the impact of pre-emptive measures on terrorist's costs. The key aspect of our findings is that, despite being members of the same category of proactive measures, retaliatory and preventive actions have differentiated effects on the incentives to join a coalition.

5. Conclusions

We build a simple model to capture key factors that influence potential strategic coalitional counterterror retaliatory effort by multiple nations that fight a common, strategic, terrorist organization. The key factors that we consider are as follows: (i) the sequential nature of strategic moves, with retaliation occurring at the last stage; (ii) the group-specific and internalized public benefits from retaliation enjoyed by coalition members; (iii) the external public benefit from retaliation enjoyed by stand-alone nations; (iv) the external backlash benefit produced by retaliation and enjoyed by the terrorist organization; and (v) the effective rate of pre-emptive counterterror measures in producing a cost to terrorist activities. Motivated by various observations of joint international retaliation triggered by terrorist attacks, we focus on retaliation by a potential counterterror coalition only. Stand-alone nations do not engage in retaliation.

Because retaliation and pre-emptive measures are members of the same family of proactive measures, retaliation becomes a viable and necessary additional weapon in the combat of terrorism when it proves to be a sufficiently different product if compared with pre-emptive measures. Retaliation is a desirable differentiated product in any of the various circumstances under which counterterror coalitions, including the grand coalition, emerge in equilibrium. We demonstrate that the grand coalition is stable depending on the factors that yield a positive net gain to any nation of being a member of the counterterror coalition relative to being a single free rider. In such circumstances, the group-specific marginal benefit from retaliation enjoyed by coalition members and the lower private marginal benefit from retaliation enjoyed by each stand-alone nation as a spillover are fundamentally important. For example, the subgame perfect equilibrium involves full cooperation whenever the group-specific marginal benefit from retaliation enjoyed by each coalition member is sufficiently large, while the external marginal benefit from retaliation enjoyed by a stand-alone nation is sufficiently small. The factors that may hinder the emergence of the grand coalition in equilibrium are backlash and the effectiveness rate of pre-emptive measures in making terrorist activities costly to the terrorist organization. The lower the effectiveness of pre-emptive measures, the more desirable retaliation becomes as a collective instrument to fight terror.

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Appendix A

Proof of Proposition 2. In order to check the internal stability of a coalition, we compare the payoff of a member m of the coalition S when it stays in the coalition:

$$\frac{s^2\beta^2}{2} + \frac{\theta^2}{2} - \theta\left(b + s^2\omega\beta - \alpha u^S\right) - \frac{1}{2}(s\alpha\theta)^2 \tag{A1}$$

to its payoff when it leaves:

$$\frac{\gamma}{2}(s-1)^2\beta^2 + \frac{\theta^2}{2} - \theta\left(b - \alpha u^{S\setminus\{m\}}\right) - \frac{1}{2}(\alpha\theta)^2 \tag{A2}$$

Therefore, a nation stays in the coalition if

$$\begin{aligned}
&\tfrac{s^2\beta^2}{2} + \tfrac{\theta^2}{2} - \theta\left(b + s^2\omega\beta - \alpha u^S\right) - \tfrac{1}{2}(s\alpha\theta)^2 \geq \tfrac{\gamma(s-1)^2\beta^2}{2} + \tfrac{\theta^2}{2} - \theta\left(b - \alpha u^{S-1}\right) - \tfrac{1}{2}(\alpha\theta)^2 \\
&\Rightarrow \beta^2\left[s^2 - \gamma(s-1)^2\right] - 2\theta\left(b + s^2\omega\beta - \alpha u^S\right) - s^2\alpha^2\theta^2 + 2\theta\left(b - \alpha u^{S-1}\right) + \alpha^2\theta^2 \geq 0 \\
&\Rightarrow \beta^2\left[s^2 - \gamma(s-1)^2\right] - 2\beta\theta\omega s^2 + 2\alpha\theta\left(u^S - u^{S-1}\right) - \alpha^2\theta^2\left(s^2 - 1\right) \geq 0
\end{aligned} \tag{A3}$$

Now, we turn to external stability. The payoff of a stand-alone nation is

$$\frac{\gamma}{2}s^2\beta^2 + \frac{\theta^2}{2} - \theta\left(b - \alpha u^S\right) - \frac{\alpha^2\theta^2}{2} \tag{A4}$$

and its payoff if it joins the coalition is

$$\frac{(s+1)^2\beta^2}{2} + \frac{\theta^2}{2} - \theta\left(b + (s+1)^2\omega\beta - \alpha u^{S+1}\right) - \frac{(s+1)^2\alpha^2\theta^2}{2} \tag{A5}$$

External stability requires

$$\frac{\gamma}{2}s^2\beta^2+\frac{\theta^2}{2}-\theta\left(b-\alpha u^S\right)-\frac{\alpha^2\theta^2}{2}\geq\frac{(s+1)^2\beta^2}{2}+\frac{\theta^2}{2}-\theta\Big(b+(s+1)^2\omega\beta-\alpha u^{S+1}\Big)-\frac{(s+1)^2\alpha^2\theta^2}{2}$$
$$\Rightarrow (s+1)^2\beta^2-\gamma s^2\beta^2-2\theta\Big(b+(s+1)^2\omega\beta-\alpha u^{S+1}\Big)-(s+1)^2\alpha^2\theta^2+2\theta\left(b-\alpha u^S\right)+\alpha^2\theta^2\leq 0 \quad \text{(A6)}$$
$$\Rightarrow \beta^2\Big[(s+1)^2-\gamma s^2\Big]-2\beta\theta\omega(s+1)^2+2\alpha\theta\left(u^{S+1}-u^S\right)-\alpha^2\theta^2\Big[(s+1)^2-1\Big]\leq 0$$

In order to evaluate the stability conditions, we need to find $u^S=\sum_{i=1}^{I}p_i^*$. We plug in the optimal values of the p_i's to obtain the following:

$$u^S=\sum_{i\notin S}p_i+\sum_{i\in S}p_i=(I-s)\alpha\theta+s(s\alpha\theta)=\alpha\theta\Big(I-s+s^2\Big) \quad \text{(A7)}$$

Given these expressions, we can write

$$u^S-u^{S-1}=\alpha\theta\left(I-s+s^2\right)-\alpha\theta\Big(I-(s-1)+(s-1)^2\Big)$$
$$=\alpha\theta\left(I-s+s^2-I+s-1-s^2+2s-1\right)=2(s-1)\alpha\theta$$
$$\text{and, similarly,} \quad \text{(A8)}$$
$$u^{S+1}-u^S=\alpha\theta\Big(I-(s+1)+(s+1)^2\Big)-\alpha\theta\left(I-s+s^2\right)$$
$$=\alpha\theta\left(I-s-1+s^2+2s+1-I+s-s^2\right)=2s\alpha\theta$$

Plugging the expressions above into the internal and external stability conditions, we obtain

$$\beta^2\Big[s^2-\gamma(s-1)^2\Big]-2\beta\theta\omega s^2+2\alpha\theta[2(s-1)\alpha\theta]-\alpha^2\theta^2\left(s^2-1\right)\geq 0$$
$$\Rightarrow \beta^2\left(s^2-\gamma s^2+2\gamma s-\gamma\right)-2\beta\theta\omega s^2+\alpha^2\theta^2\left(-s^2+4s-3\right)\geq 0 \quad \text{(A9)}$$
$$\Rightarrow \beta^2s^2-\beta^2\gamma s^2-2\beta\theta\omega s^2+2\beta^2\gamma s-\beta^2\gamma-\alpha^2\theta^2s^2+4\alpha^2\theta^2s-3\alpha^2\theta^2$$
$$\Rightarrow \left(\beta[\beta(1-\gamma)-2\theta\omega]-\alpha^2\theta^2\right)s^2+2\left(\beta^2\gamma+2\alpha^2\theta^2\right)s-\left(\beta^2\gamma+3\alpha^2\theta^2\right)\geq 0$$

and

$$\beta^2\Big[(s+1)^2-\gamma s^2\Big]-2\beta\theta\omega(s+1)^2+2\alpha\theta(2s\alpha\theta)-\alpha^2\theta^2\Big[(s+1)^2-1\Big]\leq 0$$
$$\Rightarrow \beta^2\left(s^2+2s+1-\gamma s^2\right)-2\beta\theta\omega\left(s^2+2s+1\right)+\alpha^2\theta^2\left(-s^2+2s\right)\leq 0$$
$$\Rightarrow \beta^2s^2+2\beta^2s+\beta^2-\beta^2\gamma s^2-2\beta\theta\omega s^2-4\beta\theta\omega s-2\beta\theta\omega-\alpha^2\theta^2s^2+2\alpha^2\theta^2s\leq 0, \quad \text{(A10)}$$
$$\Rightarrow \left(\beta^2-\beta^2\gamma-2\beta\theta\omega-\alpha^2\theta^2\right)s^2+\left(2\beta^2-4\beta\theta\omega+2\alpha^2\theta^2\right)s+\left(\beta^2-2\beta\theta\omega\right)\leq 0$$
$$\Rightarrow \left(\beta[\beta(1-\gamma)-2\theta\omega]-\alpha^2\theta^2\right)s^2+2\left[\beta(\beta-2\theta\omega)+\alpha^2\theta^2\right]s+\beta(\beta-2\theta\omega)\leq 0$$

respectively, which are expressions on the parameters only. □

Proof of Corollary 1. First, it is helpful to recall that $\alpha\in(0,1]$, $\beta>0$, $\gamma\in[0,2]$, $\theta>0$, and $\omega\in[0,1]$.

(i) Notice that, for $s\geq 1$,

$$as^2\geq 0$$
$$b_{\text{int}}s+c_{\text{int}}=2s\beta^2\gamma+4s\alpha^2\theta^2-\beta^2\gamma-3\alpha^2\theta^2 \quad \text{(A11)}$$
$$=(2s-1)\beta^2\gamma+(4s-3)\alpha^2\theta^2>0$$

which implies $\psi(s|\beta,\theta,\omega,\alpha)>0$. This means that internal stability is satisfied for all $s\geq 1$. As the grand coalition is externally stable by default, we conclude that it is stable.

(ii) If $a<0$, the quadratic function $\psi(s|\beta,\theta,\omega,\alpha)$ is concave. If its discriminant Δ_{int} is negative, then it has no real roots, and thus is everywhere below the x axis. In this case, no coalition is internally stable, including the grand coalition.

If $\Delta_{\text{int}}\geq 0$, we have, for $s\geq 1$:

$$as^2<0$$
$$b_{\text{int}}s+c_{\text{int}}=2s\beta^2\gamma+4s\alpha^2\theta^2-\beta^2\gamma-3\alpha^2\theta^2 \quad \text{(A12)}$$
$$=(2s-1)\beta^2\gamma+(4s-3)\alpha^2\theta^2>0$$

This means that the first term in $\psi(s|\beta,\theta,\omega,\alpha)$ is negative and quadratic in s, while the sum of the second and third terms is positive and linear in s. Thus, $\psi(s|\beta,\theta,\omega,\alpha)<0$ for large enough s. Given that $\psi(s|\beta,\theta,\omega,\alpha)$ is concave, internal stability will be satisfied for $s_{\text{int}}^-\leq s\leq s_{\text{int}}^+$ [7]. Therefore, the grand coalition will be stable if $s_{\text{int}}^-\leq I\leq s_{\text{int}}^+$. □

Proof of Corollary 2. We have already seen in the proof of Corollary 1 that (a) when the discriminant Δ_{int} is negative no coalition is internally stable, so $\Delta_{\text{int}}\geq 0$ is required; (b) under the conditions $a<0$ and $\Delta_{\text{int}}\geq 0$, internal stability holds for $s_{\text{int}}^-\leq s\leq s_{\text{int}}^+$.

External stability, on the other hand, is satisfied for large enough s, because

$$\begin{aligned}&\left(\beta[\beta(1-\gamma)-2\theta\omega]-\alpha^2\theta^2\right)s^2<0\\&2\left[\beta(\beta-2\theta\omega)+\alpha^2\theta^2\right]s+\beta(\beta-2\theta\omega)\\&=(2s+1)\beta(\beta-2\theta\omega)+\alpha^2\theta^2 s\gtreqless 0\end{aligned} \tag{A13}$$

implies that, even if the sum of the two last terms in $\varphi(s|\beta,\theta,\omega,\alpha)$ is positive, that is, $2\left[\beta(\beta-2\theta\omega)+\alpha^2\theta^2\right]s+\beta(\beta-2\theta\omega)>0$, $\varphi(s|\beta,\theta,\omega,\alpha)$ will be negative for large enough s. Moreover, as $\varphi(s|\beta,\theta,\omega,\alpha)$ is concave when $a<0$, external stability is satisfied for $s\le s^-_{ext}$ and $s\ge s^+_{ext}$.

Next, we need to show that $s^+_{ext}<s^+_{\text{int}}$, for then there will exist an s between s^+_{ext} and s^+_{int}. The algebra to obtain this result looks very complicated, but the result is incredibly simple. Using Sympy, a Python library for symbolic mathematics, we calculated the difference between s^+_{ext} and s^+_{int}, obtaining $s^+_{\text{int}}-s^+_{ext}=1$. Therefore, there exists a stable coalition s, and if $s^+_{\text{int}}<I$, this coalition is smaller than the grand coalition. □

Proof of Corollary 3. We already know from Corollary 1 that there are no internally stable coalitions when $a<0$ and $\Delta_{\text{int}}<0$. When $a<0$ and $\Delta_{\text{int}}\ge 0$, we know from the proof of Corollary 2 that a coalition of size s will be stable if $s^+_{ext}\le s\le s^+_{\text{int}}$. Thus, if $s^+_{\text{int}}<2$, there is no stable coalition larger than a singleton. □

Proof of implications of condition (ii) of Corollary 1.

Consider a as a function of β, i.e.,

$$\begin{aligned}a(\beta)&=\beta[\beta(1-\gamma)-2\theta\omega]-\alpha^2\theta^2\\&=(1-\gamma)\beta^2-2\theta\omega\beta-\alpha^2\theta^2\end{aligned} \tag{A14}$$

This quadratic function in β has the following roots:

$$\begin{aligned}\beta^+&=\frac{2\theta\omega+\sqrt{4\theta^2\omega^2+4(1-\gamma)\alpha^2\theta^2}}{2(1-\gamma)}=\frac{2\theta\omega+\sqrt{4\theta^2\left[\omega^2+(1-\gamma)\alpha^2\right]}}{2(1-\gamma)}\\&=\frac{\theta\left(\omega+\sqrt{\omega^2+(1-\gamma)\alpha^2}\right)}{(1-\gamma)}\\\beta^-&=\frac{\theta\left(\omega-\sqrt{\omega^2+(1-\gamma)\alpha^2}\right)}{(1-\gamma)}\end{aligned} \tag{A15}$$

Under the assumption that $\gamma<1$, the discriminant is positive, so there are two real roots. Moreover, $a(\beta)$ is convex and $a(\beta)<0$ when $\beta^-<\beta<\beta^+$.

When $\Delta_{\text{int}}\ge 0$ (see proof of Corollary 1), it is clear that $s^+_{\text{int}}=\left(-b_{\text{int}}-\sqrt{\Delta_{\text{int}}}\right)/2a\ge -b_{\text{int}}/2a$. Because $\lim_{\beta\to\beta^+} a=0$ (remember the constraint that $a<0$) and b_{int} increases with β, we can always find a β such that $s^-_{\text{int}}\le I\le s^+_{\text{int}}$.

If $\gamma\ge 1$, $a(\beta)$ is always negative. Notice that $s^+_{\text{int}}=\left(-b_{\text{int}}-\sqrt{\Delta_{\text{int}}}\right)/2a$ can be written as $s^+_{\text{int}}=p(\beta)/q(\beta)$, where

$$\begin{aligned}&p(\beta)=-2\left(\beta^2\gamma+2\alpha^2\theta^2\right)-\left[4\left(\beta^2\gamma+2\alpha^2\theta^2\right)^2+4\left(\beta[\beta(1-\gamma)-2\theta\omega]-\alpha^2\theta^2\right)\left(\beta^2\gamma+3\alpha^2\theta^2\right)\right]^{1/2}\\&\text{and}\\&q(\beta)=2\beta[\beta(1-\gamma)-2\theta\omega]-\alpha^2\theta^2\end{aligned} \tag{A16}$$

are polynomials of degree 2 in β. Therefore, s^+_{int} is a rational function in β whose limit is equal to the ratio of the leading coefficients of p and q. □

Notes

1. The parameter ω captures the differential benefit of attacking a nation that belongs to the coalition. If ω is negative, non-coalition members will have less incentive to free ride, making cooperation easier. By assuming a non-negative ω, we are making it harder for cooperation to take place, which increases the robustness of our coalition stability results.
2. We can show that the impact of a larger β on external stability is similar. Because the (positive) impact of retaliation on the payoff of coalition members is substantially higher than that on the payoff of stand-alone nations, it is impossible to prevent entry into a coalition smaller than the grand coalition.
3. In this paper, we do not consider the possibility that retaliation will lead to a reduction of terrorist attacks in nations that are members of the coalition.
4. The condition $\beta>2\theta\omega$ is not necessary for $\Delta_{\text{int}}\ge 0$. In fact, $\Delta_{\text{int}}>0$ in all the numerical simulations we carried out.
5. $\beta^+=2.73205\ldots$ in this case (see Appendix A for definition of β^+).
6. Notice first that, if $\gamma<1$ and $a\ge 0$, only the grand coalition is stable.
7. If $s^-_{\text{int}}=s^+_{\text{int}}=s^*_{\text{int}}$, $\psi(s|\beta,\theta,\omega,\alpha)$ has a unique real root ($\Delta_{\text{int}}=0$) and the grand coalition is stable only when $I=s^*_{\text{int}}$.

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The Hybridisation of Conflict: A Prospect Theoretic Analysis

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Abstract: Revisionist actors are increasingly operationalising a broad number of non-violent threats in their quest to change the status quo, popularly described as hybrid conflict. From a defensive point of view, this proliferation of threats compels nations to make difficult choices in terms of force posture and composition. We examine the choice process associated with this contemporary form of state competition by modelling the interactions between two actors, i.e., a defender and a challenger. As these choices are characterised by a high degree of uncertainty, we study the choice from the framework of prospect theory. This behavioural–economic perspective indicates that the defender will give a higher weight and a higher subjective value to conventional threats, inducing a higher vulnerability in the domain of hybrid deterrence. As future conflict will increasingly involve choice dilemmas, we must balance threats according to their probability of occurrence and their consequences. This article raises awareness regarding our cognitive biases when making these choices.

Keywords: hybrid threats; state competition; prospect theory; grand strategy

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1. Introduction

The re-emergence of long-term, strategic competition by so-called revisionist actors (i.e., states that are dissatisfied with the current distribution of power and that aim to reshape the world in their favour) such as China and Russia, constitutes one of the main contemporary security challenges [1]. Russia's aggressive actions in Ukraine in 2014 are generally seen as a tipping point, initiating an increase in the North Atlantic Treaty Organization's (NATO) budgets and putting great power competition back on top of the security agenda. The nature of this strategic competition is becoming increasingly complex. In addition to traditional conventional means, these revisionist actors are challenging the West by making use of a wide and varied range of threats across all operational domains. The competition in the informational (e.g., cyber and disinformation) and non-military domains, popularly known as 'hybrid threats', has created a grey zone where the traditional physical boundaries of conflict are eroded so that countries can be destabilised without a single soldier crossing the (physical) border [2–4]. These threats give rise to a number of challenges. While conventional conflict rarely takes place, hybrid threats occur continuously; they are more difficult to attribute to a perpetrator (who can always resort to the excuse of plausible deniability), and it is more difficult to assess the effects and the consequences associated with these types of threats.

From a defensive stance, the deterrence of this increasing number of threats gives rise to choice problems, as not only force posture but also force structure will have an impact on the national defence [5]. As power continues to diversify, political calculations must not only consider the classic trade-off between 'guns' and 'butter', as nations might have a limited number of resources available or other non-military spending priorities, but must also account for complements and trade-offs between 'guns' and 'guns' [5]. Hence, studying this form of state competition requires a shift in thinking.

As a more differentiated portfolio of options makes trade-offs more difficult, we venture in the strategic question: "How do we decide on allocating available budgetary means across different domains when striving to deter the wide range of threats we are confronted with?"

We study this broad research question by means of a traditional game theoretical deterrence model, resembling the interactions between a defender and a challenger that wishes to revise the status quo. Linked to today's international environment, the challenger represents a revisionist state such as Russia, China or North Korea [4,6,7]. The defender represents a liberal democracy, being a single individual state or an alliance of states such as NATO. The article is written from the point of view of the defender, which needs to decide upon the distribution of resources across domains. This strategic question involves a decision-making dilemma, as this choice could have large (political) consequences if deterrence in one of the domains should fail. We are therefore brought into the realm of prospect theory, standing as the leading framework for how people make choices under risk [8,9]. The subsequent integrating of elements of prospect theory into our game theoretic model therefore constitutes a good methodology to reveal how the defender will prioritise defensive capabilities when facing a wide series of threats.

Originating in the field of economics, a vast literature applies prospect theoretic findings to study several forms of conflict (see Section 3.2), serving as an alternative to the expected-utility theory. We are, however, the first to bridge the literature on hybrid conflict and the behavioural–economic literature on prospect theory. Moreover, we contribute to the growing literature on cross-domain deterrence (CDD), as our model incorporates the interactions between different domains (conventional and hybrid). This field of research focuses on the deterrence of asymmetric [10] and hybrid [11,12] threats, the use of threats in one domain to prevent actions in other domains (such as cyberspace) and the increasingly intertwined interactions between military threats and the growing portfolio of non-military threats in today's competitive environment [13].

The remainder of the article is structured as follows. Section 2 summarises the main characteristics of hybrid conflict. Section 3 recapitulates the main findings of prospect theory. Section 4 offers a prospect theoretic perspective on hybrid threats. Section 5 offers a preliminary (quantitative) discussion by analysing the U.S. budget composition. Section 6 summarises our findings and provides scope for follow-up research.

2. The Contemporary Nature of State Competition

Notions such as 'hybrid threats', 'non-linear warfare' [14] and 'grey zone conflict' [2] have received growing attention in recent years, especially following the events in Crimea in 2014. This article does not enter the semantic discussion of whether the changing way of state competition, rendered possible by an increase in technological progress (e.g., the 'internet of things' and the evolutions in artificial intelligence) and interconnectedness, can actually be described by a single definition. We use this umbrella term to cover a range of threats, because we believe they have some common characteristics that require further analysis in order to gain more insights into the dynamics of contemporary state competition. We refer to the terminology of hybrid threats as it has been adopted by NATO and the EU in their official strategic documents [15,16].

2.1. Characteristics of Hybrid Threats

First, hybrid threats refer to the combined and simultaneous use of a wide range of ambiguous, and often non-violent, means [17–19]. The most known and debated examples of hybrid threats are the spreading of disinformation (e.g., the Chinese and Russian spreading of disinformation during the COVID-19 health crisis), the foreign interference in elections, the use of cyber-attacks (e.g., the Solarwinds or Hafnium cyber-attacks targeting thousands of U.S. private firms), the targeting of critical infrastructure (e.g., the drone strikes by Iran's Houthi allies on Saudi Arabian refineries in 2019), the use of Special Forces to wage unconventional warfare (referring to the use of operations conducted by special forces to advise and assist foreign resistance movements to conduct a resistance warfare against their host nation or occupying force [20]), the support of extreme political parties of one's opponent with the aim of increasing political polarisation (e.g., the Russian support to EU extreme right political parties) and the use of a wide range of economic instruments to exploit interdependencies (e.g., manipulating energy prices, economic aid, the use of economic sanctions such as the Russian embargoes of Ukrainian goods during the 2014 war). The seminal work 'unrestricted warfare' by the Chinese strategists Liang and Xiangsui contains a further extensive list of tools that can be used to destabilize one's adversary [21]. The effects and consequences stemming from these tools vary widely, which immediately brings us to the second characteristic associated with these types of threats.

Revisionist actors are resorting to the aforementioned means with the aim of staying below the threshold that the attacker believes would trigger an armed response. This blurs the traditional dichotomy of peace and war and is often described as fighting in the grey zone [22] or liminal warfare [8]. These threats hence enable the revisionist to inflict losses while evading a powerful international response [19,22]. This relative reluctance of Western states to respond fiercely to hybrid threats remains partly a puzzle and is often explained by referring to the difficulty of attributing the attacks to a perpetrator with sufficient certainty [2,8]. By incorporating behavioural–economic insights, our modelling provides another innovative explanation why hybrid adversaries proceed carrying out these types of intrusions, considering Western deterrent signals as incredible.

Third, the intellectual debate on hybrid conflict requires a shift from the traditional goals of conflict. Hybrid conflict is non-linear in nature and does not involve the conquest or physical control of the opponent's territory. These threats aim to create distrust towards politicians, to polarise the public debate and to weaken the sentiment of unity. This could in the longer-term lead to a gradual change of the status quo and the balance of power [23]. Our model expounds how this deterioration in the status quo can occur.

Hybrid threats clearly constitute an attractive complement to conventional capabilities, as they have a high cost-benefit efficiency. Furthermore, the increased interconnectivity and the advances in technology continue to increase the reach, efficiency and the potential to achieve substantial effects. While the aforementioned explanatory factors for resorting to these types of threats are important, they are not the subject of this article. We argue that they are also capable of exploiting the defender's cognitive pitfalls that are associated with the psychological game of deterrence [24]. Broadening the range of threats, both conventional and hybrid, forces the defender to make allocative choices, distributing available budgetary means across defence capabilities. As this allocative choice process constitutes a decision-making dilemma, we study this question from a prospect theoretic perspective, standing as the leading framework for how people make choices under risk [9,10]. We discuss the main elements of prospect theory in the following section.

3. Prospect Theory and Decision Making under Risk

Section 3.1. highlights the main findings stemming from the empirical research on prospect theory, focusing on the seminal work of Kahneman and Tversky [25,26]. Section 3.2. briefly presents the use of prospect theory within the field of international relations.

3.1. An Introduction to Prospect Theory

Prospect theory emerged as an alternative for expected utility theory when evaluating different hypothetical choices under risk, following the seminal work of Kahneman and Tversky [25,27]. Prospect theory describes a choice process, in which available options are edited in a first phase. During the subsequent evaluation phase, the option with the highest weighted value 'V' is chosen. This value is expressed as follows and depends on two distinctive functions:

$$V = \sum_{i=1}^{n} \pi(p_i) \cdot v(x_i) \tag{1}$$

$\pi(p_i)$ represents the weighting function (see Figure 1) and measures the impact of the probability of an event on the desirability of prospects. This implies that possible outcomes are weighted by a subjective decision weight $\pi(p_i)$ instead of their objective probability (p_i). Hence, this function does not necessarily represent the objective likelihood of events but rather introduces subjective probabilities. The decision weights can consequently be influenced by other factors such as ambiguity, uncertainty and risk. The weighting function bears several properties.

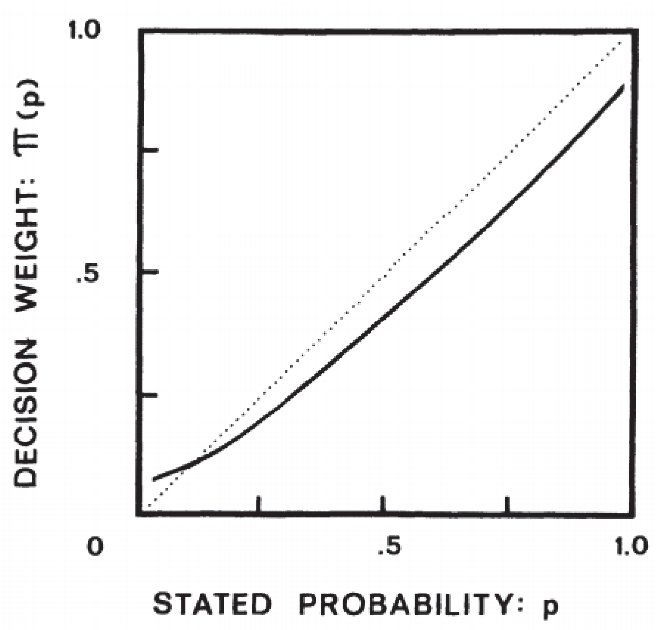

Figure 1. A hypothetical weighting function. Note: p represents the objective probability, π(p) reflects the decision weight associated with an event. Source: Kahneman and Tversky (1979).

First, the function is not well-behaved around the endpoints, reflecting the observation that individuals face difficulties when evaluating extreme probabilities. The difference between high probabilities and certainty is therefore often neglected or exaggerated. Extremely likely but uncertain outcomes are consequently often treated as being certain, also called the pseudo certainty effect. Second, sharp increases can be observed in regions with low and high probabilities. This implies that people in general tend to overweight low probability events ($\pi(p_i) > p_i$) while underweighting high probability events ($\pi(p_i) < p_i$). Third, Figure 2 shows that probabilities are lower than unity over a large range of the weighting function, leading to the principle of subadditivity, or $\pi(p_i) + \pi(1 - p_i) < 1$, implicating that decision weights do not sum to 1 when comparing two options.

$v(x_i)$ represents the value function (see Figure 2) and assigns a value to each potential outcome, reflecting the subjective value of that outcome. This function bears some distinct characteristics. First, values are measured in terms of gains and losses that stem from deviations from a reference point. In this way, people are more sensitive to changes in wealth, rather than final asset positions. The reference point often depicts the status quo but can also be a measure of the aspiration level [28]. Second, the value function is concave for gains and convex for losses, reflecting risk averse behaviour when operating in a domain of gains and a risk acceptant behaviour with respect to losses. This implies that individuals will prefer the certain outcome instead of a gamble when operating in a gains frame, even when the gamble has a higher expected utility. Individuals operating in a loss frame will on the contrary prefer a gamble in an effort to avoid certain losses, even if the expected loss is larger. Moreover, the shape of the value function reflects the characteristic of diminishing sensitivity, indicating a decreasing marginal value of both gains and losses in terms of their magnitude. Third, the value function is steeper in the domain of losses, reflecting the characteristic of loss aversion. The pain of loss is greater than the pleasure of gaining. Recent experiments in the field of neuroscience [29] indeed show that distinct neural circuits and activation patters are used when encoding and assessing gains or losses hereby confirming the asymmetric value function [30].

