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Advance in Topology and Functional Analysis

In Honour of María Jesús Chasco's 65th Birthday

Edited by
Elena Martín-Peinador, Xabier Domínguez,
T. Christine Stevens and Mikhail Tkachenko

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Mikhail Tkachenko did his Ph.D. dissertation under the supervision of Prof. Alexander Arhangel'skii at Moscow State University. Since 1993, he has been a Titular Professor in the Department of Mathematics at the Autonomous Metropolitan University (UAM-I) in Mexico City. His research interests focus on topological algebra and general topology.

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Editorial

Advance in Topology and Functional Analysis in Honour of María Jesús Chasco's 65th Birthday

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1. Introduction

We are honoured to present this Special Issue of *Axioms* with the title "Topology and Functional Analysis" to showcase recent work on this and related topics and to provide an opportunity for María Jesús Chasco's friends and colleagues to pay tribute to her mathematical career on the occasion of her 65th birthday. This issue includes significant papers dealing with topological groups, topological semi-groups and topological vector spaces. Many of them exploit the rich and fruitful interaction between topology and functional analysis, which has been a wellspring of powerful mathematical ideas and development since the early stages of both disciplines.

One of the many important programs originating within this framework can be described as borrowing some of the tools and concepts of topological vector space theory to study the structure and duality properties of Abelian topological groups. Such a viewpoint turns out to be particularly useful, for instance, when dealing with Pontryagin duality and reflexivity outside the class of locally compact groups. It should be noted that the notion of convexity admits a counterpart in the field of Abelian topological groups. Inspired by the Hahn–Banach theorem, Vilenkin introduced in [1] the notion of a quasi-convex subset of an Abelian topological group, which immediately led to the definition of locally quasi-convex groups. With these objects at hand, it is natural to extend well-known theorems from the class of locally convex spaces to the broader class of locally quasi-convex groups.

María Jesús Chasco completed her doctoral dissertation under the direction of Antonio Plans while she was working as a high-school chair. Her first research was in Hilbert space theory, and the defense of her thesis took place at the University of Zaragoza in 1985.

Soon after, she obtained a position as a Professor at the Department of Mathematics at the Engineering School of the University of Vigo, where she remained for 7 years. She was involved in multiple collaborations with her colleagues in the Department of Mathematics, and became director of the department for some time. She encouraged visits from numerous professors from other countries who contributed to creating a fruitful scientific environment, which attracted students to attend the university to write their doctoral dissertations. Concretely, she was coadvisor of the theses of Ricardo Vidal and Xabier Domínguez.

In 1997, she obtained a professorship at the University of Navarre, in Pamplona, her native town, where she has remained ever since. Her brilliant work there spanned fruitful research, pedagogical tasks that included advising the theses of Hugo Bello and Carlos Bejines and administrative positions as the Vice Dean of the Faculty of Sciences.

As a university professor, she has enjoyed the interaction with her students, who recognize her generous dedication to them and her unconditional availability to help solve their problems.

Mª Jesús has a special quality of discovering beautiful problems in mathematics and sharing ideas with colleagues. She has published more than 40 papers, mainly in topological groups, but also in functional analysis dealing with locally convex spaces and Banach spaces. Her favourite topic is duality in topological groups. She knows how to extract the best from her colleagues and discover talent wherever it is. She can easily make a team work, being a loyal friend that always speaks the truth.

She also has deep convictions which guide her actions, not falling into relativism or preconceived opinions. Friendship is one of her highest values. She enjoys travelling, art, nature and beauty wherever it is. Thus, we expect to keep sharing her enthusiastic attitude to life, her friendship and new results in mathematics for many years.

2. Overview of the Published Papers

Large-scale topology or, in other words, the study of coarse structures, is currently an important area of topology, with essential geometric and combinatorial connections [2]. In the natural coarse structures associated with any given group, the discrete subsets are exactly the so-called thin subsets: a subset X of a group G is called thin if given any finite subset F of G, one has both $Fx \cap Fy = \emptyset$ and $xF \cap yF = \emptyset$ for all but finitely many different $x, y \in X$. The paper *On factoring groups into thin subsets* (Contribution 1 by I. Protasov) is devoted to the proof of the following factorization result: Every Hausdorff nondiscrete countably infinite topological group G has two thin subsets A and B such that every $g \in G$ can be uniquely expressed as a product $g = ab$ with $a \in A$ and $b \in B$.

The contribution *Factoring continuous characters defined on subgroups of products of topological groups* (Contribution 2 by M. Tkachenko) mainly deals with extensions of characters defined on such subgroups to the whole product space. For precompact Abelian subgroups (without requiring the Hausdorff property of the spaces involved), a nice result is obtained, which includes a factorization theorem (Theorem 4). It is well known that factorization theorems constitute a powerful tool to study continuity of functions defined on products of topological spaces. The author provides interesting examples and poses the problem of whether precompact subgroups of products of paratopological Abelian groups are dually embedded (Problem 1).

In *A distinguished subgroup of compact Abelian groups* (Contribution 3), D. Dikranjan, W. Lewis, P. Loth and A. Mader consider the family of all subgroups Δ of a compact abelian group G that are compact, totally disconnected and such that G/Δ is a torus. The sum of all the subgroups with these properties is a functorial subgroup $\Delta(G)$ that is dense, zero-dimensional and such that the quotient $G/\Delta(G)$ is torsion-free and divisible. Using these ideas, the authors survey and extend earlier results on the resolution theorem for compact Abelian groups and about minimal groups.

Aggregation operators are an essential tool in science and engineering due to the ubiquitous necessity of combining several input values into a single value. In the paper *On self-aggregations of min-subgroups* (Contribution 4 by C. Bejines, S. Ardanza-Trevijano and J. Elorza), the authors study the preservation of the min-subgroup structure under aggregation functions. Min-subgroups of a group G are fuzzy sets μ with domain G satisfying $\mu(x) = \mu(x^{-1})$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for any $x, y \in G$. P. Das proved that a fuzzy set of G is a min-subgroup of G if and only if all its nonempty level sets are subgroups of G. He also introduced a natural equivalence relation between min-subgroups in terms of their level sets. The main result of this paper is the following: If G is a group and $A : [0,1]^n \to [0,1]$ is an aggregation function, then $A(\mu, \ldots, \mu)$ and μ induce the same level sets for every min-subgroup μ of G if and only if A is strictly increasing on its diagonal; that is, $A(x, \ldots, x) < A(y, \ldots, y)$ whenever $x < y$. From this result, it follows that for any min-subgroup μ of G, $A(\mu, \ldots, \mu)$ and μ belong to the same Das class whenever A is a strict t-norm or a strict t-conorm.

The notion of boundedness is a fundamental tool in many fields of mathematics, particularly in the framework of topological vector spaces. In 1959, Hejcman dealt with boundedness in uniform spaces and in topological groups [3]. The article *Bounded sets in topological spaces* (Contribution 5 by C. Bors, M. Ferrer and S. Hernández) approaches this notion for a topological space subject to the action of a monoid. They define the G-bounded sets of the topological space X, where G is a monoid that acts on X. This notion can be the seed of important developments, taking into account that the topic of actions of groups on topological spaces is nowadays a challenging one. In this paper, the authors prove, among other properties, that for a metrizable separable G-space X, the G-bounded subsets of X are completely determined by the G-bounded subsets of any dense subspace.

The paper *Distinguished property in tensor products and weak* dual spaces* (Contribution 6 by S. López-Alfonso, M. López-Pellicer and S. Moll-López) deals with locally convex spaces. Recall that a locally convex space E is distinguished if its strong dual E'_β is a barrelled space or equivalently if for every bounded subset M in $(E'_\beta)'_\beta$ there is a bounded set N in E such that $M \subset N^{\circ\circ}$. The notion of distinguished Fréchet spaces was already defined by Dieudonné and Schwartz. Here, the authors obtain distinguished properties of injective tensor products $L_p(X) \bigotimes_\varepsilon E$, where $L_p(X)$ denotes the dual space of the classical space $C_p(X)$ (the space of continuous real functions over a topological space X, endowed with the pointwise convergence topology) and E denotes a locally convex space. By imposing conditions either on X or on E, they are able to find many classes of distinguished spaces of the above-mentioned form.

The article *Aspects of differential calculus related to infinite-dimensional vector bundles and Poisson vector spaces* (Contribution 7 by H. Glöckner) deals with infinite-dimensional differential calculus. Among other questions, the differentiability properties of operator-valued maps and compositions with hypocontinuous k-linear mappings are investigated. A wide scope of applications is provided. In the field of infinite-dimensional vector bundles, these results are used to construct new bundles from given ones, such as dual bundles, topological tensor products, infinite direct sums and completions under suitable hypotheses. Another field of applications is in the class of locally convex Poisson spaces, a class defined by the author in earlier work. Roughly speaking, locally convex Poisson vector spaces are locally convex spaces E such that $E \times E$ is a $k_{\mathbb{R}}$-space, and a "Poisson bracket"—a more restrictive notion than that of a Lie bracket—is defined for the dual space of E. The differentiability results are used in this context to prove the continuity of the Poisson bracket and the continuity of the passage from a function to the associated Hamiltonian vector field.

Pro-Lie groups, which are defined as projective limits of finite-dimensional Lie groups, have been the subject of many fruitful investigations in recent years. In the article *Advances in the theory of compact groups and pro-Lie groups in the last quarter century* (Contribution 8), K. Hofmann and S. Morris provide a masterful summary that motivates and contextualizes their own contributions and those of others. Particular attention is paid to structure theorems for pro-Lie groups that are connected, almost connected or Abelian. The authors also explore the connection between pro-Lie groups and linear algebra, thereby identifying a new approach to the Hochschild–Tanaka duality of compact groups. This is one of several areas that they mention as ripe for further study.

Exploiting the connection between topological vector spaces and topological groups that were mentioned in the Introduction, an analogue of the Mackey–Arens theorem for the class of topological groups is considered in [4], a paper which initiated an extensive literature on this topic. The Mackey topology for a topological Abelian group G is defined as the finest locally quasi-convex topology which admits the same character group as G, and G is said to be a Mackey group if its original topology coincides with its Mackey topology. The class of Mackey groups includes all locally compact groups, as well as all complete metrizable ones [4]; however, there are topological Abelian groups which do not admit a Mackey topology, as was proven in [5,6]. Thus, the natural counterpart of the Mackey–Arens theorem does not hold for Abelian topological groups. As a result, it is natural to ask what is the relationship between the properties of "being a Mackey space" and "being a Mackey group" in the field of locally convex spaces. In [7], it was shown that a metrizable locally convex space might not be a Mackey group, a fact that disproved a conjecture stated in [8], (8.1). It is well known that a metrizable locally convex space carries its Mackey topology. In the paper *Normed spaces which are not Mackey groups* (Contribution 9 by S. Gabriyelyan), it is further proven that even a normed space may fail to be a Mackey group.

In an infinite dimensional topological vector space, the closed convex hull of a compact set might not be compact. Krein's theorem is an important result in this line, which can be formulated as follows: "If E is a complete locally convex space, then the closed convex hull of a weakly compact subset of E is again weakly compact". In the paper *Krein's theorem in the context of topological Abelian groups* (Contribution 10 by T. Borsich, X. Domínguez and E. Martín-Peinador), the authors interpret this result in the class of locally quasi-convex Abelian topological groups, analyze the resulting concepts and properties and expose an obstruction to the generalization of Krein's theorem to this wider context. In fact, if G denotes the family of null sequences of a compact metrizable connected group X, G has a natural group structure provided by that of $X^{\mathbb{N}}$. Under the uniform topology of $X^{\mathbb{N}}$, G becomes a complete metrizable locally quasi-convex topological group. However, the corresponding weak topology on G does not satisfy Krein's property. In other words, there exist weakly compact subsets of G whose quasi-convex hulls are not weakly compact .

A sequence (x_n) of elements of a locally convex space X is said to be absolutely summable if $\sum_n p(x_n) < \infty$ whenever p is a continuous seminorm on X or, equivalently, the Minkowski functional of an absolutely convex neighborhood of zero in X. This definition can be carried over to an arbitrary topological Abelian group G via Kaplan's generalization of Minkowski functionals. The same functionals can be then invoked to endow the group $\ell^1(G)$ of all absolutely summable sequences in G with a natural group topology. These concepts and constructions can be applied to a wide range of situations, and they often provide illuminating generalizations of the normed or the topological vector space setting. The article *On the group of absolutely summable sequences* (Contribution 11 by L. Außenhofer) contains quite a few of these generalizations; among other results, it is shown here that $\ell^1(G)$ is a Pontryagin reflexive group if G is either reflexive and metrizable or an LCA group, and $\ell^1(G)$ has the Schur property if and only if G has it.

The classical theorems by Dirichlet and Riemann on the convergence of a series of real terms can be partially generalized to much wider contexts, giving rise to a rich theory which is still being developed in a relevant way. The paper *Permutations, signs and sum ranges* (Contribution 12 by S. Chobanyan, X. Domínguez, V. Tarieladze and R. Vidal) consists mostly of a detailed survey of the advances in the sum range problem from its first formulations to the present day, including some results by M. J. Chasco and the first named author.

Along the same lines, in the paper *Series with commuting terms in topologized semigroups* (Contribution 13 by A. Castejón, E. Corbacho and V. Tarieladze), the authors present a version of the Riemann–Dirichlet unconditional convergence theorem for topologized semigroups.

The contribution *An expository lecture of María Jesús Chasco on some applications of Fubini's theorem* (Contribution 14 by A. Castejón, M. J. Chasco, E. Corbacho and V. Rodríguez de Miguel) is an elegant and powerful piece of mathematical exposition at the advanced undergraduate level, based on a masterclass given by M. J. Chasco at the University of Vigo. It contains a remarkable presentation of the Brunn–Minkowski and isoperimetric inequalities as consequences of Fubini's theorem, as well as some estimations of volumes of sections of n-dimensional balls.

3. Conclusions

The authors of the fourteen papers in this volume include friends, colleagues and collaborators of María Jesús Chasco. Ranging over many different branches of mathematics, the papers reflect the breadth of her mathematical interests. Several deal with various aspects of topological groups, but we expect that this volume will also interest specialists in general topology, functional analysis, algebra, geometry and number theory. Their quality and depth make them a fitting tribute for María Jesús Chasco's 65th birthday.

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List of Contributions:

1. Protasov, I. On Factoring Groups into Thin Subsets. *Axioms* **2021**, *10*, 89.
2. Tkachenko, M. Factoring Continuous Characters Defined on Subgroups of Products of Topological Groups. *Axioms* **2021**, *10*, 167.
3. Dikranjan, D.; Lewis, W.; Loth, P.; Mader, A. A Distinguished Subgroup of Compact Abelian Groups. *Axioms* **2022**, *11*, 200.
4. Bejines, C.; Ardanza-Trevijano, S.; Elorza, J. On Self-Aggregations of Min-Subgroups. *Axioms* **2021**, *10*, 201.
5. Bors, C.; Ferrer, M.; Hernández, S. Bounded Sets in Topological Spaces. *Axioms* **2022**, *11*, 71.
6. López-Alfonso, S.; López-Pellicer, M.; Moll-López, S. Distinguished Property in Tensor Products and Weak* Dual Spaces. *Axioms* **2021**, *10*, 151.
7. Glöckner, H. Aspects of Differential Calculus Related to Infinite-Dimensional Vector Bundles and Poisson Vector Spaces. *Axioms* **2022**, *11*, 221.
8. Hofmann, K.; Morris, S. Advances in the Theory of Compact Groups and Pro-Lie Groups in the Last Quarter Century. *Axioms* **2021**, *10*, 190.
9. Gabriyelyan, S. Normed Spaces Which Are Not Mackey Groups. *Axioms* **2021**, *10*, 217.
10. Borsich, T.; Domínguez, X.; Martín-Peinador, E. Krein's Theorem in the Context of Topological Abelian Groups. *Axioms* **2022**, *11*, 224.
11. Außenhofer, L. On the Group of Absolutely Summable Sequences. *Axioms* **2022**, *11*, 218.
12. Chobanyan, S.; Domínguez, X.; Tarieladze, V.; Vidal, R. Permutations, Signs, and Sum Ranges. *Axioms* **2023**, *12*, 760.
13. Castejón, A.; Corbacho, E.; Tarieladze, V. Series with Commuting Terms in Topologized Semigroups. *Axioms* **2021**, *10*, 237.
14. Castejón, A.; Chasco, M. J.; Corbacho, E.; Rodríguez de Miguel, V. An Expository Lecture of María Jesús Chasco on Some Applications of Fubini's Theorem. *Axioms* **2021**, *10*, 225.

References

1. Vilenkin, N.Y. The theory of characters of topological Abelian groups with boundedness given. *Izv. Akad. Nauk SSSR. Ser. Mat.* **1951**, *15*, 439–462.
2. Banakh, T.; Protasov, I. Set-theoretical problems in asymptology. *Quest. Answ. Gen. Topol.* **2022**, *40*, 67–89.
3. Hejcman, J. Boundedness in uniform spaces and topological groups. *Czech Math. J.* **1959**, *9*, 544–563. [CrossRef]
4. Chasco, M.J.; Martín-Peinador, E.; Tarieladze, V. On Mackey topology for groups. *Studia Math.* **1999**, *132*, 257–284. [CrossRef]
5. Aussenhofer, L. On the non-existence of the Mackey topology for locally quasi-convex groups. *Forum Math.* **2018**, *30*, 1119–1128. [CrossRef]
6. Gabriyelyan, S. A locally quasi-convex abelian group without a Mackey group topology. *Proc. Am. Math. Soc.* **2018**, *146*, 3627–3632. [CrossRef]
7. Gabriyelyan, S. On the Mackey topology for abelian topological groups and locally convex spaces. *Topol. Appl.* **2016**, *211*, 11–23. [CrossRef]
8. Dikranjan, D.; Martín-Peinador, E.; Tarieladze, V. Group valued null sequences and metrizable non-Mackey groups. *Forum Math.* **2014**, *26*, 723–757. [CrossRef]

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Article

On Factoring Groups into Thin Subsets

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Abstract: A subset X of a group G is called thin if, for every finite subset F of G, there exists a finite subset H of G such that $Fx \cap Fy = \emptyset$, $xF \cap yF = \emptyset$ for all distinct $x, y \in X \setminus H$. We prove that every countable topologizable group G can be factorized $G = AB$ into thin subsets A, B.

Keywords: factorizations of a group; thin subset of a group

MSC: 20F69; 54C65

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1. Introduction

Let G be a group, and $[G]^{<\omega}$ denote the set of all finite subsets of G. A subset X of is called:

- *left thin* if, for every $F \in [G]^{<\omega}$, there exists $H \in [G]^{<\omega}$ such that $Fx \cap Fy = \emptyset$ for all distinct $x, y \in X \setminus H$;
- *right thin* if, for every $F \in [G]^{<\omega}$, there exists $H \in [G]^{<\omega}$ such that $xF \cap yF = \emptyset$ for all distinct $x, y \in X \setminus H$;
- *thin* if X is left and right thin.

The notion of left thin subsets was introduced in [1]. For motivation to study left thin, right thin and thin subsets and some results and references, see Comments and surveys [2–5]. In *asymptology*, thin subsets play the part of discrete subsets (see Comments 1 and 2).

We recall that the product AB of subsets A, B of a group G is a *factorization* if $G = AB$ and each element $g \in G$ has the unique representation $g = ab$, $a \in A$, $b \in B$ (equivalently, the subsets $\{aB : a \in A\}$ are pairwise disjoint). For factorizations of groups into subsets, see [6].

Our goal is to prove the following theorem. By a countable set, we mean a countably infinite set. The group topology τ is supposed to be Hausdorff.

Theorem 1. *Let (G, τ) be a non-discrete countable topological group. Then G can be factorized $G = AB$ into thin subsets A, B.*

2. Proof

Proof of Theorem 1. Let $G = \{g_n : n < \omega\}$, $g_0 = e$, e is the identity of G, $F_n = \{g_i : i \leq n\}$.

Given two sequences $(a_n)_{n<\omega}$, $(b_n)_{n<\omega}$ in G, we denote

$$A_n = \{a_i, a_i^{-1} : i \leq n\}, \;\; B_n = \{b_i : i \leq n\}, \;\; A = \cup_{n<\omega} A_n, \;\; B = \cup_{n<\omega} B_n.$$

We want to choose $(a_n)_{n<\omega}$, $(b_n)_{n<\omega}$ so that AB is a factorization of G and A, B are thin.

Let X, Y be subsets of G. We say that XY is a *partial factorization* of G if the subsets $\{Xy : y \in Y\}$ are pairwise disjoint (equivalently, the subsets $\{Yx : x \in X\}$ are pairwise disjoint).

We put $a_0 = e$, $b_0 = e$ and suppose that $a_0, \dots, a_n$ and $b_0, \dots, b_n$ have been chosen so that the following conditions are satisfied

(1) A_nB_n is a partial factorization of G and $g_n \in A_nB_n$;

(2) $F_ib_i \cap F_jb_j = \emptyset$, $b_iF_i \cap b_jF_j = \emptyset$ for all distinct $i, j \in \{0, \dots, n\}$;

(3) $F_ia_i \cap F_ja_j = \emptyset$, $a_iF_i \cap a_jF_j = \emptyset$, $F_ia_i^{-1} \cap F_ja_j^{-1} = \emptyset$, $a_i^{-1}F_i \cap a_j^{-1}F_j = \emptyset$ and $F_ia_i^{-1} \cap F_ja_j$, $a_i^{-1}F_i \cap a_jF_j = \emptyset$ for all distinct $i, j \in \{0, \dots, n\}$;

(4) if $a_i \neq a_i^{-1}$ then $F_ia_i \cap F_ia_i^{-1} = \emptyset$, $a_iF_i \cap a_i^{-1}F_i = \emptyset$, $i \in \{0, \dots, n\}$.

We take the first element $g_m \in G \setminus A_nB_n$, put $g = g_m$ and show that there exists a symmetric neighborhood U of e such that

(5) $(A_n \cup \{x, x^{-1}\})(B_n \cup \{xg\})$ is a partial factorization for each $x \in U \setminus \{e\}$.

We choose a symmetric neighborhood V of e such that $(A_n \cup \{x, x^{-1}\})B_n$ is a partial factorization of G for each $x \in V \setminus \{e\}$.

Then we use $A_n = A_n^{-1}$, $g \in G \setminus A_nB_n$ and $e \in A_n \cap B_n$ to choose a symmetric neighborhood U of e such that $U \subset V$ and

$$(A_n \cup \{x, x^{-1}\})B_n \cap (A_n \cup \{x, x^{-1}\})xg = \emptyset,$$

equivalently, $A_nB_n \cap A_nxg = \emptyset$, $A_nB_n \cap \{x, x^{-1}\}xg = \emptyset$, $\{x, x^{-1}\}B_n \cap A_nxg = \emptyset$, $\{x, x^{-1}\}B_n \cap \{x, x^{-1}\}xg = \emptyset$ for each $x \in U \setminus \{e\}$, so we get (5). By the continuity of the group operations, the latter is possible because these 4 equalities hold for $x = e$.

If the set $\{x \in U : x^2 = e\}$ is infinite then we use (5) and choose $a_{n+1} \in U$, $a_{n+1} = a_{n+1}^{-1}$ and $b_{n+1} = a_{n+1}g$ to satisfy (1)–(3) with $n + 1$ in place of n. Otherwise, we choose $a_{n+1} \in U$, $a_{n+1} \neq a_{n+1}^{-1}$ and $b_{n+1} = a_{n+1}g$ to satisfy (1)–(4).

After ω steps, we get the desired factorization $G = AB$. □

3. Comments

1. Given a set X, a family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* on X if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X := \{(x, x) : x \in X\}$ of X;
- if E, $E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) : \exists z \ \ ((x, z) \in E,\ (z, y) \in E')\}$, $E^{-1} = \{(y, x) : (x, y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$.

Elements $E \in \mathcal{E}$ of the coarse structure are called *entourages* on X.

For $x \in X$ and $E \in \mathcal{E}$ the set $E[x] := \{y \in X : (x, y) \in \mathcal{E}\}$ is called the *ball of radius E centered at x*. Since $E = \bigcup_{x \in X}(\{x\} \times E[x])$, the entourage E is uniquely determined by the family of balls $\{E[x] : x \in X\}$. A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* of the coarse structure $\mathcal{E}$ if each set $E \in \mathcal{E}$ is contained in some $E' \in \mathcal{E}'$.

The pair $(X, \mathcal{E})$ is called a *coarse space* [7] or a *ballean* [8,9].

A subset B of X is called *bounded* if $B \subseteq E[x]$ for some $E \in \mathcal{E}$ and $x \in X$. A subset Y of X is called *discrete* if, for every $E \in \mathcal{E}$, there exists a bounded subset B such that $E[x] \cap E[y] = \emptyset$ for all distinct $x, y \in Y \setminus B$.

2. Formally, coarse spaces can be considered as asymptotic counterparts of uniform topological spaces. However, actually, this notion is rooted in *geometry, geometrical group theory* and *combinatorics* (see [7,8,10,11]).

Given a group G, we denote by $\mathcal{E}_l$ and $\mathcal{E}_r$ the coarse structures on G with the bases

$$\{\{(x, y) : x \in Fy\} : F \in [G]^{<\omega}, e \in F\}, \ \ \{\{(x, y) : x \in yF\} : F \in [G]^{<\omega}, e \in F\}$$

and note that a subset A of G is left (resp. right) thin if and only if A is discrete in the coarse space $(G, \mathcal{E}_l)$ (resp. $(G, \mathcal{E}_r)$).

