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Symmetry in Quantum Theory of Gravity

Edited by

Chris Fields and Antonino Marciano

Printed Edition of the Special Issue Published in *Symmetry*

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This is a reprint of articles from the Special Issue published online in the open access journal *Symmetry* (ISSN 2073-8994) (available at: https://www.mdpi.com/journal/symmetry/special_issues/Symmetry_quantum_theory_gravity).

For citation purposes, cite each article independently as indicated on the article page online and as indicated below:

LastName, A.A.; LastName, B.B.; LastName, C.C. Article Title. *Journal Name* **Year**, *Volume Number*, Page Range.

ISBN 978-3-0365-5391-7 (Hbk)
ISBN 978-3-0365-5392-4 (PDF)

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Editorial

Symmetry in Quantum Theory of Gravity

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Citation: Fields, C. Symmetry in Quantum Theory of Gravity. *Symmetry* **2022**, *14*, 775. https://doi.org/10.3390/sym14040775

Received: 6 April 2022
Accepted: 7 April 2022
Published: 8 April 2022

Nicolas Gisin [1,2] has emphasized that, prior to Einstein's imposition of a finite speed of light, and hence of information, in the Special Theory of Relativity, physics was nonlocal. While Newton was, as Gisin points out, rather unhappy with this situation, pre-relativistic classical physics was a physics of "instantaneous" (prior to Cantor, this must be considered an informal notion) interactions across arbitrary distances. Spacetime in this classical setting was merely a container; indeed, it was merely a set of labels. The gravitational and, later, electromagnetic interactions operating in this classical container were, moreover, in principle completely deterministic. Indeed, Tipler [3] has shown that this Laplacian, globally deterministic classical theory is equivalent to quantum theory in its Bohmian representation. Classical physics is not, of course, globally deterministic in the classroom or in practice; classical interactions are typically represented as occurring on isolated billiard tables or in isolated solar systems. It is the sigularities created by assumptions of isolation—effectively, screening of all exterior forces—that are removed by adding a Bohmian "quantum" potential to classical physics.

What, then, is the relationship between classical and quantum physics? How does this relationship depend on assumptions of locality or isolation? The difficulty of obtaining a satisfactory quantum theory of gravity is often attributed to a fundamental conflict between the demands of covariance and those of unitarity, as illustrated, for example, by discussions of the black hole information paradox [4–10]. These, as well as [3], suggest, however, that what is fundamentally at stake is the relation between classical and quantum information, or in more operational (or indeed philosophical) language, the relation between observational outcomes and the physics that they describe. Observational outcomes always characterize particular physical systems; hence this question can be rephrased as the question of whether "systems" can be considered to be both local and observer-independent.

The papers in this Special Issue all address some aspect of this latter question. The first, by Y.-Q. Gu, continues the author's previous efforts [11] to develop a fully "realist" representation—with observer-independent space and time coordinates—of spacetime that is consistent with GR by deriving a representation for the spinor connection within such a spacetime [12]. The treatment is fully geometric, employing the formalism of Clifford algebra to represent both the spinor and the spacetime. Gu suggests that, in these coordinates, the (global) cosmological constant is reproduced by the self-interaction of the (global) spinor field.

The second paper, by Illuminati, Lambiasi, and Petruzziello, extends previous work [13] on the breaking of Lorentz symmetry that results when the Heisenberg uncertainty principle is generalized to account for the effects of long-range spacetime curvature on locally-observable momenta [14]. This symmetry breaking is shown to be equivalent to one generated by a string-theoretic extension of the standard model. They provide an improved theoretical bound on the strength of this symmetry breaking, but note that this bound is still much larger than values that near-Earth observations might be expected to yield.

The paper of Moradpour, Aghababaei, and Ziaie examines the effects of generalizing the uncertainty principle from a different perspective, computing the behavior of both the Maxwell–Boltzmann distribution and its relativistic extension, the Jüttner distribution, as the long-range curvature effects are increased [15]. As expected, long-range corrections have a smaller effect on the Jüttner distribution; however, and consistent with the above, they remain far outside the bounds of current observational capabilities.

The paper of Kaparulin, Lyakhovich, and Nosyrev extends previous work [16] that seeks to achieve both stability and gauge invariance in higher-order (extended Cherns-Simons) theories with n-particle interactions [17]. Its key result is that achieving both stability and gauge invariance requires introducing a further "gauging field" that deforms the equations of motion. This additional field is interpreted as a "Higgs-like" mechanism that maintains on-shell masses.

Jack Ng's paper applies a previously developed heuristic model of spacetime foam that is compliant with the holographic principle [18] to characterize the accelerating regimes of the early (i.e., inflationary) and late (i.e., dark energy-dominated) universe [19]. Ng equates the "quanta" of spacetime foam with holographically encoded bits of information, and shows that these quanta must be distinguishable—i.e., must collectively encode macroscopic, multibit information—to generate a macroscopic spacetime. He suggests that "ordinary" bosons and fermions can be considered collective modes of these foam quanta.

The final paper, by Fields, Glazebrook, and Marcianò, develops some consequences of a previous [20] generalization of the holographic principle to arbitrary pairs of finite quantum systems in the weak interaction limit [21]. We provide formal criteria under which arbitrary interactions can be regarded as "measurements" that deploy defined quantum reference frames (QRFs) [22], and show that such interactions induce decoherent sectors on the holographic screen separating the interacting systems. We also show that joint-state separability breaks down as the QRFs deployed by the interacting systems approach functional equivalence.

While the papers collected here do not, and could not be expected to, reach a consensus on the path forward toward a satisfactory quantum theory of gravity, they do share a common theme: that nonlocality must be built somehow into coordinate systems (Gu), measurement uncertainties (Illuminati et al., Moradpour et al.), ancillary fields (Kaparulin et al.), underlying fundamental quanta (Ng) or the notion of "system" itself (Fields et al.). They thus contribute to the growing body of evidence that a local theory of observation, i.e., of classical information exchange, does not require, and may indeed be inconsistent with, a local theory of physics.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

GR	General relativity
QRF	Quantum reference frame

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Article

Theory of Spinors in Curved Space-Time

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Abstract: By means of Clifford Algebra, a unified language and tool to describe the rules of nature, this paper systematically discusses the dynamics and properties of spinor fields in curved space-time, such as the decomposition of the spinor connection, the classical approximation of the Dirac equation, the energy-momentum tensor of spinors and so on. To split the spinor connection into the Keller connection $\Upsilon_\mu \in \Lambda^1$ and the pseudo-vector potential $\Omega_\mu \in \Lambda^3$ not only makes the calculation simpler, but also highlights their different physical meanings. The representation of the new spinor connection is dependent only on the metric, but not on the Dirac matrix. Only in the new form of connection can we clearly define the classical concepts for the spinor field and then derive its complete classical dynamics, that is, Newton's second law of particles. To study the interaction between space-time and fermion, we need an explicit form of the energy-momentum tensor of spinor fields; however, the energy-momentum tensor is closely related to the tetrad, and the tetrad cannot be uniquely determined by the metric. This uncertainty increases the difficulty of deriving rigorous expression. In this paper, through a specific representation of tetrad, we derive the concrete energy-momentum tensor and its classical approximation. In the derivation of energy-momentum tensor, we obtain a spinor coefficient table $S^{\mu\nu}_{ab}$, which plays an important role in the interaction between spinor and gravity. From this paper we find that Clifford algebra has irreplaceable advantages in the study of geometry and physics.

Keywords: Clifford algebra; tetrad; spinor connection; natural coordinate system; energy-momentum tensor; local Lorentz transformation; spin-gravity interaction

PACS: 04.20.Cv; 04.20.-q; 04.20.Fy; 11.10.Ef; 11.10.-z

Citation: Gu, Y.-Q. Theory of Spinors in Curved Space-Time. *Symmetry* **2021**, *13*, 1931. https://doi.org/10.3390/sym13101931

Academic Editors: Chris Fields and Antonino Marciano

Received: 24 September 2021
Accepted: 12 October 2021
Published: 14 October 2021

1. Introduction

The Dirac equation for spinor is a magic equation, which includes many secrets of nature. The interaction between spinors and gravity is the most complicated and subtle interaction in the universe, which involves the basic problem of a unified quantum theory and general relativity. The spinor connection has been constructed and researched in many works [1–5]. The spinor field is used to explain the accelerating expansion of the universe and dark matter and dark energy [6–11]. In the previous works, we usually used spinor covariant derivative directly, in which the spinor connection takes a compact form and its physical meaning becomes ambiguous. In this paper, by means of Clifford algebra, we split the spinor connection into geometrical and dynamical parts $(\Upsilon_\mu, \Omega_\mu)$, respectively [12]. This form of connection is determined by metric, independent of Dirac matrices. Only in this representation, we can clearly define classical concepts such as coordinate, speed, momentum and spin for a spinor, and then derive the classical mechanics in detail. $\Upsilon_\mu \in \Lambda^1$ only corresponds to the geometrical calculations, but $\Omega_\mu \in \Lambda^3$ leads to dynamical effects. Ω_μ couples with the spin S^μ of a spinor, which provides location and navigation functions for a spinor with little energy. This term is also related with the origin of the magnetic field of a celestial body [12]. So this form of connection is helpful in understanding the subtle relation between spinor and space-time.

The classical theory for a spinor moving in gravitational field is firstly studied by Mathisson [13], and then developed by Papapetrou [14] and Dixon [15]. A detailed derivation can be found in [16]. By the commutator of the covariant derivative of the spinor $[\nabla_\alpha, \nabla_\beta]$, we obtain an extra approximate acceleration of the spinor as follows

$$a_\alpha(x^\mu) = -\frac{\hbar}{4m} R_{\alpha\beta\gamma\delta}(x^\mu) u^\beta(x^\mu) S^{\gamma\delta}(x^\mu), \tag{1}$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature, u^α 4-vector speed and $S^{\gamma\delta}$ the half commutator of the Dirac matrices.

Equation (1) leads to the violation of Einstein's equivalence principle. This problem was discussed by many authors [16–23]. In [17], the exact Cini–Touschek transformation and the ultra-relativistic limit of the fermion theory were derived, but the Foldy–Wouthuysen transformation is not uniquely defined. The following calculations also show that the usual covariant derivative ∇_μ includes cross terms, which is not parallel to the speed u^μ of the spinor.

To study the coupling effect of spinor and space-time, we need the energy-momentum tensor (EMT) of spinor in curved space-time. The interaction of spinor and gravity is considered by H. Weyl as early as in 1929 [24]. There are some approaches to the general expression of EMT of spinors in curved space-time [4,8,25,26]; however, the formalisms are usually quite complicated for practical calculation and different from each other. In [6–9,11], the space-time is usually Friedmann–Lemaitre–Robertson–Walker type with diagonal metric. The energy-momentum tensor $T_{\mu\nu}$ of spinors can be directly derived from Lagrangian of the spinor field in this case. In [4,25], according to the Pauli's theorem

$$\delta\gamma^\alpha = \frac{1}{2}\gamma_\beta \delta g^{\alpha\beta} + [\gamma^\alpha, M], \tag{2}$$

where M is a traceless matrix related to the frame transformation, the EMT for Dirac spinor ϕ was derived as follows,

$$T^{\mu\nu} = \frac{1}{2}\Re\langle\phi^\dagger\ (\gamma^\mu i\nabla^\nu + \gamma^\nu i\nabla^\mu)\phi\rangle, \tag{3}$$

where $\phi^\dagger\ = \phi^+\gamma$ is the Dirac conjugation, ∇^μ is the usual covariant derivatives for spinor. A detailed calculation for variation of action was performed in [8], and the results were a little different from (2) and (3).

The following calculation shows that, M is still related with $\delta g^{\mu\nu}$, and provides nonzero contribution to $T^{\mu\nu}$ in general cases. The exact form of EMT is much more complex than (3), which includes some important effects overlooked previously. The covariant derivatives operator $i\nabla_\mu$ for spinor includes components in grade-3 Clifford algebra Λ^3, which is not parallel to the classical momentum $p_\mu \in \Lambda^1$. The derivation of rigorous $T_{\mu\nu}$ is quite difficult due to non-uniqueness representation and complicated formalism of vierbein or tetrad frames. In this paper, we provide a systematical and detailed calculation for EMT of spinors. We clearly establish the relations between tetrad and metric at first, and then solve the Euler derivatives with respect to $g_{\mu\nu}$ to obtain an explicit and rigorous form of $T_{\mu\nu}$.

From the results we find some new and interesting conclusions. Besides the usual kinetic energy momentum term, we find three kinds of other additional terms in EMT of bispinor. One is the self interactive potential, which acts like negative pressure. The other reflects the interaction of momentum p^μ with tetrad, which vanishes in classical approximation. The third is the spin-gravity coupling term $\Omega_\alpha S^\alpha$, which is a higher-order infinitesimal in weak field, but becomes important in a neutron star. All these results are based on Clifford algebra decomposition of usual spin connection Γ_μ into geometrical part Υ_μ and dynamical part Ω_μ, which not only makes calculation simpler, but also highlights their different physical meanings. In the calculation of tetrad formalism, we find a new spinor coefficient table $S^{\mu\nu}_{ab}$, which plays an important role in the interaction of spinor with gravity and appears in many places.

This paper is an improvement and synthesis of the previous works arXiv:gr-qc/0610001 and arXiv:gr-qc/0612106. The materials in this paper are organized as follows: In the next section, we specify notations and conventions used in the paper, and derive the spinor connections in form of Clifford algebra. In Section 3, we set up the relations between tetrad and metric, which is the technical foundations of classical approximation of Dirac equation and EMT of spinor. We derive the classical approximation of spinor theory in Section 4, and then calculate the EMT in Section 5. We provide some simple discussions in the last section.

2. Simplification of the Spinor Connection

Clifford algebra is a unified language and efficient tool for physics. The variables and equations expressed by Clifford algebra have a neat and elegant form, and the calculation has a standard but simple procedure [12]. At first we introduce some notations and conventions used in this paper. We take $\hbar = c = 1$ as units. The element of space-time is described by

$$d\mathbf{x} = \gamma_\mu dx^\mu = \gamma^\mu dx_\mu = \gamma_a \delta X^a = \gamma^a \delta X_a, \tag{4}$$

in which γ_a stands for tetrad, and γ^a for co-frame, which satisfies the following $C\ell_{1,3}$ Clifford algebra,

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}, \qquad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \tag{5}$$

$$\gamma_\mu = f_\mu^{\ a}\gamma_a, \quad \gamma^\mu = f^\mu_{\ a}\gamma^a, \quad \eta_{ab} = \mathrm{diag}(1,-1,-1,-1). \tag{6}$$

The relation between the local frame coefficient $(f^\mu_{\ a}, f_\mu^{\ a})$ and metric is given by

$$f_\mu^{\ a} f^\mu_{\ b} = \delta^a_b, \quad f_\mu^{\ a} f^\nu_{\ a} = \delta^\nu_\mu, \quad f^\mu_{\ a} f^\nu_{\ b}\eta^{ab} = g^{\mu\nu}, \quad f_\mu^{\ a} f_\nu^{\ b}\eta_{ab} = g_{\mu\nu}. \tag{7}$$

We use the Latin characters $(a, b \in \{0,1,2,3\})$ for the Minkowski indices, Greek characters $(\mu,\nu \in \{0,1,2,3\})$ for the curvilinear indices, and $(j,k,l,m,n \in \{1,2,3\})$ for spatial indices. For local frame coefficient in matrix form $(f_\mu^{\ a})$ and $(f^\mu_{\ a})$, the curvilinear index μ is row index and Minkowski index a is column index. The Pauli and Dirac matrices in Minkowski space-time are given by

$$\sigma^a \equiv \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \tag{8}$$

$$\widetilde{\sigma}^a \equiv (\sigma^0, -\vec{\sigma}), \qquad \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3), \tag{9}$$

$$\gamma^a \equiv \begin{pmatrix} 0 & \widetilde{\sigma}^a \\ \sigma^a & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{10}$$

Since the Clifford algebra is isomorphic to the matrix algebra, we need not distinguish tetrad γ^a and matrix γ^a in algebraic calculation.

There are several definitions for Clifford algebra [27,28]. Clifford algebra is also called geometric algebra. If the definition is directly related to geometric concepts, it will bring great convenience to the study and research of geometry [12,29].

Definition 1. *Assume the element of an $n = p + q$ dimensional space-time $\mathbb{M}^{p,q}$ over $\mathbb{R}$ is given by (4). The space-time is endowed with distance $ds = |d\mathbf{x}|$ and oriented volumes dV_k calculated by*

$$d\mathbf{x}^2 = \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}\delta X^a \delta X^b, \tag{11}$$

$$dV_k = d\mathbf{x}_1 \wedge d\mathbf{x}_2 \wedge \cdots \wedge d\mathbf{x}_k = \gamma_{\mu\nu\cdots\omega} dx_1^\mu dx_2^\nu \cdots dx_k^\omega, \ \ (1 \le k \le n), \tag{12}$$

in which the Minkowski metric $(\eta_{ab}) = diag(I_p, -I_q)$, *and Grassmann basis* $\gamma_{\mu\nu\cdots\omega} = \gamma_\mu \wedge \gamma_\nu \wedge \cdots \wedge \gamma_\omega \in \Lambda^k \mathbb{M}^{p,q}$. *Then the following number with basis*

$$C = c_0 I + c_\mu \gamma^\mu + c_{\mu\nu}\gamma^{\mu\nu} + \cdots + c_{12\cdots n}\gamma^{12\cdots n}, \quad (\forall c_k \in \mathbb{R}) \tag{13}$$

together with multiplication rule of basis given in (11) *and associativity define the* 2^n*-dimensional real universal* **Clifford algebra** $C\ell_{p,q}$.

The geometrical meanings of elements $d\mathbf{x}, d\mathbf{y}, d\mathbf{x} \wedge d\mathbf{y}$ are shown in Figure 1.

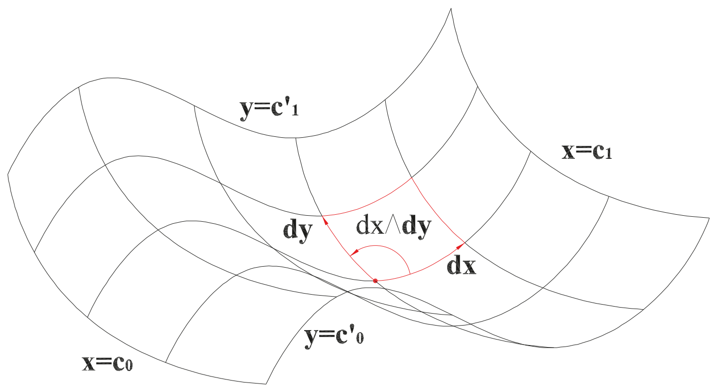

Figure 1. Geometric meaning of vectors $d\mathbf{x}, d\mathbf{y}$ and $d\mathbf{x} \wedge d\mathbf{y}$.

Figure 1 shows that the exterior product is oriented volume of the parallel polyhedron of the line element vectors, and the Grassmann basis $\gamma_{ab\cdots c}$ is just the orthonormal basis of k-dimensional volume. Since the length of a line element and the volumes of each grade constitute the fundamental contents of geometry, the Grassmann basis set becomes units to represent various geometric and physical quantities, which are special kinds of tensors.

By straightforward calculation we have [5,12,29]

Theorem 1. *For* $C\ell_{1,3}$, *we have the following useful relations*

$$I, \quad \gamma^a, \quad \gamma^{ab} = \tfrac{i}{2}\epsilon^{abcd}\gamma_{cd}\gamma^5, \quad \gamma^{abc} = i\epsilon^{abcd}\gamma_d\gamma^5, \quad \gamma^{0123} = -i\gamma^5. \tag{14}$$

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} + \gamma^{\mu\nu}, \qquad \gamma^{\mu\nu}\gamma^\omega = \gamma^\mu g^{\nu\omega} - \gamma^\nu g^{\mu\omega} + \gamma^{\mu\nu\omega}. \tag{15}$$

The above theorem provides several often used relations between the Clifford products and the Grassmann products. Since the calculations of geometric and physical quantities are mostly in the form of Clifford products, but only by expressing these forms as Grassmann products, their geometric and physical significance is clear. Thus the above transformation relations become fundamental and important.

For Dirac equation in curved space-time without torsion, we have [1–4,30],

$$\gamma^\mu(i\nabla_\mu - eA_\mu)\phi = m\phi, \qquad \nabla_\mu\phi = (\partial_\mu + \Gamma_\mu)\phi, \tag{16}$$

in which the spinor connection is given by

$$\Gamma_\mu \quad \equiv \quad \frac{1}{4}\gamma_\nu\gamma^\nu_{;\mu} = \frac{1}{4}\gamma^\nu\gamma_{\nu;\mu} = \frac{1}{4}\gamma^\nu(\partial_\mu\gamma_\nu - \Gamma^\alpha_{\mu\nu}\gamma_\alpha). \tag{17}$$

The total spinor connection $\gamma^\mu\Gamma_\mu \in \Lambda^1 \cup \Lambda^3$. Clearly, $\gamma^\mu\Gamma_\mu$ is a Clifford product, and its geometric and physical significance is unclear. Only by projecting it onto the Grassmann basis γ_a and γ_{abc}, its geometric and physical meanings become clear [12].

Theorem 2. *Dirac equation* (16) *can be rewritten in the following Hermitian form*

$$(\alpha^\mu \hat{p}_\mu - S^\mu \Omega_\mu)\phi = m\gamma^0\phi, \tag{18}$$

in which α^μ *is current operator,* $\hat{p}_\mu$ *momentum and* S^μ *spin operator,*

$$\alpha^\mu = diag(\sigma^\mu, \widetilde{\sigma}^\mu), \quad \hat{p}_\mu = i(\partial_\mu + \Upsilon_\mu) - eA_\mu, \quad S^\mu = \frac{1}{2} diag(\sigma^\mu, -\widetilde{\sigma}^\mu), \tag{19}$$

where Υ_μ *is Keller connection and* Ω_μ *Gu–Nester potential, they are respectively defined as*

$$\Upsilon_\nu \equiv \frac{1}{2} f^\mu_{\ a}(\partial_\nu f_\mu^{\ a} - \partial_\mu f_\nu^{\ a}) = \frac{1}{2}\Big[\partial_\nu(\ln\sqrt{g}) - f^\mu_{\ a}\partial_\mu f_\nu^{\ a}\Big], \tag{20}$$

$$\Omega^\alpha \equiv \frac{1}{2} f^\alpha_{\ d} f^\mu_{\ a} f^\nu_{\ b} \partial_\mu f_\nu^{\ e} \epsilon^{abcd}\eta_{ce} = \frac{1}{4\sqrt{g}}\epsilon^{\alpha\mu\nu\omega}\eta_{ab} f_\omega^{\ a}(\partial_\mu f_\nu^{\ b} - \partial_\nu f_\mu^{\ b}). \tag{21}$$

Proof. By (14) and (15), we have the following Clifford calculus

$$\begin{aligned}
\gamma^\mu\Gamma_\mu &= \frac{1}{4}\gamma^\mu\gamma^\nu(\partial_\mu\gamma_\nu - \Gamma^\alpha_{\mu\nu}\gamma_\alpha) = \frac{1}{4}(g^{\mu\nu} + \gamma^{\mu\nu})(\partial_\mu\gamma_\nu - \Gamma^\alpha_{\mu\nu}\gamma_\alpha) \\
&= \frac{1}{4}(\gamma^\mu_{;\mu} + \gamma^{\mu\nu}\partial_\mu\gamma_\nu) = \frac{1}{4}(\partial_\mu\gamma^\mu + \partial_\mu \ln(\sqrt{g})\gamma^\mu) + \frac{1}{4} f^\mu_{\ a} f^\nu_{\ b}\partial_\mu f_\nu^{\ c}\gamma^{ab}\gamma_c \\
&= \frac{1}{4}[\gamma^a\partial_\mu f^\mu_{\ a} + (f^\nu_{\ a}\partial_\mu f_\nu^{\ a})\gamma^\mu] + \frac{1}{4} f^\mu_{\ a} f^\nu_{\ b}\partial_\mu f_\nu^{\ d}\gamma^{ab}\gamma^c\eta_{cd} \\
&= \frac{1}{4} f^\mu_{\ a}\gamma^\nu(-\partial_\mu f_\nu^{\ a} + \partial_\nu f_\mu^{\ a}) + \frac{1}{4} f^\mu_{\ a} f^\nu_{\ b}\partial_\mu f_\nu^{\ d}(\eta^{bc}\gamma^a - \eta^{ac}\gamma^b + \gamma^{abc})\eta_{cd} \\
&= \frac{1}{2} f^\mu_{\ a}\gamma^\nu(\partial_\nu f_\mu^{\ a} - \partial_\mu f_\nu^{\ a}) + \frac{1}{4} f^\mu_{\ a} f^\nu_{\ b}\partial_\mu f_\nu^{\ e}\gamma^{abc}\eta_{ce} \\
&= \Upsilon_\mu\gamma^\mu + \frac{i}{2}\Omega^\alpha\gamma_\alpha\gamma^5.
\end{aligned} \tag{22}$$

Substituting it into (16) and multiplying the equation by γ^0, we prove the theorem. □

The following discussion shows that Υ_μ and Ω_μ have different physical meanings. $\partial_\mu + \Upsilon_\mu$ as a whole operator is similar to the covariant derivatives ∇_μ for vector, it only has a geometrical effect; however, Ω_μ couples with the spin of a particle and leads to the magnetic field of a celestial body [12]. $\Omega_\mu \equiv 0$ is a necessary condition for the metric to be diagonalized. If the gravitational field is generated by a rotating ball, the corresponding metric, similar to the Kerr one, cannot be diagonalized. In this case, the spin-gravity coupling term has a non-zero coupling effect.

In axisymmetric and asymptotically flat space-time we have the line element in quasi-spherical coordinate system [31]

$$d\mathbf{x} = \gamma_0\sqrt{U}(dt + Wd\varphi) + \sqrt{V}(\gamma_1 dr + \gamma_2 r d\theta) + \gamma_3\sqrt{U^{-1}} r\sin\theta d\varphi, \tag{23}$$

$$d\mathbf{x}^2 = U(dt + Wd\varphi)^2 - V(dr^2 + r^2 d\theta^2) - U^{-1}r^2\sin^2\theta d\varphi^2, \tag{24}$$

in which (U, V, W) is just functions of (r, θ). As $r \to \infty$ we have

$$U \to 1 - \frac{2m}{r}, \quad W \to \frac{4L}{r}\sin^2\theta, \quad V \to 1 + \frac{2m}{r}, \tag{25}$$

where (m, L) are mass and angular momentum of the star, respectively. For common stars and planets we always have $r \gg m \gg L$. For example, we have $m \doteq 3$ km for the sun. The nonzero tetrad coefficients of metric (23) are given by

$$\begin{cases} f_t^{\ 0} = \sqrt{U},\ f_r^{\ 1} = \sqrt{V},\ f_\theta^{\ 2} = r\sqrt{V},\ f_\varphi^{\ 3} = \frac{r\sin\theta}{\sqrt{U}},\ f_\varphi^{\ 0} = \sqrt{U}W, \\ f^t_{\ 0} = \frac{1}{\sqrt{U}},\ f^r_{\ 1} = \frac{1}{\sqrt{V}},\ f^\theta_{\ 2} = \frac{1}{r\sqrt{V}},\ f^\varphi_{\ 3} = \frac{\sqrt{U}}{r\sin\theta},\ f^t_{\ 3} = \frac{-\sqrt{U}W}{r\sin\theta}. \end{cases} \tag{26}$$

Substituting (26) into (21) or the following (54), we obtain

$$\begin{aligned} \Omega^\alpha &= f^t_{\ 0} f^r_{\ 1} f^\theta_{\ 2} f^\varphi_{\ 3}(0, \partial_\theta g_{t\varphi}, -\partial_r g_{t\varphi}, 0) \\ &= \left(Vr^2\sin\theta\right)^{-1}(0, \partial_\theta(UW), -\partial_r(UW), 0) \\ &\to \frac{4L}{r^4}(0, 2r\cos\theta, \sin\theta, 0). \end{aligned} \tag{27}$$

By (27) we find that the intensity of Ω^α is proportional to the angular momentum of the star, and its force line is given by

$$\frac{dx^\mu}{ds} = \Omega^\mu \Rightarrow \frac{dr}{d\theta} = \frac{2r\cos\theta}{\sin\theta} \Leftrightarrow r = R\sin^2\theta. \tag{28}$$

Equation (28) shows that, the force lines of Ω^α is just the magnetic lines of a magnetic dipole. According to the above results, we know that the spin-gravity coupling potential of charged particles will certainly induce a macroscopic dipolar magnetic field for a star, and it should be approximately in accordance with the Schuster–Wilson–Blackett relation [12].

For diagonal metric

$$g_{\mu\nu} = \mathrm{diag}(N_0^2, -N_1^2, -N_2^2, -N_3^2), \qquad \sqrt{g} = N_0 N_1 N_2 N_3, \tag{29}$$

where $N_\mu = N_\mu(x^\alpha)$, we have $\Omega_\mu \equiv 0$ and

$$\gamma^\mu = \left(\frac{\gamma^0}{N_0}, \frac{\gamma^1}{N_1}, \frac{\gamma^2}{N_2}, \frac{\gamma^3}{N_3}\right), \qquad \Upsilon_\mu = \frac{1}{2}\partial_\mu \ln\left(\frac{\sqrt{g}}{N_\mu}\right). \tag{30}$$

For Dirac equation in Schwarzschild metric,

$$g_{\mu\nu} = \mathrm{diag}(B(r), -A(r), -r^2, -r^2\sin^2\theta), \tag{31}$$

we have

$$\gamma^\mu = \left(\frac{\gamma^0}{\sqrt{B}}, \frac{\gamma^1}{\sqrt{A}}, \frac{\gamma^2}{r}, \frac{\gamma^3}{r\sin\theta}\right), \qquad \Upsilon_\mu = \left(1, \frac{1}{r} + \frac{B'}{4B}, \frac{1}{2}\cot\theta, 0\right). \tag{32}$$

The Dirac equation for free spinor is given by

$$i\left(\frac{\gamma^0}{\sqrt{B}}\partial_t + \frac{\gamma^1}{\sqrt{A}}(\partial_r + \frac{1}{r} + \frac{B'}{4B}) + \frac{\gamma^2}{r}(\partial_\theta + \frac{1}{2}\cot\theta) + \frac{\gamma^3}{r\sin\theta}\partial_\varphi\right)\phi = m\phi. \tag{33}$$

Setting $A = B = 1$, we obtain the Dirac equation in a spherical coordinate system. In contrast with the spinor in the Cartesian coordinate system, the spinor in the (33) includes an implicit rotational transformation [12].

3. Relations between Tetrad and Metric

Different from the cases of vector and tensor, in general relativity the equation of spinor fields depends on the local tetrad frame. The tetrad γ^α can be only determined by metric to an arbitrary Lorentz transformation. This situation makes the derivation of EMT quite complicated. In this section, we provide an explicit representation of tetrad and derive the EMT of spinor based on this representation. For convenience to check the results by computer, we denote the element by $dx^\mu = (dx, dy, dz, cdt)$ and $\delta X^a = (\delta X, \delta Y, \delta Z, c\delta T)$.

For metric $g_{\mu\nu}$, not losing generality we assume that, in the neighborhood of x^μ, dx^0 is time-like and (dx^1, dx^2, dx^3) are space-like. This means $g_{00} > 0$, $g_{kk} \le 0 (k \ne 0)$, and the following definitions of J_k are real numbers

$$J_1 = \sqrt{-g_{11}},\ J_2 = \sqrt{\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}},\ J_3 = \sqrt{-\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}},\ J_0 = \sqrt{-\det(g)}. \tag{34}$$

$$u_1 = \begin{vmatrix} g_{11} & g_{12} \\ g_{31} & g_{32} \end{vmatrix}, \quad u_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{01} & g_{02} \end{vmatrix}, \quad u_3 = \begin{vmatrix} g_{21} & g_{22} \\ g_{31} & g_{32} \end{vmatrix}, \tag{35}$$

$$v_1 = \begin{vmatrix} g_{12} & g_{13} & g_{10} \\ g_{22} & g_{23} & g_{20} \\ g_{32} & g_{33} & g_{30} \end{vmatrix},\ v_2 = \begin{vmatrix} g_{11} & g_{13} & g_{10} \\ g_{21} & g_{23} & g_{20} \\ g_{31} & g_{33} & g_{30} \end{vmatrix},\ v_3 = \begin{vmatrix} g_{11} & g_{12} & g_{10} \\ g_{21} & g_{22} & g_{20} \\ g_{31} & g_{32} & g_{30} \end{vmatrix}. \tag{36}$$

The following conclusions can be checked by computer program.

Theorem 3. *For LU decomposition of matrix* $(g_{\mu\nu})$

$$(g_{\mu\nu}) = L(\eta_{ab})L^+,\ \ (g^{\mu\nu}) = U(\eta_{ab})U^+,\ \ U = L^* = (L^+)^{-1}, \tag{37}$$

with positive diagonal elements, we have the following unique solution

$$L = (L_\mu^{\ a}) = \begin{pmatrix} -\frac{g_{11}}{J_1} & 0 & 0 & 0 \\ -\frac{g_{21}}{J_1} & \frac{J_2}{J_1} & 0 & 0 \\ -\frac{g_{31}}{J_1} & \frac{u_1}{J_1 J_2} & \frac{J_3}{J_2} & 0 \\ -\frac{g_{01}}{J_1} & \frac{u_2}{J_1 J_2} & -\frac{v_3}{J_2 J_3} & \frac{J_0}{J_3} \end{pmatrix}, \tag{38}$$

$$U = (U^\mu_{\ a}) = \begin{pmatrix} \frac{1}{J_1} & \frac{g_{21}}{J_1 J_2} & \frac{u_3}{J_2 J_3} & \frac{v_1}{J_3 J_0} \\ 0 & \frac{J_1}{J_2} & -\frac{u_1}{J_2 J_3} & -\frac{v_2}{J_3 J_0} \\ 0 & 0 & \frac{J_2}{J_3} & \frac{v_3}{J_3 J_0} \\ 0 & 0 & 0 & \frac{J_3}{J_0} \end{pmatrix}. \tag{39}$$

Theorem 4. *For any solution of tetrad (7) in matrix form* $(f_\mu^{\ a})$ *and* $(f^\mu_{\ a})$*, there exists a local Lorentz transformation* $\delta X'^a = \Lambda^a_{\ b}\delta X^b$ *independent of* $g_{\mu\nu}$*, such that*

$$(f_\mu^{\ a}) = L\Lambda^+, \qquad (f^\mu_{\ a}) = U\Lambda^{-1}, \tag{40}$$

where $\Lambda = (\Lambda^a_{\ b})$ *stands for the matrix of Lorentz transformation.*

Proof. For any solution (7) we have

$$(g_{\mu\nu}) = L(\eta_{ab})L^+ = (f_\mu^{\ a})(\eta_{ab})(f_\mu^{\ a})^+ \ \Leftrightarrow\ L^{-1}(f_\mu^{\ a})(\eta_{ab})(L^{-1}(f_\mu^{\ a}))^+ = (\eta_{ab}). \tag{41}$$

So we have a Lorentz transformation matrix $\Lambda = (\Lambda^a_{\ b})$, such that

$$L^{-1}(f_\mu^{\ a}) = \Lambda^+ \ \Leftrightarrow\ (f_\mu^{\ a}) = L\Lambda^+,\ \text{ or }\ f_\mu^{\ a} = L_\mu^{\ b}\Lambda^a_{\ b}. \tag{42}$$

Similarly we have $(f^\mu_{\ a}) = U\Lambda^{-1}$. The proof is finished. □

The decomposition (37) is similar to the Gram–Schmidt orthogonalization for vectors dx^μ in the order $dt \to dz \to dy \to dx$. In matrix form, by (37) we have $\delta X = L^+ dx$ and

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} \delta X^a \delta X^b \\ &= (L_t{}^T dt)^2 - (L_x{}^X dx + L_y{}^X dy + L_z{}^X dz + L_t{}^X dt)^2 \\ &\quad -(L_y{}^Y dy + L_z{}^Y dz + L_t{}^Y dt)^2 - (L_z{}^Z dz + L_t{}^Z dt)^2. \end{aligned} \tag{43}$$

Equation (43) is a direct result of (38), but (43) manifestly shows the geometrical meanings of the tetrad components $L_\mu^{\ a}$. Obviously, (43) is also the method of completing the square to calculate the tetrad coefficients $f_\mu^{\ a}$.

The above theorems Theorems 3 and 4 provide the solution of the Equation (7), and the geometric meaning of the solution is (4). In differential geometry, the element (4) is more fundamental than the distance formula $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, because (4) clarifies the geometric meanings of the basis vectors γ_μ and γ_a, and Clifford algebra (5) or (11) as well as Grassmann algebra (12) and (13) provide the calculating rules of the basis [12,29].