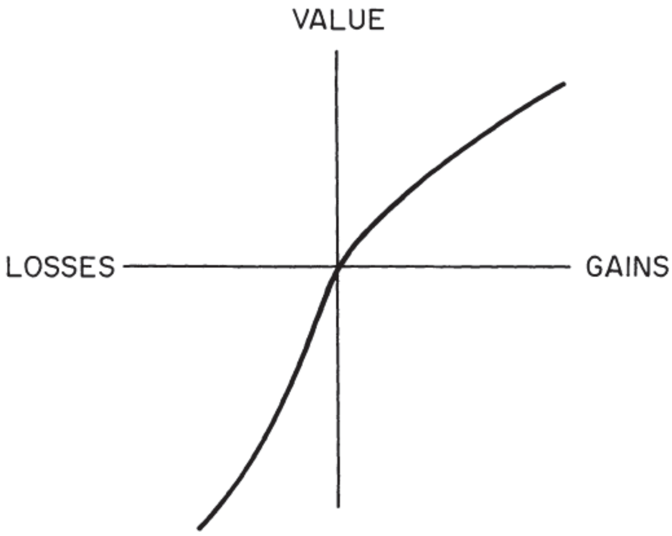

Figure 2. A hypothetical weighting function. Source: Tversky and Kahneman (1991).

3.2. The Application of Prospect Theory within the Field of International Relations

Although most of the initial research on prospect theory focuses on choices between monetary outcomes, the theory has later been applied to a wide range of decision-making problems, including the field of international relations and conflict. Next to case studies, explaining specific policy choices [31–35], the theory has been incorporated into theoretic modelling to study strategic interaction. The theory supports the revision of traditional outcomes associated with deterrence frameworks such as the chicken game [35,36], the study of great power deterrence and power cycles [37], the sequential analysis of a traditional deterrence game [38] and bargaining and ultimatum games [39]. While the list of applications of prospect theory in the field of international relations continues to grow, we are, to the best of our knowledge, the first to use the framework of prospect theory to study hybrid threats and how states respond to them.

4. Studying the Contemporary Conflict Environment from a Prospect Theoretic Perspective

We apply prospect theory to contemporary state competition, in which a defender faces a broad range of threats. Section 4.1 conceptualises this threat environment by means of a model in which the defender must deal with conventional and hybrid threats. This visualisation enables us to assess threats in terms of alternative courses of action as well as the associated outcomes and the probabilities they will occur. As these elements form the basis of 'framing' a choice problem in prospect theory [40], we heavily draw on this theory to analyse the decisions. Where Section 4.2 puts emphasis on the findings from the weighting function, Section 4.3 discusses the value function. Both functions provide corroborative insights regarding the way decision makers cope with a wide range of threats that differ strongly according to their probability of occurrence and impact. Section 4.4 discusses the challenges associated with deterring hybrid threats, by applying our findings from prospect theory.

4.1. Modelling Contemporary State Competition: The Joint Analysis of Conventional and Hybrid Threats

We present the multi-domain strategic competition between two players in Figure 3. The model is based on Balcaen et al. [41] and presents a defender (player 1) facing a challenger (player 2) that wishes to revise the status quo. We assume a unitary decision maker, in line with previous research on prospect theory within the international relations literature [9,35,38]. Further extensions such as the impact of group polarisation [42] on the decision-making process would add a further order of complexity to the model [43] and are beyond the scope of this article. It is similar to a traditional deterrence model [38,44], but allows for a variety of instruments to challenge the status quo, i.e., a combination of hybrid and conventional threats. We assume a number of simplifications. First, we differentiate between two broad categories of threats: hybrid and conventional ones, each depicted by a single branch tree. Both categories could be further expanded. The conventional domain for example could be further subdivided, making the distinction between naval, land and air forces. The hybrid branch tree could be further expanded by making a distinction between the different types of hybrid threats, e.g., disinformation campaigns, cyber-attacks, supporting proxy-forces or destabilising an opponent by means of economic coercion. We resort to this simplification because we argue that each category bears a number of similar characteristics that form the basis of the prospect theoretic analysis. Second, the deterrence game is limited to two stages and does not include retaliatory actions of the defender. We will incorporate this possibility of retaliation in Section 4.4.

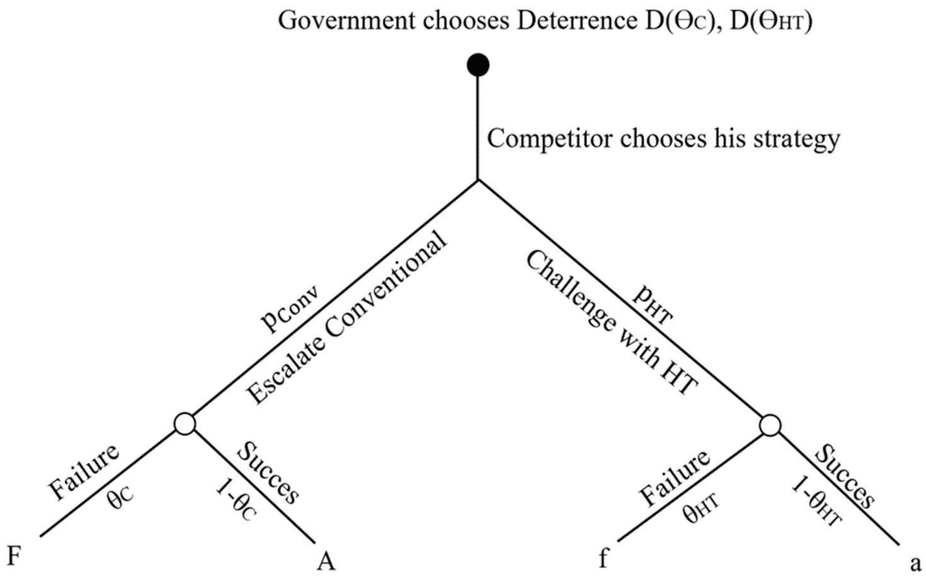

Figure 3. Interactions between a defender and a challenger (Extensive form representation). Source: simplified representation of the model presented in Balcaen et al. (2021).

The interactions between the defender and the challenger are as follows. The defender moves first and decides on the level of deterrence, making a choice between conventional deterrence $D(\theta_C)$ and capabilities that aim to deter hybrid threats $D(\theta_{HT})$. The latter can be represented as a form of *deterrence by denial* by investing in intelligence services, cyber specialists and the detection of disinformation. We specifically assume this strategy of deterrence by denial when analysing the deterrence of hybrid threats following the nature of these threats as discussed in Section 2.1, i.e., they are designed to inflict harm without justifying or provoking an armed response (i.e., punishment). We further venture in the particular discussion of deterring hybrid threats by means of a strategy of deterrence by punishment in Section 4.4. The levels of $D(\theta_{HT})$ and $D(\theta_C)$ in turn determine the challenger's perceived probabilities of failure (θ_C and θ_{HT}). These deterrence costs augment at an increasing rate in function of the failure probability. The challenger then moves and decides how to defy his opponent, with probabilities p_{Conv} (demonstrating a conventional attack) or p_{HT} (representing the use of a hybrid threat). This probability function is assumed to be continuous with $\partial p_i/\partial \theta_j > 0$, i.e., target transference implies that efforts to counter a certain type of threat increases the probability that the challenger will revert to a different type of threat to challenge the defender. The model has four potential outcomes, i.e., a failed/successful conventional attack or a failed/successful hybrid attack. The defender strives to minimise his costs, which are composed of the foregone deterrence costs $D(\theta_C)$ and $D(\theta_{HT})$ and the costs incurred as a result of an attack. Despite the fact that hybrid attacks also have the potential to inflict severe damages (e.g., by attacking vital infrastructure, shutting down the opponent's national economy), the hybrid actor generally strives to remain under the threshold that would provoke an armed response by only inflicting limited losses. If the hybrid attack fails, the defender incurs a small cost of 'f' whereas a successful hybrid entails a cost of 'a'. Conventional conflict, on the other hand, generally results in significant human, economic and material losses. Losing the conventional attack entails a large cost 'A', winning a cost 'F', with F < A. The final ordering of the potential costs for the defender are A > F > a > f.

The focus of our analysis is not on the absolute outcomes of hybrid or conventional offensives. Instead, we focus on the choice dilemma stemming from the strong contradiction between the probabilities and outcomes that characterise these two strategies. This discrepancy between the 'probabilities' and 'impacts' associated with conventional and hybrid threats is illustrated in Figure 4. Whereas the occurrence of large-scale conventional wars between two major states constitutes a HILP event (High Impact, Low Probability), hybrid attacks occur with a high probability but entail smaller effects.

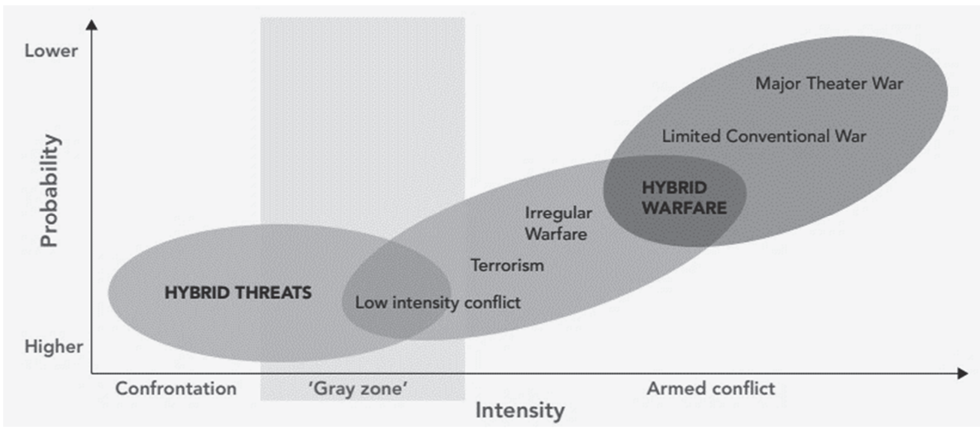

Figure 4. Probability-intensity relations across the continuum of conflict. Source: Monaghan (2019).

The defender will try to minimise his expected costs by choosing the level of deterrence $D(\theta_C)$ and $D(\theta_{HT})$. According to expected utility theory, assuming rational decision-making, the defender will balance the outcomes that are obtained by multiplying the probabilities of the different scenarios with the associated costs (the impact). However, according to prospect theory, heuristics and biases will influence the choices of the decision maker, leading to a violation of expected utility theory. The following sections incorporate the findings from prospect theory to the choice process of the defender, demonstrating the difficulty to assess the diverging prospects of conventional and hybrid conflict.

4.2. Insights from the Weighting Function

Incorporating prospect theory, the defender replaces the probabilities p_{Conv} and p_{HT} by subjective decision weights: $\pi(p_{Conv})$ and $\pi(p_{HT})$. These decision weights measure the impact of events on the desirability of prospects rather than the perceived likelihood that these events will occur [25]. This has some important implications for the weighting function.

First, as the function is assumed not to be well behaved around the endpoints, decision makers face difficulties when evaluating and responding to events that are highly likely or very unlikely. Hence, being faced with a series of threats at the extremes of the continuum of conflict complicates the decision-making process and the defining of priorities.

Second, the properties of the weighting function partially explain how we will prioritise the threats we are facing, based on the probability of occurrence. More precisely, the defender will instinctively tend to overweight low probability threats such as large-scale conventional conflict while underweighting high probability events such as the occurrence of cyber-attacks or the distribution of disinformation, or: $\pi(p_{Conv}) > p_{Conv}$ and $\pi(p_{HT}) < p_{HT}$. The overweighting of low-probability conventional conflict is further reinforced by the availability heuristic. Examples and consequences stemming from conventional conflict are widely available and come easily to mind, e.g., the images of wounded people, death or the destruction of infrastructure. They are consequently perceived as more likely than they truly are [8]. Moreover, the challenger can magnify this availability heuristic as he continues to organise nuclear tests and/or large-scale conventional exercises, by bringing its troops in a higher state of readiness and by regularly probing borders (e.g., by means of reconnaissance flights or movements of submarines). Russia, for example, gathered over 100.000 troops along the border of Ukraine and in Crimea in April 2021, signalling Putin's readiness to commit aggressive actions [45]. This signalling game further increases the subjective decision weight $\pi(p_C)$.

Third, social experiments show that the nonlinearity of the weighting function leads to a different evaluation of the complete elimination of risk as compared to the reduction of risk [25]. More specifically, individuals are willing to pay more to reduce the low probability of an event to '0' rather than obtaining the same reduction when the probability of occurring is higher. Specifically applied to our choice problem, this means that we will be more inclined to devote a higher budget to eliminate specific threats with a low probability (e.g., conventional war), while the similar reduction of threats with a higher probability (e.g., hybrid threats) is characterised by a lower 'willingness to pay'.

4.3. The Use of Hybrid Threats in Contemporary State Competition: Insights from the Value Function

In the first place, the framework of prospect theory explains why the challenger is still resorting to 'risky' actions (since the waging of hybrid attacks still entails the possibility of provoking a response that might inflict losses on behalf of the perpetrator), despite observing the deterrent measures. The challenger will defy the defender by using hybrid threats with a probability of p_{HT} since he is dissatisfied with the current status quo and perceives himself as being in a domain of losses. This motivates the challenger to risk defection as long as there is a chance that these actions will improve its situation (i.e., the benefits n_{HT} the challenger obtains). In our example, the defender can be seen as a state being satisfied with the status quo while the challenger represents a revisionist state (such as Russia, China, Iran and North Korea) that strives to change the balance of power in his favour [1,23]. Following prospect theory, deterrence becomes more difficult when potential adversaries operate in the domain of losses [35,38], as they are more willing to accept risk and to pursue confrontation. Applied to our model, this implies that higher levels of $D(\theta_C)$ and $D(\theta_{HT})$ are needed, if the defender wishes to deter the challenger across all domains. This comes at a high cost.

We now discuss how the defender assesses the different potential outcomes of the deterrence game (cfr. Figure 3). From a prospect theoretic perspective, the defender does not evaluate each outcome (A, F, a, f) as a net asset position. Instead, he assigns a subjective value to each potential outcome, i.e., the magnitude of change in relation to the asset position that serves as a reference point (in this case the status quo) [25]. We introduce the following mathematical representation of this subjective value [27,39] as it includes the variables that affect the defender's assessment of the threat environment, i.e., its degree of loss aversion (α), the degree of risk propensity (β) and the deviation from the reference point as the attack occurs (Δ). Equation (2) describes how this assessment differs upon whether the defender perceives himself as being in a domain of gains or losses, illustrating the concave (domain of gains) and the convex (domain of losses) area of the value function (cfr. Section 3.1).

$$V_i(\Delta)\begin{cases} \Delta^{\beta} & \text{for } \Delta \geq 0 \\ -\alpha(-\Delta)^{\beta} & \text{for } \Delta \leq 0 \end{cases} \quad (2)$$

Succumbing to a conventional attack involves a large negative deviation ($\Delta << 0$) from the reference point. Following Equation (2), the subjective outcome 'A' is even further reinforced (exponentially) by the factors 'α' and 'β'. Consequently, the defender experiences a great (subjective) disutility of loss when being confronted with the consequences associated with conventional conflict. This might incite the defender to reckless actions, driven by loss aversion. A challenger that resorts to hybrid threats specifically aims to avoid these reckless responses. By pursuing actions that remain below the threshold that would trigger an armed response, he refrains from provoking a substantial negative deviation ($\Delta \approx 0$) from the reference point. In doing so, he strives to avoid being confronted with a reckless defender that attempts to recover its suffered losses.

While the effects 'a' of a single hybrid attack may be small (e.g., one single piece of disinformation), the cumulative value of the losses stemming from a high number of attacks may be substantial (e.g., the long-term effects of a well-coordinated disinformation campaign such as the Russian campaign during the 2016 U.S. elections). The feeling of loss therefore depends on the framing of the reference point: do we compare our gains or losses with respect to an initial asset position prior to a series of events (i.e., a certain number of periods ago), or do we compare our situation prior to each new individual event (i.e., the period t-1)? Comparing the status quo with a reference point in the past might reveal a greater than expected (perceived) loss. Therefore, the defender's choice of reference point and prior experiences with hybrid intrusions might have an impact on the way he evaluates the outcome of a hybrid attack. There may furthermore be a difference in evaluation between the different threats. In the U.S., where the Russian electoral interference in 2016 caused a lot of turmoil, might, for example, give a higher subjective value in future similar attempts to interfere, assigning a higher value to this particular threat.

As we depart from the model of Balcaen et al. [41], we examine whether the incorporating of prospect theory leads to diverging outcomes. Overall, we find the outcomes of the original model to be strengthened. Following Equation (2), we expect the defender to assign a higher subjective value to the outcomes of conventional conflict, whereas this holds less for outcomes associated with hybrid attacks. This further magnifies the overweighting of conventional conflict we discussed in Section 4.2, encouraging the defender to maintain (or even improve) high levels of conventional deterrence. This finding has a number of implications. First, organising a high level of conventional and nuclear deterrence is costly, absorbing large budgetary resources. As stated by Kilcullen [7] (p. 140), when referring to Russia's strategy towards the West:

> They (strategic nuclear weapons), not incidentally, served as shiny objects to distract Western intelligence analysts in an area where Western countries were then obligated to continue spending money, soaking up attention and resources even as Russia's true transformation took place in the realms of asymmetric and conventional warfare.

Furthermore, this paradoxically reduces the probability that the challenger will resort to conventional war, as the failure rate of conventional conflict becomes high. As stated by the Chinese strategists Liang and Xiansui [21], the U.S. has created a trap for itself by its dominance in the conventional domain. Confronting the U.S. in a conventional conflict would be committing suicide, leading their adversaries to resort to the use of asymmetric threats. Applied to our model, the challenger will rather increase its use of hybrid threats to challenge the opponent, as the latter has a higher likelihood of success ($1 - \theta_{HT}$). This high frequency of attacks poses numerous challenges to the defender, the most important of which is the question: "Can these attacks be deterred?" We discuss this challenge in the following section.

4.4. The Credibility of Cross-Domain Deterrence by Punishment

As CDD essentially deals with the use of threats in one domain to deter an opponent from taking actions in another domain [13], we wonder whether high levels of conventional deterrence (cfr. Sections 4.2 and 4.3) could also serve to deter hybrid threats by means of 'deterrence by punishment'. We do so by extending the game with an additional round, giving the defender the possibility to respond as he encounters a hybrid attack. We examine the prospects of these response strategies by replacing the outcome 'successful hybrid attack' in Figure 3 with a new decision node. The defender could choose to simply accept the consequences of this attack ('no retaliation') or decide to 'retaliate'. This choice process is presented in Figure 5. Besides the sure losses of giving in (outcome 'a'), we include the potential outcomes of the retaliatory action and the associated payoffs for the defender.

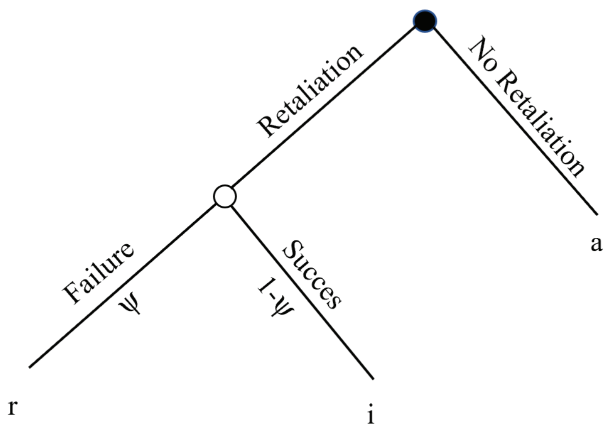

Figure 5. Retaliate or not? Choosing between two negative prospects. Source: author's own analysis.

There are two possible outcomes associated with the choice of retaliation. On the one hand, the retaliatory action could fail (taking the launching of an air raid as an example of retaliation, failing corresponds to fighter jets being intercepted or shot down), or the conflict could deteriorate even more as the challenger responds by launching counterattacks. Following Schelling [46], each act of escalation carries a degree of risk, i.e., the chance that a military action could lead to an unbearable catastrophe. This leaves the defender with the negative payoff 'r'. On the other hand, the defender's retaliatory attack could succeed, leading to gains 'i'. These gains could be interpreted as the establishment of a reputation of toughness [47], deterring future attempts to interfere in a defender's domestic country by means of hybrid threats. The defender's potential outcomes of this subgame are ordered as follows: i > 0 > a > r.

This represents a decision-making situation under risk, where the defender needs to make a choice between two negative prospects: (1) he does not retaliate and accepts the certain loss 'a' stemming from the hybrid attack or (2) he decides to retaliate and takes a gamble. He now has a $(1 - \psi)$ probability to improve its situation with the outcome 'i' and a (ψ) probability of losing even more 'r'. In an expected utility framework, the defender would retaliate if the following holds (in terms of expected utility):

$$\psi \cdot r + (1 - \psi) \cdot i \ > \ a \tag{3}$$

According to prospect theory, the probability of pursuing the risky option of retaliating will depend upon the defender's degree of risk propensity. The latter is, in turn, strongly influenced by the extent to which he perceives himself in a frame of loss (Equation (2)). When states perceive themselves in a domain of loss, loss aversion might lead them to become risk acceptant. This could result in risking open conflict in an attempt to recover the suffered losses and to restore the old status quo [36,48]. However, there appears to be disagreement in the literature regarding the magnitude of decline required before an actor perceives himself in a frame of losses, leading to risk-taking behaviour. Whereas a number of authors state that only substantial losses or serious deterioration of the status quo push an actor in the loss frame [37,49], others argue that limited losses already suffice to incite states to defect [29,34,36]. Hence, the choice for 'retaliation' or 'giving in' will depend upon the defender's evaluation of the outcome, i.e., V(a). Similarly, Berejikian [35] reasons that deterrent threats over territorial disputes are not always carried out, especially when the object of dispute has limited strategic value. Under these conditions, losing this territory will not provoke retaliation as the actor that loses the territory remains in a gains frame. Following the discussion in Section 4.3, we argue that the limited losses of hybrid attacks do not suffice to incite the defender to become so risk-acceptant that he is willing to pursue retaliatory actions that could escalate and lead to catastrophic outcomes. Consequently, the defender rather evaluates the situation as follows:

$$V(\psi \cdot r + (1 - \psi) \cdot i) < 0 < V(a) \quad (4)$$

Applied to our model, the defender will therefore remain risk averse and will prefer the certain benefits from continuous cooperation to the risks associated with the scenario of retaliation which might produce even larger losses 'r'. This consequently undermines the credible communication of deterrent threats. The failure of hybrid deterrence can be easily illustrated by real world examples. The authoritarian interference tracker [50] lists a long series of hybrid attacks that occurred since 2000, making a distinction between information manipulation, cyber operations (For example the recent 'Solarwinds' and 'Hafnium' cyber-attacks that were able to target thousands of customers and public or private firms), malign finance, civil society subversion, and economic coercion. Responses to this growing list of foreign state intrusions remain limited to economic sanctions or the expulsion of diplomats at best. NATO intended to boost its deterrence posture by claiming that article 5 can be provoked in the event of a cyber-attack [51]. Despite numerous intrusions, this has not yet occurred [52].

5. Discussion

As noted in the introduction, two broad allocative decisions (that affect the allocation of means across domains) should be taken into consideration: (1) security budgets could be increased, providing additional means to invest in complementary hybrid deterrence (i.e., implying a shift from 'butter to guns'); or (2) decision makers could, given a fixed or limited budget, decide to change the defence structure (i.e., substituting 'guns by guns'. A better terminology in the framework of our model would even be to speak in terms of the allocation between 'shields', as we are looking to defend ourselves against a wide array of distinct threats). The latter trade-off implies making priorities between domains. Insights from our modelling, including the perspective of the weighting function (Section 4.2) and the value function (Section 4.3), both indicate that a defender will value conventional deterrence more. Confirming these hypotheses empirically proves, however, to be a daunting task, as there is no (declassified) granular panel data (e.g., for all NATO countries) that provides a clear overview of the break-up of military expenditures across domains.

We therefore explored this allocative question by looking at data published by one specific country, i.e., the U.S. yearly DoD request [53]. This yearly report contains an overview of the major capital expenditures across a series of categories. These categories reflect the traditional conventional domains and certain 'new' domains such as cyber. The analysis is worthwhile, as the U.S. is the country with the highest defence expenditures [54]. We do, however, readily admit that our analysis is coarse for (at least) three reasons. First, the U.S. does not merely assume the role of a defender, but also pursues other strategic objectives. Second, we are looking at input metrics, i.e., budgetary resources devoted to security. Approaching this issue through output metrics poses even greater challenges in terms of data (it is moreover difficult to assess the 'output' of capabilities that aim to deter hybrid threats). Third, deterring hybrid threats is not a sole task for the military. In recent years, NATO and its allies have made significant efforts to provide responses to hybrid threats by establishing specialised institutions such as national cyber centres, by developing a whole of government approach (In which several agencies and ministries within a nation-state work together to counter e.g., hybrid threats [55] or by contributing to multinational centres such as the Hybrid Centre of Excellence [56]. Unfortunately, budgetary data associated with these efforts are unavailable. Hence, looking at military expenditures data only provides a partial part of the picture.

Figure 6 provides an overview of these major categorical capital expenditures over the period 2015–2021. The data shows that the conventional domains have certainly not been neglected or substituted by other domains in recent years. Both the land, air and naval domains have seen, despite the small decline in the request of 2021, a continuous increase of capital expenditures over the period 2014–2020. The maintaining of missile defence programs and tactical and strategic missiles also continues to absorb large budgetary resources. Moreover, nuclear deterrence modernisation remains a priority, costing 14 billion $ in 2020 and 28.9 billion $ in 2021. New domains such as space and cyber are however not neglected and are also steadily increased over time. This overall increase is accommodated by the increase in military expenditures.

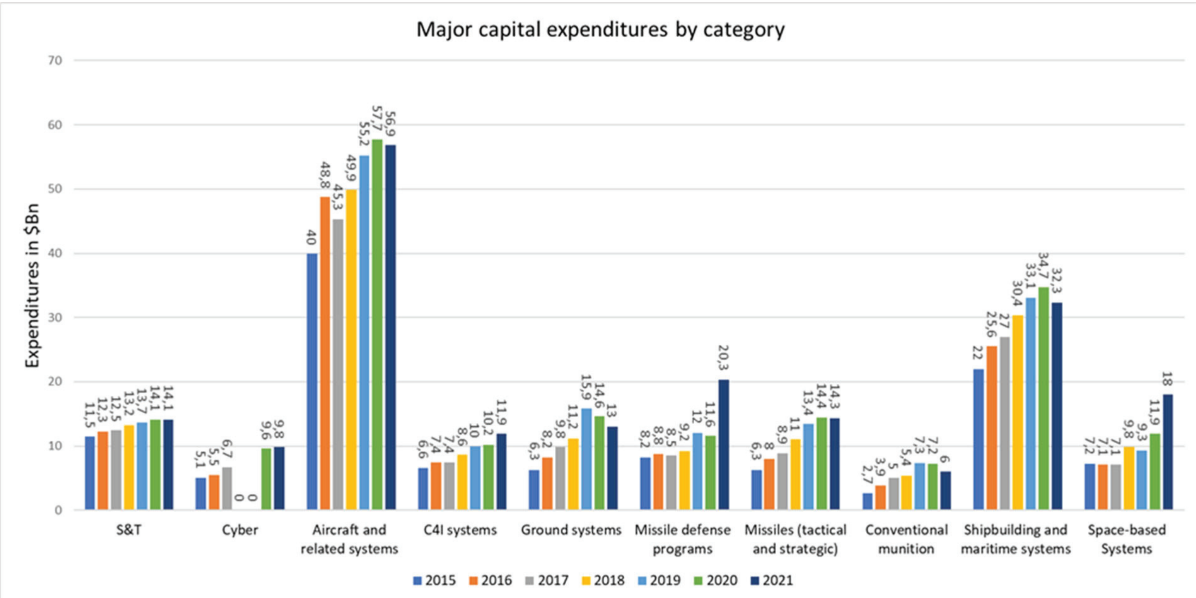

Figure 6. Major US capital expenditures by category over the period 2015–2021. Note: Values for expenditures within the cyber domain are not available for the fiscal years 2018 and 2019. Source: Own visualisation based on data from the U.S. DoD yearly budgetary request (2015–2021).

6. Conclusions and Suggestions for Future Research

The re-emergence of state competition that is being waged in an increasing number of (non-military) domains, entails difficult choices in terms of organising (cross-domain) deterrence. Not only the decision on force posture, but also on force structure could have considerable consequences upon the success rate and credibility of a nation's deterrence. We study this decision-making process by means of a deterrence model, opposing a defender and a challenger. As the defender faces a choice dilemma involving potential large strategic consequences, we incorporate findings from the leading theory of choice under risk [48], i.e., prospect theory. Both the value as the weighting function provide more insights why the defender struggles to simultaneously assess a broad range of threats that diverge strongly in terms of probability and impact. Both functions indicate that the defender will give a higher weight and a higher subjective value to HILP events. This implies that conventional deterrence remains the above-all priority. We depart from our model to offer insights with regards to one of the main research lines within the literature on cross-domain deterrence, i.e., "Can we resort to threats in one domain (in this case the conventional domain) to deter threats that are taking place in other domains (in this case hybrid threats)?" The incorporating of prospect theoretic insights within our deterrence model provides an innovative perspective why high degrees of conventional deterrence are not credible in deterring hybrid threats.

There is no counter evidence that countries are substantially depleting their conventional capabilities to make a shift to other domains (i.e., a substitution of conventional means by hybrid deterrence). States that acknowledge the consequences associated with hybrid threats (illustrated in this article by looking at the U.S.) respond by increasing their security expenditures. This does not necessarily constitute the most optimal response strategy. As the challenger expands the competition to a larger number of domains, the defender is forced to increase its security expenditures, entailing high opportunity costs. Moreover, the increasing efforts made to face hybrid challenges remain currently insufficient. This deterrence failure can be easily illustrated by looking at the large number of (successful) hybrid intrusions that continue to occur [50,57], inflicting societal unrest and economic damages. We account the low cost and limited resources required to launch certain hybrid threats as one of the main reasons why the challenger can keep the frequency and magnitude of intrusions very high. The 'Dark Web Price Index 2020' [58] provides, for example, an overview of the cost of executing certain types of cybercrimes, estimating the cost of a DDoS attack at $10 per hour. At the same time, the cost of putting a small-to medium-sized country down for an hour is estimated at $5600 per minute. We assess that this will render it more and more difficult for the defender to remain superior in all domains and to fend off all attacks at an acceptable (societal) cost. It remains to be seen who will prevail in this competitive race. As the defender must make difficult choices, this article raises awareness of our cognitive biases.