3. By [12], every countable group G has a thin subset A such that $G = AA^{-1}$. By [13], every countable topological group G has a closed discrete subset A such that $G = AA^{-1}$. For thin subsets of topological groups and factorizations into dense subsets, see [14,15].

4. Can every countable group G be factorized $G = AB$ into infinite subsets A, B? By Theorem 1, an answer to the following question could be negative only in the case of a non-topologizable group G.

On the other hand, analyzing the proof, one can see that Theorem 1 remains true if all mappings $x \longmapsto xg$, $x \longmapsto gx$, $g \in G$, $x \longmapsto x^{-1}$ and $x \longmapsto x^2$ are continuous at e. By [16], every countable group G admits a non-discrete Hausdorff topology in which all shifts and the inversion $x \longmapsto x^{-1}$ are continuous.

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References

1. Chou, C. On the size of the set of left invariant means on a semigroup. *Proc. Am. Math. Soc.* **1969**, *23*, 199–205. [CrossRef]
2. Protasov, I. Selective survey on subset combinatorics of groups. *J. Math. Sci.* **2011**, *174*, 486–514. [CrossRef]
3. Protasov, I.; Protasova, K. Resent progress in subset combinatorics of groups. *J. Math. Sci.* **2018**, *234*, 49–60. [CrossRef]
4. Protasov, I.; Slobodianiuk, S. Partitions of groups. *Math. Stud.* **2014**, *42*, 115–128.
5. Banakh, T.; Protasov, I. Set-Theoretical Problems in Asymptology. Available online: https://arxiv.org/abs/2004.01979 (accessed on 7 May 2020).
6. Szabo, S.; Sands, A. *Factoring Groups into Subsets*; CRS Press: Boca Raton, FL, USA, 2009.
7. Roe, J. *Lectures on Coarse Geometry*; Univ. Lecture Ser., 31; American Mathematical Society: Providence, RI, USA, 2003.
8. Protasov, I.; Banakh, T. *Ball Structures and Colorings of Groups and Graphs*; VNTL Publ.: Lviv, Ukraine, 2003.
9. Protasov, I.; Zarichnyi, M. *General Asymptology*; VNTL: Lviv, Ukraine, 2007.
10. De la Harpe, P. *Topics in Geometrical Group Theory*; University Chicago Press: Chicago, IL, USA, 2000.
11. Cornulier, Y.; de la Harpe, P. *Metric Geometry of Locally Compact Groups;* EMS Tracts in Mathematics; European Mathematical Society: Zürich, Switzerland, 2016.
12. Lutsenko, I. Thin systems of generators of groups. *Algebra Discret. Math.* **2010**, *9*, 108–114.
13. Protasov, I. Generating countable groups by discrete subsets. *Topol. Appl.* **2016**, *204*, 253–255. [CrossRef]
14. Protasov, I. Thin subsets of topological groups.*Topol. Appl.* **2013**, *160*, 1083–1087. [CrossRef]
15. Protasov, I.; Slobodianiuk, S. A note on factoring groups into dense subsets. *J. Group Theory* **2017**, *20*, 33–38. [CrossRef]
16. Zelenyuk, Y. On topologizing groups. *J. Group Theory* **2007**, *10*, 235–244. [CrossRef]

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Factoring Continuous Characters Defined on Subgroups of Products of Topological Groups †

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† To my colleague and friend M.J. Chasco, with great respect.

Abstract: This study is on the factorization properties of continuous homomorphisms defined on subgroups (or submonoids) of products of (para)topological groups (or monoids). A typical result is the following one: Let $D = \prod_{i\in I} D_i$ be a product of paratopological groups, S be a dense subgroup of D, and χ a continuous character of S. Then one can find a finite set $E \subset I$ and continuous characters χ_i of D_i, for $i \in E$, such that $\chi = \left(\prod_{i\in E} \chi_i \circ p_i\right) \restriction S$, where $p_i \colon D \to D_i$ is the projection.

Keywords: monoid; group; character; homomorphism; factorization; Roelcke uniformity

MSC: 22A30; 54C15; 54H11

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1. Introduction

Factorization of continuous functions defined on (dense) subspaces of topological products has a long and illustrious history, with several new ideas and discoveries. The articles [1–5] provide an excellent overview of the methodologies employed in this area of research.

The current paper is a natural extension of [6,7], in which we investigated continuous homomorphisms of subgroups (submonoids) of topological group products (monoids). We proved in those articles that in many circumstances, a continuous homomorphism $f \colon S \to H$ of a submonoid (subgroup) S of a product $D = \prod i \in ID_i$ of topological monoids (groups) to a topological monoid (group) H enables a factorization in the form

$$f = g \circ p_J \restriction S, \tag{1}$$

where J is a "small" subset of the index set I, $p_J \colon D \to D_J = \prod_{i\in J} D_i$ is the projection, and $g \colon p_J(S) \to H$ is a continuous homomorphism. If one can find a finite (countable) set J for which (1) holds true, we say that f has a *finite* (*countable*) *type*. Most of the results in [6,7] present different conditions on S and/or H under which f has a countable or even finite type. Purely algebraic aspects of this study can be found in [8].

In this article we go further and try to decompose a given continuous homomorphism $f \colon S \to H$ into a product of 'coordinate' homomorphisms, as explained below.

It follows from the Pontryagin–van Kampen duality theory that every continuous homomorphism of a product $D = \prod_{i\in I} D_i$ of compact abelian groups to the circle group $\mathbb{T}$ (called *character*) has a finite type. Hence, every continuous character of D is a linear combination of finitely many continuous characters, each of which depends on exactly one coordinate. This fact remains valid in a considerably more general situation presented by S. Kaplan in [9]:

Proposition 1. *Let χ be a continuous character of a product $\Pi = \prod_{i\in I} G_i$ of (reflexive) topological abelian groups. Then one can find pairwise distinct indices $i_1, \ldots, i_n \in I$ and continuous characters $\chi_1, \ldots, \chi_n$ of the respective groups $G_{i_1}, \ldots, G_{i_n}$ such that the equality*

$$\chi(x) = \prod_{k=1}^{n} \chi_k(x_{i_k}) \tag{2}$$

holds for each $x \in \Pi$.

An examination of the argument offered in [9] demonstrates that the 'reflexive' can be omitted from the assumptions of Proposition 1. Thus, we may reformulate the conclusion of Proposition 1 by asserting that the dual group $\Pi^\wedge$ is algebraically isomorphic to the direct sum of the factors' duals, $\bigoplus_{i\in I} D_i^\wedge$. Our objective is to extend the conclusion of Proposition 1 to a much broader class of objects, such as subgroups or submonoids of Cartesian products of monoids or paratopological groups (see Theorem 2, Corollary 1, and Theorems 4–6).

An important property of the torus $\mathbb{T}$ is that it is an *NSS group*, which means that there exists an open neighborhood of the identity in $\mathbb{T}$ containing no nontrivial subgroups. Every Lie group is an NSS group. According to ([7], Theorem 3.11), every continuous homomorphism of an arbitrary subgroup of a product of topological monoids to a Lie group has a finite type. This is an essential ingredient in several arguments presented in Section 2.

In Section 3 we complement several results from ([7], Section 2) about the continuous character of a dense submonoid S of the *P-modification* of a product $D = \prod_{i\in I} D_i$ of *topologized* monoids. We show in Proposition 3 and Example 3 that if $\varphi\colon S \to H$ is a nontrivial continuous homomorphism of S to a topologized monoid of countable pseudocharacter, then the family $\mathcal{J}(\chi)$ of the subsets J of the index set I such that φ *depend on* J is often a filter on I, and this filter can have an empty intersection, even if $S = D$ and the product $D = \mathbb{Z}(2)^\omega$ is a compact metrizable topological group (hence the P-modification of D is a discrete group).

Notation and Auxiliary Results

Let $\mathbb{C}$ be the field of complex numbers with the usual Euclidean topology. The torus $\mathbb{T}$ is identified with the multiplicative subgroup $\{z \in \mathbb{C} : |z| = 1\}$ of $\mathbb{C}$.

A *semigroup* is a nonempty set S with a binary associative operation (called *multiplication*). A semigroup with an identity is called a *monoid*. Clearly a monoid has a unique identity.

A semigroup S with some topology is said to be a *semitopological semigroup* if multiplication in S is separately continuous. This is equivalent to saying that the left and right shifts in S are continuous. If multiplication in S is jointly continuous, we say that S is a *topological semigroup*. The concept of *topological monoid* is defined similarly.

Assume that G is a semigroup (monoid, group) with a topology. If the left shifts in G are continuous, then G is called a *left topological* semigroup (monoid, group). If both left and right shifts in G are continuous, then G is said to be a *semitopological* semigroup (monoid, group). Further, if G is a group and multiplication in G is jointly continuous, we say that G is a *paratopological* group. A paratopological group with continuous inversions is a *topological* group.

A *topologized* monoid (group) is a monoid (group) with an arbitrary topology that may have no relation to multiplication in the monoid (group). We say that a left topological monoid G has *open left shifts* if for every $x \in G$, the left shift λ_x of G defined by $y \mapsto x \cdot y$ for each $y \in G$ is an open mapping of G to itself.

The *character* of an arbitrary monoid G is a (not necessarily continuous) homomorphism of G to the torus $\mathbb{T}$. The continuity of a character, if it applies, will always be specified explicitly.

In the sequel we follow the notation of Proposition 1. For every $i \in I$, let p_i be the projection of Π onto the factor G_i. Then the conclusion of the proposition is equivalent to saying that $\chi = \prod_{k=1}^{n} \chi_k \circ p_{i_k}$. It is worth noting that the projections p_i are continuous open homomorphisms, so the characters $\chi_1, \ldots, \chi_n$ are 'automatically' continuous. This assertion follows from the next simple result, which shows that for finitely many factors, the conclusion of Proposition 1 remains valid, even if the factors are topologized monoids.

Lemma 1. *Let $G = G_1 \times \cdots \times G_n$ be a product of topologized monoids and χ be a continuous homomorphism of G to a topologized semigroup K. Then there exist homomorphisms $\chi_1, \ldots, \chi_n$ of the respective monoids $G_1, \ldots, G_n$ to K such that $\chi(x) = \chi_1(x_1) \cdots \chi_n(x_n)$, for each $x = (x_1, \ldots, x_n) \in G$. This representation of χ is unique and the homomorphisms $\chi_1, \ldots, \chi_n$ are continuous.*

Proof. For every $k = 1, \ldots, n$, let e_k be the identity of G_k and p_k be the projection of G onto the factor G_k. We define a homomorphism χ_k of G_k to K by $\chi_k(y) = \chi(e_1, \ldots, y, \ldots, e_n)$ for every $y \in G_k$, where y stands at the kth position in $(e_1, \ldots, y, \ldots, e_n)$. A direct verification shows that $\chi(x) = \chi_1(x_1) \cdots \chi_n(x_n)$, for each $x = (x_1, \ldots, x_n) \in G$.

Let $\psi_1, \ldots, \psi_n$ be homomorphisms of $G_1, \ldots, G_n$, respectively, to K, satisfying $\chi(x) = \psi_1(x_1) \cdots \psi_n(x_n)$, for each $x \in G$. We fix an integer k with $1 \leq k \leq n$, and for every $y \in G_k$, consider the element $\hat{y} = (e_1, \ldots, y, \ldots, e_n) \in G$, where y stands at the kth position in $\hat{y}$. Then $\chi_k(y) = \chi(\hat{y}) = \psi_k(y)$, so $\psi_k = \chi_k$ for each $k \leq n$, and hence, the representation $\chi(x) = \chi_1(x_1) \cdots \chi_n(x_n)$ is unique.

It follows from the continuity of the homomorphism χ and the equalities $\chi_k(y) = \chi(e_1, \ldots, y, \ldots, e_n)$, where $1 \leq k \leq n$ and $y \in G_k$, that $\chi_1, \ldots, \chi_n$ are continuous. □

Let $X = \prod_{i \in I} X_i$ be the Tychonoff product of a family $\{X_i : i \in I\}$ of spaces and $a \in X$ be an arbitrary point. For every $i \in I$, the projection of X to the factor X_i is denoted by p_i. In addition, for every $x \in X$, we make

$$\text{diff}(x, a) = \{i \in I : p_i(x) \neq p_i(a)\}.$$

Then

$$\Sigma X(a) = \{x \in X : |\text{diff}(x, a)| \leq \omega\}$$

and

$$\sigma X(a) = \{x \in X : |\text{diff}(x, a)| < \omega\}$$

are dense subspaces of X which are called, respectively, the Σ*-product* and σ*-product* of the family $\{X_i : i \in I\}$ with centers at a. If every X_i is a monoid (group), we will always choose a to be the identity e of X. In the latter case, $\Sigma X(e)$ and $\sigma X(e)$ are dense submonoids (subgroups) of the product monoid (group) X and we shorten $\Sigma X(e)$ and $\sigma X(e)$ to ΣX and σX, respectively.

Assume that Z is a nonempty subset of the product $X = \prod_{i \in I} X_i$ of a family $\{X_i : i \in I\}$ of sets and $f\colon Z \to Y$ is an arbitrary mapping. We say that f *depends on* J, for some $J \subset I$, if the equality $f(x) = f(y)$ holds for all $x, y \in Z$ with $p_J(x) = p_J(y)$, where $p_J\colon X \to \prod_{i \in J} X_i$ is the projection. It is clear that if f depends on J, then there exists a mapping g of $p_J(Z)$ to Y satisfying $f = g \circ p_J \restriction Z$. Conversely, if there exists such a mapping g of $p_J(Z)$ to Y, then f depends on J.

Definition 1. *Assume that D_i is a monoid with identity e_i, where $i \in I$. For a nonempty subset J of I, we define a retraction r_J of $D = \prod_{i \in I} D_i$ by letting*

$$r_J(x)_i = \begin{cases} x_i & \text{if } i \in J; \\ e_i & \text{if } i \in I \setminus J, \end{cases}$$

for each element $x \in D$. A subset S of D is said to be retractable *if $r_J(S) \subset S$, for each $J \subset I$. If the inclusion $r_J(S) \subset S$ holds for each finite set $J \subset I$, we call S* finitely retractable.

The concept of finite retractability is used in Theorem 5.

Given a space X, we denote by PX the underlying set X with the topology whose base consists of all nonempty G_δ-sets in X. The space PX is usually referred to as the *P-modification* of X. If X is a (left) topological group (monoid), then PX with the same multiplication is also a (left) topological group (monoid).

The family of countable subsets of a given set I is denoted by $[I]^{\leq\omega}$.

2. Factoring Continuous Characters

In this section, we deal with not necessarily Hausdorff objects of topological algebra. Since a major proportion of the research articles and books on this subject treat the Hausdorffian case exclusively, we need to extend several well-known facts to non-Hausdorffian monoids and groups. We start with the following result that, informally, goes back to Graev's article ([10], pp. 52–53).

Lemma 2. *Let G be a topological group with identity e, N be the closure of the singleton $\{e\}$ in G, and $\pi\colon G \to G/N$ be the quotient homomorphism. For every continuous homomorphism $f\colon G \to H$ to a Hausdorff topological group H, there exists a unique homomorphism $g\colon G/N \to H$ satisfying $f = g \circ \pi$, and g is automatically continuous.*

Proof. Notice that N is a closed invariant subgroup of G, so the quotient topological group G/N is a T_1-space. Hence G/N is a Hausdorff. Denote by K the kernel of f. Since H is a Hausdorff, K is a closed subgroup of G. Hence, $\ker\pi = N \subset K = \ker f$. It now follows from ([11], Proposition 1.5.10) that there exists a homomorphism $g\colon G/N \to H$ satisfying $f = g \circ \pi$. Assume that a homomorphism $\tilde{g}\colon G/N \to H$ also satisfies $f = \tilde{g} \circ \pi$. If $y \in G/N$, we take an element $x \in G$ with $\pi(x) = y$. Then $g(y) = g(\pi(x)) = f(x)$, and similarly, $\tilde{g}(y) = \tilde{g}(\pi(x)) = f(x)$. Hence $\tilde{g}(y) = g(y)$ for each $y \in G/N$, so $\tilde{g} = g$. As π is open and continuous, we conclude that g is continuous. □

The pair $(G/N, \pi)$ in Lemma 2 is called the *Hausdorff reflection* of G. Abusing terminology, we usually refer to G/N as the Hausdorff reflection of G, thereby omitting the quotient homomorphism π. We also denote G/N by $T_2(G)$.

Informally speaking, the following lemma states that the functor of the Hausdorff reflection in the category of topological groups and continuous homomorphisms describes arbitrary subgroups.

Lemma 3. *Let G be a topological group with identity e, N be the closure of the singleton $\{e\}$ in G, and $\pi\colon G \to G/N$ be the quotient homomorphism. Let S be an arbitrary subgroup of G and $N_S = S \cap N$. Then the quotient group $T_2(S) = S/N_S$ is topologically isomorphic to the subgroup $\pi(S)$ of $T_2(G) = G/N$ and the restriction of π to S is an open continuous homomorphism of S onto $\pi(S)$.*

Proof. It follows from the definition of π that every closed subset C of G satisfies $C = \pi^{-1}\pi(C)$. Therefore, if the subgroup S is closed in G then $N \subset S$, $S = \pi^{-1}\pi(S)$, and the restriction of π to S is an open continuous homomorphism of S onto the subgroup $\pi(S)$ of G/N. By the first isomorphism theorem, the groups $\pi(S)$ and S/N are topologically isomorphic.

In the general case, let K be the closure of S in G. Then K is a closed subgroup of G, $N \subset K$, and by the above argument, the groups $T_2(K) = K/N$ and $\pi(K) \subset T_2(G)$ are topologically isomorphic. Hence it suffices to verify that the group $T_2(S)$ is topologically isomorphic in relation to the subgroup $\pi(S)$ of K/N. To this end we show that the restriction of π to S is an open homomorphism onto the subgroup $\pi(S)$ of K/N. Let U be a nonempty open set in K and $V = U \cap S$. Since $K = \pi^{-1}\pi(K)$ and $N \subset K$, the set U satisfies the equality $U = \pi^{-1}\pi(U)$. Hence the set $\pi(U) \cap \pi(S) = \pi(U \cap S) = \pi(V)$ is open in $\pi(S)$. Thus, $\pi{\restriction}S$ is an open homomorphism of S onto $\pi(S)$ whose kernel is $S \cap N$, so the groups $T_2(S)$ and $\pi(S)$ are topologically isomorphic. □

Let us recall that the *precompact Hausdorff reflection* of a given topological group G is a pair (H, φ_G), where H is a precompact Hausdorff topological group and $\varphi_G\colon G \to H$ is a continuous homomorphism, such that for every continuous homomorphism $g\colon G \to K$ to a Hausdorff precompact topological group K, there exists a continuous homomorphism

$h\colon H \to K$ satisfying $g = h \circ \varphi_G$. Every topological group G has a precompact Hausdorff reflection and this reflection is unique up to topological isomorphism [12]. The homomorphism φ_G is referred to as *universal* for G.

Lemma 4. *Let S be a dense subgroup of a topological group G and (H, φ_G) be the precompact Hausdorff reflection of G. Let $T = \varphi_G(S)$ and $\psi = \varphi_G{\restriction}S$. Then (T, ψ) is the precompact Hausdorff reflection of the group S.*

Proof. Since H is a precompact Hausdorff topological group, so is its dense subgroup T. Therefore it suffices to verify that the continuous onto homomorphism $\psi\colon S \to T$ is universal for S. Let $g\colon S \to K$ be a continuous homomorphism to a precompact Hausdorff group K. The completion of K, say, ϱK, is a compact Hausdorff topological group. Hence the group ϱK is complete. Since S is dense in G, g extends to a continuous homomorphism $g^*\colon G \to \varrho K$. By the universality of φ_G, there exists a continuous homomorphism $h^*\colon H \to \varrho K$ such that $g^* = h^* \circ \varphi_G$. Let h be the restriction of h^* to T. Then $g = g^*{\restriction}S = h^* \circ \varphi_G{\restriction}S = h^* \circ \psi = h \circ \psi$. This proves the universality of ψ for S. □

A subgroup S of a topological abelian group G is said to be *dually embedded* in G if every continuous character of S extends to a continuous character of G. The next lemma is well known in the special case of Hausdorff topological groups ([13], Lemma 2.2).

Lemma 5. *Every subgroup S of a precompact topological abelian group G is dually embedded in G.*

Proof. Let e be the identity of G and N be the closure of the singleton $\{e\}$ in G. Additionally, let $p\colon G \to G/N$ be the quotient homomorphism. Since G is precompact, the pair $(G/N, p)$ is the precompact Hausdorff reflection of G. Let S be a subgroup of G. Denote by K the closure of S in G. It follows from the definition of N that $N \subset K$ and $K = p^{-1}p(K)$, so $K/N \cong p(K)$ and $(p(K), q)$ is the precompact Hausdorff reflection of K, where $q = p{\restriction}K$. Since S is dense in K, Lemma 4 implies that $(q(S), q{\restriction}S) = (p(S), p{\restriction}S)$ is the precompact Hausdorff reflection of S.

Let χ be a continuous character of S. There exists a continuous character λ of the subgroup $T = p(S)$ of the precompact Hausdorff group G/N such that $\chi = \lambda \circ p{\restriction}S$. By ([13], Lemma 2.2), T is dually embedded in the Hausdorff precompact abelian group G/N, so λ extends to a continuous character λ^* of G/N. Hence $\chi^* = \lambda^* \circ p$ is an extension of χ to a continuous character of G and S is dually embedded in the group G. □

The following fact complements Lemma 5 in the non-abelian case.

Lemma 6. *Every dense subgroup S of an arbitrary topological group G is dually embedded in G.*

Proof. Let (H, φ_G) be the precompact Hausdorff reflection of the group G. We put $T = \varphi_G(S)$ and $\psi = \varphi_G{\restriction}S$. By Lemma 4, the pair (T, ψ) is the precompact Hausdorff reflection of S.

Let χ be a continuous character of S. Then there exists a continuous character χ_T of T such that $\chi = \chi_T \circ \psi$. Since the group H is precompact and Hausdorffian, it follows from ([13], Lemma 2.2) that T is dually embedded in H. Hence, χ_T extends to a continuous character λ of H. Thus, $\chi^* = \lambda \circ \varphi_G$ is a continuous character of G which extends χ. □

Lemma 6 is not valid for *closed* subgroups of Hausdorff topological groups. In fact, even a compact subgroup of a separable metrizable topological abelian group can fail to be dually embedded ([11], Example 9.9.61).

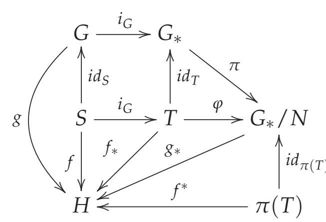
According to Proposition 3.6.12 of [11], a continuous homomorphism of a dense subgroup S of a Hausdorff topological group G to a complete Hausdorffian topological group H extends to a continuous homomorphism of G to H. Below we generalize this fact by showing that it remains valid for dense subgroups of arbitrary paratopological groups. Our argument makes use of the *topological group reflection* of a paratopological group (see [14]).

Theorem 1. *Let S be a dense subgroup of a paratopological group G and $f\colon S \to H$ be a continuous homomorphism of S to a complete Hausdorff topological group H. Then f extends to a continuous homomorphism $g\colon G \to H$.*

Proof. Let $i_G\colon G \to G_*$ be the identity mapping of G onto the topological group reflection G_* of G. It follows from ([14], Theorem 12) that the subgroup $T = i_G(S)$ of G_* is topologically isomorphic to the topological group reflection S_* of S, so we can identify the groups T and S_* algebraically and topologically.

Since H is a topological group, there exists a continuous homomorphism $f_*\colon T \to H$ satisfying $f = f_* \circ i_G \restriction S$. It follows from the continuity of i_G that T is a dense subgroup of G_*. However, the groups G_* and T may fail to be Hausdorffian.

To reduce our further argument to the case of Hausdorff groups, we denote by N the closure of the singleton $\{e_G\}$ in G_* and consider the quotient homomorphism $\pi\colon G_* \to G_*/N$. Then the quotient group G_*/N is the Hausdorff reflection of G_*. By Lemma 3, the subgroup $\pi(T)$ of G_*/N is the Hausdorff reflection of T and the homomorphism $\varphi = \pi \restriction T$ of T onto $\pi(T)$ is open and continuous. Since the group H is Hausdorffian, Lemma 2 implies the existence of a continuous homomorphism $f^*\colon \pi(T) \to H$ satisfying the equality $f_* = f^* \circ \varphi$. Notice that T is dense in G_* and $\pi(T)$ is dense in G_*/N. Therefore, by ([11], Corollary 3.6.17), f_* extends to a continuous homomorphism $g_*\colon G_*/N \to H$ (we use the completeness of H here).



Then $g = g_* \circ \pi \circ i_G$ is a continuous homomorphism of G to H which extends f. This proves the theorem. □

We complement Theorem 1 in Proposition 2 by considering continuous homomorphisms defined on dense submonoids of topological monoids.

Example 1. Closed subgroups of completely regular paratopological groups need not be dually embedded. Hence Theorem 1 does not extend to closed subgroups of paratopological groups.