For LU decomposition (39), we define a **spinor coefficient table** by

$$S_{ab}^{\mu\nu} \equiv \begin{pmatrix} 0 & -U_1^{\{\mu} U_2^{\nu\}} & -U_1^{\{\mu} U_3^{\nu\}} & -U_1^{\{\mu} U_0^{\nu\}} \\ U_2^{\{\mu} U_1^{\nu\}} & 0 & -U_2^{\{\mu} U_3^{\nu\}} & -U_2^{\{\mu} U_0^{\nu\}} \\ U_3^{\{\mu} U_1^{\nu\}} & U_3^{\{\mu} U_2^{\nu\}} & 0 & -U_3^{\{\mu} U_0^{\nu\}} \\ U_0^{\{\mu} U_1^{\nu\}} & U_0^{\{\mu} U_2^{\nu\}} & U_0^{\{\mu} U_3^{\nu\}} & 0 \end{pmatrix} = -S_{ba}^{\mu\nu}, \tag{44}$$

in which

$$U_a^{\{\mu} U_b^{\nu\}} = \frac{1}{2}(U^\mu_{\ a} U^\nu_{\ b} + U^\nu_{\ a} U^\mu_{\ b}) = U_b^{\{\mu} U_a^{\nu\}}. \tag{45}$$

$S_{ab}^{\mu\nu} = U_a^{\{\mu} U_b^{\nu\}} \operatorname{sign}(a-b) = -S_{ba}^{\mu\nu}$ is not a tensor for indices (a, b), it is symmetrical for Riemann indices (μ, ν) but anti-symmetrical for Minkowski indices (a, b). For diagonal metric we have $S_{ab}^{\mu\nu} \equiv 0$. It should be stressed again, $S_{ab}^{\mu\nu}$ is not a tensor for indices (a, b); however, for any local Lorentz transformation $\delta X' = \Lambda \delta X$, if taking (44) as the proper values and setting Lorentz transformation

$$(S'^{\mu\nu}_{ab}) = \Lambda^* (S^{\alpha\beta}_{cd}) \Lambda^{-1}, \qquad \Lambda^* \equiv (\Lambda^{-1})^+,$$

then $S_{ab}^{\mu\nu}$ becomes a tensor for indices (a, b).

By representation of (38), (39) and relation (40), we can check the following results by straightforward calculation.

Theorem 5. *For tetrad (40), we have*

$$\frac{\partial f_\alpha^{\ n}}{\partial g_{\mu\nu}} = \frac{1}{4}(\delta^\mu_\alpha f^\nu_{\ m} + \delta^\nu_\alpha f^\mu_{\ m}) \eta^{nm} + \frac{1}{2} S_{ab}^{\mu\nu} f_\alpha^{\ a} \eta^{nb}. \tag{46}$$

$$\frac{\partial f^\alpha_{\ a}}{\partial g_{\mu\nu}} = -\frac{1}{4}(f^\mu_{\ a} g^{\alpha\nu} + f^\nu_{\ a} g^{\mu\alpha}) - \frac{1}{2} S_{ab}^{\mu\nu} f^\alpha_{\ n} \eta^{nb}. \tag{47}$$

Or equivalently,

$$\frac{\partial \gamma_\alpha}{\partial g_{\mu\nu}} = \frac{1}{4}(\delta^\mu_\alpha \gamma^\nu + \delta^\nu_\alpha \gamma^\mu) + \frac{1}{2} S_{ab}^{\mu\nu} f_\alpha^{\ a} \gamma^b. \tag{48}$$

$$\frac{\partial \gamma^\alpha}{\partial g_{\mu\nu}} = -\frac{1}{4}(g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu) - \frac{1}{2} S_{ab}^{\mu\nu} f^\alpha_{\ n} \gamma^a \eta^{nb}. \tag{49}$$

Or equivalently,

$$\delta\gamma_\alpha = \frac{1}{2}\gamma^\beta(\delta g_{\alpha\beta} + S^{\mu\nu}_{ab} f_\alpha{}^a f_\beta{}^b \delta g_{\mu\nu}), \tag{50}$$

$$\delta\gamma^\lambda = -\frac{1}{2}g^{\lambda\beta}\gamma^\alpha(\delta g_{\alpha\beta} + S^{\mu\nu}_{ab} f_\alpha{}^a f_\beta{}^b \delta g_{\mu\nu}) = -g^{\lambda\alpha}\delta\gamma_\alpha. \tag{51}$$

For any given vector A^μ, we have

$$A^\alpha \frac{\partial\gamma_\alpha}{\partial g_{\mu\nu}} = \frac{1}{4}(A^\mu\gamma^\nu + A^\nu\gamma^\mu) + \frac{1}{4}S^{\mu\nu}_{ab}(A^a\gamma^b - A^b\gamma^a), \tag{52}$$

$$A_\alpha \frac{\partial\gamma^\alpha}{\partial g_{\mu\nu}} = -\frac{1}{4}(A^\mu\gamma^\nu + A^\nu\gamma^\mu) + \frac{1}{4}S^{\mu\nu}_{ab}(A^a\gamma^b - A^b\gamma^a). \tag{53}$$

In (46)–(53), we set $\frac{\partial\gamma^\alpha}{\partial g_{\mu\nu}} = \frac{\partial\gamma^\alpha}{\partial g_{\nu\mu}} = \frac{1}{2}\frac{d\gamma^\alpha}{dg_{\mu\nu}}$ for $\mu \neq \nu$ to obtain the tensor form. $\frac{d}{dg_{\mu\nu}}$ is the total derivative for $g_{\mu\nu}$ and $g_{\nu\mu}$. $S^{\mu\nu}_{ab}$ is transformed from (44).

The following derivation only use the property $S^{\mu\nu}_{ab} = -S^{\mu\nu}_{ba}$. For Ω^α, we have

$$\Omega^d = \frac{1}{4}\epsilon^{abcd} f^\alpha_{\ a} S^{\mu\nu}_{bc}\partial_\alpha g_{\mu\nu}, \qquad \Omega^\alpha = -\frac{1}{4}\epsilon^{dabc} f^\alpha_{\ d} f^\beta_{\ a} S^{\mu\nu}_{bc}\partial_\beta g_{\mu\nu}. \tag{54}$$

4. The Classical Approximation of Dirac Equation

In this section, we derive the classical mechanics for a charged spinor moving in gravity, and disclose the physical meaning of connections Υ_μ and Ω_μ. By covariance principle, the Dirac Equation (18) is valid and covariant in any regular coordinate system; however, in order to obtain the energy eigenstates of a spinor we need to solve the Hamiltonian system of quantum mechanics, and in order to derive its classical mechanics we need to calculate the spatial integrals of its Noether charges such as coordinates, energy and momentum. These computations cannot be realized in an arbitrary coordinate system, but must be performed in a coordinate system with realistic global simultaneity; that is, we need the Gu's natural coordinate system (NCS) [12,32]

$$ds^2 = g_{tt}dt^2 - \bar{g}_{kl}dx^k dx^l, \quad d\tau = \sqrt{g_{tt}}dt = f_t{}^0 dt, \quad dV = \sqrt{\bar{g}}d^3x. \tag{55}$$

in which ds is the proper time element, $d\tau$ the Newton's absolute cosmic time element and dV the absolute volume element of the space at time t. NCS generally exists and the global simultaneity is unique. Only in NCS we can clearly establish the Hamiltonian formalism and calculate the integrals of Noether charges. In NCS, we have

$$f_t{}^0 = \sqrt{g_{tt}}, \quad f^t_{\ 0} = \frac{1}{\sqrt{g_{tt}}}, \quad \gamma_t = \sqrt{g_{tt}}\gamma_0, \quad \gamma^t = \frac{1}{\sqrt{g_{tt}}}\gamma^0. \tag{56}$$

Then by (20) we obtain

$$\Upsilon_\mu = \frac{1}{2}\left(\partial_t \ln\sqrt{\bar{g}},\ f_k{}^a\partial_j f^j_{\ a} + \partial_k \ln\sqrt{\bar{g}}\right), \quad \Upsilon^t = g^{tt}\Upsilon_t, \quad \Upsilon^k = -\bar{g}^{kl}\Upsilon_l. \tag{57}$$

In NCS, to lift and lower the index of a vector means $\Omega^t = g^{tt}\Omega_t$, $\Omega^k = -\bar{g}^{kl}\Omega_l$.

More generally, we consider the Dirac equation with electromagnetic potential eA^μ and nonlinear potential $N(\check{\gamma}) = \frac{1}{2}w\check{\gamma}^2$, where $\check{\gamma} = \phi^+\gamma_0\phi$. Then (18) can be rewritten in Hamiltonian formalism

$$i\alpha^t\nabla_t\phi = \mathbf{H}\phi, \qquad \mathbf{H} = -\alpha^k\hat{p}_k + e\alpha^t A_t + S^\mu\Omega_\mu + (m - N')\gamma_0, \tag{58}$$

where $\mathbf{H}$ is the Hamiltonian or energy of the spinor, $\alpha^t = f^t_{\ 0}\alpha^0 = (\sqrt{g_{tt}})^{-1}$ and $\nabla_\mu = \partial_\mu + \Upsilon_\mu$. Since $d\tau = f_t{}^0 dt$ is the realistic time of the universe, only $i\alpha^t\nabla_t = i\partial_\tau$ is the true energy operator for a spinor. g_{tt} represents the gravity, and it cannot be generally merged into $d\tau$ as performed in a semi-geodesic coordinate system.

In traditional quantum theory, we simultaneously take coordinate, speed, momentum and wave function of a particle as original concepts. This situation is the origin of logical confusion. As a matter of fact, only **wave function** ϕ is independent concept and dynamical Equation (58) is fundamental in logic. Other concepts of the particle should be defined by ϕ and (58). Similarly to the case in flat space-time [33], we define some classical concepts for the spinor.

Definition 2. *The coordinate $\vec{X}$ and speed $\vec{v}$ of the spinor is defined as*

$$X^k(t) \equiv \int_{R^3} x^k |\phi|^2 \sqrt{\bar{g}} d^3x = \int_{R^3} x^k q^t \sqrt{g} d^3x, \quad v^k \equiv \frac{d}{d\tau} X^k = f^t{}_0 \frac{d}{dt} X^k, \tag{59}$$

where R^3 stands for the total simultaneous hypersurface, $q^\mu = \phi^+ \alpha^\mu \phi = \check{\alpha}^\mu$ is the current.

By definition (59) and current conservation law $q^\mu_{;\mu} = (\sqrt{g})^{-1} \partial_\mu (q^\mu \sqrt{g}) = 0$, we have

$$\begin{aligned} v^j &= f^t{}_0 \int_{R^3} x^j \partial_t (q^t \sqrt{g}) d^3x = -f^t{}_0 \int_{R^3} x^j \partial_k (q^k \sqrt{g}) d^3x \\ &= f^t{}_0 \int_{R^3} q^j \sqrt{g} d^3x \to \int_{R^3} q^j \sqrt{\bar{g}} d^3x. \end{aligned} \tag{60}$$

Since a spinor has only a very tiny structure, together with normalizing condition $\int_{R^3} q^t \sqrt{g} d^3x = 1$, we obtain the classical point-particle model for the spinor as [33]

$$q^\mu \to u^\mu \sqrt{1 - v^2} \delta^3(\vec{x} - \vec{X}), \quad v^2 = \bar{g}_{kl} v^k v^l, \quad u^\mu = \frac{dX^\mu}{ds} = \frac{v^\mu}{\sqrt{1 - v^2}}, \tag{61}$$

where the Dirac-δ means $\int_{R^3} \delta^3(\vec{x} - \vec{X}) \sqrt{\bar{g}} d^3x = 1$.

Theorem 6. *For any Hermitian operator $\hat{P}$, $P \equiv \int_{R^3} \sqrt{\bar{g}} \phi^+ \hat{P} \phi d^3x$ is real for any ϕ. We have the following generalized Ehrenfest theorem,*

$$\frac{dP}{dt} = \Re \int_{R^3} \sqrt{g} \phi^+ \Big(\alpha^t \partial_t \hat{P} - i f^t{}_0 [\hat{P}, f_t{}^0] \mathbf{H} + i [\mathbf{H}, \hat{P}] \Big) \phi d^3x, \tag{62}$$

where $\Re$ means taking the real part.

Proof. By (57) and (58), we have

$$\begin{aligned} \frac{dP}{dt} &= \frac{d}{dt} \int_{R^3} \sqrt{\bar{g}} \phi^+ \hat{P} \phi d^3x \\ &= \Re \int_{R^3} \sqrt{\bar{g}} (\phi^+ (\partial_t \hat{P}) \phi + i (i\partial_t \phi)^+ \hat{P} \phi - i \phi^+ \hat{P} (i \partial_t \phi) + \phi^+ \hat{P} \phi \partial_t \ln \sqrt{\bar{g}}) d^3x \\ &= \Re \int_{R^3} \sqrt{\bar{g}} \Big(\phi^+ (\partial_t \hat{P}) \phi + i f_t{}^0 (\mathbf{H} \phi)^+ \hat{P} \phi - i \phi^+ \hat{P} (f_t{}^0 \mathbf{H} \phi) \Big) d^3x \\ &= \Re \int_{R^3} \sqrt{g} \phi^+ \Big(\alpha^t \partial_t \hat{P} - i f^t{}_0 [\hat{P}, f_t{}^0] \mathbf{H} + i [\mathbf{H}, \hat{P}] \Big) \phi d^3x \\ &\quad + \Re \int_{R^3} \sqrt{g} \phi^+ (\partial_k \alpha^k + \alpha^k \partial_k \ln \sqrt{g} - 2\alpha^k \Upsilon_k) \hat{P} \phi d^3x \\ &= \Re \int_{R^3} \sqrt{g} \phi^+ \Big(\alpha^t \partial_t \hat{P} - i f^t{}_0 [\hat{P}, f_t{}^0] \mathbf{H} + i [\mathbf{H}, \hat{P}] \Big) \phi d^3x. \end{aligned} \tag{63}$$

Then we prove (62). The proof clearly shows the connection Υ^μ has only geometrical effect, which cancels the derivatives of $\sqrt{g}$. Obviously, we cannot obtain (62) from the conventional definition of spinor connection Γ_μ. □

Definition 3. *The 4-dimensional momentum of the spinor is defined by*

$$p^{\mu} = \Re \int_{R^3} (\phi^+ \hat{p}^{\mu} \phi) \sqrt{\bar{g}} d^3 x. \tag{64}$$

For a spinor at energy eigenstate, we have classical approximation $p^{\mu} = mu^{\mu}$, where m defines the classical **inertial mass** *of the spinor.*

Theorem 7. *For momentum of the spinor $p_{\mu} = \Re \int_{R^3} \sqrt{\bar{g}} \phi^+ \hat{p}_{\mu} \phi d^3 x$, we have*

$$\frac{d}{d\tau} p_{\mu} = f^t{}_0 \Re \int_{R^3} \sqrt{g} (eF_{\mu\nu} q^{\nu} + \check{S}^a \partial_{\mu} \Omega_a - \partial_{\mu} N - \phi^+ (\partial_{\mu} \alpha^{\nu}) \hat{p}_{\nu} \phi) d^3 x, \tag{65}$$

in which

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \qquad \check{S}^a = \phi^+ S^a \phi. \tag{66}$$

Proof. Substituting $\hat{P} = \hat{p}_{\mu}$ and $\mathbf{H}\phi = \alpha^t i \nabla_t \phi$ into (62), by straightforward calculation we obtain

$$\begin{aligned} \frac{d}{d\tau} p_{\mu} &= f^t{}_0 \Re \int_{R^3} \sqrt{g} \phi^+ \Big(-e\alpha^t \partial_t A_{\mu} - (\partial_{\mu} \alpha^t) i \nabla_t + \alpha^k \partial_k \hat{p}_{\mu} \Big) \phi d^3 x \\ &\quad + f^t{}_0 \Re \int_{R^3} \sqrt{g} \phi^+ \Big(\partial_{\mu} (-\alpha^k \hat{p}_k + e\alpha^t A_t + S^{\nu} \Omega_{\nu} - N' \gamma_0) \Big) \phi d^3 x \\ &= f^t{}_0 \Re \int_{R^3} \sqrt{g} (eF_{\mu\nu} q^{\nu} + \phi^+ \partial_{\mu} (S^{\nu} \Omega_{\nu}) \phi - \partial_{\mu} N) d^3 x - K_{\mu}, \end{aligned} \tag{67}$$

in which

$$K_{\mu} = f^t{}_0 \Re \int_{R^3} \sqrt{g} \phi^+ (\partial_{\mu} \alpha^{\nu}) \hat{p}_{\nu} \phi d^3 x. \tag{68}$$

By $S^{\mu} \Omega_{\mu} = S^a \Omega_a$, we prove the theorem. □

For a spinor at particle state [33], by classical approximation $q^{\mu} \to v^{\mu} \delta^3 (\vec{x} - \vec{X})$ and local Lorentz transformation, we have

$$\int_{R^3} eF_{\mu\nu} q^{\nu} \sqrt{g} d^3 x \to f_t{}^0 eF_{\mu\nu} u^{\nu} \sqrt{1 - v^2}, \tag{69}$$

$$\int_{R^3} \phi^+ S^a \phi (\partial_{\mu} \Omega_a) \sqrt{g} d^3 x \to f_t{}^0 \bar{S}^a \partial_{\mu} \Omega_a \sqrt{1 - v^2} = f_t{}^0 \partial_{\mu} (\bar{S}^a \Omega_a) \sqrt{1 - v^2}, \tag{70}$$

$$\begin{aligned} \int_{R^3} \partial_{\mu} N \sqrt{g} d^3 x &= \int_{R^3} \partial_{\mu} (N \sqrt{g}) d^3 x - \int_{R^3} N \Gamma^{\nu}_{\mu\nu} \sqrt{g} d^3 x \\ &\to \delta^t_{\mu} \frac{d}{dt} (f_t{}^0 \varpi \sqrt{1 - v^2}) - f_t{}^0 \Gamma^{\nu}_{\mu\nu} \varpi \sqrt{1 - v^2}, \end{aligned} \tag{71}$$

in which the proper parameters $\bar{S}^a = \int_{R^3} \phi^+ S^a \phi d^3 X$ is almost a constant, $\bar{S}^a$ equals to $\pm \frac{1}{2} \hbar$ in one direction but vanishes in other directions. $\varpi = \int_{R^3} N d^3 X$ is scale dependent. Then (65) becomes

$$\frac{d}{ds} p_{\mu} \to eF_{\mu\nu} u^{\nu} + \partial_{\mu} (\bar{S}^{\nu} \Omega_{\nu}) + \varpi \left(\Gamma^{\alpha}_{\mu\alpha} - \delta^t_{\mu} \frac{d}{dt} \zeta \right) - \frac{K_{\mu}}{\sqrt{1 - v^2}}, \tag{72}$$

where $\zeta = \ln(f_t{}^0 \varpi \sqrt{1 - v^2})$.

Now we prove the following classical approximation of K_{μ},

$$K_{\mu} \to -\frac{1}{2} (\partial_{\mu} g_{\alpha\beta}) m u^{\alpha} u^{\beta} \sqrt{1 - v^2}. \tag{73}$$

For *LU* decomposition of metric, by (47) we have

$$\frac{\partial f^\nu_a}{\partial g_{\alpha\beta}} = -\frac{1}{4}(f^\alpha_a g^{\nu\beta} + f^\beta_a g^{\alpha\nu}) - \frac{1}{2}S^{\alpha\beta}_{ab} f^\nu_n \eta^{nb}, \tag{74}$$

where $S^{\mu\nu}_{ab} = -S^{\mu\nu}_{ba}$ is anti-symmetrical for indices (a,b). Thus we have

$$\begin{aligned}(\partial_\mu \alpha^\nu)\hat{p}_\nu &= \partial_\mu g_{\alpha\beta}\frac{\partial f^\nu_a}{\partial g_{\alpha\beta}}\alpha^a \hat{p}_\nu = \partial_\mu g_{\alpha\beta}\Big(-\frac{1}{4}(\alpha^\alpha \hat{p}^\beta + \alpha^\beta \hat{p}^\alpha) - \frac{1}{2}S^{\alpha\beta}_{ab} f^\nu_n \eta^{nb}\alpha^a \hat{p}_\nu\Big) \\ &= -\frac{1}{4}\partial_\mu g_{\alpha\beta}\Big((\alpha^\alpha \hat{p}^\beta + \alpha^\beta \hat{p}^\alpha) + 2S^{\alpha\beta}_{ab}\alpha^a \hat{p}^b\Big). \end{aligned} \tag{75}$$

For classical approximation we have

$$\check{\alpha}^a = \phi^+ \alpha^a \phi \to v^a \delta^3(\vec{x} - \vec{X}), \quad \hat{p}^b \phi \to m u^b \phi, \quad S^{\alpha\beta}_{ab} = -S^{\alpha\beta}_{ba}. \tag{76}$$

Substituting (76) into (75), we obtain

$$\int_{R^3} \sqrt{g}\phi^+ (\partial_\mu \alpha^\nu)\hat{p}_\nu \phi d^3x \to -\frac{1}{2} f_t{}^0 (\partial_\mu g_{\alpha\beta}) p^\alpha u^\beta \sqrt{1-v^2}. \tag{77}$$

So (73) holds.

In the central coordinate system of the spinor, by relations

$$\Gamma^\nu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}), \quad \frac{d}{d\tau} g_{\mu\nu} = \sqrt{1-v^2} u^\alpha \partial_\alpha g_{\mu\nu}, \tag{78}$$

it is easy to check

$$g_{\mu\nu}\Gamma^\nu_{\alpha\beta} p^\alpha u^\beta \sqrt{1-v^2} - p^\nu \frac{dg_{\mu\nu}}{d\tau} = -\frac{1}{2}(\partial_\mu g_{\alpha\beta}) p^\alpha u^\beta \sqrt{1-v^2}. \tag{79}$$

Substituting (79) into (73) we obtain

$$K_\mu \to g_{\mu\nu}\Gamma^\nu_{\alpha\beta} p^\alpha u^\beta \sqrt{1-v^2} - p^\nu \frac{dg_{\mu\nu}}{d\tau}. \tag{80}$$

Substituting (80) and $ds = \sqrt{1-v^2}d\tau$ into (72), we obtain **Newton's second law** for the spinor

$$\frac{d}{ds} p^\mu + \Gamma^\mu_{\alpha\beta} p^\alpha u^\beta = g^{\alpha\mu}\Big(eF_{\alpha\beta}u^\beta + \varpi(\Gamma^\beta_{\alpha\beta} - \delta^t_\alpha \frac{d}{dt}\ln\zeta) + \partial_\alpha(\bar{S}^\nu \Omega_\nu)\Big). \tag{81}$$

The classical mass m weakly depends on speed v if $\varpi \neq 0$.

By the above derivation we find that Newton's second law is not as simple as it looks, because its universal validity depends on many subtle and compatible relations of the spinor equation. A complicated partial differential equation system (58) can be reduced to a 6-dimensional dynamics (59) and (81) is not a trivial event, which implies the world is a miracle designed elaborately. If the spin-gravity coupling potential $S_\mu \Omega^\mu$ and nonlinear potential ϖ can be ignored, (81) satisfies 'mass shell constraint' $\frac{d}{dt}(p^\mu p_\mu) = 0$ [33,34]. In this case, the classical mass of the spinor is a constant and the free spinor moves along geodesic. In some sense, only vector potential is strictly compatible with Newtonian mechanics and Einstein's principle of equivalence.

Clearly, the additional acceleration in (81) $\Omega_\mu \in \Lambda^3$ is different from that in (1), which is in Λ^2. The approximation to derive (1) $\hbar \to 0$ may be inadequate, because $\hbar$ is a universal constant acting as unit of physical variables. If $\varpi = 0$, (81) obviously holds in all coordinate system due to the covariant form, although we derive (81) in NCS; however, if $\varpi > 0$ is large enough for dark spinor, its trajectories will manifestly deviate from geodesics, so the dark halo in a galaxy is automatically separated from ordinary matter. Besides, the nonlinear potential is scale dependent [12].

For many body problem, dynamics of the system should be juxtaposed (58) due to the superposition of Lagrangian,

$$i\alpha^t(\partial_t + \Upsilon_t)\phi_n = \mathbf{H}_n\phi_n, \quad \mathbf{H}_n = -\alpha^k\hat{p}_k + e\alpha^t A_t + (m_n - N'_n)\gamma_0 + \Omega_\mu S^\mu. \tag{82}$$

The coordinate, speed and momentum of n-th spinor are defined by

$$\vec{X}_n(t) = \int_{R^3} \vec{x} q^t_n \sqrt{g} d^3x, \quad \vec{v}_n = \frac{d}{d\tau}\vec{X}_n, \quad p^\mu_n = \Re \int_{R^3} \phi^+_n \hat{p}^\mu \phi_n \sqrt{\bar{g}} d^3x. \tag{83}$$

The classical approximation condition for point-particle model reads,

$$q^\mu_n \to u^\mu_n \sqrt{1 - v^2_n}\delta^3(\vec{x} - \vec{X}_n), \quad u^\mu_n \equiv \frac{dX^\mu_n}{ds_n} = (1, \vec{v}_n)/\sqrt{1 - v^2_n}. \tag{84}$$

Repeating the derivation from (72) to (76), we obtain classical dynamics for each spinor,

$$\frac{d}{ds_n} p^\mu_n + \Gamma^\mu_{\alpha\beta} p^\alpha_n u^\beta_n = g^{\alpha\mu}\left(e_n F_{\alpha\beta} u^\beta_n + \varpi_n(\Gamma^\beta_{\alpha\beta} - \delta^t_\alpha \frac{d}{dt}\ln\zeta_n) + \partial_\alpha(\bar{S}^\nu\Omega_\nu)\right). \tag{85}$$

5. Energy-Momentum Tensor of Spinors

Similarly to the case of metric $g_{\mu\nu}$, the definition of Ricci tensor can also differ by a negative sign. We take the definition as follows

$$R_{\mu\nu} \equiv \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\mu\Gamma^\alpha_{\nu\alpha} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}, \qquad R = g^{\mu\nu}R_{\mu\nu}. \tag{86}$$

For a spinor in gravity, the Lagrangian of the coupling system is given by

$$\mathcal{L} = \frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_m, \quad \mathcal{L}_m = \Re\langle\phi^+\alpha^\mu\hat{p}_\mu\phi\rangle - \phi^+\Omega_\mu S^\mu\phi - m\phi^+\gamma^0\phi + N, \tag{87}$$

in which $\kappa = 8\pi G$, Λ is the cosmological constant, and $N = \frac{1}{2}w\check{\gamma}^2$ the nonlinear potential. Variation of the Lagrangian (87) with respect to $g_{\mu\nu}$, we obtain Einstein's field equation

$$G^{\mu\nu} + \Lambda g^{\mu\nu} + \kappa T^{\mu\nu} = 0, \qquad G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -\frac{\delta(R\sqrt{g})}{\sqrt{g}\delta g_{\mu\nu}}. \tag{88}$$

where $\frac{\delta}{\delta g_{\mu\nu}}$ is the Euler derivatives, and $T^{\mu\nu}$ is EMT of the spinor defined by

$$T^{\mu\nu} = -2\frac{\delta(\mathcal{L}_m\sqrt{g})}{\sqrt{g}\delta g_{\mu\nu}} = -2\frac{\partial\mathcal{L}_m}{\partial g_{\mu\nu}} + 2(\partial_\alpha + \Gamma^\gamma_{\alpha\gamma})\frac{\partial\mathcal{L}_m}{\partial(\partial_\alpha g_{\mu\nu})} - g^{\mu\nu}\mathcal{L}_m. \tag{89}$$

By detailed calculation we have

Theorem 8. *For the spinor ϕ with nonlinear potential $N(\check{\gamma})$, the total EMT is given by*

$$T^{\mu\nu} = \frac{1}{2}\Re\langle\phi^+(\alpha^\mu\hat{p}^\nu + \alpha^\nu\hat{p}^\mu + 2S^{\mu\nu}_{ab}\alpha^a\hat{p}^b)\phi\rangle + g^{\mu\nu}(N'\check{\gamma} - N) + K^{\mu\nu} + \widetilde{K}^{\mu\nu}, \tag{90}$$

$$K^{\mu\nu} = \frac{1}{2}\epsilon^{abcd}\check{S}_d\left(\frac{1}{2}f^\beta_a S^{\mu\nu}_{bc} g^{\lambda\kappa} + \frac{\partial(f^\beta_a S^{\mu\nu}_{bc})}{\partial g_{\lambda\kappa}} - \frac{\partial(f^\beta_a S^{\lambda\kappa}_{bc})}{\partial g_{\mu\nu}}\right)\partial_\beta g_{\lambda\kappa}, \tag{91}$$

$$\widetilde{K}^{\mu\nu} = \frac{1}{4}\epsilon^{abcd}S^{\mu\nu}_{cd}(\partial_a\check{S}_b - \partial_b\check{S}_a), \qquad \check{S}_\mu \equiv \phi^+ S_\mu\phi. \tag{92}$$

Proof. The Keller connection $i\Upsilon_\alpha$ is anti-Hermitian and actually vanishes in $\Re\langle\phi^+\alpha^\alpha\hat{p}_\alpha\phi\rangle$. By (89) and (53), we obtain the component of EMT related to the kinematic energy as

$$\begin{aligned} T_p^{\mu\nu} &\equiv -2\frac{\delta}{\delta g_{\mu\nu}}\Re\langle\phi^+\alpha^\alpha\hat{p}_\alpha\phi\rangle = -2\Re\langle\phi^+\left(\frac{\partial\alpha^\alpha}{\partial g_{\mu\nu}}\right)(i\partial_\alpha - eA_\alpha)\phi\rangle \\ &= \frac{1}{2}\Re\langle\phi^+(\alpha^\mu\hat{p}^\nu + \alpha^\nu\hat{p}^\mu + 2S_{ab}^{\mu\nu}\alpha^a\hat{p}^b)\phi\rangle, \end{aligned} \tag{93}$$

where we take A_μ as independent variable. By (54) we obtain the variation related with spin-gravity coupling potential as

$$\frac{\partial(\phi^+\Omega^d S_d\phi)}{\partial g_{\mu\nu}} = \frac{1}{4}\epsilon^{abcd}\check{S}_d\frac{\partial(f^\alpha_{\ a}S_{bc}^{\lambda\kappa})}{\partial g_{\mu\nu}}\partial_\alpha g_{\lambda\kappa}, \tag{94}$$

$$\begin{aligned} (\partial_\alpha + \Gamma^\beta_{\alpha\beta})\frac{\partial(\phi^+\Omega^d S_d\phi)}{\partial(\partial_\alpha g_{\mu\nu})} &= \frac{1}{4}\epsilon^{abcd}(\partial_\alpha + \Gamma^\beta_{\alpha\beta})(f^\alpha_{\ a}S_{bc}^{\mu\nu}\check{S}_d) \\ &= \frac{1}{4}\epsilon^{abcd}\left[S_{bc}^{\mu\nu}\partial_a\check{S}_d + \check{S}_d\left(\frac{\partial(f^\alpha_{\ a}S_{bc}^{\mu\nu})}{\partial g_{\lambda\kappa}} + \frac{1}{2}f^\alpha_{\ a}S_{bc}^{\mu\nu}g^{\lambda\kappa}\right)\partial_\alpha g_{\lambda\kappa}\right]. \end{aligned} \tag{95}$$

Then we have the EMT for term $\Omega^\mu\check{S}_\mu$ as

$$T_s^{\mu\nu} = -2\frac{\partial(\Omega^d\check{S}_d)}{\partial g_{\mu\nu}} + 2(\partial_\alpha + \Gamma^\beta_{\alpha\beta})\frac{\partial(\Omega^d\check{S}_d)}{\partial(\partial_\alpha g_{\mu\nu})} = K^{\mu\nu} + \widetilde{K}^{\mu\nu}. \tag{96}$$

Substituting Dirac Equation (18) into (87), we get $\mathcal{L}_m = N - N'\check{\gamma}$. For nonlinear potential $N = \frac{1}{2}w\check{\gamma}^2$, we have $\mathcal{L}_m = -N$. Substituting all the results into (89), we prove the theorem. □

For EMT of compound systems, we have the following useful theorem [12].

Theorem 9. *Assume matter consists of two subsystems I and II, namely $\mathcal{L}_m = \mathcal{L}_I(\phi) + \mathcal{L}_{II}(\psi)$, then we have*

$$T^{\mu\nu} = T_I^{\mu\nu} + T_{II}^{\mu\nu}. \tag{97}$$

If the subsystems I and II have not interaction with each other, namely,

$$\frac{\delta}{\delta\psi}\mathcal{L}_I(\phi) = \frac{\delta}{\delta\phi}\mathcal{L}_{II}(\psi) = 0, \tag{98}$$

then the two subsystems have independent energy-momentum conservation laws, respectively,

$$T_{I;\nu}^{\mu\nu} = 0, \qquad T_{II;\nu}^{\mu\nu} = 0. \tag{99}$$

For classical approximation of EMT, we have $\phi^+S_{ab}^{\mu\nu}\alpha^a\hat{p}^b\phi \to S_{ab}^{\mu\nu}u^a p^b = 0$. By the symmetry of the spinor, the proper value $\int_{R^3}\widetilde{K}^{\mu\nu}d^3X = 0$. By the structure and covariance, we should have

$$K^{\mu\nu} = k_1\check{S}_\alpha\Omega^\alpha g^{\mu\nu} + k_2(\Omega^\mu\check{S}^\nu + \Omega^\nu\check{S}^\mu), \tag{100}$$

where k_1, k_2 are constants to be determined. By (82), we find that the energy of spin-gravity interaction is just $\check{S}^\mu\Omega_\mu$. Besides, if $A^\mu = 0$, the spinor is an independent system and its energy-momentum conservation law $T_{;\nu}^{\mu\nu} = 0$ holds, so its classical approximation should give (81) as $F_{\mu\nu} = 0$. This means we have $k_1 = 1$ and $k_2 = 0$, or equivalently $K_t^t = \check{S}^\mu\Omega_\mu$.

For the classical approximation of (90), by the summation of energy we have the total EMT as

$$T^{\mu\nu} \to [mu^\mu u^\nu + (\bar{S}^\alpha \Omega_\alpha + \bar{\varpi}) g^{\mu\nu}] \sqrt{1 - v^2} \delta^3(\vec{x} - \vec{X}). \tag{101}$$

$\bar{\varpi} > 0$ acts like negative pressure, which is scale dependent. If the metric is diagonalizable, then we have $\Omega_\mu \equiv 0$, so the term $\bar{S}^\alpha \Omega_\alpha$ vanishes in cosmology.

Some previous works usually use one spinor to represent matter field. This may be not the case, because spinor fields only has a very tiny structure. Only to represent one particle by one spinor field, the matter model can be comparable with general relativity, classical mechanics and quantum mechanics [11,12,33]. By the superposable property of Lagrangian, the many body system should be described by the following Lagrangian

$$\mathcal{L}_m = \sum_n \Big(\Re \langle \phi_n^+ \alpha^\mu \hat{p}_\mu \phi_n \rangle - \check{S}_n^\mu \Omega_\mu - m_n \check{\gamma}_n + N_n \Big). \tag{102}$$

The classical approximation of EMT becomes

$$T^{\mu\nu} \to \sum_n \Big[m_n u_n^\mu u_n^\nu + (\bar{\varpi}_n + \check{S}_n^\alpha \Omega_\alpha) g^{\mu\nu} \Big] \sqrt{1 - v_n^2} \delta^3(\vec{x} - \vec{X}_n), \tag{103}$$

which leads to the EMT for average field of spinor fluid as follows

$$T^{\mu\nu} = (\rho + P) U^\mu U^\nu + (W - P) g^{\mu\nu}. \tag{104}$$

The self potential becomes negative pressure W, which takes the place of cosmological constant Λ in Einstein's field equation. W has very important effects in astrophysics [12].

6. Discussion and Conclusions

From the calculation of this paper, we can find that Clifford algebra is indeed a unified language and efficient tool to describe the laws of nature. To represent the physical and geometric quantities of Clifford algebra, the formalism is neat and elegant and the calculation and derivation are simple and standard. The decomposition of spinor connection into Υ_μ and Ω_μ by Clifford algebra, not only makes the calculation simpler, but also highlights their different physical meanings. $\Upsilon_\mu \in \Lambda^1$ only corresponds to geometric calculations similar to the Levi–Civita connection, but $\Omega_\mu \in \Lambda^3$ results in physical effects. Ω_μ is coupled with the spin of spinor field, which provides position and navigation functions for the spinor, and is the origin of the celestial magnetic field. $\Omega_\mu \equiv 0$ is a necessary condition of the diagonalizablity of metric, which seems to be also sufficient.