Taking the theoretic model proposed in this article as a starting point, we acknowledge that further qualitative and quantitative research is required to test and improve our understanding of hybrid threats and to optimise our policy responses. We offer two specific recommendations, in line with previous empirical research on prospect theory in a context of conflict. First, future research could confront test subjects (e.g., decision makers such as politicians, or regular citizens if we want to assess how the public evaluates these threats) with hybrid threat scenarios that involve hypothetical policy responses and different outcomes. This methodology, in line with earlier research on other forms of conflict [9,42,59], allows to identify the degree of loss aversion and the types of events that trigger a response, i.e., which events produce a subjective feeling of loss that makes us more risk acceptant? It might be particularly interesting to compare policy responses against various reference points that are framed differently, i.e., with respect to the asset position at the beginning of a series of choices (going back in time) or with respect to the asset position at each individual choice. Second, as hybrid attacks occur frequently, we can conduct large-N statistical analyses [29]. This is, however, resource and time consuming. Although a (non-exhaustive) list of hybrid attacks can be easily obtained (e.g., the authoritarian interference tracker), this is not the case for the policy responses. Identifying these responses requires additional qualitative research such as the analysis of policy documents, statements and interviews. A particular challenge lies in the identification of the reference point prior to being exposed to hybrid threats [43]. These findings can help to increase the credibility of our resolve towards hybrid threats and the delineation of red lines in current great power state competition.

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Validating Game-Theoretic Models of Terrorism: Insights from Machine Learning

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Abstract: There are many competing game-theoretic analyses of terrorism. Most of these models suggest nonlinear relationships between terror attacks and some variable of interest. However, to date, there have been very few attempts to empirically sift between competing models of terrorism or identify nonlinear patterns. We suggest that machine learning can be an effective way of undertaking both. This feature can help build more salient game-theoretic models to help us understand and prevent terrorism.

Keywords: machine learning; terrorism; game theory

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1. Introduction

Game-theoretic models of terrorism are a useful tool in understanding the interactions between states and terrorist groups, the organization of terror groups, and the coordination of counterterrorism efforts [1]. These models provide insights and testable hypotheses. Yet, far too often, many of these hypotheses remain untested. Even when model-generated hypotheses are tested, the focus is on the effect of a particular theory-generated variable on, say, the likelihood of terrorism. This testing may explore causal channels. However, because empirical evidence using traditional econometric channels is not organized to check for relative salience, the importance of a correlation or even causal effect relative to other such effects is unknown. This inability of traditional econometric techniques to check for relative salience in an organized way makes it hard to sift among competing theoretical models. This sort of sifting is essential for policy. A nation plagued by terrorist attacks needs to know which theoretical model provides the largest counterterrorism impact.

Further, it is essential to know how a particular variable may affect terrorism. Game-theoretic models have a key strength. They show comparative static or even dynamic (particularly in evolutionary models) equilibrium shifts. Thus, variables that affect terrorism may do so in nonlinear ways. Traditional econometric tests focused on parametric point estimates are not built to pick up these equilibrium shifts. Nonlinearity, of course, can often be imposed in econometrics. However, this forces the researcher to guess where these nonlinearities may be (squaring the variable, for example, defines a particular shape on a relationship that may or may not be accurate).

Data problems also plague traditional econometric tests of game-theoretic models. Terrorism is, thankfully, rare. However, empirically, this requires heroic assumptions about the distribution of data when making inferences. Even without the rarity aspect, hypotheses testing for significance requires assumptions about the underlying distribution of data that are swept under the rug. Game theoretic models highlight strategic interaction between agents, which are often endogenous. Thus, assumptions about the distribution of data are necessary to estimate efficient and unbiased estimators. Then, there is the issue of model specification. Model specification is often subject to a researcher's explicit and implicit biases. All of this contributes to charges of p hacking in academic research [2].

We suggest that the full power of game theoretical insights can be validated by machine learning. This is particularly important for science in cases such as the study of terrorism, where randomized control trials are impossible or unethical. Therefore, a key contribution of this paper is to introduce the emerging methodology of machine learning to the game-theoretic study of terrorism that can, to a great extent, overcome the limitations of classical regression-based methods [3].

This paper will identify a methodology to identify a robust list of factors that contribute to an increased risk of terrorism and would inform the government on precisely what information to monitor in order to be able to anticipate terrorist events before they occur and would hence contribute to the design of counterterrorism policy at the strategic level. This set of variables can be the starting point for further causal analysis [4]. Our approach will rank variables by predictive importance. Counterterrorism policy is by definition something whose effect happens in the future. Thus, more predictively important variables can be better candidates for policy.

Predictively important variables are not necessarily causal. However, all causal variables should predict. We can identify variables that do not predict well. This reduces the likelihood that these variables are causal. Thus, theoretical models that suggest such variables matter for terrorism are less likely to be explanations for terrorism. To the best of our knowledge this sort of approach is new in the literature on terrorism.

Game-theoretic models predict such nonlinear relationships in comparative static settings. We use machine learning technology to develop partial dependence plots that let the data reveal how predictive variables affect terrorism. This feature makes machine learning an essential vehicle for exploring nonlinear relationships between a policy variable of interest and its effect on the likelihood of terrorism. Moreover, because our algorithms are theory-agnostic, we can let the data speak to actual relationships that can iteratively help us build better game-theoretic models.

We lay down some conceptual foundations about terrorism in Section 2. Section 3 describes the machine learning techniques we use. We describe our data in Section 4. We report our results in Section 5. In Section 6 we provide examples of how our results can be helpful for validating game theoretic models. Section 7 concludes.

2. Conceptual Foundations

The game-theoretic approach to terrorism tries to identify and deter terrorists through a cost–benefit lens that highlights the deep interaction between attacker and defender. Terrorism is a choice for successful rebellions (e.g., in Algeria, Israel, and Cyprus;) [5,6]. Deterrence involves greater policing/punishment and policies to increase the opportunity costs of terrorism at the tactical level [7,8]. However, the very act of deterrence elicits a response [9]. For example, attackers' and defenders' efforts may be complementary, which implies that improving military defense may be counterproductive [10].

The choice of terrorism is also a consequence of the nature of the target. Terrorists will substitute away from hard targets, suggesting that piecemeal policies that focus on some targets at the expense of others may be unproductive [11]. The nature of the target drives even the type of terrorist attack, such that harder targets elicit more suicide attacks in the context of a club goods model [12]. Moreover, increased military aid creates a moral hazard problem in recipient countries who now have an incentive to have terrorists attack them [13].

These lines of research show that terrorism is not a thing in itself; it is a tactical choice driven by context. Further, the relationship between attackers and defenders is constantly changing. A priori, there is no reason to believe that these changes have a linear pattern: Enders and Hoover, for example, empirically show a nonlinear relationship between income and terrorism [14]. However, despite the nonlinear relationships predicted by game-theoretic attacker–defender models, most empirical tests, if any, only provide information on the significance of point estimates. This is insufficient for policymakers since potential underlying nonlinearities may make average point estimates unhelpful. A deer hunter shooting a foot to the left of the deer and a foot to the right of the deer but claiming he shot the deer on average is correct but will go hungry.

Current empirical research has tended to identify the "correlates" of terrorism and has largely failed to identify a consistent set of such correlations. Thus, predicting terrorist attacks has so far mainly been speculative. Machine learning algorithms can provide scientifically cross-validated predictions of the likelihood of a terrorist attack to provide national security agencies with an abbreviated, cross-validated list of variables (i.e., policy levers) that can best identify and hopefully deter terrorism. Machine learning techniques identify the most predictive variables among those. These algorithms then identify validated data-driven relationships between a predictively important covariate of terrorism and the likelihood of terrorism. This approach helps develop better models because they are theoretically agnostic. This agnosticism can help sift between theoretical models—a good model should be able to predict robustly. At the same time, predicting the likelihood of terrorist attacks provides meaningful intelligence for preventing terrorism.

3. Machine Learning

Machine learning (ML) methods are a growing set of methods for predicting and classifying various outcomes. These approaches have two applications: validating policy recommendations and testing theory [15]. Policymakers need to understand the potential effect of a policy before it is implemented, by definition, a matter of prediction. A theory, too, must be able to predict behavior. Machine learning is not a silver bullet, but it can help with these issues.

Further, the machine learning techniques we use do not require assumptions about the underlying distributions of the variables and the error terms. Thus, statistical issues arising out of problems such as endogeneity may be less relevant in these prediction models. For example, say we can identify a highly predictive variable, say X, for terrorism. The predictive value alone, shorn of endogeneity considerations, suggests that policy and academic research should focus on understanding the relationship between X and terrorism. This investigation would include how other variables may influence X as well. Thus, machine learning is a good place to start an investigation as well. Just because X predicts terrorism does not mean it is causal. However, if it is a good predictor then there must be something about X that deserves further scrutiny. By the same token, variables that do not predict terrorism can hardly be causal. A causal variable, by definition, should be predictive. Theoretical models that highlight variables that fail to predict are therefore unlikely to be good explanations for terrorism. This logic allows us to eliminate nonpredictive variables from consideration as casual factors. This process of reasoning provides a path for eliminating theoretical models that are unlikely to causally explain terrorism.

From a policy perspective, predictive analysis has a more direct affect. Say the predictive variable X is, upon further econometric analysis, is also found to be causal. Then, X can potentially be a policy lever because we know X predictably causes terrorism. Therefore, manipulating X can potentially reduce terrorism. Thus, machine learned prediction analysis can supplement econometric techniques for policy analysis.

Everything we noted above can be done using econometrics. However, econometrics requires assumptions about the underlying distribution of the variables. The concomitant endogeneity and specification problems and potential solutions are both susceptible to bias and a source of competing explanations for terrorism. For example, a particular theoretical model might suggest an empirical link between a variable and terrorism that can be tested. Such a test may even reveal a causal link with the right instrument. Yet, without a sense of the predictive salience of this link relative to other competing links we can have no idea whether this causal link is good explanation for terrorism. This is particularly an issue for game theoretic models because these by definition highlight endogenous strategic interactions. Machine learning models, by focusing on accurate prediction even in the presence of endogeneity are particularly suited for the empirical investigation of game theoretic models.

This paper suggests that validated ML techniques can help determine whether a particular theoretical model of terrorism has predictive salience relative to others. In the process, we address some problems inherent in interpreting machine-learned results.

We will build an empirical model using several parametric and nonparametric ML techniques (classical regression, Poisson regression, artificial neural network, regression tree, bootstrap aggregating, boosting, and random forest) to measure how and how well publicly available economic, geographic, and institutional variables *predict* the frequency and severity of terror attacks [16]. The first step in this process will be to identify the machine learning approach that best predicts terrorism.

Next, using the best technique, we will identify the most important variables for predicting terrorism. This process can help validate the predictive salience of a theoretical model relative to others.

Finally, we plot the partial dependency plots for terrorism to show how each variable impacts terrorism across the distribution of its values. This technique is important because game-theoretic analysis gives us reason to believe that many of the correlates of terrorism have nonlinear impacts. Partial dependence plots also help us interpret results more meaningfully.

ML techniques identify tipping points in the range of a particular variable that may place a country at a lower or higher risk of terrorism. We illustrate these tipping points using partial dependence plots, which show how the incidence and severity of terror attacks fluctuate across each variable's observed values. Further, by identifying the variables that have the *most* predictive power, we could help develop a framework to distinguish between competing theoretical explanations of terrorism. Suppose, for instance, political models of terrorism may suggest that terrorism may be a tactic employed by disenfranchised groups with little or no voice in government. In contrast, economic models may suggest that groups employ terrorism as a signal of credibility to gain a seat at the negotiating table against the regime when it divvies up rents from resource wealth. Suppose ML methodologies rank democracy as a better predictor of terrorism than primary commodities exports, for example. In that case, we can assume that the political model may be a better explanation of terrorism than the economic model, or vice versa. Moreover, this approach can eliminate correlates of conflict that do not predict terrorism. Presumably, correlates that do not predict well cannot be considered as variables that cause terrorism. Such culling also helps build better specified and more precise models.

Our ML approach will help us better understand causal patterns explaining terrorism. Moreover, we offer a better understanding of how to predict terrorism, which will help policymakers design counterterrorist policies. The remainder of this section outlines the prediction algorithms we use to predict the aggregate terror risk for a country. Readers who are familiar with these algorithms—or will be bored by a technical description of them!—may skip to the results section. Those looking for a more detailed description of the algorithms may consult their coverage by [16].

3.1. Classical and Other Regression Analysis

Using given data from a learning sample, $L = \{(y_1, \boldsymbol{x}_1), \ldots (y_N, \boldsymbol{x}_N)\}$, any prediction function, $d(\boldsymbol{x}_i)$, maps the vector of input variables, $\boldsymbol{x}$, into the output variable (the number of terror attacks), y. An effective prediction algorithm seeks to define parameters that minimize an error function such as the mean absolute deviations or mean squared error, over the predictions. In linear regression models, $d(\boldsymbol{x}_i)$ is simply a linear function of the inputs. A linear model with the MSE error function yields the ordinary least squares (OLS) regression model:

$$R_{OLS}(d) = \frac{1}{n}\sum_{i=1}^{N}(y_i - d(\boldsymbol{x}_i))^2,$$

where $d(\boldsymbol{x}_i) = \boldsymbol{x}_i\beta$ is a linear function of the inputs.

Although OLS can sometimes yield good predictions (on average, the *best* prediction among all linear models, in fact), it has some undesirable properties in the case of predicting terror attacks. Specifically, since a large number of cases in our sample experience no terror attacks at all, while some of them experience very large numbers of attacks, we will expect the OLS model to predict *negative* numbers of terror attacks for some observations—which is nonsense.

As an alternative, one corrects this problem by estimating a Poisson regression, which will estimate the average number of terror attacks conditional on the inputs, x, to be an exponential function of a linear combination of the inputs expressed as:

$$\lambda = E(y \mid x) = e^{\beta \mathbf{x}}.$$

This means that the probability of observing a specific number of terror attacks will be:

$$p(y|\mathbf{x}) = \frac{e^{yx\beta}e^{-e^{x\beta}}}{y!}.$$

The Poisson model then proceeds by estimating the parameters to maximize the likelihood function for this Poisson probability distribution.

While these more sophisticated regression methods successfully purge the bias from the individual parameter estimates that might result from overdispersion, they do so to the detriment of the model's overall predictive accuracy. Alternative approaches, which ensure a relatively high degree of accuracy while also avoiding nonsensical predictions, use nonparametric tree methods or combinations of trees to predict the number of terror attacks.

3.2. Artificial Neural Networks (ANNs)

A feedforward artificial neural network is a series of binary regression models connecting each of the K input variables to M hidden nodes, over which, in the case of a regression problem such as ours (as opposed to a classification problem in the case of a binary target variable), a linear regression connects the hidden nodes to the output we hope to predict in the final layer. The logistic function is the usual activation function in the first layer, but in general any sigmoid function will have the desired properties. In a regression problem such as ours, the final layer usually contains only one output; the same is true for classification problems involving a binary output. For classification problems involving multinomial outputs, there can be any number of outputs. Thus, this methodology is quite flexible. Hence, with one output node, an ANN estimates $K \cdot M$ parameters.

We present a diagram of a simple ANN for predicting terror attacks in Figure 1. In the figure, each connection corresponds to a weight for each *input* variable ($I1, \ldots, I5$) and bias (constant) terms ($B1$ and $B2$) into the hidden nodes ($H1$ and $H2$), or into the output node (O1). In the diagram, we show a two-layer neural network (the inputs do not count as a layer) with five inputs, two hidden nodes, and a constant. The link function connecting the hidden layer to the outputs, and which is not explicitly shown, is linear.

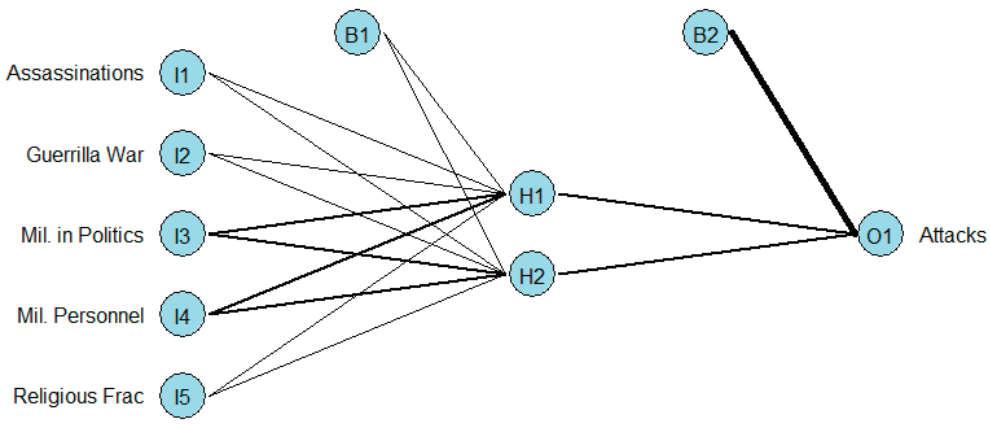

Figure 1. ANN Diagram Example.

Using a least-squares objective, the estimation of the ANN minimizes:

$$R(\alpha, \beta; \boldsymbol{x}) = \sum_{i=1}^{N} (y_i - f(\alpha, \beta; \boldsymbol{x}_i))^2,$$

where $f(\alpha, \beta; \boldsymbol{x}_i) = Z\beta$ connects the hidden layer to the output, and $Z = \frac{1}{1+\exp(x\alpha)}$ is the logit function connecting the inputs to the hidden layer. Using the first order conditions with respect to the parameters for the hidden layer, α, and the parameters to the output layer, β, the estimation finds the solution according to a gradient descent rule:

$$\beta_m^{r+1} = \beta_m^r - \sum_{i=1}^{N} \frac{\partial R}{\partial \beta_m^r} - \lambda \beta_m^r,$$

where l is called the "weight decay" and acts as a penalty on the parameter and effectively restricts the parameters towards zero to avoid "overfitting" the model to the learning sample.

ANNs often perform well in situations where the interplay between input components is more important than any of their values. As such, they are often used in image and pattern recognition problems. We estimate the network using the *nnet* package implemented in *R* [17]. This implementation uses a single hidden layer (in which we used 100 nodes and 100 iterations). This work used all default options, save for specifying that the final layer should be linear. The initial weights were chosen randomly, and the goal function was the sum of the squared errors.

3.3. Regression Trees

Classification and regression trees (CART) diagnose and predict outcomes by finding binary splits in the input variables to optimally divide the sample into subsamples with successively higher levels of accuracy in the output variable, y. Therefore, unlike linear models, where the parameters are linear coefficients on each input variable, the parameters of the tree models are "if–then" statements that split the dataset according to the observed values of the inputs. We provide only a brief summary of tree construction as it pertains to our objectives [18].

More specifically, a tree, T, has four main parts:

1. Binary splits to splits in the inputs that divide the subsample at each node, t;
2. Criteria for splitting each node into additional "child" nodes, or including it in the set of terminal nodes, T^*;
3. A decision rule, $d(\boldsymbol{x})$, for assigning a predicted output value to each terminal node;
4. An estimate of the predictive quality of the decision rule, d.

The first step is achieved at each node by minimizing a measure of impurity. The most common measure of node impurity, and the one we use for our tree algorithms, is the mean square error, denoted $\hat{R}(d) = \frac{1}{n} \sum_{i=1}^{N} (y_i - d(\boldsymbol{x}_i))^2$. Intuitively, this method searches for the cutoff in each input that minimizes errors, then selecting which input yields the greatest improvement in node impurity using its optimal splitting point.

Then, a node is declared to be terminal if one of the following conditions is met: (1) that the best split fails to improve the node impurity by more than a predetermined minimum improvement criterion; or (2) the split creates a "child" node that contains fewer observations than the minimum allowed (Note that there is a tradeoff here: setting lower values for the minimum acceptable margin of improvement or the minimum number of observations in a child node will lead to a more accurate prediction (at least within the sample the model uses to learn). However, improving the accuracy of the algorithm within the sample may lead to overfitting in the sense that the model will perform more poorly out-of-sample). At each terminal node, the decision rule assigns observations with a predicted outcome based on some measure of centrality. In the case of count (number of terror attacks or fatalities) or continuous (amount of property damage) outcomes, centrality is usually the mean of the observations conditional on reaching that node.

The predictive quality of the rule is also evaluated using the *mean square error*, $\hat{R}(d) = \frac{1}{n}\sum_{i=1}^{N}(y_i - d(\boldsymbol{x}_i))^2$. This misclassification rate is often cross-validated by splitting the sample several times and re-estimating the misclassification rate each time to obtain an average misclassification of all of the cross-validated trees.

3.3.1. Boosting Algorithms

Iteratively re-estimating or combining ensembles of trees by averaging their predictions can often improve the accuracy of a tree algorithm. Boosting algorithms, bootstrap aggregating (bagging), and random forests all predict outcomes using ensembles of classification trees. The basic idea of these algorithms is to improve the predictive strength of a "weak learner" by iterating the tree algorithm many times by either modifying the distribution by reweighting the observations (boosting), randomly resampling a subset of the learning sample (bagging), or randomly sampling subsets of the input variables (random forest). These approaches then either classify the outcomes according to the outcome of the "strongest" learner once the algorithm achieves the desired error rate (boosting), or according to the outcome of a vote by the many trees (bagging).

Boosting has been proposed to augment the strength of a "weak learner" (an algorithm that predicts poorly) [19,20]. Specifically, for a given distribution $\mathcal{D}$ of importance values assigned to each observation in L, and for a given desired error, $\widetilde{R}$, and failure probability, φ, a *strong learner* is an algorithm that has a sufficiently high probability (at least $1 - \varphi$) of achieving an error rate no higher than $\widetilde{R}$. A weak learner has a lower probability (less than $1 - \varphi$) of achieving the desired error rate. Boosting algorithms for classification create a set of M classifiers, $F = (f_1, \ldots, f_M)$ that progressively reweight the importance of each observation based on whether the previous classifier predicted it correctly or incorrectly. Modifications of the boosting algorithm for classification have also been developed for regression trees [21,22].

Starting with a $\mathcal{D}_1 = (1/N, \ldots, 1/N)$, suppose that our initial classifier, $f_1 = T$ (single-tree CART, for example), is a "weak learner" in that the misclassification rate, $\hat{R}(d)$ is greater than the desired maximum desired misclassification rate, $\widetilde{R}$. Next, for all observations in the learning sample, recalculate the distribution weights for the observations as:

$$\mathcal{D}_2 = \frac{\mathcal{D}_1(i)}{Z_2} \times \begin{cases} \frac{\hat{R}_1(d)}{1-\hat{R}_1(d)} & if\ d_1(\boldsymbol{x}_i) = y_i \\ 1 & otherwise \end{cases},$$

where Z_m is a scaling constant that forces the weights to sum to one.

The final decision rule for the boosting algorithm is to categorize the outcomes according to $d(\boldsymbol{x}) = \underset{y \in Y}{\operatorname{argmax}} \sum_{m:d_m(\boldsymbol{x})=y} \log\left(\frac{1-\hat{R}_m(d)}{\hat{R}_m(d)}\right)$. Using this decision rule and its corresponding predictions, we calculate the estimate of the misclassification rate in the same way as in step (4) of the single tree algorithm.

3.3.2. Bootstrap Aggregating (Bagging)

The bagging method proposed by [23] takes random resamples, $\{L^{(M)}\}$, from the learning sample *with replacement* to create M samples using only the observations from the learning sample. Each of these samples will contain N observations—the same as the number of observations in the full training sample. However, in any one bootstrapped sample, some observations may appear twice (or more), others not at all. Note that the probability that a single observation is selected in each draw from the learning set is $1/N$. Hence, sampling with replacement, the probability that it is completely left out of any given bootstrap sample is $(1 - 1/N)^N$. For large samples this tends to $1/e$. The probability that an observation will be completely left out of all M bootstrap samples, then, is $(1 - 1/N)^{NM}$. The bagging method then adopts the rules for splitting and declaring nodes to be terminal described in the previous section to build M classification trees.

To complete steps (3) and (4), bagging needs a way of aggregating the information of the predictions from each of the trees. The way that bagging (and, as we will soon see, a random forest) does this for class variables is through *voting*. For *classification trees* (categorical output variables), the voting processes each observation through all of the M trees that was constructed from each of the bootstrapped samples to obtain that observation's predicted class for each tree. Note that the observations under consideration could be from the in-sample learning set or from outside the sample (the test set). The predicted class for the entire model, then, is equal to the mode prediction of all of the trees. For *regression trees* (continuous output variables), the voting process calculates the mean of the predicted values for all of the bootstrapped trees. Finally, the bagging calculates the redistribution estimate in the same way as it did for the single classification tree, using the predicted class based on the voting outcome.

3.3.3. Random Forests

Like bagging, a random forest is a tree-based algorithm that uses a voting rule to determine the predicted class of each observation. However, whereas the bagging randomizes the selection of the observations for each tree, a random forest may randomize over multiple dimensions of the classifier [24]. The most common dimensions for randomizing the trees are selecting the input variables for the node of each tree and the observations included for constructing each of the trees. We briefly describe the construction of the trees for the random forest ensemble below.

A random forest is a collection of tree decision rules, $\{d(\boldsymbol{x}, \Theta_m), m = 1, \ldots, M\}$, where Θ_m is a random vector specifying the observations and inputs that are included at each step of the construction of the decision rule for that tree. To construct a tree, the random forest algorithm takes to following steps:

i Randomly select $n \leq N$ observations from the learning sample;
ii At the "root" node of the tree, select $k \in K$ inputs from $\boldsymbol{x}$;
iii Find the split in each variable selected in (ii) that minimizes the mean square error at that node and select the variable/split that achieves the minimal error;
iv Repeat the random selection of inputs and optimal splits in (ii) and (iii) until some stopping criteria (minimum improvement, minimum number of observations, or maximum number of levels) is met.

The bagging method described in the previous subsection is in fact a special case of a random forest where, for each tree, Θ_m, of a random selection of $n = N$ observations from the learning sample with replacement (and each observation having a probability of being selected in each draw equal to $1/N$) and sets the number of inputs to select at each node, k, equal to the full length of the input vector, K so that all of the variables are considered at each node.

3.4. Validation and Testing of Predictive Accuracy

Once we have built our learning algorithm, the next issue is to evaluate the validity of our error estimates and the predictive strength of our models. Error estimates ($R[d]$) can sometimes be misleading if the model we are evaluating is overfitted to the learning sample. These error estimates can be tested out-of-sample or cross-validated using the learning sample.

To test the out-of-sample validity, we simply split the full dataset into two random subsets of *countries*: the first, known as the *learning sample* (or training sample) contains the countries and observations that will build the models; the second, known as the *test sample*, will test the out-of-sample predictive accuracy of the models. The out-of-sample error rates will indicate which models and specifications perform best, and will help reveal if any of the models are overfitted.

To validate the error rates, machine learning uses either hold-out validation or cross-validation. In our study, we have used hold-out validation, which involves training the models using one portion (in our case 70% selected at random) of the dataset. The algorithm then tests the learned model by measuring the mean square error between the predicted value and the actual value in the 30% of the data unseen by it. A model with an acceptably low error rate in the sample unseen by it is presumably a good predictive model. This out of sample test also guards against overfitting. An overfitted model may be highly accurate in the learning sample but it would be unlikely to predict well in the test sample.

4. Data

As a first step in analyzing some preliminary data on terrorism, we have predicted the number of terror attacks using each of the seven models described above (OLS regression, Poisson regression, regression tree, random forest, bagging, and boosting). For our specification, we have included 69 input (or explanatory) variables that cover most of the ones discussed in Gassebner and Luechinger's survey of the empirical literature on conflict [25].

We measure our output (or "dependent") variable, Terror Attacks, as the total number of terror attacks in a country in the last five years. This variable comes from the Global Terror Database published by the University of Maryland and covers 1970–2014. When we combine all of the variables, our sample covers 1975–2014, since some entire data sources, such as the Database of Political Institutions, do not become available until 1975. To maintain the spirit of "prediction" in our model, we then consider our input ("explanatory") variables as five-year lagged averages of the preceding five years. Moreover, we only consider the variables at nonoverlapping five-year intervals so that none of the same information is contained in consecutive time intervals in our sample. In this sense, at any given point in time, policymakers will be able to use our model to predict whether a country will likely experience a greater or lesser number of terror incidents in the next five years. Moreover, this approach reduces the risk of endogeneity; the past can potentially affect the future, but it seems unlikely that the future can affect the past. In addition, this lagging reduces the risk of collider bias among the potential predictors if one were to interpret partial dependence plots causally. Collider bias happens when the target variable (Y, terrorism here) and a variable of theoretical interest (say T) affects a third variable, say X, in the model. In that case, if the researcher is interested in justifying a causal relationship between T and Y, X should be taken out of the model specification. Placing the target variable in the future helps justify that there can be no such relationship. We do *not* interpret our partial dependence plots causally.

From the Cross-National Time Series [26] we take the numbers of assassinations, demonstrations, government crises, guerrilla warfare incidents, purges, riots, and strikes as measures of underlying low-level social instability. We also take the number of cabinet changes and executive changes as measured of political instability, and the effectiveness of the legislature as a measure of political legitimacy.

From the Database of Political Institutions [27] we take the number of checks on power; executive and legislative indices of electoral competition; legislative, government, and opposition fractionalization indices; government Herfindahl index; and government polarization index as measures of the concentration (or not) of power and accountability (or not) within the government. We then include the changes in veto players, the existence of electoral fraud, executive tenure, the presence of a military executive, and political stability and executive power measures. Finally, we include plurality voting and proportional representation as indicators of structural differences in electoral rules.

Next, we take several indices of government quality from the International Country Risk Guide [28]. It is important to remember that, for each of the ICRG indices, a higher value always coincides with "better" outcomes on this dimension of institutional quality. For example, in the case of the "internal conflict" (or "external conflict") index, a higher value for the index somewhat counterintuitively corresponds to less conflict. The same can be said for "ethnic tensions", "religious tensions" and "military in politics"—in each of these cases, higher values relate to less of the (bad) thing that the variable name implies. That being said, we include the following indices from the ICRG: the bureaucratic quality and corruption indices as measures of the transparency of government; ethnic tensions, external conflict, internal conflict, law and order, and religious tensions as measures of the levels of latent (or open) social hostility, and the government's ability to ease those hostilities; government stability and investment profile indices as measures of the government's credibility in carrying out stated policies and refraining from expropriation; and democratic accountability and military in politics indices as a measure of the legitimacy and responsiveness of the regime to the public's preferences. We also add the Polity2 index and regime durability from the Polity IV Project as additional measures of legitimacy and responsiveness.

As measures of economic and cultural divisions within society, we include measures of income inequality and ethnic and religious fractionalization. The former comes from the Standardized World Income Inequality Database [29]. The latter come from [30], which in turn come from the *Atlas Naroda Mira* [31].

Finally, we include numerous measures of economic human development from the World Development Indicators from the World Bank. They are: aid and development assistance; arms exports and imports; public education and health spending; female labor force participation; foreign direct investment (FDI); fuel exports; gross domestic product (GDP) per capita; government consumption; the stock of foreign born immigrants; infant mortality; the inflation rate in consumer prices; life expectancy; literacy; military expenditures; military personnel; population and its rate of growth; portfolio investment; primary, secondary, and tertiary school enrollment rates; social contributions; telephones per 100,000 people; the unemployment rate; urban population; and the youth dependency ratio.