Proof. Let $\mathbb{S}$ be the Sorgenfrey line endowed with the usual topology and addition. Clearly $\mathbb{S}$ is a regular (even hereditarily normal) paratopological group. Additionally, let $\Delta = \{(x, -x) : x \in \mathbb{S}\}$ be the *second diagonal* of $\mathbb{S} \times \mathbb{S}$. It is well known and easy to verify that the subgroup Δ is discrete and closed. Hence every character of Δ is continuous and Δ can be identified with the real line $\mathbb{R}_d$ endowed with the discrete topology. On the one hand, an easy calculation shows that the family of characters of Δ has the cardinality $\mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$, where $\mathfrak{c} = 2^{\omega}$. On the other hand, the groups $\mathbb{S}$ and $\mathbb{S} \times \mathbb{S}$ are separable, so there are at most $\mathfrak{c}^{\omega} = \mathfrak{c}$ continuous characters of $\mathbb{S} \times \mathbb{S}$. Therefore, not every character of Δ extends to a continuous character of $\mathbb{S} \times \mathbb{S}$. In other words, Δ fails to be dually embedded in $\mathbb{S} \times \mathbb{S}$. It is also clear that not every character of Δ admits the representation described in Lemma 1 (or in Theorem 2 that follows). □

The next result is a considerable generalization of Proposition 1.

Theorem 2. *Let $D = \prod_{i\in I} D_i$ be a product of paratopological groups and S be a subgroup of D. Assume that for every finite set $F \subset I$, the subgroup $p_F(S)$ of $D_F = \prod_{i\in F} D_i$ is dually embedded in D_F, where $p_F\colon D \to D_F$ is the projection. Then for every continuous character χ of S, one can find a finite set $E \subset I$ and continuous characters χ_i of $p_i(S)$, for $i \in E$, such that $\chi = \big(\prod_{i\in E} \chi_i \circ p_i\big)\restriction S$.*

Proof. By Corollary 3.12 in [7], one can find a finite set $E \subset I$ and a continuous character χ_E of $p_E(S)$ such that $\chi = \chi_E \circ p_E\restriction S$, where $p_E\colon D \to \prod_{i\in E} D_i$ is the projection. By the assumptions of the theorem, $T = p_E(S)$ is a dually embedded subgroup of $D_E = \prod_{i\in E} D_i$. Hence χ_E extends to a continuous character ψ of D_E. According to Lemma 1, for every $i \in E$, there exists a continuous character ψ_i of G_i such that $\psi = \prod_{i\in E} \psi_i \circ q_i$, where $q_i\colon D_E \to D_i$ is the projection. Let $p_i\colon D \to D_i$ be the projection, for each $i \in E$. Since $p_i = q_i \circ p_E$ and $\chi = \psi \circ p_E\restriction S$, we conclude that the required equality $\chi = \big(\prod_{i\in E} \chi_i \circ p_i\big)\restriction S$ is valid. □

Example 1 explains why in Theorem 2, we require the projections of a subgroup $S \subset D$ to finite subproducts to be dually embedded, though this does not exclude the possibility that the theorem be valid for *arbitrary* subgroups of products of (para)topological groups. Later, in Example 2, we will show that such a generalization of Theorem 2 is impossible, even if the factors of the product $D = \prod_{i\in I} D_i$ are topological groups.

By Theorem 1, a dense subgroup of a paratopological group is dually embedded. Hence the next corollary is immediate from Theorem 2.

Corollary 1. *Let $D = \prod_{i\in I} D_i$ be a product of paratopological groups, S be a dense subgroup of D, and χ be a continuous character of S. Then one can find a finite set $E \subset I$ and continuous characters χ_i of D_i, for $i \in E$, such that $\chi = \big(\prod_{i\in E} \chi_i \circ p_i\big)\restriction S$, where $p_i\colon D \to D_i$ is the projection.*

The next example shows that the conditions on S for 'dual embedding' in Theorem 2 and 'dense' in Corollary 1 are essential.

Example 2. *There exist countably infinite, metrizable topological abelian groups G_1 and G_2, and a closed discrete subgroup Δ of the product $\Pi = G_1 \times G_2$ such that $p_1(\Delta) = G_1$, $p_2(\Delta) = G_2$, and the only continuous character of the group Π is the trivial one. Here p_1 and p_2 are projections of Π onto G_1 and G_2, respectively. In particular, the trivial character of Δ is the only one representable in the form described in Corollary 1.*

Proof. Let G be a countable, infinite Boolean group. Then G is the direct sum of countable copies of the group $\mathbb{Z}(2) = \{0, 1\}$, so G is as in item (2) of Lemma 0 in [15]. Therefore, Theorem′ on page 22 of [15] implies that G admits a metrizable topological group topology τ_1 such that the only continuous character of $G_1 = (G, \tau_1)$ is the trivial one.

Our first observation is that the group G_1 is not precompact — otherwise continuous characters of G_1 would separate elements of G_1. Since every non-zero element of the countable group G_1 has order 2, one can apply ([16], Theorem 5.28) to find an open neighborhood U of zero e_1 in G_1 and a (necessarily discontinuous) automorphism f of the group G_1 such that $f(U) \cap U = \{e_1\}$. In other words, the group G_1 is *self-transversal.*

Let $\tau_2 = \{f(V) : V \in \tau_1\}$ be the image of the topology τ_1 under the automorphism f and $G_2 = (G, \tau_2)$. Then f is a topological isomorphism of G_1 onto G_2 and the only continuous character of G_2 is the trivial one. By Lemma 1 the product group $\Pi = G_1 \times G_2$ has the same property. Denote by Δ the subgroup $\{(x, x) : x \in G\}$ of the group Π. It is clear that $p_1(\Delta) = G_1$ and $p_2(\Delta) = G_2$. The set $O = U \times f(U)$ is open in Π and it follows from our choice of the set U that the intersection $O \cap \Delta$ contains only the identity element of $G_1 \times G_2$. Hence the subgroup Δ of Π is discrete and closed. It is clear that every character of Δ is continuous, and that the only character of Δ that can be expressed in the form presented in Corollary 1 is the trivial one. □

Since the subgroup Δ of the group $G_1 \times G_2$ in Example 2 is discrete, we see that Corollary 1 is not valid for locally compact subgroups of products of topological groups. However, it is valid for *precompact* abelian subgroups of product groups.

First, we present a well-known result from [17] often called the *Comfort–Ross duality* for precompact topological abelian groups. We denote the family of all characters of an abstract group G to the torus $\mathbb{T}$ by $Hom(G,\mathbb{T})$. Clearly, the pointwise multiplication of characters in $Hom(G,\mathbb{T})$, $(\chi_1 \cdot \chi_2)(x) = \chi_1(x) \cdot \chi_2(x)$, makes it an abelian group.

Theorem 3. *For every abelian group G, there exists a natural (i.e., functorial) monotone bijection between the family of precompact topological group topologies on G and the subgroups of the group $Hom(G,\mathbb{T})$.*

'Monotone' in Theorem 3 means that a finer precompact topological group topology on G corresponds to a bigger subgroup of $Hom(G,\mathbb{T})$. For more details on this correspondence, see [17].

In the following theorem we do not impose any separation restrictions on the factors D_i:

Theorem 4. *Let C be a precompact abelian subgroup of a product $D = \prod_{i\in I} D_i$ of topological groups and χ be a continuous character of C. Then one can find a finite set $E \subset I$ and continuous characters χ_i of $p_i(C)$, for $i \in E$, such that $\chi = \left(\prod_{i\in E} \chi_i \circ p_i\right)\restriction C$, where $p_i\colon D \to D_i$ is the projection.*

Proof. The projection $p_i(C)$ is a precompact abelian subgroup of the group D_i, for each $i \in I$. We can assume, therefore, that each factor $D_i = p_i(C)$ is a precompact abelian group. Then D is also a precompact topological abelian group. For every $i \in I$, let $D_i^\wedge$ be group of continuous characters of D_i. By ([17], Theorem 1.2), the topology of D_i is initial with respect to $D_i^\wedge$. Consider the family

$$\mathcal{A} = \{\chi \circ p_i : i \in I,\ \chi \in D_i^\wedge\}.$$

Then each element of $\mathcal{A}$ is a continuous character of D, so $\mathcal{A} \subset D^\wedge$. Let H be the subgroup of $D^\wedge$ generated by $\mathcal{A}$. Every element χ of H has the form

$$\chi = \prod_{k=1}^{n} \chi_k \circ p_{i_k}, \tag{3}$$

where $i_1, \ldots, i_n$ are pairwise distinct elements of I and $\chi_k \in D_{i_k}^\wedge$ for each $k = 1, \ldots, n$. It is clear that the topology of D is initial with respect to H. Since C is a topological subgroup of D, the family of restrictions $H_C = \{\chi\restriction C : \chi \in H\}$ generates the topology of C. Notice that H_C is a subgroup of $C^\wedge \cap Hom(C,\mathbb{T})$, so Theorem 3 implies that $H_C = C^\wedge$. The latter equality, together with (3), implies the required conclusion. □

Problem 1. *Does Theorem 4 extend to precompact subgroups of products of paratopological abelian groups?*

The main difficulty in solving Problem 1 is the fact that the topological group reflection of a subgroup C of a paratopological abelian group D can have a strictly finer topology than the topology of C inherited from D_*. In other words, Lemma 4 cannot be extended to paratopological groups. Even the very special case of Problem 1, where C is a precompact subgroup of the product of two (precompact) paratopological groups, is not clear.

The following result extends a well-known property of continuous homomorphisms of topological groups to a more general case when the domain of a homomorphism is a dense *submonoid* of a topological monoid with open shifts. First we recall the notions of Roelcke uniformity and Roelcke completeness in topological groups.

Let G be a topological group and $\mathcal{N}(e)$ be the family of open neighborhoods of the identity e in G. For every $U \in \mathcal{N}(e)$, the set

$$O_U = \{UxU : x \in G\}$$

is an open entourage of the diagonal in $G \times G$ and the family $\{O_U : U \in \mathcal{N}(e)\}$ constitutes a base for a compatible uniformity on G, say, $\mathcal{V}_G$, which is called the *Roelcke uniformity* of G (see [11], Section 1.8). If the uniform space $(G, \mathcal{V}_G)$ is complete, we say that the group G is *Roelcke-complete*.

Proposition 2. *Let S be a dense submonoid of a topological monoid D with open shifts. Then every continuous homomorphism $f\colon S \to K$ to a Roelcke-complete Hausdorff topological group K extends to a continuous homomorphism $f^*\colon D \to K$.*

Proof. Let $\mathcal{N}(e)$ be the family of open neighborhoods of the identity e in D. We denote by $\mathcal{Q}$ the *quasi-Roelcke* uniformity of D whose base consists of the sets

$$Q_V = \{(x, y) \in D \times D : Vx \cap yV \neq \emptyset \neq Vy \cap xV\},$$

where $V \in \mathcal{N}(e)$ (see [18]). It is easy to see that the topology of D generated by $\mathcal{Q}$ is weaker than the original topology of D. Additionally, let $\mathcal{V}_K$ be the Roelcke uniformity of the group K.

Consider a continuous homomorphism $f\colon S \to K$ to a Roelcke-complete Hausdorff topological group K with identity e_K. We claim that f is uniformly continuous considered as a mapping of $(S, \mathcal{Q}{\restriction}S)$ to $(K, \mathcal{V}_K)$. To this end, take an arbitrary symmetric element $U \in \mathcal{N}(e_K)$ and choose an element $W \in \mathcal{N}(e_K)$ such that $W^2 \subset U$. Then $\overline{W} \subset U$. By the continuity of f, we can find an element $V \in \mathcal{N}(e)$ satisfying $f(V \cap S) \subset W$. We are yet to verify that $(f(x), f(y)) \in O_U$ whenever $(x, y) \in Q_V \cap S^2$, or equivalently, $(f \times f)(Q_V \cap S^2) \subset O_U$.

Let $(x, y) \in Q_V \cap S^2$. Then $Vx \cap yV \neq \emptyset$ and $Vy \cap xV \neq \emptyset$. Since S is dense in D and the sets Vx and yV are open in D, we can choose a point $z \in S \cap Vx \cap yV$. It follows from the continuity of shifts in D and the density of $S \cap V$ in V that for $z \in Vx \subset \overline{(S \cap V) \cdot x}$, the closure is taken in D. As $z \in S$, we see that z is in the closure of $(S \cap V) \cdot x$ in S. Hence $f(z) \in \overline{f(V \cap S) \cdot f(x)} = \overline{f(V \cap S)} \cdot f(x)$, by the continuity of f; the closure is taken in K. Since $\overline{f(V \cap S)} \subset \overline{W} \subset U$, the latter implies that $f(z) \in Uf(x)$. A similar argument, starting with $z \in yV$, shows that $f(z) \in f(y)U$. Thus $f(z) \in Uf(x) \cap f(y)U \neq \emptyset$, whence $f(y) \in Uf(x)U^{-1} = Uf(x)U$. This implies that $(f(x), f(y)) \in O_U$ and proves the uniform continuity of f as a mapping of $(S, \mathcal{Q}{\restriction}S)$ to $(K, \mathcal{V}_K)$.

Since the space $(K, \mathcal{V}_K)$ is complete, f extends to a uniformly continuous mapping $f^*\colon (D, \mathcal{Q}) \to (K, \mathcal{V}_K)$. It follows from the density of S in D and the Hausdorffness of K that f^* is a homomorphism. □

Corollary 2. *Let S be a dense submonoid of a topological monoid D with open shifts. Then every continuous homomorphism $f\colon S \to K$ to a locally compact topological group K extends to a continuous homomorphism $f^*\colon D \to K$.*

Proof. According to Proposition 2 it suffices to verify that every locally compact topological group K is Roelcke-complete. The latter fact is immediate since for every compact neighborhood U of the identity in K, every Cauchy filter ξ in the uniform space $(K, \mathcal{V}_K)$ has an element contained in the compact set UxU, for some $x \in K$. Hence ξ converges to an element of K and $(K, \mathcal{V}_K)$ is complete, where $\mathcal{V}_K$ is the Roelcke uniformity of K. □

Now we apply Proposition 2 in a less obvious way.

Theorem 5. *Let S be a dense submonoid of a product $D = \prod_{i \in I} D_i$ of topological monoids with open shifts and $f\colon S \to K$ be a continuous homomorphism to a Lie group K. If S is either finitely* *retractable or open in D, then f extends to a continuous homomorphism $f^*\colon D \to K$. Hence, one can find a finite set $E \subset I$ and continuous homomorphisms $\chi_i\colon D_i \to K$ for $i \in E$, such that $f^*(x) = \prod_{i\in E} \chi_i(x_i)$ for each $x = (x_i)_{i\in I} \in D$.*

Proof. Depending on whether S is finitely retractable or open, we apply, respectively, Theorem 2.12 or Theorem 3.8(b) of [7] to conclude that f depends on a finite set $E \subset I$. In either case, there exists a continuous homomorphism $g\colon p_E(S) \to K$ satisfying $f = g \circ p_E{\restriction}S$, where p_E is the projection of D to $D_E = \prod_{i\in E} D_i$. Then $p_E(S)$ is a dense submonoid of D_E and D_E is a topological monoid with open shifts, by ([7], Lemma 3.5). Hence we are entitled to apply Proposition 2 to the homomorphism g. Hence, there exists a continuous homomorphism $g^*\colon D_E \to K$ extending g. According to Lemma 1 we can find continuous homomorphisms $\chi_i\colon D_i \to K$ for $i \in E$ such that $g(y) = \prod_{i\in E} \chi_i(y_i)$, for each $y = (y_i)_{i\in E}$. Then $f^* = g^* \circ p_E$ is a continuous homomorphism of D to K extending f and satisfying $f^*(x) = \prod_{i\in E} \chi_i(x_i)$, for each $x \in D$. This implies the required equality for the homomorphism f. □

According to ([7], Theorem 5), every continuous homomorphism $f\colon S \to K$ of an arbitrary subgroup S of a product D of topological monoids to a Lie group K has a *finite type*, i.e., can be represented as the composition of the projection p_E of S to a finite subproduct D_E of D and a continuous homomorphism of $p_E(S)$ to K. Therefore, by arguing as in the proof of Theorem 5 and applying Proposition 2 we deduce the following:

Theorem 6. *Let $D = \prod_{i\in I} D_i$ be a product of topological monoids with open shifts, S be a dense subgroup of D, and $f\colon S \to K$ be a continuous homomorphism to a Lie group K. Then f extends to a continuous homomorphism $f^*\colon D \to K$, so one can find a finite set $E \subset I$ and continuous homomorphisms $\chi_i\colon D_i \to K$, for $i \in E$, such that $f^*(x) = \prod_{i\in E} \chi_i(x_i)$ for each $x = (x_i)_{i\in I} \in D$.*

3. More on Continuous Homomorphisms of *P*-Modifications of Products and Their Dense Submonoids

First we introduce notation which is used in this section and clarifies our aim.

Let $X = \prod_{i\in I} X_i$ be the product of a family $\{X_i : i \in I\}$ of sets, Z be a subset of X, and $f\colon Z \to Y$ be a mapping. Denote by $\mathcal{J}(f)$ the family of all sets $J \subset I$ such that f depends on J. Our main concern is to determine the properties of the family $\mathcal{J}(f)$. For example, one can ask whether $\mathcal{J}(f)$ is a filter or whether it has minimal, by inclusion, elements, or even the smallest element. It has been shown by W. Comfort and I. Gotchev in [19–21] that the family $\mathcal{J}(f)$ can have quite a complicated set-theoretic structure, even if X is a Cartesian product of topological spaces and f is a continuous mapping to a space Y. It is worth mentioning that the thorough study of the family $\mathcal{J}(f)$ was motivated by a somewhat simpler question on whether $\mathcal{J}(f)$ had a countable element $J \subset I$. The reader can find an extensive bibliography related to this question in the aforementioned articles and in the earlier survey article [22] by M. Hušek.

It turns out that the intersection of the family $\mathcal{J}(f)$, denoted by J_f, admits a clear description in terms of f. We say that an index $i \in I$ is *f-essential* if there exist points $x, y \in Z$ such that $\mathrm{diff}(x, y) = \{i\}$ and $f(x) \neq f(y)$. Let E_f be the set of all f-essential indices in I. By Proposition 2.2 in [23], $J_f = E_f = \bigcap \mathcal{J}(f)$. In particular, the set J_f is empty if and only if no index $i \in I$ is f-essential.

Below we present a useful fact which is not valid for arbitrary dense subgroups of the topological group PD, the P-modifications of the product $D = \prod_{i\in I} D_i$ of topologized monoids D_i, not even if the factors D_i are finite discrete groups (see [6], Example 1).

Proposition 3. *Let $D = \prod_{i\in I} D_i$ be a Cartesian product of topologized monoids, S be a submonoid of D with $\Sigma D \subset S$, and $\varphi\colon PS \to H$ be a nontrivial continuous homomorphism of the P-modification of S to a topologized monoid H of countable pseudocharacter. Then the family*

$$\mathcal{J}(\varphi) = \{J \subset I : \varphi \text{ depends on } J\}$$

is a filter on the index set I.

Proof. Since the subspace PS of PD is a P-space, the homomorphism $\varphi\colon PS \to PH$ remains continuous (see, e.g., [6], Lemma 6). Notice that PH is a discrete space. Therefore, we can assume that H carries the discrete topology. By applying ([6], Proposition 2), we find a countable subset E of I and a continuous homomorphism φ_E of $p_E(S) \subset PD_E$ to H such that $\varphi = \varphi_E \circ p_E \restriction S$, where $p_E\colon D \to D_E = \prod_{i\in E} D_i$ is the projection. It follows from $\Sigma D \subset S$ that $p_E(S) = D_E$. Hence $\bar{\varphi} = \varphi_E \circ p_E$ is a continuous homomorphism of PD to H. It follows from the definition of $\bar{\varphi}$ that this homomorphism depends on E. Furthermore, if $\bar{\varphi}$ depends on F, for some $F \subset I$, then so does φ. It is now clear that $\mathcal{J}(\bar{\varphi}) = \mathcal{J}(\varphi)$.

Therefore, we can assume without loss of generality that φ is a continuous character of $PD = S$. Assume that $J_1 \subset J_2 \subset I$ and $J_1 \in \mathcal{J}(\varphi)$. Then there exists a mapping $g\colon D_{J_1} = \prod_{i\in J_1} D_i \to H$ satisfying $\varphi = g \circ p_{J_1}$, where $p_{J_1}\colon PD \to PD_{J_1}$ is the projection. Clearly g is a homomorphism. Since the projection p_{J_1} is open, the homomorphism g is continuous. Therefore, g is a continuous homomorphism of PD_{J_1} to H. Let $p^{J_2}_{J_1}$ be the projection of D_{J_2} to D_{J_1}. Then $\varphi = g \circ p_{J_1} = g \circ p^{J_2}_{J_1} \circ p_{J_2} = f \circ p_{J_2}$, where $f = g \circ p^{J_2}_{J_1}$ is a continuous homomorphism of PD_{J_2}. Hence, φ depends on J_2 and $J_2 \in \mathcal{J}(\varphi)$.

Let J_1 and J_2 be arbitrary elements of $\mathcal{J}(\varphi)$. It is easy to see that $\ker p_{J_1} \subset \ker\varphi$ and $\ker p_{J_2} \subset \ker\varphi$. Put $J = J_1 \cap J_2$. Then

$$\ker p_J = \ker p_{J_1} \cdot \ker p_{J_2} \subset \ker\varphi \neq D.$$

In particular, $J \neq \emptyset$ (we identify $p_\emptyset$ with the constant mapping of D to the identity e_D of D). It follows from the inclusion $\ker p_J \subset \ker\varphi$ that there exists a homomorphism $h\colon D_J \to H$ satisfying $\varphi = h \circ p_J$ (see [24], Theorem 1.48 or [6], Lemma 2). We conclude that $J \in \mathcal{J}(\varphi)$.

To sum up, the family $\mathcal{J}(\varphi)$ is a filter. □

The reader can find several results about continuous homomorphisms or characters defined on dense submonoids and subgroups of Cartesian (equivalently, *Tychonoff*) products in [6,7]. On many occasions, the conclusions there are stronger than the one in Proposition 3.

It is natural to ask whether the filter $\mathcal{J}(\varphi)$ in Proposition 3 contains a minimal by inclusion element. The next example answers this question in the negative, even if S is the P-modification of the compact metrizable group $\mathbb{Z}(2)^\omega$ (so S is discrete). Notice that the continuous characters of the compact group $\mathbb{Z}(2)^\omega$ are described in Proposition 1.

Example 3. Let the group $G = \mathbb{Z}(2)^\omega$ carry the discrete topology. There exist a non-trivial character χ of G and a decreasing sequence $\{J_n : n \in \omega\}$ of infinite subsets of ω with empty intersection such that χ depends on J_n, for each $n \in \omega$. Hence the filter $\mathcal{J}(\chi)$ does not have minimal elements.

Proof. Let $J_n = \omega \setminus \{0, 1, \ldots, n\}$, for each $n \in \omega$. Denote by $\mathbf{1}$ the point of $\mathbb{Z}(2)^\omega$ all coordinates of which are equal to 1. Additionally, let

$$H_n = \{x \in \mathbb{Z}(2)^\omega : x(i) = 0 \text{ for each } i \in J_n\}.$$

Clearly, H_n is a subgroup of G and $H_n \subset H_{n+1}$, for each $n \in \omega$. Hence $H = \bigcup_{n=0}^\infty H_n$ is also a subgroup of G. Since $\mathbf{1} \notin H$, there exists a character χ of G such that $\chi(H) = \{1\}$ and $\chi(\mathbf{1}) = -1$. It is immediate from the definition that χ depends on J_n, for each $n \in \omega$. Since $\bigcap_{n=0}^\infty J_n = \emptyset$, the family $\mathcal{J}(\chi)$ has no smallest element. Taking into account that $\mathcal{J}(\chi)$ is a filter (see Proposition 3), we infer that it does not contain minimal elements either. □

Since the subgroup H of G in the proof of Example 3 is dense in $G = \mathbb{Z}(2)^\omega$ provided the latter group is endowed with the usual Tychonoff product topology, the above character χ is discontinuous on the compact group $\mathbb{Z}(2)^\omega$. It turns out that considering the Tychonoff product topology improves the situation greatly — the family $\mathcal{J}(\chi)$ always has a finite *minimal* (by inclusion) element, for each continuous character χ of an arbitrary subgroup G of a product of left topological groups. This conclusion can be recovered using techniques from [9] in the special case where G itself is a product of *topological* groups, but the reader can find a direct argument in the more general Proposition 2.1 of [7].