In the theoretical analysis of the spinor equation and its classical approximation, we must use Gu's natural coordinate system with realistic cosmic time. This is a coordinate system with universal applicability and profound philosophical significance, which can clarify many misunderstandings about the concept of space-time. The energy-momentum tensor of the spinor field involves the specific representation of the tetrad. Through the LU decomposition of metric, we set up the clear relationship between the frame and metric, and then derive the exact EMT of spinor. In the derivation, we discover a new non-tensor spinor coefficient table $S_{ab}^{\mu\nu}$, which has some wonderful properties and appears in many places in the spinor theory, but the specific physical significance needs to be further studied.

We usually use limits such as $\hbar \to 0$ and $c \to \infty$ in classical approximation of quantum mechanics. In some cases, such treatment is inappropriate. $(\hbar, c)$ are constant units for physical variables, how can they take limits? In the natural unit system used in this paper or the dimensionless equations, we do not even know where the constants are. We can only make approximations such as $|v| \ll c$ or (61) while the average radius of the spinor is much smaller than its moving scale. Most paradoxes and puzzles in physics are caused by such ambiguous statements or overlapping concepts in different logical systems. A detailed discussion of these issues is given in [12,33].

This paper clearly shows how general relativity, quantum mechanics and classical mechanics are all compatible. Newton's second law is not as simple as it looks, its universal validity depends on many subtle and compatible relations of the spinor equation as shown in Section 4. A complicated Dirac equation of spinor can be reduced to a 6-dimensional ordinary differential dynamics is not a trivial event, which implies that the world is a miracle designed elaborately. In fact, all the fundamental physical theories can be unified in the following framework expressed by the Clifford algebra [12,33]:

$\mathbf{A_1}$. *The element of space-time is described by*

$$d\mathbf{x} = \gamma_\mu dx^\mu = \gamma_a \delta X^a, \tag{105}$$

where the basis γ_a and γ_μ satisfy the $C\ell_{1,3}$ Clifford algebra (5).

$\mathbf{A_2}$. *The dynamics for a definite physical system takes the form as*

$$\gamma^\mu \partial_\mu \Psi = \mathcal{F}(\Psi), \tag{106}$$

where $\Psi = (\psi_1, \psi_2, \cdots, \psi_n)^T$, and $\mathcal{F}(\Psi)$ consists of some Clifford numbers of Ψ, so that the total equation is covariant.

$\mathbf{A_3}$. *The dynamic equation of a physical system satisfies the action principle*

$$S = \int \mathcal{L}(\Psi, \partial\Psi)\sqrt{g} d^4 x, \tag{107}$$

where the Lagrangian $\mathcal{L} \in \mathbb{R}$ is a superposable scalar.

$\mathbf{A_4}$. *Nature is consistent, i.e., for all solutions to* (106) *we always have*

$$\Psi(\mathbf{x}) \in L^\infty(\mathbb{M}^{1,3}). \tag{108}$$

Funding: This research received no external funding.

Acknowledgments: It is my pleasure to acknowledge James M. Nester for his enlightening discussions and encouragement. I once encountered the difficulty in the derivation of the energy-momentum tensor. He suggested to me to learn Clifford algebra, which is the key to solving the problem. This paper was improved and refined as suggested by the two reviewers, and the author thanks them so much.

Conflicts of Interest: The author declares no conflict of interest.

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Spontaneous Lorentz Violation from Infrared Gravity

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Abstract: In this paper, we investigate a novel implication of the non-negligible spacetime curvature at large distances when its effects are expressed in terms of a suitably modified form of the Heisenberg uncertainty relations. Specifically, we establish a one-to-one correspondence between this modified uncertainty principle and the Standard Model Extension (SME), a string-theoretical effective field theory that accounts for both explicit and spontaneous breaking of Lorentz symmetry. This tight correspondence between string-derived effective field theory and modified quantum mechanics with extended uncertainty relations is validated by comparing the predictions concerning a deformed Hawking temperature derived from the two models. Moreover, starting from the experimental bounds on the gravity sector of the SME, we derive the most stringent constraint achieved so far on the value of the free parameter in the extended Heisenberg uncertainty principle.

Keywords: extended uncertainty principle; Standard Model Extension; Lorentz violation

Citation: Illuminati, F.; Lambiase, G.; Petruzziello, L. Spontaneous Lorentz Violation from Infrared Gravity. *Symmetry* **2021**, *13*, 1854. https://doi.org/10.3390/sym13101854

Academic Editors: Chris Fields, Antonino Marciano and Jerzy Kowalski Glikman

Received: 7 July 2021
Accepted: 20 August 2021
Published: 3 October 2021

1. Introduction

Since the establishment of quantum mechanics and general relativity, there has been a constantly growing effort to merge quantum and gravitational effects at arbitrary energy scales in a complete and consistent theoretical framework. This effort has produced several plausible candidates for a quantum theory of the gravitational interaction, such as string theory [1,2], loop quantum gravity [3,4], and doubly special relativity [5–7]. The study of the interplay between quantum and gravitational effects appears to be particularly important at the currently reachable energies, as this is the regime where we have hope that physical predictions can factually be probed in tabletop laboratory tests. Therefore, the analysis of low-energy, infrared (IR) quantum gravitational manifestations may represent a promising starting point for the possible construction of ultraviolet (UV) complete models of quantum gravity. In this respect, an additional motivation for the study of IR phenomena stems from the UV/IR duality discovered for the first time in the context of the AdS/CFT correspondence [8–10].

In the present paper, we focus on IR gravity to exhibit how Lorentz symmetry is affected by the non-negligible spacetime curvature at large distances. In a quantum mechanical setting, such a feature can be incorporated by extending the Heisenberg uncertainty principle (HUP) with the addition of a position-dependent correction that introduces a non-vanishing minimal uncertainty in momentum [11–15] and provides a form of an extended uncertainty principle (EUP). Among different versions of the EUP, the best known one includes a universal modification that is geometric-independent; indeed, such a generalization of the standard uncertainty relations arises naturally when merging quantum mechanics and general relativity. In other scenarios, the version of the EUP that is accounted for entails a dependence on the intrinsic geometric properties of the underlying background curvature; for more details along this direction, see Refs. [16–20]. For the sake of completeness, it is worth mentioning that several works inspired by string theory [21–24] also allow for the existence of a momentum-dependent correction to the HUP; such an extension is known as the generalized uncertainty principle (GUP) (for several development of this subject, see Refs. [25–38] and therein).

In order to quantify the breakdown of Lorentz symmetry induced by the EUP, we consider a string-theoretical effective field theory according to which any operator appearing in the Standard Model (SM) Lagrangian is contracted with Lorentz-violating fields [39,40]. This model is known as the Standard Model Extension (SME), and we will focus in particular on its gravity sector [41,42] in the non-relativistic limit [43]. Specifically, along the line of Ref. [44], in the following, we establish a one-to-one correspondence between the predictions of the EUP and of the SME concerning the deformation of the Hawking temperature for a Schwarzschild black hole:

$$T_H = \frac{\hbar c^3}{8\pi k_B G M}\,. \tag{1}$$

In so doing, we essentially relate the free deformation parameter arising in the framework of the EUP with the Lorentz-violating fields contained in the SME Lagrangian, thereby explaining spontaneous Lorentz symmetry breaking in terms of large-scale effects. Moreover, by relying on the experimental bounds associated with the SME gravity sector, we manage to derive novel constraints on the EUP free deformation parameter, which are more stringent than the previously known ones [15].

The paper is organized as follows: in Section 2, we briefly give an overlook of the main aspects of EUP together with a heuristic derivation of the modified Hawking temperature. The same procedure is carried out for the SME in Section 3, whereas Section 4 contains the theoretical and numerical comparison between the two predictions. Finally, Section 5 contains the concluding remarks and discussion.

2. Extended Uncertainty Principle and Modified Hawking Temperature

Starting from the Heisenberg uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}\,, \tag{2}$$

one can incorporate the influence of spacetime curvature at large distances by adding a position-dependent term in the r.h.s. of Equation (2), namely [11,13–15]

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + \alpha \Delta x^2\right), \tag{3}$$

with α being the inverse of a squared length and $\alpha \Delta x^2 \ll 1$. One possible interpretation of the free deformation parameter α is to conceive it as a function of the cosmological constant Λ in a (anti-) de Sitter space [45]. However, in greater generality, it can simply be seen as a consequence of the intrinsic spacetime curvature at large cosmological distances that enforces a limit to the precision with which to resolve the momentum of a point particle [15]. In turn, this fact implies that there exists a minimal uncertainty for p proportional to the constant α, i.e., $\Delta p_{min} \simeq \hbar\sqrt{|\alpha|}$. Note that the parameter α does not have to be necessarily constant; indeed, it may be a function of spacetime position or even a stochastic variable. Similar considerations have already been addressed in the context of GUP [30,38] and they equally hold true for the EUP currently investigated.

The above inequality (3) can be straightforwardly derived from the deformed canonical commutation relation:

$$\left[\hat{X}, \hat{P}\right] = i\hbar\left(\mathbf{1} + \alpha \hat{X}^2\right). \tag{4}$$

From the previous equation, one can deduce a simple representation of $\hat{X}$ and $\hat{P}$ in terms of auxiliary operators $\hat{x}$ and $\hat{p}$ for which the standard canonical commutation relation holds (i.e., $[\hat{x}, \hat{p}] = i\hbar$). Specifically:

$$\hat{X} = \hat{x}\,, \qquad \hat{P} = \left(1 + \alpha \hat{x}^2\right)\hat{p} - 2i\alpha\hbar\hat{x}\,, \tag{5}$$

where the second term in the r.h.s. of the second expression ensures that $\hat{P} = \hat{P}^{\dagger}$. The three-dimensional analysis of the above framework would allow for the emergence of spatial non-commutativity [11,12], since one can immediately verify that $[\hat{X}_j, \hat{X}_k] \neq 0$. Nevertheless, we can safely ignore all the non-commutative corrections associated with the operators $\hat{X}_j$, as they would depend on higher powers of α, i.e.,

$$\hat{X}_j = \hat{x}_j + \mathcal{O}(\alpha^2)\,, \tag{6}$$

which, in the present, case are neglected. We refer to Refs. [11,12] for further mathematical details.

To evaluate the deformed Hawking temperature of a Schwarzschild black hole of mass M, we follow some simple heuristic considerations as outlined, e.g., in Refs. [27,46]. The starting point is the natural assumption that the position uncertainty of the photons just after they have been emitted by the black hole is proportional to the Schwarzschild radius, namely $\Delta x \simeq \gamma r_s$, with γ to be fixed by requiring consistency with the standard picture in the limit $\alpha \to 0$. Under these circumstances, from Equation (3), we obtain

$$\Delta p \simeq \frac{\hbar c^2}{4\gamma GM}\left[1 + 4\alpha \frac{G^2 M^2}{c^4}\gamma^2\right]. \tag{7}$$

Now, we can express the characteristic energy of the emitted photons Δpc in terms of the temperature of the radiation in compliance with the equipartition theorem [27,46], by virtue of which $\Delta pc \simeq k_B T$. Finally, the equation for the α-deformed Hawking temperature T_{EUP} reads

$$T_{EUP} = \frac{\hbar c^3}{4\gamma k_B GM}\left[1 + 4\alpha \frac{G^2 M^2}{c^4}\gamma^2\right], \tag{8}$$

which, in terms of the Hawking temperature T_H, reads:

$$T_{EUP} = T_H\left[1 + 16\alpha\pi^2 \frac{G^2 M^2}{c^4}\right], \tag{9}$$

where we have set $\gamma = 2\pi$ in order to recover the original Hawking result in the limit $\alpha \to 0$. For a thorough discussion on the above correspondence, we refer the reader to Ref. [47].

3. Standard Model Extension and Modified Hawking Temperature

The Standard Model Extension is a generalization of the Standard Model of particle physics, which predicts both explicit and spontaneous Lorentz symmetry breaking. Motivated by string-theoretical arguments [39,40], the SME enlarges the SM domain by contracting any SM field with Lorentz-violating operators that give rise to new phenomenology. Although the Standard Model Extension was originally conceived to extend the Standard Model only, later on, the gravitational interaction was also added, with the ensuing Lorentz-violating coefficients.

For our purposes, we are interested in investigating the minimal SME gravity sector [41–44], which includes exclusively Lorentz-violating operators of mass dimension three or four. In particular, by denoting with S_{EH} and S_m the Einstein–Hilbert and the matter action, respectively, the minimal SME total gravitational action reads [43]

$$S = S_{EH} + S_m + S_{LV}\,, \qquad S_{LV} = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\left(-uR + s^{\mu\nu}R^T_{\mu\nu} + t^{\mu\nu\rho\lambda}C_{\mu\nu\rho\lambda}\right), \tag{10}$$

where S_{LV} denotes the effective Lorentz symmetry-breaking action derived from string theory, R is the Ricci scalar, $R^T_{\mu\nu}$ is the traceless Ricci tensor, $C_{\mu\nu\rho\lambda}$ is the Weyl conformal tensor, and u, $s^{\mu\nu}$ and $t^{\mu\nu\rho\lambda}$ are the Lorentz-violating effective fields.

In the regime of the post-Newtonian (PPN) approximation [48,49], a Schwarzschild-like solution of the linearized field equations for the minimal SME can be found [43], and it is given by

$$ds^2 = f(r)c^2dt^2 - \frac{1}{f(r)}dr^2 - r^2d\Omega^2\,, \tag{11}$$

where

$$f(r) = 1 - \frac{2GM}{rc^2}\left[1 + \bar{s}^{ij}g_{ij}(\theta,\phi)\right]. \tag{12}$$

In the above, $\bar{s}^{ij}$ denote the vacuum expectation values of the fields s^{ij} and $g_{ij}(\theta,\phi)$ are functions of the angular coordinates, for which the inequality $g_{ij}(\theta,\phi) \leq 1$ holds regardless of the choice for θ and ϕ. The absence of $\bar{u}$ in Equation (12) is related to the fact that a non-vanishing value for such parameter only amounts to a scaling of the PPN metric [43], and thus it can be set to zero. On the other hand, the disappearance of $\bar{t}^{\mu\nu\rho\lambda}$ is a typical feature occurring in the post-Newtonian SME gravitational phenomenology known as "t puzzle", and it has been extensively discussed in several works (see, for instance, Refs. [50–56]).

In order to evaluate the deformed Hawking temperature T_{SME} arising in the SME framework, we must compute [44,57,58]

$$T_{SME} = \frac{\hbar c}{4\pi k_B}\frac{df(r)}{dr}\bigg|_{r=r_0}, \tag{13}$$

where r_0 solves the equation $f(r_0) = 0$, thereby denoting the horizon radius. At this point, a straightforward calculation yields

$$T_{SME} = T_H\left[1 - \bar{s}^{ij}g_{ij}(\theta,\phi)\right], \tag{14}$$

which has to be compared with Equation (9) derived in the EUP framework. Before concluding this section, it is worth pointing out that the temperature (14) is anisotropic, as it explicitly depends on the angular position. However, such an observation does not undermine the validity of our main goal; indeed, Equation (9) might be anisotropic as well, since α is not bound to be a constant.

4. Comparison and Consistency Conditions

We will now look at the relations that are required in order to achieve consistency between the predictions (9) and (14) and thus relate the large-scale effects of spacetime curvature with spontaneous breaking of the Lorentz symmetry. The comparison of the deformed Hawking temperatures deduced from the two distinct physical settings shows that the two are consistent provided that the following identification holds:

$$\alpha = -\frac{c^4}{16\pi^2G^2M^2}\bar{s}^{ij}g_{ij}(\theta,\phi)\,, \tag{15}$$

for fixed values of θ and ϕ. Therefore, as already argued, the magnitude of α is not constant, but it varies with the angular displacement, thus giving rise to an anisotropic T also for the EUP-corrected Hawking temperature. Since the $g_{ij}(\theta,\phi)$ can always be taken as positive quantities [44], the sign of α strictly depends on the sign of the Lorentz-violating coefficients

$\bar{s}^{ij}$. Additionally, by means of consistency arguments that do not require Lorentz invariance, it is known that α correctly characterizes an EUP associated with an expanding universe if and only if $\alpha < 0$ [13,14,45]. Therefore, to allow agreement with experimental evidence, assuming $g_{ij}(\theta, \phi) \simeq 1 \ \forall i, j$ [44], we must necessarily impose

$$\sum_{i,j} \bar{s}^{ij} > 0, \tag{16}$$

which realizes a novel bound on the admissible values of the Lorentz-violating coefficients. Of course, by virtue of Equation (15), the bounds holding for the quantities $\bar{s}^{ij}$ in turn imply bounds on the EUP free deformation parameter that lead to an extremely significant improvement with respect to the existing constraints [15] on the possible values of α.

In Table 1, we report the known bounds on $\bar{s}^{ij}$, the ensuing constraints that they entail on $\sqrt{|\alpha|}$ by virtue of Equation (15) and the corresponding experimental frameworks used to determine each bound.

Table 1. Estimated bounds on the EUP parameter.

Experiments	Bounds on $\lvert\bar{s}^{ij}\rvert$	Bounds on $\sqrt{\lvert\alpha\rvert}$
Geodetic effect ($M = M_\oplus$) [59]	$\lvert\bar{s}^{ij}\rvert \lesssim 10^{-3}$	$\sqrt{\lvert\alpha\rvert} \lesssim 6.37 \times 10^{-2}\,\mathrm{m}^{-1}$
Gravity Probe B ($M = M_\oplus$) [59]	$\lvert\bar{s}^{ij}\rvert \lesssim 10^{-4}$	$\sqrt{\lvert\alpha\rvert} \lesssim 2.01 \times 10^{-2}\,\mathrm{m}^{-1}$
Frame dragging ($M = M_\oplus$) [59]	$\lvert\bar{s}^{ij}\rvert \lesssim 10^{-7}$	$\sqrt{\lvert\alpha\rvert} \lesssim 6.37 \times 10^{-4}\,\mathrm{m}^{-1}$
Gravimetry ($M = M_\oplus$) [59]	$\lvert\bar{s}^{ij}\rvert \lesssim 10^{-10}$	$\sqrt{\lvert\alpha\rvert} \lesssim 2.01 \times 10^{-5}\,\mathrm{m}^{-1}$
Lunar laser ranging ($M = M_\oplus$) [59,60]	$\lvert\bar{s}^{ij}\rvert \lesssim 10^{-12}$	$\sqrt{\lvert\alpha\rvert} \lesssim 2.01 \times 10^{-6}\,\mathrm{m}^{-1}$
Torsion pendulum ($M = M_\oplus$) [59]	$\lvert\bar{s}^{ij}\rvert \lesssim 10^{-15}$	$\sqrt{\lvert\alpha\rvert} \lesssim 6.37 \times 10^{-8}\,\mathrm{m}^{-1}$
Perihelion precession ($M = M_\odot$) [59]	$\lvert\bar{s}^{ij}\rvert \lesssim 10^{-9}$	$\sqrt{\lvert\alpha\rvert} \lesssim 1.98 \times 10^{-10}\,\mathrm{m}^{-1}$
Binary pulsar ($M = 2.8\,M_\odot$) [59,61]	$\lvert\bar{s}^{ij}\rvert \lesssim 10^{-11}$	$\sqrt{\lvert\alpha\rvert} \lesssim 7.05 \times 10^{-12}\,\mathrm{m}^{-1}$
Solar-spin precession ($M = M_\odot$) [59]	$\lvert\bar{s}^{ij}\rvert \lesssim 10^{-13}$	$\sqrt{\lvert\alpha\rvert} \lesssim 1.98 \times 10^{-12}\,\mathrm{m}^{-1}$

The bounds on $\sqrt{|\alpha|}$ derived from the geodetic effect and Gravity Probe B are similar to the ones typically encountered in phenomenological works on this topic [15]. Consequently, we note that all the other results contained in Table 1 considerably strengthen the constraint on α. Specifically, the inequality $\sqrt{|\alpha|} \lesssim 1.98 \times 10^{-12}\,\mathrm{m}^{-1}$ provides the best current bound on the EUP free deformation parameter α, and further refinement of the experimental sensitivity may allow for an even more stringent constraint.

It is worth observing that, should one regard α as emergent from a non-vanishing cosmological constant in de Sitter space, we would have $\alpha = -\Lambda/3 \simeq -3.66 \times 10^{-53}\,\mathrm{m}^{-2}$, which, in the framework of a near-Earth experiment, would correspond to $|\bar{s}^{ij}| \simeq 9.06 \times 10^{-54}$. This is in line with the expectation that Lorentz-violating coefficients should be indeed extremely small corrections to Lorentz-symmetric physics [43].

5. Concluding Remarks

In this work, we have investigated the consequences of relating large-scale effects ascribable to the non-negligible spacetime curvature and the spontaneous Lorentz symmetry breaking as described by the gravity sector of the Standard Model Extension. The relation is obtained by requiring consistency between the different modifications of the Hawking temperature predicted by the SME and by a quantum mechanical model endowed with an extended uncertainty principle deformed due to spacetime curvature effects. Investigating the consequences of the consistency relations imposed between the SME and the EUP, we have shown how to significantly enhance the existing bounds on the EUP curvature-induced deformation parameter starting from the experimental constraints on the Lorentz-violating coefficients that enter the gravity sector of the SME. This simple comparison points at the possibility, in suitable settings, of probing high-energy quantum physics via low-energy gravitational effects. To some extent, the idea underlying the present study shares the same philosophy characterizing the well-known AdS/CFT correspondence [8], as it may potentially provide some further insight towards a full understanding of the UV/IR duality.

Author Contributions: L.P. performed calculations and drafted the paper. F.I. and G.L. supervised the work. All authors discussed the results and edited the paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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Brief Report

A Note on Effects of Generalized and Extended Uncertainty Principles on Jüttner Gas

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Abstract: In recent years, the implications of the generalized (GUP) and extended (EUP) uncertainty principles on Maxwell–Boltzmann distribution have been widely investigated. However, at high energy regimes, the validity of Maxwell–Boltzmann statistics is under debate and instead, the Jüttner distribution is proposed as the distribution function in relativistic limit. Motivated by these considerations, in the present work, our aim is to study the effects of GUP and EUP on a system that obeys the Jüttner distribution. To achieve this goal, we address a method to get the distribution function by starting from the partition function and its relation with thermal energy which finally helps us in finding the corresponding energy density states.

Keywords: generalized uncertainty principle; extended uncertainty principle; Jüttner distribution

Citation: Moradpour, H.; Aghababaei, S.; Ziaie, A.H. A Note on Effects of Generalized and Extended Uncertainty Principles on Jüttner Gas. *Symmetry* **2021**, *13*, 213. https://doi.org/10.3390/sym13020213

Academic Editor: Chris Fields
Received: 25 December 2020
Accepted: 26 January 2021
Published: 28 January 2021

1. Introduction

A general prediction of any quantum gravity theory is the possibility of the existence of a minimal length in nature, known as the Planck length, below which no other length can be observed. It is commonly believed that in the vicinity of the Planck length, the smooth structure of spacetime is replaced by a foamy structure due to quantum gravity effects [1–3]. Therefore, the Planck scale can be regarded as a separation line between classical and quantum gravity regimes. There is a general consensus that in the scale of this minimal size, the characteristics of different physical systems would be altered. For instance, the introduction of a minimal length scale results in a generalization of the Heisenberg uncertainty principle (HUP) in such a way that it incorporates gravitationally induced uncertainty, postulated as the generalized uncertainty principle (GUP) [4]. In fact, the HUP breaks down for energies near the Planck scale, i.e., when the Schwarzschild radius is comparable to the Compton wavelength and both are close to the Planck length. This deficiency is removed by revising the characteristic scale through the modification of HUP to GUP.

In recent decades, numerous studies on the effects of GUP in various classical and quantum mechanical systems have been performed [5–30]. Uncertainty in momentum is also bounded from below and it is proposed that its minimum is non-zero, a proposal which modifies HUP to the extended uncertainty principle (EUP) [31–35]. In the presence of EUP and GUP, the general form of modified HUP is proposed as

$$\Delta x \Delta p \geq \frac{\hbar}{2}\Big(1 + \alpha(\Delta x)^2 + \eta(\Delta p)^2 + \gamma\Big), \tag{1}$$

in which α, η, and γ are positive deformation parameters [35,36]. It should be noted that there is another formulation of GUP and EUP [37], and also that extended forms of HUP like GUP may break the fundamental symmetries such as Lorentz invariance and CPT [38].

On the other hand, it is known that heavy ions can be accelerated to very high kinetic energies constituting an ensemble of ideal gas with relativistic velocities in large particle accelerators [39]. In such high energy regimes, minimal length effects may appear and could have their own influences on the statistics of ideal gases. Therefore, particle accelerators could provide a setting to examine the phenomena related to short-distance physics [40,41]. Based on minimum observable length, the quantum gravity implications on the statistical properties of ideal gases have been investigated in many studies, see, e.g., [42–45] and references therein. In the framework of GUP: (i) deformed density of states and an improved definition of the statistical entropy have been introduced in [46,47], (ii) Maxwell–Boltzmann statistics have been investigated in [48], and (iii) employing Maxwell–Boltzmann statistics, the thermodynamics of relativistic ideal gas has also been analyzed in [49]. In the same manner, there have been some studies on the deformation of statistical concepts in the EUP framework [50,51].

Jüttner distribution is a generalization of Maxwell–Boltzmann statistics to the relativistic regimes, which appears in high energy physics. Since quantum gravity is a high energy physics scenario, its statistical effects may be more meaningful in the framework of Jüttner distribution function compared to the Maxwell–Boltzmann distribution [52]. Here, our main aim is to study the effects of GUP and EUP on Maxwell–Boltzmann and Jüttner distributions and density of states in energy space. To achieve this goal, we begin by providing an introductory note on Maxwell–Boltzmann and Jüttner distributions. We then address a way to find these functions by starting from the partition function of the system. The effects of GUP and EUP on these statistics are also studied in the subsequent sections, respectively. The last section is devoted to a summary of the work.

2. The Maxwell–Boltzmann and Jüttner Distribution Functions

We begin by considering an ideal gas composed of non-interacting particles and set the units so that $\omega_0 = 2\pi\hbar = 1$, where ω_0 denotes the fundamental volume of each cell in the two-dimensional phase-space. This value of ω_0 originates from the well-known commutation relation between canonical coordinates x and p, and indeed, it is the direct result of HUP [22]. Therefore, any changes in HUP can affect ω_0.

2.1. Non-Relativistic Gas

Let us consider a 3-dimensional classical gas consisting of N identical non-interacting particles of mass m with $E = mv^2/2$, where E and v denote the energy and velocity of each particle, respectively. At temperature T, the Maxwell–Boltzmann (MB) distribution function is given by

$$f_{MB}(v,\beta) = Z_{MB} \exp\left(-\frac{\beta m v^2}{2}\right), \tag{2}$$

where Z_{MB} is a normalization constant, and $\beta \equiv 1/K_B T$ with K_B being the Boltzmann constant. In terms of E, we have

$$f_{MB} = 4\pi v^2(E) f_{MB}(v(E),\beta)\frac{dv}{dE} = Z_{MBE} E^{\frac{1}{2}} \exp(-\beta \mathrm{E}), \tag{3}$$

in which Z_{MBE} is a new normalization constant and $E^{\frac{1}{2}}$ denotes the density of states with energy E. The normalization constants can be calculated using the normalization constraint

$$\int_0^\infty f_{MB}(v,\beta) d^3v = \int_0^\infty f_{MB}(E,\beta) dE = 1. \tag{4}$$

The extremum of $f_{MB}(E,\beta)$ is located at $E = 1/2\beta \equiv E_{MB}^{ext}$ or equally at velocity $v = 1/\sqrt{\beta m} \equiv v_{MB}^{ext}$. One can also evaluate the partition function of the mentioned gas (with Hamiltonian $H = p^2/2m$) as

$$Q_N = \frac{1}{N!}\underbrace{\int \ldots \int}_{3N} \exp(-\beta H) d^{3N}x d^{3N}p = \frac{\left(Q_1^{NR}\right)^N}{N!}, \tag{5}$$

where

$$Q_1^{NR} = \int \exp(-\beta H) d^3x d^3p = V\left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}}, \tag{6}$$

denotes the single partition function of a nonrelativistic gas and V refers to the total volume of the system. In this manner, the corresponding thermal energy per particle (U) takes the form

$$U^{NR} = \int_0^\infty E f_{MB}(E,\beta) dE = -\frac{\partial \ln Q_1^{NR}}{\partial \beta} = \frac{3}{2\beta} = 3E_{MB}^{ext}. \tag{7}$$

Although the use of Equation (7) dates back to before the discovery of special relativity theory by Einstein, the ultra-relativistic expression of E produces interesting results in this framework [53].

2.2. Relativistic Gas

In the relativistic situations, where $E = \sqrt{p^2c^2 + m^2c^4}$ in which c denotes the light velocity and m is the rest mass, one can employ Equation (5) to get

$$\begin{aligned} Q_1^R &= Q_1^{NR}\Psi(\sigma), \\ \Psi(\sigma) &= \frac{i^{\frac{3}{2}} m^3 c^6 \sqrt{\frac{\pi}{2}} H_2^{(1)}(i\sigma)}{(i\sigma)^{\frac{5}{2}}}, \end{aligned} \tag{8}$$

as the partition function of a single particle [52,54,55]. Finally, we obtain the thermal energy per particle as

$$U^R = \frac{1}{\beta}\left[1 - i\sigma \frac{H'^{(1)}_2(i\sigma)}{H_2^{(1)}(i\sigma)}\right] = -\frac{\partial}{\partial \beta} \ln Q_1^R. \tag{9}$$

In the above equations, $\sigma = \beta m c^2$, $H_n^{(j)}(i\sigma)$ is the n-th order Hankel function of the j-th kind, and prime denotes a derivative with respect to the argument of the function. The above results were first reported in 1911 by Jüttner [52], who attempted to calculate the energy of a relativistic ideal gas using the conventional theory of relativistic statistical mechanics. According to Jüttner's results, a comprehensive study of a 3-dimensional relativistic system requires the Jüttner distribution (f_J)

$$f_J(\gamma,\beta) = Z_J(\gamma^2 - 1)^{\frac{1}{2}} \gamma \exp(-\beta m \gamma), \tag{10}$$

instead of MB distribution (f_{MB}) [56–61]. Jüttner distribution is indeed the relativistic extension of generalized isotropic MB distribution when $E(p) = m\gamma(p)c^2$. Here, Z_J is the normalization constant and $\gamma = \frac{1}{\sqrt{1-v^2}}$ refers to the Lorentz factor, where the units have been set so that $c = 1$. In terms of E, simple calculations give us Jüttner distribution as

$$f_J(E,\beta) = Z_{JE} a_J(E) \exp(-\beta E), \tag{11}$$

where $a_J(E) = E\sqrt{E^2 - m^2}$ denotes the density of states in energy representation, and in terms of v one finds

$$f_J = (v, \beta) = Z_{JV}\left(\frac{1}{\sqrt{1-v^2}}\right)^5 \exp\left(-\frac{m\beta}{\sqrt{1-v^2}}\right), \tag{12}$$

in which Z_{JE} and Z_{JV} are new normalization constants [57]. These constants can be evaluated using the normalization condition

$$\begin{aligned} &\int_1^\infty f_J(\gamma,\beta)d\gamma = \int_m^\infty f_J(E,\beta)dE = 1 \\ &= \int_0^1 f_J(\gamma(v),\beta)\tfrac{d\gamma}{dv}dv = \int_0^1 f_J(v,\beta)d^3v, \end{aligned} \tag{13}$$

which clearly states that

$$f_J(v,\beta) = \frac{1}{4\pi v^2} f_J(\gamma(v),\beta)\frac{d\gamma}{dv}. \tag{14}$$

It is finally useful to note that the extremum of $f_J(v,\beta)$ is located at $v = \sqrt{1-\left(\frac{\beta m}{5}\right)^2} \equiv v_J^{ext}$ leading to $E_J^{ext} = \frac{5}{\beta}$, a solution which is valid only when $\beta m < 5$. There are also other proposals for Jüttner function ($f_J(\gamma,\beta)$) [56–61], but the standard form Equation (11) considered in this paper is confirmed by some previous studies [58–60]. The corresponding thermal energy per particle (i.e., $\langle\gamma\rangle$) (or equally, the ratio U/N in Equation (12)) can also be obtained by using $f_J(\gamma,\beta)$, as

$$U^R \equiv m\langle\gamma\rangle = \int_m^\infty E f_J(E,\beta)dE = -\frac{\partial}{\partial\beta}\ln Q_1^R. \tag{15}$$

Although Equations (7) and (15) are simple examples, they confirm that the mean value of energy (or equally, thermal energy) can be calculated by using either the partition function or the distribution function. Moreover, employing these equations, one can find the distribution functions whenever the partition function is known. Indeed, if the phase-space geometry is deformed, then the partition function will also be modified. Therefore, one can find the corresponding modified MB and Jüttner distributions by directly using Equations (7) and (15) for the non-relativistic and relativistic cases, repetitively.

3. Generalized Uncertainty Principle, Partition and Distribution Functions

In the units of $\hbar = c = 1$, the relation $[x_k, p_l] = i\delta_{kl}$ is the standard commutation relation between the canonical coordinates x and p. This relation leads to HUP in the framework of quantum mechanics and is the backbone of calculating ω_0 [48,53]. Thus, the volume element d^3xd^3p changes whenever different coordinates (commutation relations) are used [42–44,48]. For GUP, we have [14,31]

$$\Delta X \Delta P \geq \frac{1}{2}\Big[1 + \eta(\Delta P)^2 + \ldots\Big], \tag{16}$$

where η denotes the GUP parameter and it is based on modified commutation relations

$$\begin{aligned} &[X_k, P_l] = i\big(\delta_{kl}(1+\eta P^2) + \eta' P_k P_l\big), \\ &[P_k, P_l] = 0, \\ &[X_k, X_l] = i\tfrac{2\eta - \eta' + (2\eta+\eta')\eta P^2}{1+\eta P^2}(P_k X_l - P_l X_k), \end{aligned} \tag{17}$$

where $k, l = 1, 2, 3$ for a 3-dimensional space [43,62]. P and X are generalized coordinates which are not necessarily equal to the canonical coordinates p and x. In this manner, assuming $\eta' = 0$ and η is independent of $\hbar$, one finds

$$d^3xd^3p \to \frac{d^3Xd^3P}{(1+\eta P^2)^3}, \tag{18}$$

which must be considered as the volume element in X-P space instead of d^3xd^3p [42–44,48]. This means that the density of states in the X-P phase space is affected by the factor of $(1+\eta P^2)$ [42]. In this situation, the single particle partition function can also be found as

$$Q_1^{GUP} = \int \exp(-\beta H(P,X)) \frac{d^3Xd^3P}{(1+\eta P^2)^3}, \tag{19}$$

where $H(P,X)$ denotes Hamiltonian in generalized coordinates [42,43,50]. The corresponding thermal energy (U^{GUP}) can be calculated using the relation

$$U^{GUP} = -\frac{\partial}{\partial \beta} \ln Q_1^{GUP}, \tag{20}$$

along with Equation (19), which finally gives

$$U^{GUP} = \frac{\int H(P) \exp(-\beta H(P)) \frac{d^3P}{(1+\eta P^2)^3}}{\int \exp(-\beta H(P)) \frac{d^3P}{(1+\eta P^2)^3}}. \tag{21}$$

In obtaining this equation, the fact that $H(\equiv E)$ is independent of X has been used which cancels integration over d^3X. Indeed, the density of states in phase-space is changed under the shadow of GUP [42,43,48], a result which affects the distribution function.

For a single free particle with $H = \frac{P^2}{2m}$, the ideal gas law is still valid, and therefore

$$Q_1^{NR,GUP} = Q_1^{NR} I\left(\frac{2m\eta}{\beta}, 3\right), \tag{22}$$

while the explicit form of the function $I\left(\frac{2m\eta}{\beta}, 3\right)$ can be followed in [44] and Q_1^{NR} is introduced in Equation (6). The effects of GUP are stored in $I\left(\frac{2m\eta}{\beta}, 3\right)$, and in the limit of $\eta \to 0$, one gets the ordinary single partition function of a free particle. Correspondingly, the partition function of a single free relativistic particle can also be evaluated using $H^2 = P^2 + m^2$ in Equation (19). By doing so one finds

$$Q_1^{R,GUP} = \int \exp\left(-\beta\sqrt{P^2+m^2}\right) \frac{d^3Xd^3P}{(1+\eta P^2)^3}, \tag{23}$$

for which the solution reads

$$Q_1^{R,GUP} = Q_1^{R}\left(1 - \eta\frac{15}{2}\frac{1}{\beta m}\right), \tag{24}$$

when $m \gg \frac{1}{\beta}$ [50].