Rather than exhaustively describing the distributional characteristics and justifying the inclusion of each variable, we kindly refer the reader to visit Gassebner and Luechinger's survey and the references therein to the various studies that have already provided such a description and justification [25]. For readers interested in some of the characteristics of the observed data in our sample, we have included the descriptive statistics for all 69 variables in Table 1.

We can see from the table that each of our explanatory variables has omitted values to varying degrees. The tree-based methods (single trees, boosting, bagging, and random forest) can automatically exploit the full information available by using surrogate information or using the median or mode at that branch of a tree as a best guess the value of a missing data point. Standard parametric methods (in our case Poisson regression and neural networks) do not do this automatically, and regression methods that do (such as full-information maximum likelihood), might do so in ways that give different imputations of the missing data.

To resolve this, we preprocess our data using random forest imputation. The basic idea is that we consider a covariate that does not have missing data (in our case conflict), and perform a random forest model to predict that variable (instead of the true variable of interest since that would be "cheating" for running the full model). Next, whenever the algorithm encounters a missing value at any tree node, the imputation substitutes the median or mode for that variable and continues with the subsequent splits. Therefore, the imputed values in each tree exploit the full complement of conditional distribution for that variable based on that tree. Averaging over all of the trees, we obtain imputed values for missing data points that uses as much relevant data about the conditional distribution of the variable as possible. It also has the advantage of creating imputed values that are naturally bounded by the domains of the observed data. Parametric methods such as multiple imputation estimate parameters based on an assumed distribution for the missing variables, and depending on the sensitivity of the parameters and the distributions of the covariates, may lead to extreme values outside of the logical bounds for a given variable (e.g., negative income).

Table 1. Variables and descriptive statistics.

Variable	Source	Obs	Mean	Std. Dev.	Min.	Max.
Terror Attacks	GTD	6411	86.88	375.81	0	10,701
Assassinations	CNTS	5318	0.21	0.84	0.00	18.50
Cabinet Changes	CNTS	5310	0.44	0.37	0.00	3.50
Demonstrations	CNTS	5318	0.52	1.15	0.00	14.00
Effectiveness of Leg.	CNTS	5297	1.74	0.94	0.00	3.00
Executive Changes	CNTS	5310	0.19	0.28	0.00	3.00
Government Crises	CNTS	5318	0.13	0.27	0.00	2.67
Guerrilla Warfare	CNTS	5318	0.12	0.32	0.00	2.60
Purges	CNTS	5318	0.03	0.13	0.00	2.50
Riots	CNTS	5318	0.31	1.05	0.00	18.20
Strikes	CNTS	5318	0.12	0.34	0.00	3.40
Changes in Veto Players	DPI	4838	0.12	0.15	0.00	1.00
Checks on Power	DPI	4831	2.52	1.60	1.00	17.00
Exec. Electoral Comp.	DPI	4850	5.15	2.08	1.00	7.00
Executive Years in Office	DPI	4859	7.93	7.68	1.00	45.00
Electoral Fraud	DPI	4214	0.14	0.32	0.00	1.00
Government Frac	DPI	4428	0.19	0.25	0.00	1.00
Government Herfindahl	DPI	4428	0.82	0.25	0.02	1.00
Government Polarization	DPI	4673	0.36	0.69	0.00	2.00
Legislative Frac.	DPI	4419	0.46	0.30	0.00	1.00
Leg. Electoral Comp.	DPI	4855	5.41	2.00	1.00	7.00
Military Executive	DPI	4856	0.21	0.39	0.00	1.00
Opposition Frac	DPI	3362	0.45	0.27	0.00	1.00
Plurality Voting	DPI	3877	0.68	0.46	0.00	1.00
Proportional Rep.	DPI	3474	0.58	0.49	0.00	1.00
Bureaucratic Quality	ICRG	3376	2.11	1.19	0.00	4.00
Corruption	ICRG	3376	3.08	1.35	0.00	6.00
Democratic Accountability	ICRG	3376	3.64	1.62	0.00	6.00
Ethnic Tensions	ICRG	3376	3.91	1.44	0.00	6.00
External Conflict	ICRG	3376	9.48	2.22	0.00	12.00
Government Stability	ICRG	3376	7.45	2.10	1.00	11.50
Internal Conflict	ICRG	3376	8.61	2.62	0.03	12.00
Investment Profile	ICRG	3376	6.94	2.34	0.08	12.00
Law and Order	ICRG	3376	3.60	1.48	0.25	6.00
Military in Politics	ICRG	3376	3.66	1.80	0.00	6.00
Religious Tensions	ICRG	3376	4.54	1.35	0.00	6.00
Polity2	Polity IV	4520	1.16	7.26	−10.00	10.00
Regime Durability	Polity IV	4569	23.99	28.73	0.00	198.00
Ethnic Fractionalization	Reynal-Querol	4749	0.45	0.28	0.01	0.96
Religious Fractionalization	Reynal-Querol	4749	0.28	0.23	0.00	0.78
Income Inequality (Gini)	SWIID	3350	38.52	9.87	16.49	69.35
Area	WDI	6110	682,865	1,717,163	2	16,400,000
Off. Aid & Dev. Assistance	WDI	4045	0.08	0.11	−0.01	0.76
Arms Exports	WDI	1703	0.01	0.08	0.00	1.50
Arms Imports	WDI	3976	0.04	0.12	0.00	3.32
Education Spending	WDI	3436	4.45	2.32	0.59	44.30
Foreign Direct Investment	WDI	4602	2.80	4.72	−32.30	72.50
Female Labor Force Part.	WDI	3293	50.12	17.55	9.20	90.80
Fuel Exports	WDI	3875	16.82	28.33	0.00	100.00
GDP per Capita	WDI	4807	9560.35	16,016.19	65.64	141,000.00
Government Consumption	WDI	4538	16.47	6.87	3.37	84.50
Health Spending	WDI	2647	3.48	2.21	0.01	18.36
Immigrant Stock	WDI	4975	8.07	13.75	0.03	86.80
Infant Mortality	WDI	5103	48.11	40.72	2.18	174.00
Inflation	WDI	4168	32.94	254.00	−17.60	6522.40
Life Expectancy	WDI	5074	64.79	10.58	24.30	82.50
Literacy Rate	WDI	1549	73.42	23.01	10.90	100.00
Military Expenditures	WDI	2995	2.74	3.03	0.09	48.60
Military Personnel	WDI	3092	1.88	2.23	0.06	35.80
Population	WDI	5190	30.94	116.87	8.82	1316.00
Population Growth	WDI	5190	1.80	1.44	−4.84	15.50
Portfolio Investment	WDI	4000	0.01	0.16	−0.02	4.88
Primary Enrollment	WDI	4763	97.05	22.35	15.80	208.00
Secondary Enrollment	WDI	4407	60.84	33.35	2.13	155.60
Social Contributions	WDI	1203	17.11	15.02	0.00	59.97
Telephones	WDI	5127	14.70	18.58	0.01	103.42
Tertiary Enrollment	WDI	4135	18.62	19.36	0.00	99.20
Unemployment	WDI	3007	9.03	6.78	0.20	59.50
Urban Population	WDI	5190	50.33	24.51	4.18	100.00
Youth Dependency	WDI	5000	62.07	23.94	19.44	114.40

5. Results

5.1. Predictive Quality

Table 2 reports the predictive quality of each of the models using the 70 variables. The best models we see to predict the overall number of terror attacks are the single regression tree, random forest, and bagging predictors, which reduce the overall MSE in the learning sample by about 64%, and 63%, and 59%, respectively, compared to the unconditional sample mean. An average of all of the models' predicted values (which sometimes provides a better prediction, especially in cases of classification) improves the MSE by about 49%. In comparison, OLS regression improves the MSE by about 26%. However, as we might expect, the trees that use random bootstrapping (bagging and random forest) predict considerably better out of sample, with a test sample MSE reduction of 71% and 70% of the total MSE, respectively. Of particular interest here is the fact that these models achieve a significant reduction in the MSE despite the exclusion of the lagged number of terror attacks in our model since the pre-existing level of violence has been shown to be one of the strongest predictors of current and future violence in studies of conflict [32]. We exclude lagged terror attacks because we are partly looking to predict (a reason to include), but also looking to select a model to build theories and test causal effects (subsequent analyses). Lagged attacks would improve the prediction but would explain *so much* of the variation that we are not left with much to select a model on

Table 2. MSEs for the various learning models.

	Learning Sample		**Test Sample**	
	MSE	**% Decrease**	**MSE**	**% Decrease**
OLS Regression	107,708.05	25.71%	98,119.17	26.12%
Poisson Regression	151,539.85	−4.52%	139,385.78	−4.96%
Neural Network	144,695.12	0.20%	132,389.28	0.31%
Regression Tree	52,038.41	64.11%	80,182.62	39.62%
Boosting Predictor	141,677.19	2.28%	129,790.58	2.27%
Bagging Predictor	59,866.71	58.71%	40,202.12	69.73%
Random Forest	54,271.19	62.57%	38,504.85	71.01%
Average of All Predictors	74,564.39	48.57%	76,391.30	42.48%
Total MSE	144,987.24		132,802.82	

It is worth noting that the Poisson regression model, which tends to yield more valid estimates of causal effects, actually *increases* the MSE of the predictor compared to a prediction based on the simple sample mean. This is not quite the case for the neural network model, but we can see that the neural network and boosting models predict relatively poorly both in and out of sample.

5.2. Variable Importance

Table 3 reports the variable importance levels (measured as the percentage of the total reduction in MSE that is attributed to that variable) based on the single regression tree, boosting, bagging, and random forest models, which predicts conflict the best, although different algorithms or different runs of the same algorithm may identify different sets of predictors [15]. Theoretically agnostic algorithms may choose a predictive variable one time and another at a different time if they are predictive substitutes. The risk for this happening is reduced for algorithms such as random forests, bagging, or boosting because the algorithm learns by taking multiple subsamples and averaging the results. We take this one step further by averaging the variable importance results across several algorithms to give us a sense of confidence in the stability of the variable importance ranking.

Table 3. Variable importance rankings.

Variable	Tree	Bagging	Boosting	Forest	Average
Assassinations	7.618	24.930	62.966	12.388	14.979
Guerrilla War	2.677	10.735	30.698	9.436	7.616
Military Personnel	15.482	4.166	0.000	2.555	7.401
Religious Frac	12.386	4.761	0.000	3.218	6.788
Military Politics	12.682	1.913	1.082	3.529	6.042
Health Spending	3.765	4.390	2.499	3.548	3.901
Year	1.882	5.704	0.000	3.888	3.825
Population	0.947	3.568	0.394	5.562	3.359
Exec Yrs in Office	6.441	1.940	0.000	1.315	3.232
Fuel Exports	6.193	1.455	0.000	1.227	2.958
Dem Accountability	5.222	1.243	0.000	1.411	2.625
Effectiveness of Leg	0.000	3.104	0.000	3.041	2.048
Aid & Assistance	2.528	0.973	0.000	1.826	1.775
Gini	0.981	2.106	0.000	2.083	1.723
Tertiary Enrollment	2.053	0.752	0.000	2.023	1.609
Female LFPR	0.000	1.226	2.361	3.213	1.480
Portfolio Investment	0.000	2.695	0.000	1.600	1.432
Area	1.858	1.194	0.000	0.837	1.297
Arms Imports	1.425	1.402	0.000	1.009	1.279
Strikes	0.662	1.369	0.000	1.711	1.247
Ethnic Tension	0.733	0.467	0.000	2.409	1.203
Checks	1.702	1.248	0.000	0.615	1.188
Internal Conflict	0.969	0.125	0.000	2.257	1.117
Telephones	1.882	0.435	0.000	0.697	1.005
Law Order	1.322	0.560	0.000	0.966	0.950
GDP pc	0.235	0.842	0.000	1.736	0.938
Urban Population	1.710	0.404	0.000	0.645	0.920
Ethnic Frac	0.469	1.336	0.000	0.885	0.897
Polity 2	0.000	1.355	0.000	1.244	0.866
Investment Prof	0.321	1.019	0.000	1.169	0.836
Legislative Frac	1.425	0.532	0.000	0.549	0.835
Riots	0.307	0.729	0.000	1.382	0.806
Primary Enrollment	1.425	0.170	0.000	0.687	0.761
Arms Exports	0.000	1.233	0.000	0.950	0.728
Demonstrations	0.179	0.464	0.000	1.511	0.718
Unemployment	0.000	0.813	0.000	1.310	0.708
Religious Tension	0.000	0.300	0.000	1.601	0.634
Infant Mortality	0.000	0.773	0.000	1.046	0.606
Secondary Enrollment	0.000	0.848	0.000	0.765	0.537
Immigrant Stock	0.016	0.672	0.000	0.816	0.501
Reg Durability	0.248	0.547	0.000	0.609	0.468
Gov Consumption	0.000	0.369	0.000	0.824	0.398
Gov Stability	0.618	0.089	0.000	0.441	0.383
Corruption	0.075	0.472	0.000	0.584	0.377
Life Expectancy	0.000	0.242	0.000	0.889	0.377
Youth Dependency	0.000	0.286	0.000	0.842	0.376
FDI	0.000	0.372	0.000	0.719	0.363
Fraud	0.346	0.106	0.000	0.421	0.291
Opposition Frac	0.000	0.304	0.000	0.560	0.288
Inflation	0.207	0.326	0.000	0.300	0.278
External Conflict	0.259	0.140	0.000	0.428	0.276
Bureaucratic Qual	0.000	0.277	0.000	0.501	0.259
Leg. Elec. Comp.	0.248	0.192	0.000	0.295	0.245
Exec. Elec. Comp.	0.000	0.262	0.000	0.465	0.242
Literacy Rate	0.000	0.143	0.000	0.574	0.239
Population Growth	0.000	0.394	0.000	0.302	0.232
Proportional Rep	0.000	0.368	0.000	0.326	0.231
Social Contributions	0.000	0.173	0.000	0.515	0.229
Military Expend	0.167	0.212	0.000	0.277	0.219
Purges	0.331	0.051	0.000	0.179	0.187
Education Spending	0.000	0.243	0.000	0.304	0.182
Military Exec	0.000	0.085	0.000	0.316	0.134
Government Herfindahl	0.000	0.082	0.000	0.235	0.106
PluralityVoting	0.000	0.093	0.000	0.196	0.096
Gov Polarization	0.000	0.063	0.000	0.198	0.087
Government Frac	0.000	0.093	0.000	0.087	0.060
Cabinet Changes	0.000	0.028	0.000	0.085	0.038
Changes in Vetoes	0.000	0.065	0.000	0.018	0.028
Executive Changes	0.000	0.020	0.000	0.044	0.021
Government Crises	0.000	−0.050	0.000	−0.191	−0.080

Here, we see that the first five variables in the list account for close to one-third (about 31 percent) of the overall improvement in the random forest model's MSE. We also see that the single strongest predictor of current levels of terrorism is a history of assassinations in that country, which accounts for about 12% of the total reduction in the MSE in the random forest model, and 25% of the reduction for the bagging model and 63% of the reduction for the boosting model. The second strongest predictor, guerrilla war, accounts for about 10%

of the MSE reduction for the bagging and forest models and over 30% of the decrease for the boosting model.

After regime-directed violence, two of the following three strongest predictors involve the extent to which the military engages with everyday life and politics. Military personnel and the military in politics index account for almost 15% of the reduction in MSE combined, on average (slightly more in the single tree, somewhat less in the bagging and forest models, and not in the boosting algorithm). In between these measures of military engagement, we see religious fractionalization to account for about 7% of the variation on average. Rounding out the top ten predictors are health spending (3.9% of the MSE), time trend (3.8%), population (3.6%), executive tenure (3.2%), and fuel exports (2.6%).

At this point, our algorithmic approach suggests we have a group of variables that predict terrorism quite well. Moreover, we have identified the top predictors of terrorism. The reader will note that many of the variables identified by the literature do not have predictive salience [25]. Indeed, many of the variables highlighted in the literature, such as investment profile, bureaucratic quality, or religious tensions, have very little predictive salience. This culling helps us identify the kinds of theoretical models that can help us better understand terrorism. For example, the joint importance of guerilla war and military personnel is quite high and suggests that terrorism may be best understood as a tactical choice in asymmetric warfare rather than an outcome of institutional deficiencies in bureaucratic quality or lack of economic opportunity. This sort of explanation lends credence to the argument that a war on terror is strategically empty—just as a war on the blitzkrieg or the pincer movement, both tactical choices, would be strategically empty. However, such explanations are also predicated on the nature of the relationships between the top predictors and terrorist attacks. We turn to identify just such relations next, highlighting a methodological approach that is particularly in tune with the nonlinear relationships predicted by game-theoretic models.

5.3. The Nonlinear Relationship between Greater Security and Terrorism

The next step is to analyze *how* each of the variables impacts aggregate terror risk. To do this, we use a *partial dependence plot* mapping the possible values of the input variable of interest onto the observed incidence of terror attacks. Partial dependence plots display the marginal effect of variable x_k conditional on the observed values of all of the other variables, $(x_{1,-k}, x_{2,-k}, \ldots\ x_{n,-k})$. Specifically, it plots the graph of the function:

$$\hat{f}(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{x}_k, \boldsymbol{x}_{i,-k}),$$

where the summand, $f(\boldsymbol{x}_k, \boldsymbol{x}_{i,-k})$, is simply the observed outcome of the number of terror attacks.

This section focuses on three partial dependence plots that highlight game-theoretic models of terrorism that suggest that any fundamental understanding of terrorism should be understood as a tactical choice by rebel organizations.

Figure 2 shows that guerrilla warfare increases terrorism. While there may be some overlap between guerrilla warfare and terrorism, agencies that make national security policies tend to define them as distinct phenomena. Hence, in some cases, we might think of terrorism and guerrilla warfare as different tactics employed by rebel groups towards similar ends [33]. Moreover, guerilla warfare predicts terrorism five years out.

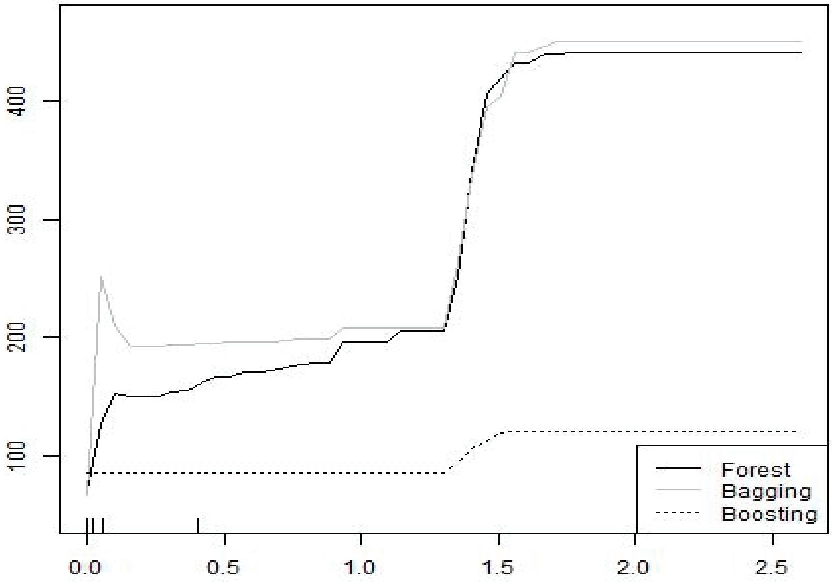

Figure 2. Partial dependence plot: guerrilla warfare.

Thus, in Figure 3, more military personnel also translate into more terror attacks on average, though these averages mask a u-shaped relationship. Last, in Figure 4, we notice that increased military involvement *in politics reduces* aggregate terror risk. Taken together, and in the absence of the predictive salience of such institutional variables such as bureaucratic quality and investment profile that capture elements of state capacity, we can grope toward a model of terrorism rooted in the understanding of a specific kind of state capacity. The nonlinear relationships embedded in this understanding suggest that game-theoretic models where equilibrium switching is possible due to interactions between agents are better suited than traditional neoclassical utility maximization approaches.

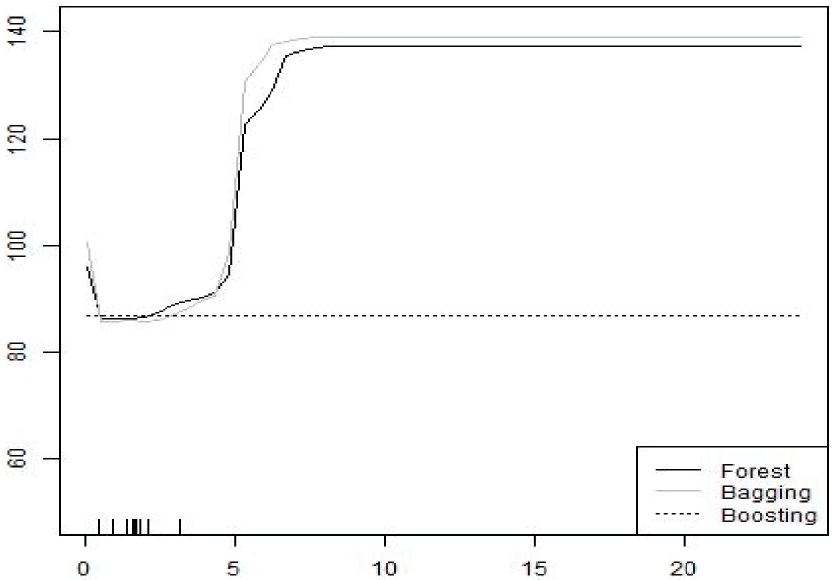

Figure 3. Partial dependence plot: military personnel.

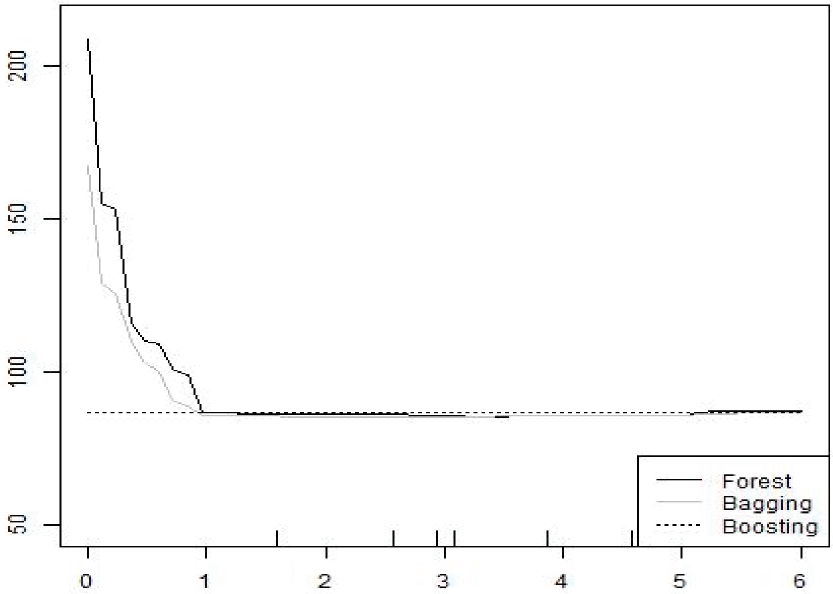

Figure 4. Partial dependence plot: military in politics.

State capacity (or the lack thereof) is a reasonably standard explanation for conflict [34]. The theoretical basis for an empirical understanding of this relationship lies in three concepts: military capacity, administrative quality, and institutional coherence.

Ref. [35] suggests that military personnel and expenditures, bureaucratic quality measures, and popular institutional measures such as polity or those reported by ICRG have construct and theoretical validity as a measure of the three elements of state capacity. We have all these variables as part of our predictive algorithm. Nevertheless, of these three, it appears that military capacity is most salient for understanding terrorism. Thus, our algorithm has been pretty specific about what kind of theoretical models are more likely explanations for terrorist attacks. This suggests that models better understand why terrorism happens [13]. The how matters as well.

There is a clear equilibrium switch in the number of terrorist attacks as guerilla warfare intensifies. However, there is an optimum level of intensity beyond which the number of terrorist attacks is stable. Again, this is the result suggested by game-theoretic models where equilibrium switches can be, for example, a consequence of changes in the payoffs. An econometric point parametric estimate would never capture these breakpoints unless, out of sheer coincidence, the researcher imposes the assumption of such a breakpoint. However, point estimates can be particularly misleading, for example, in the case of military personnel. An average effect captured in a point estimate would merely show a positive relationship, rather than the nuance where (initially at least) increasing military personnel reduces terrorism, thus suggesting a cost-minimizing optimum amount of military personnel. Nevertheless, we also have the somewhat counterintuitive but ultimately plausible result that hardening targets by increasing the number of military personnel elicits more terrorist attacks than substitute attacks away from these targets. This sort of result is reminiscent of security dilemmas rather than Beckerian policing models.

On the other hand, we cannot completely throw out institutional coherence as a predictor of terrorism. Military dictatorships can control terrorist attacks better. This result provides an interesting counterpoint to the argument that military regimes are more vulnerable to terrorism [36].

6. Game-Theoretic Model Validation

Others have suggested that machine learning can help validate theoretical models because they are designed to test whether a model is predictive or not [15,37]. A good theoretical model should be able to predict behavior.

Standard econometric approaches to testing models are particularly fraught when it comes to testing game-theoretic models because endogeneity is a feature rather than a bug in game-theoretic models. For example, terrorists respond to counterterrorism by changing their behavior, which in turn suggests changes in counterterrorism. Thus, any econometric approach to terrorism must be cautious to avoid endogeneity-driven estimation biases. Many of these methods reduce the predictive value of a model (for example, many causal studies have very low R-squares). Yet, as we noted above, a good theoretical model should also be able to predict. Predictive machine learning can help determine whether a causal variable is also predictive. A causal variable that is also predictive can help convince academics and policymakers of the salience of a theoretical model.

Partial dependence plots can capture equilibrium shifts to capture comparative static effects of game-theoretic models. We discuss this aspect quite extensively in the previous section. However, variable importance can help us sift through models of terrorism to identify more predictive variable specifications. We highlight three examples within subsets of the game-theoretic literature to emphasize this point.

One strand of the game theoretic literature focuses on group cohesion. Future uncertainty generated by increased counter terrorism can lead to rebel group splintering, thereby increasing the risk of terrorism as these splinter groups jockey for survival [38]. Figure 3 highlights just such a result; an increase in military personnel does indeed predict an increase in the number of terrorist attacks. Further, the first two most predictive variables, guerrilla warfare and assassinations, also predict an increase in the number of terrorist attacks. Guerrilla warfare and assassinations also point to significant political uncertainty. This suggests that political uncertainty may be an important predictor of terrorist attacks, possibly by affecting group cohesion. These findings would suggest a deeper, and causal, dive into understanding how rebel group splintering in the face of political uncertainty may affect terrorism. That is to say, machine-learning can be a first step toward finding explanations of terrorism in conjunction with game-theoretic models and causal econometric analysis.

Counterterrorism efforts require global coordination. For example, destroying a terrorist training ground may require the US to take action in North Africa or the Middle East. Theoretically, military aid to a country that hosts a terrorist organization creates a disincentive to remove the terrorist problem [13]. In addition, terrorism is a tactical choice for a rebel group when facing a formidable state that the rebels do not want to provoke too much [33].

Both these models suggest that military strength should be a predictor of terrorist attacks. Our algorithm identifies the size of the military as one of the most important predictors of terrorism. As noted in Figure 3, an increase in the size of the military predicts an initial rise in terrorist acts as expected by both the game-theoretic modelsthat predict an increase in the intensity of terrorist attacks, particularly suicide attacks, when targets harden [12]. Nevertheless, further increases in the size of the military keep the risk of terror attacks elevated without increasing terrorist attacks, a potential benefit for a host country receiving military aid. That is, military size increases terrorism at first and then levels off, tracking the prediction from Bapat's model (see Figure 2, p. 311 in [13]).

Equilibrium may also shift from guerrilla warfare to terrorism as a function of the accuracy of a state's military action [33]. If terrorist tactics are more provocative, the probability of a terrorist attack increases. On the flip side, if guerrilla action is more provocative then the probability of guerilla warfare increases. The point is that as the degree of provocation changes there is an equilibrium switch from guerilla warfare to terrorism. Our result in Figure 3 identifies just such an equilibrium switch to increased terrorism as the intensity of guerilla warfare increases. First of all, this means that equilibrium switches to more terrorism are related to guerilla warfare. Thus, our result in Figure 3 supports Carter's model prediction. However, our result also suggests that, if Carter's model is a true reflection of reality, then as guerilla warfare intensifies there is some change in the underlying parameters in a way that makes terrorist action more provocative. Thus, our results suggest that there may be a relationship between guerilla action and provocation that changes the likelihood of terrorism. This space may bear further theoretical investigation.

Terrorist organizations need to survive to achieve their goals. A strand of the game-theoretic literature is devoted to understanding how terrorist organizations recruit and retain members while overcoming incentive compatibility problems when secrecy is essential.

De Mesquita's game-theoretic model suggests that counterterrorism efforts that reduce economic opportunity can increase terrorist mobilization [39]. In any case, terrorist organizations will put more resources into terrorism (presumably leading to more successful attacks) when they recruit and retain higher-ability terrorists. Therefore, the BDM (2005) model would suggest that, empirically, countries with better economic opportunities would have fewer terrorist attacks. Moreover, he notes that his model suggests, among other things, that ethnically divided societies would see more terrorist attacks and that development aid may reduce terrorism (presumably by increasing economic opportunity).

Our results fails to validate many of the predictions of this model [39]. For example, the variable investment profile includes contract enforcement and risk of expropriation by the state. These variables are components of economic freedom or opportunity. For example, the risk of expropriation reduces the likelihood of economic growth [40]. However, this variable is not an important predictor of terrorism. Moreover, neither ethnic divisions (as measured by the ethnic tensions variable) nor development aid are important predictors of terrorism.

Presumably terrorist organizations mobilize to perpetrate terrorist attacks. However, while factors such as the lack of economic opportunity may indeed affect mobilization, it seems highly unlikely that they affect terrorist attacks. If economic opportunity was important for mobilization it should be able to predict terrorism since terrorism is the purpose for mobilization. This brings into question the role of economic opportunity in explaining mobilization.

Our examples in this section suggest that some game-theoretic models generate validated predictions while others do not. Now all of these models may be causal. Our algorithms make no claims for causality. Yet, if a model is a generalizable explanation of reality, then its predictions should be validated empirically. On this criterion, all models cannot be treated equally. Further, we show how partial dependence plots, by highlighting nonlinear relationships, can help validate game theoretic models that very typically generate hypotheses with nonlinear patterns. Further, these results are data-driven and therefore unbiased by assumptions about any particular theoretical concern. Consequently, empirical results that are consistent with theoretical consequences provide an unbiased validation. Last, once again because our results are data-driven rather than based on theoretical assumptions, they can give us hints about what areas need a theoretical structure. Of course, empirical validation of this new theory should give rise to even more spaces that need theory in an iterative process that slowly erases gaps in knowledge.