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References

1. Mazur, S. On continuous mappings on Cartesian products. *Fundam. Math.* **1952**, *39*, 229–238. [CrossRef]
2. Miščenko, A. Some theorems on product spaces. *Fundam. Math.* **1966**, *58*, 259–284. (In Russian)
3. Engelking, R. On functions defined on Cartesian products. *Fundam. Math.* **1966**, *59*, 221–231. [CrossRef]
4. Noble, N.; Ulmer, M. Factoring functions on Cartesian products. *Trans. Am. Math. Soc.* **1972**, *163*, 329–340. [CrossRef]
5. Hušek, M. Mappings from products. Topological Structures II. *Math. Cent. Tracts* **1979**, *115*, 131–145.
6. Tkachenko, M. Factoring continuous homomorphisms defined on submonoids of products of topologized monoids. *Axioms* **2019**, *8*, 86. [CrossRef]
7. Tkachenko, M. Continuous homomorphisms defined on (dense) submonoids of products of topological monoids. *Axioms* **2020**, *9*, 23, doi:10.3390/axioms9010023 [CrossRef]
8. Bergman, G.M. Homomorphisms on infinite direct products of groups, rings and monoids. *Pac. J. Math.* **2016**, *274*, 451–495. [CrossRef]
9. Kaplan, S. Extension of the Pontrjagin duality I: Infinite products. *Duke Math. J.* **1948**, *15*, 649–658. [CrossRef]
10. Graev, M.I. Theory of topological groups I. Norms and metrics on groups. Complete groups. Free topological groups. *Uspekhy Mat. Nauk* **1950**, *5*, 3–56. (In Russian)
11. Arhangel'skii, A.V.; Tkachenko, M.G. *Topological Groups and Related Structures*; van Mill, J., Ed.; Atlantis Press: Paris, France; Amsterdam, The Netherlands, 2008; xiv + 781p, ISBN 978-90-78677-06-2. [CrossRef]
12. He, W.; Xiao, Z. The τ-precompact Hausdorff group reflection of topological groups. *Bull. Belg. Math. Soc. Simon Stevin* **2018**, *25*, 107–120. [CrossRef]
13. Galindo, J.; Tkachenko, M.; Bruguera, M.; Hernández, C. Reflexivity in precompact groups and extensions. *Topol. Appl.* **2014**, *163*, 112–127. [CrossRef]
14. Tkachenko, M. Group reflection and precompact paratopological groups. *Topol. Algebra Appl.* **2013**, *1*, 22–30. [CrossRef]
15. Ajtai, M.; Havas, I.; Komlós, J. Every group admits a bad topology. In *Studies in Pure Mathematics*; Springer: Basel, Switzerland, 1983; pp. 21–34.
16. Błaszczyk, A.; Tkachenko, A. Transversal, T_1-independent, and T_1-complementary topologies. *Topol. Appl.* **2017**, *230*, 308–337. [CrossRef]
17. Comfort, W.W.; Ross, K.A. Topologies induced by groups of characters. *Fund. Math.* **1964**, *55*, 283–291. [CrossRef]
18. Banakh, T.; Ravsky, A. Each regular paratopological group is completely regular. *Proc. Am. Math. Soc.* **2017**, *145*, 1373–1382. [CrossRef]
19. Comfort, W.W.; Gotchev, I. Continuous mappings on subspaces of products with the κ-box topology. *Topol. Appl.* **2009**, *156*, 2600–2608. [CrossRef]
20. Comfort, W.W.; Gotchev, I.S. Functional dependence on small sets of indices. *Sci. Math. Jpn.* **2009**, *69*, 363–377.
21. Comfort, W.W.; Gotchev, I.S. Cardinal Invariants for κ-Box Products: Weight, Density Character and Suslin Number. *Diss. Math.* **2016**, *516*, 41. [CrossRef]
22. Hušek, M. Continuous mappings on subspaces of products. *Symp. Math.* **1976**, *17*, 25–41.
23. Tkachenko, M. Factoring continuous mapping defined on subspaces of topological products. *Topol. Appl.* **2020**, *281*, 107198, doi:10.1016/j.topol.2020.107198 [CrossRef]
24. Carruth, J.H.; Hildebrant, J.A.; Koch, R.J. *The Theory of Topological Semigroups*; Marcel Dekker, Inc.: New York, NY, USA; Basel, Switzerland, 1983; Volume I.

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A Distinguished Subgroup of Compact Abelian Groups

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Abstract: Here "group" means additive abelian group. A compact group G contains δ–subgroups, that is, compact totally disconnected subgroups Δ such that G/Δ is a torus. The canonical subgroup $\mathbf{\Delta}(G)$ of G that is the sum of all δ–subgroups of G turns out to have striking properties. Lewis, Loth and Mader obtained a comprehensive description of $\mathbf{\Delta}(G)$ when considering only finite dimensional connected groups, but even for these, new and improved results are obtained here. For a compact group G, we prove the following: $\mathbf{\Delta}(G)$ contains $\mathrm{tor}(G)$, is a dense, zero-dimensional subgroup of G containing every closed totally disconnected subgroup of G, and $G/\mathbf{\Delta}(G)$ is torsion-free and divisible; $\mathbf{\Delta}(G)$ is a functorial subgroup of G, it determines G up to topological isomorphism, and it leads to a "canonical" resolution theorem for G. The subgroup $\mathbf{\Delta}(G)$ appeared before in the literature as $\mathrm{td}(G)$ motivated by completely different considerations. We survey and extend earlier results. It is shown that td, as a functor, preserves proper exactness of short sequences of compact groups.

Keywords: full free subgroup; (locally) compact abelian group; Pontryagin Duality; totally disconnected; 0-dimensional; precompact; functorial subgroup; quasi-torsion element; minimal group; totally minimal group; exotic torus

MSC: 20K15; 22K45; 22C05

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1. Introduction

The topological groups studied in this paper are mainly the Pontryagin duals of discrete abelian groups with some emphasis on the duals of torsion-free groups. The latter are exactly the compact connected abelian groups. Non-compact topological groups prominently appear in Section 7.

The result ([1], Proposition 8.15, p. 416) deals with the existence of compact totally disconnected subgroups Δ of a compact group G such that G/Δ is a torus. These δ–***subgroups*** enter into the Resolution Theorem for compact abelian groups ([1], Theorem 8.20, p. 420, see also Section 6). The duals of the short exact sequences $\Delta \rightarrowtail G \twoheadrightarrow T$ where G is a compact group, Δ is a δ–subgroup of G and thus T is a torus, are precisely the exact sequences $F \rightarrowtail A \twoheadrightarrow D$ where A is a discrete group, F is a free subgroup of A and D is a torsion group. This suggests the study of the ***full free subgroups*** F of A, i.e., the free subgroups of A with torsion quotient. Let $\mathcal{F}(A)$ denote the family of all full free subgroups of A and let $\mathcal{D}(G)$ denote the family of all δ–subgroups of the compact group G. In Theorem 1, a comprehensive description of $\mathcal{F}(A)$ is established, and by duality a similarly comprehensive description of $\mathcal{D}(G)$ is obtained (Theorem 6). In fact, there is an anti-isomorphism of semi-lattices $\delta : \mathcal{F}(A) \to \mathcal{D}(G)$ where $G = A^\wedge$ (Theorem 5).

The canonical subgroup $\mathbf{\Delta}(G) := \sum \mathcal{D}(G)$ of G, referred to as "Fat Delta", has interesting properties:

(FD1) It contains $\mathrm{tor}(G)$, is dense in G, and $G/\mathbf{\Delta}(G)$ is torsion-free and divisible (Theorem 6(2),(4),(6) and Theorem 10(2)).
(FD2) If G is not totally disconnected, then $\mathbf{\Delta}(G)$ is a proper subgroup of G, and hence is not locally compact (Proposition 6(1)).
(FD3) $\mathbf{\Delta}(G)$ is zero-dimensional (Theorem 19), and contains every closed totally disconnected subgroup of G (Proposition 5).
(FD4) Fat Delta is a functorial subgroup in the sense that for any morphism $f : G \to H$ we have $f[\mathbf{\Delta}(G)] \subseteq \mathbf{\Delta}(H)$ (Corollary 3, Proposition 10(1)), moreover $f[\mathbf{\Delta}(G)] = \mathbf{\Delta}(H)$ if f is surjective (Proposition 10(2)).
(FD5) The Fat Delta of a product is the product of the Fat Deltas of the factors (Theorem 10(4), Proposition 10(4)).
(FD6) If $G = A^\wedge$ is a compact group, then $\mathbf{\Delta}(G) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ (see Theorem 10(1) for a more rigorous formulation).
(FD7) $\mathbf{\Delta}(G)$ determines G up to topological isomorphism (Theorem 12).

The group $\mathbf{\Delta}(G)$ coincides with $\mathrm{tor}(G)$ if and only if $G = T \times E$ with T a finite dimensional torus and E a bounded group (Theorem 9). We obtain a "canonical" resolution theorem (Theorem 15) for a compact abelian group G where the canonical $\mathbf{\Delta}(G)$ replaces a random δ–subgroup.

In [2], the case of connected compact groups of finite dimension was studied; here we generalize to arbitrary compact abelian groups of any dimension, but even in the case of finite dimension, our results on Fat Delta surpass by far those in [2].

Furthermore, Fat Delta, defined differently, in greater generality, and called $\mathrm{td}(G)$, previously appeared in the literature ([3], pp. 127–128, [4]). We quote, elaborate, and extend results from earlier works as follows.

In Section 7, we provide a different 'projective' characterization of $\mathrm{td}(G)$ (see Proposition 9(1)) and various applications of $\mathbf{\Delta}(G) = \mathrm{td}(G)$. It is proved that td, as a functor, preserves proper exactness of short sequences of compact groups (Corollary 4). The interest in the subgroup $\mathrm{td}(G)$ of compact groups (see Definition 4) was triggered by the intensive research on the Open Mapping Theorem since the early seventies of the last century [3–15] (see Definition 5 for the relevant properties and Theorem 17 for criteria for the inheritance of these properties from dense subgroups). Section 7.3 is focused on the topological p-Sylow subgroups $\mathrm{td}_p(G)$ of $\mathrm{td}(G)$.

The characterization (FD6) of Fat Delta for compact groups first appeared in ([6], (2), p. 217) and ([3], Proposition 4.1.4).

In Section 8, we discuss some open problems.

In a forthcoming paper [16], we extend the characterization (FD6) to larger classes of topological abelian groups (e.g., subgroups of LCA groups). To this end, we introduce there a new series of functorial subgroups in TAG, related to $\mathrm{td}(G)$ and $\mathrm{td}_p(G)$, and consider alternative definitions of Fat Delta for non compact groups.

2. Notation and Background

Our reference on abelian groups is [17]. As a rule $A, B, C, D, E, \dots$ denote discrete groups and $G, H, K, L, \dots$ are used to denote topological groups. Unless otherwise stated, p is an arbitrary prime number. If $\mathcal{C}$ is a category of groups, then "A is a $\mathcal{C}$–group" and "$A \in \mathcal{C}$" means that A is an object of $\mathcal{C}$. By $A \leq B$ we mean that A is a sub-object of B when $A, B \in \mathcal{C}$. We will deal with the following categories:

- The category AG of discrete abelian groups with morphisms algebraic homomorphisms, $\cong$ denoting isomorphism in this category, also called algebraic isomorphism;
- TAG is the category of topological abelian groups with morphisms continuous algebraic homomorphisms, $\cong_t$ denoting isomorphism in this category;
- LCA is as usual the full subcategory of TAG consisting of locally compact Hausdorff groups.

We will use $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ while $\mathbb{P}$ denotes the set of all prime numbers. Furthermore, $\mathbb{R}$ denotes the additive group of real numbers, $\mathbb{Z}$ the integers and

$\mathbb{T}$ the additively written circle group $\mathbb{R}/\mathbb{Z}$ equipped with the compact quotient topology. A ***torus*** is a topological group isomorphic with a power $\mathbb{T}^{\mathfrak{m}}$ where $\mathfrak{m}$ is any cardinal.

The torsion subgroup (p-torsion subgroup) of an abelian group G is denoted by $\mathrm{tor}(G)$ ($\mathrm{tor}_p(G)$, respectively). We have $\mathrm{tor}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z} \leq \mathbb{T}$ with the subspace topology, and $\mathrm{tor}_p(\mathbb{T}) \leq \mathbb{Q}/\mathbb{Z}$ with the subspace topology. We use $\mathbb{Z}(p^\infty) := \mathrm{tor}_p(\mathbb{T}) = \{m/p^n + \mathbb{Z} \mid m, n \in \mathbb{N}_0\}$ in agreement with ([1], p. 27).

The m–socle of a group X is $X[m] := \{x \in X \mid mx = 0\}$ and the socle of X is $\mathrm{Soc}(X) = \bigoplus_{p\in\mathbb{P}} X[p]$. By μ_m^X we denote multiplication by m in X. For a subgroup, Y of X and $m \in \mathbb{N}$, define

$$m_X^{-1}Y = \{x \in X \mid mx \in Y\}, \text{ equivalently, } m_X^{-1}Y/Y = (X/Y)[m].$$

This concept is used to construct larger full free subgroups from given full free subgroups.

In the following discussion of divisible hulls, $\mathbb{Z}(p^\infty)$ is the discrete quasi-cyclic group ([17], p. 16).

The group D is a ***divisible hull*** of A if D is divisible and A is an essential subgroup of D, equivalently, if D/A is a torsion group and $\bigoplus_{p\in\mathbb{P}} D[p] \subseteq A$. Divisible hulls exist for any group and divisible groups are direct sums of copies of $\mathbb{Q}$ and of $\mathbb{Z}(p^\infty)$, $p \in \mathbb{P}$ ([17], p. 136).

The $\mathbb{Z}$–adic topology of $\mathbb{Z}$ (having as a local base at 0 the filter base $\{n\mathbb{Z} : n \in \mathbb{N}\}$) will be denoted by $\nu_\mathbb{Z}$. We denote by $G^\wedge$ the Pontryagin dual of a TAG–group G, while $\widehat{G}$ is reserved for the completion of G. In particular, $\widehat{\mathbb{Z}}$ is the completion of $(\mathbb{Z}, \nu_\mathbb{Z})$ and $\widehat{\mathbb{Z}}_p$ is the completion of $\mathbb{Z}$ in the p-adic topology.

For topological groups G, H we will deal with $\mathrm{cHom}(G, H)$, the set of all continuous homomorphisms from G to H. Throughout, we assume that the groups of morphisms $\mathrm{cHom}(G, H)$ carry the compact-open topology. We will use the notation of ([1], p. 337), so recall that the sets $W(C, U) = \{f \in \mathrm{cHom}(G, H) \mid f[C] \subseteq U\}$ where C is compact in G and U is open in H, form a basis for the topology of $\mathrm{cHom}(G, H)$.

By $\mathrm{c}(G)$ we denote the 0–component of G and by $\mathrm{a}(G)$ the arc component of $0 \in G$. A Hausdorff topological group G is ***zero-dimensional*** if G has a base of clopen sets. Clearly, every linearly topologized group is zero-dimensional and every zero-dimensional group is totally disconnected. Recall that a group is linearly topologized if it possesses a neighborhood basis at 0 consisting of subgroups.

Lemma 1 ([1], E8.6, p. 414)**.** *Let G be a locally compact abelian group. Then G is totally disconnected if and only if G is zero-dimensional.*

A topological abelian group G is said to be ***precompact*** if its completion is compact. It is a well-known and deep fact that a topological abelian group G is precompact if and only if the topology of G is generated by its continuous characters, which means that the characters $\chi \in G^\wedge$ separate the points of G and the injective (continuous) diagonal map $G \to \prod_{\chi\in G^\wedge} \chi[G] \leq \mathbb{T}^{G^\wedge}$ is an embedding ([3]).

Proposition 1. *Let G be a topological abelian group and let G_i, $i \in I$, be a family of topological groups. Then*

$$\mathrm{cHom}(G, \textstyle\prod_{i\in I} G_i) \cong_\mathrm{t} \prod_{i\in I} \mathrm{cHom}(G, G_i).$$

Proof. Let $\pi_j : \prod_{i\in I} G_i \to G_j$ be the projections. Then

$$\pi : \mathrm{cHom}(G, \textstyle\prod_{i\in I} G_i) \to \prod_{i\in I} \mathrm{cHom}(G, G_i) : \pi(f) = (\ldots, \pi_i \circ f, \ldots).$$

is the restriction of the well-known algebraic isomorphism. Evidently, $\pi(f) = (\ldots, \pi_i \circ f, \ldots) \in \prod_{i\in I} \mathrm{cHom}(G, G_i)$, so π is well-defined.

To show that π is continuous, consider the generic open neighborhood $V = \prod_{i\in I} V_i$, $V_i = W(C, U_i)$, of $0 \in \prod_{i\in I} \mathrm{cHom}(G, G_i)$ where J is a finite subset of I, $C \subseteq G$ is compact, U_j, $j \in J$, is an open neighborhood of $0 \in G_j$, $\forall\, i \notin J : U_i = G_i$. Then $W := W(C, \prod_{i\in I} V_i)$ is an open neighborhood of $0 \in \prod_{i\in I} \mathrm{cHom}(G, \prod_{i\in I} G_i)$ and $\pi[W] \subseteq V$.

To show that π is open, we consider a basic open subset U of $\mathrm{Hom}(A, \prod_{i\in I} G_i)$, i.e., $U = W(C, \prod_i U_i)$ where C is compact in G and $\prod_i U_i$ is open in $\prod_{i\in I} G_i$, i.e., there is a finite subset J of I such that $\forall\, i \in J : U_i$ is open in G_i and $\forall\, i \notin J : U_i = G_i$. Then $\forall\, i \in J : W(C, U_i)$ is open in $\mathrm{Hom}(G, G_i)$ and $\forall\, i \notin J : W(C, U_i) = W(C, G_i) = G_i$. Hence, $\prod_{i\in I} W(C, U_i)$ is open in $\prod_{i\in I} \mathrm{Hom}(G, G_i)$ and it is easily checked that $\pi[W(C, \prod_i U_i)] = \prod_{i\in I} W(C, U_i)$ showing that π is an open map. □

Let A be a discrete group and G any topological group. Then, the compact open topology on $\mathrm{Hom}(A, G)$ coincides with the subspace topology of $\mathrm{Hom}(A, G) \subseteq G^A$ where G^A carries the product topology (=topology of point-wise convergence). This is well-known and is easily seen noting that the compact subsets of A are exactly the finite subsets.

Let G and H be topological groups. Recall ([18], p. 1) that $\alpha \in \mathrm{cHom}(G, H)$ is ***proper*** if α is open onto its range. A short exact sequence $K \rightarrowtail G \twoheadrightarrow H$ is ***proper*** if both maps are proper. Embeddings of subgroups are examples of proper monomorphisms, and proper epimorphisms are quotient maps. For a subgroup H of an abelian group G, we denote by $H \overset{\mathrm{ins}}{\rightarrowtail} G$ the inclusion homomorphism, a proper map.

In ([1], Proposition 1.17, p. 12) Proposition 2 is proved for $G = \mathbb{T}$ in which case $\prod_{i\in I} \mathrm{Hom}(A_i, G)$ is compact and it is easy to show that Φ is a quotient map.

Proposition 2. *Let A_i, $i \in I$, be a family of discrete abelian groups, G a topological abelian group. Then*

$$\Phi : \prod_{i\in I} \mathrm{Hom}(A_i, G) \overset{\rightarrow}{\cong}_{\mathrm{t}} \mathrm{Hom}\left(\bigoplus_{i\in I} A_i, G\right) : (\Phi((\ldots, f_i, \ldots)))\left(\sum_{i\in I} a_i\right) = \sum_{i\in I} f_i(a_i).$$

Proof. Let $\mathrm{ins}_i : A_i \to \bigoplus_{i\in I} A_i$ be the insertions belonging to the direct sum. The map Φ is the standard algebraic isomorphism and

$$\Phi^{-1} : \mathrm{Hom}(\textstyle\bigoplus_{i\in I} A_i, G) \to \prod_{i\in I} \mathrm{Hom}(A_i, G) : \Phi^{-1}(f) = (\ldots, f \circ \mathrm{ins}_i, \ldots).$$

We first show that Φ^{-1} is continuous. By definition of the product topology, Φ^{-1} is continuous if and only if $\pi_i \circ \Phi^{-1} : \mathrm{Hom}(\bigoplus_{i\in I} A_i, G) \to \mathrm{Hom}(A_i, G)$ is continuous where $\pi_i : \prod_{j\in I} \mathrm{Hom}(A_j, G) \to \mathrm{Hom}(A_i, G)$ is the projection belonging to the product. Let U be an open neighborhood of $0 \in G$ and let F be a finite subset of A_i. Then $W := W(F, U)$ is a generic neighborhood of $0 \in \mathrm{Hom}(A_i, G)$. As $F \subseteq \bigoplus_{i\in I} A_i$, the set $W' = \{f \in \mathrm{Hom}(\bigoplus_{i\in I} A_i, G) \mid f[F] \subseteq U\}$ is an open neighborhood of $0 \in \mathrm{Hom}(\bigoplus_{i\in I} A_i, G)$. Evidently $\Phi^{-1}[W'] \subseteq W$.

We show next that Φ is continuous. Let F be a finite subset of $\bigoplus_{i\in I} A_i$ and U an open neighborhood of $0 \in G$. Then $W = W(F, U)$ is a generic open neighborhood of $0 \in \mathrm{Hom}(\bigoplus_{i\in I} A_i, G)$. Then there is a finite subset J of I such that $F \subseteq \bigoplus_{i\in J} A_i$. Furthermore, for $j \in J$, there exist finite sets $B_j \subseteq A_j$ such that $F \subseteq \sum_{j\in J} B_j$. For $i \notin J$, let $B_j = \{0\}$. There exists an open neighborhood V of $0 \in G$ such that $V^{|J|} \subseteq U$. Then $\prod_{i\in I} W(B_i, V)$ is an open neighborhood of $0 \in \prod_{i\in I} \mathrm{Hom}(A_i, G)$. We claim that $\Phi[\prod_{i\in I} W(B_i, V)] \subseteq W(F, U)$. Let $f = (f_i) \in \prod_{i\in I} W(B_i, V)$ and $b = \sum_{j\in J} b_j \in F$. Then $(\Phi(f))(b) = \sum_{j\in J} f_j(b_j) \in V^{|J|} \subseteq U$. □

The following is surely well-known.

Lemma 2. *Let G and H be topological abelian groups and $\varphi : G \to H$ a surjective homomorphism with kernel K.*

(1) *Suppose that φ is continuous and K is dense in G. Then H is indiscrete.*
(2) *Suppose that φ is an open map and H is indiscrete. Then K is dense in G.*
(3) *Suppose that H is indiscrete and* $\mathrm{cHom}(G, H)$ *is endowed with the compact-open topology. Then* $\mathrm{cHom}(G, H)$ *is indiscrete.*

Proof. (1) Suppose that C is a non-empty closed subset of H. Then $\varphi^{-1}[C]$ is closed in G containing K. As K is dense in G it follows that $\varphi^{-1}[C] = G$. It follows that $C = H$. Hence, the only closed sets in H are H and $\emptyset$, so H is indiscrete.

(2) Let $x \in G$ and $U = -U$ a symmetric open neighborhood of $0 \in G$. Then $\varphi[U]$ is non-empty and open in H and as H is indiscrete, $\varphi[U] = H$. Hence, there is $y \in U$ such that $\varphi(y) = \varphi(x)$ and so $x - y = z \in K$ and $z \in x + U$ showing that K is dense in G.

(3) The open sets of $\mathrm{cHom}(G, H)$ are the sets of the form $W := W(C, U) = \{f \in \mathrm{cHom}(G, H) \mid f[C] \subseteq U\}$ where C is a compact subset of G and U is an open subset of H. By hypothesis $U = \emptyset$ or $U = H$. Whatever C may be, in the first case $W = \emptyset$ and in the second case $W = \mathrm{cHom}(G, H)$. □

3. The Meet Semi-Lattice $\mathcal{F}(A)$ of Full Free Subgroups in AG

The following notation relating an arbitrary group A with its torsion-free quotient $A_0 := A/\operatorname{tor}(A)$ will be used throughout.

Let $A \in \mathrm{AG}$ and let $\varphi_0 : A \to A_0$ be the natural epimorphism. For future use we record the short exact sequence

$$\mathrm{E}_0 : \operatorname{tor}(A) \overset{\text{ins}}{\rightarrowtail} A \overset{\varphi_0}{\twoheadrightarrow} A_0.$$

It is well-known that $\mathbb{Q}A_0 := \mathbb{Q} \otimes_{\mathbb{Z}} A_0 \cong \mathbb{Q} \otimes_{\mathbb{Z}} A$ is a $\mathbb{Q}$–vector space containing $A_0 \cong \mathbb{Z} \otimes_{\mathbb{Z}} A_0$ as an essential subgroup. Thus $\mathbb{Q}A_0$ is a divisible hull of A_0. The ***rank*** of A is the dimension of $\mathbb{Q}A_0$: $\mathrm{rk}(A) := \mathrm{rk}(A_0) := \dim_{\mathbb{Q}}(\mathbb{Q}A_0)$.

For $F \in \mathcal{F}(A)$, set $F_0 := \varphi_0[F] = \frac{F \oplus \operatorname{tor}(A)}{\operatorname{tor}(A)} \cong F$. Then $\mathrm{rk}(A) = \mathrm{rk}(F) = \mathrm{rk}(F_0)$.

In the literature, the dimension of a compact abelian group is defined in several equivalent ways. The cardinal $\dim(G) = \mathrm{rk}(G^\wedge)$ will serve for the purposes of this article.

For every prime p we define the ***p*-rank of** A by $\mathrm{rk}_p(A) := \dim_{\mathbb{Z}/p\mathbb{Z}}(A[p])$.

A discrete divisible group D is determined up to isomorphism by the invariants $\mathrm{rk}_p(D)$ counting the summands isomorphic to $\mathbb{Z}(p^\infty)$ and $\mathrm{rk}(D/\operatorname{tor}(D))$ counting the summands isomorphic to $\mathbb{Q}$. See ([17], Chapter 4) for details.

Lemma 3. *If A is a torsion-free group, then* $\mathrm{rk}_p(A/pA) \leq \mathrm{rk}(A)$.

Proof. It suffices to check that if $\{b_1, \ldots, b_n\}$ is a linearly independent subset of A/pA, where $b_i = a_i + pA$, $a_i \in A$, then $\{a_1, \ldots, a_n\}$ is linearly independent in A. Assume that $m_1 a_1 + \cdots + m_n a_n = 0$, with $m_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, n$. As A is torsion-free, we can assume without loss of generality that $\gcd(p, m_j) = 1$ for some $j = 1, 2, \ldots, n$. After projecting in A/pA we obtain $m_1 b_1 + \cdots + m_n b_n = 0$. By the choice of $\{b_1, \ldots, b_n\}$ this gives $m_i = p\mathbb{Z}$ for all i. This contradicts $\gcd(p, m_j) = 1$. □

Now we see that the p-ranks of a compact connected group G of finite dimension are bounded from above by $\dim(G)$.