3.1. Maxwell–Boltzmann Statistics

Bearing in mind the recipe which led to the expression for $f_{MB}(E,\beta)$, one can get the modified MB distribution in the X-P space as

$$\begin{aligned} f_{MB}^{GUP}(E,\beta) &= 4\pi P^2(E) \frac{\exp(-\beta E)}{(1+\eta P^2(E))^3} \frac{dP}{dE} \\ &= Z_{MBE}^{GUP} \frac{E^{\frac{1}{2}} \exp(-\beta E)}{(1+2\eta m E)^3}, \end{aligned} \tag{25}$$

where Z_{MBE}^{GUP} denotes the normalization constant in the presence of GUP. The thermal energy then reads

$$U^{GUP} = \int_0^\infty E f_{MB}^{GUP}(E,\beta) dE. \tag{26}$$

One can also find the normalization constant Z_{MBE}^{GUP} as

$$Z_{MBE}^{GUP} = \left[\int_0^\infty \frac{E^{\frac{1}{2}} \exp(-\beta E)}{(1+2\eta m E)^3} dE \right]^{-1}, \tag{27}$$

which is equal to Z_{MBE} in the limit where $\eta \to 0$. Obviously, the MB distribution $f_{MB}(E,\beta)$, is recovered through Equation (25) at the appropriate limit of $\eta = 0$. For the density of states in HUP framework we have $a_{MB}(E) = \sqrt{E}$. This relation is modified in the presence of GUP effects and thus, the density of states will take the following form

$$a_{MB}^{GUP}(E) = \frac{\sqrt{E}}{(1+2\eta m E)^3}, \tag{28}$$

which is in agreement with the results of [18]. The extremum of $f_{MB}^{GUP}(E,\beta)$ is also located at

$$\varepsilon_{MB}^{ext} = \frac{1+10m\eta E_{MB}^{ext}}{4m\eta} \left(\sqrt{1 + \frac{8m\eta E_{MB}^{ext}}{(1+10m\eta E_{MB}^{ext})^2}} - 1 \right), \tag{29}$$

which clearly indicates $\varepsilon_{MB}^{ext} \to E_{MB}^{ext}$ whenever $\eta \to 0$. In Figure 1, the effects of GUP on the distribution function in MB statistics are shown where the temperature is considered to be constant ($\beta = 1$).

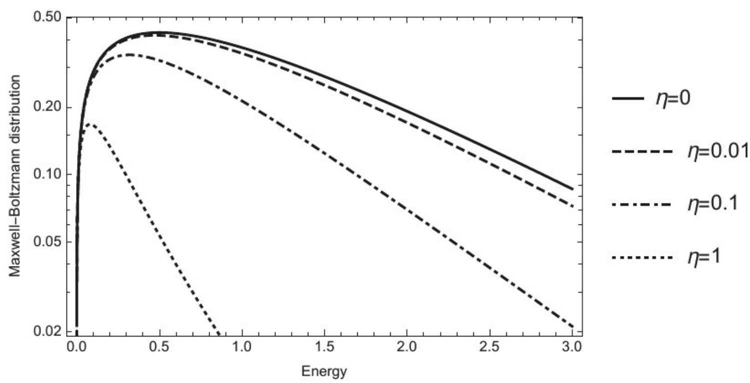

Figure 1. Maxwell–Boltzmann (MB) distribution versus energy for $\eta = 0.5, 1, 1.5$. The ordinary MB distribution is denoted by the solid curve. Here, we have set the units so that $\hbar = c = K_B = 1$.

3.2. Jüttner Statistics

In the relativistic situation, where $H = \sqrt{P^2 + m^2} (\equiv E)$, following the above recipe, we get the modified Jüttner distribution as

$$f_J^{GUP}(E,\beta) = Z_{JE}^{GUP} \frac{E\sqrt{E^2 - m^2} \exp(-\beta E)}{(1+\eta[E^2 - m^2])^3}, \tag{30}$$

which recovers $f_J(E,\beta)$ in the limit where $\eta \to 0$. Here, Z_{JE}^{GUP} is also a normalization constant which can be calculated by utilizing the normalization constraint $\int_m^\infty f_J^{GUP}(E,\beta) dE = 1$. We also find

$$a_J^{GUP}(E) = \frac{E\sqrt{E^2 - m^2}}{(1+\eta[E^2 - m^2])^3}, \tag{31}$$

as the density of states in Jüttner statistics in the presence of GUP. Figure 2 shows the behavior of $f_J^{GUP}(E,\beta)$ for some positive values of η parameter.

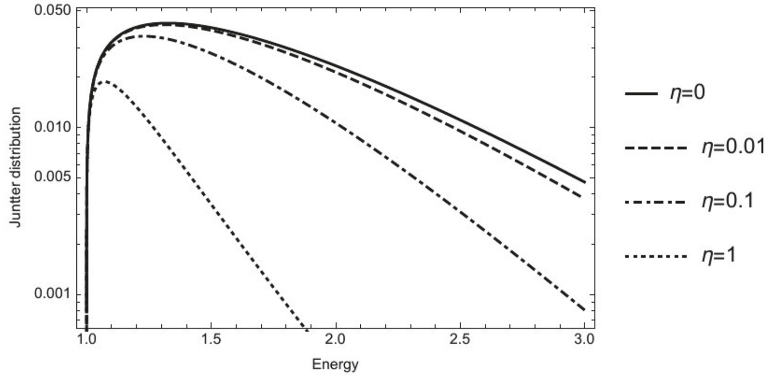

Figure 2. Behavior of Jüttner distribution against energy for $\eta = 0.5, 1, 1.5$. The solid curve belongs to the ordinary Jüttner distribution and we have set the units so that $\hbar = c = K_B = 1$.

4. Extended Uncertainty Principle, Partition and Distribution Functions

The modified Heisenberg algebra in the EUP framework can be recast into the following form

$$[X_i, P_j] = i(\delta_{ij} + \alpha X_i X_j), \tag{32}$$

where α is a small positive parameter known as the EUP parameter. In the limit of $\alpha \to 0$, the canonical commutation relation of the standard quantum mechanics is recovered. Based on the commutation relation Equation (32), the HUP is modified by

$$(\Delta X_i)(\Delta P_i) \geq \frac{1}{2}\left[1 + \alpha(\Delta X_i)^2 + \ldots\right], \tag{33}$$

which leads to a non-zero minimum uncertainty in momentum as $(\Delta P_i)_{\min} = \sqrt{\alpha}$. Here, we apply the coordinate representation of the operators X_i and P_i expressed as

$$\begin{aligned} X_i &= x_i, \\ P_i &= (\delta_{ij} + \alpha x_i x_j) p_j, \end{aligned} \tag{34}$$

where x_i and p_j satisfy the standard commutation relation of ordinary quantum mechanics. This representation yields the following commutation relation for the momentum operator

$$[P_i, P_j] = i\alpha(x_i p_j - p_i x_j). \tag{35}$$

In the *X*-*P* space, the modified volume element

$$\frac{d^3X d^3P}{(1 + \alpha X^2)^3}, \tag{36}$$

should be considered instead of $d^3x d^3p$ [20]. We then proceed to consider the consequences of such a modification in calculating the partition and distribution functions. For a single particle, the partition function in *X*-*P* space can be found as

$$Q_1^{EUP} = \int \exp(-\beta H(P, X)) \frac{d^3X d^3P}{(1 + \alpha X^2)^3}, \tag{37}$$

whence we get the corresponding thermal energy as

$$U^{EUP} = -\frac{\partial}{\partial \beta} \ln Q_1^{EUP}. \tag{38}$$

The above expression can also be combined with Equation (37) to give Equation (21). For the free non-relativistic and relativistic particles, one finds

$$Q_1^{NR,EUP} = V_{eff}(\alpha, r)\left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}} = \frac{V_{eff}(\alpha, r)}{V} Q_1^{NR}, \tag{39}$$

and

$$Q_1^{R,EUP} = \frac{V_{eff}(\alpha, r)}{V} Q_1^{R}, \tag{40}$$

respectively, where we have defined $V_{eff}(\alpha, r) = \int_0^r \frac{d^3X}{(1+\alpha X^2)^3}$ as the effective volume, and in the limit of $\alpha \to 0$, the usual volume V is recovered. Since $V_{eff}(\alpha, r)$ is independent of β, the thermal energy related to EUP is the same as what we obtained in Equations (7) and (9), respectively. Consequently, EUP does not affect the Maxwell–Boltzmann and Jüttner distribution functions, because the corresponding effective volume has no dependence on β.

5. Conclusions

The Jüttner function is the relativistic version of MB distribution and is proper for studying relativistic (high energy) systems. On the other hand, the minimal length comes into play in the realms of high energy physics. Hence, compared with MB distribution, the study of its effects on Jüttner distribution would be more meaningful. Thus, our attempt in the present work was to address an algorithm with the help of which, one can get the distribution function, starting from the partition function. Motivated then by the abovementioned arguments, we studied the effects of GUP and EUP (two aspects of quantum gravity) on Jüttner distribution and the corresponding density of states in energy space. We also addressed the consequence of applying our approach to the MB distribution in order to find the density of states Equation (28) which is in agreement with previous reports [42,48], a result which confirms our approach. The results of our study are summarized in Tables 1 and 2 for the non-relativistic and relativistic regimes, respectively.

Table 1. Non-relativistic ideal gas ($2\pi\hbar = 1$).

	HUP	GUP	EUP
The volume of phase space element	1	$(1+\eta P^2)^3$	$(1+\alpha X^2)^3$
Density of States	$\sqrt{E}$	$\frac{\sqrt{E}}{(1+2m\eta E)^3}$	$\sqrt{E}$
Single Partition Function	$V\left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}}$	$V\left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}} I\left(\frac{2m\eta}{\beta}, 3\right)$	$V_{eff}(\alpha, r)\left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}}$

Table 2. Relativistic ideal gas ($2\pi\hbar = 1$).

	HUP	GUP	EUP
The volume of phase space element	1	$(1+\eta P^2)^3$	$(1+\alpha X^2)^3$
Density of States	$E\sqrt{E^2-m^2}$	$\frac{E\sqrt{E^2-m^2}}{(1+\eta(E^2-m^2))^3}$	$E\sqrt{E^2-m^2}$
Single Partition Function	$V\left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}}\Psi(\sigma)$	$V\left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}}\Psi(\sigma)\left(1-\eta\frac{15}{2}\frac{1}{\beta m}\right)$, when $m \gg \frac{1}{\beta}$	$V_{eff}(\alpha, r)\left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}}\Psi(\sigma)$

It is obvious from Figures 1 and 2, that the effects of the existence of a non-zero minimal length ($\eta \neq 0$) on distribution functions become more sensible as energy increases. This means that the probability of achieving high energy states when $\eta \neq 0$ is smaller than the $\eta = 0$ case. It is also worth mentioning that though there exist some proposals to test observable effects of the minimal length [63], the Planck scale is currently far beyond our reach. Since by comparing the Planck energy ($\approx 10^{16}$ TeV) [64] to the energy achieved in the Large Hadron Collider ($\approx$10 TeV) [65], or the Planck length ($\approx 10^{-35}$ m) to the uncertainty within the position of the LIGO mirrors ($\approx 10^{-18}$ m) [66] or the Planck time ($\approx 10^{-44}$ s) to the shortest light pulse produced in laboratory($\approx 10^{-17}$ s) [67], we observe that we are at best 15 orders of magnitude away from achieving the Planck scale. In this regard, future developments within these experimental setups are expected in order to search for the footprints of GUP effects in nature.

Finally, regarding the results reported in [68] and [69] the usefulness of Tsallis distribution function in high energy physics is expected. In line with these results, some researchers study the possibility of describing the distribution of transverse momentum in the Large Hadron Collider and Relativistic Heavy Ion Collider, employing the Tsallis distribution, expressed as [70–73]

$$f_T(q,\beta) = Z_T[1-(1-q)\beta E]^{\frac{1}{1-q}}. \tag{41}$$

Here Z_T and q denote the normalization constant and non-extensivity parameter, respectively. Although utilizing our approach to investigate the effects of GUP and EUP on Equation (41) is straightforward, such a study needs more careful analysis owing to the issues raised by [38] which states a criterion on the domains of validity of Maxwell–Boltzmann, Jüttner, and Tsallis distributions as a special high energy phenomenon. Therefore, it can be considered as an attractive topic for future studies.

Author Contributions: All authors have the same contribution. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The study did not report any data.

Conflicts of Interest: The authors declare no conflict of interest.

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Extended Chern–Simons Model for a Vector Multiplet

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Abstract: We consider a gauge theory of vector fields in $3D$ Minkowski space. At the free level, the dynamical variables are subjected to the extended Chern–Simons (ECS) equations with higher derivatives. If the color index takes n values, the third-order model admits a $2n$-parameter series of second-rank conserved tensors, which includes the canonical energy–momentum. Even though the canonical energy is unbounded, the other representatives in the series have a bounded from below the 00-component. The theory admits consistent self-interactions with the Yang–Mills gauge symmetry. The Lagrangian couplings preserve the energy–momentum tensor that is unbounded from below, and they do not lead to a stable non-linear theory. The non-Lagrangian couplings are consistent with the existence of a conserved tensor with a 00-component bounded from below. These models are stable at the non-linear level. The dynamics of interacting theory admit a constraint Hamiltonian form. The Hamiltonian density is given by the 00-component of the conserved tensor. In the case of stable interactions, the Poisson bracket and Hamiltonian do not follow from the canonical Ostrogradski construction. Particular attention is paid to the "triply massless" ECS theory, which demonstrates instability even at the free level. It is shown that the introduction of extra scalar field, serving as Higgs, can stabilize the dynamics in the vicinity of the local minimum of energy. The equations of motion of the stable model are non-Lagrangian, but they admit the Hamiltonian form of dynamics with a Hamiltonian that is bounded from below.

Keywords: higher-derivative theories; extended Chern–Simons; Hamiltonian formalism; stability

Citation: Kaparulin, D.S.; Lyakhovich, S.L.; Nosyrev, O.D. Extended Chern–Simons Model for a Vector Multiplet. *Symmetry* **2021**, *13*, 1004. https://doi.org/10.3390/sym13061004

Academic Editors: Antonino Marciano and Chris Fields

Received: 30 April 2021
Accepted: 1 June 2021
Published: 3 June 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

1. Introduction

The higher-derivative theories are well known for their better convergency properties at classical and quantum levels and wider symmetry. In most instances, these advantages come at the price of dynamic instability, which is the typical problem for the models in this class. At the classical level, the solutions to the equations of motion demonstrate runaway behavior ("collapse"). At the quantum level, ghost poles appear in the propagator, and so the unitarity of dynamics is an issue. These peculiarities follow from a single fact: the canonical energy is unbounded in every non-singular higher derivative theory. For a review of the problem, we cite recent articles [1–3] and references therein. The canonical energy of singular models can be bounded. The examples include $f(R)$-theories of gravity [4–7]. These models do not demonstrate instability. The stability problem for constrained higher-derivative theories is discussed in [8]. In the majority of interesting (constrained or unconstrained) models, the instability has a special form: the classical dynamics are regular (no "collapse"), but the canonical energy is unbounded from below. The Pais–Uhlenbeck oscillator [9], Podolsky [10] and Lee–Wick [11,12] electrodynamics, ECS theory [13] and conformal gravity [14] are examples. With regard to the current studies, we refer readers to [15–18] and references therein. In all these models, the quantum theory is expected to be well defined [2,19], but the application of the canonical quantization procedure based on the Ostrogradski procedure (The canonical Hamiltonian formalism for the non-singular higher-derivative theories was first proposed in [20]. Its generalization for singular theories was developed in [21]: see also [22]. For recent studies, see [23,24].) leads to a model with a spectrum of energy that is unbounded from below; that is why the stability and unitarity of quantum dynamics is the most important issue.

Recently, it has been recognized that the higher-derivative dynamics can be stable at the classical and quantum levels even if the canonical energy of the model is unbounded. The articles [25,26] explain the stability of Pais–Uhlenbeck theory from the perspective of the existence of the Hamiltonian form of dynamics with a bounded (non-canonical) Hamiltonian. The quantization of the model with an alternative Hamiltonian leads to a quantum theory with a well-defined vacuum state with the lowest energy. In [27,28], the authors use a special PT-symmetry for the construction of stable quantum mechanics in the higher-derivative oscillator model. In [1–3,29], the authors state that the non-linear higher derivative models may exhibit well-defined classical dynamics without "collapsing" trajectories with runaway behavior. In the latter articles, the existence of stable classical dynamics serves as a necessary prerequisite for the construction of a well-defined quantum theory. With regard to the studies of field theories, we refer readers to [30,31]. All the models mentioned in this paragraph have one common feature: they admit alternative (in contrast to the canonical energy) conserved quantities that are bounded from below. Quantum stability is achieved if the model admits the Hamiltonian form of dynamics, with the bounded conserved quantity being the Hamiltonian. This means that the stable higher-derivative theory is characterized by two ingredients: the bounded conserved quantity and the Poisson bracket on its phase space that brings the equations of motion to the Hamiltonian form with a Hamiltonian bounded from below.

In [32], the stability is studied in the class of models of the derived type. At the free level, the wave operator of the theory is given by a polynomial (characteristic polynomial) in another formally self-adjoint operator of lower order. It has been shown that each derived theory admits a series of conserved tensors, which includes the canonical energy–momentum [33]. The number of entries in the series grows with the order of the characteristic polynomial. Even though the canonical energy is unbounded, the other conserved tensors in the series can be bounded from below [33]. The bounded conserved quantity stabilizes the classical dynamics of the theory. The quantum stability is explained by the existence of the Lagrange anchor, which relates a bounded conserved quantity with the invariance of the model with respect to the time translations. (The concept of the Lagrange anchor was proposed in [34] in the context of the quantization of non-Lagrangian field theories. Later, it was recognized that the Lagrange anchor connects symmetries and conserved quantities [35].) In the first-order formalism, the Lagrange anchor defines the Poisson bracket [36]. The Hamiltonian is given by the conserved quantity in the series and is expressed in terms of phase-space variables. The linear higher-derivative theory is stable if a bounded conserved quantity, related to time-translation symmetry, is admitted by the model [32]. In [37,38], the authors consider the problem of the consistent deformations of symmetries and conserved quantities that preserve the stability of dynamics in the class of derived-type models. In all cases, interaction vertices do not come from the least-action principle with higher derivatives, but the equations of motion admit the Hamiltonian form of dynamics. The main difficulty of the procedures cited above is that they do not automatically preserve gauge invariance. This restricts their applications in gauge systems.

The ECS model [13] is the simplest gauge theory of the derived type. In current studies, it often serves as a prototype of the class of gauge theories with higher derivatives; see, e.g., [39–41]. The stability of the ECS model was first studied in [33]. It has been observed that the theory of order p admits a $p-1$-parameter series of second-rank conserved tensors, which includes the canonical energy–momentum. The canonical energy of the model is always unbounded from below. The other tensors in the series may have a 00-component that is bounded from below. The stability conditions for the theory are determined by the structure of the roots of the characteristic polynomial [33]. The theory is stable if all the nonzero roots of the characteristic polynomial are real and different and the zero root has a multiplicity of one or two. The stability of the ECS theory is confirmed in [42].

The ECS model was shown to be a multi-Hamiltonian at the free level in [43]. The series of Hamiltonians includes the canonical (Ostrogradski) Hamiltonian, which is unbounded, and the other representatives, which are bounded or unbounded depending on the model parameters. If the Hamiltonian is bounded, this ensures the stability of the model at both classical and quantum levels. The model admits the inclusion of stable non-Lagrangian interactions with scalar, fermionic and gravitational fields that preserve a selected representative in the series of conserved quantities of free model [38,43,44]. However, the gauge symmetry is abelian in the sector of the vector field in all these examples.

The concept of the consistency of interaction between the gauge fields was developed in [45]. The consistent couplings between gauge vectors were studied in [46,47]. It was shown that the Yang–Mills vertex is unique in the covariant setting. The Lagrangian self-interactions of the gauge vector multiplet subjected to the ECS equations at the free level were reconsidered in [42]. The most general consistent non-linear theory has been proven to have the Yang–Mills gauge symmetry, and the Lagrangian is given by the covariantization of the free ECS Lagrangian. The interacting model demonstrates Ostrogradski instability at the non-linear level in all instances, even though the free theory is stable. This result implies a no-go theorem for stable Lagrangian interactions in the ECS model. The conclusion is not surprising because the Lagrangian couplings preserve the conservation law of canonical energy, being unbounded already at the free level. The inclusion of non-Lagrangian interactions can solve the issue of the dynamic stability at the interacting level, because such couplings preserve bounded conserved quantities. However, the problem of the construction of stable and consistent non-Lagrangian (self-)couplings (stable or unstable) has not been studied in the ECS model in the literature before.

In the current work, we present a class of stable self-interactions in the theory of vector multiplets. The free fields are subjected to the ECS equations of the third order. The interaction is (in general) non-Lagrangian. The non-linear equations of motion are consistent with Yang–Mills gauge symmetry. A selected second-rank conserved tensor of the free model is preserved by the coupling. Depending on the values of coupling constants, this can be bounded or unbounded. The Lagrangian interaction vertex in [42] is unstable. The non-Lagrangian coupling can be consistent with the existence of the bounded conserved tensor. The equations of motion admit the Hamiltonian form of dynamics. On the shell, the Hamiltonian density is given by the 00-component of the conserved tensor. For Lagrangian interactions, the canonical formalism with the unbounded Ostrogradski Hamiltonian is reproduced. The bounded Hamiltonians do not follow from the Ostrograski procedure, and thus we find non-canonical Hamiltonian formalism in this case. In all the instances, the Poisson bracket is a non-degenerate tensor, so the model admits a Hamiltonian action principle. The quantization of the first-order action with the bounded Hamiltonian leads to a stable quantum theory with a well-defined vacuum state.

In the case of resonance (multiple roots of a charlatanistic polynomial), the dynamics of the theory are unstable even at the free level. The inclusion of self-interaction does not improve the stability properties of dynamics of this model because the conserved quantities have the same structure in the free and non-linear models. In the current article, a model with the third-order zero resonance root is of interest. The free field is subjected to the "triply massless" Chern–Simons (CS) equations. The wave operator of free equations is given by a cube of the CS operator. To stabilize the dynamics at the non-linear level, we apply the "Higgs-like" mechanism described in [48]. Introducing an extra scalar, we generate a coupling such that the energy of the interacting theory obtains a local minimum for a solution with nonzero values of dynamical variables. The motions of small fluctuations in the vicinity of this solution are stable because the non-linear theory has no resonance. The dynamics of fluctuations admit the Hamiltonian form, with the Hamiltonian being given by the positive definite quadratic form of the fields. This means that the dynamics of the theory with a resonance can be stabilized by the inclusion of an appropriate interaction.

The article is organized as follows. In Section 2, we consider the third-order ECS model for a vector multiplet. Particular attention is paid to the structure of symmetries, conserved quantities of the theory and the stability of the dynamics. In Section 3, we propose stable self-interactions between the multiplet of vector fields with the Yang–Mills gauge symmetry. The general model in the class model preserves a selected conserved tensor of the free theory, which can be bounded or unbounded from below depending on the values of coupling constants. The stable interactions are inevitably non-Lagrangian. In Section 4, we construct the constrained Hamiltonian formalism for the non-linear model. The density of the Hamiltonian is given by the 00-component of the conserved tensor, being expressed in terms of the phase-space variables. In Section 5, we consider the issue of the stability of "triply massless" ECS theory. We propose the class of consistent couplings, with the scalar field stabilizing the dynamics in the vicinity of equilibrium position. The equations of motion are non-Lagrangian, but they admit the constrained Hamiltonian form with a bounded Hamiltonian. The concluding section discusses the potential applications of the model that are considered in the article.

2. Higher-Derivative Chern–Simons Model

We consider the most general third-order gauge theory of the vector multiplet $A^a = A^a{}_\mu(x)dx^\mu$, $\mu = 0,1,2$, $a = 1,2,\ldots,n$ in three-dimensional Minkowski space. The action functional of the model reads

$$S[A(x)] = \frac{1}{2}\int A^a{}_\mu(\alpha_1 F^{a\mu} + \alpha_2 G^{a\mu} + \alpha_3 K^{a\mu})d^3x\,. \tag{1}$$

where the real numbers α_1, α_2 and α_3 are model parameters. Without loss of generality, we assume that the coefficient at the higher derivative term is nonzero, $\alpha_3 \neq 0$. The vectors $F^a{}_\mu$, $G^a{}_\mu$ and $K^a{}_\mu$ denote the (generalized) field strengths of the potential $A^a{}_\mu$,

$$F^a{}_\mu = \varepsilon_{\mu\nu\rho}\partial^\nu A^{a\rho}\,, \qquad G^a{}_\mu = \varepsilon_{\mu\nu\rho}\partial^\nu F^{a\rho}\,, \qquad K^a{}_\mu = \varepsilon_{\mu\nu\rho}\partial^\nu G^{a\rho}\,, \tag{2}$$

where the Levi–Civita symbol $\epsilon_{\mu\nu\rho}, \epsilon_{012} = 1$ is antisymmetric with respect to indices. All the tensor indices are raised and lowered by the Minkowski metric. We use the mostly positive convention for the signature of metrics throughout the paper. The summation is implied over the isotopic indices $a = 1,\ldots,n$ repeated at one level unless otherwise stated. For $\alpha_3 = \alpha_1 = 1$, $\alpha_2 = 0$, $n = 1$, the action functional Equation (1) was first proposed in the article [13]. We refer to the model Equation (1) as the ECS theory for a vector multiplet.

The least-action principle for the functional Equation (1) brings us to the following Lagrange equations for the field A_μ:

$$\frac{\delta S}{\delta A^{a\mu}} \equiv \alpha_1 F^a{}_\mu + \alpha_2 G^a{}_\mu + \alpha_3 K^a{}_\mu = 0\,, \qquad a = 1,\ldots,n\,. \tag{3}$$

These equations have the derived form [33] because the wave operator is a polynomial in the CS operator $*d$. The symbol $*$ denotes the Hodge star operator, and d is the de Rham differential. For the exposition of differential form theory concepts, we refer readers to the book [49]. The structure of the Poincare group representation, being described by the Equation (3), is determined by the roots of the characteristic polynomial

$$M(\alpha; z) = \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3\,. \tag{4}$$

The polynomial $M(\alpha; z)$ follows from Equation (3) after the formal replacement of the CS operator $*d$ by the complex-valued variable z. The following cases are distinguished in [33].

(1a) $\alpha_1 \neq 0\,, (\alpha_2)^2 - 4\alpha_1\alpha_3 > 0\,.$ The characteristic polynomial has a zero root and two different nonzero real roots. Equation (3) describes two massive spin-1 self-dual fields. (The theory of a massive spin-1 field being subjected to the self-duality condition was proposed in [50]. For the theory of representations of the 3*D* Poincare group, we refer readers to [51–53]). A zero root corresponds to the CS mode, which is a pure gauge;

(1b) $\alpha_1 \neq 0\,, (\alpha_2)^2 - 4\alpha_1\alpha_3 < 0$. The characteristic polynomial has a zero root and two complex roots. The case is similar to (1a), but the masses of vector modes are complex. The Poincare group representation is non-unitary;

(2a) $\alpha_2 \neq 0, \alpha_1 = 0$. The characteristic polynomial has multiple two zero roots and a simple nonzero root. The set of subrepresentations includes a massless spin-1 field and a massive spin-1 mode subjected to the self-duality condition;

(2b) $\alpha_2 \neq 0, (\alpha_2)^2 - 4\alpha_1\alpha_3 = 0$. The characteristic polynomial has multiple two nonzero roots and a simple zero root. The subrepresentations describe a "doubly massive" massive mode and a gauge CS mode. The Poincare group representation is non-unitary;

(3) $\alpha_1 = \alpha_2 = 0$. The characteristic polynomial has multiple three zero roots. Equation (3) describes the "triply" massless extended CS theory. The Poincare group representation is non-unitary.

As we see, the ECS model describes the unitary representation of the Poincare group if all the nonzero roots of the characteristic equation are different, and the zero root has a multiplicity of one or two. The dynamical degrees of freedom include the massive spin-1 vector subjected to the self-duality condition and/or spin-1 massless field, meeting the $3D$ Maxwell equations. In all instances, the model has two local physical degrees of freedom (four physical polarizations).

The action function Equation (1) is preserved by the $2n$-parameter series of infinitesimal transformations:

$$\delta_{\xi;\beta} A^a{}_\mu = -\varepsilon_{\mu\nu\rho}\xi^\rho (\beta^a{}_1 F^{a\rho} + \beta^a{}_2 G^{a\rho})\,, \qquad a = 1, \ldots, n, \tag{5}$$

(no summation in a). The transformation parameters are the constant vector ξ^μ and real numbers $\beta^a{}_k$, $a = 0, \ldots, n$, $k = 1, 2$. The series Equation (5) includes the space–time translations with the independent parameters for the individual vector $A^a{}_\mu$ in the multiplet and higher-order transformations, whose value is determined by $\beta^a{}_2$. The Noether theorem associates symmetries Equation (5) with the $2n$-parameter series of second-rank conserved tensors, which has the form

$$\Theta^{\mu\nu}(\beta; \alpha) = \sum_{a=1}^{n} (\beta^a{}_1 \Theta^{a\mu\nu}{}_1(\alpha) + \beta^a{}_2 \Theta^{a\mu\nu}{}_2(\alpha))\,, \tag{6}$$

where

$$\Theta^{a\mu\nu}{}_1(\alpha) = \alpha_3 (G^{a\mu} F^{a\nu} + G^{a\nu} F^{a\mu} - g^{\mu\nu} G^a{}_\rho F^{a\rho}) + \alpha_2 (F^{a\mu} F^{a\nu} - \frac{1}{2} g^{\mu\nu} F^a{}_\rho F^{a\rho})\,; \tag{7}$$

$$\Theta^{a\mu\nu}{}_2(\alpha) = \alpha_3 (G^{a\mu} G^{a\nu} - \frac{1}{2} g^{\mu\nu} G^a{}_\rho G^{a\rho}) - \alpha_1 (F^{a\mu} F^{a\nu} - \frac{1}{2} g^{\mu\nu} F^a{}_\rho F^{a\rho}) \tag{8}$$

(no summation in a). The quantities $\Theta^{a\mu\nu}{}_1$ and $a = 1, \ldots, n$ Equation (7) represent the canonical energy–momentum tensors of individual fields in the vector multiplet $A^a{}_\mu$. The tensors $\Theta^{a\mu\nu}{}_2$ and $a = 1, \ldots, n$ Equation (8) are other conserved quantities. The total number of independent conserved tensors in the free theory is $2n$ because each field in the multiplet admits two different symmetries.

The 00-component of the conserved tensor Equation (6) reads

$$\begin{aligned} \Theta^{00}(\beta; \alpha) = \frac{1}{2} \sum_{a=1}^{n} \Big\{ & \beta_2 \alpha_3 (G^{a0} G^{a0} + G^{ai} G^{ai}) + 2\beta_1 \alpha_3 (G^{a0} F^{a0} + G^{ai} F^{ai}) + \\ & + (\beta_1 \alpha_2 - \beta_2 \alpha_1)(F^{a0} F^{a0} + F^{ai} F^{ai}) \Big\}\,. \end{aligned} \tag{9}$$

The summation over the index $i = 1, 2$ repeated at one level is implied. The quadratic form Equation (9) can be reduced to the principal axes as follows:

$$\Theta^{00}(\beta;\alpha) = \sum_{a=1}^{n} \left\{ \frac{\alpha_3(X^{a0}X^{a0} + X^{ai}X^{ai})}{2\beta^a{}_2} + \frac{C(\beta^a;\alpha)(F^{a0}F^{a0} + F^{ai}F^{ai})}{2\beta^a{}_2} \right\}. \tag{10}$$

where $X^a_\mu = \beta^a{}_1 F^a{}_\mu + \beta^a{}_2 G_\mu$, and the following notation is used:

$$C(\beta^a;\alpha) = -(\beta^a{}_2)^2\alpha_1 + \beta^a{}_2\beta^a{}_1\alpha_2 - (\beta^a{}_1)^2\alpha_3\,. \tag{11}$$

In Section 4, we see that $X^{a0}X^{a0} + X^{ai}X^{ai}$ and $F^{a0}F^{a0} + F^{ai}F^{ai}$ depend on different initial data (See the detailed discussion in [43]). In this case, the 00-component Equation (9) is bounded from below if the coefficients at squares are positive:

$$\beta^a{}_2\alpha_3 > 0, \qquad C(\beta^a;\alpha) > 0\,, \qquad a = 1,\ldots,n\,. \tag{12}$$

We ignore the case of the semi-positive quadratic form because the degenerate conserved quantities do not ensure the stability of dynamics. Relations Equation (12) are consistent if and only if the model parameters $\alpha_1, \alpha_2, \alpha_3$ meet conditions (1a) and (2a) of classification on pages 4–5. In cases (1b), (2b) and (3) of this classification, the conditions in Equation (12) are inconsistent. This means that the free model Equation (1) is stable if the wave Equation (3) describes a unitary representation of the Poincare group. This result is quite natural because the theories that correspond to the unitary representations of the Poincare group should have stable dynamics.

3. Stable Interactions

In this section, we present an example of stable self-interactions in the model Equation (1). The interactions are non-Lagrangian. The dynamics of the non-linear theory are determined by the equations of motion. The interactions are associated with the deformations of free equations of motion that preserve the gauge symmetries and gauge identities of the free model. The interaction is consistent if the number of physical degrees of freedom is preserved by coupling. For details of the concept of the consistency of interaction in the class of unnecessary Lagrangian theories, we refer readers to [54].

We begin the construction of a non-linear theory by assuming that the dynamical fields take values in the Lie algebra of a semisimple Lie group with the generators $t^a, a = 1,\ldots,n$,

$$\mathcal{A}_\mu = A^a{}_\mu(x)t^a dx^\mu\,, \qquad [t^a, t^b] = f^{abc}t^c\,, \qquad \mathrm{tr}(t^a t^b) = \delta^{ab}\,. \tag{13}$$

The covariant analogs of the (generalized) field strength vectors Equation (2) are defined as follows:

$$\mathcal{F}_\mu = \epsilon_{\mu\nu\rho}(\partial^\nu \mathcal{A}^\rho + \frac{1}{2}[\mathcal{A}^\nu, \mathcal{A}^\rho])\,, \qquad \mathcal{G}_\mu = \epsilon_{\mu\nu\rho}D^\nu\mathcal{F}^\rho\,, \qquad \mathcal{K}_\mu = \epsilon_{\mu\nu\rho}D^\nu\mathcal{G}^\rho. \tag{14}$$

The vectors $\mathcal{F}_\mu$, $\mathcal{G}_\mu$, $\mathcal{K}_\mu$ lie in the Lie algebra of a semisimple Lie group, and D stands for the covariant derivative

$$D_\mu = \partial_\mu + [\mathcal{A}_\mu, \cdot\,]\,. \tag{15}$$

The vector $\mathcal{F}_\mu$ represents a dual of the standard Yang–Mills strength tensor. The generalized field strengths $\mathcal{G}_\mu$ and $\mathcal{K}_\mu$ are other covariant quantities that involve second and third derivatives of $\mathcal{A}_\mu$.

We consider the interactions that are polynomial in the gauge invariants of Equation (14) and do not involve the highest derivative in the non-linear part. In this setting, the most general self-consistent non-linear theory is determined by the following equations of motion:

$$\mathbb{T}^\mu = \alpha_3\mathcal{K}^\mu + \alpha_2\mathcal{G}^\mu + \alpha_1\mathcal{F}^\mu - \frac{\alpha_3{}^2}{2C(\beta;\alpha)}\epsilon^{\mu\nu\rho}[\beta_1\mathcal{F}_\nu + \beta_2\mathcal{G}_\nu, \beta_1\mathcal{F}_\rho + \beta_2\mathcal{G}_\rho] = 0\,. \tag{16}$$

We prove the uniqueness of this coupling in Appendix A. The model parameters are the real numbers $\alpha_k, k = 1,2,3$, β_l, $l = 1,2$ and $f^{abc}, a, b, c = 1,\dots,n$. The constants α_1, α_2 determine the free limit of the Equation (16). The numbers β_1 and β_2 distinguish admissible couplings with the same gauge group. Throughout this section and below, we assume that $C(\beta;\alpha) \neq 0$. The equations of motion Equation (16) come from the least-action principle if $\beta_1 = 1, \beta_2 = 0$. In this case, the action functional reads

$$S[\mathcal{A}(x)] = \frac{1}{2}\mathrm{tr}\int \mathcal{F}_\mu(\alpha_1\mathcal{A}_\mu + \alpha_2\mathcal{F}^\mu + \alpha_3\mathcal{G}^\mu)d^3x\,. \tag{17}$$

This action functional was first derived in [42] (The higher-derivative Yang–Mills theory with a similar structure of the Lagrangian has been known for a long time [55]). The same paper tells us that Equation (17) is the most general form of consistent self-interaction in the gauge theory of vector fields. This means that the most general consistent Lagrangian coupling Equation (17) is included in the model Equation (16). If the parameter β_2 is nonzero, Equation (16) does not follow from the least-action principle for any functional with higher derivatives. The variational principle with auxiliary fields still exists, even if the higher-derivative model is non-Lagrangian. In the last case, the theory Equation (16) admits consequent quantization and the establishment of a relationship between symmetries and conserved quantities.