7. Conclusions

In this paper, we highlight two aspects of machine learning that can supplement game-theoretic analysis. First, we can sift among competing theoretical models in a theoretically agnostic way to identify those models which have the most predictive salience. A good theoretical model should be able to make predictions. Here, our algorithm suggests that models predicting economic opportunity, development assistance, and ethnic tensions may not be predictively salient. In contrast, those that predict a more formidable target would elicit more terrorist attacks so are predictively salient.

Game-theoretic models, by their very nature, highlight endogenous relationships driven by strategic interactions. Machine learning algorithms, by focusing on predictive accuracy instead of tests of significance, can identify whether a variable is predictive or not even if it is endogenous with the target variable, terrorism. To the extent that causal variables should be predictive, identifying predictive variables can help jumpstart the search for causal links. This process is made more efficient because we can eliminate variables that are unlikely to be causal because they are not predictive in an empirical framework that is unbiased by endogeneity problems.

Second, game-theoretic approaches often predict nonlinear relationships between variables where equilibriums switch in comparative static scenarios. The partial dependence plots generated by machine learning algorithms can identify these nonlinearities and equilibrium switches in a theoretically agnostic way. Partial dependence plots are, therefore, a particularly suitable testing methodology for game-theoretic comparative statics.

Thus, machine learning techniques can reduce bias and help find better explanations for terrorism. This is important for formulating better counterterrorist policies. These techniques have other benefits as well. For example, they can impute missing data and predictively validate the imputation, and they do not require heroic assumptions about the underlying distribution of data.

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games

Article

Algorithm for Computing Approximate Nash Equilibrium in Continuous Games with Application to Continuous Blotto

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Abstract: Successful algorithms have been developed for computing Nash equilibrium in a variety of finite game classes. However, solving continuous games—in which the pure strategy space is (potentially uncountably) infinite—is far more challenging. Nonetheless, many real-world domains have continuous action spaces, e.g., where actions refer to an amount of time, money, or other resource that is naturally modeled as being real-valued as opposed to integral. We present a new algorithm for approximating Nash equilibrium strategies in continuous games. In addition to two-player zero-sum games, our algorithm also applies to multiplayer games and games with imperfect information. We experiment with our algorithm on a continuous imperfect-information Blotto game, in which two players distribute resources over multiple battlefields. Blotto games have frequently been used to model national security scenarios and have also been applied to electoral competition and auction theory. Experiments show that our algorithm is able to quickly compute close approximations of Nash equilibrium strategies for this game.

Keywords: continuous game; national security; Blotto game; imperfect information

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1. Introduction

Successful algorithms have been developed for computing approximate Nash equilibrium strategies in a variety of finite game classes, even classes that are challenging from a computational complexity perspective. For example, an algorithm that was recently applied for approximating Nash equilibrium strategies in six-player no-limit Texas hold'em poker defeated strong human professional players [1]. This is an extremely large extensive-form game of imperfect information. Even solving three-player perfect-information strategic-form games is challenging from a theoretical complexity perspective; it is PPAD-hard[1] to compute a Nash equilibrium in two-player general-sum and multiplayer games, and it is widely believed that no efficient algorithms exist [2–4]. Strong algorithms have also been developed for stochastic games, even with multiple players and imperfect information [5]. Stochastic games have potentially infinite duration but a finite number of states and actions.

Continuous games are fundamentally different from finite games in several important ways. The first is that they are not guaranteed to have a Nash equilibrium; Nash's theorem only proved the existence of a Nash equilibrium in finite games [6]. A second challenge is that we may not even be able to represent mixed strategies in continuous games, as they correspond to probability distributions over a potentially (uncountably) infinite pure strategy space. So even if a game has a Nash equilibrium, we may not even be able to represent it, let alone compute it. Equilibrium existence results and algorithms have been developed for certain specialized classes; however, there are still many important game classes for which these results do not hold. Even two-player zero-sum games remain a challenge. For example, the fictitious play algorithm has been proven to converge to Nash equilibrium for finite two-player zero-sum games (and certain classes of multiplayer and nonzero-sum games), but this result does not extend to continuous games [7].

[1] PPAD stands for "Polynomial Parity Arguments on Directed graphs".

A strategic-form game consists of a finite set of players $N = \{1, \ldots, n\}$, a finite set of pure strategies S_i for each player i, and a real-valued utility for each player for each strategy vector (aka *strategy profile*), $u_i : S_1 \times \ldots \times S_n \to \mathbb{R}$. A two-player game is called *zero sum* if the sum of the payoffs for all strategy profiles equals zero, i.e., $u_1(s_1, s_2) + u_2(s_1, s_2) = 0$ for all $s_1 \in S_1, s_2 \in S_2$.

A *mixed strategy* σ_i for player i is a probability distribution over pure strategies, where $\sigma_i(s_{i'})$ is the probability that player i plays $s_{i'} \in S_i$ under σ_i. Let Σ_i denote the full set of mixed strategies for player i. A strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a *Nash equilibrium* if $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$ for all $\sigma_i \in \Sigma_i$ for all $i \in N$, where σ_{-i}^* denotes the vector of the components of strategy σ^* for all players excluding i. It is well known that a Nash equilibrium exists in all finite games [6]. In practice, all that we can hope for in many games is the convergence of iterative algorithms to an approximation of Nash equilibrium. For a given candidate strategy profile σ^*, define $\epsilon(\sigma^*) = \max_{i \in N} \max_{\sigma_i \in \Sigma_i} \left[u_i(\sigma_i, \sigma_{-i}^*) - u_i(\sigma_i^*, \sigma_{-i}^*) \right]$. The goal is to compute a strategy profile σ^* with as small a value of ϵ as possible (i.e., $\epsilon = 0$ indicates that σ^* comprises an exact Nash equilibrium). We say that a strategy profile σ^* with value ϵ constitutes an *ϵ-equilibrium*. For two-player zero-sum games, there are algorithms with bounds on the value of ϵ as a function of the number of iterations and game size, and for different variations ϵ is proven to approach zero in the limit at different worst-case rates (e.g., [8]).

If σ_i^1 and σ_i^2 are two mixed strategies for player i and $p \in (0, 1)$, then we can consider mixed strategy $\sigma_i' = p\sigma_i^1 + (1 - p)\sigma_i^2$ in two different ways. The first interpretation, which is the traditional one, is that σ_i' is the mixed strategy that plays pure strategy $s_i \in S_i$ with probability $p\sigma_i^1(s_i) + (1 - p)\sigma^2(s_i)$. Thus, σ_i' can be represented as a single mixed strategy vector of length $|S_i|$. A second interpretation is that σ_i' is the mixed strategy that with probability p selects an action by randomizing according to the probability distribution σ^1, and with probability $1 - p$ selects an action by randomizing according to σ^2. Using this interpretation implementing σ_i' requires storing full strategy vectors for both σ^1 and σ^2, though clearly the result would be the same as in the first case.

In extensive-form imperfect-information games, play proceeds down nodes in a *game tree*. At each node x, the player function $P(x)$ denotes the player to act at x. This player can be from the finite set N or an additional new player called Chance or Nature. Each player's nodes are partitioned into *information sets*, where the player cannot distinguish between the nodes at a given information set. Each player has a finite set of available actions at each of the player's nodes (note that the action sets must be identical at all nodes in the same information set because the player cannot distinguish the nodes). When play arrives at a leaf node in the game tree, a terminal real-valued payoff is obtained for each player according to utility function u_i. Nash equilibrium existence and computational complexity results from strategic-form games hold similarly for imperfect-information extensive-form games; e.g., all finite games are guaranteed to have a Nash equilibrium, two-player zero-sum games can be solved in polynomial time, and equilibrium computation for other game classes is PPAD-hard.

Randomized strategies can have two different interpretations in extensive-form games. Note that a *pure strategy* for a player corresponds to a selection of an action for each of that player's information sets. The classic definition of a *mixed strategy* in an extensive-form game is the same as for strategic-form games: a probability distribution over pure strategies. However, in general the number of pure strategies is exponential in the size of the game tree, so a mixed strategy corresponds to a probability vector of exponential size. By contrast, the concept of a *behavioral strategy* in an extensive-form game corresponds to a strategy that assigns a probability distribution over the set of possible actions at each of the player's information sets. Since the number of information sets is linear in the size of the game tree, representing a behavioral strategy requires only storing a probability vector of size that is linear in the size of the game tree. Therefore, it is much preferable to work with behavioral strategies than mixed strategies, and algorithms for extensive-form games generally operate on behavioral strategies. Kuhn's theorem states that in any finite extensive-form game with perfect recall, for any player and any mixed strategy, there exists a behavioral strategy that induces the same distribution over terminal nodes as the mixed strategy against all opponent strategy profiles [9]. The converse is also true. Thus, mixed strategies are still functionally equivalent to behavioral strategies, despite the increased complexity of representing them.

Continuous games generalize finite strategic-form games to the case of (uncountably) infinite strategy spaces. Many natural games have an uncountable number of actions; for example, games in which strategies correspond to an amount of time, money, or space. One example of a game that has recently been modeled as a continuous game in the AI literature is computational billiards, in which the strategies are vectors of real numbers corresponding to the orientation, location, and velocity at which to hit the ball [10].

Definition 1. *A continuous game is a tuple* $G = (N, S, U)$ *where*

- $N = \{1, 2, 3, \ldots, n\}$ *is the set of players*
- $S = (S_1, \ldots, S_n)$*, where each* S_i *is a (compact) metric space corresponding to the set of strategies of player* i
- $U = (u_1, \ldots, u_n)$*, where* $u_i : S_1 \times \ldots \times S_n \to \mathbb{R}$ *is the utility function of player* i

Mixed strategies are the space of Borel probability measures on S_i. The existence of a Nash equilibrium for any continuous game with continuous utility functions can be proven using Glicksberg's generalization of the Kakutani fixed point theorem [11]. The result is stated formally in Theorem 1 [12]. In general, there may not be a solution if we allow non-compact strategy spaces or discontinuous utility functions. We can define extensive-form imperfect-information continuous games similarly to that for finite games, with analogous definitions of mixed and behavioral strategies.

Theorem 1. *Consider a strategic-form game in which the strategy spaces* S_i *are nonempty compact subsets of a metric space. If the payoff functions* u_i *are continuous, there exists a (mixed strategy) Nash equilibrium.*

While this existence result has been around for a long time, there has been very little work on practical algorithms for computing equilibria in continuous games. One interesting class of continuous games for which algorithms have been developed is *separable games* [13]; however, this imposes a significant restriction on the utility functions, and many interesting continuous games are not separable. Additionally, algorithms for computing approximate equilibria have been developed for several other classes of continuous games, including simulation-based games [14], graphical tree-games [15], and continuous poker models [16]. The continuous Blotto game that we consider does not fit in any of these classes, and in fact has discontinuous utility functions, so we cannot apply Theorem 1 or these algorithms.

2. Continuous Blotto Game

The Blotto game is a type of two-player zero-sum game in which the players are tasked to simultaneously distribute limited resources over several objects (or battlefields). In the classic version of the game, the player devoting the most resources to a battlefield wins that battlefield and the gain (or payoff) is then equal to the total number of battlefields won. The Blotto game was first proposed and solved by Borel in 1921 [17] and has been frequently applied to national security scenarios. It has also been applied as a metaphor for electoral competition, with two political parties devoting money or resources to attract the support of a fixed number of voters: each voter is a "battlefield" that can be won by one party. The game also finds application in auction theory where bidders must make simultaneous bids [18].

Initial approaches derived analytical solutions for special cases of the general problem. Borel and Ville proposed the first solution for three battlefields [19], and Gross and Wagner generalized this result for any number of battlefields [20]. However, they assumed that colonels have the same number of troops. Roberson computed optimal strategies of the Blotto games in the continuous version of the problem where all of the battlefields have the same weight, for models with both symmetric and asymmetric budgets [21]. Hart considered the discrete version, again when all battlefields have equal weight, and solved it for certain special cases [22]. It was not until 2016 that the first algorithm was provided to solve the general version of the game. Initially a polynomial-time algorithm that involved solving exponential-sized linear programs was presented [23], which was later improved to a linear program of polynomial size [24]. These polynomial-time algorithms are for the discrete version of the game; however, no general algorithm has been devised for the original continuous Blotto game. As described earlier, there are many challenges present for solving continuous games that do not exist for finite games, even for two-player zero-sum games.

Most of the prior approaches solve perfect-information versions of the game in which all players have public knowledge of the values of the battlefields. Adamo and Matros studied a Blotto game in which players have incomplete information about the other player's resource budgets [25]. Kovenock and Roberson studied a model where the players are subject to incomplete information about the battlefield valuations [26]. In both of these works, all players are equally uninformed about the parameters. Recently some work has provided analytical solutions for certain settings with asymmetric information, in which both players know the values of the battlefields but one player knows their order while the other player only knows a distribution over the possible orders [27,28]. This model is an imperfect-information game in which player 1 must select a strategy without knowing the order, while player 2 can select a different mixed strategy conditional on the actual order. We study and present an algorithm for the asymmetric imperfect-information continuous version of the Blotto game, which is perhaps the most challenging variant. Note that our approach also applies to the perfect information version as well.

A continuous Blotto game is a tuple $G = (N, F, O, p, v, B, S, \delta, u)$:

- Set of players $N = \{1, 2\}$
- Set $F = \{1, 2, \ldots, |F|\}$ of battlefields
- Set of slots $Q = F$
- Set O of outcomes, which is a subset of the set of permutations of elements of F, where $o(q)$ denotes the battlefield in slot q for $o \in O$, $q \in Q$. Let $M = |O|$.
- Probability mass function p with $p(o)$ for each $o \in O$
- Positive real value v_f for each battlefield $f \in F$
- Positive real-valued budget B_i for each player $i \in N$
- Pure strategy space of player 1 S_1 is $\{(x_q) \in \mathbb{R}^{|Q|} | \sum_q x_q = B_1, x_q \geq 0 \; \forall q \in Q\}$. Let $s_1(q)$ denote the probability of selecting slot q for $s_1 \in S_1$.
- Pure strategy space of player 2 S_2 is $\{(x_{o,q}) \in \mathbb{R}^{|O||Q|} | \sum_q x_{o,q} = B_2, x_{o,q} \geq 0 \; \forall o \in O \; \forall q \in Q\}$. Let $s_2(o, q)$ denote the probability of selecting slot q under outcome o for $s_2 \in S_2$.
- $\delta \in \mathbb{R} > 0$
- Utility function $u_1(s_1, s_2) = \sum_o p(o) \sum_q C_1(s_1(q), s_2(o, q))$ for $s_1 \in S_1, s_2 \in S_2$, where
  - $C_1(s_1(q), s_2(o, q)) = v_{o(q)}$ if $s_1(q) \geq s_2(o, q) + \delta$,
  - $C_1(s_1(q), s_2(o, q)) = -v_{o(q)}$ if $s_1(q) \leq s_2(o, q)$,
  - $C_1(s_1(q), s_2(o, q)) = 0$ otherwise
- Utility function $u_2(s_1, s_2) = \sum_o p(o) \sum_q C_2(s_1(q), s_2(o, q))$ for $s_1 \in S_1, s_2 \in S_2$, where
  - $C_2(s_1(q), s_2(o, q)) = -v_{o(q)}$ if $s_1(q) \geq s_2(o, q) + \delta$,
  - $C_2(s_1(q), s_2(o, q)) = v_{o(q)}$ if $s_1(q) \leq s_2(o, q)$,
  - $C_2(s_1(q), s_2(o, q)) = 0$ otherwise

Each player must select a real-valued amount of resources to put on the battlefield in slot $q \in Q$, subject to the constraint that the total does not exceed the player's budget B_i.

Player 1 does not know the outcome o, which defines the order of the battlefields; they only know that the outcome is $o \in O$ with probability $p(o)$. Player 2 knows the order and is able to condition their strategy on this additional information. For each slot q, if player 1 uses an amount of resources $s_1(q)$ that exceeds player 2's amount $s_2(o,q)$ by at least δ, then player 1 "wins" the battlefield $o(q)$ in slot q and receives its value $v_{o(q)}$ (and player 2 receives $-v_{o(q)}$); if $s_2(o,q) \geq s_1(q)$ then player 2 wins $v_{o(q)}$ and player 1 loses $v_{o(q)}$; otherwise, both players get zero. This game is clearly zero sum because player 1 and player 2's payoff sum to zero for each situation.

Note that the utility function is discontinuous: payoffs for a given slot can shift abruptly between $v_{o(q)}$, 0, and $-v_{o(q)}$ with arbitrarily small changes in the strategies. This means that Theorem 1 does not apply, and the game is not necessarily guaranteed to have a Nash equilibrium. The game does also not fall into the specialized classes of games such as separable games for which prior algorithms have been developed. Note that often the Blotto game is presented without the δ term; typically player 1 wins the battlefield if $s_1(q) > s_2(o,q)$, and player 2 wins if $s_2(o,q) \geq s_1(q)$. We add in the δ term because our algorithm involves the invocation of an optimization solver, and optimization algorithms typically cannot handle strict inequalities. We can set δ to a value very close to zero.

3. Algorithm

Fictitious play is an iterative algorithm that is proven to converge to Nash equilibrium in two-player zero-sum games (and in certain other game classes), though not in general for multiplayer or non-zero-sum games [7,29]. While it is not guaranteed to converge in multiplayer games, it has been proven that if it does converge, then the average of the strategies played throughout the iterations constitute an equilibrium [30]. Fictitious play has been successfully applied to approximate Nash equilibrium strategies in a three-player poker tournament to a small degree of approximation error [5,31]. More recently, fictitious play has also been used to approximate equilibrium strategies in multiplayer auction [32,33] and national security [34] scenarios. Fictitious play has been demonstrated to outperform another popular iterative algorithm, counterfactual regret minimization, in convergence to equilibrium in a range of multiplayer game classes [35].

In classical fictitious play, each player plays a best response to the average strategies of his opponents thus far. Strategies are initialized arbitrarily (typically they are initialized to be uniformly random). Then each player uses the following rule to obtain the average strategy at time t:

$$\sigma_i^t = \left(1 - \frac{1}{t}\right)\sigma_i^{t-1} + \frac{1}{t}\sigma_i'^t,$$

where $\sigma_i'^t$ is a best response of player i to the profile σ_{-i}^{t-1} of the other players played at time $t-1$. The final strategy output after T iterations σ^T is the average of the strategies played in the individual iterations (while the best response $\sigma_i'^t$ is the strategy actually played at iteration t).

The classical version of fictitious play involves representing two strategies per player; the current strategy σ_i^t and the current best response $\sigma_i'^t$. Note that once we compute the next round strategy σ_i^{t+1} from σ_i^t and $\sigma_i'^{t+1}$, we no longer need to maintain either σ_i^t or $\sigma_i'^{t_1}$ in memory. We interpret σ_i^t as a single mixed strategy that selects action s_j with probability $\left(1 - \frac{1}{t}\right)\sigma_i^{t-1}(s_j) + \frac{1}{t}\sigma_i'^t(s_j)$.

An alternative, and seemingly nonsensical, way to implement fictitious play would be to separately store each of the pure strategies that are played $\sigma_i'^t$, rather than to explicitly average them at each step. Using this representation, the best response can be computed by selecting the pure strategy that maximizes the average (or sum) of the utilities against $\sigma_{-i}'^0, \ldots, \sigma_{-i}'^{t-1}$. This method of implementing fictitious play seems nonsensical for several reasons. First, it involves picking a strategy that maximizes the sum of utilities against t different opponent strategies as opposed to maximizing the utility against a single strategy. And second, it involves storing t pure strategies for each player, which would require using significantly more memory than the original approach when t exceeds $|S_i|$. Despite these clear drawbacks, nonetheless it is apparent that this approach is still equivalent to the original approach and results in the same sequence of strategies being played. When the algorithm is applied to an imperfect-information game, we can view it as operating with mixed as opposed to behavioral strategies (in contrast to prior algorithms for solving imperfect-information games). We refer to this new approach as "Redundant fictitious play" due to the fact that it "redundantly" stores all of the strategies played individually instead of storing them as a single mixed strategy. Redundant fictitious play is depicted in Algorithm 1.

Algorithm 1 Redundant fictitious play for two-player games

Inputs: Number of iterations T

Initialize strategy arrays $S_1[T], S_2[T]$
$S_1[0], S_2[0] \leftarrow$ InitialValues()
$v_1^*[0] \leftarrow u_1(S_1[0], S_2[0])$
$v_2^*[0] \leftarrow u_2(S_1[0], S_2[0])$
for $t = 1$ to T **do**
  $S_1[t] \leftarrow BestResponse1(Mix(S_2, 0, t-1))$
  $S_2[t] \leftarrow BestResponse2(Mix(S_1, 0, t-1))$
  $\epsilon_1[t] \leftarrow u_1(S_1[t], Mix(S_2, 0, t-1)) - v_1^*[t-1]$
  $\epsilon_2[t] \leftarrow u_2(Mix(S_1, 0, t-1), S_2[t]) - v_2^*[t-1]$
  $\epsilon[t] \leftarrow \max_i \epsilon_i[t]$
  $v_1^*[t] \leftarrow u_1(Mix(S_1, 0, t), Mix(S_2, 0, t))$
  $v_2^*[t] \leftarrow u_2(Mix(S_1, 0, t), Mix(S_2, 0, t))$

In Algorithm 1, we store T strategies for each player, where T is the total number of iterations. We can initialize strategies arbitrarily for the first iteration (e.g., to uniform random). For all subsequent iterations the strategy $S_i[t]$ is a pure strategy best response to a strategy of the opponent.[2] The notation $Mix(S_i, 0, t-1)$ refers to the mixed strategy for player i that plays strategy $S_i[u]$ with probability $\frac{1}{t}$, for $0 \leq u \leq t-1$; that is, it mixes uniformly over the strategies $S_i[0], \ldots, S_i[t-1]$. The algorithm then computes the game value to player i under the current iteration strategies as well as the *exploitability* of each player (difference between best response payoff and game value). This determines the maximum amount that each player can gain by deviating from the strategies; we can then say that the strategies computed at iteration $t-1$ constitute an ϵ_t-equilibrium, where $\epsilon_t = \max_i \epsilon_i[t]$.

Now, suppose that G is a continuous game and no longer a finite game. Assuming that we initialize the strategies $S_i[0]$ to be pure strategies, all of the strategies $S_i[t]$ are now pure strategies and the algorithm does not need to represent any mixed strategies. This is very useful, since for continuous games a mixed strategy may be a probability distribution that puts weight on infinitely many pure strategies and cannot be compactly represented. However, pure strategies can typically be represented compactly in continuous games. For example, if the strategy spaces are compact subsets of $\mathbb{R}^n$, then each pure strategy corresponds to a vector of n real numbers, which can be easily represented assuming that n is not too large. For example in continuous Blotto player 1 must select an amount of resource to use for each of $|F|$ battlefields, and therefore storing a pure strategy requires storing $|F|$ real numbers, which is easy to do. Thus, Redundant Fictitious Play can be feasibly applied to continuous games, while the classical version cannot.

The only remaining challenge for continuous games is the best response computation, which may be challenging for certain complex utility functions. However, for the common assumptions that the pure strategy spaces are compact and the utility functions are continuous, this optimization is typically feasible to compute.

[2] Note that there always exists at least one pure-strategy best response to any mixed strategy.

For the continuous Blotto game, we present optimization formulations for computing player 1 and 2's best response below. Both of these are mixed integer linear programs (with a polynomial number of variables and constraints). Note that we are able to construct efficient best response procedures for this game despite the fact that the utility function is discontinuous.

Player 1's best response function is the following, where X_q is a variable denoting the amount of resources put on slot q, and $Y_{t,o,q}$ is the amount of resources put on slot q under outcome o by player 2's fixed strategy at iteration t:
Maximize $\sum_t \sum_o \sum_q \left(p(o) \cdot b_{t,o,q} \cdot v_{o(q)} \right)$ subject to:

$$b_{t,o,q} = 1 \rightarrow X_q \geq Y_{t,o,q} + \delta \text{ for all } t, o, q \tag{1}$$
$$\sum_q X_q = B_1 \tag{2}$$
$$0 \leq X_q \leq B_1 \text{ for all } q \tag{3}$$
$$b_{t,o,q} \text{ binary in } \{0, 1\} \text{ for all } t, o, q \tag{4}$$

The constraints in Equation (1) are called indicator constraints and state that if the binary variable $b_{t,o,q}$ has value equal to 1, then the linear constraint $X_q \geq Y_{t,o,q} + \delta$ must hold. Indicator constraints are supported by many integer-linear program optimization solvers, such as CPLEX and Gurobi. We could additionally impose indicator constraints $b_{t,o,q} = 0 \rightarrow X_q \leq Y_{t,o,q}$; however, these are unnecessary and would significantly increase the size of the problem. To see the correctness of the procedure, suppose that $X_q \geq Y_{t,o,q} + \delta$ and $\sum_q X_q = B_1$ but that $b_{t,o,q} = 0$. Then the objective clearly increases by setting $b_{t,o,q} = 1$ instead to include the additional term $p(o) \cdot b_{t,o,q} \cdot v_{o(q)}$. So there cannot exist another solution satisfying the budget and indicator constraints with higher objective value.

While player 1 must assume that the outcome is distributed according to p, player 2 is aware of the outcome and therefore can condition their strategy on it. Therefore, player 2 solves a separate optimization for each value of $o \in O$ to compute the best response to the strategy of player 1.

Player 2's best response function given outcome $o \in O$ is the following, where Y_q is a variable denoting the amount of resources put on slot q and $X_{t,q}$ is the amount of resources put on slot q according to player 1's fixed strategy at iteration t:
Maximize $\sum_t \sum_q \left(b_{t,q} \cdot v_{o(q)} \right)$ subject to:

$$b_{t,q} = 1 \rightarrow Y_q \geq X_{t,q} \text{ for all } t, q \tag{5}$$
$$\sum_q Y_q = B_2 \tag{6}$$
$$0 \leq Y_q \leq B_2 \text{ for all } q \tag{7}$$
$$b_{t,q} \text{ binary in } \{0, 1\} \text{ for all } t, q \tag{8}$$

Correctness of player 2's best response function follows by similar reasoning to that of player 1's. Player 1's best response optimization has $T'M|Q|$ binary variables $b_{t,o,q}$, where T' is the current algorithm iteration and $M = |O|$ denotes the number of outcomes, and $|Q|$ continuous variables X_q. Since the number of indicator constraints is also $T'M|Q|$, the size of the formulation is $O(TM|Q|) = O(TM|F|)$, which is polynomial in all of the input parameters. Similarly, player 2 must solve M optimizations, each one with size $O(T'|Q|)$. Note that in practice this algorithm could be parallelized by solving each of these $M + 1$ optimizations simultaneously on separate cores as opposed to solving them sequentially (in our implementation we solve them sequentially). However, since player 1's optimization is much larger than each of player 2's, the bottleneck step is player 1's optimization, and such a parallelization may not provide a significant reduction in the runtime.

Note that as we run successive iterations of Algorithm 1, the size of these optimization problems becomes larger, since the opponent's strategy is a mixture over t pure strategies, where t is the current algorithm iteration. We have seen that the number of variables and constraints scales linearly in t. Therefore, we expect earlier iterations of the algorithm to run significantly faster than later iterations. We will see the exact magnitude of this disparity in the experiments in Section 4. A potential solution to this issue would be to include an additional parameter K in Algorithm 1. Instead of computing a best response to the mixture over all t of the opponent's pure strategies, a subset of K of them is selected by sampling and a best response is computed just to a uniform mixture over the pure strategies in the sampled subset. This sampling would occur for each iteration, so a potentially different subset of size K would be selected at each iteration. This would ensure that the complexity of the best response computations remains constant over all iterations and does not become intractable for later iterations. This approach would be unbiased and produces the same result in expectation over the sampling outcomes. However, it may lead to high variance in results and lead to poor convergence in practice. Perhaps this could be mitigated by performing multiple runs of the sampling algorithm in parallel and selecting the run with lowest value of ϵ.

Note that Algorithm 1 can be applied to extensive-form imperfect-information games in addition to simultaneous strategic-form games (in fact the continuous Blotto game that we apply it to has imperfect information for player 1, since player 1 does not know the value of o while player 2 does). As long as pure strategies can be represented and best responses can be computed efficiently (which are both the case for imperfect-information games), the algorithm can be applied. Also note that while we presented the algorithm just for a two-player game, it can also be run on multiplayer games (just as for standard fictitious play). The best response computations are still just a single agent optimization problem given fixed strategies for the opposing players. In fact, fictitious play has been demonstrated to obtain successful convergence to Nash equilibrium in a variety of multiplayer settings [35], despite the fact that it is not guaranteed to converge to Nash equilibrium in general for games that are not two-player zero-sum.

We can compute $v_1^*[t]$ and $\epsilon_1[t]$ for Algorithm 1 in the continuous Blotto game using the procedures depicted in Algorithms 2 and 3 (and analogously for $v_2^*[t]$ and $\epsilon_2[t]$).

Algorithm 2 Procedure to compute $v_1^*[t]$ in continuous Blotto

$v_1^*[t] \leftarrow 0$
for $t_1 = 0$ to t **do**
    for $t_2 = 0$ to t **do**
        for $o \in O$ **do**
            for $q \in Q$ **do**
                if $S_1[t_1](q) \geq S_2[t_2](o,q) + \delta$ **then**
                    $v_1^*[t] \leftarrow v_1^*[t] + p(o)v_{o(q)}$
                else if $S_1[t_1](q) \leq S_2[t_2](o,q)$ **then**
                    $v_1^*[t] \leftarrow v_1^*[t] - p(o)v_{o(q)}$
$v_1^*[t] \leftarrow \frac{v_1^*[t]}{(t+1)^2}$
return $v_1^*[t]$

Algorithm 3 Procedure to compute $\epsilon_1[t]$ in continuous Blotto

$\epsilon_1[t] \leftarrow 0$
for $t_2 = 0$ to $t-1$ **do**
  for $o \in O$ **do**
    for $q \in Q$ **do**
      if $S_1[t](q) \geq S_2[t_2](o,q) + \delta$ **then**
        $\epsilon_1[t] \leftarrow \epsilon_1[t] + p(o)v_{o(q)}$
      else if $S_1[t](q) \leq S_2[t_2](o,q)$ **then**
        $\epsilon_1[t] \leftarrow \epsilon_1[t] - p(o)v_{o(q)}$
$\epsilon_1[t] \leftarrow \frac{\epsilon_1[t]}{t}$
$\epsilon_1[t] \leftarrow \epsilon_1[t] - v_1^*[t-1]$
return $\epsilon_1[t]$

4. Experiments

We experimented on a game with three battlefields f_1, f_2, f_3, with values $v_1 = 0.7$, $v_2 = 0.2$, $v_3 = 0.1$, and three outcomes (each with probability $\frac{1}{3}$):

- Outcome 1 has f_1 in slot 1, f_2 in slot 2, f_3 in slot 3.
- Outcome 2 has f_3 in slot 1, f_1 in slot 2, f_2 in slot 3.
- Outcome 3 has f_2 in slot 1, f_3 in slot 2, f_1 in slot 3.