Corollary 1. *Let A be a discrete torsion-free group of finite rank n. Then* $\mathrm{rk}_p(A^\wedge) = \mathrm{rk}_p(A/pA) \leq n$.

Proof. Clearly, $G = A^\wedge$ is a compact connected group with $\dim(G) = n$. The socle $G[p]$ of G is the kernel of μ_p^G, the multiplication by p in G, and hence closed and therefore compact.

We have the proper exact sequence $G[p] \rightarrowtail G \overset{\mu_p^G}{\twoheadrightarrow} G$ which gives the proper exact sequence

$G^\wedge \overset{\mu_p^{G^\wedge}}{\rightarrowtail} G^\wedge = A^{\wedge\wedge} = A \twoheadrightarrow A/pA \cong G[p]^\wedge$. Hence, $\mathrm{rk}_p(G[p]^\wedge) = \mathrm{rk}_p(A/pA) \leq \mathrm{rk}(A) = n < \infty$, by Lemma 3. Thus, $G[p] \cong G[p]^\wedge$ and $\mathrm{rk}_p(G) = \mathrm{rk}_p(G[p]) = \mathrm{rk}_p(G[p]^\wedge) \leq n$. □

We first illuminate the abundance of full free subgroups in a group.

Lemma 4. *Let* $\mathrm{tor}(A) \neq A \in \mathrm{AG}$. *Then the following hold.*

(1) $\{a_i \mid i \in I\}$ *is a linearly independent set in A if and only if* $\{a_i + \mathrm{tor}(A) \mid i \in I\}$ *is a linearly independent set in* A_0. *Moreover,* $\{a_i \mid i \in I\}$ *is maximal linearly independent if and only if* $\{a_i + \mathrm{tor}(A) \mid i \in I\}$ *is maximal linearly independent.*

(2) *If* $\{a_i \mid i \in I\}$ *is a (maximal) linearly independent set in A and* $\forall\, i \in I : t_i \in \mathrm{tor}(A)$, *then* $\{a_i + t_i \mid i \in I\}$ *is a (maximal) linearly independent subset of A.*

(3) *Every linearly independent set extends to a maximal linearly independent set. In particular, every torsion-free element in A is contained in a maximal linearly independent subset.*

(4) *If* $\{a_i \mid i \in I\}$ *is a maximal linearly independent subset of A, then* $F = \bigoplus_{i\in I} \mathbb{Z}a_i$ *is a full free subgroup of A. Conversely, if* $F = \bigoplus_{i\in I} \mathbb{Z}a_i$ *is a full free subgroup of A, then* $\{a_i \mid i \in I\}$ *is a maximal linearly independent subset of A.*

(5) *If* $F \in \mathcal{F}(A)$, *then* $F_0 \cong F$ *and* $A_0/F_0 \cong A/(F \oplus \mathrm{tor}(A))$, $F_0 \in \mathcal{F}(A_0)$, *and* $\varphi_0^{-1}[F_0] = F \oplus \mathrm{tor}(A)$.

(6) *Given* $F_0 \in \mathcal{F}(A_0)$, *there exists* $F \in \mathcal{F}(A)$ *such that* $\varphi_0[F] = F_0$ *and* $\varphi^{-1}[F_0] = F \oplus \mathrm{tor}(A)$. *If* $F, F' \in \mathcal{F}(A)$ *and* $F_0 = F_0'$, *then there is* $\varphi \in \mathrm{Hom}(F, \mathrm{tor}(A))$ *such that* $F' = \{\varphi(x) + x \mid x \in F\}$. *Note that* $\mathrm{Hom}(F, \mathrm{tor}(A)) \cong \mathrm{tor}(A)^{\mathrm{rk}(A)}$

(7) *A maximal linearly independent subset of* A_0 *is a* $\mathbb{Q}$*–basis of* $\mathbb{Q}A_0$.

(8) *If* $\{v_i \mid i \in I\}$ *is a* $\mathbb{Q}$*–basis of* $\mathbb{Q}A_0$, *then there exist positive integers* m_i *such that* $\forall\, i \in I$: $m_i v_i \in A$ *and* $F = \bigoplus_{i\in I} \mathbb{Z}(m_i v_i)$ *is a full free subgroup of A.*

Proof. Maximal linearly independent subsets exist by Zorn's Lemma.

(6) Suppose that $F, F' \in \mathcal{F}(A)$ and $F_0 = F_0'$. Then $F \oplus \mathrm{tor}(A) = F' \oplus \mathrm{tor}(A)$. By ([19], Lemma 1.1.3, p. 6) there exists $\varphi \in \mathrm{Hom}(F, \mathrm{tor}(A))$ such that $F' = \{\varphi(x) + x \mid x \in F\}$.

The rest consists of easy and well-known observations. □

We always assume that $A_0 \neq \{0\}$, i.e., we assume that A is not a torsion group. The dual $T^\wedge$ of a torsion group T is a compact totally disconnected group.

Theorem 1. *For* $A \in \mathrm{AG}$, *the family* $\mathcal{F} := \mathcal{F}(A)$ *has the following properties.*

(1) *Let* $F, F' \in \mathcal{F}$. *Then* $F \cap F' \in \mathcal{F}$.

(2) *If* $F \in \mathcal{F}$, $F' \leq F$ *and* F/F' *is a torsion group, then* $F' \in \mathcal{F}$.

(3) *If* $F \in \mathcal{F}$, *then* $\forall\, m \in \mathbb{N} : mF \in \mathcal{F}$ *and* $\bigcap_m mF = \{0\}$.

(4) $\bigcap \mathcal{F} = \{0\}$. *If* $A \neq \mathrm{tor}(A)$ *then* $\bigcup \mathcal{F} = A \setminus \mathrm{tor}(A)$ *and* $\sum \mathcal{F} = A$.

(5) $\mathcal{F}$ *is a meet semi-lattice with meet* $\cap$.

(6) *Let* $F \in \mathcal{F}$. *Then* $\forall\, m \in \mathbb{N} : m_A^{-1}F = F' \oplus A[m]$ *for some* $F' \in \mathcal{F}$, *and* $(F' \oplus A[m])/F = (A/F)[m]$. *If A is torsion-free, then* $m_A^{-1}F \in \mathcal{F}$.

Proof. (1) Certainly $F \cap F'$ is free as subgroup of free groups. The map $A/(F \cap F') \to A/F \oplus A/F' : a + (F \cap F') \mapsto (a + F, a + F')$ is well-defined and injective. Hence, $A/(F \cap F')$ is torsion.

(2) and (3) are trivial.

(4) It follows from (3) that $\bigcap \mathcal{F} = \{0\}$. If $A \neq \mathrm{tor}(A)$, then $\bigcup \mathcal{F} = A \setminus \mathrm{tor}(A)$ is evident, and it follows from Lemma 4(2) that $\sum \mathcal{F} = A$.

(5) Follows from (1).

(6.1) We first assume that A is torsion-free. Then the multiplication μ_m^A is injective, and $\mu_m^A : A \to mA$ is an isomorphism. Thus, $m^{-1}F = (\mu_m^A)^{-1}[mA \cap F]$ is free since F is free. As $F \subseteq m_A^{-1}F$ it follows that $m_A^{-1}F \in \mathcal{F}$.

(6.2) Recall that $F_0 \in \mathcal{F}(F_0)$ and by (6.1) $F_0 \subseteq m_{A_0}^{-1}F_0 \in \mathcal{F}(A_0)$ with $(m_{A_0}^{-1}F_0)/F_0 = (A_0/F_0)[m]$. It is straightforward to check that $\varphi_0[m_A^{-1}F] \subseteq m_{A_0}^{-1}F_0$ and it follows that

$\varphi_0[m_A^{-1}F]$ is free. Hence, the epimorphism $\varphi_0 : m_A^{-1}F \to \varphi_0[m_A^{-1}F]$ splits with kernel $m_A^{-1}F \cap \mathrm{tor}(A) = \mathrm{tor}(A)[m]$. Hence, $m_A^{-1}F = F' \oplus \mathrm{tor}(A)[m]$ for some free group $F' \cong \varphi_0[m_A^{-1}F]$.

It remains to show that A/F' is a torsion group. As A_0/F_0 is torsion and $F_0 \subseteq \varphi_0[m_A^{-1}F]$, we see that $A_0/\varphi_0[m_A^{-1}F]$ is a torsion group. Let $a \in A$. Then there is $k \in \mathbb{N}$ such that $k\varphi_0(a) \in \varphi_0[m_A^{-1}F] = \varphi_0[F']$. Hence, $k\varphi_0(a) = \varphi_0(ka) = \varphi_0(b)$ for some $b \in F'$ and $ka - b \in \mathrm{Ker}(\varphi_0) = \mathrm{tor}(A)$. Thus, there is $k' \in \mathbb{N}$ such that $k'ka = k'b \in F'$. Finally, $(F' \oplus A[m])/F = (m_A^{-1}F)/F = (A/F)[m]$. □

Remark 1. *In general, $\mathcal{F}(A)$ is not closed under finite sums, so $\mathcal{F}(A)$ may not be a lattice, and therefore, $A = \sum \mathcal{F}(A)$ may not be the directed union (direct limit) of its members. However, for $A = A_0$, using Theorem 1(6) (with $\mathrm{tor}(A) = \{0\}$), given $F \in \mathcal{F}(A)$, also the larger $m_A^{-1}F$ is a full free subgroup, and as A/F is a torsion group, we obtain an ascending chain*

$$F = (1!)_A^{-1}F \subseteq (2!)_A^{-1}F \subseteq \cdots \subseteq (m!)_A^{-1}F \subseteq ((m+1)!)_A^{-1}F \subseteq \cdots$$

of full free subgroups of A whose union is A.

In the case of a torsion-free group A of finite rank, the quotients A/F for $F \in \mathcal{F}(A)$ are somewhat alike ([2], Theorem 3.5(9)). For arbitrary rank there is a great variety of quotients A/F.

Proposition 3. *Let A be an abelian group of infinite rank $\mathfrak{m}$. Let $F \in \mathcal{F}(A)$. Then $\mathrm{rk}(F) = |F| = \mathfrak{m}$. Let T be any torsion group that is $\mathfrak{m}$–generated. Then there is an epimorphism $\varphi : F \twoheadrightarrow T$ with $F_\varphi := \mathrm{Ker}(\varphi) \in \mathcal{F}(A)$, and there is an exact sequence $T \rightarrowtail A/F_\varphi \twoheadrightarrow A/F$. Moreover, $\mathrm{rk}_p(A/F) \leq \mathfrak{m}$.*

Proof. Routine and simple. □

In the case of infinite rank, the sum of two full free subgroups need not be free, as shown by Jim Reid (([20], Theorem 2.2)):

Theorem 2. *Let A be a torsion-free group of infinite rank.*

(a) *(([20], Theorem 2.2) and its proof) Given a free subgroup F of A with $\mathrm{rk}(F) = \mathrm{rk}(A)$, there is a second free subgroup F_1 such that $A = F + F_1$.*
(b) *([20], Corollary 3.5) There exists a full free subgroup F_0 of A such that A/F_0 is divisible (A is "quotient divisible").*

One can deduce from (a) that in a torsion-free group A of infinite rank every non-free subgroup of torsion index is the sum of two full free subgroups.

Definition 1. *An abelian group A is **$\mathcal{F}$-summable** if for any $F_1, F_2 \in \mathcal{F}(A)$ also $F_1 + F_2 \in \mathcal{F}(A)$.*

Theorem 2(a) yields:

Theorem 3. *$A \in \mathrm{AG}$ is $\mathcal{F}$-summable if and only if A is either torsion-free of finite rank or is free of arbitrary rank.*

Proof. If $\mathrm{tor}(A) \neq \{0\}$, then there are full free subgroups whose sum contains torsion elements (Lemma 4(2)). So a summable group must be torsion-free.

Suppose that A is torsion-free and $\mathcal{F}$-summable of infinite rank. Then A is the sum of two free subgroups and hence of two full free subgroups. As A is summable, it is free. The converse is clear.

If the torsion-free group A has finite rank, then full free subgroups are finitely generated and finitely generated torsion-free subgroups are free. □

4. The Semi-Lattices $\mathcal{F}(A)$ and $\mathcal{D}(G)$

Let $A \in \mathrm{AG}$ and $G = A^\wedge$. Then G is compact, not necessarily connected. Let $F \in \mathcal{F}(A)$. Then $F \overset{\mathrm{ins}}{\rightarrowtail} A \overset{\alpha}{\twoheadrightarrow} A/F$ is exact where α is the natural epimorphism. Therefore,

$$(A/F)^\wedge \overset{\alpha^\wedge}{\rightarrowtail} G \overset{\mathrm{restr}}{\twoheadrightarrow} F^\wedge$$

is exact, where $F^\wedge$ is a torus isomorphic to $\mathbb{T}^{\mathrm{rk}(A)}$ and $\alpha^\wedge[(A/F)^\wedge]$ is a compact totally disconnected subgroup of G. Hence, $\alpha^\wedge[(A/F)^\wedge] \in \mathcal{D}(G)$. We obtained the mapping

$$\mathcal{F}(A) \to \mathcal{D}(G) : F \mapsto \alpha^\wedge\big[(A/F)^\wedge\big]. \tag{1}$$

Let G be a compact group and $A = G^\wedge$. Then A is a possibly mixed group. Let $\Delta \in \mathcal{D}(G)$. Then $\Delta \overset{\mathrm{ins}}{\rightarrowtail} G \overset{\beta}{\twoheadrightarrow} G/\Delta$ is exact where β is the natural epimorphism and G/Δ is a torus. Therefore,

$$(G/\Delta)^\wedge \overset{\beta^\wedge}{\rightarrowtail} A \overset{\mathrm{restr}}{\twoheadrightarrow} \Delta^\wedge$$

is exact, where $\Delta^\wedge$ is a torsion group and $\beta^\wedge[(G/\Delta)^\wedge]$ is a full free subgroup of A. Hence, $\beta^\wedge[(G/\Delta)^\wedge] \in \mathcal{F}(A)$. We obtained

$$\mathcal{D}(G) \to \mathcal{F}(A) : \Delta \mapsto \beta^\wedge\big[(G/\Delta)^\wedge\big]. \tag{2}$$

Lemma 5. *Let $G \in \mathrm{LCA}$ and H a closed subgroup of G. The sequence $H \overset{\mathrm{ins}}{\rightarrowtail} G \overset{\varphi}{\twoheadrightarrow} G/H$ is exact in* LCA. *Hence,*

$$(G/H)^\wedge \overset{\varphi^\wedge}{\rightarrowtail} G^\wedge \overset{\mathrm{restr}}{\twoheadrightarrow} H^\wedge \tag{3}$$

is exact in LCA, *and $\varphi^\wedge[(G/H)^\wedge] = (G^\wedge, H)$.*

Proof. Suppose first that $\chi \in (G^\wedge, H)$. Then $\chi[H] = \{0\}$, so $\chi \restriction_H = 0$. Hence, χ is in the kernel of the restriction map in (3), i.e., $\chi \in \mathrm{Ker(restr)} = \alpha^\wedge[(A/F)^\wedge]$. Conversely, if $\chi \in \mathrm{Ker(restr)}$, then $\chi[H] = \{0\}$ and $\chi \in (G^\wedge, H)$. □

For a general topological abelian group G, the family $\mathrm{Lat}(G)$ of closed subgroups is a lattice with the operations $C_1 \wedge C_2 = C_1 \cap C_2$ and $C_1 \vee C_2 = \overline{C_1 + C_2}$. There also exist greatest lower bounds and least upper bounds for infinite families: Let $\mathcal{C}$ be a family of closed subgroups of G. Then $\bigcap \mathcal{C}$ is a closed subgroup of G and $\bigcap \mathcal{C} = \bigwedge \mathcal{C}$. The subgroup $\overline{\sum \mathcal{C}}$ is closed and $\overline{\sum \mathcal{C}} = \bigvee \mathcal{C}$. See ([1], p. 361).

We will establish that $\mathcal{F}(A)$ and $\mathcal{D}(A^\wedge)$ are anti-isomorphic semi-lattices. To do so, we use results of ([1], p. 351) where we find annihilators $H^\perp$ defined as follows.

For $G \in \mathrm{LCA}$, we have the pairing $G^\wedge \times G \to \mathbb{T} : (\chi, g) \mapsto \chi(g)$. For a subset X of G, we define the ***annihilator*** $X^\perp$ ***of*** $X \subseteq G$ ***in*** $G^\wedge$ by $X^\perp := (G^\wedge, X) = \{\chi \in G^\wedge \mid \chi[X] = 0\}$ while for $Y \subseteq G^\wedge$, we define $Y^\perp = \{g \in G \mid \forall \rho \in Y : \rho(g) = 0\}$

Note that $X^{\perp\perp} \subseteq G$ is not the same as $(G^{\wedge\wedge}, X^\perp) = (G^{\wedge\wedge}, (G^\wedge, X))$. However, they are topologically isomorphic:

Lemma 6. *Let $A \in \mathrm{LCA}$. Then, for $X \subseteq A$, the natural evaluation isomorphism $\eta_A : A \to A^{\wedge\wedge}$ restricts to an isomorphism $X^{\perp\perp} \to (A^{\wedge\wedge}, X^\perp)$, $\eta_A\big[X^{\perp\perp}\big] = (A^{\wedge\wedge}, X^\perp)$. In particular, $X^{\perp\perp}$ is a full free subgroup of A if and only if $(A^{\wedge\wedge}, X^\perp)$ is a full free subgroup of $A^{\wedge\wedge}$.*

Proof. We need to check that $\eta_A[X^{\perp\perp}] = (A^{\wedge\wedge}, X^\perp)$. Let $a \in A$. Then
$\eta_A(a) \in (A^{\wedge\wedge}, X^\perp) \iff \forall \chi \in X^\perp : \eta_A(a)(\chi) = \chi(a) = 0 \iff a \in X^{\perp\perp}$. □

We rely on the basic properties of annihilators ([1], pp. 351–362), in particular see ([1], Theorem 7.64, p. 392); alternatively see ([21], pp. 270–275).

Theorem 4 ([1], Theorem 7.64(iv),(v), (vi), p. 392)**.** *Let $G \in \mathrm{LCA}$. Then $H \mapsto H^\perp = (G^\wedge, H)$ with $H^{\perp\perp} = H$, is a lattice anti-isomorphism between $\mathrm{Lat}(G)$ and $\mathrm{Lat}(G^\wedge)$. In particular, $H \subseteq K$ if and only if $K^\perp \subseteq H^\perp$.*

If $H \rightarrowtail G \twoheadrightarrow G/H$ is proper exact in LCA, then

$$(G/H)^\wedge \cong_t H^\perp \quad \text{and} \quad H^\wedge \cong_t G^\wedge/H^\perp.$$

Theorem 5. *Let $A \in \mathrm{AG}$ and $G = A^\wedge$. The lattice anti-isomorphism $H \mapsto H^\perp$ of Theorem 4 restricts to an anti-isomorphism of semi-lattices $\delta : \mathcal{F}(A) \to \mathcal{D}(G)$. In particular we have:*

- *$\mathcal{D}(G)$ is a join semi-lattice with join $+$.*
- *$\forall F, F_1, F_2 \in \mathcal{F}(A) : \delta(F) = F^\perp \in \mathcal{D}(G)$; if $F_1 \subseteq F_2$, then $\delta(F_2) \subseteq \delta(F_1)$; $\delta(F_1 \cap F_2) = \delta(F_1) + \delta(F_2)$.*

Proof. By Theorem 4 we only need to show that $\delta(\mathcal{F}(A)) = \mathcal{D}(G)$.

Let $F \in \mathcal{F}(A)$. Then $F^\perp = (G, F)$ by definition, and $(G, F) = \alpha^\wedge[(A/F)^\wedge] \in \mathcal{D}(G)$ by Lemma 5 and (1). So δ is well defined.

Let $\Delta \in \mathcal{D}(G)$. By (2) $\beta^\wedge[(G/\Delta)^\wedge] \in \mathcal{F}(A^{\wedge\wedge})$ and by Lemma 6 $\Delta^\perp \in \mathcal{F}(A)$ and $\delta(\Delta^\perp) = \Delta^{\perp\perp} = \Delta$. □

We now establish, for a compact group $G = A^\wedge$, the properties of $\mathcal{D}(G)$ corresponding to the properties of $\mathcal{F}(A)$. Recall that for any $m \in \mathbb{N}$ and any subgroup Y of X, we have $m_X^{-1}Y = \{x \in X \mid mx \in Y\}$.

Definition 2. *For a compact abelian group G set $\mathbf{\Delta}(G) := \sum \mathcal{D}(G)$.*

We collect here some properties of the subgroup $\mathbf{\Delta}(G)$, "Fat Delta".

Theorem 6. *Let $G = A^\wedge$, $A \in \mathrm{AG}$. The family $\mathcal{D} := \mathcal{D}(G)$ has the following properties.*

(1) $\mathcal{D}$ is a join semi-lattice with join $+$. Hence, $\mathbf{\Delta}(G) = \bigcup \mathcal{D}$.
(2) $\mathbf{\Delta}(G)$ is dense in G, while $\bigcap \mathcal{D} = \{0\}$ if $c(G) \neq \{0\}$, otherwise $\bigcap \mathcal{D} = G$.
(3) Let $\Delta = \delta(F)$ and $\Delta' = \delta(F')$ and assume that $\Delta \subseteq \Delta'$. Then $F' \subseteq F$ and $\Delta'/\Delta \cong_t (F/F')^\wedge$.
(4) If $\Delta \in \mathcal{D}$, then $m_G^{-1}\Delta \in \mathcal{D}$ for any $m \in \mathbb{N}$. Hence, $\mathrm{tor}(G/\Delta) \subseteq \mathbf{\Delta}/\Delta$ and $\mathrm{tor}(G) \subseteq \mathbf{\Delta}(G)$.
(5) Let $\Delta \in \mathcal{D}$ and $m \in \mathbb{N}$. Then there is $\Delta' \in \mathcal{D}$ such that $m\Delta = \Delta' \cap mG$. If A is torsion-free, then $m\Delta \in \mathcal{D}$.
(6) $G/\mathbf{\Delta}(G)$ is torsion-free.

Proof. (1) Theorem 5 establishes the semi-lattice property. As $\mathcal{D}$ is closed under finite sums, we have $\sum \mathcal{D} = \bigcup \mathcal{D}$.

(2) By ([1], Theorem 7.64(vii), p. 392) $\overline{\bigcup \mathcal{D}} = (G, \bigcap \mathcal{F}(A)) = (G, 0) = G$ and $\bigcap \mathcal{D} = (G, \sum \mathcal{F}(A))$. If $A \neq \mathrm{tor}(A)$, then $\sum \mathcal{F}(A) = A$, by Theorem 1(4). So, $\overline{\bigcup \mathcal{D}} = (G, A) = \{0\}$ in this case. If G is totally disconnected, then $\mathcal{D} = \{G\}$, so $\bigcap \mathcal{D} = G$.

(3) We have the following commutative diagram with natural maps and exact rows

$$\begin{array}{ccccc} & & & & F/F' \\ & & & & \downarrow \text{ins} \\ F' & \xrightarrow{\text{ins}} & A & \xrightarrow{\varphi'} & A/F' \\ \downarrow \text{ins} & & \downarrow = & & \downarrow \psi \\ F & \xrightarrow{\text{ins}} & A & \xrightarrow{\varphi} & A/F \end{array} \quad \text{and its dual} \quad \begin{array}{ccccc} (A/F)^\wedge & \xrightarrow{\varphi^\wedge} & G & \xrightarrow{\text{restr}} & F^\wedge \\ \downarrow \psi^\wedge & & \downarrow = & & \downarrow \text{restr} \\ (A/F')^\wedge & \xrightarrow{(\varphi')^\wedge} & G & \xrightarrow{\text{restr}} & (F')^\wedge \\ \downarrow \text{restr} & & & & \\ (F/F')^\wedge & & & & \end{array}$$

We conclude that

$$\frac{\Delta'}{\Delta} = \frac{(\varphi')^\wedge[(A/F')^\wedge]}{\varphi^\wedge[(A/F)^\wedge]} \cong_{\mathrm{t}} \frac{(A/F')^\wedge}{\psi^\wedge[(A/F)^\wedge]} \cong_{\mathrm{t}} \left(\frac{F}{F'}\right)^\wedge$$

(4) Let $\Delta = F^\perp = (G, F) \in \mathcal{D}$ with $F \in \mathcal{F}(A)$. If $m \in \mathbb{N}$, then $m_G^{-1}\Delta = m_G^{-1}(G, F) = (G, mF)$ (cf. [21], Lemma 6.4.14, p. 274), so since $mF \in \mathcal{F}(A)$ we have $m_G^{-1}\Delta \in \mathcal{D}$.

(5) Let $\Delta = \delta(F)$ for some $F \in \mathcal{F}(A)$. By Theorem 1(3) we know that $m_A^{-1}F = F' \oplus A[m]$ for some $F' \in \mathcal{F}(A)$. Using ([21], Lemma 6.4.13, p. 27) and ([21], Lemma 6.4.15, p. 27) we obtain $m\Delta = m\delta(F) = \overline{m(G, F)} = (G, m_A^{-1}F) = (G, F' + A[m]) = (G, F') \cap (G, A[m]) = \delta(F') \cap \overline{mG}$. Furthermore, $\delta(F)$ and G are both compact and hence, so are $m\delta(F)$ and mG, therefore closed, and equal to the closures.