The concept of interaction consistency for theories that are not necessarily Lagrangian has been developed in [54]. This paper tells us that the non-Lagrangian interaction is consistent if the non-linear theory admits the same number of (i) gauge symmetries, (ii) gauge identities and (iii) physical degrees of freedom as a free model. All these facts are easily verified. At first, the equations of motion Equation (16) are preserved by the Yang–Mills gauge symmetry:

$$\delta_\zeta\mathcal{A}_\mu = D_\mu\zeta\,, \qquad \delta_\zeta\mathbb{T}_\mu = [\,\zeta\,,\mathbb{T}_\mu]\,, \tag{18}$$

where $\zeta = (\zeta^a(x), a = 1,\dots,n)$ is the gauge transformation parameter. The free model Equation (17) is preserved by the standard gradient gauge symmetry, $\delta_\zeta {A^a}_\mu = \partial_\mu\zeta^a$. As required, the gauge symmetry Equation (18) is given by the deformation of the gradient gauge symmetry of free model. The important difference is that the gauge symmetry Equation (18) is non-abelian. Thus, the inclusion of interaction tends towards the model with non-abelian gauge symmetry. We have an obvious set of gauge identities between the Equation (16),

$$\mathbb{D}_\mu\mathbb{T}^\mu = 0\,, \qquad \mathbb{D}_\mu = D_\mu + \frac{\alpha_3}{C(\beta;\alpha)}[\beta_1\mathcal{F}_\mu + \beta_2\mathcal{G}_\mu, \cdot\,]\,. \tag{19}$$

Again, the leading term of the gauge identity is given by the free contribution. This agrees with the concept of interaction consistency. At the final step of analysis, we verify that the physical degrees of freedom number is preserved by the coupling. Equation (8) of [54] expresses the number of physical degrees of freedom via the orders of gauge symmetries, gauge identities and equations of motion in the involutive of dynamics. The systems Equations (3) and (16) are involutive, and they have equal orders of gauge symmetries, gauge identities and equations of motion. Thus, they have to possess the same number of physical degrees of freedom. All the above implies that the non-Lagrangian interaction Equation (16) is consistent for the general values of the parameters β, α.

The theory Equation (16) admits a symmetric conserved tensor of second rank of the following form:

$$\begin{aligned}\mathbf{\Theta}^{\mu\nu}(\beta;\alpha) = \mathrm{tr}\Big\{&\beta_2\alpha_3(\mathcal{G}^\mu\mathcal{G}^\nu - \frac{1}{2}g^{\mu\nu}\mathcal{G}_\rho\mathcal{G}^\rho) + \beta_1\alpha_3(\mathcal{G}^\mu\mathcal{F}^\nu + \mathcal{G}^\nu\mathcal{F}^\mu - \\ &-g^{\mu\nu}\mathcal{G}_\rho\mathcal{F}^\rho) + (\beta_1\alpha_2 - \beta_2\alpha_1)(\mathcal{F}^\mu\mathcal{F}^\nu - \frac{1}{2}g^{\mu\nu}\mathcal{F}_\rho\mathcal{F}^\rho)\Big\}\,,\end{aligned} \tag{20}$$

where $\alpha_k, k = 1,2,3$ and $\beta_l, l = 1,2$ are the model parameters. The divergence of the quantity $\Theta^{\mu\nu}(\beta;\alpha)$ reads

$$\partial_\nu \Theta^{\mu\nu} = -\mathrm{tr}(\epsilon^{\mu\nu\rho}(\beta_1\mathcal{F}_\nu + \beta_2\mathcal{G}_\nu)\mathbb{T}_\rho)\,. \tag{21}$$

Expression Equation (20) is the covariantization of a selected representative in the conserved tensor series: Equation (6),

$$\beta^a{}_1 = \beta_1\,, \qquad \beta^a{}_2 = \beta_2\,, \qquad a = 1,\ldots,n\,. \tag{22}$$

As is seen, the model Equation (16) represents the class of deformations of free ECS Equation (3) that preserves a selected representative in the series Equation (6) of conserved quantities at the non-linear level. It is important to note that the other representatives in the series Equation (20) are no longer conserved in the non-linear theory Equation (16). This happens because the parameters $\beta_k, k = 1,2$ in the conserved tensor Equation (20) are unambiguously fixed by the interaction.

As far as stability is concerned, the 00-component of the tensor Equation (20) is relevant. The latter reads

$$\begin{aligned}\Theta^{00}(\beta;\alpha) = \mathrm{tr}\Big\{&\frac{1}{2}\beta_2\alpha_3(\mathcal{G}^0\mathcal{G}^0 + \mathcal{G}^i\mathcal{G}^i) + \beta_1\alpha_3(\mathcal{G}^0\mathcal{F}^0 + \mathcal{G}^i\mathcal{F}^i) + \\ &+\frac{1}{2}(\beta_1\alpha_2 - \beta_2\alpha_1)(\mathcal{F}^0\mathcal{F}^0 + \mathcal{F}^i\mathcal{F}^i)\Big\}\,.\end{aligned} \tag{23}$$

The conserved tensor is a bilinear form in (generalized) strengths $\mathcal{F}_\mu$, $\mathcal{G}_\mu$. The application of the quadratic form Equation (23) to the principal axes reveals that the model is stable if

$$\beta_2\alpha_3 > 0\,, \qquad C(\beta;\alpha) > 0\,. \tag{24}$$

These conditions can be consistent or inconsistent depending on the values of the model parameters α_k, $k = 1,2,3$ and β_l, $l = 1,2$. The Lagrangian interaction vertex Equation (17) does not meet stability requirements. This confirms the instability of the variational coupling proposed in [42]. The stable interactions in the class of models Equation (16) are inevitably non-Lagrangian. The similar form of stability conditions Equation (12), Equation (24) at the free and interacting cases implies that the linear and non-linear dynamics are stable or unstable simultaneously. In the class of theories of stable dynamics at the linear level, Equation (16) determines a class of non-linear models that preserve a selected bounded conserved quantity in the series Equation (6).

Now, we can return to the special case $C(\beta;\alpha) = 0$, which is excluded in Equation (16). The conserved quantity Equations (6) and (22) are a degenerate quadratic form of the initial data. The 00-component of the free conserved tensor can be bounded from below, but its degeneracy implies the existence of zero vectors. The motion of the system in the degenerate direction can be infinite, and the corresponding conserved quantity appears to be irrelevant to stability even at the free level. The formula Equation (16) prevents the construction of interacting theories that preserve the conserved tensors with a semi-definite 00-component.

4. Hamiltonian Formalism

In this section, we construct a constrained Hamiltonian formalism for the higher-derivative theory Equation (16).

Let us first explain what we understand by the constrained Hamiltonian formalism for the system of not necessarily Lagrangian field equations with higher derivatives. A general fact is that the higher-derivative system can be reduced to the first order by the introduction of extra fields that absorb the time derivatives of original dynamical variables. The set of original dynamical variables and extra fields is denoted by $\{\varphi^I(\boldsymbol{x},t), \lambda^A(\boldsymbol{x},t)\}$, where $t = x^0$

is the time, and $\boldsymbol{x} = (x^1, x^2)$ stands for space coordinates. The multi-indices A, I label the phase-space variables. The first-order equations are said to be Hamiltonian if there exists a Hamiltonian function $\mathcal{H}(\varphi^I, \boldsymbol{\nabla}\varphi^I, \boldsymbol{\nabla}^2\varphi^I, ...)$ ($\boldsymbol{\nabla}$ stands for derivatives by the space $\boldsymbol{x}$) and a Poisson bracket $\{\varphi^I(\boldsymbol{x}), \varphi^J(\boldsymbol{y})\}$ such that the equations constitute a constrained Hamiltonian system; i.e.,

$$\begin{gathered}\dot{\varphi}^I(\boldsymbol{x}) = \{\varphi^I(\boldsymbol{x}), \textstyle\int \mathcal{H}d\boldsymbol{y}\}, \quad \theta_A(\varphi^I(\boldsymbol{x}), \boldsymbol{\nabla}\varphi^I(\boldsymbol{x}), \dots) = 0\,; \\ \mathcal{H} = \mathcal{H}_0(\varphi^J(\boldsymbol{x}), \boldsymbol{\nabla}\varphi^J(\boldsymbol{x}), \dots) + \lambda^A(\boldsymbol{x})\theta_A(\varphi^I(\boldsymbol{x}), \boldsymbol{\nabla}\varphi^I(\boldsymbol{x}), \dots)\,.\end{gathered} \tag{25}$$

where the dot denotes the derivative by time

$$\dot{\varphi}^I = \frac{d\varphi^I}{dx^0}\,. \tag{26}$$

The system is multi-Hamiltonian if there exists a series of Hamiltonian $\mathcal{H}_\beta$ and Poisson brackets $\{\varphi^I(\boldsymbol{x}), \varphi^J(\boldsymbol{y})\}_\beta$ parameterized by constraints $\beta_1, \dots, \beta_k$ such that they determine the same equations of motion for the dynamical fields. The existence of the constrained Hamiltonian formulation is not guaranteed for a system of general non-Lagrangian equations of motion. In particular, the Hamiltonian formulation for the ECS theory may not exist, at least for certain combinations of model parameters.

We begin the construction of the Hamiltonian formalism with the reduction of order of Equation (16). The space components of the field strength $\mathcal{F}_i, i = 1,2$, and generalized field strength of second order $\mathcal{G}_i, i = 1,2$, are chosen as extra fields. By construction, they absorb the first and second time derivatives of space components of the vector field $\mathcal{A}_i, i = 1,2$,

$$\begin{gathered}\mathcal{F}_i = \epsilon_{ij}(\dot{\mathcal{A}}_j - \partial_i\mathcal{A}_0 + [\mathcal{A}_0, \mathcal{A}_j])\,, \\ \mathcal{G}_i = -\ddot{\mathcal{A}}_i + \partial_i\dot{\mathcal{A}}_0 - [\dot{\mathcal{A}}_0, \mathcal{A}_i] - [\mathcal{A}_0, \dot{\mathcal{A}}_i] + [\mathcal{A}_0, \mathcal{A}_i - \partial_i\mathcal{A}_0 + [\mathcal{A}_0, \mathcal{A}_i]] - \\ -\epsilon_{ij}\epsilon_{kl}\partial_j(\partial_k\mathcal{F}_l + 1/2[\mathcal{A}_k, \mathcal{A}_l]) + \epsilon_{ij}\epsilon_{kl}[\mathcal{A}_j, \partial_k\mathcal{F}_l + 1/2[\mathcal{A}_k, \mathcal{A}_l]]\,.\end{gathered} \tag{27}$$

where $\epsilon_{ij}, \epsilon_{12} = 1$ is the $2D$ Levi–Civita symbol. The Latin indices i, j run over the values $1, 2$. Summation over Latin indices repeated at the same level is implied. As can be seen from the equations, in Equation (27), the fields $\mathcal{F}_i$ absorb the first time derivatives of space components of original vector field $\mathcal{A}_i$. The vector $\mathcal{G}_i$ involves the second time derivatives of $\mathcal{A}_i$. The time components $\mathcal{F}_0, \mathcal{G}_0$ of vectors Equation (27) are functions of $\mathcal{A}_0$, $\mathcal{A}_i$, $\mathcal{F}_i$ and $\mathcal{G}_i$, with no new combinations of time derivatives being involved:

$$\mathcal{F}_0 \equiv \epsilon_{ij}(\partial_i\mathcal{A}_j + [\mathcal{A}_i, \mathcal{A}_j])\,, \qquad \mathcal{G}_0 \equiv \epsilon_{ij}(\partial_i\mathcal{F}_j + [\mathcal{A}_i, \mathcal{G}_j])\,. \tag{28}$$

In the remaining part of the article, we associate the quantities $\mathcal{F}_0$, $\mathcal{G}_0$ with their expressions in terms of the phase-space variables $\mathcal{A}_0$, $\mathcal{A}_i$ and $\mathcal{F}_i$.

Substituting the extra variables in Equation (27) into Equation (16), we obtain the first-order equations for the fields $\mathcal{A}_i$, $\mathcal{F}_i$, $\mathcal{G}_i$:

$$\dot{\mathcal{A}}_i = \partial_i\mathcal{A}_0 - [\mathcal{A}_0, \mathcal{A}_i] - \epsilon_{ij}\mathcal{F}_j\,; \tag{29}$$

$$\dot{\mathcal{F}}_i = \partial_i\mathcal{F}_0 + [\mathcal{A}_i, \mathcal{F}_0] - [\mathcal{A}_0, \mathcal{F}_i] - \epsilon_{ij}\mathcal{G}_j\,; \tag{30}$$

$$\begin{gathered}\dot{\mathcal{G}}_i = \partial_i\mathcal{G}_0 + \frac{\alpha_2}{\alpha_3}\epsilon_{ij}\mathcal{G}_j + \frac{\alpha_1}{\alpha_3}\epsilon_{ij}\mathcal{F}_j + [\mathcal{A}_i, \mathcal{G}_0] - [\mathcal{A}_0, \mathcal{G}_i] + \\ + \frac{{\alpha_3}^2}{C(\beta;\alpha)}[\beta_1\mathcal{F}_0 + \beta_2\mathcal{G}_0, \beta_1\mathcal{F}_i + \beta_2\mathcal{G}_i]\,.\end{gathered} \tag{31}$$

The evolutionary Equations (29)–(31) are supplemented by the constraint

$$\Theta \equiv \epsilon_{ij}(\alpha_1\partial_i\mathcal{A}_j + \alpha_2\partial_i\mathcal{F}_j + \alpha_3\partial_i\mathcal{G}_j + [\mathcal{A}_i, \frac{1}{2}\alpha_1\mathcal{A}_j + \alpha_2\mathcal{F}_j + \alpha_3\mathcal{G}_j] - \\ - \frac{\alpha_3}{2C(\beta;\alpha)}[\beta_1\mathcal{F}_i + \beta_2\mathcal{G}_i, \beta_1\mathcal{F}_j + \beta_2\mathcal{G}_j]) \approx 0\,. \tag{32}$$

The Equations (16), (29)–(32) are equivalent. Evolutionary Equations (29) and (30) express the auxiliary fields $\mathcal{F}_i$ and $\mathcal{G}_i$ in terms of time derivatives of the original vector potential $\mathcal{A}_\mu$. Solving them with respect to the unknown $\mathcal{F}_i$, $\mathcal{G}_i$, we obtain the relations in Equation (27). Equations (31) and (32) represent the space and time components of the original higher-derivative system in Equation (16), where all the higher derivatives of the vector potential are expressed in terms of extra variables $\mathcal{F}_i$, $\mathcal{G}_i$. The left-hand side of Equation (32) does not involve time derivatives, so we have the constraint $\Theta \approx 0$. The sign $\approx$ means the equality modulo constraint in Equation (32). Once the time evolution is preserved (see the gauge identity, Equation (19)), no secondary constraints are imposed on the fields.

We associate the on-shell Hamiltonian $\mathcal{H}_0$ with the 00-component of the conserved tensor in Equation (20). In terms of the phase-space variables $\mathcal{A}_i$, $\mathcal{F}_i$ and $\mathcal{G}_i$, it reads

$$\mathcal{H}_0 = \frac{1}{2}\mathrm{tr}\Big\{\beta_2\alpha_3(\mathcal{G}^0\mathcal{G}^0 + \mathcal{G}^i\mathcal{G}^i) + 2\beta_1\alpha_3(\mathcal{G}^0\mathcal{F}^0 + \mathcal{G}^i\mathcal{F}^i) + \\ + (\beta_1\alpha_2 - \beta_2\alpha_1)(\mathcal{F}^0\mathcal{F}^0 + \mathcal{F}^i\mathcal{F}^i)\Big\}\,. \tag{33}$$

where the functions $\mathcal{F}^0$ and $\mathcal{G}^0$ are defined in Equation (28). The on-shell Hamiltonian $\mathcal{H}_0$ depends on the free model parameters $\alpha_k, k = 1,2,3$ and coupling constants $\beta_l, l = 1,2$. Off-shell, the Hamiltonian is a sum of the on-shell part of Equation (33) and a linear combination of constraints. We chose the following ansatz for the total Hamiltonian:

$$\mathcal{H} = \mathcal{H}_0 + \mathrm{tr}\Big[\frac{C(\beta;\alpha)}{\beta_2\alpha_2 - \beta_1\alpha_3}\mathcal{A}_0 - \frac{\beta_2\beta_1\alpha_3}{\beta_2\alpha_2 - \beta_1\alpha_3}\mathcal{F}_0 - \frac{{\beta_2}^2\alpha_3}{\beta_2\alpha_2 - \beta_1\alpha_3}\mathcal{G}_0\Big]\Theta\,, \tag{34}$$

where β, α are constant parameters. The on-shell vanishing terms included in the Hamiltonian do not contribute to the equations of motion of gauge-invariant quantities. The equations of motion do alter for non-gauge-invariant variables. As we are attempting to determine the Hamiltonian and Poisson bracket by literally reproducing the first-order Equation (29)–(32) of Equation (16), the on-shell vanishing contributions are kept under control in Equation (34).

The Hamiltonian is well-defined if the parameters β and α are subject to the following conditions:

$$C(\beta;\alpha) \neq 0\,, \qquad \beta_1\alpha_3 - \beta_2\alpha_2 \neq 0. \tag{35}$$

These relations have a clear origin. The first condition in this set ensures that the on-shell Hamiltonian is a non-degenerate quadratic form of the phase-space variables $\mathcal{A}_i$, $\mathcal{F}_i$, $\mathcal{G}_i$. This requirement is reasonable because the degenerate Hamiltonian cannot generate the evolution of all physical degrees of freedom. The second relation of Equation (35) ensures that the numerical factor at the Lagrange multiplier $\mathcal{A}_0$ is non-singular. This is necessary to reproduce the correct gauge transformations for all the dynamical variables. We also note that the obstructions to the existence of the Hamiltonian remain valid in the free limit. This means that the inclusion of an interaction by the scheme presented in Section 3 does not restrict the class models that admit the Hamiltonian formulation. In other words, every theory in the class of Equation (16) admitting the Hamiltonian formulation with the Hamiltonian Equation (34) in the free limit is Hamiltonian at the interacting level. Hereafter, we assume that the relations in Equation (35) are satisfied. The Hamiltonian is on-shell bounded from below if the conditions in Equation (24) are satisfied. In this case, we expect to construct the constrained Hamiltonian formulation with a bounded Hamiltonian. The corresponding quantum theory has a good chance to be stable.

Now, let us find the Poisson bracket between the fields $\mathcal{A}_i$, $\mathcal{F}_i$ and $\mathcal{G}_i$ that leads Equations (29)–(32) to adhere to the Hamiltonian form Equation (25) with the Hamiltonian Equation (34). Comparing the right-hand sides of Equations (25), (29)–(31), we obtain the following system of algebraic equations:

$$\Big\{A_i, \int \mathcal{H}(\boldsymbol{y})d\boldsymbol{y}\Big\} \approx \partial_i\mathcal{A}_0 - [\mathcal{A}_0,\mathcal{A}_i] - \epsilon_{ij}\mathcal{F}_j\,; \tag{36}$$

$$\Big\{F_i, \int \mathcal{H}(\boldsymbol{y})d\boldsymbol{y}\Big\} \approx \partial_i\mathcal{F}_0 + [\mathcal{A}_i,\mathcal{F}_0] - [\mathcal{A}_0,\mathcal{F}_i] - \epsilon_{ij}\mathcal{G}_j\,; \tag{37}$$

$$\begin{aligned}\Big\{G_i, \int \mathcal{H}(\boldsymbol{y})d\boldsymbol{y}\Big\} \approx{}& \partial_i\mathcal{G}_0 + \frac{\alpha_2}{\alpha_3}\epsilon_{ij}\mathcal{G}_j + \frac{\alpha_1}{\alpha_3}\epsilon_{ij}\mathcal{F}_j + [\mathcal{A}_i,\mathcal{G}_0] - [\mathcal{A}_0,\mathcal{G}_i] + \\ &+\frac{{\alpha_3}^2}{C(\beta;\alpha)}[\beta_1\mathcal{F}_0+\beta_2\mathcal{G}_0,\beta_1\mathcal{F}_i+\beta_2\mathcal{G}_i]\,.\end{aligned} \tag{38}$$

In Equations (36)–(38), the sign $\approx$ means the equality modulo constraint seen in Equation (32). The Poisson bracket defined by these equations depends on five independent arguments: the free model parameters α_3, α_2 and α_1 and coupling constants β_2 and β_1. The bracket has the following form:

$$\{\mathcal{G}^a{}_i(\boldsymbol{x}),\mathcal{G}^b{}_j(\boldsymbol{y})\} = \frac{\beta_1{\alpha_2}^2 - \beta_1\alpha_1\alpha_3 - \beta_2\alpha_2\alpha_1}{{\alpha_3}^2C(\beta;\alpha)}\epsilon_{ij}\delta^{ab}\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y})\,; \tag{39}$$

$$\{\mathcal{F}^a{}_i(\boldsymbol{x}),\mathcal{G}^b{}_j(\boldsymbol{y})\} = \frac{\beta_2\alpha_1 - \beta_1\alpha_2}{\alpha_3C(\beta;\alpha)}\epsilon_{ij}\delta^{ab}\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y})\,; \tag{40}$$

$$\{\mathcal{F}^a{}_i(\boldsymbol{x}),\mathcal{F}^b{}_j(\boldsymbol{y})\} = \{\mathcal{A}^a{}_i(\boldsymbol{x}),\mathcal{G}^b{}_j(\boldsymbol{y})\} = \frac{\beta_1}{C(\beta;\alpha)}\epsilon_{ij}\delta^{ab}\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y})\,; \tag{41}$$

$$\{\mathcal{A}^a{}_i(\boldsymbol{x}),\mathcal{F}^b{}_j(\boldsymbol{y})\} = \frac{-\beta_2}{C(\beta;\alpha)}\epsilon_{ij}\delta^{ab}\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y})\,; \tag{42}$$

$$\{\mathcal{A}^a{}_i(\boldsymbol{x}),\mathcal{A}^b{}_j(\boldsymbol{y})\} = 0\,. \tag{43}$$

where $\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y}) = \delta(x^1-y^1)\delta(x^2-y^2)$ is the $2D$ δ-function in the space coordinates. The Poisson bracket in Equations (39)–(43) is a covariant generalization of its free analog, which is derived in [43]. This result is not surprising. The free limit of expressions in Equations (39)–(43) is determined by the linear model, while the Poisson brackets with a polynomial dependence on fields contradict the structure of Equation (25). In the last case, the Poisson bracket with the Hamiltonian involves higher powers of fields than the right-hand side of first-order Equations (29)–(31).

The Poisson bracket in Equations (39)–(43) is a non-degenerate tensor, so it has an inverse, being a symplectic two-form. The latter defines a Hamiltonian action functional

$$\begin{aligned}S_h = \int \Big\{\mathrm{tr}\Big(&\frac{C(\beta;\alpha)}{\beta_2\alpha_2-\beta_1\alpha_3}\varepsilon_{ij}(\alpha_1\mathcal{A}_i+2\alpha_2\mathcal{F}_i+2\alpha_3\mathcal{G}_i)\dot{\mathcal{A}}_j + \frac{{\beta_1}^2\alpha_3}{\beta_2\alpha_2-\beta_1\alpha_3}\times \\ &\times\varepsilon_{ij}(\beta_1\mathcal{F}_i+\beta_2\mathcal{G}_i)(\beta_1\mathcal{F}_j+\beta_2\mathcal{G}_j)\Big) - \mathcal{H}\Big\}dxdt\,,\end{aligned} \tag{44}$$

where $\mathcal{H}$ denotes the Hamiltonian Equation (34). In the case $\beta_2 = 0\,, \beta_1 = 1$, Equation (44) reproduces the Ostrogradski action for the variational model Equation (17),

$$S_c = \int \Big\{ \mathrm{tr}(\epsilon_{ij}(\alpha_1 \mathcal{A}_i + 2\alpha_2 \mathcal{F}_i + 2\alpha_3 \mathcal{G}_i)\dot{\mathcal{A}}_j - \epsilon_{ij}\mathcal{F}_i\dot{\mathcal{F}}_j) - \mathcal{H}_c \Big\} dxdt\,, \tag{45}$$

where $\mathcal{H}_c$ is the canonical Hamiltonian,

$$\mathcal{H}_c = \frac{1}{2}\mathrm{tr}\Big(\alpha_3(\mathcal{G}^0\mathcal{F}^0 + \mathcal{G}^i\mathcal{F}^i) + \alpha_2(\mathcal{F}^0\mathcal{F}^0 + \mathcal{F}^i\mathcal{F}^i) + \mathcal{A}_0\Theta\Big)\,. \tag{46}$$

In the free case, the Hamiltonian formulation Equation (44) was first proposed in [43]. Later, it was re-derived in [42] by the direct application of the Ostrogradski procedure to the action functional Equation (1). The comparison of Equations (12) and (35) implies that the Hamiltonian Equation (34) is on-shell unbounded for all the Lagrangian interactions. For non-Lagrangian interactions, the Hamiltonian Equation (34) can be on-shell bounded or unbounded depending on the value of model parameters. If the Hamiltonian is on-shell bounded, the interacting theory is stable at the quantum level. To our knowledge, the ECS theory Equation (16) is the first higher-derivative model with a non-abelian gauge symmetry admitting an alternative Hamiltonian formulation with a bounded Hamiltonian. This means that the concept of the stabilization of dynamics by means of an alternative Hamiltonian formalism applies beyond the linear level.

In the free limit, the Hamiltonian Equation (34) and Poisson bracket in Equations (39)–(43) depend on the parameters β_1 and β_2, while the equations of motion in Equations (29)–(32) do not. This means that the free ECS theory Equation (1) is a multi-Hamiltonian theory. The general representative of the series of Hamiltonian formulations does not follow from the Ostrogradski procedure. The reason is that the bounded and unbounded Hamiltonian cannot be connected by the change of coordinates in the phase space. The number of entries in the free Hamiltonian series equals $2n$, where n is the number of color indices. The on-shell Hamiltonian is given by the 00-component Equation (9) of the conserved tensor Equation (6). The Poisson bracket between the phase-space variables $A^a{}_i$, $F^a{}_i$,$G^a{}_i$ is determined by the formulas Equations (39)–(43), but the parameters β_1, β_2 are replaced by $\beta^a{}_1$, $\beta^a{}_2$ for each field in the multiplet. At the non-linear level, a selected Poisson bracket, as shown in Equations (39)–(43), is preserved. If the respective Hamiltonian is bounded, the Poisson bracket cannot follow from the Ostrogradski construction; thus, the Hamiltonian formalism for the stable non-linear theories does not follow from the canonical formulation.

In the conclusion of this section, we present the Poisson brackets between the constraints Equations (39)–(43):

$$\{\Theta^a(\boldsymbol{x}), \Theta^b(\boldsymbol{y})\} = \frac{\beta_2\alpha_2 - \beta_1\alpha_3}{C(\beta;\alpha)} f^{abc}\Theta^c(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{47}$$

The Hamiltonian is gauge-invariant,

$$\{\Theta^a(\boldsymbol{x}), \mathcal{H}_0(\boldsymbol{y})\} = 0\,, \qquad a = 1, \dots, n\,. \tag{48}$$

As can be seen, the first-order model, Equation (44), is a gauge theory of a special form in the whole range, Equation (35), of admissible values of the parameters α, β. Equation (44) determines the least-action principle for the model. The quantization of this theory can be performed by means of the well-known procedures [56]. In all instances with a Hamiltonian that is bounded from below, we expect the stability of the model. This means that Equation (16) determines a higher-derivative theory, which can be stable at the classical and quantum level. Because of the existence of first-order formalism, the model is as good as the Lagrangian theories. In particular, this admits consequent quantization and correspondence between symmetries and conserved quantities. The Hamiltonian serves as a quantity that is associated to the invariance of the model in Equation (44) with respect to the time translations. The Hamiltonian Equation (34) is on-shell bounded if the conditions in Equation (24) are satisfied.

5. Resonance Case

In case of resonance (positions (2b), (3) of classification on pages 4–5), the non-linear theory Equation (3) admits the Hamiltonian form of dynamics, but the Hamiltonian is unbounded in all instances (conditions Equation (24) are inconsistent). Thus, the model has to be considered to be unstable. In this section, we demonstrate that the dynamics of the theory with resonance can be stabilized by means of the inclusion of an interaction with an extra dynamical field. We apply the "Higgs-like" mechanism, which was first proposed in the context of study of the "doubly massless" generalized Podolsky electrodynamics in the paper [48]. Here, we use it in the theory with non-abelian gauge symmetry for the first time. We mostly consider the model (Equation (16)) with the third-order resonance. We chose the following values for the parameters of free theory, Equation (1): $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = -1$. This choice does not restrict the generality of our work, because the constant α_3 accounts for the possibility of multiplication of equations of motion by an overall factor. The wave operator of the free model (Equation (3)) appears to be cube of the CS operator, so we have a sort of "triply massless" extended theory.

We begin the construction of the interaction by extending the set of dynamical variables by a real scalar field $\phi(x)$. The non-linear theory of the vector multiplet $\mathcal{A}_\mu(x)$ and scalar field $\phi(x)$ is determined by the equations of motion, which have the following form:

$$\begin{aligned} \mathbb{T}_\mu = \epsilon_{\mu\nu\rho}\Big\{D^\nu[(\widetilde{\gamma}^2\phi^2-1)\mathcal{G}+\gamma^2\phi^2\mathcal{F}]^\rho + \frac{(\widetilde{\gamma}^2\phi^2-1)^2}{2\beta_1{}^2}[\beta_1\mathcal{F}^\nu+\beta_2\mathcal{G}^\nu,\beta_1\mathcal{F}^\rho+\beta_2\mathcal{G}^\rho]\Big\} = 0\,, \\ \mathbb{T} = \Big\{\partial_\mu\partial^\mu - (m^2 - \widetilde{\gamma}^2\frac{(\beta_1\mathcal{F}_\rho+\beta_2\mathcal{G}_\rho)(\beta_1\mathcal{F}^\rho+\beta_2\mathcal{G}^\rho)}{\beta_1{}^2} + \phi^2\Big\}\phi = 0\,. \end{aligned} \tag{49}$$

where the vectors $\mathcal{F}_\mu$ and $\mathcal{G}_\mu$ are defined in Equation (2), and the abbreviation $\widetilde{\gamma}^2 = \gamma^2\beta_2/\beta_1$ is used. The constants β_2, β_1 and γ, m are model parameters and are real numbers. Throughout this section, we assume that $m, \gamma > 0$ and $\beta_1 \neq 0$. The option $\beta_1 = 0$ is not admissible because the self-coupling Equation (16) between the vector fields becomes inconsistent in this case. The value β_2 can be an arbitrary real number (positive, negative or zero). Only positive values of β_2, β_1 lead to stable couplings. This justifies the notation $\widetilde{\gamma}^2$ for the quantity $\gamma^2\beta_2/\beta_1$. Without loss of generality, we put a unit coefficient at ϕ^3-term. An overall factor at the ϕ^3-vertex can be absorbed by the scaling of the scalar field, $\phi \mapsto \lambda\phi$, with the appropriate $\lambda \neq 0$.

Equation (49) has a clear meaning. The first line of the system in Equation (49) describes the motion of the vector multiplet. The linear term in the fields in the equations corresponds to the "triply massless" ECS theory. The coupling includes the self-interaction term of Equation (16) and an extra contribution involving the scalar field. It is convenient to think that Equation (49) follows from Equation (16) after the formal redefinition of the model parameters:

$$\alpha_1 = 0\,, \qquad \alpha_2 = \gamma^2\phi^2\,, \qquad \alpha_3 = \widetilde{\gamma}^2\phi^2 - 1\,. \tag{50}$$

The characteristic polynomial in (Equation (4)) for the model reads

$$M(\alpha; z) = \gamma^2\phi^2 z^2 + (\widetilde{\gamma}^2\phi^2 - 1)z^3\,. \tag{51}$$

If the scalar field is set to a nonzero constant from the outset, we obtain Equation (16) with the second-order resonance for the zero root (case (2a) of classification on pages 4–5). As explained above, the corresponding model describes a massless spin-1 field and a massive spin-1 vector subjected to the self-duality condition. It is stable at the classical and quantum levels. The second equation in the system of Equation (49) describes the motion of the scalar field. This equation includes ϕ^3-coupling, which ensures the existence of a nonzero stationary solution for $\mathcal{A}_\mu = 0$. This means that ϕ serves as the Higgs field in the model of Equation (49).

The interaction defined by Equation (49) is consistent. The equations are preserved by the standard Yang–Mills gauge symmetry in Equation (18). The scalar field is preserved by the gauge transformation. The gauge identity has a slightly different form:

$$\mathbb{D}_\mu \mathbb{T}^\mu = 0\,, \qquad \mathbb{D}_\mu = D_\mu + \frac{\widetilde{\gamma}^2\phi^2 - 1}{{\beta_1}^2}[\beta_1\mathcal{F}_\mu + \beta_2\mathcal{G}_\mu, \cdot\,]\,. \tag{52}$$

We note that the gauge generator involves the scalar field explicitly. The model has the same number of physical degrees of freedom as the free theory because the orders of equations of motion in Equation (49), gauge identities in Equation (52) and gauge symmetries in Equation (18) are preserved by coupling. Equations (49) do not follow from the least-action principle for a function with higher derivatives unless $\beta_2 = 0$. In the case of Lagrangian coupling, the action principle reads

$$S[\phi(x), \mathcal{A}(x)] = \frac{1}{2}\int \Big\{\mathcal{F}_\mu(\mathcal{G}^\mu + \gamma^2\phi^2\mathcal{F}^\mu) + \partial_\mu\phi\partial^\mu\phi + m^2\phi^2 - \frac{1}{2}\phi^4\Big\}d^3x\,. \tag{53}$$

The dynamics of the Lagrangian theory are unstable. In the case of non-Lagragian couplings, the dynamics of the model can be stable. In the remaining part of this section, we address an issue of the construction of the constrained Hamiltonian formalism with a bounded Hamiltonian for the higher-derivative Equation (49). The existence of such a formalism implies the classical and quantum stability of the model. To avoid repetition, we outline the most crucial steps of the construction, while the details are provided above.

We introduce a special notation for the linear combination of generalized strengths $\mathcal{F}_\mu, \mathcal{G}_\mu$ entering the free part of Equation (49),

$$\mathcal{W}_\mu = (\widetilde{\gamma}^2\phi^2 - 1)\mathcal{G}_\mu + \gamma^2\phi^2\mathcal{F}_\mu\,. \tag{54}$$

We choose the space components of the vectors $\mathcal{F}_i$ and $\mathcal{W}_i$, $i = 1,2$ Equation (27) as variables absorbing the first and second derivatives of the original dynamical field $\mathcal{A}_\mu$. In the sector of the scalar field, we introduce the canonical momentum $\pi = \dot{\phi}$. We consider $\mathcal{W}_0$ as a special notation for the combination Equation (54) of the derivatives of phase-space variables $\mathcal{A}_\mu, \mathcal{F}_i$. In terms of the variables ϕ, π $\mathcal{A}_\mu$, $\mathcal{F}_i$, $\mathcal{G}_i$, the first-order equations eventually read

$$\dot{\mathcal{A}}_i = \partial_i\mathcal{A}_0 - [\mathcal{A}_0, \mathcal{A}_i] - \epsilon_{ij}\mathcal{F}_j\,; \tag{55}$$

$$\dot{\mathcal{F}}_i = \partial_i\mathcal{F}_0 + [\mathcal{A}_i, \mathcal{F}_0] - [\mathcal{A}_0, \mathcal{F}_i] - \epsilon_{ij}\frac{\mathcal{W}_j - \gamma^2\phi^2\mathcal{F}_j}{\widetilde{\gamma}^2\phi^2 - 1}\,; \tag{56}$$

$$\dot{\mathcal{W}}_i = \partial_i\mathcal{W}_0 + [\mathcal{A}_i, \mathcal{W}_0] - [\mathcal{A}_0, \mathcal{W}_i] + \epsilon_{ij}[\beta_2\mathcal{W}_0 - \beta_1\mathcal{F}_0, \beta_2\mathcal{W}_j - \beta_1\mathcal{F}_j]\,; \tag{57}$$

$$\dot{\pi} = \Big\{\partial_i\partial_i - m^2 + \widetilde{\gamma}^2\frac{(\beta_2\mathcal{W}_\mu - \beta_1\mathcal{F}_\mu)(\beta_2\mathcal{W}^\mu - \beta_1\mathcal{F}^\mu)}{(\widetilde{\gamma}^2\phi^2 - 1)^2} + \phi^2\Big\}\phi\,; \tag{58}$$

$$\dot{\phi} = \pi\,. \tag{59}$$

The evolutionary Equations (55)–(59) are supplemented by the constraint

$$\Theta \equiv \epsilon_{ij}\Big(\partial_i\mathcal{W}_j + [\mathcal{A}_i, \mathcal{W}_j] + \frac{1}{2{\beta_2}^2}[\beta_2\mathcal{W}_i - \beta_1\mathcal{F}_i, \beta_2\mathcal{W}_j - \beta_1\mathcal{F}_j]\Big) = 0\,. \tag{60}$$

The constraint conserves the on-shell property if the identity Equation (52) is taken into account. The higher-derivative system Equation (49) follows from

Equations (55)–(57) and (60) if all the extra variables are expressed in terms of the derivatives of original dynamical fields $\phi, \mathcal{A}$.