We assume that player 2 observes the outcome while player 1 does not. We used a budget $B_1 = 10$ for player 1 and $B_2 = 7$ for player 2. We used $\delta = 0.0001$. We used the default feasibility tolerance in Gurobi, which is 1.0×10^{-6}. We ran our algorithm for 5000 iterations and computed ϵ_i for each player every 10 iterations. Recall that we defined the exploitability of the computed strategies at iteration t as $\epsilon[t] = \max_i \epsilon_i[t]$. The experiments did not use any sampling and computed the best response against the opponent's full mixed strategy at each iteration using the mixed-integer linear programs described in Section 3. We used the parallel version of Gurobi's mixed integer linear programming solver [36] with six cores on a laptop.

The results are shown in Figure 1. It took slightly under 25,000 s (around 6.9 h) to run 5000 iterations of our algorithm. The final strategies had an exploitability of 0.0307 for player 1 and 0.0292 for player 2, indicating that the strategies constitute an ϵ-equilibrium for $\epsilon = 0.0307$. (After 5000 additional iterations ϵ decreased further to 0.021.) The exploitability values are not monotonically decreasing, and the lowest value in these experiments was actually obtained with $\epsilon = 0.0259$ at iteration 4480. The expected value for player 1 in the final strategies is -0.10969. The exploitability fell below 0.05 for the first time after 1759.4 s (29.3 min), obtaining $\epsilon = 0.0494$ on iteration 1400. From the figure we can also see that the runtimes varied for the different iterations, as expected (nearly half of the 5000 iterations were completed in the first 5000 s).

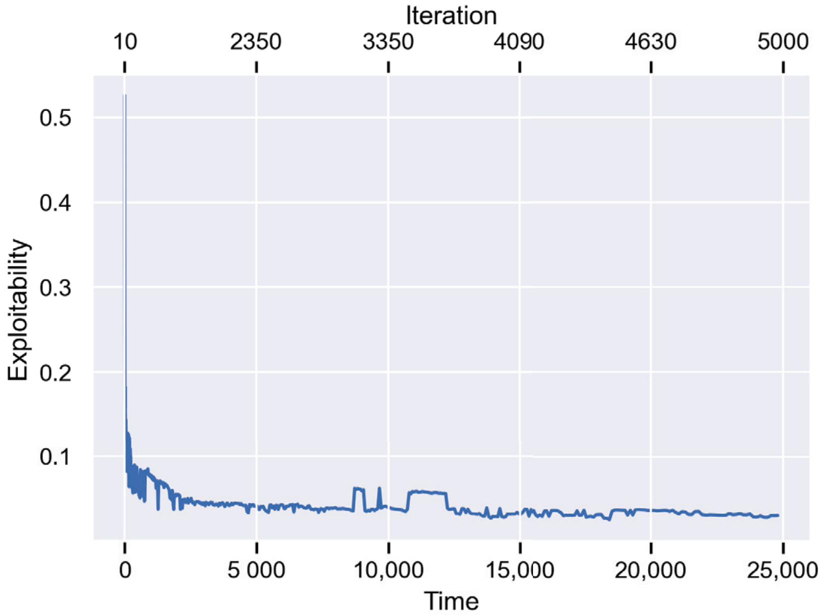

Figure 1. Exploitability (ϵ) vs. runtime (seconds) and algorithm iteration for continuous imperfect-information Blotto game.

5. Conclusion

We presented a new algorithm for computing Nash equilibrium in a broad class of continuous games. The algorithm is based on integrating a novel variant of fictitious play in which the strategies from all iterations are stored with custom best response functions. Solving continuous games is particularly challenging as a Nash equilibrium is not even guaranteed to exist and mixed strategies may put weight on infinitely many pure strategies; yet for many realistic games it is more natural to model strategies as subsets of real numbers than as integers. We implemented our algorithm on a continuous imperfect-information model of the Blotto game, a well-studied model of resource allocation with applications to national security. We created a new mixed-integer linear program formulation for the best response function. We demonstrated that the algorithm converged quickly to an ϵ-equilibrium for ϵ equal to 0.03 after 5000 iterations of the algorithm (several hours), which corresponds to 30% of the minimum battlefield value. While the Blotto game has been studied analytically and efficient algorithms have been developed for the discrete case, this is the first algorithm for solving the continuous case.

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Article

The Path of Terror Attacks

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Abstract: This paper derives a dynamic path of ongoing terror attacks as a function of terrorists' capacity and a target government's counterterror capacity. The analysis provides several novel insights and characterizations. First, the effect of counterterror policy is limited. Second, proactive counterterror policy affects the depreciation (fatigue) of terrorists' capacity, and defensive counterterror policy limits the worst-case scenario. Third, fluctuations in the time path of attacks are a function of terrorists' time preferences and adjustment costs of changing tactics, which are policy invariant. Indeed, in our model, the oscillations of terror attacks occur irrespective of the government's counterterror stance. Fourth, collective action inefficiencies associated with the underprovision of proactive counterterror policies and overprovision of defensive ones are further exacerbated by our finding that proactive counterterror policy is the more effective of the two. Hence, the more effective policy is underprovided.

Keywords: terror cycles; terror paths; counterterror policy; conflict dynamics; asymmetric conflict

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1. Introduction

Terrorist attacks follow cyclical paths, in terms of both tactics used and measurable consequences such as casualties (e.g., Enders et al. [1] (1992); Enders and Sandler [2] (2000); Feichtinger et al. [3] (2001); Faria [4] (2003); Das [5] (2008); Feichtinger and Novak [6] (2008)). Knowledge of the path's determinants is essential for designing efficient counterterror policies, since it identifies both the main variables and parameters associated with government counterterror tactics, and the terrorists' rational use of resources, and gives a time horizon for terror campaigns and the duration of terror organizations. We introduce and analyze a game-theoretic dynamic model generating an explicit cyclical path of terror attacks and the time adjustment for changing tactics.

The stocks of terror and counterterror capacities co-determine the outcome of the necessarily asymmetric conflict between terrorists and target government (The concept of an organization's stock of terror capacity abstracts from issues regarding recruitment and training of militants explicitly examined in Faria and Arce [7,8] (2005, 2012a), Calkins et al. [9] (2008), Udwadia, Leitman, and Lambertini [10] (2006), and Faria [11] (2014). Following Kaplan et al. [12] (2005), the stock of terror capacity is a broad notion constituting the human, physical, and monetary resources used to launch terror attacks. "A terror organization's stock of terror capacity can usefully be viewed as an accumulation of the potential to plan and carry out terror attacks" (Keohane and Zeckhauser [13] (2003, p. 204)). Terrorists' threat capacity includes anything of value to the terrorist, including, but not limited to, its organization, its possessions, a physical or nonphysical commodity, and an information set (Hausken [14] (2008)).

The targeted government's interests lie not so much in terrorists' capacity as in eliminating its effects. In Keohane and Zeckhauser [13] (2003), the analysis of terror capacity is multi-period; however, only the government acts, as the stock of terror capacity is assumed to follow Brownian motion with positive drift. In Hausken's [14] (2008) analysis of terrorists' resource capacity, both terrorists and the government interact strategically, albeit for two periods. Here, the interaction between terrorists and government is both strategic and ongoing.

Terrorism is a form of asymmetric conflict, with asymmetry appearing in our model in two ways. First, we adopt a Stackelberg or leader–follower framework where the government leads and terrorists follow. Consequently, the target government maximizes its payoff, understanding terrorists' strategy will be a best reply to its counterterror policy. Second, the costs of terror and counterterror actions respect Richardson–Lanchester line-of-fire (in)efficiencies associated with asymmetric dynamic conflict (e.g., Avenhaus and Fichtner [15] (1984); Strickland [16] (2011); MacKay [17] (2015); Kress [18] (2020)). Specifically, terrorists' costs of investing in capacity are a function of their investment only, because terrorists have no problems in identifying or locating the target government. That is to say, the interdependence between terrorists and government does not arise on the cost side of terrorists' payoffs. Instead, government counterterror policy affects the benefit side of terrorists' payoffs by partially determining the probability and consequences of a successful attack. By contrast, the government's costs are proportional to the product of its level of investment and terrorists' capacity because the clandestine nature of terror operations creates an asymmetry in the terrorists' favor. In particular, terrorists must be found before their capacity for terror can be targeted.

2. The Model

The model is essentially a stock of (counter)terror capacity competition between the terrorists and targeted government; therefore, the natural framework is dynamic game theory. Define k as the stock of terror capacity and K the stock of counterterror capacity. The stock of terror capacity includes resources terrorists accumulate to support their cause: "a network of supporters; financial capacity; weapons, explosives, and materiel; destructive know-how; a communications network; the tacit approval or even active encouragement of a state or states; trained personnel; and a sufficient number of recruits willing to risk prison or death. The mix of resources may vary greatly from organization to organization, but some accumulated capacity is essential for terror activity" (Keohane and Zeckhauser [13] (2003, pp. 203–4).

For failed states (e.g., Barros et al. [19] (2008) and George [20] (2016)), such as Somalia, $K < k$ holds, while for poor and disorganized regions (e.g., Faria [21] (2008)), such as Russia's interaction with Chechenia, $K \approx k$; for Europe and North America, $K > k$. We focus on the asymmetric case where $K > k$.

2.1. Deriving the Path of Terrorists' Capacity

Our solution concept for each period of our infinite horizon game is Stackelberg equilibrium. Dockner et al. [22] (2000, pp. 135–141) provide an overview of this solution. As the follower, in each state, t, terrorists observe the government's counterterror strategy. As such, the leader (government) anticipates that the follower selects its best reply to the leader's strategy and the leader maximizes its payoff accordingly. Therefore, we start by analyzing the terrorists' problem first in order to derive the terrorist's best reply to the government's strategy at time t.

The terrorists rationally employ their resources to efficiently attack the target government. Function $A(\cdot)$ measures the consequence of terror attacks, including the logistical likelihood of success. Arce [23] (2019) provides measurements of $A(\cdot)$ in terms of disability-adjusted lives lost to terror tactics ranging from suicide bombings to combined firearms/explosives attacks to vehicular assaults. Moreover, the target government's investment in counterterror capacity in the previous period, ΔK_{t-1}, reduces the number or lethality of terror attacks in the current period; i.e., $A(\cdot)$ decreases in ΔK_{t-1}. The benefit portion of the terrorists' payoff therefore takes the form $A(\Delta K_{t-1})k_t$, where $A'(\Delta K_{t-1}) < 0$, and k_t is the attack resources stockpiled by terrorists (e.g., Hausken and Zhuang [24,25] 2011a, b). Coefficient $A(\Delta K_{t-1})$ on k_t indicates the efficacy of the terrorists' stock is a negative function of the government's counterterror policy (Berman and Gavious [26] (2007) analyze a model in which the government chooses cities in which to maximize security, through the location choice of facilities that provide support in case of a terrorist attack.). As such, strategic interdependence arises on the benefit side of terrorists' payoffs.

Turning to the terrorists' costs, they face increasing costs of adjusting their stock of terror capacity, an assumption present in dynamic models of capital investment dating back to at least Gould [27] (1968). Hence, $c(i_t)$, $c'(i_t) > 0$, and $c''(i_t) > 0$, where i_t is the gross investment in k. Adjustment costs correspond to the implicit opportunity costs of foregone terrorism owing to the use of an organization's resources to alter its terror capacity. Such a cost structure implies investment in terror capacity can result in a spectacular attack, but the size or number of terrorist attacks is not unlimited. From the perspective of Richardson's [28] (1939) dynamics of conflict, terrorists cannot engage in an arms race with governments.

The rate of change of the stock of terrorists' capacity is given by

$$\Delta k_t \equiv k_{t+1} - k_t = i_t - \delta k_t \tag{1}$$

where $\delta < 1$ is the terrorists' rate of capacity depreciation ("fatigue" in the parlance of Richardson's (1939) conflict dynamics).

We assume the purchase price of a unit of stock of terrorism capacity is constant and equal to 1. The terrorists' payoff at a point in time is $A(\Delta K_{t-1})k_t - i_t - c(i_t)$. As foreshadowed in the introduction, asymmetric conflict is captured by terrorists' costs being a function of the terrorists' investment, i_t, and not the counterterror investment of the government, I_t, or the government's stock of counterterror capacity, K_t. Moreover, in Richardson–Lanchester approaches to asymmetric dynamic conflict, government targets are "in the open" for terrorists (e.g., MacKay [17] (2015)). By contrast, the government's cost structure, given below, reflects the fact that terrorists are rarely in the open.

The present value of the terrorists' payoffs is

$$\sum_{t=0}^{\infty} \frac{1}{(1+\tau)^t} [A(\Delta K_{t-1})k_t - i_t - c(i_t)] \tag{2}$$

where τ is the terrorists' rate of time preference or its impatience. Terrorists choose the level of investment over time, i_t, to maximize (2) subject to (1), taking the path of the government's counterterror capacity, K_t, as given. The Lagrangian, $\mathcal{L}$, for the terrorists' maximization problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \frac{1}{(1+\tau)^t} \{A(\Delta K_{t-1})k_t - i_t - c(i_t) + q_t[i_t + (1-\delta)k_t - k_{t+1}]\} \tag{3}$$

where the $q_t{}'s$ are the Lagrange multipliers corresponding to the identity in Equation (1) for the evolution of the stock of terror capacity, Δk_t, given i_t for $t = 0, 1, 2, \ldots, \infty$. We denote the Lagrange multipliers by the lower-case q, rather than the more common λ or μ, owing to their relationship to Tobin's (marginal) q. Tobin's q measures the internal value capacity generates for an organization relative to its replacement cost. When $q > 1$, the returns on investment in capacity exceed its costs. Here, q_t is the value to the terrorists of an additional unit of capacity at time t; q_t is the shadow price of Δk_t at the end of period t. Note that the constraint is contained within the braces of $\mathcal{L}$, implying shadow prices, q_t, $t = 0, 1, 2, \ldots, \infty$, are measured in t-period values rather than in present values. Each q_t is discounted by $\frac{1}{(1+\tau)^t}$ in $\mathcal{L}$.

The first-order conditions for the terrorists with respect to i_t are

$$\frac{\partial \mathcal{L}}{\partial i_t} = 0 \Longrightarrow q_t = 1 + c'(i_t) \tag{4}$$

For period $t+1$ this is

$$q_{t+1} = 1 + c'(i_{t+1}). \tag{5}$$

Equations (4) and (5) are Tobin's q for terrorists' investment in terror capacity. In the steady state, the value of q corresponds to the cost of acquiring a unit of capacity (fixed at 1) plus marginal adjustment costs. Given the prices of a unit of capacity, P_{k_t} and $P_{k_{t+1}}$, the investment rule associated with Tobin's q is as follows: investment takes place, $i_t > 0$, if $q_t / P_{k_t} > 1$ (similarly, $i_{t+1} > 0$ if $q_{t+1} / P_{k_{t+1}} > 1$). With prices P_{k_t} and $P_{k_{t+1}}$ normalized to 1, Equations (4) and (5) are consistent with the terrorists' positive level of investment in capacity.

The first-order condition for the path of terror capacity is

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\frac{1}{(1+\tau)^t} q_t + \frac{1}{(1+\tau)^{t+1}} [A(\Delta K_t) + (1-\delta) q_{t+1}] = 0 \tag{6}$$

Inserting the values of q_t and q_{t+1} from (4) and (5) into this relationship yields

$$1 + c'(i_t) = \frac{A(\Delta K_t) + (1-\delta)[1 + c'\,(i_{t+1})]}{(1+\tau)} \tag{7}$$

Equation (7) is the Euler equation for terrorists' capacity investment, capturing the optimal choice between investment today and investment tomorrow when both investments are interior. The terrorists equate the cost of an additional unit of terror capacity in the current period, which is fixed at 1, plus the adjustment costs, to the discounted value of the sum of (i) the return on increased capacity in the next period, $A(\Delta K_t)$, and (ii) the fatigued (depreciated) unit of additional capacity in the next period, along with the associated savings in adjustment costs. The term $A(\Delta K_t)$ reflects the strategic interdependence of the value of an attack, as it is a function of the government's counterterror policy, ΔK_t.

Assuming a simple convex adjustment cost function, $c(i_t) = 0.5 c i_t^2$, from Equations (4), (5), and (7) we have

$$1 + c i_{t+1} = (1-\delta)^{-1}[(1+\tau)(1 + c i_t) - A(\Delta K_t)] \tag{8}$$

Inserting Equation (1) for $i_t = k_{t+1} - k_t(1-\delta)$ into (8) yields

$$k_{t+2} - k_{t+1} = [\tau + \delta - \delta c(1-\delta) k_{t+1} + (1+\tau) c(k_{t+1} - (1-\delta) k_t) - A(\Delta K_t)] \tag{9}$$

Equation (9) characterizes the optimal path of terrorists' capacity as a function of the government's counterterror policy, ΔK_t.

2.2. Examining the Government's Counterterror Policy

We now address the government's problem. Safety from terrorism is a public good, and a government's constituents often hold it accountable for the provision of this public good or lack thereof (Müller [29] (2011); Arce [30] (2020)). Consequently, the government maximizes society's net safety, given by the difference between total safety, $S(K_t)$, and cost term $k_t I_t$. The product $k_t I_t$ captures governments' difficulties in targeting terrorists. As terrorists are clandestine by definition, the cost of targeting terrorists is proportional to the stock of terror capacity. Terrorists must be found prior to being targeted, as is the case in models of dynamic conflict where governments "fire blindly" into an "area" defined by k_t. Kress [18] (2020) calls it, "firing into the brown." By contrast, government targets have to be in the open for terrorism to influence an audience beyond the immediate victims, in agreement with the objective of terrorism in standard definitions of the phenomenon.

The present value of the government's objective function is

$$\sum_{t=0}^{\infty} \frac{1}{(1+\gamma)^t} [S(K_t) - k_t I_t] \tag{10}$$

where γ is the government's rate of time preference (impatience).

In a Stackelberg solution, in each state, t, the government takes the terrorists' best reply function, given by Equation (9), and the rate of change of its stock of counterterror capacity:

$$K_t = I_t - \overline{\delta} K_t \quad (11)$$

as dynamic constraints. Where, $\overline{\delta}$ is the depreciation rate (fatigue) of the government's stock of counterterror capacity.

The Lagrangian, Γ, for the government's maximization problem is

$$\Gamma = \sum_{t=0}^{\infty} \frac{1}{(1+\gamma)^t} \{ S(K_t) - k_t I_t + Q_t [I_t + (1-\overline{\delta})K_t - K_{t+1}] + \mu_t [(1+\tau)(1+ci_t) - A(\Delta K_t) - (1+\delta)(1+ci_{t+1})] \} \quad (12)$$

where the $Q_t{}'s$ are the Lagrange multipliers corresponding to the value to the government of an additional unit of counterterror capacity formation at time t, as given in Equation (11). The use of the letter Q for the Lagrange multipliers for the capacity formation constraints in Equation (12) is again as an indicator of the relationship between these Lagrange multipliers and Tobin's marginal q for government investment in counterterror capacity. The $\mu_t{}'s$ are the Lagrange multipliers measuring the effect on the government of an additional unit of terrorists' capacity formation by terrorists at time t, as given in in Equation (8). As such, the $\mu_t{}'s$ are expected to take negative values because additional terror capacity is detrimental to safety. Once again, the Lagrange multipliers, $Q_0, Q_1, Q_2, \ldots; \mu_0, \mu_1, \mu_2, \ldots$, and associated constraints are measured as t-period values (i.e., contained within the braces of Γ). They are discounted each period by $\frac{1}{(1+\gamma)^t}$.

The first-order condition for the government's maximization problem with respect to I_t is

$$\frac{\partial \Gamma}{\partial I_t} = 0 \Rightarrow Q_t = k_t \ (\Rightarrow \text{ for period } t+1 : Q_{t+1} = k_{t+1}) \quad (13)$$

Unlike Tobin's q for terrorists, Tobin's q for the government, Q_t, exhibits strategic interdependence because it is a function of terrorists' current capacity, k_t. Intuitively, from Equation (13), if $k_t = 0$ then $Q_t = 0$ and, by Tobin's q, the government does not invest in counterterror capacity: $I_t = 0$. This is similarly the case for k_{t+1}, Q_{t+1}, and I_{t+1}. Moreover, by the investment rule for Tobin's q, investment takes place only if $Q_t > 1$. Consequently, if k_t is nominal, i.e., $k_t < 1$, then the government does not invest in counterterror capacity either ($k_t < 1 \Rightarrow Q_t < 1 \Rightarrow I_t = 0$), because the government's (shadow) cost of reducing terror capacity exceeds terrorists' current capacity. As a result, small terror groups/capacities fly below the government's radar as they do not trigger a government response. Continuing:

$$\frac{\partial \Gamma}{\partial i_t} = 0 \Rightarrow \mu_{t+1} = \frac{(1-\delta)(1+\gamma)}{(1+\tau)} \mu_t; \text{ and} \quad (14)$$

$$\frac{\partial \Gamma}{\partial K_{t+1}} = 0 \Rightarrow S'(K_{t+1}) + Q_{t+1}\left(1-\overline{\delta}\right) + \mu_{t+1}\left(A'(\Delta K_{t+1})\right) = (1+\gamma)[Q_t + \mu_t A'(\Delta K_t)] \quad (15)$$

To simplify, we assume safety function $S(K_t) = \overline{S} K_t{}^{\sigma}$. The properties of adjustment costs, $c(i_t)$, allow for spectacular terror attacks, but asymmetry precludes terrorists from engaging in an arms race with the government. As such, let $\overline{A}$ be the maximum potential effect of a terrorist attack, and the parameter g the marginal efficiency of the growth of government's counterterror capacity in curbing terror attacks. It follows that $A(\Delta K_t) = \overline{A} - g\Delta K_t$ and $A'(\Delta K_t) = -g$. Equation (15) becomes

$$\sigma \overline{S} K_{t+1}^{\sigma-1} + Q_{t+1}\left(1-\overline{\delta}\right) - \mu_{t+1} g = (1+\gamma)[Q_t - \mu_t g] \quad (16)$$

2.3. Elimination of Terrorists' Threat

Equation (16) can be further reduced by applying $Q_t = k_t$ and $Q_{t+1} = k_{t+1}$ from Equation (13) and steady-state conditions $K_{t+1} = K_t = K^*$; $\mu_{t+1} = \mu_t = \mu^*$. As discussed above, μ^* must be negative, since an increase in the terror capacity must decrease the optimal value of the government's objective function. Without loss of generality, let $\mu^* = -1$.

Our first major result is highlighted by the following proposition:

Proposition 1. *Only governments who are more impatient than terrorists (i.e., $\gamma > \tau$) find it in their interests to attempt to fully eliminate terrorists' capacity.*

Proof of Proposition 1. The long-run equilibrium level of K^* necessary to fully eliminate terrorists' capacity, i.e., the value of K^* yielding $k = 0$, is obtained from Equation (16) in the steady state and is given by:

$$K^*_{k=0} = \left(\frac{\sigma\overline{S}}{g\gamma}\right)^{\frac{1}{1-\sigma}} \tag{17}$$

For an interior solution, $K^*_{k=0} > 0$, by Equation (14), the steady state where $\mu_{t+1} = \mu_t = \mu^*$ yields $\frac{1+\tau}{1+\gamma} = 1 - \delta < 1$. Consequently, only governments who are more impatient than terrorists (i.e., $\gamma > \tau$) find it in their interests to attempt to fully eliminate terrorists' capacity. □

Yet, it unlikely that targeted governments are less impatient than terrorists because the lifespan of terrorist organizations is relatively short (Vittori [31] (2009; Faria and Arce [32] (2012b); Gaibulloev and Sandler [33] (2013)). Consequently, when governments are the more patient of the two, they do not find it optimal to set K^* such that $k = 0$. Instead, patient governments treat terrorism as an ongoing phenomenon. The ongoing interaction between governments and terrorists is the subject of the following section.

3. Ongoing Terrorism and the Dynamic Path of Terror Capacity

An ongoing terrorist threat occurs when the government is more patient than terrorists. In order to characterize the dynamic path of terror capacity, we analyze the terrorists' (follower's) response by substituting Equation (17) into Equation (9). From Equation (11), when $I_t \neq K^*_{k=0}$, terrorists' capacity follows the following dynamic path:

$$k_{t+2} - k_{t+1} = [c(1-\delta)]^{-1}\left[\tau + \delta - \delta c(1-\delta)k_{t+1} + (1+\tau)c(k_{t+1} - (1-\delta)k_t) - \overline{A} + g(I_t - \overline{\delta}K^*)\right] \tag{18}$$

The presence of terms k_{t+2}, k_{t+1}, and k_t in Equation (18) implies the path of terrorists' capacity is a second-order linear difference equation. Solving the equation involves dividing it into two parts: a particular solution and a homogenous solution. We begin by deriving a solution particular to the steady state: $k_{t+2} = k_{t+1} = k_t \neq 0$. The *particular solution*, k_p, is

$$k_p = \left(\frac{\overline{A} - g(I_t - K^*) - \tau - \delta}{c\delta(\tau + \delta)}\right) \tag{19}$$

The second part of the solution to Equation (18) is the *homogenous solution*, so named because it corresponds to the case where the constant term in Equation (18), $\overline{A} - g(I_t - K^*) - \tau - \delta$, is zero. The trivial solution takes the form $k_{t+2} = k_{t+1} = k_t = 0$. The nontrivial solution is typically derived by assuming the homogenous solution takes the same form as the solution to a first-order difference equation: $k_t = \lambda^t$, where $\lambda \neq 0$ is an unknown constant interpreted as an eigenvalue (characteristic root). Setting $\overline{A} - g(I_t - K^*) - \tau - \delta = 0$ and substituting $k_t = \lambda^t$ into Equation (18) gives the following:

$$\lambda^t\left[\lambda^2 - \left(1 - \delta + \frac{1+\tau}{1-\delta}\right)\lambda + (1+\tau)c\right] = 0 \tag{20}$$

where the term in brackets is the characteristic equation for Equation (18). The homogeneous solution has two roots:

$$\lambda_1 = \frac{\left(1-\delta+\frac{1+\tau}{1-\delta}\right)+\sqrt{\left(\delta-1-\frac{1+\tau}{1-\delta}\right)^2-4(1+\tau)c}}{2} \tag{21}$$

$$\lambda_2 = \frac{\left(1-\delta+\frac{1+\tau}{1-\delta}\right)-\sqrt{\left(\delta-1-\frac{1+\tau}{1-\delta}\right)^2-4(1+\tau)c}}{2} \tag{22}$$

As δ is the rate of terror capacity depreciation (fatigue), it follows that $\delta < 1$ and both roots are complex. The complex roots can be written as conjugate pairs $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, where, from Equation (20), $\alpha = \frac{\left(\delta-1-\frac{1+\tau}{1-\delta}\right)}{2}$ and $\beta = \frac{\sqrt{4(1+\tau)c-\left(\delta-1-\frac{1+\tau}{1-\delta}\right)^2}}{2}$. The complex conjugate pairs correspond to the solutions $k_t = \lambda_1^t = (\alpha+i\beta)^t$ and $k_t = \lambda_2^t = (\alpha-i\beta)^t$. Adopting polar representations, $\lambda_1 = \alpha + i\beta = \sqrt{\alpha^2+\beta^2}\cdot(\cos\theta + i\sin\theta)$ and $\lambda_2 = \alpha - i\beta = \sqrt{\alpha^2+\beta^2}\cdot(\cos\theta - i\sin\theta)$, where $\sin\theta = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$ and $\cos\theta = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$.

For these complex roots, the two corresponding homogenous solutions, $k_h^{(1)}$ and $k_h^{(2)}$, are

$$k_h^{(1)} = \left(\sqrt{(1+\tau)c}\right)^t(\cos\theta t + i\sin\theta t) \text{ and } k_h^{(2)} = \left(\sqrt{(1+\tau)c}\right)^t(\cos\theta t - i\sin\theta t) \tag{23}$$

By the superposition principle, if $k_h^{(1)}$ and $k_h^{(2)}$ are solutions to a homogenous difference equation, then so is

$$k_h = C_1 k_h^{(1)} + C_2 k_h^{(2)} \tag{24}$$

where C_1 and C_2 are arbitrary constants.

Since k_t must be a real number, the homogenous solution must be a real number. As $k_h^{(1)}$ and $k_h^{(2)}$ are imaginary, then it must be the case that C_1 and C_2 are imaginary as well. Hence, C_1 and C_2 can also be expressed as complex conjugates:

$$C_1 = \hat{\alpha} + i\hat{\beta} = \sqrt{\hat{\alpha}^2+\hat{\beta}^2}\cdot(\cos\theta + i\sin\theta);\ C_2 = \hat{\alpha} - i\hat{\beta} = \sqrt{\hat{\alpha}^2+\hat{\beta}^2}\cdot(\cos\theta - i\sin\theta) \tag{25}$$

By substituting these values for C_1 and C_2 into Equation (24), Goldberg [34] (1986, p. 140) provides the steps for reducing Equation (24) to

$$k_h = 2\hat{C}_1\left(\sqrt{(1+\tau)c}\right)^t\cos(\theta t + \hat{C}_2), \tag{26}$$

where $\hat{C}_1$ and $\hat{C}_2$ are arbitrary real constants and θ is the same as before. Interestingly, the homogenous solution depends only on the terrorists' impatience, τ, and adjustment costs, c.

We now present the main result characterizing the dynamic path of terrorists' capacity accumulation. The complete solution for the path of terrorists' capacity requires combining the homogenous solution, k_h, with the particular solution, k_p (Intuitively, if k_t and k_p are solutions to a difference equation with nonzero constant term, then $k_t - k_p$ is a solution to the homogenous version of the difference equation). That is, $k_t = k_h + k_p$, yielding

$$k_t = 2\hat{C}_1\left(\sqrt{(1+\tau)c}\right)^t\cos(\theta t + \hat{C}_2) + \underbrace{\left(\frac{\overline{A} - g(I_t - K^*) - \tau - \delta}{c\delta(\tau+\delta)}\right)}_{k_p} \tag{27}$$

As the cosine function oscillates, the path of terror capacity exhibits a fluctuating pattern periodic in nature. The path is a stepped fluctuation of discrete points (it is only smooth for continuous time), oscillating between values above and below $k_p = \left(\frac{\overline{A}-g(I_t-K^*)-\tau-\delta}{c\delta(\tau+\delta)}\right)$. What matters in our context is convergence, as determined by the term $\left(\sqrt{(1+\tau)c}\right)^t$. Three possibilities emerge.