(6) Let $x \in G$. If $mx \in \mathbf{\Delta}(G)$ for some $m \in \mathbb{N}$, then $mx \in \Delta$ for some $\Delta \in \mathcal{D}$. Then $x \in m_G^{-1}\Delta \in \mathcal{D}$ by (4), thus $x \in \mathbf{\Delta}(G)$. Therefore $G/\mathbf{\Delta}(G)$ is torsion-free. □

The fact that linearly independent sets can be enlarged to maximal linearly independent sets has the following dual.

Proposition 4. *Let $G = A^\wedge$ be a compact abelian group of infinite dimension.*

Suppose that Θ is a subgroup of G such that G/Θ is a torus of dimension $\mathfrak{m}$. Then Θ contains some $\Delta \in \mathcal{D}(G)$ and $\mathfrak{m} \le \dim(G)$.

Proof. $\Theta = E^\perp$ for some subgroup E of A (Theorem 4). We claim that E is a free subgroup of A. From $\Theta \overset{\mathrm{ins}}{\rightarrowtail} G \twoheadrightarrow T$ where T is a torus of dimension $\mathfrak{m}$, we conclude the exact sequence $T^\wedge \rightarrowtail A^{\wedge\wedge} \overset{\mathrm{restr}}{\twoheadrightarrow} \Theta^\wedge$. By Lemma 5 $T^\wedge \cong (A^{\wedge\wedge}, \Theta)$ and by Lemma 6 $(A^{\wedge\wedge}, \Theta) \cong (A, \Theta) = E^{\perp\perp} = E$. As $T^\wedge$ is free of rank $\mathfrak{m}$ as the dual of a torus, so is E and $\mathfrak{m} \le \mathrm{rk}(A)$. Let F be a full free subgroup containing E. Then $\Delta := F^\perp \in \mathcal{D}(G)$ and $\Theta = E^\perp \supseteq F^\perp = \Delta$. □

Let $G = A^\wedge$. We next study the connection between $\mathcal{D}(G)$, $\mathcal{D}(\mathrm{c}(G))$, $\mathbf{\Delta}(G)$, and $\mathbf{\Delta}(\mathrm{c}(G))$. Given A, let $T = \mathrm{tor}(A)$ and let $F \in \mathcal{F}(A)$. Then, we obtain the following commutative diagram with exact rows and its dual.

$$\begin{array}{ccccc} & & F & \xrightarrow{\varphi_0} & F_0 \\ & & \downarrow{\scriptstyle \mathrm{ins}} & & \downarrow{\scriptstyle \mathrm{ins}} \\ T & \xrightarrow{\mathrm{ins}} & A & \xrightarrow{\varphi_0} & A_0 \\ \downarrow{\scriptstyle \psi_T} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_0} \\ \frac{T\oplus F}{F} & \xrightarrow{\mathrm{ins}} & \frac{A}{F} & \xrightarrow{\varphi_0} & \frac{A_0}{F_0} \end{array} \quad \text{with dual} \quad \begin{array}{ccccc} \left(\frac{A_0}{F_0}\right)^\wedge & \xrightarrow{\varphi_0^\wedge} & \left(\frac{A}{F}\right)^\wedge & \xrightarrow{\mathrm{restr}} & \left(\frac{T\oplus F}{F}\right)^\wedge \\ \downarrow{\scriptstyle \psi_0^\wedge} & & \downarrow{\scriptstyle \psi^\wedge} & & \downarrow{\scriptstyle \psi_T^\wedge} \\ A_0^\wedge & \xrightarrow{\varphi_0^\wedge} & G & \xrightarrow{\mathrm{restr}} & T^\wedge \\ \downarrow{\scriptstyle \mathrm{restr}} & & \downarrow{\scriptstyle \mathrm{restr}} & & \\ F_0^\wedge & \xrightarrow{\varphi_0^\wedge} & F^\wedge & & \end{array}$$

We now set

- $\Delta := \psi^\wedge[\left(\frac{A}{F}\right)^\wedge]$
- $G_0 := \varphi_0^\wedge[A_0^\wedge]$
- $\Delta_0 := (\varphi_0^\wedge \circ \psi_0^\wedge)[\left(\frac{A_0}{F_0}\right)^\wedge] = (\psi^\wedge \circ \varphi_0^\wedge))[\left(\frac{A_0}{F_0}\right)^\wedge] \subseteq G$

and obtain

$$\begin{array}{ccccc} \Delta_0 & \xrightarrow{\mathrm{ins}} & \Delta & \xrightarrow{\mathrm{restr}_\Delta} & T^\wedge \\ \downarrow{\scriptstyle \mathrm{ins}} & & \downarrow{\scriptstyle \mathrm{ins}_\Delta} & & \downarrow{\scriptstyle =} \\ G_0 & \xrightarrow{\mathrm{ins}} & G & \xrightarrow{\mathrm{restr}} & T^\wedge \\ \downarrow{\scriptstyle \mathrm{restr}} & & \downarrow{\scriptstyle \mathrm{restr}_G} & & \\ F^\wedge & \xrightarrow{=} & F^\wedge & & \end{array} \tag{4}$$

We have the following easy consequences.

Theorem 7. *Let $G = A^\wedge$ where $A \in \mathrm{AG}$ and consider* (4)*, where $T = \mathrm{tor}(A)$.*

(1) $G_0 = \varphi_0^\wedge[A_0^\wedge] = \mathrm{tor}(A)^\perp$ *coincides with the* 0*–component* $\mathrm{c}(G)$ *of* G.
(2) $\mathrm{c}(G)$ *is divisible and so, algebraically,* $G \cong \mathrm{c}(G) \oplus T^\wedge$ *and* $G \cong_{\mathrm{t}} \mathrm{c}(G) \oplus T^\wedge$ *if and only if* A *splits, i.e.,* $A \cong A_0 \oplus T$.
(3) $\Delta_0 = \Delta \cap \mathrm{c}(G)$. *Thus* $\mathcal{D}(\mathrm{c}(G)) = \{D \cap \mathrm{c}(G) \mid D \in \mathcal{D}(G)\}$, $\mathbf{\Delta}(\mathrm{c}(G)) = \mathbf{\Delta}(G) \cap \mathrm{c}(G)$, *and* $\mathbf{\Delta}(\mathrm{c}(G))$ *is closed in* $\mathbf{\Delta}(G)$.
(4) $G = \Delta + \mathrm{c}(G)$ *and* $\mathbf{\Delta}(G) = \Delta + \mathbf{\Delta}(\mathrm{c}(G))$.
(5) $\mathrm{c}(G)/\Delta_0 \cong_{\mathrm{t}} G/\Delta$ *and* $\Delta/\Delta_0 \cong_{\mathrm{t}} G/\mathrm{c}(G) \cong_{\mathrm{t}} T^\wedge$.
(6) *With the established notation* $\mathbf{\Delta}(\mathrm{c}(G))$ *is divisible and hence algebraically a direct summand of* $\mathbf{\Delta}(G)$.
(7) *There is a topological isomorphism* $\mathrm{tor}(A)^\wedge \cong_{\mathrm{t}} \frac{\Delta}{\Delta_0} \to \frac{\mathbf{\Delta}(G)}{\mathbf{\Delta}(\mathrm{c}(G))}$.

Proof. (1) As A_0 is torsion-free, its dual G_0 is connected and G/G_0 is totally disconnected. Hence, G_0 is the 0–component of G: $\mathrm{c}(G) = G_0$. The equality $\varphi_0^\wedge[A_0^\wedge] = \mathrm{tor}(A)^\perp$ follows from Lemma 5.

(2) $\mathrm{c}(G)$ is divisible as the dual of a torsion-free group. The rest is evident.

(3) It follows from the definition that $\Delta_0 \subseteq \Delta \cap \mathrm{c}(G)$. On the other hand, let $x \in \Delta \cap \mathrm{c}(G)$. Then $0 = \mathrm{restr}_G(x) = (\mathrm{restr}_G \circ \mathrm{ins}_\Delta)(x) = \mathrm{restr}_\Delta(x)$, hence, $x \in \Delta_0$. This proves the equality $\Delta_0 = \Delta \cap \mathrm{c}(G)$.

The topological isomorphism $A_0^\wedge \to \mathrm{c}(G) = \varphi_0^\wedge[A_0^\wedge]$ maps the family $\mathcal{D}(A_0^\wedge)$ onto $\mathcal{D}(\mathrm{c}(G))$. Thus, the annihilator $(A_0^\wedge, (F \oplus T)/T)$, a typical member of $\mathcal{D}(A_0^\wedge)$, is mapped onto $\Delta_0 = (G, F) \cap \mathrm{c}(G)$, a typical member of $\mathcal{D}(\mathrm{c}(G))$. Therefore, $\mathcal{D}(\mathrm{c}(G)) = \{D \cap \mathrm{c}(G) \mid D \in \mathcal{D}(G)\}$ and $\mathbf{\Delta}(\mathrm{c}(G)) = \mathbf{\Delta}(G) \cap \mathrm{c}(G)$.

(4) We have $\Delta + \mathrm{c}(G) = \psi^\wedge[(A/F)^\wedge] + \varphi_0^\wedge[A_0^\wedge] = (G, F) + (G, T) = (G, F \cap T) = G$. By (4) we have $\Delta + \mathbf{\Delta}(\mathrm{c}(G)) = \Delta + (\mathrm{c}(G) \cap \mathbf{\Delta}(G)) = (\Delta + \mathrm{c}(G)) \cap \mathbf{\Delta}(G) = \mathbf{\Delta}(G)$.

(5) Follows immediately from (3) and (4).

(6) $\mathrm{c}(G)$ is divisible and $\mathbf{\Delta}(\mathrm{c}(G))$ is pure in $\mathrm{c}(G)$. Hence, $\mathbf{\Delta}(\mathrm{c}(G))$ is divisible.

(7) We have the following commutative diagram with exact row and natural maps:

$$
\begin{array}{ccccc}
\Delta_0 & \xrightarrow{\ \mathrm{ins}\ } & \Delta & \xrightarrow{\ \alpha\ } & \frac{\Delta}{\Delta_0} \\
\downarrow{\scriptstyle \mathrm{ins}} & & \downarrow{\scriptstyle \mathrm{ins}} & & \downarrow{\scriptstyle \xi} \\
\mathbf{\Delta}(\mathrm{c}(G)) & \xrightarrow{\ \mathrm{ins}\ } & \mathbf{\Delta}(G) & \longrightarrow & \frac{\mathbf{\Delta}(G)}{\mathbf{\Delta}(\mathrm{c}(G))}
\end{array}
$$

The map ξ is injective because $\Delta \cap \mathbf{\Delta}(\mathrm{c}(G)) = \Delta_0$. By (5), $\mathbf{\Delta}(G) = \Delta + \mathbf{\Delta}(\mathrm{c}(G))$, so ξ is surjective. To show that ξ is continuous, let U be open in $\mathbf{\Delta}(G)/\mathbf{\Delta}(\mathrm{c}(G))$. By commutativity of the right square in the diagram, $W := \{x \in \Delta \mid \xi(\alpha(x)) \in U\}$ is open in Δ, thus $\xi^{-1}[U] = \alpha[W]$ is open in Δ/Δ_0. Therefore, ξ is continuous. Since Δ/Δ_0 is compact, we conclude that ξ is a topological isomorphism. □

A number of results on free subgroups are worth dualizing.

Theorem 8. *Let $G = A^\wedge$ be a compact abelian group of infinite dimension.*

(1) *Suppose that* D *is a closed subgroup of* G *such that* G/D *is a torus. Then there exists* $\Delta \in \mathcal{D}(G)$ *such that* $D \cap \Delta \cap \mathrm{c}(G) = 0$. *In particular, for every* $\Delta \in \mathcal{D}(G)$*, there is* $\Delta' \in \mathcal{D}(G)$ *such that* $\Delta \cap \Delta' \cap \mathrm{c}(G) = 0$.
(2) *There exists* $\Delta \in \mathcal{D}(G)$ *such that* $\Delta_0 = \Delta \cap \mathrm{c}(G) \in \mathcal{D}(\mathrm{c}(G))$ *is torsion-free.*

Proof. (1) Let $F = D^\perp$. Then $F \cong_{\mathrm{t}} (G/D)^\wedge$ is a free subgroup of A. So $\varphi_0[F]$ is free in A_0. By Theorem 2(a), there exists a free subgroup F_1 of A_0 such that $A_0 = F + F_1$. By enlarging F_1 if necessary we may assume that F_1 is maximal, i.e., full free. There exists a (full) free subgroup F_2 of A such that $\varphi_0[F_2] = F_1$. Then $A = F + F_2 + \mathrm{tor}(A)$ and $0 = F^\perp \cap F_2^\perp \cap \mathrm{tor}(A)^\perp$. The claim is established by setting $\Delta' = F_2^\perp$.

(2) By Theorem 2(b), there exists a full free subgroup F_0 of A_0 such that A_0/F_0 is divisible. There exists $F \in \mathcal{F}(A) : \varphi_0[F] = F_0$ and $\frac{\mathrm{tor}(A)\oplus F}{F} \rightarrowtail \frac{A}{F} \twoheadrightarrow \frac{A_0}{F_0}$ is exact. By duality $(A_0/F_0)^\wedge \rightarrowtail (A/F)^\wedge \twoheadrightarrow \mathrm{tor}(A)^\wedge$. As A_0/F_0 is divisible, its dual is torsion-free ([1], corollary 8.5, p. 410), so $\Delta_0 := (A_0/F_0)^\wedge \in \mathcal{D}(A_0^\wedge)$ is torsion-free. Modulo embeddings $\Delta_0 \subseteq \Delta = (A/F)^\wedge \in \mathcal{D}(A)$ and $\Delta_0 = \Delta \cap \mathrm{c}(G)$. □

Corollary 2. *Let $G = A^\wedge$ be a compact connected abelian group of infinite dimension, i.e., A is torsion-free of infinite torsion-free rank.*

(1) Suppose that D is a subgroup of G such that G/D is a torus. Then there exists a subgroup D' of G such that $D \cap D' = 0$ and G/D' is a torus. In particular, for every $\Delta \in \mathcal{D}(G)$, there is $\Delta' \in \mathcal{D}(G)$ such that $\Delta \cap \Delta' = 0$.

(2) There exists a torsion-free $\Delta \in \mathcal{D}(G)$.

We can easily settle the question when $\mathbf{\Delta}(G)$ is as small as possible, i.e., $\mathbf{\Delta}(G) = \mathrm{tor}(G)$.

Theorem 9. *Let $G = A^\wedge$ be a compact abelian group. Then $\mathbf{\Delta}(G) = \mathrm{tor}(G)$ if and only if $G \cong_{\mathrm{t}} \mathbb{T}^n \times E$ where $n \in \mathbb{N}_0$ and E is bounded.*

Proof. We only need to consider the consequences of $\mathbf{\Delta}(G)$ being a torsion group. As $\mathbf{\Delta}(G) = \sum \mathcal{D}(G)$, this occurs if and only if every $\Delta \in \mathcal{D}(G)$ is a torsion group. Since Δ is compact, it must be bounded torsion. Furthermore, we use that for every $F \in \mathcal{F}(A)$, the dual $(A/F)^\wedge$ is topologically isomorphic to some $\Delta \in \mathcal{D}(G)$, so a bounded torsion group.

(a) Assume first that A is torsion-free. By Corollary 2(2) we have $n := \mathrm{rk}(A) < \infty$. Now pick an arbitrary $F \in \mathcal{F}(A)$. Since $(A/F)^\wedge$ is a bounded torsion group, so is A/F, hence, $mA \subseteq F$ for some $m \in \mathbb{N}$, so $A \cong mA$ is free of rank n and $G \cong_{\mathrm{t}} \mathbb{T}^n$.

(b) In the general situation, by Theorem 7(3), $\mathbf{\Delta}(\mathrm{c}(G))$ must be a torsion group and hence by (b), $A/\mathrm{tor}(A)$ must be free of finite rank. So $A = F \oplus \mathrm{tor}(A)$ for some finite rank free subgroup F of A, thus $G \cong_{\mathrm{t}} H \times E$ with $H \cong_{\mathrm{t}} \mathbb{T}^n$ and $E \cong_{\mathrm{t}} \mathrm{tor}(A)^\wedge \cong_{\mathrm{t}} (A/F)^\wedge$. For the latter group to be torsion, it must be bounded. □

Remark 2. *The dual concept (in the category sense of reversing arrows) of $\mathcal{F}(A)$ is the family $\mathcal{K}(A) := \{\mathrm{Ker}(\psi) \mid \psi \in \mathrm{Hom}(A, F), F \text{ free}\}$. It is easy to see that $\mathcal{K}(A)$ is closed under finite intersections and $\mathfrak{K}(A) = \bigcap \mathcal{K}(A) = \bigcap\{\mathrm{Ker}(\psi) \mid \psi \in \mathrm{Hom}(A, \mathbb{Z})\}$ is a fully invariant subgroup of A that has no free direct summands. Mostly we have $\mathfrak{K}(A) = A$, e.g., if A is divisible or torsion. We call A **free-reduced** if $\mathrm{Hom}(A, \mathbb{Z}) = \{0\}$, equivalently, if A has no free direct summands. Evidently,* A is free-reduced if and only if $G = A^\wedge$ is torus-free. *For a compact group G, let $\mathcal{T}(G) = \{T \mid T \text{ is a torus subgroup of } G\}$.*

For any $A \in$ AG, and a short exact sequence $K \overset{\mathrm{ins}}{\rightarrowtail} A \overset{\varphi_F}{\twoheadrightarrow} F$ where F is free, it follows that $F^\wedge \overset{\varphi_F^\wedge}{\rightarrowtail} G := A^\wedge \overset{\mathrm{restr}}{\twoheadrightarrow} K^\wedge$ is exact in LCA. *Hence, $\varphi_F^\wedge[F^\wedge] = K^\perp$ is a torus subgroup of G and we have a map*

$$\kappa : \mathcal{K}(A) \to \mathcal{T}(G) : \kappa(K) = K^\perp.$$

As for $\mathcal{F}(A)$ and $\mathcal{D}(G)$ it follows that κ is a bijective map satisfying $\kappa(K_1 \cap K_2) = \kappa(K_1) + \kappa(K_2)$ and $K_1 \subseteq K_2$ if and only if $\kappa(K_2) \subseteq \kappa(K_1)$. In particular, $\mathfrak{T}(G) := \sum \mathcal{T}(G) = \bigcup \mathcal{T}(G)$ and $\kappa(\mathfrak{K}(A)) = (\mathfrak{K}(A))^\perp = \mathfrak{T}(G)$. This recaptures most of the results of ([1], p.p 440, 441). The group $A/\mathfrak{K}(A)$ need not be free and the dual group $\mathfrak{T}(G)$ need not be a torus.

A theorem of K. Stein ([17], Corollary 8.3, p. 114) says that every countable torsion-free group A_0 has a decomposition $A_0 = F \oplus \mathfrak{K}(A_0)$ where F is free. The duals of countable torsion-free groups are exactly the compact connected metric groups ([1], pp. 447–450).

Remark 3. *One can ask further whether a compact group has other connected factors of dimension* 1 *(so-called solenoids of which* $\mathbb{T}$ *is an example). For finite dimensional connected compact groups this leads to the "Main Decomposition" that was derived in [22].*

5. The Fat Delta of Compact Groups

So far we know from Theorem 6 that for any compact abelian group G,

- $\mathbf{\Delta}(G) = \sum \mathcal{D}(G) = \bigcup \mathcal{D}(G)$,
- $\mathbf{\Delta}(G)$ is dense in G,
- $\mathrm{tor}(G) \subseteq \mathbf{\Delta}(G)$ and $G/\mathbf{\Delta}(G)$ is torsion-free.
- $\mathbf{\Delta}(\mathrm{c}(G))$ is divisible.

In this section, we will establish further properties of Fat Delta. We start with a preliminary observation.

Lemma 7. *Let G and H be compact abelian groups and let $\alpha : G \to H$ be a continuous epimorphism. Then we have:*

(1) If G is totally disconnected, then so is H.
(2) If G is a torus, then so is H.

Proof. (1) Since α is surjective, the adjoint map $\alpha^\wedge : H^\wedge \to G^\wedge$ is injective ([23], (24.38), p. 392). Assume that G is totally disconnected. Then $G^\wedge$ is torsion, thus $H^\wedge$ is torsion and therefore H is totally disconnected (see [23], (24.26), p. 385).

(2) Now suppose G is a torus. Then $G^\wedge$ is free, so since subgroups of free groups are free, $H^\wedge$ is free. Thus, H is a torus. □

In general, Fat Delta does not contain every totally disconnected subgroup. However, it contains all closed totally disconnected subgroups:

Proposition 5. *Let G be a compact abelian group and D a closed totally disconnected subgroup of G. Then $D \subseteq \mathbf{\Delta}(G)$. Thus $\mathbf{\Delta}(G)$ is the subgroup of G generated by all closed totally disconnected subgroups of G.*

Proof. Choose $\Delta \in \mathcal{D}(G)$. Then $\Delta + D$ is compact ([23], (4.4), p. 17), and the natural map $\alpha : \Delta \times D \to \Delta + D$ is a continuous epimorphism. By Lemma 7(1), $\Delta + D$ is totally disconnected. Now consider the continuous epimorphism $\beta : G/\Delta \to (G/\Delta)/[(\Delta + D)/\Delta] \cong_t G/(\Delta + D)$ (see [23], (5.35), p. 45). Since G/Δ is a torus, so is $G/(\Delta + D)$ by Lemma 7(2). This means that $\Delta + D \in \mathcal{D}(G)$, so $D \subseteq \Delta + D \subseteq \mathbf{\Delta}$ as claimed. □

The significance of Proposition 5 is that it shows that for a compact group G Fat Delta $\mathbf{\Delta}(G)$ coincides with the subgroup $\mathrm{td}(G)$ that is defined and motivated by totally different considerations (see Definition 4 and Proposition 9(2)). For the sake of easy reference, we list the results that could be proved easily in the present context but are proved in greater generality in the exhaustive study of $\mathrm{td}(G)$ in Section 7.

Proposition 6. *Let G be a compact abelian group. Then the following are true.*

(1) $\mathbf{\Delta}(G)$ is zero-dimensional, in particular totally disconnected (Theorem 19). Consequently, if G is not totally disconnected, then $G \neq \mathbf{\Delta}(G)$ and hence $\mathbf{\Delta}(G)$ is not a locally compact subgroup of G.
(2) Any countable extension of $\mathbf{\Delta}(G)$ is zero-dimensional (in particular totally disconnected) as well (Proposition 11).

$\mathbf{\Delta}(A^\wedge) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$

We first establish some background.

Lemma 8. *Let G, H, K be topological abelian groups.*

(1) *Suppose that H is a topological subgroup of K such that for all $f \in \mathrm{cHom}(G,K)$ we have $f[G] \subseteq H$. Let $\mathrm{ins} : H \to K$ be the insertion. Then $\mathrm{ins}_* : \mathrm{cHom}(G,H) \to \mathrm{cHom}(G,K) : \mathrm{ins}_*(f) = \mathrm{ins} \circ f$ is a topological isomorphism.*

(2) *Suppose that $H \overset{\alpha}{\rightarrowtail} K \overset{\beta}{\twoheadrightarrow} L$ is a short exact sequence in TAG, α is proper, and G is some other topological group. Then*

$$\mathrm{cHom}(G,H) \overset{\alpha_*}{\rightarrowtail} \mathrm{cHom}(G,K) \overset{\beta_*}{\to} \mathrm{cHom}(G,L), \text{ where } \alpha_*(f) = \alpha \circ f,\ \beta_*(f) = \beta \circ f,$$

is an exact sequence in TAG and α_* is proper. The map β_* is not claimed to be surjective.

(3) *Let $A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\twoheadrightarrow} C$ be a short exact sequence of discrete groups and let G be a divisible topological group. Then*

$$\mathrm{cHom}(C,G) \overset{\beta^*}{\rightarrowtail} \mathrm{cHom}(B,G) \overset{\alpha^*}{\twoheadrightarrow} \mathrm{cHom}(A,G), \text{ where } \beta^*(f) = f \circ \beta,\ \alpha^*(f) = f \circ \alpha,$$

is an exact sequence of topological groups. In addition, β^* is proper.

(4) *For a discrete torsion group $T = \bigoplus_{p\in\mathbb{P}} \mathrm{tor}_p(T)$, we have $\mathrm{cHom}(T,\mathbb{Q}/\mathbb{Z}) \cong_{\mathrm{t}} T^\wedge$, the topological isomorphism being ins_*, and $T^\wedge \cong_{\mathrm{t}} \prod_{p\in\mathbb{P}}(\mathrm{tor}_p(T))^\wedge$ where $(\mathrm{tor}_p(T))^\wedge \cong_{\mathrm{t}} \mathrm{Hom}(\mathrm{tor}_p(T),\mathbb{Z}(p^\infty))$.*

Proof. (1) It is evident that ins_* is bijective and maps $W(C, H\cap V)$ onto $W(C,V)$ where C is compact in G and V is open in K.

(2) By standard discrete homological algebra

$$\mathrm{Hom}(G,H) \overset{\alpha_*}{\rightarrowtail} \mathrm{Hom}(G,K) \overset{\beta_*}{\to} \mathrm{Hom}(G,L) \to \mathrm{Ext}(G,H)$$

is exact in AG. Let $f \in \mathrm{cHom}(G,H)$. Then $\alpha_*(f) = \alpha \circ f$ is continuous and α_* is well-defined. Similarly, $\beta_* : \mathrm{cHom}(G,K) \to \mathrm{cHom}(G,L)$ is well-defined.

To show that α_* is continuous, let C be a compact subset of G and let U_K be an open neighborhood of $0 \in K$. Then $V := W(C,U_K)$ is a basic open neighborhood of $0 \in \mathrm{cHom}(G,K)$. It follows that $U := W(C,\alpha^{-1}[U_K])$ is an open neighborhood of $0 \in \mathrm{cHom}(G,H)$ and α_* maps U into V as is easily checked.