The model in Equation (49) admits a second-rank symmetric conserved tensor:

$$\begin{aligned}\mathbf{\Theta}^{\mu\nu}(\beta;\alpha) = \mathrm{tr}\Big\{ & \frac{\widetilde{\gamma}^2\phi^2 - 1}{\beta_2}\Big((\beta_1\mathcal{F}^\mu + \beta_2\mathcal{G}^\mu)(\beta_1\mathcal{F}^\nu + \beta_2\mathcal{G}^\nu) - \frac{1}{2}g^{\mu\nu}(\mathcal{F}_\rho + \mathcal{G}_\rho)(\mathcal{F}^\rho + \mathcal{G}^\rho)\Big) + \\ & + \frac{\beta_2{}^2}{\beta_1}\Big(\mathcal{F}^\mu\mathcal{F}^\nu - \frac{1}{2}g^{\mu\nu}\mathcal{F}_\rho\mathcal{F}^\rho\Big)\Big\} + \partial^\mu\phi\partial^\nu\phi + g^{\mu\nu}\big(-\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{4}\phi^4\big);\end{aligned} \tag{61}$$

$$\partial_\nu\mathbf{\Theta}^{\mu\nu}(\beta;\alpha) = -\mathrm{tr}(\epsilon^{\mu\nu\rho}(\beta_1\mathcal{F}_\nu + \beta_2\mathcal{G}_\nu)\mathbb{T}_\rho) + \partial^\mu\phi\cdot\mathbb{T}\,. \tag{62}$$

The Hamiltonian is given by a sum of the 00-component of the conserved tensor and is expressed in terms of variables ϕ, π, $\mathcal{A}_\mu$, $\mathcal{F}_i$, $\mathcal{W}_i$ Equations (27) and (54) and a constraint term:

$$\begin{aligned}\mathcal{H} = \mathrm{tr}\Big\{ & \frac{(\beta_2\mathcal{W}^0 - \beta_1\mathcal{F}^0)(\beta_2\mathcal{W}^0 - \beta_1\mathcal{F}^0) + (\beta_2\mathcal{W}^i - \beta_1\mathcal{F}^i)(\beta_2\mathcal{W}^i - \beta_1\mathcal{F}^i)}{2(\widetilde{\gamma}^2\phi^2 - 1)\beta_2} + \\ & + \frac{\beta_1{}^2}{2\beta_2}(\mathcal{F}^0\mathcal{F}^0 + \mathcal{F}^i\mathcal{F}^i) - (\mathcal{A}_0 - \beta_1\mathcal{F}_0 + \beta_2\mathcal{W}_0)\Theta\Big\} + \\ & + \frac{1}{2}(\pi\pi + \partial^i\phi\partial^i\phi) - \frac{1}{2}m^2\phi^2 + \frac{1}{4}\phi^4\,.\end{aligned} \tag{63}$$

where the quantities $\mathcal{F}_0, \mathcal{W}_0$ denote abbreviations of Equations (28) and (54), and the numbers m, γ are model parameters. The Hamiltonian is on-shell bounded if

$$\widetilde{\gamma}^2\phi^2 - 1 > 0\,, \qquad \beta_2 > 0\,. \tag{64}$$

These conditions involve the scalar field ϕ. Once the initial value of ϕ is a Cauchy datum for Equation (49), the Hamiltonian cannot be globally bounded. However, the Hamiltonian is given by a positive definite quadratic form in the variables $\mathcal{F}$, $\mathcal{G}$ in the range $|\widetilde{\gamma}\phi| > 1$ of values of the scalar field. This corresponds to the case of a stability island.

The Poisson bracket between the fields ϕ, π, $\mathcal{A}_\mu$, $\mathcal{F}_i$ and $\mathcal{G}_i$ is determined by the following system of equations:

$$\Big\{\mathcal{A}_i, \int \mathcal{H}(\boldsymbol{y})d\boldsymbol{y}\Big\} \approx \partial_i\mathcal{A}_0 - [\mathcal{A}_0, \mathcal{A}_i] - \epsilon_{ij}\mathcal{F}_j\,; \tag{65}$$

$$\Big\{\mathcal{F}_i, \int \mathcal{H}(\boldsymbol{y})d\boldsymbol{y}\Big\} \approx \partial_i\mathcal{F}_0 + [\mathcal{A}_i, \mathcal{F}_0] - [\mathcal{A}_0, \mathcal{F}_i] - \epsilon_{ij}\frac{\mathcal{W}_j - \gamma^2\phi^2\mathcal{F}_j}{\widetilde{\gamma}^2\phi^2 - 1}\,; \tag{66}$$

$$\Big\{G_i, \int \mathcal{H}(\boldsymbol{y})d\boldsymbol{y}\Big\} \approx \partial_i\mathcal{W}_0 + [\mathcal{A}_i, \mathcal{W}_0] - [\mathcal{A}_0, \mathcal{G}_i] + \frac{\epsilon_{ij}}{\beta_1{}^2}[\beta_2\mathcal{W}_0 - \beta_1\mathcal{F}_0, \beta_2\mathcal{W}_j - \beta_1\mathcal{F}_j]\,. \tag{67}$$

$$\Big\{\pi, \int \mathcal{H}(\boldsymbol{y})d\boldsymbol{y}\Big\} \approx \partial_i\partial_i\phi + \Big(m^2 - \widetilde{\gamma}^2\frac{(\beta_2\mathcal{W}_\mu - \beta_1\mathcal{F}_\mu)(\beta_2\mathcal{W}^\mu - \beta_1\mathcal{F}^\mu)}{(\widetilde{\gamma}^2\phi^2 - 1)^2} - \phi^2\Big)\phi\,; \tag{68}$$

$$\Big\{\phi, \int \mathcal{H}(\boldsymbol{y})d\boldsymbol{y}\Big\} \approx \pi. \tag{69}$$

All the equalities are considered modulo terms that remove the modulo constraint of Equation (60). The solution for Equations (65)–(69) reads

$$\{\mathcal{W}^a{}_i(\boldsymbol{x}), \mathcal{W}^b{}_j(\boldsymbol{y})\} = \{\mathcal{F}^a{}_i(\boldsymbol{x}), \mathcal{W}^b{}_j(\boldsymbol{y})\} = \{\mathcal{A}^a{}_i(\boldsymbol{x}), \mathcal{A}^b{}_j(\boldsymbol{y})\} = 0\,; \tag{70}$$

$$\{\mathcal{A}^a{}_i(x), \mathcal{W}^b{}_j(y)\} = -\{\mathcal{F}^a{}_i(x), \mathcal{F}^b{}_j(y)\} = \frac{1}{\beta_1}\epsilon_{ij}\delta^{ab}\delta^{(2)}(x-y)\,; \tag{71}$$

$$\{\mathcal{A}^a{}_i(x), \mathcal{F}^b{}_j(y)\} = \frac{\beta_2}{{\beta_1}^2}\epsilon_{ij}\delta^{ab}\delta^{(2)}(x-y)\,; \tag{72}$$

$$\{\phi(x), \pi(y)\} = \delta^{(2)}(x-y)\,. \tag{73}$$

All the Poisson brackets between ϕ, π and $\mathcal{A}_i$, $\mathcal{F}_i$ and $\mathcal{G}_i$ vanish. The relations of Equations (70)–(72) can be obtained from Equations (39)–(42) after the substitution of Equation (50). We note that the Poisson bracket between the fields $\mathcal{A}_i$, $\mathcal{F}_i$, $\mathcal{W}_i$ is constant, even though the theory involves extra fields π, ϕ.

Now, we can demonstrate the phenomenon of dynamic stabilization by means of the "Higgs-like" mechanism proposed in [48]. Equation (49) admits a nonzero stationary solution:

$$\mathcal{A}(x) = \mathcal{F}(x) = \mathcal{G}(x) = 0\,, \qquad \pi(x) = 0\,, \qquad \phi(x) = \pm m\,. \tag{74}$$

Introducing the notation $\phi^*(x) = \phi(x) \mp m$ and expanding Equation (49) in the vicinity of this vacua, we obtain the linearized equations for the fields

$$\begin{gathered}\mathbb{T}_\mu = (m^2\widetilde{\gamma}^2 - 1)\mathcal{K}_\mu + m^2\gamma^2\mathcal{G}_\mu + \ldots = 0\,,\\ \mathbb{T} = (\partial_\mu\partial^\mu + 5m^2)\phi^* + \ldots = 0\,.\end{gathered} \tag{75}$$

The dots denote the terms that are at least quadratic in the fields. As we see, the dynamics of the field $\mathcal{A}$ are described by the higher-derivative of Equation (3) with $\alpha_3 = m^2\widetilde{\gamma}^2 - 1$, $\alpha_2 = m^2\gamma^2$, $\alpha_1 = 0$. In this case, the system has a second-order resonance for the zero root, which does not affect the stability of the dynamics, at least at the free level. The on-shell Hamiltonian reads

$$\begin{aligned}\mathcal{H} = \operatorname{tr}\Big\{&\frac{1}{2(m^2\widetilde{\gamma}^2-1)\beta_2}((\beta_1\mathcal{W}^0 - \beta_1\mathcal{F}^0)(\beta_1\mathcal{W}^0 - \beta_1\mathcal{F}^0) + (\beta_2\mathcal{W}^i - \beta_1\mathcal{F}^i)(\beta_2\mathcal{W}^i - \beta_1\mathcal{F}^i)) +\\ &+\frac{{\beta_1}^2}{2\beta_2}(\mathcal{F}^0\mathcal{F}^0 + \mathcal{F}^i\mathcal{F}^i) - (\mathcal{A}_0 - \beta_1\mathcal{F}_0 + \beta_2\mathcal{W}_0)\Theta\Big\} +\\ &+\frac{1}{2}(\pi\pi + \partial^i\phi^*\partial^i\phi^* + 5m^2(\phi^*)^2) + \ldots\end{aligned} \tag{76}$$

The dots denote the terms that are at least cubic in the fields. The quadratic part of the Hamiltonian is on-shell bounded if $m\widetilde{\gamma} > 1$. This means that the dynamics of small fluctuations in the vicinity of vacua in Equation (74) are stable. Once local stability is sufficient for the construction of stable quantum theory, the "Higgs-like" mechanism can stabilize the dynamics of the non-linear theory of Equation (16) with the third-order resonance.

6. Conclusions

The results of the article demonstrate that the inclusion of non-Lagrangian couplings can solve the problem of the inclusion of stable interactions in higher-derivative models. Equation (16) introduces a two-parameter series of couplings in the theory of vector fields with Yang–Mills gauge symmetry. The dynamics of the model are stable if the coupling parameters meet the conditions in Equation (24). To our knowledge, Equation (16) represents a first example of a stable non-linear higher derivative model with non-abelian gauge symmetry whose canonical energy is unbounded from below in the free limit (Here, we mean the class of models that demonstrate the Ostrogradski instability at the free level. In the $f(R)$-theories of gravity [4,5], the canonical energy is bounded from below in the free approximation). The stable couplings do not follow from the least-action principle, but they admit the Hamiltonian form of dynamics with a Hamiltonian that is bounded from below. The presence of the Hamiltonian action principle allows the consecutive construction of quantum theory, which is expected to be free of ghosts. The model Equation (16) can be considered as the stable $3D$ generalization of the conventional higher derivative Yang–Mills symmetry [55]. The theory admits the inclusion of stable interactions with the scalar, spinor and gravitational fields in a similar manner to [38,43,44]. The constructed models can serve as the $3D$ higher-derivative generalizations of scalar electrodynamics, quantum chromodynamics and gravity with higher-derivative matter. These theories can mimic the actual $4D$ models that describe the real world. Finally, the procedure of the inclusion of non-Lagrangian couplings can be generalized to the case of arbitrary space–time dimensions. This will lead to the construction of new models of fundamental interactions with the non-Lagrangian equations of motion for the fields. In future studies, these models can be applied to unsolved problems such as the physics beyond the standard model [57] or dark matter [58].

The study of the "triply massless" ECS provides a new insight into the well-known problem of the stability of Pais–Uhlenbeck-type theories with equal frequencies [9,28]. Even though the theories with resonance are usually considered as unstable even at the free level, our results propose a constructive procedure of the stabilization of dynamics by the inclusion of coupling with an extra field. Equations of motion of the model are given in Equation (49). The dynamics are metastable in the vicinity of the energy minimum, while global stability in the model with resonance is impossible. A similar phenomenon is observed in higher-derivative gravities [31]. The ECS model provides a simple setting for the study of the metastability phenomenon of dynamics in the class of gauge theories. The conclusions of the study can be applied to gravity models of interest. We also mention that the inclusion of an interaction in the model with resonance does not change the gauge symmetry of vector fields, which remains of the Yang–Mills form. The gauge symmetry-preserving deformations of equations of motion are known as "gaugings" [59,60]. The results of this article tell us that the "gaugings" are essential in the context of the stabilization of higher-derivative dynamics with resonance. This application of this special type of interactions has never been considered before. Further studies will allow the construction of a new class of stable vector-scalar models with higher derivatives.

Author Contributions: Conceptualization, D.S.K. and S.L.L.; methodology, D.S.K. and S.L.L.; validation, D.S.K., S.L.L. and O.D.N.; investigation, D.S.K. and O.D.N.; writing—original draft preparation, D.S.K. and O.D.N.; writing—review and editing, D.S.K.; supervision, S.L.L.; All authors have read and agreed to the published version of the manuscript.

Funding: The part of the work concerning the construction of stable interactions in the ECS model (Sections 2 and 3) is supported by the Ministry of Science and Higher Education of the Russian Federation, Project No. 0721- 2020-0033. The construction of the Hamiltonian formalism (Sections 4 and 5) was conducted with the support of the Russian Science Foundation grant 18-72-10123 in association with the Lebedev Physical Institute of RAS. The work of SLL was partially supported by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS".

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank A.A. Sharapov for valuable discussions regarding this work. We also benefited from the valuable comments of three anonymous referees that helped us to improve the initial version of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

Abbreviations

The following abbreviations are used in this manuscript:

CS	Chern–Simons
ECS	Extended Chern–Simons

Appendix A. Uniqueness of Consistent Interaction Vertex

In this Appendix, we demonstrate the fact of the uniqueness of the interaction vertex in Equation (16) in the class of Poincare-covariant couplings that are polynomial in the invariants $\mathcal{F}_\mu, \mathcal{G}_\mu$ without higher derivatives. We apply the method of the inclusion of not necessarily Lagrangian consistent interactions in [54].

Let us first explain the concept of interaction consistency in the class of non-Lagrangian field theories. The dynamics of the theory are determined by a system of partial differential equations (equations of motion) imposed onto the dynamical fields $\varphi^I(x)$:

$$\mathbb{T}_a(\varphi^I(x), \partial\varphi^I(x), \partial^2\varphi^I(x), \ldots) = 0\,. \tag{A1}$$

The equations of motion do not necessarily follow from the least-action principle for any functional $S[\varphi(x)]$, so I and a may run over different sets. For the free model, the left-hand side of Equation (A1) is supposed to be linear in the fields. The interactions are associated with the deformations of system Equation (A1) by non-linear terms. The equations of motion for the theory with coupling are polynomial in the fields

$$\mathbb{T}_a = \mathbb{T}^{(0)}{}_a + \mathbb{T}^{(1)}{}_a + \mathbb{T}^{(2)}{}_a + \ldots = 0\,. \tag{A2}$$

where $\mathbb{T}^{(0)}$, $\mathbb{T}^{(1)}$ and $\mathbb{T}^{(2)}$ are linear, quadratic and cubic in the dynamical variables. Throughout the section, the system Equation (A1) (or, equivalently, Equation (A2)) is supposed to be involutive. The concept of involution implies that Equation (A1) has no differential consequences of lower order (hidden integrability conditions). The ECS Equation (16) is involutive.

The defining relations for gauge symmetries and gauge identifies in the system of partial derivative Equation (A1) read

$$\delta_\epsilon\varphi^I = R^I{}_\alpha\epsilon^\alpha, \qquad \delta_\epsilon\mathbb{T}_a|_{\mathbb{T}=0} = 0\,; \tag{A3}$$

$$L^a{}_A\mathbb{T}_a \equiv 0\,. \tag{A4}$$

where $R^I{}_\alpha, L^a{}_A$ are certain differential operators. For non-Lagrangian equations, the gauge symmetries and gauge identities are not related to each other, so the multi-indices A, α are different. In the perturbative setting, the gauge symmetry and gauge identity generators are supposed to be polynomial in fields

$$R^I{}_\alpha = R^{(0)I}{}_\alpha + R^{(1)I}{}_\alpha + R^{(2)I}{}_\alpha + \ldots\,, \qquad L^a{}_A = L^{(0)a}{}_A + L^{(1)a}{}_A + L^{(2)a}{}_A + \ldots \tag{A5}$$

where $R^{(0)}, L^{(0}, R^{(1)}, L^{(1)}$ and $R^{(2)}, L^{(2)}$ are field-independent, linear and quadratic in the dynamical variables. The interaction is consistent if all the gauge symmetries and gauge identities of the free model are preserved by the deformation of equations of motion. Equation (8) of paper [54] provides a simple formula for the computation of the number of physical degrees of freedom based on the orders of derivatives involved in the equations of models, gauge symmetries and gauge identities. The free and non-linear theory must have the same number of physical degrees of freedom.

Relations in Equations (A3) and (A4) imply the following consistency conditions for $\mathbb{T}^{(k)}$, $R^{(k)}$, $L^{(k)}$ and $k = 0, 1, 2, \ldots$:

$$R^{(0)I}{}_{\alpha}\partial_I \mathbb{T}^{(0)}{}_a = 0\,; \tag{A6}$$

$$R^{(0)I}{}_{\alpha}\partial_I \mathbb{T}^{(1)}{}_a + R^{(1)I}{}_{\alpha}\partial_I \mathbb{T}^{(0)}{}_a = 0\,; \tag{A7}$$

$$R^{(0)I}{}_{\alpha}\partial_I \mathbb{T}^{(2)}{}_a + R^{(1)I}{}_{\alpha}\partial_I \mathbb{T}^{(1)}{}_a + R^{(2)I}{}_{\alpha}\partial_I \mathbb{T}^{(0)}{}_a = 0 \tag{A8}$$

(where the symbol ∂_I denotes a variational derivative with respect to the field φ^I);

$$L^{(0)a}{}_A \mathbb{T}^{(0)}{}_a = 0\,; \tag{A9}$$

$$L^{(0)a}{}_A \mathbb{T}^{(1)}{}_a + L^{(1)a}{}_A \mathbb{T}^{(0)}{}_a = 0\,; \tag{A10}$$

$$L^{(0)a}{}_A \mathbb{T}^{(2)}{}_a + L^{(1)a}{}_A \mathbb{T}^{(1)}{}_a + L^{(2)a}{}_A \mathbb{T}^{(0)}{}_a = 0\,. \tag{A11}$$

Equations (A6) and (A9) determine the gauge symmetry and gauge identity generators $R^{(0)}$, $L^{(0)}$ of the free theory. These quantities are usually given from the outset. Relations in Equations (A7) and (A10) determine the first-order corrections to the equations of motion $T^{(1)}$, gauge symmetry generators $R^{(1)}$ and gauge identity generators $L^{(1)}$. Relations of Equations (A7) and (A10) determine the second-order corrections to the equations of motion $T^{(2)}$, gauge symmetry generators $R^{(2)}$ and gauge identity generators $L^{(2)}$. The procedure of interaction construction can be extended to the third and higher orders. Once the most general covariant ansatz is applied for $T^{(k)}$, $k = 1, 2, \ldots$ the procedure of Equations (A6)–(A11), … allows a complete classification of consistent interactions in a given field theory. An important subtlety of this procedure is that some lower-order couplings can be inconsistent at the higher orders of perturbation theory. The first critical step is the extension of the first-order (quadratic) interaction vertex to the second order of perturbation theory.

Equation (3) determines the left-hand side of the free ECS equations $\mathbb{T}^{(0)}$:

$$\mathbb{T}^{(0)a}{}_{\mu} = \alpha_1 F^a{}_{\mu} + \alpha_2 G^a{}_{\mu} + \alpha_3 K^a{}_{\mu} = 0\,. \tag{A12}$$

The gauge symmetries and gauge identities are defined by the gradient and divergence operators:

$$R^{(0)}{}_{\mu} = \partial_{\mu}\,, \qquad L^{(0)\mu} = \partial^{\mu}\,. \tag{A13}$$

The free gauge identity of Equation (A6) and free gauge transformation read

$$\partial_{\mu}\mathbb{T}^{(0)\mu} \equiv 0\,, \qquad \delta_{\epsilon} A^a{}_{\mu} = \partial_{\mu}\zeta^a\,, \tag{A14}$$

where ζ values are gauge transformation parameters. We consider the Poincare-covariant interactions that are expressed in terms of gauge covariants $\mathcal{F}_{\mu}$, $\mathcal{G}_{\mu}$, $\mathcal{K}_{\mu}$ Equation (14) with no higher-derivative terms being included in the coupling. In this case, the equations of motion are automatically preserved by the Yang–Mills gauge symmetry (Equations (A6)–(A8) are satisfied). Consistent interaction vertices of first and second orders are selected by the conditions in Equations (A10)–(A11). We elaborate on this problem below.

We assume that the equations of motion are polynomial in gauge covariants $\mathcal{F}_\mu$, $\mathcal{G}_\mu$, $\mathcal{K}_\mu$. The linear term is given by the covariantization of the free Equation (3). The most general covariant first-order interaction vertex without higher-derivatives reads

$$\begin{gathered}\mathbb{T}^{(0)}{}_\mu+\mathbb{T}^{(1)}{}_\mu=\\=\alpha_1\mathcal{F}_\mu+\alpha_2\mathcal{G}_\mu+\alpha_3\mathcal{K}_\mu+\epsilon_{\mu\nu\rho}\Big(\frac{1}{2}k_1[\mathcal{F}^\nu,\mathcal{F}^\rho]+k_2[\mathcal{F}^\nu,\mathcal{G}^\rho]+\frac{1}{2}k_3[\mathcal{G}^\nu,\mathcal{G}^\rho]\Big)\,,\end{gathered}\tag{A15}$$

where $k_l, l=1,2,3$ are constants. The covariant divergence of equations of motion reads

$$\begin{gathered}D_\mu\mathbb{T}^\mu=-\frac{1}{\alpha_3}[k_2\mathcal{F}_\mu+k_3\mathcal{G}_\mu,\mathbb{T}^\mu]+\\+\frac{1}{\alpha_3}({\alpha_3}^2-\alpha_3k_1+\alpha_2k_2-\alpha_1k_3)[\mathcal{F}_\mu,\mathcal{G}^\mu]+(k_2^2-k_3k_1)\varepsilon_{\mu\nu\rho}[\mathcal{F}^\mu,[\mathcal{F}^\nu,\mathcal{G}^\rho]]\,.\end{gathered}\tag{A16}$$

The gauge identity is satisfied in the first-order approximation in Equation (A10) if the coefficients $k_l, l=1,2,3$ satisfy the relation

$${\alpha_3}^2-\alpha_3k_1+\alpha_2k_2-\alpha_1k_3=0\,.\tag{A17}$$

The general solution to this equations reads

$$k_1=-\frac{{\beta_1}^2{\alpha_3}^2}{C(\beta;\alpha)}+\alpha_1\beta_3\,,\qquad k_2=-\frac{\beta_2\beta_1{\alpha_3}^2}{C(\beta;\alpha)}\qquad k_3=-\frac{{\beta_2}^2{\alpha_3}^2}{C(\beta;\alpha)}+\alpha_3\beta_1\,,\tag{A18}$$

where β_1, β_2, β_3 are coupling parameters. The parameters β_1,β_2 determine the coupling vertex of Equation (16) (two constants determine a single coupling because the ratio β_1/β_2 is relevant). The constant β_3 is responsible for another interaction vertex, which is consistent at the first order of perturbation theory. The interaction vertex of Equation (16) is self-consistent, with no higher-order corrections required for the equations of motion. The other coupling needs cubic corrections to the equations of motion in the fields. To prove the uniqueness of the interaction of Equation (16), we should demonstrate that the ansatz of Equations (A15) and (A18) is inconsistent at the second order of perturbation theory for $\beta_3\neq 0$.

The most general second-order covariant interaction vertex reads

$$\begin{gathered}\mathbb{T}^{(2)}{}_\mu=l_1[\mathcal{F}_\nu,[\mathcal{F}_\mu,\mathcal{F}^\nu]]+l_2[\mathcal{F}_\nu,[\mathcal{G}_\mu,\mathcal{F}^\nu]]+l_3[\mathcal{F}_\nu,[\mathcal{F}_\mu,\mathcal{G}^\nu]]+\\+l_4[\mathcal{G}_\nu,[\mathcal{F}_\mu,\mathcal{G}^\nu]]+l_5[\mathcal{G}_\nu,[\mathcal{G}_\mu,\mathcal{F}^\nu]]+l_6[\mathcal{G}_\nu,[\mathcal{G}_\mu,\mathcal{G}^\nu]]\,,\end{gathered}\tag{A19}$$

where $l_p, p=\overline{1,6}$ are constants. The covariant divergence of equations of motion reads (only cubic terms are written out)

$$\begin{gathered}D_\mu\mathbb{T}^\mu=\frac{1}{\alpha_3}\epsilon_{\mu\nu\rho}[\mathcal{S}^{\mu\nu},\mathbb{T}^\rho]+\frac{1}{2}C_1(k;l)\varepsilon_{\mu\nu\rho}[\mathcal{F}^\mu,[\mathcal{F}^\nu,\mathcal{G}^\rho]]+\frac{1}{2}C_2(k;l)\varepsilon_{\mu\nu\rho}[\mathcal{G}^\mu,[\mathcal{G}^\nu,\mathcal{F}^\rho]]+\\+\frac{l_1}{2}[\mathcal{F}_\mu,[\mathcal{F}_\nu,D^\mu\mathcal{F}^\nu+D^\nu\mathcal{F}^\mu]]+\frac{l_3-l_2}{2}[\mathcal{G}_\mu,[\mathcal{F}_\nu,D^\mu\mathcal{F}^\nu+D^\nu\mathcal{F}^\mu]]+\frac{2l_2-l_3}{2}[\mathcal{F}_\mu,[\mathcal{G}_\nu,D^\mu\mathcal{F}^\nu+D^\nu\mathcal{F}^\mu]]+\\+\frac{l_5}{2}[\mathcal{G}_\mu,[\mathcal{G}_\nu,D^\mu\mathcal{F}^\nu+D^\nu\mathcal{F}^\mu]]+\frac{l_3}{2}[\mathcal{F}_\mu,[\mathcal{F}_\nu,D^\mu\mathcal{G}^\nu+D^\nu\mathcal{G}^\mu]]+\frac{l_5-l_4}{2}[\mathcal{F}_\mu,[\mathcal{G}_\nu,D^\mu\mathcal{G}^\nu+\\+D^\nu\mathcal{G}^\mu]]+\frac{2l_4-l_5}{2}[\mathcal{G}_\mu,[\mathcal{F}_\nu,D^\mu\mathcal{G}^\nu+D^\nu\mathcal{G}^\mu]]+\frac{l_6}{2}[\mathcal{G}_\mu,[\mathcal{G}_\nu,D^\mu\mathcal{G}^\nu+D^\nu\mathcal{G}^\mu]]\,.\end{gathered}\tag{A20}$$

Here, the following notation is used:

$$C_1(k;l) = k_2{}^2 - k_1 k_3 - 3l_1 - \frac{\alpha_1}{\alpha_3} l_4 - \frac{\alpha_1}{\alpha_3} l_5 - \frac{\alpha_1}{\alpha_3} l_6 \,, \qquad C_2(k;l) = 3l_2 + l_3 - \frac{\alpha_2}{\alpha_3} l_5 - \frac{\alpha_1}{\alpha_3} l_6 \,; \tag{A21}$$

$$\begin{gathered} \mathcal{S}^{\mu\nu} = l_3[\mathcal{F}^\mu, [\mathcal{F}^\nu, \cdot]] + l_4[\mathcal{G}^\mu, [\mathcal{F}^\nu, \cdot]] + l_6[\mathcal{G}^\mu, [\mathcal{G}^\nu, \cdot]] - l_4[[\mathcal{F}^\mu, \mathcal{F}^\nu], \cdot] - \\ - l_5[[\mathcal{F}^\mu, \mathcal{G}^\nu], \cdot] - l_6[[\mathcal{G}^\mu, \mathcal{G}^\nu], \cdot] \,. \end{gathered} \tag{A22}$$

The interaction is consistent at the second order of perturbation theory if the right-hand side of this expression causes the modulo free Equation (3) to vanish. The critical observation is that the expressions of the form $[\mathcal{X}_\mu, [\mathcal{Y}_\nu, D^\mu \mathcal{Z}^\nu + D^\nu \mathcal{Z}^\mu]]$, where $\mathcal{Y}_\mu, \mathcal{Z}_\mu, \mathcal{Y}_\mu = \mathcal{F}_\mu$ or $\mathcal{G}_\mu$ represent on-shell independent combinations of fields and their derivatives. Once they are made to vanish, we conclude

$$k_2{}^2 - k_3 k_1 = 0 \,, \qquad l_p = 0 \,, \qquad p = \overline{1,6} \,. \tag{A23}$$

The general solution to these equations has the form (A18) with $\beta_3 = 0$. Taking account of this fact, the interaction (16) is unique in the class of covariant couplings without higher derivatives.

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Communication

Holographic Foam Cosmology: From the Late to the Early Universe

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Abstract: Quantum fluctuations endow spacetime with a foamy texture. The degree of foaminess is dictated by black hole physics to be of the holographic type. Applied to cosmology, the holographic foam model predicts the existence of dark energy with critical energy density in the current (late) universe, the quanta of which obey infinite statistics. Furthermore, we use the deep similarities between turbulence and the spacetime foam phase of strong quantum gravity to argue that the early universe was in a turbulent regime when it underwent a brief cosmic inflation with a "graceful" transition to a laminar regime. In this scenario, both the late and the early cosmic accelerations have their origins in spacetime foam.

Keywords: spacetime foam; holography; dark energy; cosmic inflation; infinite statistics; turbulence

Citation: Ng, Y.J. Holographic Foam Cosmology: From the Late to the Early Universe *Symmetry* **2021**, *13*, 435. https://doi.org/10.3390/sym13030435

Academic Editor: Chris Fields

Received: 19 February 2021
Accepted: 3 March 2021
Published: 8 March 2021

1. Introduction

There are two cosmic accelerations that we are aware of: a brief inflationary acceleration [1–5] in the early universe and the present ("late" universe") acceleration [6,7] that is attributed to dark energy. Normally they are treated independently and separately; but there are also some works [8,9] that consider both regimes of accelerated expansions. We think it is conceptually necessary and aesthetically pleasing to trace both cosmic accelerations to a common cause in fundamental theory. Following Wheeler [10–12], we believe that space is composed of an ever-changing geometry and topology called spacetime foam and that the foaminess is due to quantum fluctuations of spacetime. We argue for a scenario in which spacetime foam is the origin of both cosmic accelerations.

The outline of this paper is as follows. We begin with a brief review of the holographic spacetime foam model. In Section 2, we use the quantum uncertainty principle coupled with black-hole physics to show that spacetime is indeed foamy and the degree of foaminess is consistent with the holographic principle; we further argue that there necessarily exists a dark sector in the universe. In Section 3, we apply the holographic spacetime foam to cosmology (with the corresponding cosmology called holographic foam cosmology (HFC) [13–16]) and argue for the existence of dark energy with critical energy density in the present universe and its quanta obey infinite statistics, also known as quantum Boltzmann statistics. In Section 4, we use the deep similarities between the physics of turbulence and the universal geometric properties of the holographic spacetime foam to heuristically argue that the early universe was in a turbulent phase during which the universe underwent a brief cosmic inflationary acceleration. Section 5 contains our concluding remarks, especially with respect to the naturalness and inevitability of inflation in the early universe.

We will use the subscript "P" to denote Planck units (with, e.g., $l_P \equiv (\hbar G/c^3)^{1/2} \sim 10^{-33}$ cm being the Planck length). Furthermore, for simplicity, $\hbar$, c, and the Boltzmann constant k_B are often put equal to unity.

2. Holographic Spacetime Foam

One manifestation of spacetime fluctuations is in the induced uncertainties in any distance measurement. Consider the following gedanken experiment [17] to measure the distance l between a clock at one point and a mirror at another. By sending a light signal from the clock to the mirror in a timing experiment, we can determine the distance. The quantum uncertainty in the positions of the clock and the mirror introduces an inaccuracy δl. Let us concentrate on the clock (of mass m). If it has a linear spread δl when the light signal leaves the clock, then its position spread grows to $\delta l + \hbar l (mc\delta l)^{-1}$ when the light signal returns to the clock, with the minimum uncertainty at $\delta l = (\hbar l/mc)^{1/2}$. Hence one concludes that $\delta l^2 \gtrsim \frac{\hbar l}{mc}$. One can supplement this requirement with a limit from general relativity [18,19], viz., δl must be larger than the Schwarzschild radius Gm/c^2 of the clock, yielding $\delta l \gtrsim \frac{Gm}{c^2}$ (henceforth we will neglect multiplicative constants of order unity), the product of which with the bound from quantum fluctuations finally gives [18,20,21]

$$\delta l \gtrsim (l l_P^2)^{1/3} = l_P \left(\frac{l}{l_P}\right)^{1/3}. \tag{1}$$

This bound on δl can also be derived by the following method which provides additional valuable insights. Consider a spherical volume of radius l over the amount of time $T = 2l/c$ it takes light to cross the volume. One way to map out the geometry of this spacetime region [22] is to fill the space with clocks, exchanging signals with other clocks and measuring the signals' times of arrival. This process of mapping the geometry is a sort of computation; hence the total number of operations is bounded by the Margolus-Levitin theorem [23], which stipulates that the rate of operations for any computer cannot exceed the amount of energy E that is available for computation divided by $\pi\hbar/2$. To avoid collapsing the region into a black hole, the total mass M of clocks must be less than $lc^2/2G$, corresponding to the upper bound on energy density

$$\rho \sim \frac{l/G}{l^3} = (l l_P)^{-2}. \tag{2}$$

Together, these two limits imply that the total number of operations that can occur in a spatial volume of radius l for a time period $2l/c$ is no greater than $\sim (l/l_P)^2$. (Here and henceforth we set $c = 1 = \hbar$). To maximize spatial resolution, each clock must tick only once during the entire time period. Furthermore, if we regard the operations partitioning the spacetime volume into "cells", then on the average each cell occupies a spatial volume no less than $\sim l^3/(l^2/l_P^2) = l l_P^2$, yielding an average separation between neighboring cells no less than $l^{1/3} l_P^{2/3}$. This spatial separation is interpreted as the average minimum uncertainty in the measurement of a distance l, that is, $\delta l \gtrsim l^{1/3} l_P^{2/3}$, in agreement with the result Equation (1) obtained above.

We can now heuristically derive the holographic principle. Since, on the average, each cell occupies a spatial volume of $(\delta l)^3 \lesssim l l_P^2$, a spatial region of size l can contain no more than $l^3/(l l_P^2) = (l/l_P)^2$ cells. Thus, this spacetime foam model corresponds to the case of maximum number of bits of information l^2/l_P^2 in a spatial region of size l, that is allowed by the holographic principle [24–27]. Accordingly, we will refer to this spacetime foam model (corresponding to $\delta l \gtrsim l^{1/3} l_P^{2/3}$) as the holographic spacetime foam model.