Case 1: $\sqrt{(1+\tau)c} > 1$. Here, k_t oscillates with ever-increasing amplitude, implying a divergent and explosive path of terrorists' capacity accumulation. Such an outcome is not possible because terrorists' limited resources are the defining feature of terrorism as asymmetric conflict; i.e., $K_t \gg k_t$. Alternatively, for the situation of failed states, $K_t < k_t$, this case identifies when terrorists win.

Case 2: $\sqrt{(1+\tau)c} = 1$. The solution is an equilibrium solution. Here, k_t oscillates (about k_p) with constant amplitude.

Case 3: $\sqrt{(1+\tau)c} < 1$. Here, k_t oscillates with monotonic-decreasing amplitude and converges to k_p as $t \to \infty$. This holds iff $c < \frac{1}{1+\tau}$.

Cases 2 and 3 are relevant for the present study because terrorists generally do not have the resources to engage in an arm's race with targeted governments. Several novel observations arise from the characterization of the dynamics of the capacity accumulation path given in Equation (27). First, $c \leq \frac{1}{1+\tau}$ relates terrorists' adjustment (opportunity) cost of investing in new terror capacity (foregone terrorism) to terrorists' discount factor. In particular, patient terrorists can exhibit a variety of tactics over their lifespan because their patience (low τ) allows for the associated higher adjustment costs. By contrast, impatient terrorists will not forestall attacks in order to accommodate the adjustment costs associated with a portfolio of tactics. No direct counterterror policy prescription follows from the $c \leq \frac{1}{1+\tau}$ characterization, as neither τ nor c are policy variables. They are, instead, the terrorists' primitives. Impatience term τ stems from the terrorists' time preferences, and counterterror policy has no effect on adjustment costs, c, which are measured in terms of the attacks terrorists are willing to forgo to adjust their stock of terror capacity.

Second, a lull in terror activity need not be indicative of successful counterterror policy. Instead, it can be due to patient terrorists undergoing the adjustment costs associated with a forthcoming wave of new tactics. For example, Enders and Sandler [2] (2000, p. 323) employ time series analysis to show that "authorities should focus on anticipating upturns in incidents involving casualties following fairly length lulls of greater than two years". Moreover, the authors identify the period immediately prior to the yet-to-occur events of September 11, 2001 as being the longest lull on record. The tactical innovation of simultaneous coordinated skyjackings to employ airliners as weapons during 9/11 reveals Al Qaeda's willingness to undertake the adjustment costs stemming from its meticulous preparations prior to the attacks. By contrast, the spate of vehicular assaults incited by ISIS during the late 2010′s required little in the way of adjustment costs; as instructions were distributed online, many of the vehicles were stolen or rented, and the operatives were at arms-length (Siqueira and Arce [35] 2020).

Proposition 2. *Fluctuations in the time path of attacks are a function of terrorists' time preferences and adjustment costs of changing tactics, which are policy-invariant.*

Proof of Proposition 2. The government has no control over the oscillatory component of the time path of terrorism, as the terms surrounding the cosine function in Equation (27), c and τ, are the terrorists' primitives.

Fourth, the time path characterized in Equation (27) is akin to a time-variant system with k_p as the input and k_t the output. This begs the question as to the government's degree of control (over k_p). In the steady state (i.e., $\Delta K_t = 0 \Rightarrow I_t = K^*$), k_p reduces to

$$k_p^* = \frac{\overline{A}-\tau-\delta}{c\delta(\tau+\delta)} \tag{28}$$

Counterterror policies traditionally fall into two broad categories: proactive and defensive (e.g., Frey [36] (2004); Arce and Sandler [37] (2005); Sandler and Siqueira [38] (2006); Bandyopadhyay and Sandler [39] (2011); Bier and Hausken [40] (2011)). Proactive policies include attacking terrorists' training grounds and freezing the assets of supporting organizations. Proactive policies directly target the stock of terror assets, i.e., they increase fatigue term δ. By contrast, defensive policies, such as hardening targets and controlling the inflow of potential terrorists' immigrants or refugees, limit the upper bound on terrorists' capacity, $\overline{A}$ (For alternatives to defensive strategies see (Frey and Luechinger [41] 2003)). Moreover, the literature on the collective action problems associated with proactive and defensive policies most often treats terrorists as a passive third party. Equation (28) provides the direct link between proactive and defensive counterterror policies and the actions of terrorists themselves. In particular, $k_p = 0$ when the mix of defensive and proactive counterterror policies satisfies $\overline{A} - \delta = \tau$. *Counterterror policy is formulated with reference to terrorists' impatience*. Under these circumstances, terrorists' capacity is not zero but instead oscillates about $k_p = 0$ if the government has the requisite winningness, resources, intelligence, etc., to set $\overline{A} - \delta = \tau$ by decreasing $\overline{A}$ via defensive policy and increasing δ via proactive policy. □

Fifth, in the absence of the requirements sufficient to set $\overline{A} - \delta = \tau$, government control over the time path of terror capacity via counterterror policy is characterized by

$$\frac{\partial k_p^*}{\partial \overline{A}} = \frac{1}{c\delta(\tau+\delta)} \tag{29}$$

$$\frac{\partial k_p^*}{\partial \delta} = \frac{-c\delta(\tau+\delta) - \left(\overline{A} - \tau - \delta\right)[c(\tau+\delta) + c\delta\tau]}{[c\delta(\tau+\delta)]^2} \tag{30}$$

An increase in defensive counterterror policy decreases $\overline{A}$, leading to a decrease in the stock of terror capacity given by Equation (29). Similarly, an increase in proactive counterterror policy increases δ, leading to a decrease in the stock of terror capacity given by Equation (30).

Sixth, from the perspective of international collective action and the coordination of counterterror policy (e.g., Faria et al. [42] 2020), defensive counterterror policies are strategic complements and proactive ones are strategic substitutes (Sandler and Siqueira [38] (2006); Faria et al. [43] (2017).) Accordingly, governments overuse defensive policies and underprovide proactive ones. The characterizations given in Equations (29) and (30) shed further light on these inefficiencies, giving rise to the following proposition.

Proposition 3. *Proaction is both underprovided and more effective compared with defensive counterterror policy.*

Proof of Proposition 3. From Equations (29) and (30), $\left|\frac{\partial k_p^*}{\partial \delta}\right| > \left|\frac{\partial k_p^*}{\partial \overline{A}}\right|$. □

Corollary. *At the same time, neither proactive nor defensive counterterror policies affect the ebb and flow of terrorists' actions; the amplitude of terrorism is determined by terrorists' primitives c and τ. Hence, the fluctuations in our model are consistent with terrorists deliberately undertaking what appear to be uncertain (time-variant) actions on their part.*

4. Conclusions

This paper considers a dynamic game of terror and counterterror capacity accumulation between terrorists and a target government in order to provide a full characterization of their ongoing asymmetric conflict in terms of the time path of terrorists' capacity. The term "ongoing conflict" is used because the necessary condition for governments' willingness to attempt to fully eliminate terrorists' capacity is for the government to be more impatient than the terrorists. As it is well known that terrorist groups are short-lived relative to their target governments (excluding failed or organizationally disadvantage states), the necessary condition on relative time preferences is unlikely to be met. Consequently, targeted governments purposefully treat terrorism as an ongoing phenomenon.

Within this context, an advantage of dynamic models producing time paths of terror activity is they can be "tested, evaluated, and improved upon through the use of actual field data" (Strickland [16] (2011, p. 161)). While such an exercise is a future research direction, it is beyond the scope of the present analysis.

At the same time, the analysis provides several novel insights. For example, governments' resignation to terrorism's persistence is not akin to an "optimal negative externality" argument, such as occurs for pollution abatement. In the case of pollution abatement, the presence of diminishing marginal social benefits and increasing marginal social costs leads to a positive level of (optimal) pollution (Mishan [44] 1974). By contrast, only the base accumulation of terror capacity around which oscillations occur is determined by policy. We characterize how proactive counterterror capacity affects the depreciation (fatigue) of terrorists' capacity and how defensive counterterror policy limits the worst-case scenario. The effectiveness of such policies is a function of terrorists' primitives (time preferences and adjustment costs of changing tactics), which are policy-invariant.

Accordingly, the effect of counterterror policy is limited. Oscillations in the time path of terror capacity are a function of terrorists' primitives. Consequently, the amplitude of terrorism only converges to zero in the long run. Once again, such dampening is determined by terrorists' primitives, rather than counterterror policy. As such, terrorists' willingness to make the impatience–adjustment cost tradeoff is the root determinant of their longevity.

Counterterror policy is therefore plagued by inefficiencies and paradoxes. The ebb and flow of terror tactics results from terrorists trading impatience for improved tactics and their associated adjustment costs. Such lulls in activity and changes in tactics are observationally equivalent to terrorists substituting tactics in response to defensive counterterror policies (e.g., security screening in airports) and yet they may have absolutely nothing to do with counterterror tactics. Indeed, in our model, the oscillations occur *irrespective* of the government's counterterror stance. In addition, collective action inefficiencies associated with the underprovision of proactive counterterror policies and overprovision of defensive ones are further exacerbated by our finding that proactive counterterror policy is the more effective of the two. Hence, the more effective policy is underprovided. This is a novel characterization of counterterror policy relative to the extant literature.

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EU Demand for Defense, 1990–2019: A Strategic Spatial Approach

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Abstract: For 1990–2019, this study presents two-step GMM estimates of EU members' demands for defense spending based on alternative spatial-weight matrices. In particular, EU spatial connectivity is tied to EU membership status, members' contiguity, contiguity and power projection, inverse distance, and arms trade. At a Nash equilibrium, our EU demand equations are derived explicitly from a spatially based game-theoretical model of alliances. Myriad spatial linkages among EU members provide a robust free-riding finding, which differs from the spatial and non-spatial literature on EU defense spending. Even though the EU applies common trade policies and allows for unrestricted labor movement among members, members' defense responses adhered to those of a defense alliance. Moreover, EU defense spending exhibits positive responses to GDP and transnational terrorist attacks, and a negative response to population. During the sample period, EU members did not view Russia as a military threat.

Keywords: European Union (EU), spatial autoregression and connectivity; alliance; strategic free riding; Nash equilibrium

JEL Classification: D74; H41; C21

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1. Introduction

First introduced by Olson and Zeckhauser [1], the economic theory of alliances is a game-theory-based representation that continues to exert major influences on empirical studies of the demand for military expenditure (ME) (see, e.g., Douch and Solomon [2]; Dudley and Montmarquette [3]; Hilton and Vu [4]; Kim and Sandler [5]; McGuire and Groth [6]; Smith [7,8]). That theory emphasizes the nonrival and nonexcludable benefits that one ally's defense provision confers on other allies. Defense is nonrival among allies because one ally's consumption of defense-derived deterrence of potential adversaries does not detract, in the least, from the deterrence gained by other allies from each ally's defense provision. Non-excludability of benefits characterizes defense since, once provided, all allies receive the associated benefits when allied countries are united through common interests (e.g., joint infrastructure, resident citizens, resource supply lines, trade, and foreign direct investment). The two publicness properties of defense result in alliance free riding in which allies do not fully reveal their true preferences for defense by relying to some extent on the defense provision (spillovers) of their allies [1,9]. Consequently, free riding motivates an ally to reduce its defense spending in response to collective increases in that of the other allies [10,11]. This negative relationship between defense spending and defense spillovers provides an easy-to-implement test of the free-riding prediction. The more one ally views its defense as a substitute for the ME of its allies, the greater is the downward slope of the defense reaction curve in terms of spillovers. Another theoretical prediction of alliance theory is that large, rich allies carry a disproportionately large defense burden for poor allies in terms of their GDP devoted to defense (i.e., ME/GDP) [1,12]. If, however, defense spending gives rise to alliance-wide and country-specific, jointly produced outputs, then the anticipated negative reaction to defense spillovers and the expected disproportionate burden sharing may be curbed or even reversed [13–15]. In the former case, free riding is reduced.

The primary purpose of the current study is to present spatial-based, two-step generalized method of moments (GMM) estimates for European Union (EU) members' demands for ME during 1990–2019 and a post-2007 subperiod. Our spatial autoregression (SAR) estimates correspond to seven spatial-weight matrices that capture EU membership, members' contiguity with one another, members' contiguity with Russia, inverse distance between countries' capitals, arms trade among countries, and two alternative power projection representations. Even though the EU is not a traditional defense alliance that commits members to view an attack on one as an attack on all with the pledge of a collective response as does Article 5 of North Atlantic Treaty Organization (NATO), the EU faces concerns that may evoke a coordinated response to a security threat. Hence, EU members' defense possesses publicness properties made more poignant by the nearness of countries that raise free-rider concerns and justifies the economics of alliances as a theoretical foundation for defense demands [16]. Previous nonspatial investigations of European countries' defense spending do not uncover free riding due to quite different modeling assumptions, especially in regard to defense spillovers (e.g., [2,17,18]).

The current study is most similar to Xiaoxin and Bo [19] who also apply a spatial approach to a sample of European countries for 2000–2018. Like the current study, Xiaoxin and Bo [19] consider spatial connectivity in terms of contiguity, inverse distance, and arms trade. However, crucial differences distinguish the two investigations. First, the current study has ME, and not ME/GDP, as its dependent variable. As a consequence, we uncover evidence of EU free riding in terms of ME, while they find defense burdens responding positively to the defense burdens of other European countries, consistent with defense burden convergence. Second, the current analysis accounts for US defense spending and power projection, which is not the case for Xiaoxin and Bo [19]. Third, unlike Xiaoxin and Bo [19], we only include EU members in the sample after they join the EU. We are interested in the EU members' behavior and not that of European countries in general, since we view the EU institution as an important driver of defense responses. Our set of countries during each year differs from that of Xiaoxin and Bo [19] because non-EU countries, not part of NATO, such as Belarus, Ukraine, and others, are excluded from all runs in the current study. Fourth, the threat variables greatly differ between the two studies. Fifth, we present strong evidence of free riding that is not solely based on arms trade as in Xiaoxin and Bo [19]. The current study is complementary to their interesting analysis; both analyses have different insights to offer.

The present study possesses some noteworthy findings about EU members' defense spending. Based on seven measures of spatial linkage, there is robust evidence of defense free riding among EU members during the last three decades. Free riding is particularly strong after 2007. Additionally, free riding characterizes EU countries when augmented by some non-EU NATO allies. For alternative threat measures involving Russia, EU countries are not viewing Russia as a threat. EU defense spending generally responds negatively to population, consistent with social welfare spending crowding out defense spending as population size increases. Transnational terrorist attacks against EU assets (i.e., people and property) consistently increase EU defense spending when Russian ME is included as a threat variable. We also show that US power projection supports EU free riding, even though the United States does not belong to the EU.

The remainder of the paper contains six sections. A brief literature review is contained in Section 2. Section 3 presents some relevant background on the EU, including how it expanded over the years and its security concerns. In Section 4, we indicate the game-theoretic model behind our EU defense demand equations. Section 5 offers our empirical methodology and data sources, while Section 6 interprets the empirical findings. Concluding remarks and policy implications are gathered in Section 7.

2. Literature Review

In recent years, spatial econometric techniques are fruitfully applied to the estimation of defense demand in alliance and non-alliance settings. For instance, Flores [20], Goldsmith [21], and Skogstad [22] investigate spatial-based defense demands for 168, 120, and 124 countries, respectively, while George and Sandler [23] examine defense demands for the NATO alliance. The first three studies use ME/GDP as the measure of defense demand, while the fourth study casts ME as its measure of defense demand. Another noteworthy spatial analysis of defense demand is by Yesilyurt and Elhorst [24] who investigate the determinants of defense burdens (ME/GDP) for 144 countries using four spatial econometric models and eight different spatial-weight matrices.

The studies employing ME/GDP as the dependent variable generally find a positive response of one country's defense burden to those of other spatially tied or neighboring countries—a result in potential opposition to free riding. By contrast, George and Sandler [23] estimate a robust negative relationship between an ally's ME and the aggregate ME of other spatially linked allies, consistent with free riding. Their approach and the one applied here derive the set of allies' defense demand equations from a constrained optimization that accounts for resources, defense spillovers, and other strategic considerations, so that ME is implicitly defined by the first-order conditions (FOCs) of each country's decision maker. By contrast, articles with ME/GDP as the dependent variable replace ME/GDP for defense spending without clearly showing how this switch follows from the underlying theory—e.g., what utility function or modeling assumptions are consistent with defense burden, not ME, being the dependent variable.

The literature's estimated positive relationship among countries' defense burdens may arise from the tendency for these burdens to converge to a common value [25,26]. Once converged or nearing convergence, defense burdens are apt to rise and fall in sync, making for a positive burden-sharing relationship among allies. That positive relationship may also arise from an alliance mandating allies' adherence to having a set percentage of GDP devoted to ME—e.g., the 2014 Wales Summit two percentage of GDP rule for NATO. Additionally, a country's defense burden is affected by its defense spending and GDP, which can mask the causal determinants of ME demand. Our methodology is to stay with the more traditional approach of casting ME as the dependent variable and GDP as one of the independent variables [4,6].

Spatial econometric representations of defense demand permit a variety of spatial connectivities. For a 1991 cross section, Goldsmith [21] includes contiguity among countries and the inverse distance between countries' capitals as the key measures of spatial connectivities. With inverse distance, a larger separation between countries' capitals results in smaller weights being placed on their defense efforts because one country's forces would take longer to be redeployed to defend the other country's assets, resulting in less defense spillovers. Flores [20] allows for spatial weights based on contiguity, alliance membership, and set distances, while Skogstad [22] includes spatial weights based on contiguity, inverse distance, and power projection. Both of these empirical analyses involve cross sections. For a panel of European countries, Xiaoxin and Bo [19] employ spatial connectivity based on contiguity, inverse distance, and arms trade. For NATO, George and Sandler's [23] spatial weights stem from contiguity, inverse distance, and power projection for a panel covering 1968–2015 and select subperiods.

3. On the EU

In 1951, the Treaty of Paris established the European Coal and Steel Community (ECSC) to regulate some industrial outputs in its six members (Belgium, France, Italy, Luxembourg, the Netherlands, and West Germany). ECSC was the first supranational organization to link sovereign European states to foster economic coordination. European integration progressed further with the Treaty of Rome in 1957 and the creation of the European Economic Community (EEC) with its six inaugural members being those of the ECSC [27] (The background facts for this paragraph come from the EU Enlargement Factsheet [27]. Unified Germany replaced West Germany in 1990). From EEC's outset, enlargement was allowed for European countries that respect the rule of law, the principles of democracy, the preservation of civil liberties, the protection of human rights, the furtherance of equality, and the rights of minorities. Until 2019, there have been seven enlargement waves that raised membership from 6 to 28 countries. Denmark, Ireland, and the UK joined in 1973; Greece entered in 1981; Spain and Portugal became members in 1986; Austria, Finland, and Sweden joined in 1995; Cyprus, the Czech Republic, Estonia, Hungary, Latvia, Lithuania, Malta, Poland, Slovakia, and Slovenia enlisted in 2004, Bulgaria and Romania entered in 2007; and Croatia joined in 2013. The UK left the EU in 2020, but was a member during the years covered by our empirical study. Another eight countries are in the accession process, but were not members in 2019. For the purposes of the baseline empirical analysis, we consider 28 member countries according to their accession date if admitted after 1990.

The EEC was initially intended to promote greater economic and political cooperation in Europe among members. Economic cooperation involves the elimination of trade barriers within the EEC and the establishment of a common trade policy with non-EEC members, which meant that the EEC started as a customs union. At first, political cooperation took the form of a common agricultural policy. The Treaty on European Union (also known as the Maastricht Treaty) was enacted in 1991 and entered into force in November 1993 when the EEC changed its name to the EU. Political union within the EU was enhanced over time by supranational infrastructure in the form of the Council of the EU, the European Commission, European Court of Justice, and the European Parliament. The latter's legislators are elected by member countries' voters. Within the EU, political policy stances are taken with respect to the environment (e.g., climate change), human rights, employment practices, justice, foreign affairs, and security. When the Schengen Agreement of 1985 allowed for the unrestricted movement of EU citizens, the EU transformed itself from a customs union to a common market.

Since 1990, the EU faces many common security challenges including nearby internal conflicts in Bosnia and Herzegovina during 1992–1995 among Serbs, Croats, and Muslims, and an armed conflict in the late 1990s between ethnic Serbs and Albanians. If those conflicts had not been quelled, they could have spread to neighboring countries and affected EU commerce and other interests. Even the Chechen conflict spilled over to the EU in the form of transnational terrorist attacks (e.g., Belgium, France, Germany, Spain, and the UK), which raised security issues in terms of efforts to coordinate counterterrorism policies and to share intelligence. Coordination is needed because enhanced counterterrorism actions taken by one EU member may transfer a planned attack to a less-secure neighbor country [28].

Starting in 2011, civil wars in Libya and Syria brought security externalities to the EU through refugee flows, disrupted resource flows, and transnational terrorism. In the case of Libya, NATO-led operations to depose Muammar Qaddafi resulted in a failed state that continues to pose security threats to EU interests. To address nearby internal conflicts in the Middle East, Africa, and Central Asia, the EU must engage in and support peacekeeping operations (PKOs) through the United Nations and other regional organizations.

The rising nationalism of the Putin regime poses an external threat, especially for EU countries near or adjoining Russia. The threat is more poignant in light of Russia's annexation of Crimea in 2014, after which Russian-based rebel incursions in eastern Ukraine added to EU security worries. However, the question remains whether the most at-risk EU countries are responding appropriately to this potential threat with increased defense pending.

4. Spatial-Based Theoretical Model of EU Member's Defense Demand

We formulate a theoretical model for an EU member's demand for defense spending based on the aggregate ME of other members, threat proxies, and other country-specific exogenous considerations. When relating a country's ME to those of other countries, we primarily consider the members of a particular common market such as the EU. The EU's Common Security and Defense Policy (CSDP) is tied integrally to internal and external security threats of member states [29] in which the CSDP promotes EU military coordination, EU-supported peacekeeping activities, EU-NATO cooperation, defense industrial development, counterterrorism cooperation, and other EU security policies. EU peacekeeping activities may involve the deployment of forces drawn from member states or the funding of peacekeeping operations by other regional organizations (e.g., the African Union) [30]. Although the EU is not a defensive alliance, EU members' security is joined by geography, similar security interests, economic concerns, and common threats. That connectivity means that an EU member's defense demand is dependent, in part, on other members' ME, so that EU membership is a basis for specifying members' strategic defense demands, analogous to a military alliance.

We present an underlying theoretical model for an EU member's defense demand that is sufficiently general to allow for alternative spatial considerations arising from members' geographical location vis-à-vis one another and that of a potential adversary [20,22,23,31,32]. Additionally, the model permits EU members' ME to be influenced by non-EU NATO allies (e.g., Norway, Turkey, and the United States), transnational terrorist attacks, and arms-trade connectivity. Alternative spatially weighted defense spillovers allow the defense demand representation to be particularized to myriad connectivity scenarios.

We commence with an N-member common market scenario where each member's ME decision is made by a unitary entity (i.e., the country's executive decision maker), who maximizes country i's social welfare, U^i, by allocating i's national income, I^i, between real defense spending, q^i, and real consumption, c^i, in country i. The unit price of defense is given by p, while that of consumption is normalized to equal 1. Thus, each common market member faces a resource or budget constraint,

$$I^i = c^i + pq^i \tag{1}$$

where national income is equated to gross domestic product, GDP.

Defense spending within the common market implies a degree of publicness in which one member's ME may augment the security of other members, not unlike a defense alliance [1,3,7,8,11,33]. However, those defense benefits may vary based on the providing country's location relative to the benefit-recipient country or countries, so that nearer recipients obtain more defense spillovers than more distant ones. That anticipation is particularly true of conventional land forces, which can be deployed more quickly among nearby countries.

A key component of the defense demand model is security spillovers or spill-ins, stemming from the ME of the other $N-1$ common market countries. Defense spillovers, Q_{-i}, of member i are denoted by

$$Q_{-i} = \sum_{k \neq i}^{N} \delta^k q^k \tag{2}$$

where $k = 1, \ldots, N$ and $k \neq i$, so that the spatially weighted defense provision of the other member countries is included. The δ^k weights can assume many alternative forms [20–24]. If only membership in the common market determines defense benefit spillovers, then $\delta^k = 1$ for all members, and 0 for nonmembers. In that scenario, location is not driving spillovers and each member's defense provision is perfectly substitutable, consistent with purely public defense spending [34]. If border contiguity determines defense spillovers among common market members, then $\delta^k = 1$ for member countries sharing a land or water border with country i, and $\delta^k = 0$ for members not contiguous with member i. Like Skogstad [22], we can expand the contiguity measure of spillovers to assign a weight of 1 to noncontiguous allied countries with a marked ability to project their power. Within the EU, such countries may include the UK, France, and Germany. Outside the EU, a unit weight may be given to the United States given its massive ability to project power and its security commitment to many EU members. If, instead, spillovers are dependent on spatial propinquity, then δ^k may equal the inverse distance between the capitals of countries i and k. The inverse-distance weight allows closer countries to derive more defense spillovers from one another [21]. Alternatively, the defense spillover weight may be tied to the extent of arms trade among market members, where larger arms trade flows between two countries enhance their derived defense spillovers [19]. Those flows may, in part, lift spillovers owing to countries' military forces becoming more interoperable, thereby bolstering joint defense deployment. Such arms trade linkages also tie the security interests of the trading members together.

To complete the Nash-equilibrium demand derivation, we consider i's social welfare function (To ensure sufficiency, the social welfare function is assumed to be strictly quasi-concave.),

$$U^i = U^i\left(c^i, q^i + Q_{-i}, \mathbf{X}^i, \mathbf{T}^i\right), \quad i = 1, \ldots, N, \tag{3}$$

where i's decision maker's perceived welfare rises at a diminishing rate with private consumption and the common market's aggregate defense spending, $Q = q^i + Q_{-i}$. Thus, we assume that $U_c^i = \partial U^i/\partial c^i > 0$, $U_Q^i = \partial U^i/\partial Q > 0$, $U_{cc}^i = \partial^2 U^i/\partial c^{i2} < 0$, and $U_{QQ}^i = \partial^2 U^i/\partial Q^2 < 0$, which are common economic assumptions. To ensure the satisfaction of second-order conditions (SOCs), we assume that $U_{cQ}^i > 0$ so that private consumption and aggregate defense are Edgeworth-Pareto complements enhancing one another's marginal utility. We also note that

$$\partial U^i/\partial q^k = U_{q^k}^i = \delta^k U_Q^i > 0, \tag{4}$$

when $\delta^k \neq 0$, indicating that the defense spending of other common market members are plain complements [35,36]. In (3), the vector $\mathbf{X}^i$ contains exogenous determinants of country i's defense spending that include the country's population and trade openness. The threat vector $\mathbf{T}^i$ reduces social welfare and includes transnational terrorist attacks against country i's interests as well as proxies for an enemy's defense spending. The latter may assume the form of i's contiguity with the enemy.

By substituting the budget constraint in (1) and the spillover constraint in (2) into the social welfare function in Equation (3), we can express the representative EU member's maximization problem as:

$$\max_{q^i} U^i\left(I^i - pq^i, q^i + \sum_{k\neq i}^{N} \delta^k q^k, \mathbf{X}^i, \mathbf{T}^i\right),\ i = 1, \ldots, N, \tag{5}$$

for which each member treats the defense spillovers as a parameter, consistent with Nash-equilibrium levels of defense demand for the collective of member countries. (Unlike Douch and Solomon [2] or Smith [7,8], we do not assume an intermediate security function, because doing so will not affect the reduced-form demand for defense). In (5), each common market ally is choosing its best response defense spending in relationship to those of the other common market countries, thereby constituting a Nash equilibrium from which no country would unilaterally change its defense spending decision. The resulting FOCs associated with (5) are:

$$-pU_c^i + U_Q^i = 0, i = 1, \ldots, N. \tag{6}$$

By implicitly differentiating (6), we have:

$$\frac{\partial q^i}{\partial q^k} = -\frac{-p\delta^k U_{cQ}^i + \delta^k U_{QQ}^i}{p^2 U_{cc}^i - 2pU_{cQ}^i + U_{QQ}^i} < 0,\ i, k = 1, \ldots, N,\ i \neq k, \tag{7}$$

which indicates strategic substitutes [35–37] so that each country reacts negatively to increases in the defense spending of another common market country when $\delta^k \neq 0$. The denominator is negative by the SOCs, while the numerator is positive when multiplied by the minus sign. A trade-off between q^i and the aggregate defense spending of the rest of the common market, Q_{-i}, is also negative with $-pU^i_{cQ} + U^i_{QQ}$ in the numerator of dQ_{-1}/dq^i and the SOCs in the denominator, again indicating strategic substitutes. In the aggregate case, the only running variable is $i = 1, \ldots, N$.

The FOCs in (6) can be transformed into the following set of implicitly defined defense demands for the common market members:

$$q^i = q^i\left(I^i, p, \sum_{k\neq i}^{N} \delta^k q^k, \mathbf{X}^i, \mathbf{T}^i\right),\ i = 1, \ldots, N. \quad (8)$$

There are a number of issues to highlight with respect to these defense demands. First, they correspond to a static Nash-equilibrium set of demands for the EU countries or a suitably defined subset based on the spatial weights. Second, given the explicit theoretical framework for defense demand above, defense corresponds to ME and not to a defense burden (i.e., ME/GDP). Third, if defense is a normal good with a positive income elasticity as generally believed and found in the literature, then defense demand responds positively to GDP [34]. Fourth, by necessity, the relative price of defense is normalized to equal 1 because of a lack of defense price data [2,7,8]. Any bias from this normalization disappears provided the prices of defense goods inflate at the same rate as those of civilian goods, which was the case in a Stockholm International Peace Research Institute (SIPRI) [38] study. However, Solomon [39,40] finds a greater relative inflation rate for Canadian defense goods compared to non-defense goods, where price data were available. Nevertheless, we must drop p because of lack of relative defense price data for EU member countries. Fifth, given similar security concerns of EU members, we anticipate that country i's defense spending responds negatively to defense spillovers (i.e., $dQ_{-1}/dq^i < 0$), indicative of strategic-substitute-driven free riding. Sixth, as a country-specific factor in vector $\mathbf{X}^i$, population may increase country i's defense demand when its decision maker places great importance on protecting the people from external threat. If, instead, larger populations create the need for more social spending that siphons off funds from defense, then defense demand may respond negatively to population [41]. A larger population may also reduce defense demand by allowing the country to substitute relatively cheaper manpower for defense capital [42]. The true or net influence of population on defense demand is an empirical question owing to opposing forces. As another country-specific factor, trade openness (imports and exports as a share of GDP) is likely to have a negative influence on defense spending when greater trade augments linkages among countries that reduce the gain from conflict given greater interrelated interests [19]. Finally, threat factors—transnational terrorist attacks, Russian ME, or proximity to Russia—may increase defense demand. On the contrary, transnational terrorist attacks may not induce a larger defense spending response in the presence of other non-military means for countering terrorism through law enforcement, intelligence, homeland security spending, or the counterterrorism actions of INTERPOL [43]. In the case of Russia, this may not be true if EU members do not view Russia as a security threat.