We show next that our sequence is exact at $\mathrm{cHom}(G,K)$. As $\beta_* \circ \alpha_* = (\beta\circ\alpha)_* = 0$ we have $\mathrm{Im}(\alpha_*) \subseteq \mathrm{Ker}(\beta_*)$. To show that $\mathrm{Ker}(\beta_*) \subseteq \mathrm{Im}(\alpha_*)$, let $f \in \mathrm{Ker}(\beta_*)$. By the discrete exactness there exist $g \in \mathrm{Hom}(G,H)$ such that $f = \alpha \circ g$. To conclude, we need to show that g is continuous. To do so let U be open in H. By assumption α is proper, hence, there is an open set $V \subseteq K$ such that $\alpha[U] = \alpha[H] \cap V$. Then $U \subseteq \alpha^{-1}[\alpha[H]\cap V] = \alpha^{-1}[V]$ and actually $U = \alpha^{-1}[V]$. In fact, let $x \in H$ such that $\alpha(x) \in V \cap \alpha[H] = \alpha[U]$. So there exists $x' \in U$ such that $\alpha(x) = \alpha(x')$ and as α is injective, $x = x' \in U$. We now get that $g^{-1}[U] = g^{-1}[\alpha^{-1}[V]] = f^{-1}[V]$ is open in G, showing that g is continuous.

It remains to show that α_* is proper. Let C be compact in G and U open in H. Then $W(C,U)$ is a generic open set in $\mathrm{cHom}(G,H)$ and $W(C,V)$, where $\alpha[U] = \alpha[H] \cap V$, is open in $\mathrm{cHom}(G,K)$. We claim that

$$\alpha_*[W(C,U)] = \alpha_*[\mathrm{cHom}(G,H)] \cap W(C,V).$$

Let $f \in W(C,U)$. Then $f[C] \subseteq U$ and hence $\alpha_*(f)[W(C,U)] \subseteq \alpha_*[\mathrm{cHom}(G,H)] \cap W(C,V)$ because $(\alpha\circ f)[C] \subseteq \alpha[U] \subseteq V$.

Now let $g \in \alpha_*[\mathrm{cHom}(G,H)] \cap W(C,V)$. Then there is $f \in \mathrm{cHom}(G,H)$ such that $g = \alpha \circ f$. We show that $f \in W(C,U)$. In fact, $(g[C] \subseteq V) \Longrightarrow ((\alpha[f[C]] \subseteq \alpha[H] \cap V = \alpha[U]$. As α is injective it follows that $f[C] \subseteq U$, i.e., $f \in W(C,U)$.

(3) By discrete abelian group theory we have that $\mathrm{Hom}(C,G) \overset{\beta^*}{\rightarrowtail} \mathrm{Hom}(B,G) \overset{\alpha^*}{\twoheadrightarrow} \mathrm{Hom}(A,G) \to \mathrm{Ext}(C,G)$ is exact and $\mathrm{Ext}(C,G) = \{0\}$ as G is divisible. We have $\mathrm{cHom} = \mathrm{Hom}$ as A,B,C are discrete, so the exactness of the claimed sequence is clear.

We need to show that β^* and α^* are continuous when the Hom groups are given the compact-open topology.

To show that β^* is continuous, let K be a compact (=finite) subset of B and let U_G be an open neighborhood of $0 \in G$. Then $V := W(K, U_G)$ is a generic open neighborhood of $0 \in \mathrm{cHom}(B, G)$. Hence, $U := W(\beta[K], U_G)$ is an open neighborhood of $0 \in \mathrm{cHom}(C, G)$. Let $f \in U$. Then $f[\beta[K]] = \beta^*(f)[K] \subseteq U_G$, i.e., $\beta^*(f) \in V$ showing that β^* is continuous. Similarly, α^* is continuous.

It remains to show that β^* is proper. The set $V := W(K, U_G)$, where K is compact (=finite) in C and U_G open in G, is a generic open subset of $\mathrm{cHom}(C, G)$. As β is surjective, there is a finite subset K' of B such that $\beta[K'] = K$. Then $U := W(K', U_G)$ is an open subset of $\mathrm{cHom}(B, G)$. We claim that $\beta^*[V] = U \cap \beta^*[\mathrm{cHom}(C, G)]$. In fact, $f \in V$ means that $f[K] \subseteq U_G$ and hence $\beta^*(f)[K'] = f[\beta[K']] = f[K] \subseteq U_G$, so $\beta^*[V] \subseteq U \cap \beta^*[\mathrm{cHom}(C, G)]$. To show equality, let $g \in U \cap \beta^*[\mathrm{cHom}(C, G)]$. Then there exists $f \in \mathrm{cHom}(C, G)$ such that $g = f \circ \beta$ and $U_G \supset g[K'] = (f \circ \beta)[K'] = f[\beta[K']] = f[K]$, so $f \in V$.

(4) By Lemma 8(1) we have $\mathrm{cHom}(T, \mathbb{Q}/\mathbb{Z}) \cong_{\mathrm{t}} \mathrm{cHom}(T, \mathbb{T}) \cong_{\mathrm{t}} T^\wedge$. By Proposition 2 $T^\wedge \cong_{\mathrm{t}} \prod_{p \in \mathbb{P}} (\mathrm{tor}_p(T))^\wedge$, and again Lemma 8(1) entails $(\mathrm{tor}_p(T))^\wedge \cong_{\mathrm{t}} \mathrm{Hom}(\mathrm{tor}_p(T), \mathbb{Z}(p^\infty))$. □

Lemma 9. *Let K, X, Y, K', X', Y' be topological abelian groups. It is assumed that the diagram*

$$\begin{array}{ccccc} K & \xrightarrow{\text{ins}} & X & \xrightarrow{\alpha} & Y \\ \downarrow{\scriptstyle \xi_K} & & \downarrow{\scriptstyle \xi_X} & & \downarrow{\scriptstyle \eta} \\ K' & \xrightarrow{\text{ins}} & X' & \xrightarrow{\beta} & Y' \end{array}$$

is commutative, all maps are continuous, its rows are exact, ξ_X is proper, β is a quotient map, i.e., β is open, and ξ_K is an isomorphism. Then α is a quotient map.

Proof. Let U be open in X. As ξ is proper, there is an open set V of X' such that $\xi_X[U] = \xi_X[X] \cap V$. We claim that $\alpha[U] = \eta^{-1}[\beta[V]]$ that is open in Y.

First let $x \in U$. Then $\eta(\alpha(x)) = \beta(\xi_X(x)) \in \beta[V]$, hence, $\alpha(x) \in \eta^{-1}\beta[V]$.

Now suppose that $y \in \eta^{-1}[\beta[V]] \subseteq Y$. Then $\eta(y) = \beta(v)$ for some $v \in V$. There exists $x \in X$ such that $\alpha(x) = y$. Hence, $\beta(\xi_X(x)) = \eta(\alpha(x)) = \beta(v)$, and thus, $v - \xi_X(x) \in \mathrm{Ker}(\beta)$. It follows that there exists $k \in K$ such that $\xi_K(k) = v - \xi_X(x)$ and so $\xi_X(k + x) = v \in \xi[X] \cap V = \xi[U]$. As ξ is injective it follows that $k + x \in U$ and $\alpha(k + x) = \alpha(x) = y$. □

We have the proper short exact sequence of topological groups

$$\mathrm{E} : \mathbb{Q}/\mathbb{Z} \overset{\text{ins}}{\rightarrowtail} \mathbb{T} \overset{\gamma}{\twoheadrightarrow} \mathbb{R}/\mathbb{Q}$$

where, as usual, $\mathbb{T}$ is the quotient group of $\mathbb{R}$, $\mathbb{Q}/\mathbb{Z}$ the subgroup of $\mathbb{T}$, and $\mathbb{R}/\mathbb{Q}$ carries the quotient topology which is indiscrete as $\mathbb{Q}$ is dense in $\mathbb{R}$ (Lemma 2(1)).

Let A be a discrete group and F a full free subgroup of A of rank $\mathfrak{m} := \mathrm{rk}(A)$. We have exact sequences $F \overset{\text{ins}}{\rightarrowtail} A \overset{\varphi_F}{\twoheadrightarrow} A/F$ where A/F is a torsion group. We obtain a diagram as follows.

$$\begin{array}{ccccc} \mathrm{Hom}(A/F, \mathbb{Q}/\mathbb{Z}) & \overset{\varphi_F^*}{\rightarrowtail} & \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) & \overset{\text{ins}^*}{\twoheadrightarrow} & \mathrm{Hom}(F, \mathbb{Q}/\mathbb{Z}) \\ \downarrow{\scriptstyle \text{ins}_*} & & \downarrow{\scriptstyle \text{ins}_*} & & \downarrow{\scriptstyle \text{ins}_*} \\ \mathrm{Hom}(A/F, \mathbb{T}) & \overset{\varphi_F^*}{\rightarrowtail} & \mathrm{Hom}(A, \mathbb{T}) & \overset{\text{ins}^*}{\twoheadrightarrow} & \mathrm{Hom}(F, \mathbb{T}) \\ & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} \\ & & \mathrm{Hom}(A, \mathbb{R}/\mathbb{Q}) & \overset{\text{ins}^*}{\twoheadrightarrow} & \mathrm{Hom}(F, \mathbb{R}/\mathbb{Q}) \end{array} \quad (5)$$

(1) By standard discrete homological algebra the diagram is commutative and rows and columns are exact.
(2) All the domains of the Hom groups carry the discrete topology, hence cHom = Hom in all cases.
(3) All Hom groups in the diagram carry the compact-open topology. It follows from Lemma 8(2) that all the maps $(\cdot)_*$ are continuous. It follows from Lemma 8(3) that all the maps $(\cdot)^*$ are continuous.
(4) By Lemma 8(1) the left most ins_* is a topological isomorphism.
(5) By Lemma 8(2) columns 2 and 3 are exact in TAG.
(6) By Lemma 8(3) the three rows are exact in TAG.
(7) The situation of Lemma 9 matches the top part of (5) and we conclude that $\text{ins}^* : \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(F, \mathbb{Q}/\mathbb{Z})$ is a quotient map. It is easy to see that $\text{ins}^* : \text{Hom}(A, \mathbb{R}/\mathbb{Q}) \to \text{Hom}(F, \mathbb{R}/\mathbb{Q})$ is an isomorphism. Since both groups are indiscrete (by Lemma 2(3)), this is a topological isomorphism.

Theorem 10. *Let $G = A^\wedge$ where $A \in \text{AG}$ has torsion-free rank $\mathfrak{m}$. Then:*

(1) $\mathbf{\Delta}(G) = \text{ins}_*[\text{Hom}(A, \mathbb{Q}/\mathbb{Z})] \subseteq G$ *where* $\text{ins} : \mathbb{Q}/\mathbb{Z} \to \mathbb{T}$.
(2) $G/\mathbf{\Delta}(G) \cong \mathbb{R}^{\mathfrak{m}}$. *Algebraically,* $\text{c}(G) = \mathbf{\Delta}(\text{c}(G)) \oplus K$ *where* $K \cong \mathbb{R}^{\mathfrak{m}}$.
(3) *If $G_i = A_i^\wedge$ where $A_i \in \text{AG}$ $(i \in I)$, then* $\mathbf{\Delta}(\prod_{i\in I} G_i) \cong_{\text{t}} \prod_{i \in I} \mathbf{\Delta}(G_i)$.

Proof. (1) Row 2 of (5) implies that $\varphi_F^*(\text{Hom}(A/F, \mathbb{T})) \subseteq \text{ins}_*[\text{Hom}(A, \mathbb{Q}/\mathbb{Z})]$ where $\Delta = \varphi_F^*(\text{Hom}(A/F, \mathbb{T}))$ is a delta subgroup of G. As F was arbitrary it follows that $\mathbf{\Delta}(G) \subseteq \text{ins}_*[\text{Hom}(A, \mathbb{Q}/\mathbb{Z})]$. It remains to show that $\mathbf{\Delta}(G) \supset \text{ins}_*[\text{Hom}(A, \mathbb{Q}/\mathbb{Z})]$.

Let $f \in \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ and set $K = \text{Ker}(f)$. Then $f[A] \subseteq \mathbb{Q}/\mathbb{Z}$ is a torsion group, so A/K is a torsion group and any full free subgroup F of K is a full free subgroup of A. Let F be so given. Then $g : A/F \to \mathbb{Q}/\mathbb{Z} : g(a + F) = f(a)$ is a well-defined homomorphism and $f = g \circ \varphi_F = \varphi_F^*(g)$, so $f \in \Delta \subseteq \mathbf{\Delta}(G)$.

(2) We have algebraic isomorphisms $G/\mathbf{\Delta}(G) \cong \text{Hom}(A, \mathbb{R}/\mathbb{Q}) \cong \text{Hom}(F, \mathbb{R}/\mathbb{Q}) \cong (\mathbb{R}/\mathbb{Q})^{\mathfrak{m}} \cong \mathbb{R}^{\mathfrak{m}}$, the first isomorphism granted by exactness of column 2 of (5), the second isomorphism by (7), and the remaining isomorphism is easy to see. The group $\mathbf{\Delta}(\text{c}(G))$ is divisible as observed earlier (see Theorem 6(6)), so it is algebraically a direct summand of $\text{c}(G)$. Since $\text{rk}(A_0) = \mathfrak{m}$, applying the above argument to $c(G) = A_0^\wedge$ we deduce that $c(G)/\mathbf{\Delta}(c(G)) \cong (\mathbb{R}/\mathbb{Q})^{\mathfrak{m}}$. Therefore, we have $\text{c}(G) = \mathbf{\Delta}(\text{c}(G)) \oplus K$, with $K \cong c(G)/\mathbf{\Delta}(c(G)) \cong (\mathbb{R}/\mathbb{Q})^{\mathfrak{m}} \cong \mathbb{R}^{\mathfrak{m}}$.

(3) Set $A = \bigoplus_{i \in I} A_i$ and $G = \prod_{i\in I} G_i$. Then we have $G = \prod_{i \in I} A_i^\wedge \cong_{\text{t}} A^\wedge$, so $G \cong_{\text{t}} A^\wedge$ and $\mathbf{\Delta}(G) \cong_{\text{t}} \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \cong_{\text{t}} \prod_{i\in I} \text{Hom}(A_i, \mathbb{Q}/\mathbb{Z}) \cong_{\text{t}} \prod_{i \in I} \mathbf{\Delta}(G_i)$. □

From now on, we will identify $\mathbf{\Delta}(G)$ with $\text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ if $G = A^\wedge$ is compact. The next corollary, establishing that $\mathbf{\Delta}$ is a functorial subgroup and showing that $\mathbf{\Delta}$, as a functor, preserves exactness of short sequences of compact groups, will be reproved in greater generality in Proposition 10.

Corollary 3. *(1) Let G and H be compact abelian groups and $g \in \text{cHom}(G, H)$. Then $g(\mathbf{\Delta}(G)) \subseteq \mathbf{\Delta}(H)$; in particular $\mathbf{\Delta}(G)$ is fully invariant in G and if $G \leq H$, then $\mathbf{\Delta}(G) \leq \mathbf{\Delta}(H)$.*
(2) Let G, H, K be compact abelian groups. Suppose that $G \rightarrowtail H \twoheadrightarrow K$ is a short exact sequence in TAG. *Then $\mathbf{\Delta}(G) \rightarrowtail \mathbf{\Delta}(H) \twoheadrightarrow \mathbf{\Delta}(K)$ is a short exact sequence in* TAG.

Proof. (1) Without loss of generality $G = A^\wedge$, $H = B^\wedge$ and $g = f^\wedge = f^*$ for some $f \in \text{Hom}(B, A)$. Then $\mathbf{\Delta}(G) = \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \xrightarrow{g} \text{Hom}(B, \mathbb{Q}/\mathbb{Z}) = \mathbf{\Delta}(H)$.

(2) Without loss of generality $G = A^\wedge, H = B^\wedge, K = C^\wedge$ and $G \rightarrowtail H \twoheadrightarrow K$ is the dual of $C \rightarrowtail B \twoheadrightarrow A$. Then $\text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \rightarrowtail \text{Hom}(B, \mathbb{Q}/\mathbb{Z}) \twoheadrightarrow \text{Hom}(C, \mathbb{Q}/\mathbb{Z})$ is an exact sequence of topological groups (Lemma 8(3)). □

The next proposition shows how Fat Delta can be used to recognize finite-dimensional compact groups.

Proposition 7. *Let G be a compact abelian group and $\Delta \in \mathcal{D}(G)$ with $G/\Delta \cong_t \mathbb{T}^\kappa$, where $\kappa = \dim(G) = \mathrm{rk}(G^\wedge)$. Then $\mathbf{\Delta}(G)/\Delta \cong (\mathbb{Q}/\mathbb{Z})^\kappa$. In particular, $\mathbf{\Delta}(G)/\Delta = \mathrm{tor}(G/\Delta)$ if and only if G is finite-dimensional.*

Proof. To the proper short exact sequence $\Delta \rightarrowtail G \twoheadrightarrow G/\Delta$ apply Corollary 3(2) to deduce that $\mathbf{\Delta}(G)/\Delta \cong \mathbf{\Delta}(G/\Delta)$, due to the fact that $\Delta = \mathbf{\Delta}(\Delta)$. Since $G/\Delta \cong_t \mathbb{T}^\kappa$, we have $\mathbf{\Delta}(G)/\Delta \cong \mathbf{\Delta}(\mathbb{T}^\kappa) \cong_t (\mathbf{\Delta}(\mathbb{T}))^\kappa = (\mathbb{Q}/\mathbb{Z})^\kappa$ by Theorem 10(3). Since $(\mathbb{Q}/\mathbb{Z})^\kappa$ is torsion precisely when $\kappa < \infty$, we are done. □

We will use the following well-known result below.

Theorem 11 ([24], page 86, Corollary 8.48)**.** *Let G, C be Hausdorff abelian groups, assume that C is complete, H is a dense subgroup of G. Then every morphism $f : H \to C$ has a unique extension $\overline{f} : G \to C$.*

Theorem 12. *Let G and H be compact abelian groups. Then $G \cong_t H$ if and only if $\mathbf{\Delta}(G) \cong_t \mathbf{\Delta}(H)$.*

Proof. (a) Suppose $\phi : G \to H$ is an isomorphism of topological groups. By Corollary 3 applied to ϕ and ϕ^{-1}, we obtain $\phi(\mathbf{\Delta}(G)) = \mathbf{\Delta}(H)$, hence $\mathbf{\Delta}(G) \cong_t \mathbf{\Delta}(H)$.

(b) Let $f : \mathbf{\Delta}(G) \to \mathbf{\Delta}(H)$ be an isomorphism of topological groups. The group H is compact, hence complete, and $\mathbf{\Delta}(G)$ is dense in G. Hence, there is a unique extension morphism $\overline{f} : G \to H$ of f. Similarly, we have the unique continuous extension $\overline{f^{-1}} : H \to G$ of $f^{-1} : \mathbf{\Delta}(H) \to \mathbf{\Delta}(G)$. The morphism $\overline{f^{-1}} \circ \overline{f} : G \to G$ extends $\mathrm{id}_{\mathbf{\Delta}(G)} : \mathbf{\Delta}(G) \to \mathbf{\Delta}(G)$ which is also extended by id_G, hence, by uniqueness we have $\overline{f^{-1}} \circ \overline{f} = \mathrm{id}_G$. Similarly, $\overline{f} \circ \overline{f^{-1}} = \mathrm{id}_H$. Hence, $(\overline{f})^{-1} = \overline{f^{-1}}$ is continuous. □

Theorem 12 and Corollary 3 imply that $G \mapsto \mathbf{\Delta}(G)$ is a category equivalence on the category of compact abelian groups to the category of all $\mathbf{\Delta}(G)$. This calls for *a useful characterization of the class of topological groups that appear as $\mathbf{\Delta}(G)$ for some compact abelian group G.* So far we can say the following. If D is a topological group such that $D \cong_t \mathbf{\Delta}(G)$ for some compact group G, then the following are true.

(1) D is totally disconnected and zero-dimensional (Proposition 11).
(2) The completion $\widehat{D}$ of D is compact (i.e., D is precompact), $D = \mathbf{\Delta}(\widehat{D})$ and $\mathrm{tor}(D) = \mathrm{tor}(\widehat{D})$.
(3) D contains a directed family $\mathcal{D}$ of compact totally disconnected subgroups such that $D = \bigcup \mathcal{D}$.
(4) $D \in \mathrm{LCA}$ if and only if $\widehat{D}$ is totally disconnected, and, if so, $D = \widehat{D}$ is compact.
(5) D is totally minimal (Theorem 19).

Given a group D with all the required properties, we would have $\mathbf{\Delta}(\widehat{D}) \cong_t D$, i.e., the completion functor is the inverse of the functor $\mathbf{\Delta}$.

Theorem 12 and the preceding discussion suggest to study the structure of $\mathbf{\Delta}(G)$ for a given compact group G. We will attempt this below in the simplest possible case of solenoids. A ***solenoid*** is a compact connected group of dimension 1, i.e., the dual of a torsion-free group of rank 1. To do so, we will use a simple result on divisible hulls of discrete groups and Lemma 10 on divisible hulls of certain products of groups.

Lemma 10. *Let P be a set of prime numbers, X_p be discrete groups and $X = \prod_{p\in P} X_p$. For each $p \in P$, let D_p be a divisible hull of X_p. Let $D := \prod_{p\in P} D_p$. Assume that each D_p/X_p is a p-primary group. Let $\mathrm{D}(X)$ be a subgroup of D containing X such that $\mathrm{D}(X)/X = \mathrm{tor}(D/X)$. Then*

(1) $\mathrm{D}(X)$ *is a divisible hull of* X,
(2) $\mathrm{D}(X) = \prod^{\mathrm{loc}}_{p\in P}(D_p, X_p) := \{(d_p) \in D \mid d_p \in X_p$ *for almost all* $p \in P\}$,

(3) $\mathrm{D}(X)/X \cong \bigoplus_{p\in P} D_p/X_p$.

Proof. (1) Clearly D is divisible as a product of divisible groups and

$$D/\mathrm{D}(X) \cong (D/X)/(\mathrm{D}(X)/X) \cong (D/X)/(\mathrm{tor}(D/X))$$

is torsion-free, hence $\mathrm{D}(X)$ is pure in D and therefore divisible. It remains to show that X is essential in $\mathrm{D}(X)$. For any prime q, we have $D[q] = \prod_{p\in P} D_p[q] \le \prod_{p\in P} X_p = X$. Indeed, for $q \ne p \in P$ we have $D_p[q] \le X_p$ because D_p/X_p is p-primary, while $D_q[q] \subseteq X_q$ because D_q is the divisible hull of X_q.

(2) Let $(d_p) \in \mathrm{D}(X)$. Then $m(d_p) \in X$ for some $m \ne 0$ which requires that $\forall\, p \in P$: $md_p \in X_p$. Our hypotheses imply that $d_p \in X_p$ for all those p that do not divide m. So $\mathrm{D}(X) \subseteq \prod^{\mathrm{loc}}_{p\in P}(D_p, X_p)$ and equality is evident.

(3) The map $\xi : \mathrm{D}(X) \to \bigoplus_{p\in P} D_p/X_p : \xi((d_p)) = \sum_{p\in P} d_p + X_p$ is evidently well-defined, surjective, and $\mathrm{Ker}(\xi) = X$. □

Torsion-free groups A with $\mathrm{rk}(A) = 1$, ***rank-one groups*** for short, are discussed and classified in ([17], Chapter 12, Section 1). These are exactly the groups isomorphic with additive subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$. Types are equivalence classes $[(h_p)_{p\in\mathbb{P}}]$ of "height sequences" $(h_p)_{p\in\mathbb{P}}$ where $0 \le h_p \le \infty$. Two height sequences are equivalent if they differ only at finitely many places where both sequences have finite entries. For the precise definition of type see Lemma 11(1) or ([17], p. 409, 411).

Two rank-one groups are isomorphic if and only if their types are equal.

Lemma 11 displays a representative rank-one group, its type, and dual solenoid. For a prime p, we define $\frac{1}{p^\infty}\mathbb{Z} := \left\langle \frac{1}{p^k}\mathbb{Z} \mid k \in \mathbb{N} \right\rangle$.

Lemma 11. *(1)* *Let $\mathbb{Z} \le A \le \mathbb{Q}$. Then there exist values h_p such that*

$$A = \left\langle \frac{1}{p^{h_p}}\mathbb{Z} \mid p \in \mathbb{P}, 0 \le h_p \le \infty \right\rangle, \frac{A}{\mathbb{Z}} \cong \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^{h_p}), \text{ and } \mathrm{tp}(A) = [(h_p)_{p\in\mathbb{P}}].$$

For $\mathbb{P}_\infty := \{p \mid h_p = \infty\}$, one has $p \in \mathbb{P}_\infty$ if and only if $pA = A$.