3. From Spacetime Foam to Dark Energy

As a corollary to the above discussion, we can now give a heuristic argument [13,22,28] on why the universe cannot contain ordinary matter only. Start by assuming the universe (of size l) has only ordinary matter. According to the statistical mechanics for ordinary matter at temperature T, energy scales as $E \sim l^3 T^4$ and entropy goes as $S \sim l^3 T^3$. Black hole physics can be invoked to require $E \lesssim \frac{l}{G} = \frac{l}{l_P^2}$. Then it follows that the entropy S and hence also the number of bits I (or the number of degrees of freedom) on ordinary matter are bounded by $\lesssim (l/l_P)^{3/2}$. We can repeat verbatim the argument given in Section 2 to conclude that, if only ordinary matter exists, $\delta l \gtrsim \left(\frac{l^3}{(l/l_P)^{3/2}}\right)^{1/3} = l^{1/2}l_P^{1/2}$ which is much greater than $l^{1/3}l_P^{2/3}$, the result found above from our analysis of the gedanken experiment and implied by the holographic principle. Thus, there must be other kinds of matter/energy with which the universe can map out its spacetime geometry to a finer spatial accuracy than is possible with the use of only conventional ordinary matter. We conclude that a dark sector necessarily exists in the universe!

The above discussion leads to the prediction of dark energy. To see that, let us now generalize this discussion for a static spacetime region with low spatial curvature to the case of the recent/present universe by substituting l by $1/H$, where H is the Hubble parameter [13,28]. Equation (2) yields the cosmic energy density $\rho \sim \left(\frac{H}{l_P}\right)^2 \sim (R_H l_P)^{-2} \sim 10^{-120}M_P^4$ with R_H being the Hubble radius. Next, recall that we have also shown that the universe contains $I \sim (R_H/l_P)^2$ bits of information ($\sim 10^{120}$ for the current epoch) [13]. Hence the average energy carried by each of these bits or quanta is $\rho R_H^3/I \sim R_H^{-1}$. These long-wavelength bits or "particles" (quanta of spacetime foam) carry negligible kinetic energy. (Alternatively one can interpret these quanta as constituents of dark energy, contributing a more or less uniformly distributed cosmic energy density and hence acting as a dynamical effective cosmological constant $\Lambda \sim H^2$.) Note: Such long-wavelength quanta can hardly be called particles. We will simply call them "particles" in quotation marks. Since pressure (energy density) is given by kinetic energy minus (plus) potential energy, a negligible kinetic energy means that the pressure of the unconventional energy is roughly equal to minus its energy density, leading to accelerating cosmic expansion, in agreement with observation [6]. This scenario is very similar to that of quintessence [29,30], but it has its origin in the holographic spacetime foam [15,31].

How do these long-wavelength quanta differ from ordinary particles? Consider $N \sim (R_H/l_P)^2$ such "particles" in volume $V \sim R_H^3$ at $T \sim R_H^{-1}$, the average energy carried by each "particle". If these "particles" obey Boltzmann statistics, the partition function $Z_N = (N!)^{-1}(V/\lambda^3)^N$ gives the entropy of the system $S = N[ln(V/N\lambda^3) + 5/2]$, with thermal wavelength $\lambda \sim T^{-1} \sim R_H$. However, then $V \sim \lambda^3$, so S becomes negative unless $N \sim 1$ which is equally nonsensical. A simple solution is to stipulate that the N inside the log in S, i.e, the Gibbs factor $(N!)^{-1}$ in Z_N, must be absent. (This means that the N "particles" are distinguishable!) Then the entropy is positive: $S = N[ln(V/\lambda^3) + 3/2] \sim N$. Now, the only known consistent statistics in greater than 2 space dimensions without the Gibbs factor is the quantum Boltzmann statistics, also known as infinite statistics [32–34]. Thus, we conclude that the "particles" constituting dark energy obey infinite statistics, rather than the familiar Fermi or Bose statistics [28,35]. For completeness, let us list some of the properties of infinite statistics [32–34]. A Fock realization of infinite statistics is given by $a_k a_l^\dagger = \delta_{k,l}$. It is known that particles obeying infinite statistics are distinguishable, and importantly their theories are non-local [34,36] (to be more precise, the fields associated with infinite statistics are not local, neither in the sense that their observables commute at spacelike separation nor in the sense that their observables are pointlike functionals of the fields). Their quanta are extended (consistent with what we show above for dark energy). The number operator and Hamiltonian, etc., are both nonlocal and nonpolynomial in the field operators. This property of non-locality will be useful later in the discussion of the early universe. However, we should note that TCP theorem and cluster decomposition still hold; and quantum field theories with infinite statistics remain unitary [34].

4. From Spacetime Foam and Turbulence to Cosmic Inflation

To date, we have applied HFC to the present and recent cosmic eras (with $\rho \sim 10^{-120}M_P^4$). However, what about the early universe (with $\rho \sim 10^{-8}M_P^4$)? Actually the discussion in the preceding section has already given us some helpful hints [13] especially with respect to inflation [1–5] in the early universe. For example: (1) The flatness problem is largely solved because, according to HFC, the cosmic energy is of critical density.

(2) It is quite possible that HFC provides sufficient density perturbation as the model contains the essence of a k-essence model. (Furthermore, the horizon problem may also be solved since spacetime foam physics is essentially quantum black hole physics and thus is closely related to wormhole physics which can be used [37] to solve the horizon problem.) Nevertheless one important aspect of the early universe appears to be missing in HFC. It is connected to the expectation (supported by Wheeler's insight [38]) that, due to quantum fluctuations, spacetime, when probed at very small scales, as is the case for the early universe, will appear very complicated—something akin in complicity to a chaotic turbulent froth. So, was spacetime turbulent in the early universe? To this question we now sketch a positive response.

First let us show the deep similarities between the problem of quantum gravity and turbulence [39,40]. The connection between them can be traced to the role of diffeomorphism symmetry in classical gravity and the volume preserving diffeomorphisms of classical fluid dynamics. In the case of irrotational fluids in three spatial dimensions, the equation for the fluctuations of the velocity potential can be written in a geometric form [41] of a harmonic Laplace–Beltrami equation: $\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b\varphi) = 0$, where the effective space time metric has the canonical ADM form $ds^2 = \frac{\rho_0}{c}[c^2dt^2 - \delta_{ij}(dx^i - v^i dt)(dx^j - v^j dt)]$, with c being the sound velocity. In this expression for the metric, it is apparent that the velocity of the fluid v^i plays the role of the shift vector N^i in the canonical Dirac/ADM treatment of Einstein gravity: $ds^2 = N^2dt^2 - h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$. Hence in the fluid dynamics context, $N^i \to v^i$, and a fluctuation of v^i would imply a fluctuation of the shift vector (and hence a fluctuation of the spacetime metric) and vice versa.

Next, recall that the onset of turbulence can be predicted by the dimensionless Reynolds number $Re = Lv/\nu$, where v is the velocity field, L is a characteristic scale and ν is the kinematic viscosity which is given by the product of the mean free path $\tilde{l}$ and an effective velocity factor $\tilde{v}$. At low Reynolds number, fluid flow is laminar, i.e., smooth; at high Reynolds number, the flow becomes turbulent. A characteristic signature for turbulence is the formation of eddies at various scales. The energy cascades from large-scale structures to smaller scale ones. The scale at which molecular diffusion becomes important and viscous dissipation of energy transpires is the Kolmogorov length which we will denote as ℓ. Let us note that, in fully developed turbulence in three spatial dimensions, Kolmogorov scaling specifies the behavior of n-point correlation functions of the fluid velocity. The scaling [42,43] follows from the assumption of constant energy flux, $\frac{v^2}{t} \sim \varepsilon$, where v stands for the velocity field of the flow, and the single length scale ℓ is given as $\ell \sim v \cdot t$. This implies that $v \sim (\varepsilon\,\ell)^{1/3}$, consistent with the experimentally observed two-point function $\langle v^i(\ell)v^j(0)\rangle \sim (\varepsilon\,\ell)^{2/3}\delta^{ij}$ (the famous two-thirds law, which yields, via a one-dimensional Fourier transformation, the energy scaling $E(k) \sim k^{-5/3}$, Kolmogorov's seminal "5/3" law for the energy distribution in the turbulent fluid.) Now recall our discussion above on distance fluctuations $\delta\ell \sim \ell^{1/3}\ell_P^{2/3}$, and define the velocity as $v \sim \frac{\delta\ell}{t_c}$, since the natural characteristic time scale is $t_c \sim \frac{\ell_P}{c}$, then it follows that $v \sim c\left(\frac{\ell}{\ell_P}\right)^{1/3}$. (The implication is that at short distance, the spacetime is a chaotic and stochastic fluid in a turbulent regime with the Kolmogorov length l. The energy cascades are a property of the spacetime foam.) It is now obvious that a Kolmogorov-like scaling [42,43] in turbulence has been obtained. This interpretation of the Kolmogorov scaling in the quantum gravitational setting implies that the quantum fluctuation phase of strong quantum gravity in the early universe could be governed by turbulence.

The discussion above is for the case of very large Reynolds number. However, was Re actually large enough in the early universe to set off turbulence? For the purpose of comparison and illustration, let us recall that, in conventional cosmology at time, say, 10^{-35} s, vL in the numerator of Re is roughly given by the product of $v \sim 10^{-2}c$ and $L \sim 10^8 l_P$ [44–46]. For HFC close to Planckian time, the denominator of Re is given by $\nu \sim cl_P$ since, in that regime, momentum transport could only be due to Planckian dynamics. For the discussion to follow, let us note that, relatively speaking, the effective velocity factor

$\tilde{v}$ in ν is not that different from the v factor in the numerator of Re. The onset of turbulence was due to the smallness of the length scale (which plays the role of an effective mean free path) in the denominator of Re, viz., $\tilde{l} = l_P << L \sim 10^8 l_P$ that was mainly responsible for yielding a large Re in the (very) early universe.

There remains one crucial hurdle to overcome. If, as we suggest, turbulence in the early universe was related to inflation, we have to confront the "graceful" exit problem: How to get a small enough Re to transit to the laminar phase and to end inflation in the process? It is here the nonlocality property enjoyed by the quanta of spacetime foam (due to the fact that they obey infinite statistics) came to the rescue since the length scale $\tilde{l}$ in ν would naturally and eventually extend to the order of L. (Compare with the case of dark energy discussed in Section 3.) This would yield a small enough Re to suppress turbulence and to naturally end inflation.

5. Conclusions

We have sketched a scenario in which both the late and the early cosmic accelerations have a common origin and can be traced to spacetime foam. The case for dark energy in the current/recent ("late") universe was proposed before [13,28], while the case for cosmic inflation in the early Univerese is the main focus of this paper. One attractive feature of our present proposal is that the scheme is very economical, involving no arbitrary or fine-tuned parameters. It is also natural in that inflation was inevitable as turbulence set off by the Planckian dynamics was inevitable in the early universe. The scheme also provides a rationale for why inflation lasted only briefly (say, $\sim 10^{-33}$ s) as the turbulent phase was quickly terminated due to the nonlocal (extended) property of the quanta of spacetime foam. Of course it is important to check if this scenario is supported by more quantitative arguments and calculations. In passing we should also mention that it will be of great interest to see if our scenario for inflation discussed above can at least mitigate some of the criticism [47,48] against the inflation paradigm.

We conclude with two observations the first of which involves the crucial question of whether our approach yields enough e-folds of inflation to solve the myriad of cosmological problems. The following heuristic argument would seem to say probably there were. Let us follow the folklore: at the end of inflation, the energy stored in the quanta of spacetime foam would be converted into hot ordinary particles (as well as dark matter). Since the Grand Unification era is around $\sim 10^{-34}$ s, which, as an order-of-magnitude estimate, should also mark the end of inflation, giving enough (say $\gtrsim 65$) e-folds of inflation. This argument may be strengthened, if, as has been proposed [49], quantum gravity can actually be the origin of (ordinary-) particle statistics, and that infinite statistics (the statistics obeyed by quanta of spacetime foam) is the underlying statistics. In that case, ordinary particles that obey Bose or Fermi statistics are actually some sort of collective degrees of freedom of "particles" of infinite statistics. (See [50] for a discussion of such a construction.) Thus, arguably, at the end of inflation, quanta of spacetime foam could be converted into ordinary particles as required.

We end this paper with our second observation. Our aesthetically pleasing scenario can be compared to a recent model [8] of quintessential inflation based on the assumption that the slow roll parameter has a Lorentzian form as a function of the number of e-folds. Its form corresponds to the vacuum energy both in the inflationary (with $\rho \sim 10^{-8} M_P^4$) and the dark energy (with $\rho \sim 10^{-120} M_P^4$) epochs which are treated symmetrically. In this model the inflationary scale is exponentially amplified while the dark energy scale is suppressed, producing a curious cosmological see-saw mechanism. In the present work, the two cosmic accelerations are also attributed to a single mechanism; but they are related by some sort of turbulent-laminar duality (or chaotic-smooth complementarity). It would be interesting to see if our scheme can be approximated by an effective theory in which a similar ansatz for the slow roll parameter naturally arises as in the see-saw model [8]. However, a full investigation may require a non-perturbative treatment of quantum gravity, involving a truly non-local field theory of "particles" obeying infinite statistics. (We note that when the see-saw model is realized in the context of a single scalar field, the extracted potential of the scalar field is fairly complicated. Can this complication be a reflection of non-perturbative quantum gravity involving non-local quanta of spacetime foam?)

Funding: This research was partly supported by the Bahnson Fund and the Kenan Professors Research Fund of the University of North Carolina at Chapel Hill.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: I thank David Benisty and Eduardo Guendelman for a useful correspondence. I am grateful to the Bahnson Fund and the Kenan Professors Research Fund of the University of North Carolina at Chapel Hill for partial financial support.

Conflicts of Interest: The author declares no conflict of interest.

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Reference Frame Induced Symmetry Breaking on Holographic Screens

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Abstract: Any interaction between finite quantum systems in a separable joint state can be viewed as encoding classical information on an induced holographic screen. Here we show that when such an interaction is represented as a measurement, the quantum reference frames (QRFs) deployed to identify systems and pick out their pointer states induce decoherence, breaking the symmetry of the holographic encoding in an observer-relative way. Observable entanglement, contextuality, and classical memory are, in this representation, logical and temporal relations between QRFs. Sharing entanglement as a resource requires a priori shared QRFs.

Keywords: black hole; contextuality; decoherence; quantum error-correcting code; quantum reference frame; system identification; channel theory

Citation: Fields, C.; Glazebrook, J.F.; Marcianò, A. Reference Frame Induced Symmetry Breaking on Holographic Screens. *Symmetry* **2021**, *13*, 408. https://doi.org/10.3390/sym13030408

Academic Editor: Kazuharu Bamba

Received: 9 February 2021
Accepted: 27 February 2021
Published: 3 March 2021

1. Introduction

The holographic principle (HP) states, in its covariant formulation, that for any finite spacelike boundary $\mathcal{B}$, open or closed, the classical, thermodynamic entropy $S(L(\mathcal{B}))$ of any light-sheet $L(\mathcal{B})$ of $\mathcal{B}$ satisfies:

$$S(L(\mathcal{B})) \leq A(\mathcal{B})/4\,, \tag{1}$$

where $A(\mathcal{B})$ is the area of $\mathcal{B}$ in Planck units [1]. The HP was motivated by the Bekenstein bound on the thermodynamic entropy of a black hole (BH), and has traditionally been interpreted as a bound on the thermodynamic entropy of, and hence the classical information encodable on, an independently-defined surface $\mathcal{B}$, e.g., the stretched horizon of a BH [2,3]; see [1,4] for reviews.

We can, however, also view (1) from a more general perspective, as a fundamental principle of information geometry that associates a (finite) minimal surface $\mathcal{B}$ with any bit string of (finite) entropy S, and hence with any classical channel of width S bits. Such a channel can be constructed, without loss of generality, as follows: Let $U = AB$ be a finite, closed quantum system, assume separability, $|AB\rangle = |A\rangle|B\rangle$ over any time interval of interest, and write the interaction:

$$H_{AB} = \beta^k k_B T^k \sum_i^N \alpha_i^k M_i^k, \tag{2}$$

where $k = A$ or B, the M_i^k are N Hermitian operators with eigenvalues in $\{-1, 1\}$, the $\alpha_i^k \in [0,1]$ are such that $\sum_i^N \alpha_i^k = 1$, k_B is Boltzmann's constant, T^k is k's temperature, and $\beta^k \geq \ln 2$ is an inverse measure of k's thermodynamic efficiency that depends on the internal dynamics H_k. At each time step, A obtains exactly N bits of information about B from this channel and vice versa, entirely independently of the internal dynamics H_A and H_B. With this construction, we can state the following generalized holographic principle (cf. [5] Thm. 1):

GHP: If but only if a pair of finite quantum systems A and B have a separable joint state $|AB\rangle = |A\rangle|B\rangle$, there is a finite spacelike surface $\mathcal{B}$, with area $A(\mathcal{B}) \geq A(\mathcal{B})_{min} = 4\ln 2 N l_P^2$, N the dimension of H_{AB} and l_P the Planck length, that implements H_{AB} as a classical channel.

This GHP is a purely information-theoretic principle that makes no reference to any spatial embedding of A or B. We show in [5] that it holds for any spatial embedding of A and B allowed by special relativity. As $\mathcal{B}$ is ancillary to the interaction H_{AB}, we will be unconcerned with the spatial scale of $\mathcal{B}$; in systems with low energy densities, we can expect $A(\mathcal{B}) \gg A(\mathcal{B})_{min}$.

The form of the Hamiltonian (2) has two immediately-apparent symmetries. First, the number N of transferred bits is fixed; hence Equation (2) is symmetric as a channel. The holographic screen $\mathcal{B}$ "looks the same" and encodes the same information, N bits, from either A's or B's perspective. Second, the terms $\alpha_i^k M_i^k$ of Equation (2) can be re-arranged in any order. If we view $\mathcal{B}$ as implemented by an ancillary array of non-interacting qubits as in [5,6], these qubits can be permuted arbitrarily; hence the state of $\mathcal{B}$ is invariant under the symmetric group S_N.

Here we consider the system A to be an "observer" of B, and study apparent, observer-relative symmetry breaking on $\mathcal{B}$ induced by the implementation of one or more quantum reference frames (QRFs) by the internal Hamiltonian H_A. The role of reference frames in physical theory is to allow observations made at different times and/or places to be compared. While in classical physics reference frames are often treated in abstracto, in quantum theory they must be considered to be physically implemented, and hence as QRFs; meter sticks, clocks, and gyroscopes are canonical examples [7]. Sharing an external QRF, e.g., a Cartesian frame, across either space or time requires the observers involved to implement an equivalent internal QRF [8]; hence all QRFs deployed by A can, without loss of generality, be considered to be implemented by H_A (cf. [9,10]).

We begin in Section 2 below by briefly reviewing some consequences of separating systems A and B by a holographic screen $\mathcal{B}$; such separation prevents, in particular, measurements by A of the entanglement entropy of B. We then introduce in Section 3 an explicit, fully general, and semantically-rich category-theoretic formalism with which to specify the QRFs deployed by any observer, focussing first on QRFs employed for system identification (Section 3.1) and then considering QRFs employed for pointer measurements (Section 3.2). We show that sequential pointer measurements break the S_N symmetry of the screen $\mathcal{B}$, inducing decoherence (Section 3.3). For illustration, we turn to the particular case of measuring Hawking quanta from a BH, showing explicitly that no experiment can demonstrate entanglement between a BH and a distant free quantum of radiation (Section 3.4). We close this section by showing that the free-energy costs of deploying a QRF induce coarse-graining (Section 3.5). These results provide the background required to prove, in Section 4, that sharing entanglement as a resource requires a priori shared, entangled QRFs, and then to prove, in Section 5, that whether two observers share QRFs is finite Turing undecidable. We close in Section 6 by showing that writing classical memories to a screen $\mathcal{B}$ creates phase correlations that further disrupt the S_N symmetry of the screen. These results together suggest that, far from being "an apparent law of physics that stands by itself" [1], the HP in its generalized GHP form is central to quantum information theory.

2. Instantaneous Interactions across $\mathcal{B}$

To write the Hamiltonian (2), we require the joint state $|AB\rangle$ to be separable; it is this separability that makes $\mathcal{B}$ a classical channel. Distinguishing the classical entropy (S) from the entanglement entropy ($\mathcal{S}$),we can restate the GHP in summary form as:

Lemma 1. *If systems A and B are separated by a finite holographic screen $\mathcal{B}$, the entanglement entropy of the joint state $\mathcal{S}(|AB\rangle) = 0$.*

Proof. By the definition of $\mathcal{B}$; see [5], Thm. 1 showing that any finite-bandwidth classical communication channel can be represented as a finite holographic screen for details. □

Lemma 1 immediately rules out transfers of quantum information across $\mathcal{B}$; hence A has access to neither the internal Hamiltonian H_B or the entanglement entropy $\mathcal{S}(B) =_{def} max(\mathcal{S}(|B_1B_2\rangle))$ over tensor-product decompositions $B_1B_2 = B$. It is clear, moreover, that Equation (2) can hold only if the Hilbert space dimensions $dim(\mathcal{H}_A), dim(\mathcal{H}_B) \geq N$, where equality holds only if A and B contain no "hidden" degrees of freedom that do not, over any time interval under consideration, contribute to H_{AB}. Hence A cannot place upper limits on either the dimension $dim(\mathcal{H}_B)$ or the entanglement entropy $\mathcal{S}(B)$ of B. We will assume in what follows that B is "large," $dim(\mathcal{H}_B) \gg N$, and has near-maximal entanglement entropy $\mathcal{S}(B) \approx dim(\mathcal{H}_B)/2$; this is effectively the assumption that H_{AB} only minimally perturbs $|B\rangle$. As with this assumption the full state $|B\rangle$ is not observable by A, we will use the notation $|B\rangle^A$ to indicate the "observed" (by A) partial state of B that is encoded on $\mathcal{B}$. By (2), this observed state $|B\rangle^A$ is an eigenstate of H_{AB} with dimension N.

2.1. Example: Scattering

Consider a scattering process mediated by a gauge boson, as shown in Figure 1a. Both incoming and outgoing joint states are asymptotically separable, so the exchanged information is encoded on a holographic screen $\mathcal{B}$. Ignoring charge and spin, the encoded information specifies a classical momentum transfer $\Delta\vec{p}$. No quantum information is transferred across $\mathcal{B}$; in particular, the scattering processes transfers no information about the entanglement entropy $\mathcal{S}(B)$ to A (Figure 1b).

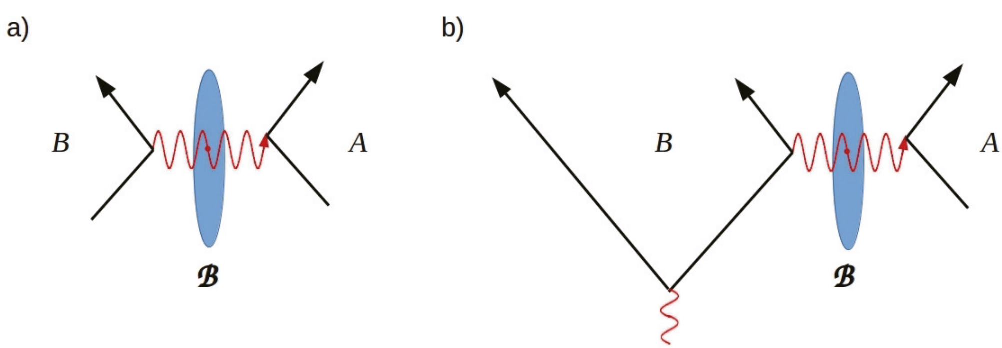

Figure 1. (**a**) A gauge boson transfers asymptotically-classical momentum information across a holographic screen $\mathcal{B}$. (**b**) The scattering process transfers no information about the entanglement entropy $\mathcal{S}(B)$.

Writing the Hamiltonian as in (2) requires the dimension N of the observed state $|B\rangle^A$ to be finite and hence the momentum transfer $\Delta\vec{p}$ to be discrete. Discrete values of $\Delta\vec{p}$ reflect the discrete cost of information in units of $\hbar$. In a experimental setting in which $\Delta\vec{p}$ is measured at some asymptotically-distant location, the dimension N, and hence the number of (ancillary) qubits required to represent $\mathcal{B}$ as a channel, is set operationally by the resolution of the detector. In this case $\Delta\vec{p}$ is the measured pointer value, and a full description of the interaction requires specification of the QRF employed for system identification as discussed in Section 3 below.

2.2. Example: Hawking Radiation

For an asymptotic observer A, coupled pair annihilation and production near a holographic screen $\mathcal{B}$, with one positive- and one negative-energy mode transiting the screen (Figure 2a) is indistinguishable from a scattering event at $\mathcal{B}$ (Figure 2b). Hence far from a black hole (BH), Hawking radiation from the BH is indistinguishable from scattering from the stretched horizon of the BH. Lemma 1 forbids any modes detectable by A from carrying information about the entanglement entropy $\mathcal{S}(B)$ as discussed above; hence the only observable entropy of B is the classical, thermodynamic entropy $S(L(\mathcal{B}))$ given by Equation (1).

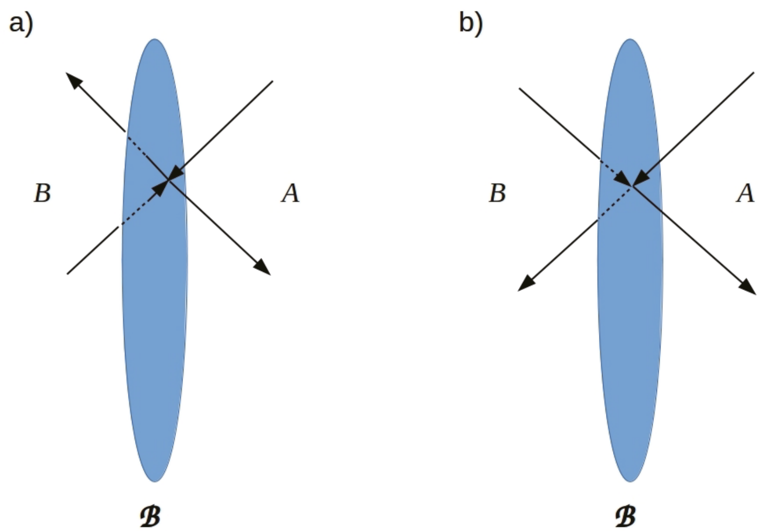

Figure 2. (**a**) Hawking pair annihilation-production near a BH is asymptotically indistinguishable from (**b**) symmetric scattering from the stretched horizon.

The distinction between the thermodynamic entropy $S(L(\mathcal{B}))$ and the entanglement entropy $\mathcal{S}(B)$ for B a BH, and hence $\mathcal{B}$ the stretched horizon, has been recently clarified from a number of perspectives [11–14], showing in particular that preserving unitarity does not require a firewall [15] to prevent detection of excess entanglement. Considering the outgoing states to be measured by some observer requires specifying a QRF as noted above; we consider this issue in the particular case of Hawking quanta further in Section 3.4 below.

2.3. Symmetry across $\mathcal{B}$ Corresponds to "Free Choice" of QRFs

A QRF deployed by A, i.e., implemented by H_A, corresponds to a set of observables held fixed while other observables are allowed to vary [6,8] as discussed more explicitly in Section 3 below. It thus corresponds to a subset of the M_i^A. Associative groupings of the M_i^A in Equation (2) are clearly independent of associative groupings of the M_i^B; hence choices of QRF by A have no bearing on choices of QRF by B or vice versa. Equivalently, swapping the labels A and B has no effect on Equation (2).

This "free choice" of QRFs corresponds to a the absence of superdeterminist correlations between A and B. Such correlations implement entanglement [16,17] so are forbidden if $|AB\rangle = |A\rangle|B\rangle$. We discuss the effects of locally breaking this free-choice symmetry in Section 4 below.

3. Reference Frame Induced Decoherence

3.1. QRFs for System Identification

From A's perspective, the partial state $|B\rangle^A$ encoded on $\mathcal{B}$ is pure: it encodes all of the information about B that is accessible, even in principle, to A. Mixed or decoherent states (we will use these terms interchangeably), in contrast, always indicate a lack of access to information that is in principle accessible: a state ρ^S of S is decoherent if there is some non-null system E such that $\rho^S = \mathrm{Tr}_E \rho^{SE} = \mathrm{Tr}_E |SE\rangle\langle SE|$. In this case, E is the purifier or "environment" of the S and $|SE\rangle$ is the purification of ρ^S by E (see [18–20] for extended discussions). That such a purifying E exists physically, not just formally, for any mixed ρ^S is a fundamental assumption of quantum theory [21,22], sometimes stated as an explicit axiom [23]. From an operation perspective, E comprises degrees of freedom that interact with those of S but that cannot be, or at least are not observed when ρ^S is measured. If ρ^S is, for example, the state of a particle beam, E would include the degrees of freedom of the ion source, the magnetic fields, the ambient vacuum in the beam lines, etc.

The minimal setting employed here avoids the circularity that arises when a system-environment decomposition $B = SE$ is stipulated a priori [24–27] by forcing S to be identified by some QRF implemented by A. As shown in [6], any QRF can be specified by a cocone diagram (CCD), a category-theoretic construction comprising a hierarchical arrangement of Barwise-Seligman Information Flow binary classifiers/classifications $\mathcal{A}_\alpha$ [28] as depicted in Figure 3. These classifiers represent observables in context; namely, each classifier is a conceptual representation $\mathcal{A}_\alpha = \langle event_\alpha, (condition, context)_\alpha, valuation_\alpha \rangle$ (essential details of the concepts and constructions are recalled in Appendix A, and in particular Appendix A.1 for the latter concept) (More generally, these classifiers can be seen as a triad of: (i) events (atomic, observed or experienced, imposition of boundaries, etc.); (ii) conditions/contents/influences as paired with contexts/measurements/detectors; and (iii) valuation. Again see Appendix A.1 for the formal details). Each classifier $\mathcal{A}_\alpha$ is valued in $\{-1, 1\}$ in accordance with its associated operator M^A_α implementing "yes/no questions" as intrinsic to Equation (2) ([6], Section 3.2). In this sense, $\mathcal{A}_\alpha$ may be alternatively regarded as an eigen-classifier for M^A_α.

To construct the CCD, we select a subset $\{M^A_k, \dots M^A_n\}$ of measurement operators and assign to each a binary classifier $\mathcal{A}_k, \dots, \mathcal{A}_n$, respectively, with each requiring a fixed value, $+1$ or -1, from the corresponding operator; the $\mathcal{A}_k \dots \mathcal{A}_n$ thus specify a fixed bit string as input to the CCD. Further binary classifiers, each of which can be thought of as a classical logic gate, are added to form "hidden layers" $\mathcal{C}$; the maps between classifiers are "infomorphisms" as defined in [28] that satisfy the diagram-commutativity requirements for a cocone (see [29,30] for general category-theoretic definitions, Reference [31] for discussion of the cocone as a general representation of complex conditionals, Reference [32] for applications, examples, and discussion of the obvious analogy with artificial neural networks, and [6] for summaries of the relevant definitions as they apply in the current context). As shown in [6], a CCD exists over a subset $\mathcal{A}_k \dots \mathcal{A}_n$ of classifiers if and only if the corresponding subset $\{M^A_k, \dots M^A_n\}$ of binary-valued operators all mutually commute (see [33] for a formal proof). The colimit $\mathbf{C}$ of the CCD encodes the classical "output" of the QRF as a bit string. As formulated in [33], the CCD then becomes manifestly a scale-free context-dependent architecture. Operationally, it can be thought of as a "deep" neural network with re-entrant connections [32]. In the general case these connections are implemented by quantum processes (i.e., by H_A); the intermediate classifiers at each layer $\mathcal{C}$ then implement measurements, with the general form of (2), of the outcomes of these processes.

The channel implemented by the qubits $q_1 ... q_N$ is free of classical noise by definition: there is no external system to provide a noise source [6,8]. It is evident in Figure 3, however, that this channel transmits a fixed classical bit string, e.g., (1, 1, 1, ..., 1) from B to A if but only if A and B share a z axis. Hence we have:

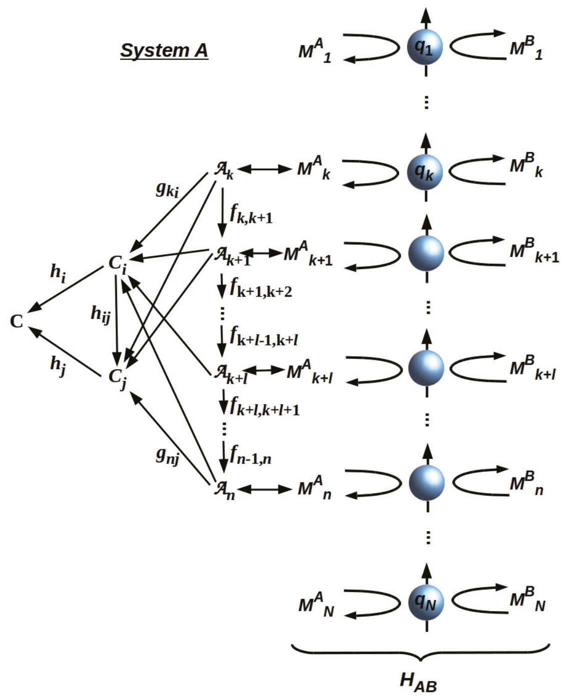

Figure 3. A cocone diagram (CCD) is a commuting diagram depicting maps (infomorphisms) f_{ij} between (eigen-)classifiers $\mathcal{A}_i$ and $\mathcal{A}_j$, maps g_{kl} from the $\mathcal{A}_k$ to one or more channels $\mathcal{C}_l$ over subsets of the $\mathcal{A}_i$, and maps h_l from channels $\mathcal{C}_l$ to the colimit **C** (cf. Equation (6.7) of [32]). Such a CCD can be associated (double-headed arrows) with any subset of binary operators $M_k^A ... M_n^A$ provided that these operators all mutually commute. The CCD specifies, in this case, a classical algorithm implemented by H_A. The complete set of operators M_i^A and M_i^B in (2) together with the array of mutually noninteracting qubits $q_1 ... q_N$ (i.e., the screen $\mathcal{B}$) implement the classical channel between A and B. Free choice of QRFs by A and B corresponds to independent, free choice of z axis by A and B at each qubit. Note that should the CCD fail to commute (in which case the colimit becomes undefined), then the $\mathcal{A}_i$ are considered as "non-co-deployable" (observables), and their corresponding distributed system exhibits intrinsic contextuality ([33], Section 7).

Lemma 2. *The channel implemented by a holographic screen between A and B is free of quantum noise if and only if A and B share QRFs.*

Proof. If A and B share QRFs, each channel qubit is prepared and then measured in the same basis. As the preparation—measurement cycle is effectively instantaneous, prepared and measured outcomes must be the same up to measurement resolution. If A and B do not share QRFs, the preparation and measurement bases for each channel qubit may be arbitrarily different, introducing arbitrary phase rotations, i.e., quantum noise, between B's preparation and A's measurement. □

Shared QRFs correspond to einselection of a preferred basis for decoherence [18,22] at $\mathcal{B}$. We show in Section 4 below that Lemma 2 strongly restricts classical communication, and hence execution of local operations, classical communication (LOCC) protocols [7,34] by spacelike separated observers.

3.2. Reference and Pointer Measurements

The CCD in Figure 3 has a natural physical interpretation: it specifies the hierarchy of logical constraints that must be satisfied to identify the outcome values produced by the operators $M_k^A \dots M_n^A$, and hence the components $k \dots n$ of the pure state $|B\rangle^A$, as the observed (effective or virtual) state ρ^S of an observed (effective or virtual) system S. The state ρ^S of any such S has by convention two components, a time-varying pointer state ρ^P that is of interest as a measurement outcome, and the remaining reference state ρ^R that by remaining fixed over the macroscopic time required for multiple cycles of pointer measurements enables the re-identification of the single, fixed system S with pointer state ρ^P. The pointer state ρ^P here includes not just the traditional "pointer(s)" of S, but also any adjustable "settings" of S that may vary during a sequence of measurements. The reference state ρ^R, in contrast, specifies the fixed properties of the system S that fix its identity and hence allow re-identification over time. If S is a macroscopic item of apparatus, for example, these include both the exterior size, shape, color, brand name, and location required to pick the apparatus out, e.g., by visual search, from the cluttered background of the laboratory as a whole, as well as the internal structural and functional properties that enable it to serve as a measurement device, i.e., as a QRF [10].