5. Data and Methodology

5.1. Empirical Methodology

Based on our theoretical model, a country's ME is a function of the aggregate ME of other countries in an implicit or explicit alliance. This means that the spillover term is an important explanatory variable in the demand equation for ME. However, a straightforward inclusion of the spillover term in an ordinary least squares (OLS) regression can cause estimation problems due to endogeneity [44,45]. To address this concern, we use

SAR models, a class of spatial models, used in the empirical literature to account for the spillover terms [23,24,30].

The SAR representation of the demand for ME can be depicted as:

$$Y_{it} = \rho \sum_{k \neq i} w_{ikt} Y_{kt} + \beta \mathbf{X}_{it} + \mu_i + \tau_t + \varepsilon_{it}, \quad (9)$$

where Y_{it} is the log of ME for country i during year t. In (9), $\sum_{k \neq i} w_{ikt} Y_{kt}$ represents defense spillovers, corresponding to the spatially weighted sum of ME of other EU countries. Additionally, Y_{kt} is the log of ME for country k in year t, and w_{ikt} is the spatial weight representing the relative connectivity between countries i and j in year t. In spatial models, the spillover term is also called the spatial lag of the dependent variable because it is comparable to the time-lagged variable in a time-series model. While the time-lagged term captures the temporal dependence between two time periods, the spatial-lagged term represents the spatial dependence between a given geographical unit and other units in the dataset. The relative strength of the spatial dependence among units is captured by the spatial weight, w_{ikt}. $\mathbf{X}_{it}$ is the vector of time-varying controls that influence country i's ME, including its GDP, population, and trade openness. In (9), ρ and β represent the estimated spatial spillover and control coefficients, respectively. μ_i and τ_t correspond to country- and year-level fixed effects, respectively. Finally, ε_{it} is the idiosyncratic error term.

Generally, spatial weights (w_{ikt}) could be geographical (e.g., distance between members), political (e.g., alliance or common market membership), or economic (e.g., trade between two members) connectivity measures. In our study, we initially utilize five such weights. First, we assign a weight of 1 to all EU members. Second, we use contiguity, which assigns a value of 1 to member k's ME if members i and k share borders, and 0 otherwise. Third, we employ the inverse of the distance between the capital cities of two members. Fourth, we consider arms trade as a spatial weight where δ^k denotes the value of all transfers of major conventional weapons between members i and k. Members might place more importance on the ME of other members with whom they have an active military trade relationship. Fifth, we allow for a spatial weight measure that assigns a value of 1 to each EU member that shares its borders with Russia. Given the recent geo-political tensions between Russia and the EU, an EU member may view the ME of allies contiguous to Russia as a sign of an emerging security threat. Following Neumayer and Plümper [31], and George and Sandler [23], we do not row-standardize the spatial weights. Under row-standardization, for each year, the spatial weight of a given country is divided by the sum of spatial weights of all other sample countries. As explained in George and Sandler [23], row-standardization imposes implicit assumptions that are theoretically undesirable for the study of any military alliance or common market that changes its membership size over time. With row-standardization, EU expansion would diminish the ME influence of key EU members that spend more on defense as the membership expands.

Because of endogeneity concerns, we estimate our SAR models using the two-step efficient GMM estimator, where the spatial lags of the explanatory variables (except ME spillovers) are used as external instruments for the spatial lag of the ME variable [46]. All regressions are estimated using standard errors that are robust to heteroskedasticity and autocorrelation.

5.2. Data

For the empirical analysis, we include all EU countries for 1990–2019, based on their year of accession to the EU if after 1990. Our time frame choice coincides with the formation of the Russian federation, which EU countries may view as a military threat. Early 1990s also witnessed the emergence of religious fundamentalist terrorism and an associated wave of transnational terrorism that influenced national security policies of many EU countries [47].

Our dependent variable, the log of ME, is drawn from the SIPRI [48] Extended Military Expenditure Database, which records consistent ME time series data for most countries during 1949–2019. To foster its reliable and consistent country-level ME estimates, SIPRI [48] relies on government documents, international statistics, journals, and news reports. In the current study, ME is measured in constant 2018 US dollars. Independent variables include GDP, population, and trade openness, drawn from the World Development Indicators (WDI) [49]. Like ME, GDP is measured in constant 2018 US dollars. We use the GDP deflator variable provided by WDI to calculate GDP values in constant US dollars. In particular, WDI relies on the World Bank and OECD National Accounts data to calculate the deflator values. The trade openness variable corresponds to the sum of total imports and exports as a share of GDP. The number of transnational terrorist attacks is drawn from the International Terrorism: Attributes of Terrorist Events (ITERATE) data set [50] (Mickolus et al. 2020), which records observations on transnational terrorist events by venue country during 1968–2019.

As discussed earlier, the construction of the spatial-lagged variables requires data on various measures of spatial weights. The data on distance between capital cities are from Gleditsch and Ward [51]; data on the contiguity relationships between countries are obtained from Correlates of War (COW) Direct Contiguity Dataset [52,53]. Finally, spatial weights, representing the value of arms trade between countries, are drawn from the SIPRI Arms Transfer Database [54], which contains information on all transfers of major conventional weapons from 1950 to 2019. Specifically, we apply the trend-indicator value (TIV), a measure intended to represent the transfer of military resources based on the known unit production costs of a core set of weapons [54]. During the sample period, the average TIV between two countries, which engaged in some form of military trade, is 73.23 million dollars. In the absence of row-standardization, the straightforward inclusion of such large TIV values as spatial weights results in very large spillover terms with small difficult-to-interpret coefficients. Hence, for the purpose of spatial weighting, we divide all TIV values by 1,000,000, thereby representing the arms-trade spatial weights in millions of dollars.

6. Empirical Results

Tables 1–5 report two-step GMM estimates of the determinants of the demand for ME of EU countries from 1990 to 2019. Country and year fixed effects are included, except when EU membership weights are used to construct the spatial-lagged term (In that model, we omit the year fixed effects because their inclusion, when equal weights (e.g., EU membership) are applied, renders the estimates inconsistent). In each of the models, we display the adjusted R^2, Kleibergen-Paap LM statistic for testing instrument strength, and Hansen J-statistic for testing the overidentifying restrictions for instruments.

Table 1 reports five models where the only difference involves the type of spatial-lagged (SL) variables included. For all SL of ME terms in Table 1, the coefficients are spillover elasticities, given the log of the dependent variable and the log of the spillover measures. The coefficients of the five spatial-lagged terms depict a significant and negative influence on the log of ME, indicating that EU countries reduce their defense spending in response to increases in the weighted MEs of other EU members. That result holds for Model 1 where equal unit weights are assigned to EU members. Despite EU not being a traditional defense alliance, its members show clear evidence of free riding with regard to their military spending decisions. Since EU members are quite politically homogenous, geographically close, and confront similar international security threats, EU countries apparently view their fellow members' ME as strategic substitutes for their own ME.

The spatial-lagged term in Model 2 captures how an EU country's percentage change in ME responds to percentage changes in the ME of other EU members with which it shares land or water borders. The negative coefficient of that spillover term, albeit at 10 percent significance level, suggests the presence of free riding among contiguous EU members. A similar finding is reported for an inverse-distance-weighted spatial-lagged term in Model 3, further supporting the free-riding outcome among EU members. The defense-spending responsiveness to inverse distance is greater than for the previous two models.

Table 1. EU military expenditure (ME), 1990–2019.

Variable	(1)	(2)	(3)	(4)	(5)
SL of ME (EU membership)	−0.0005 ***				
	(−4.06)				
SL of ME (contiguity)		−0.00107 *			
		(−1.92)			
SL of ME (inverse distance)			−0.165 **		
			(−2.54)		
SL of ME (contiguity with Russia)				−0.745 ***	
				(−7.80)	
SL of ME (arms trade)					−0.00000344 **
					(−2.11)
Ln (GDP)	0.856 ***	0.875 ***	0.907 ***	0.580 ***	0.500 ***
	(7.81)	(6.57)	(7.38)	(4.80)	(2.81)
Ln (Population)	−1.080 ***	−1.180 ***	−1.042 ***	−0.295	−1.080 **
	(−2.63)	(−3.02)	(−2.83)	(−1.05)	(−2.37)
Terrorist attacks	0.00125 *	0.000216	0.0000260	0.000233	0.000780 *
	(1.83)	(0.49)	(0.06)	(0.63)	(1.72)
Trade Openness	−0.00104	−0.0001	−0.00008	0.0003	−0.0045 ***
	(−1.42)	(−0.13)	(−0.11)	(0.53)	(−3.92)
Year FE	NO	YES	YES	YES	YES
Adjusted R^2	0.994	0.994	0.995	0.997	0.389
Kleibergen-Paap LM test statistic	106.0	25.57	22.45	37.06	17.94
Hansen J statistic (prob > χ^2)	0.344	0.0425	0.0406	0.996	0.272
N	628	612	628	628	277

Significance levels (SL): * $p < 0.10$, ** $p < 0.05$, and *** $p < 0.01$. t-statistics in parentheses. All models include country fixed effects. Italics: variables.

Table 2. EU military expenditure, 2008–2019.

Variable	(1)	(2)	(3)	(4)	(5)
SL of ME (EU membership)	−0.00264 **				
	(−2.28)				
SL of ME (contiguity)		−0.009 ***			
		(−3.44)			
SL of ME (inverse distance)			−1.917 ***		
			(−3.66)		
SL of ME (contiguity with Russia)				−0.643 ***	
				(−4.91)	
SL of ME (arms trade)					0.0000008
					(0.06)
Ln (GDP)	1.469 ***	1.211 ***	1.149 ***	0.889 ***	1.052 ***
	(8.05)	(5.06)	(5.59)	(4.33)	(6.48)
Ln (Population)	−1.398 *	−1.558 **	−1.623 **	−0.485	−0.988 *
	(−1.90)	(−1.99)	(−2.36)	(−0.79)	(−1.93)
Terrorist attacks	−0.00240	−0.007 **	−0.007 **	−0.005 **	−0.0015
	(−0.73)	(−2.37)	(−2.47)	(−2.35)	(−0.34)
Trade Openness	−0.00208	−0.00167	−0.00102	−0.000312	−0.00213
	(−1.55)	(−1.05)	(−0.67)	(−0.29)	(−1.60)
Year FE	NO	YES	YES	YES	YES
Adjusted R^2	0.994	0.994	0.995	0.712	0.631
Kleibergen-Paap LM test statistic	72.04	9.500	18.99	26.24	7.874
Hansen J statistic (prob > χ^2)	0.00103	0.348	0.0489	0.0894	0.892
N	331	319	331	331	138

Significance levels: * $p < 0.10$, ** $p < 0.05$, and *** $p < 0.01$. t-statistics in parentheses. All models include country fixed effects.

Table 3. EU and non−EU NATO countries' military expenditure, 1990–2019.

Variable	(1)	(2)	(3)	(4)	(5)
SL of ME (EU membership)	−0.0004 ***				
	(−3.46)				
SL of ME (contiguity)		−0.001 **			
		(−2.28)			
SL of ME (inverse distance)			−0.391 ***		
			(−3.19)		
SL of ME (contiguity with Russia)				−0.656 ***	
				(−6.29)	
SL of ME (arms trade)					−0.000005 ***
					(−4.04)
Ln (GDP)	0.590 ***	0.638 ***	0.600 ***	0.523 ***	0.514 ***
	(4.91)	(5.80)	(5.23)	(4.51)	(4.45)
Ln (Population)	−0.197	−0.204	−0.249	0.114	−0.658
	(−0.50)	(−0.58)	(−0.68)	(0.36)	(−1.55)
Terrorist attacks	0.000783	−0.000605	−0.000578	−0.000398	0.000489
	(1.54)	(−0.95)	(−0.91)	(−0.61)	(1.10)
Trade Openness	−0.00126 *	−0.000894	−0.000454	−0.000479	−0.00547 ***
	(−1.89)	(−1.28)	(−0.71)	(−0.83)	(−4.74)
Year FE	NO	YES	YES	YES	YES
Adjusted R^2	0.994	0.995	0.995	0.996	0.538
Kleibergen-Paap LM test statistic	131.3	44.06	22.74	35.64	17.42
Hansen J statistic (prob > χ^2)	0.297	0.773	0.0627	0.290	0.376
N	796	780	796	791	338

Significance levels: * $p < 0.10$, ** $p < 0.05$, and *** $p < 0.01$. t-statistics in parentheses. All models include country fixed effects.

Table 4. EU military expenditure, 1990–2019 (Alternate spatial weights).

Variable	(1)	(2)	(3)	(4)	(5)	(6)
	1990–2019		2008–2019		1990–2019 (EU and non-EU NATO countries)	
SL of ME (US + contiguity)	−0.001 **		−0.01 ***		−0.001 **	
	(−2.00)		(−3.70)		(−2.28)	
SL of ME (US + UK + France + contiguity)		−0.001 **		−0.01 ***		−0.001 **
		(−2.00)		(−3.70)		(−2.27)
Ln (GDP)	0.876 ***	0.876 ***	1.462 ***	1.458 ***	0.637 ***	0.636 ***
	(7.03)	(7.02)	(8.33)	(8.26)	(5.88)	(5.87)
Ln (Population)	−1.172 ***	−1.171 ***	−2.165 ***	−2.095 ***	−0.215	−0.213
	(−3.14)	(−3.14)	(−3.15)	(−3.02)	(−0.62)	(−0.62)
Terrorist attacks	0.000217	0.000220	−0.005 **	−0.005 **	−0.0006	−0.000604
	(0.50)	(0.51)	(−2.03)	(−2.05)	(−0.94)	(−0.94)
Trade Openness	−0.000134	−0.000131	−0.0009	−0.0008	−0.0008	−0.000839
	(−0.20)	(−0.19)	(−0.61)	(−0.55)	(−1.31)	(−1.31)
Year FE	YES	YES	YES	YES	YES	YES
Adjusted R^2	0.995	0.995	0.995	0.995	0.995	0.995
Kleibergen-Paap LM test statistic	26.00	25.97	10.11	10.11	44.09	44.13
Hansen J statistic (prob > χ^2)	0.0418	0.0382	0.00652	0.00674	0.769	0.781
N	628	628	331	331	796	796

Significance levels: ** $p < 0.05$ and *** $p < 0.01$. t-statistics in parentheses. All models include country fixed effects.

Table 5. EU military expenditure, 1990–2019 (alternate threat measure).

Variable	(1)	(2)	(3)	(4)	(5)
SL of ME (EU membership)	−0.0004 ***				
	(−2.85)				
SL of ME (contiguity)		−0.0017 ***			
		(−3.08)			
SL of ME (inverse distance)			−0.25 ***		
			(−3.45)		
SL of ME (US + contiguity)				−0.002 ***	
				(−3.11)	
SL of ME (US + UK + France + contiguity)					−0.002 ***
					(−3.11)
Ln (GDP)	0.908 ***	0.799 ***	0.879 ***	0.807 ***	0.802 ***
	(7.73)	(7.67)	(7.70)	(7.82)	(7.75)
Ln (Population)	−1.113 ***	−1.287 ***	−1.144 ***	−1.235 ***	−1.234 ***
	(−2.67)	(−2.98)	(−2.73)	(−2.91)	(−2.91)
Terrorist attacks	0.00116 **	0.00130 **	0.00135 **	0.00131 **	0.00130 **
	(2.47)	(2.11)	(2.32)	(2.06)	(2.04)
Trade Openness	−0.00123	−0.000792	−0.00114	−0.000948	−0.0009
	(−1.64)	(−0.89)	(−1.45)	(−1.21)	(−1.20)
Russian ME	−0.0173	−0.0566 **	−0.0421	−0.0574 **	−0.0562 **
	(−0.61)	(−2.08)	(−1.47)	(−2.10)	(−2.06)
Year FE	NO	YES	YES	YES	YES
Adjusted R^2	0.994	0.993	0.994	0.994	0.994
Kleibergen-Paap LM test statistic	92.94	21.55	40.30	21.67	21.79
Hansen J statistic (prob > χ^2)	0.0723	0.853	0.0216	0.817	0.714
N	604	588	604	604	604

Significance levels: ** $p < 0.05$ and *** $p < 0.01$. t-statistics in parentheses. All models include country fixed effects.

In Model 4, the spillover term represents how an EU member's ME responds to percentage changes in the ME of other EU members that share their borders with Russia. If EU countries perceive Russia as a major security threat, they are expected to react positively to the ME of EU members in close proximity to Russia. In contrast to that prior, the significant and negative coefficient of the spatial-lagged term in Model 4 implies that EU members actually reduce their ME in response to an increase in the ME of EU members contiguous to Russia. This suggests that EU countries do not consider Russia as a major security threat for the sample post-Cold War period, at least in terms of their ME response. This appears true despite Russia's annexation of Crimea in 2014, its conflict in Chechnya and elsewhere, and the recent rise of Russian nationalism. Finally, in Model 5, we use the value of arms trade among EU members to capture spatial spillovers, based on the assumption that an EU member places more weight on the ME of its EU arms-trading partners. The spillover term's coefficient is again negative and significant, thereby increasing the confidence in our previous free-riding results. Apparently, larger arms trade within the EU cements military ties or connectivity. That is, EU members can more readily view other members' arsenals as substitutable to their own arsenal if they share similar types of armaments that are interoperable and of a more current vintage.

Next, we consider the estimated coefficients of the control variables in Table 1. Consistent with the theoretical expectations, the log of GDP shows a significant and positive association with the log of ME in all models. That positive income elasticity indicates that defense is a normal, but inelastic, good with values that range from 0.5 to 0.9 over the five models. The population elasticity is significant and negative in all models except Model 4, with values less than –1. As discussed earlier, more populous countries need to allocate more money to domestic social welfare programs, thereby redirecting resources away from their military budgets. A population elasticity of less than –1 reflects a robust reallocation to social welfare concerns as population increases, consistent with EU countries wanting to cash in on the peace dividend after the Cold War. The number of transnational terrorist attacks has no significant association with ME, except in Models 1 and 5, where the anticipated positive coefficients are marginally significant at the 10 percent level. Trade openness does not significantly influence the demand for ME except in Model 5, where the negative coefficient suggests that greater trade openness limits the need for defense, as countries are more dependent on one another. The large values of the Kleibergen-Paap LM test statistic indicate that the instruments are strong. Moreover, the Hansen *J*-statistic rejects the null hypothesis of overidentifying restrictions at the 5 percent level, except for Models 2 and 3.

The global financial crisis of 2007–2008 had significant impacts on the defense spending trajectories of most EU countries. In the post-crisis years, many EU countries had to spend greater sums on various economic stimulus packages and social welfare programs, thereby diverting resources away from their military budgets. In 2009 alone, European defense expenditure fell by about 3 percent and continued to decline steadily until 2013 [48]. To test whether this decline in defense spending influenced EU members' free-riding behavior, we re-estimate the demand determinants of ME during 2008–2019, the post-crisis years, using the same five spatial weights.

As shown in Table 2, four of the five spatial-lagged ME spillover terms, the exception being the arms-trade-weighted spatial-lagged term, display negative and significant influences on the log of EU members' ME, indicating that defense free-riding behavior characterized the post-crisis years. The fall in arms trade among EU members may be behind the insignificance of the arms-trade spatial-lagged term in Table 2. The four significant spatial spillover elasticities are similar in values to the corresponding elasticities for 1990–2019. To test whether similar behavior characterized the pre-crisis period, we also estimate EU defense demand equations for 1990–2007. We could not, however, uncover any evidence of free riding in that earlier period, thus suggesting that EU members' free-riding behavior during the entire sample period was primarily driven by the post-2007 years (The results are available upon request). Turning to the control variables, we see that GDP and population variables generally display more elastic results than for the entire period covered by Table 1. In particular, income is now elastic with values greater than 1 in four of the five models. Additionally, population is generally tied to a larger negative response after 2007, consistent with the greater need to turn to social welfare programs after the financial crisis. Unlike the results for 1990–2019, the number of transnational terrorist attacks has a significant negative effect on EU members' ME. The direction of that effect is unanticipated since transnational attacks on an EU members' assets at home or abroad, both of which are measured in the data set, should raise their defense spending. This unanticipated finding may be driven by declines in transnational terrorist incidents in Europe and throughout the world as borders were made more secure following 9/11. Transnational terrorist attacks became more prevalent in countries hosting terrorist groups [47]. Such foreign-venue attacks are likely to elicit a smaller proactive response than home attacks, which pose a larger existential threat to the country. The unanticipated result may follow from counterterrorism measures by law enforcement, intelligence agencies, and INTERPOL assuming a greater importance, thus making the need for counterterrorism military actions less relevant.

Next, we estimate the ME demand equations for EU and non-EU NATO countries combined. The non-EU NATO countries included in the analysis are Albania, Montenegro, Macedonia, Norway, Turkey, and the United States. For the first three countries, we account for when they joined NATO. Since both NATO and EU have many common member countries and share various geo-political characteristics and threats, this pooling is justified. Moreover, we explore how EU members' free-riding behavior changed in response to the largest military spender in NATO—the United States—being included during the sample 1990–2019 period. As shown in Table 3, the free-riding behavior associated with the spatial lags of ME is amazingly consistent after adding non-EU NATO countries to the sample.

The only quantitative difference concerns the inverse-distance, spatial-lagged ME term, which displays a smaller free-riding elasticity. This may be due to greater distance between allies with the addition of the United States. A noteworthy change with the larger sample is that the population variable is no longer significant. That suggests that for non-EU NATO countries, there is no evidence of population size affecting social welfare crowding out ME spending. Both Turkey and the United States are populous countries that have, relative to EU countries, maintained their defense spending, which may have weakened the earlier influence of population on ME. In Table 3, the coefficients of transnational terrorist attacks and trade openness are mostly insignificant, similar to Table 1.

In Table 4, we use alternative spatial weights for 1990–2019 to capture how EU allies respond to power projections by the US and contiguous allies, and by the US, the UK, France, and contiguous allies. Even though the US is separated by an ocean from Europe, the US' huge power-projection capabilities in terms of aircraft carriers and transport planes make it similar to a contiguous country. The relatively high ME of those three powerful countries means that they are uniquely capable of exerting influence on the ME decisions of EU countries, particularly in terms of free riding. Models 1 and 2 report estimates for EU countries for 1990–2019, while Models 3 and 4 present estimates for EU countries for 2008–2019. Results for EU and non-EU NATO countries combined are reported in Models 5 and 6 for 1990–2019. Across all models, the spatial-lagged spillover terms are significant and negative, further supporting free riding and strategic substitutes. Those findings highlight that the EU relies partly on US power-projection abilities. The coefficients for the control variables are generally consistent with those from Tables 1–3, implying that defense spending is income normal and that population exerts a negative social welfare trade-off for Models 1–4. Transnational terrorist attacks are negative and significant for just 2008–2019, and trade openness is not significant.

Table 5 contains results for spatial regressions where Russian ME replaces Russian contiguity-weighted, spatial-lagged term as a threat measure. In addition, the two spatial-lagged power-projection ME terms are applied instead of the spatial-lagged arms-trade term. Unlike Model 4 in Tables 1–3, the threat measure (Russian ME) is a control in addition to five designated spillover terms. In Table 5, all five spillover elasticities are negative and significant, indicative of free riding. In Models 2, 3, and 5, EU members reduce their ME in response to an increase in Russian ME, which provides further evidence that EU countries are not apparently perceiving Russia as a major security threat. Other findings are consistent with the previous results—namely, GDP displays a robust positive elasticity and population shows a robust negative elasticity less that –1. A notable difference in the runs concerns transnational terrorist attacks, which indicate a significant positive influence on EU defense spending in all five models as originally predicted. As in earlier runs, trade openness has an insignificant effect on EU defense spending.

Up to this point, standard controls for the estimation of defense demand are used. At the urging of a reviewer, we estimate a set of robustness runs, displayed in Table 6, that allows for additional controls, such as the presence of a military industry, the number of armed conflicts, and the extent of globalization. The dummy variable measuring the presence of a military industry is constructed using the SIPRI Arms Industry Database [55]. That database provides information on both public and private arms-producing and military services companies in a sample country. For 1990–2019, we also allow for the number of armed conflict incidents that a sample country participated in, drawn from the Uppsala Conflict Data Program/Peace Research Institute Oslo (UCDP/PRIO) armed conflict dataset [56,57]. Finally, we add the KOF Globalization Index, which measures the economic, social, and political dimensions of globalization from 1970 to 2019 [58,59].

Table 6. EU military expenditure, 1990–2019 (robustness checks).

Variable	(1)	(2)	(3)	(4)	(5)
SL of ME (EU membership)	−0.000252 **				
	(−2.36)				
SL of ME (contiguity)		−0.000724 *			
		(−1.84)			
SL of ME (inverse distance)			−0.118 *		
			(−1.82)		
SL of ME (contiguity with Russia)				−0.830 ***	
				(−10.70)	
SL of ME (arms trade)					−0.00000381 **
					(−2.14)
Ln (GDP)	0.787 ***	0.793 ***	0.791 ***	0.512 ***	0.511 ***
	(7.55)	(6.74)	(6.84)	(5.42)	(2.78)
Ln (Population)	−1.189 ***	−0.776 **	−0.494	0.0662	−1.085 **
	(−3.32)	(−2.08)	(−1.35)	(0.25)	(−2.43)
Terrorist attacks	0.000886 **	0.000434	0.000251	0.000473	0.000880 **
	(1.97)	(1.29)	(0.77)	(1.64)	(2.07)
Presence of military industry	−0.0342	−0.0444	−0.0414	−0.0148	
	(−0.69)	(−0.90)	(−0.82)	(−0.35)	
No. of conflicts	0.0295	−0.0246	−0.0235	−0.0293	−0.0341
	(0.44)	(−0.55)	(−0.54)	(−0.70)	(−0.63)
KOF Globalization Index	−0.0103 ***	0.0159 ***	0.0140 ***	0.0112 ***	0.0173 *
	(−2.77)	(3.59)	(3.10)	(2.81)	(1.75)
Year FE	NO	YES	YES	YES	YES
Adjusted R^2	0.278	0.421	0.417	0.659	0.401
Kleibergen-Paap LM test statistic	9.535	5.233	5.295	35.62	5.913
Hansen J statistic (prob > χ^2)	0.339	0.0902	0.00727	0.749	0.372
N	600	585	600	600	267

Significance levels: * $p < 0.10$, ** $p < 0.05$, and *** $p < 0.01$. *t*-statistics in parentheses. All models include country fixed effects.

In Table 6, we introduce the three new determinants of the demand for ME and check whether their inclusion changes the results for the five spillover terms. Those new controls are added to the baseline models in Table 1. As shown in Table 6, the ME of EU members shows clear evidence of free riding (i.e., negative spillover values) even after the inclusion of additional controls, thereby increasing confidence in our initial results. Moreover, ME remains income normal, while population generally reduces ME, consistent with crowding-out. The KOF Globalization Index shows a significant and negative association with ME in Model 1. However, when we include year fixed effects to control for temporal shocks (Models 2–5), the globalization coefficient is significant and positive, suggesting that EU members with high degrees of economic, social, and political globalization tend to spend more on ME. That result is not surprising since countries that are more globalized show very high rates of economic growth [58]. Armed conflicts and the presence of a military industry have no significant effect on the ME of EU countries.

7. Concluding Remarks

Based on myriad spatial-linkage measures, our spatial estimates of EU military spending during the post-Cold War era provide a consistent, robust message that differs from past non-spatial and spatial EU defense spending estimates [2,17–19]. Most notably, free riding, indicative of strategic substitutes, characterizes EU members' ME for all seven spatial connectivity measures of this study. Free riding is particularly strong in the post-2007 period following the great recession, as social welfare programs assume a greater importance. The inclusion of non-EU NATO members, especially the United States, in the sample bolsters free riding under various empirical formulations. Even though the EU does not pledge its members to come to one another's defense in times of exigency as in a traditional military alliance, common market members act like allies who consider other members' ME substitutable for their own ME, leading to the prevalence of free riding. Additionally, we find that defense is an income normal good whose elasticity generally exceeds 1 during 2008–2019. Moreover, population displays an elasticity less than –1, indicative of social welfare demands crowding out defense as population increases. EU members' defense spending for threat measures is consistent with Russia not being viewed by members as a threat under alternative empirical specifications during 1990–2019. Finally, transnational terrorist attacks indicate a robust threat only when Russian ME is a control.

Our findings suggest some policy conclusions. First, if NATO were to dissolve or to decrease in importance, the EU appears to be in no position to rectify the alleged free-riding problem in Europe because EU members are content, like NATO members, to view other EU members' defense spending as strategic substitutes under a wide range of spatial connectivity assumptions. Thus, NATO's mission to protect Europe from external threats cannot be readily replaced by the EU. Second, since 1990, the EU appears more focused on cashing in on the post-Cold War peace dividend and on diverting defense spending to social welfare expenditure following the great recession. In the post-COVID period, this diversion of defense spending to social welfare is anticipated to continue for EU members as they address the economic downturn. If common defense is to assume a greater role in the EU, then members must change their thinking about threats facing them. Despite Russian aggression in Ukraine and elsewhere, this change in mindset has not yet occurred. Any such change must come from within the EU, perhaps motivated by even more aggressive actions by Russia. Third, if the EU were to assume a greater importance as a military alliance, then France, Germany, and the UK would need to raise their defense provision greatly to replace the dominant defense spending of the United States. However, our study still shows EU free riding on the United States, a non-EU member. As long as this reliance on the United States persists, the EU is unlikely to change its defense spending patterns. Fourth, with civil wars in North Africa, sub-Saharan Africa, and the Middle East, the EU faces unrest elsewhere that causes disruptions in terms of refugee flows, trade, and resource supply lines. To confront such challenges, the EU must begin to address its endemic free-riding problem, which may well require increased pledges of defense spending, like the recent 2 percent rule adopted by NATO in terms of defense burdens. Fifth, those civil wars mean that the EU must better coordinate its efforts at peacekeeping in nearby conflict-challenged regions. Such coordination has a long way to go, indicated by the eventual need to rely on US air power and munitions during a NATO-supported ouster of Qaddafi during 2011. Sixth, to be equipped to address peacekeeping outside of Europe, the EU members must build up their capacity to project power through greater transport aircraft and ships (e.g., aircraft carriers).

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