Following the notation of [6], we indicate by $\{M_i^P\}$ and $\{M_j^R\}$ the disjoint subsets of $M_k^A \dots M_n^A$ that measure ρ^P and ρ^R, respectively. Call the dimensions of these components $N^P + N^R = N^S$. As ρ^R serves as a fixed reference, clearly $\forall i, j, [M_i^P, M_j^R] = 0$. Pointer state measurements, however, generically do not commute; adjustable apparatus settings, in particular, are useful only to the extent that they do not commute with pointer readings. Hence generically, $\exists i, j,\ [M_i^P, M_j^P] \neq 0$. Call the set of mutually-commuting subsets of $\{M_i^P\}$, and hence of classifiers $\mathcal{A}_i^P$, $\{P_i\}$; in this case a state ρ_i^S is computed by a CCD over RP_i. This decomposition is shown in a simplified notation in Figure 4. Clearly under these conditions the joint state ρ^S must be separable as $\rho^R \rho^{P_i}$, i.e., the system components R and P must be mutually decoherent.

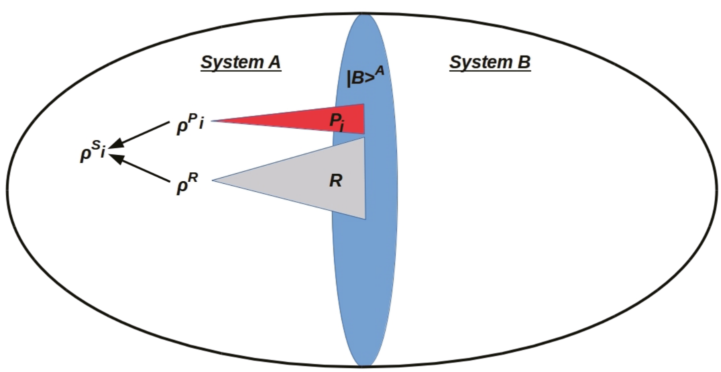

Figure 4. A cocone diagram (CCD) computing an effective (or virtual) "system state" ρ^S comprises classifier channels computing an effective pointer state ρ^{P_i} and an effective reference state ρ^R (cf. [6]). These channels define the effective "subsystems" R and P_i comprising S. The CCD acts on the pure physical state $|B\rangle^A$ encoded by H_{AB} on the holographic screen $\mathcal{B}$ (blue) separating A from B. The computation represented by the CCD is implemented by the internal dynamics H_A.

3.3. Sequential Pointer Measurements Induce Decoherence

State transitions $\mathcal{G}_{ij} : \rho_i^S(t) \to \rho_j^S(t + \Delta t)$, although associative and invertible, in general do not commute, and have a set of multiple identities; hence they can be represented as elements of a groupoid [35,36] $(\{\mathcal{G}_{ij}\}, \circ)$ such that $\mathcal{G}_{ij} \circ \mathcal{G}_{ji} \neq \mathcal{G}_{ji} \circ \mathcal{G}_{ij}$ if and only if $[M_i^P, M_j^P] \neq 0$ [6]. The action of $(\{\mathcal{G}_{ij}\}, \circ)$ on this set of system states, indexed by a macroscopic discrete time τ, is illustrated in Figure 5 (for the formal definition of the action of a groupoid on a set, see, e.g., ([36], Section 10.4)).

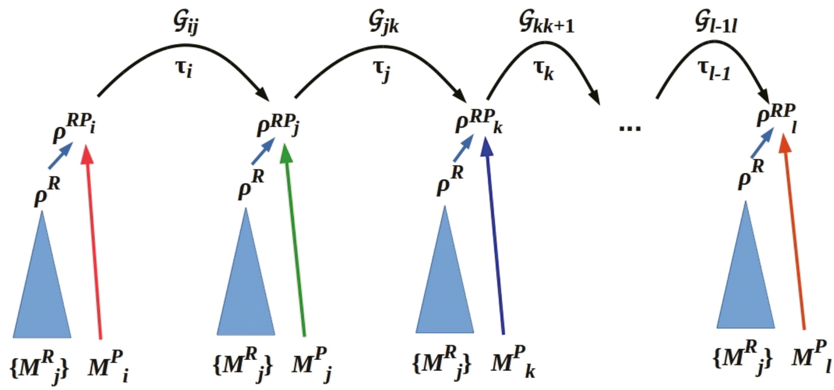

Figure 5. A sequence of CCDs identifying R (blue triangles) and measuring pointer components $P_i, P_j, P_k \dots P_l$. Transitions between CCDs are implemented by groupoid elements, e.g., $\mathcal{G}_{ij}$ and labeled by discrete macroscopic times τ_i. The operators M^P_i can equally well be generalized to subsets $\{M^P\}_i$ of mutually-commuting pointer-state observables.

A reference state ρ^R computed by a CCD R from the outcome values of a set of operators M^R_j is, effectively, a logical constraint on the identities of the qubits q_j that the M^R_j measure. Hence we have:

Lemma 3. *In any system AB characterized by (2), fixing a reference state ρ^R over a macroscopic time interval τ locally breaks the S_N symmetry of the holographic screen $\mathcal{B}$ encoding the eigenvalues of H_{AB}.*

Proof. Suppose the M^R_j measure the states of $N^R = n - k$ qubits as shown in Figure 3; we can neglect P and assume that the other qubits constitute the environment and are swap-symmetric. Holding ρ^R fixed for τ is holding the N^R outcomes of the M^R_j fixed for τ; $\forall j, M^R_j|q_j(t)\rangle = |1\rangle$ or $|-1\rangle$ for t within τ. This cannot be guaranteed if q_j is swapped for some environmental q_i with an unconstrained state; hence any such swap must be forbidden by a "selection rule." This breaks the S_N symmetry on $\mathcal{B}$. □

Lemma 3 is in fact obvious: the CCD R assigns each of the physical degrees of freedom q_j a specific role in the computation of ρ^R, one that an arbitrary qubit in an arbitrary state cannot satisfy. The qubits q_j are classically phase-locked by R, while the phases of the environmental qubits can vary freely, preserving their swap symmetry. The CCD R effectively divides $\mathcal{B}$ into two (not necessarily simply connected) regions, one in which the qubits are classically phase-correlated and the other in which they are not. Any such division induces decoherence between non-swap-symmetric and swap-symmetric qubits. These conditions equally hold for any CCD measuring a pointer state ρ^{P_i}.

Lemma 3 associates decoherence with system identification as a necessary prerequisite of pointer-state measurement. As Zurek emphasized ([22] p. 1794),

> [T]he formulation of the measurement problem and its resolution through the appeal to decoherence require a universe split into systems. Yet, it is far from clear how one can define systems given an overall Hilbert space 'of everything' and the total Hamiltonian.

Subsequent work demonstrated that no preferred decomposition of an overall Hilbert space or its Hamiltonian could legitimately be assumed a priori [37,38], rendering all formulations of decoherence that assumed an a priori $B = SE$ decomposition circular (see [24–27] for relevant discussion). By characterizing all observations as mediated by a holographic screen $\mathcal{B}$, the GHP localizes the $B = SE$ decomposition to the observer's QRFs [5,6]. All systems S and states ρ^S are, therefore, virtual in the precise sense of computational processes implemented on underlying, observationally inaccessible hardware [39]: the observer A itself with its Hamiltonian H_A.

3.4. Example: Mass and Hawking radiation QRFs for a BH

Suppose A employs a local QRF R_{BH} (e.g., a local sample of the ambient photon field) to measure both the position x and the mass M of a distant BH B and a particle detector R_r to measure the momentum $\vec{p}$ of one or more quanta of radiation. Her task, familiar from discussions of horizon complementarity [40] and the firewall paradox [15], is to determine whether her local quantum r is a Hawking quantum r_H from B (see [11–14] for relevant discussion). As illustrated in Figure 6, answering this question requires a QRF R_H that associates $\vec{p}$ with an identified Hawking quantum.

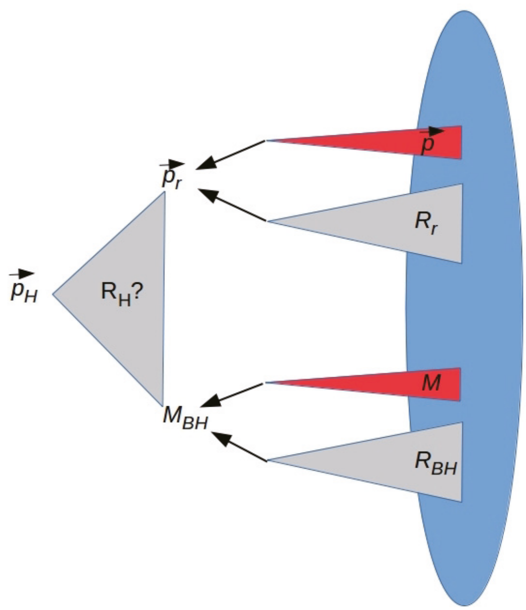

Figure 6. Identifying a local quantum of radiation as a Hawking quantum r_H from a distant BH requires a local Hawking QRF R_H. Lemma 3 rules this out.

Lemma 3 shows that the required R_H cannot be implemented, even in principle: determining that the BH has lost information requires observation over macroscopic time, inducing decoherence on $\mathcal{B}$. Hence not only is A prevented by $\mathcal{B}$ from obtaining information about the BH entanglement entropy $\mathcal{S}(B)$ (Lemma 1); she cannot obtain entanglement information about identified systems if distinct QRFs are required for their identification. The entanglement entropy $\mathcal{S}(|B>|r>)$ is, in particular, experimentally inaccessible even in principle. Horizon complementarity is, therefore, not required to prevent observations of no-cloning violations by Hawking quanta; such observations cannot be made because the QRF needed to identify an observable BH as the source of the quanta is unavailable. Thought experiments in which observers measure entanglement entropies before and after falling into a BH, as employed in, e.g., [15], are unrealizable even in principle.

The limitation imposed by Lemma 3 generalizes, via the ER = EPR hypothesis, to any system with a spatially-distributed purifier, e.g., an "octopus" BH topologically connected to its entangled Hawking quanta by ER bridges [11,41]. The complete system state is pure but unobservable in principle, as the QRFs required to localize the spatially-distributed components would induce decoherence separately on each component. This problem of QRF-induced decoherence in spatially-distributed purifiers is similarly relevant to treatments of potential entanglement effects surviving the inflationary epoch, e.g., [42–45]. Bell-type communication protocols, e.g., [46–49], circumvent this problem by employing classical communication, treated as an a priori preferred QRF, to provide localization information as discussed in Section 4 below.

3.5. Computation and Memory Costs Induce Coarse-Graining

Provided their intermediate states are not recorded to a persistent, classical memory, logically reversible computations can in principle be performed without net energetic cost; logically irreversible computations, in contrast, cost at least ln2 $k_B T$ per bit [50,51]. What is of interest in practice, however, is the incremental cost of computation, including the cost of writing intermediate states to a classical memory, even if the computational step is to be reversed later. An observation can only be considered to have been "made" if the result is written to a classical memory from which it can later reported, e.g., classically communicated to another observer [52]. A system S can, in particular, only be regarded as "observed" at a time t if its reference state $\rho^R(t)$ is written to a classical memory from which it can be reported. System identification over macroscopic τ clearly requires writing to and reading from such a memory as discussed further in Section 6 below.

The free energy required to fund the incremental cost of computing and recording must be supplied by what Landauer [50] called the "non-information-bearing degrees of freedom" of the computer and/or its environment, even if this free energy is repaid in part later. Viewed on the output side, i.e., in terms of A's action on B, these non-information-bearing degrees of freedom exhaust the waste heat generated by the computing and recording processes. This distinction between information-bearing and non-information-bearing degrees of freedom breaks S_N symmetry as discussed above. As shown in [6], this symmetry breaking can be expressed thermodynamically as the requirement that $\beta^R, \beta^P > \beta^E \geq \ln 2$, where β^E is the efficiency of the operators M_k^E acting on E. The environment E provides, in other words, the incremental free energy required to irreversibly identify R and measure P. The fuel value $\beta^E k_B T$ is independent of the bit value (+1 or -1) of the outcome; hence these outcomes are "non-information-bearing" for the computation implemented by H_A. They therefore retain full permutation symmetry, justifying the trace over their joint state.

Any classical computation can be performed reversibly, e.g., with Toffoli gates, and any reversible computation can be performed with some unitary operator [46]. The only obligate classical steps in computing either ρ^R or ρ^{P_j} are, therefore, the initial step of writing the "input" outcomes of the M_j^R and the selected M_i^P onto the relevant classifiers and the final step of writing the time-stamped joint state specification ρ^{RP_j} on a classical memory. The criterion of classicality for the memory is operational: the time-stamped state specification must be reportable at any later time without disturbing other processes. For a perfectly efficient system, the free energy required to write each (Reference, Pointer) outcome ρ^{RP_j} to memory is, therefore:

$$\Delta H_j \geq \ln 2(N^R + N^P + N^\rho + N^\tau) k_B T, \tag{3}$$

where $N^R + N^P$ is the number of bits required to record the inputs, N^ρ is the number of bits required to record ρ^{RP_j}, N^τ is the number of bits required to record the timestamp. This incremental ΔH_j must be supplied by E during each interval τ_j, so (3) places a lower limit $N^E \geq N^R + N^P + N^\rho + N^\tau$ on the number of qubits of E and therefore on the total area $A(\mathcal{B})$ of the holographic screen $\mathcal{B}$. In any realistic system thermodynamic efficiency is less than ideal; hence $\beta^E > \ln 2$ and $A(\mathcal{B})$ is correspondingly larger.

As $N^R + N^P$ remains fixed, ΔH_j is minimized as $N^\rho + N^\tau \rightarrow 0$, i.e., as classical memory is coarse-grained relative to $\mathcal{B}$. We can, therefore, generically expect QRFs to encode high redundancy over states $|B\rangle^A$ mapped to the same ρ^{RP_j}, i.e., we can expect any CCD implementing a QRF to include logical OR gates and hence to lose information. Optimal coarse-graining jointly minimizes the cost of memory and the cost of redundancy. Furthermore, any QRF that is coarse-grained engenders redundancy and can be considered as a quantum error correcting code (QECC) [46]. This is relevant to the discussion in Section 3.4 above: a QEEC can be used to reconstruct local effective field theory observables, which as pointed out in [53], are applicable to BH states whose entanglement entropy falls short of saturating the Bekenstein-Hawking bound. Such local observables are designed to protect coherence in the Hilbert space of codes by correcting errors due to the emission of Hawking quanta, by entangling radiation within other regions of Hilbert space and inducing entanglement swaps that increase the entanglement entropy of the BH interior over time. As discussed in Section 3.4 above, such postulated entanglement swaps are unobservable even in principle.

4. Reference Frame Induced Entanglement

Communication protocols that employ shared entanglement depend on shared QRFs, e.g., shared z-axis QRFs for s_z measurements [8]. This suggests that the shared entanglement is in fact induced by the shared QRFs, a suggestion consistent with the general observer-dependence of entanglement [37,38].

Consider a Bell protocol described in the lab frame, as shown in Figure 7. An entangled state is distributed from a source to Alice and Bob, who remain spacelike-separated throughout the protocol's operation. They are free to adjust their detector settings during the interval Δt_{set}. Following data processing (the interval Δt_{proc}), Bob sends his classically-encoded measurement outcomes to Alice via a classical channel. Alice can then compute the joint statistics, obtaining a Bell-inequality violation and hence an observation of entanglement at t_{meas}. Alice's ability to compute the joint statistics, and hence to observe entanglement, critically depends on two assumptions. First, Alice must know the code that Bob employs to encode his results; effectively, they must "speak the same language". Second, the communication from Bob to Alice must be classical, i.e., must not involve a quantum measurement [7]. If the communication is not considered classical, i.e., if Bob sends Alice a QRF with which he is entangled, Alice must identify the transmitted QRF in order to measure its state, inducing decoherence as discussed in Section 3.4 above. These two assumptions are operationally equivalent: Alice scrambling Bob's message by decohering a transmitted quantum state has the same effect as Alice scrambling Bob's classical coding scheme by employing, e.g., an obsolete one-time pad. In either case, Alice does not "understand" Bob's encoded results and her subsequent statistical analysis is meaningless [33].

The assumption of classical communication is, effectively, the assumption of a preferred pointer measurement that returns the content of the communicated message without requiring prior identification, via a separate measurement, of the physical medium, i.e., the QRF, via which the message has been transmitted. Alice, in other words, does not have to identify Bob to receive his message, just as Wigner does not, in his famous thought experiment, have to identify his friend to receive his friend's observational outcomes [54]. Hence assuming classical communication is assuming an a priori shared QRF [8]. This breaks the free-choice symmetry across $\mathcal{B}$ as discussed in Section 2.3 above; if $\mathcal{B}$ is considered a qubit array as in Figure 3, the assumption of classical communication is the assumption that A and B use identical z-axis QRFs on a subset of qubits as confirmed by Lemma 2 above. Call this subset of qubits *message*; the observed states $|message\rangle^A$ and $|message\rangle^B$ are superdeterministically correlated. Choosing a decomposition that identifies the shared QRFs shows that $|message\rangle = |message\rangle^{AB}$ is a single pure state. That such a pure state exists is the operational meaning of the requirement that Alice and Bob "share a language" for classical communication.

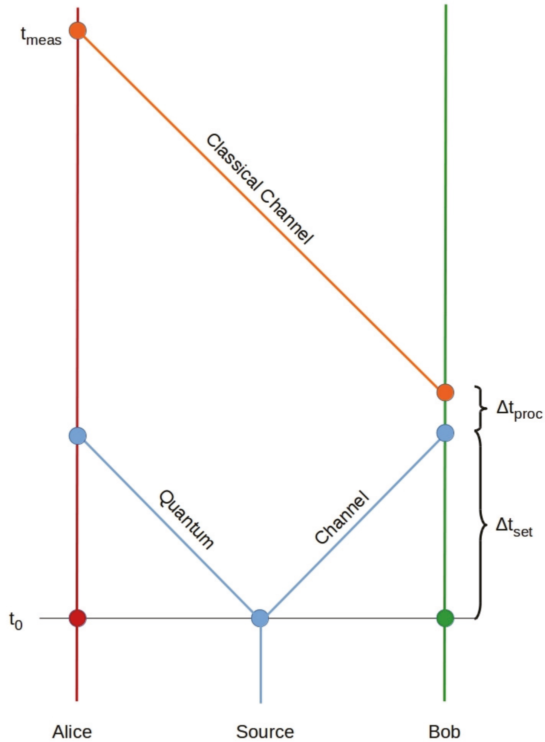

Figure 7. A typical Bell protocol described in the lab frame. Sharing of measurements results via a classical channel is required to observe a Bell-inequality violation. If Alice's interaction with Bob's message is viewed as an ordinary quantum measurement, the entanglement disappears as in Section 3.4 above.

Redescribing the Bell protocol in the frame of the entangled state, as illustrated in Figure 8, makes both the shared QRF and its entangled state manifest. Hence we have:

Theorem 1. *Sharing entanglement requires shared entanglement.*

Proof. Spacelike-separated Alice and Bob can observe entanglement only if they can compare their observational outcomes. By Lemma 2, this requires an a priori shared QRF. Classical transfer of a QRF also requires an a priori shared QRF [8]; hence the shared QRF can only be shared as an entangled state. □

Superdeterminist correlations, i.e., absence of free choice of QRFs, is a general feature of LOCC communication protocols. In the Bell protocol, Alice's response to the bit string received from Bob is predetermined by the requirement of a shared QRF. Other protocols superdetermine the "choices" made during Δt_{set}, and hence the outcomes observed. Entanglement-enabled secure communication protocols, for example, require Alice and Bob to deploy QRFs and execute measurement on an otherwise-uncharacterized qubit in the order specified by the protocol. These protocols avoid decoherence, and hence enable quantum communication, by rendering Alice's and Bob's QRFs effectively entangled for the duration of the protocol. Here again, avoiding decoherence is equivalent, operationally, to sharing a language in which, e.g., protocol instructions are classically communicated.

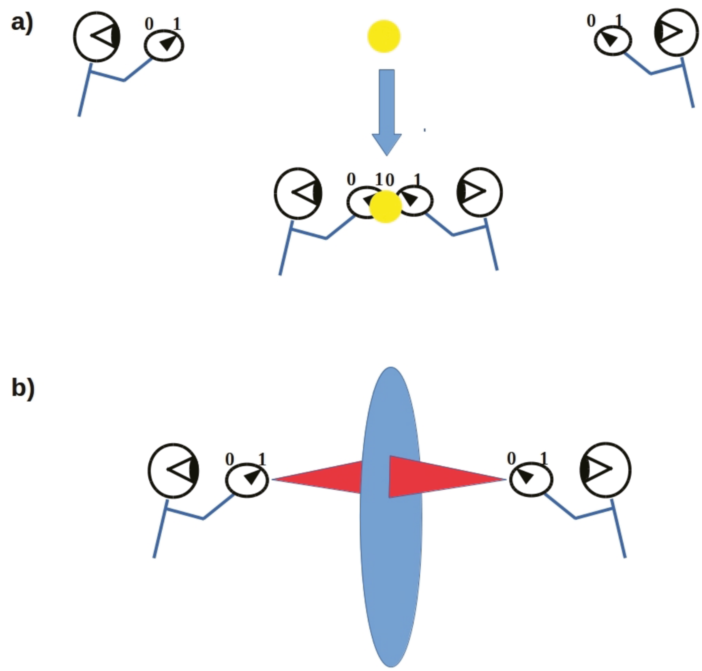

Figure 8. (**a**) A Bell protocol in the frame of the entangled state (yellow circle). Alice and Bob collide at t_{meas}, at which time they share, and together measure, the entangled state. (**b**) This is equivalent to Alice and Bob sharing an entangled QRF that reports consistent pointer outcomes to each observer.

The special role played by classical communication in LOCC protocols has been investigated previously in extended Wigner's friend experiments in which the outcomes of classical communication between pairs of "observers" and "friends" are contrasted with the outcomes of quantum measurements of "friends" by "observers" [55,56]. These experiments have been interpreted as showing, subject to an assumption of no superdeterminism, that observed events cannot be regarded as observer-independent. By treating all observed events as relative to observer-implemented QRFs, we show here that the assumption of classical communication between observers, widely regarded as physically inconsequential prior to [55,56], is to enforce local superdeterminism.

5. Reference Frame Induced Contextuality

Contextuality and entanglement are conceptually equivalent [57]. For a fixed P, switching between QRFs over M_j^R and M_k^T, where $[M_j^R, M_k^T] \neq 0$ for at least one pair j, k induces contextuality, i.e., no non-contextual probability distribution consistent with the Kolmogorov axioms and hence with Dutch-book coherence can be defined over the combined set of outcomes [6,33].

Consistent with the findings of [55,56], no Kolmogorov-consistent, non-contextual probability distribution can be defined over the combined outcomes of Alice's and Bob's observations unless it can be demonstrated, for the relevant P, and for R and T the QRFs deployed by Alice and Bob, respectively, that $\forall j, k$, $[M_j^R, M_k^T] = 0$. This cannot, however, be demonstrated by any observer of Alice and Bob, as no such observer has, by Lemma 1, access to the internal Hamiltonians H_{Alice} or H_{Bob}.

This result can be stated in more formal terms of undecidability.

Theorem 2. *Whether arbitrarily-chosen QRFs R and T compute the same function f is finite Turing undecidable.*

Proof. Let f designate the function computed by R, i.e., the function computed by the CCD representing R; we then ask whether T computes this same f. Whether an arbitrarily chosen computer computes any non-trivial function f cannot, however, be decided by any finite Turing machine [58]. Hence whether T computes f is finite Turing undecidable. □

As shown in [33], contextuality induced by non-commuting QRFs renders the Frame problem, the problem of circumscribing the degrees of freedom that do not change their values as the result of an action, e.g., a measurement [59] unsolvable even in domains with small numbers of degrees of freedom (cf. [60]).

Theorems 1 and 2 have as an obvious corollary:

Corollary 1. *Whether two observers share an entangled state is finite Turing undecidable.*

Hence whether Alice and Bob have successfully completed a quantum communication protocol is finite Turing undecidable.

6. Writing and Reading Classical Memories

As noted earlier, a sequence of observations made over macroscopic time is only reportable at some later time if the observations have been recorded on a classical memory. Distinguishing measurements made at different times requires, moreover, some method of distinguishing the memories. We therefore assume that the bit string composing each memory record includes a substring encoding a time stamp τ_j, which we take to be generated by the groupoid action of the $\mathcal{G}_{ij}$. Considering this classical memory to be implemented by H_A would prima facie require internal decoherence, i.e., disrupt the purity of $|A\rangle$. This can be avoided if A is regarded as writing all classical memories on $\mathcal{B}$. As the result to be written to memory is coarse-grained compared to the input from which it was generated (Section 3.5), only a relatively small number of qubits on $\mathcal{B}$ need be devoted to memory.

Reversing the arrows in a CCD yields the dual construction, a cone diagram (CD) with the single source classifier the limit over the bottom-level classifiers [32]. A CD can be constructed to encode any finite bit string on an underlying bit array, i.e., to write any finite bit string to memory. Regarding each memory bit as a preparation instruction for a corresponding qubit on $\mathcal{B}$, we can represent a memory-write operation to $\mathcal{B}$ as in Figure 9.

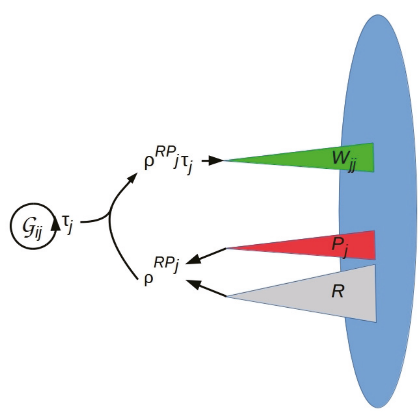

Figure 9. A CD W_{jj} (green triangle) specifies a memory-write operation of the time-stamped state (ρ^{RP_j}, τ_j) to $\mathcal{B}$. The timestamp τ_j is generated by the groupoid action $\mathcal{G}_{ij}$.

Reading the memory reverses the arrows on the memory-write CD W_{jj} to a CCD, i.e., a QRF for retrieving the time-stamped value (ρ^{RP_j}, τ_j). Writing readable memory records on $\mathcal{B}$ imposes phase correlations across time on $\mathcal{B}$; such correlations obviously further disrupt the S_N symmetry of $\mathcal{B}$. Reading and rewriting memory records also imposes an energetic cost as in Section 3.

7. Conclusions

We have investigated, in this paper, a generalization of the HP in which interactions H_{AB} between finite quantum systems A and B that maintain a separable joint state are represented as exchanges of information across a holographic screen $\mathcal{B}$. While the role of $\mathcal{B}$ is ancillary to the action of H_{AB}, the permutation symmetry of $\mathcal{B}$ is broken when the internal Hamiltonian H_A is considered to implement QRFs that identify "systems" and measure their states. This symmetry breaking induces decoherence of identified systems by forcing the "environment" that remains to serve as both free energy source and waste heat sink. Observable entanglement, contextuality, and classical memory are, in this representation, logical and temporal relations between QRFs implemented by H_A.

It is natural to interpret the holographic screen $\mathcal{B}$ not merely as ancillary, but as a "physical" space, i.e., a stretched horizon, separating A from B. In this context, broken permutation symmetries on $\mathcal{B}$ become broken exchange symmetries between points in the $2+1$ spacetime defined by $(\mathcal{B}, \tau)$, τ a characteristic "macroscopic" time scale for H_{AB} as above (see [61] for details). From A's observational perspective, exchange-inequivalent regions of $(\mathcal{B}, \tau)$ correspond to coarse-grained, decoherent "systems" while exchange-symmetric regions are "empty space" that supplies free energy and exhausts waste heat. That the GHP itself forces these virtual decoherent systems to obey gauge symmetries is shown in [61].

It is tempting to speculate that a third spatial dimension is induced when, but only when, A implements QRFs capable of identifying single systems across time while varying pointer observables such as size, shape, and color, and that inertial mass is a QRF representing the observable response of an identified system to actions by A. Whether the fundamental symmetries of space, time, and matter, or even all of physics can be completely reconstructed "within" an observer A, and hence viewed as a computation implemented by H_A, remains to be determined. The fact that physics is done by physicists, systems that appear to interact with their environments via a Hamiltonian of the form (2), suggests that such a reconstruction is possible; possible routes forward are discussed in [62–64].

It is, finally, increasingly being suggested that the entropic structure of a BH may be more than a phenomenological correlate of its mass, possibly providing a route toward defining mass [65], specifying nontrivial internal topological structure [11], or even generating the phenomenology of dark energy or dark matter [66]. Theoretical investigation of the QRFs implemented by a BH may, therefore, offer exciting future developments.

Author Contributions: Conceptualization, C.F., J.F.G., A.M.; writing—original draft preparation, C.F.; writing—review and editing, C.F., J.F.G., A.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Federico and Elvia Faggin Foundation (CF), the Shanghai Municipality, Grant No. KBH1512299 (AM), and by Fudan University, Grant No. JJH1512105 (AM).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data reported.

Acknowledgments: JFG wishes to thank J. McLennan for discussions on related topics.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

BH	Black Hole
CCD	Cocone Diagram
CD	Cone Diagram
EPR	Einstein-Podolsky-Rosen
ER	Einstein-Rosen
GHP	Generalized Holographic Principle
LOCC	Local Operations, Classical Communication
QECC	Quantum Error-Correcting Code
QRF	Quantum Reference Frame

Appendix A. The Basics of Channel Theory Information Flow and Context Dependency

The Channel Theory of [28] introduces the idea of a "classifier" (or "classification") as accommodating a "context" in terms of its constituent "tokens" in some language and the "types" to which they belong.

Definition A1. *A classifier $\mathcal{A}$ is a triple $\langle Tok(\mathcal{A}), Typ(\mathcal{A}), \models_{\mathcal{A}}\rangle$ where $Tok(\mathcal{A})$ is a set of "tokens", $Typ(\mathcal{A})$ is a set of "types", and $\models_{\mathcal{A}}$ is a "classification" relation between tokens and types.*

Note that this definition specifies a classifier/classification as an object in the category of Chu spaces [67–69] where '$\models_{\mathcal{A}}$' is realized by a satisfaction relation valued in some set **K** (with no structure assumed). The arrows (morphisms) between classifiers are specified by the following:

Definition A2. *Given two classifiers $\mathcal{A} = \langle Tok(\mathcal{A}), Typ(\mathcal{A}), \models_{\mathcal{A}}\rangle$ and $\mathcal{B} = \langle Tok(\mathcal{B}), Typ(\mathcal{B}), \models_{\mathcal{B}}\rangle$, an infomorphism $f : \mathcal{A} \to \mathcal{B}$ is a pair of maps $\overrightarrow{f} : Tok(\mathcal{B}) \to Tok(\mathcal{A})$ and $\overleftarrow{f} : Typ(\mathcal{A}) \to Typ(\mathcal{B})$ such that $\forall b \in Tok(\mathcal{B})$ and $\forall a \in Typ(\mathcal{A})$, $\overrightarrow{f}(b) \models_{\mathcal{A}} a$ if and only if $b \models_{\mathcal{B}} \overleftarrow{f}(a)$.*

Information is inherently a physical mode of distinctions and relationships between them, and not simply a reduction to a quantity of bits as it would be for Shannon information that passively neglects the substance of reasoning. Rather, it instead conforms to the set of logical constraints as imposed by Definition A2. An infomorphism as a mapping between classifiers provides the basic building blocks towards constructing multi-level, quasi-hierarchical classification systems. Such a framework of information transfer is indicative of causation, which itself may be viewed as a form of computation in view of the regular relations in a distributed system [70]. References [6,32,33] bring to the forefront many examples, and applications of the above concepts that include probability distributions, Bayesian belief networks, event space structures, formal concept analysis, and fuzzy relationships (as further relevant to this issue, let us point out that the Sorkin model of spacetime causal sets [71,72] has been interpreted in terms of classifiers (Chu spaces) in [73] (reviewed in [32])). In particular, Reference [33] focuses on orders of contextuality with ramifications to active inference and to the Frame Problem.

The specifics of transmitting information via classifiers and infomorphisms lead, in [28], to defining the idea of an information channel over classifiers, the most general of which leads to the categorical notion of a cocone with the core **C** the colimit of all possible upward-going structure-preserving maps from the classifiers $\mathcal{A}_i$. Such a colimit core, when it exists, can be regarded as a classifier which embraces the totality of information that is common to the component classifiers $\mathcal{A}_i$. The resulting structure is a cocone diagram (CCD) as exemplified Figure 3. Within such a framework, the means by which channels encode sets of mutual constraints between classifiers is regulated by a local logic as presented formally in ([28], Ch. 12) (reviewed in [32,33]). Basically, the idea is that the types of a (regular) theory specify the logical structure of a given situation. A local logic is essentially a classifier having a (regular) theory along with a subset of tokens that satisfy all constraints of the theory as specified by a sequent (a sequent $M \models_{\mathcal{A}} N$ of a classifier $\mathcal{A}$ is a pair of subsets M, N of $Typ(\mathcal{A})$ such that $\forall\, x \in Tok(\mathcal{A})$, $x \models_{\mathcal{A}} M \Rightarrow x \models_{\mathcal{A}} N$). An infomorphism preserving this additional logical structure is then promoted to a logic infomorphism. In short then, a local logic "identifies" the token(s) satisfying all of the types, the logic infomorphisms are those infomorphisms that transfer token-identification information between local logics, and the channels comprise sets of (logic) infomorphisms encoding mutual constraints that assemble multiple identified tokens. As demonstrated in [74], a sequent of a theory can be weakened to a conditional probability such that a CCD becomes a network of hierarchical Bayesian inference, as reviewed and formulated in [32,33] (cf. [75]), and whose structure is compatible with the variational free energy principle as the latter is fundamental to the precision of perceptual inference [76] (the sequent relation can be weakened by requiring only that if $x \models_{\mathcal{A}} M$, there is some probability $Prob(N|M)$ that $x \models_{\mathcal{A}} N$. Essentially it is how a conditional probability interprets the logical implication "$\Rightarrow$" [77]).

Appendix A.1. Example: Observables in Context

One fundamental example incorporating "context", instrumental in [33] has the following Chu space ingredients: Consider the following countable (in practice, finite) sets:

(i) A a set of "events" (in the general sense of the term, e.g., as observed value combinations or atomic), as related to
(ii) a set B of conditions specifying "objects/contents" or "influences," and
(iii) a set R of contexts (or, in certain instances, a set of "detectors", "measurements" or "methods").

The set B can be decomposed as $B = B^M \cup B^C$ (disjoint union), where B^M contains "objects/contents" or "degrees of freedom" that are observed or measured in some event $a \in A$, and B^C contains what is not observed in the events in A. This leads to defining a 'large' space,

$$X := B \times R = (B^M \cup B^C) \times R, \tag{A1}$$

in assuming that A, B and R are subsets of the same (even larger) probability space $\mathcal{P}$ (We do not make any assumptions about corresponding types of probability distributions (e.g., discrete versus continuous) in relationship to $\mathcal{P}$. Neither do we specify the nature of random variables, nor the possible orders of "connectedness" (of distributions)). Thus, based on this data we consider the classifier,

$$\mathcal{A} = \langle A, X, \models_{\mathcal{A}} \rangle, \tag{A2}$$

as comprising observables in context, where as in Section 3, the classification relation '$\models_{\mathcal{A}}$' is realized by the Chu space valuation in the set $\mathbf{K} = \{-1, 1\}$. Notably, in [33], '$\models_{\mathcal{A}}$' can be realized for an inferential process by the conditional probability $p(a|x) = p(a|\{b, c\})$, whenever defined, for $a \in A, b \in B$ and $c \in R$, and which for suitable indexing, leads to an information flow of hierarchical Bayesian inference within a CCD [33]. The background to the results in Section 5 here can be found in ([33], Section 7). In particular, ([33], Th. 7.1) states the criteria for intrinsic contextuality (non-co-deployable observables) in terms of noncommutativity of a CCD. Note that the above classifier (Chu space) formulism of contextuality is very general. Special cases of the set $X = B \times R$ are the sets of binary random variables labelled by a measurement (contents-context) system as basic to the theory of Contextuality-by-Default [78,79]. Much amounts to the question of determining the nature of an empirical model e relative to how a probability distribution can be obtained as the marginals of a global probability distribution on the outcomes to all measurements. For example, e is said to be contextual in [80] if the corresponding probability distribution cannot be obtained by such global means. This has a compatible interpretation in terms of the non-existence of a global section of a sheaf defined relative to a "measurement cover" in [81]. These methods of studying contextuality are also very general, and as for those of [33], can extend beyond quantum theory to such disciplines as linguistics and psychology. To see the explicit connections between these various approaches would indeed be a worthwhile undertaking.

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ISBN 978-3-0365-5392-4