(2) *Let $\Sigma = A^\wedge$ and $\Delta := (A/\mathbb{Z})^\wedge$. Then (with a harmless identification) $\Delta \in \mathcal{D}(\Sigma)$, and $\Delta \cong_{\mathrm{t}} \prod_{p\in\mathbb{P}} \widehat{\mathbb{Z}}(p^{h_p})$ where $\widehat{\mathbb{Z}}(p^\infty) = \widehat{\mathbb{Z}}_p$ is the group of p-adic integers and $\widehat{\mathbb{Z}}(p^{h_p}) = \mathbb{Z}(p^{h_p})$ is the cyclic group of order p^{h_p} for $h_p < \infty$. Furthermore, Σ and $\mathbf{\Delta}(\Sigma)$ are divisible, $\mathbf{\Delta}(\Sigma)/\Delta \cong \mathbb{Q}/\mathbb{Z}$, and $\mathrm{tor}(\Sigma) \subseteq \mathbf{\Delta}(\Sigma)$.*

(3) $\mathrm{Soc}(\Sigma) = \bigoplus_{p\notin\mathbb{P}_\infty} \mathbb{Z}(p)$ *and* $\mathrm{tor}(\Sigma) = \bigoplus_{p\notin\mathbb{P}_\infty} \mathbb{Z}(p^\infty)$.

Proof. (1) Given $p \in \mathbb{P}$ either A contains every fraction $1/p^k$ (in which case $h_p = \infty$) or A contains a smallest fraction $1/p^{h_p}$. These fractions generate A and determine the type of A. (The h_p are the "p-heights" of $1 \in A$.)
To prove the last assertion, note that $pA = A$ implies $h_p = \infty$. Conversely, if $h_p = \infty$, then $\frac{1}{p^{h_p}}\mathbb{Z} = \langle \frac{p}{p^k}\mathbb{Z} \mid k \in \mathbb{N} \rangle = p\left(\frac{1}{p^{h_p}}\mathbb{Z}\right)$, thus $pA = A$.

(2) Σ is divisible by ([1], Corollary 8.5, p. 410). By Theorem 6(6) it follows that $\mathbf{\Delta}(\Sigma)$ is pure in Σ and hence is also divisible. By Proposition 7 $\mathbf{\Delta}(\Sigma)/\Delta \cong \mathbb{Q}/\mathbb{Z}$ and by Theorem 6(4) $\mathrm{tor}(\Sigma) \subseteq \mathbf{\Delta}(\Sigma)$. The rest is clear.

(3) By Corollary 1, $\mathrm{rk}_p(\Sigma) = \mathrm{rk}_p(A/pA) \le 1$.

According to (1), $\mathrm{rk}_p(A/pA) > 0$ if and only if $A \ne pA$, i.e., when $h_p < \infty$. Hence, $\mathrm{rk}_p(\Sigma) > 0$ if and only if $h_p < \infty$ (i.e., when $p \notin \mathbb{P}_\infty$) and in this case $\mathrm{rk}_p(\Sigma) = 1$. This proves that $\mathrm{Soc}(\Sigma) = \bigoplus_{p\notin\mathbb{P}_\infty} \mathbb{Z}(p)$. As $\mathrm{tor}(\Sigma)$ is divisible, it is the divisible hull of $\bigoplus_{p\notin\mathbb{P}_\infty} \mathbb{Z}(p)$ and so $\mathrm{tor}(\Sigma) = \bigoplus_{p\notin\mathbb{P}_\infty} \mathbb{Z}(p^\infty)$. □

We illustrate the situation with some special cases.

Example 1. *(1) For a first concrete example, let* $A_1 = \sum_{p\in\mathbb{P}} \frac{1}{p}\mathbb{Z}$ *and* $\Sigma_1 = A_1^{\wedge}$*. Then* $\mathrm{tp}(A_1) = [(1,1,\ldots)]$*, and* $\mathbf{\Delta}(\Sigma_1)$ *is the divisible hull of* $\Delta = \prod_{p\in\mathbb{P}} \mathbb{Z}(p)$ *and* $\mathbf{\Delta}(\Sigma_1) = \prod^{\mathrm{loc}}_{p\in\mathbb{P}}(\mathbb{Z}(p^\infty), \mathbb{Z}(p))$*.*

(2) Next let $A_2 = \mathbb{Q}$*. Then* $\mathrm{tp}(A_2) = [(\infty,\infty,\ldots)]$*,* $\Sigma_2 = \mathbb{Q}^{\wedge}$ *is torsion-free,* $\Delta = \prod_{p\in\mathbb{P}} \widehat{\mathbb{Z}}_p$ *and* $\mathbf{\Delta}(\Sigma_2)$ *is the divisible hull of* Δ*, so* $\mathbf{\Delta}(\Sigma_2) = \prod^{\mathrm{loc}}_{p\in\mathbb{P}}(\widehat{\mathbb{Q}}_p, \widehat{\mathbb{Z}}_p)$ *where* $\widehat{\mathbb{Q}}_p = \frac{1}{p^\infty}\widehat{\mathbb{Z}}_p$ *is the additive group of p-adic numbers.*

(3) For $A_3 = \mathbb{Z}$*,* $\mathrm{tp}(A_3) = [(0,0,\ldots)]$*,* $\Sigma_3 = \mathbb{Z}^{\wedge} = \mathbb{T}$*,* $\Delta = \{0\}$*, but* $\mathrm{Soc}(\Sigma_3) = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p) \subseteq \mathbf{\Delta}(\Sigma_3)$*,* $\mathbf{\Delta}(\Sigma_3)$ *is the divisible hull of* $\mathrm{Soc}(\Sigma_3)$*, so* $\mathbf{\Delta}(\Sigma_3) = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty) = \mathbb{Q}/\mathbb{Z} = \mathrm{tor}(\Sigma_3)$*.*

Note that Δ is a particular δ–subgroup of Σ. Sometimes (e.g., (1), (2)), but not always (e.g., (3)), $\mathbf{\Delta}(\Sigma)$ is the divisible hull of Δ. In the general case additional δ–subgroups must be employed.

Proof. (1) In this case, $\forall\, p \in \mathbb{P} : h_p = 1$. By Lemma 11(1) $\mathbf{\Delta}(\Sigma_1)$ is the divisible hull of Δ, and the rest follows from Lemma 10.

(2) $\Sigma_2 = \mathbb{Q}^{\wedge}$ is torsion-free, $\mathrm{Soc}(\Sigma_2) = \{0\} \subseteq \Delta$, $A/\mathbb{Z} = \mathbb{Q}/\mathbb{Z} = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)$, $\Delta := (A/\mathbb{Z})^{\wedge} = \prod_{p\in\mathbb{P}} \widehat{\mathbb{Z}}_p$, and $\mathbf{\Delta}(\Sigma)$ is the divisible hull of Δ.

(3) Clear. □

The next theorem deals with the general case. The relevance of the final assertion will become clear in Section 7.3 (see Definition 9 and Example 4, see also Problem 2).

Theorem 13. *Let* $A = \sum_{p\in\mathbb{P}} \frac{1}{p^{h_p}}\mathbb{Z}$*. Define* $\Sigma = A^{\wedge}$ *and* $\mathbb{P}_\infty$ *as above, and let*

$$\mathbb{P}_{\mathrm{fin}} := \{p \mid 0 < h_p < \infty\} \text{ and } \mathbb{P}_0 := \{p \mid h_p = 0\}.$$

Then $\mathbf{\Delta}(\Sigma)$ *is the divisible hull of* $\prod_{p\in\mathbb{P}_\infty} \widehat{\mathbb{Z}}_p \oplus \prod_{p\in\mathbb{P}_{\mathrm{fin}}} \mathbb{Z}(p^{h_p}) \oplus \bigoplus_{p\in\mathbb{P}_0} \mathbb{Z}(p)$*, so*

$$\mathbf{\Delta}(\Sigma) = \prod^{\mathrm{loc}}_{p\in\mathbb{P}_\infty} (\mathbb{Q}_p, \widehat{\mathbb{Z}}_p) \oplus \prod^{\mathrm{loc}}_{p\in\mathbb{P}_{\mathrm{fin}}} (\mathbb{Z}(p^\infty), \mathbb{Z}(p^{h_p})) \oplus \bigoplus_{p\in\mathbb{P}_0} \mathbb{Z}(p^\infty). \tag{6}$$

Moreover, $\mathrm{Soc}(\Sigma)$ *is dense in* Σ *if and only if* $\mathbb{P}_0$ *is infinite.*

Proof. $\Delta = \prod_{p\in\mathbb{P}_\infty} \widehat{\mathbb{Z}}_p \oplus \prod_{p\in\mathbb{P}_{\mathrm{fin}}} \mathbb{Z}(p^{h_p}) \subseteq \mathbf{\Delta}(\Sigma)$ and $\mathbf{\Delta}(\Sigma)$ is not the divisible hull of Δ if $\mathbb{P}_0 \neq \emptyset$. However, (Lemma 11) $D := \Delta \oplus \bigoplus_{p\in\mathbb{P}_0} \mathbb{Z}(p) \subseteq \mathbf{\Delta}(\Sigma)$ and $\mathrm{Soc}(\mathbf{\Delta}(\Sigma)) \subseteq \mathrm{Soc}(\Sigma) \subseteq D$. Hence, $\mathbf{\Delta}(\Sigma)$ is the divisible hull of D. Apply Lemma 10.

Recall that $\mathrm{Soc}(\Sigma) = \bigoplus_{p\in\mathbb{P}_0} \mathbb{Z}(p) \oplus \bigoplus_{p\in\mathbb{P}_{\mathrm{fin}}} \mathbb{Z}(p)$, where $\mathbb{Z}(p) = \Sigma[p]$ when the latter is non-trivial. Let $\phi : \Sigma \to \Sigma/\Delta = \mathbb{T}$. We claim that

$$\phi(\bigoplus_{p\in\mathbb{P}_0} \mathbb{Z}(p)) = \bigoplus_{p\in\mathbb{P}_0} \mathbb{T}[p]. \tag{7}$$

Indeed, if $t = \phi(x) \in (\Sigma/\Delta)[p]$ for some $x \in \Sigma$ and $p \in \mathbb{P}_0$, then $h_p = 0$ and $pt = 0$ in Σ/Δ, so $px \in \Delta$. It follows from the above description of Δ that Δ is p-divisible for $p \in \mathbb{P}_0$. Hence, $px = pz$ where $z \in \Delta$. Then $px - pz = 0$, so $x - z = \Sigma[p] = \mathbb{Z}(p)$. Therefore, $t = \phi(x) = \phi(x - z)$. This proves (7).

If $\mathbb{P}_0$ is infinite, $\bigoplus_{p\in\mathbb{P}_0} \mathbb{T}[p]$ is dense in $\mathbb{T}$, hence (7) implies that the compact subgroup $\Sigma_1 := \overline{\bigoplus_{p\in\mathbb{P}_0} \Sigma[p]}$ of Σ satisfies $\phi(\Sigma_1) = \overline{\bigoplus_{p\in\mathbb{P}_0} \mathbb{T}[p]} = \mathbb{T}$. Hence, $1 = \dim\mathbb{T} \leq \dim\Sigma_1 \leq \dim\Sigma = 1$ and consequently, $\dim\Sigma_1 = \dim\Sigma = 1$, hence $\dim\Sigma/\Sigma_1 = \dim\Sigma - \dim\Sigma_1 = 0$. Since Σ/Σ_1 is connected, this implies $\Sigma_1 = \Sigma$.

If $\mathbb{P}_0$ is finite, then Σ_1 is finite, while $\Sigma_2 = \overline{\bigoplus_{p\in\mathbb{P}_{\text{fin}}} \Sigma[p]} \leq \Delta$. Therefore, using again Lemma 11(3),

$$\overline{\text{Soc}(\Sigma)} = \overline{\bigoplus_{p\in\mathbb{P}_0} \Sigma[p] + \bigoplus_{p\in\mathbb{P}_{\text{fin}}} \Sigma[p]} = \overline{\Sigma_1 + \bigoplus_{p\in\mathbb{P}_{\text{fin}}} \Sigma[p]} = \Sigma_1 + \Sigma_2 \leq \Sigma_1 + \Delta \neq \Sigma,$$

since $\Sigma_1 + \Delta$ is a totally disconnected, while Σ is connected. □

Remark 4. *By Theorem 12, two compact groups are isomorphic if and only if their Fat Deltas are isomorphic as topological groups. A classification of Fat Deltas amounts to a classification of compact groups. A compact group is just the completion of its Fat Delta. Solenoids indicate the problems ahead.*

For any solenoid Σ, we have $\text{rk}_p(\Sigma) = 0$ for $p \in \mathbb{P}_\infty$, $\text{rk}_p(\Sigma) = 1$ for $p \in \mathbb{P}_{fin} \cup \mathbb{P}_0$. For the concrete examples $\text{rk}(\boldsymbol{\Delta}(\Sigma_1)) = 2^{\aleph_0}$, $\text{rk}(\boldsymbol{\Delta}(\Sigma_2)) = 2^{\aleph_0}$, while $\text{rk}(\boldsymbol{\Delta}(\Sigma_3)) = 0$. In general $\text{rk}(\boldsymbol{\Delta}(\Sigma)) = 2^{\aleph_0}$ except that $\text{rk}(\boldsymbol{\Delta}(\Sigma)) = 0$ for $\Sigma = \mathbb{Z}^\wedge = \mathbb{T}$. The algebraic invariants of $\boldsymbol{\Delta}(\Sigma)$ are the same for many non-isomorphic solenoids Σ. So the topological isomorphism class of $\boldsymbol{\Delta}(\Sigma)$, and hence of Σ, is in no way determined by these invariants. To distinguish between two Fat Deltas that are algebraically isomorphic one needs to know their topology. The description (6) *involves the types of Σ. It may help in determining the topology of $\boldsymbol{\Delta}(\Sigma)$. Conversely, knowing the topology of $\boldsymbol{\Delta}(\Sigma)$ should make it possible to recapture the type of Σ.*

6. Resolutions

The Resolution Theorem, a structure theorem for compact abelian groups, first appeared in [25] and later in an extended form in ([1], Theorem 8.20, p. 420), where it got its name.

Definition 3. *Recall that the "Lie algebra" of G, $\mathfrak{L}(G)$, defined as $\mathfrak{L}(G) = \text{cHom}(\mathbb{R}, G)$, is a real topological vector space via the stipulation $(rf)(x) := f(rx)$ where $f \in \mathfrak{L}(G)$ and $r, x \in \mathbb{R}$, and carries the topology of uniform convergence on compact sets ([1], Definition 5.7, p. 117, Proposition 7.36, p. 373). For every morphism $\varphi : G \to H$ in* TAG*, one obtains a morphism $\mathfrak{L}(\varphi) : \mathfrak{L}(G) \to \mathfrak{L}(H)$ in the category $\text{TAG}_\mathbb{R}$ of real topological vector spaces by letting $\mathfrak{L}(\varphi)(f) := \varphi \circ f$ for $f \in \mathfrak{L}(G)$. This defines a functor $\mathfrak{L} : \text{TAG} \to \text{TAG}_\mathbb{R}$ with the following useful properties:*

- *(i) ([1], Proposition 7.38(i), p. 374) $\mathfrak{L}(G) = \mathfrak{L}(c(G))$ and $\mathfrak{L}$ commutes with products, i.e., $\mathfrak{L}(\prod_i G_i) \cong_t \prod_i \mathfrak{L}(G_i)$.*
- *(ii) ([1], Proposition 7.38(ii), p. 374) If $\varphi : G \to H$ is a morphism in* TAG*, then $\mathfrak{L}(\varphi)$ is injective, whenever* $\text{Ker}\,\varphi$ *is totally disconnected;*
- *(iii) ([1], Corollary 8.19, p. 419) if G is a compact group and $\Delta \in \mathcal{D}(G)$ with $G/\Delta = \mathbb{T}^\mathfrak{m}$, then, with $\varphi : G \to G/\Delta$, $\mathfrak{L}(\varphi) : \mathfrak{L}(G) \to \mathfrak{L}(G/\Delta) = \mathbb{R}^\mathfrak{m}$ is a topological isomorphism. The last equality is in fact a topological isomorphism obtained as composition of two others. The first one is the isomorphism $\mathfrak{L}(\mathbb{T}^\mathfrak{m}) \cong_t \mathfrak{L}(\mathbb{T})^\mathfrak{m}$ from (i). The second one is $\mathfrak{L}(\mathbb{T}) \cong_t \mathbb{R}$, that can be obtained from the obvious equality $\mathfrak{L}(\mathbb{T}) = \mathbb{R}^\wedge$, by letting $\rho : \mathbb{R} \to \mathfrak{L}(\mathbb{T}) : \rho(r)(x) = rx + \mathbb{Z}$ for $r, x \in \mathbb{R}$.*

The exponential map *is the morphism $\exp_G : \mathfrak{L}(G) \to G$ defined by $\exp(\chi) = \chi(1)$ ([1], p. 372). It "commutes" with morphisms $\varphi : G \to H$ in* TAG*, i.e., $\varphi \circ \exp_G = \exp_H \circ \mathfrak{L}(\varphi)$. This means that $\exp = (\exp_G)_{G\in\text{TAG}}$ is a natural transformation from the functor $\mathfrak{L}$ to the identity functor of* TAG*. For further properties of the "Lie algebra" $\mathfrak{L}(G)$ and the "exponential morphism" see ([1] Proposition 7.38, p. 374, Theorem 7.66, p. 395)). In particular, $\exp_\mathbb{T} : \mathfrak{L}(\mathbb{T}) \to \mathbb{T}$ is defined by $\exp_\mathbb{T}(\rho(r)) = r + \mathbb{Z}$ for $r \in \mathbb{R}$ and ρ as in (iii) above.*

We can now recall the original Resolution Theorem.

Proposition 8 ([25], Proposition 2.2)**.** *For a compact abelian group G there is a compact zero-dimensional subgroup Δ of G such that the homomorphism*

$$\varphi : \Delta \times \mathfrak{L}(G) \to G : \varphi((d,\chi)) = d + \exp(\chi)$$

satisfies the following conditions:

(1) *φ is continuous, surjective, and open, i.e., is a quotient morphism.*
(2) $\mathrm{Ker}(\varphi)$ *is algebraically and topologically isomorphic to* $\Gamma := \exp^{-1}[\Delta]$*, and Γ is a closed totally disconnected subgroup of $\mathfrak{L}(G)$. In particular, it does not contain any nonzero vector spaces.*
(3) $\varphi[\{0\} \times \mathfrak{L}(G)] = \exp[\mathfrak{L}(G)]$ *is dense in* $\mathrm{c}(G)$*, the identity component of G.*

In the above notation, one can prove also that $\exp[\mathfrak{L}(G)] = \mathrm{a}(G)$, the path connected component of 0, while $\mathrm{c}(G) = \overline{\mathrm{a}(G)}$ ([1], Theorem 8.30, p. 430 and Theorem 8.4, p. 409).

We first revisit the classical Resolution Theorem for compact connected groups of finite dimension with substantial additions as we determine the kernel of the resolution map φ explicitly up to topological isomorphism (see (4)).

Theorem 14 (Resolution Theorem)**.** *Let G be a compact abelian group of finite dimension $n :=$* $\dim(G)$*. For $\Delta \in \mathcal{D}(G)$ define $\varphi : \Delta \times \mathfrak{L}(G) \to G$ by $\varphi(d,\chi) = d + \exp(\chi)$ for $(d,\chi) \in \Delta \times \mathfrak{L}(G)$. Then:*

(1) *φ is surjective, continuous, and open.*
(2) $\Gamma := \mathrm{Ker}(\varphi) = \{(-\exp(\chi),\chi) \mid \chi \in \exp^{-1}[\Delta]\}$*. The projection $\Delta \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ maps Γ isomorphically onto* $\exp^{-1}[\Delta]$*, so* $\Gamma \cong_{\mathrm{t}} \exp^{-1}[\Delta]$*. Furthermore,* $\exp^{-1}[\Delta]$ *is a closed totally disconnected subgroup of $\mathfrak{L}(G)$.*
(3) $\mathfrak{L}(G) \cong_{\mathrm{t}} \mathbb{R}^n$*, in particular* $\dim_{\mathbb{R}}(\mathfrak{L}(G)) = n$*;*
(4) *$\Gamma \cong_{\mathrm{t}} \mathbb{Z}^n$ where $\mathbb{Z}^n$ carries the discrete topology, i.e., the subspace topology in $\mathbb{R}^n$.*
(5) $\exp[\exp^{-1}[\Delta]] = \Delta \cap \mathrm{a}(G)$ *is dense in Δ.*

Proof. (1) and (2) are part of ([1], Theorem 8.20, p. 420).

(3) Follows from (iii).

(4) By (2) the projection $\Delta \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ induces a continuous epimorphism

$$G \cong_{\mathrm{t}} \frac{\Delta \times \mathfrak{L}(G)}{\Gamma} \twoheadrightarrow \frac{\mathfrak{L}(G)}{\exp^{-1}[\Delta]}$$

hence $\frac{\mathfrak{L}(G)}{\exp^{-1}[\Delta]}$ is compact. By ([1], Theorem A1.12.(i), p. 715) and (3), there is a basis $\{e_i\}$ of $\mathfrak{L}(G) \cong_{\mathrm{t}} \mathbb{R}^n$, i.e., $\mathfrak{L}(G) = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n$, such that $\exp^{-1}[\Delta] \cong_{\mathrm{t}} \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_p \oplus \mathbb{Z}e_{p+1} \oplus \cdots \oplus \mathbb{Z}e_{p+q}$ and $\frac{\mathfrak{L}(G)}{\exp^{-1}[\Delta]} \cong_{\mathrm{t}} \mathbb{T}^q \oplus \mathbb{R}^{n-p-q}$. As $\exp^{-1}[\Delta]$ is totally disconnected we have $p = 0$, and as $\mathfrak{L}(G)/\exp^{-1}[\Delta]$ is compact, $0 = n - p - q = n - q$ and it follows that $q = n$.

(5) It is routine to verify that $\exp[\exp^{-1}[\Delta]] = \Delta \cap \mathrm{a}(G)$. Set $\mathbb{Z}_\Delta := \Delta \cap \mathrm{a}(G)$. It is easily seen that $\Gamma \subset \mathbb{Z}_\Delta \times \mathfrak{L}(G) \subset \overline{\mathbb{Z}_\Delta} \times \mathfrak{L}(G) \subset \Delta \times \mathfrak{L}(G)$. We obtain the exact sequence

$$\frac{\overline{\mathbb{Z}_\Delta} \times \mathfrak{L}(G)}{\Gamma} \overset{\mathrm{ins}}{\rightarrowtail} \frac{\Delta \times \mathfrak{L}(G)}{\Gamma} \overset{f}{\twoheadrightarrow} \frac{\Delta \times \mathfrak{L}(G)}{\overline{\mathbb{Z}_\Delta} \times \mathfrak{L}(G)} \cong_{\mathrm{t}} \frac{\Delta}{\overline{\mathbb{Z}_\Delta}}$$

Here $\frac{\Delta \times \mathfrak{L}(G)}{\Gamma} \cong_{\mathrm{t}} G$ is connected, hence $\Delta/\overline{\mathbb{Z}_\Delta}$ is connected as well, by the surjectivity of f. On the other hand, $\Delta/\overline{\mathbb{Z}_\Delta}$ is totally disconnected because Δ, being compact and totally disconnected is profinite ([1], Theorem 1.34, p. 22), and quotients of profinite groups are profinite ([26], Proposition 2.2.1(a), p. 28), and in particular totally disconnected. This is possible only when the quotient $\Delta/\overline{\mathbb{Z}_\Delta}$ is trivial. Therefore, $\overline{\mathbb{Z}_\Delta} = \Delta$. □

Remark 5. *(a) For the torus $G = \mathbb{T}^n$ one has $\mathfrak{L}(G) = \mathbb{R}^n$, so the Resolution theorem applied to G is simply the covering homomorphism $\varphi : \mathbb{R}^n \to \mathbb{T}^n$ if one takes $\Delta = 0$ (in general Δ must be a finite subgroup of $\mathbb{T}^n$).*

(b) Using the fact that $\exp[\mathfrak{L}(G)] = \mathrm{a}(G)$, the covering map φ could be replaced by the surjective, continuous, and open map $\psi : \Delta \times \mathrm{a}(G) \to G : \varphi(d, x) = d + x$, for $(d, x) \in \Delta \times \mathrm{a}(G)$ which has the advantage that now both groups Δ and $\mathrm{a}(G)$ are subgroups of G. One has to take into account that the map $\exp_G : \mathfrak{L}(G) \to \mathrm{a}(G)$ need not be injective. More precisely, $\mathfrak{K}(G) = \mathrm{Ker}(\exp)$ is trivial precisely when G is torus-free. However, even when G is torus-free, this map is only a continuous isomorphism that need not be a homeomorhism.

(c) As an application of Theorem 14 we obtain a nice presentation of the solenoid $\Sigma_2 = \mathbb{Q}^\wedge$ from Example 1 (2). As shown there, Σ_2 has a delta subgroup $\Delta = \widehat{\mathbb{Z}} = \prod_{p\in\mathbb{P}} \widehat{\mathbb{Z}}_p$ and $\Sigma_2/\Delta \cong \mathbb{T}$. So by Definition 3 (iii), $\mathfrak{L}(\Sigma_2) \cong_t \mathbb{R}$. Hence, Theorem 14 gives a resolution $\varphi : \Delta \times \mathbb{R} \to \Sigma_2$, with $\Gamma = \ker\varphi \cong_t \mathbb{Z}$ and $\Delta \cap a(\Sigma_2) = \langle\chi_1\rangle \cong \mathbb{Z}$, where $\chi_1 : \mathbb{Q} \to \mathbb{T}$ is defined by $\chi_1(x) = x + \mathbb{Z}$ for $x \in \mathbb{Q}$.

The same representation can also be obtained directly by standard use of Pontryagin duality. Indeed, let $\mathbf{1} = (1_p)_{p\in\mathbb{P}} \in \Delta$ and $u = (\mathbf{1}, -1) \in \Delta \times \mathbb{R}$. Then $\langle u\rangle \cong_t \mathbb{Z}$ and $K = (\Delta \times \mathbb{R})/\langle u\rangle$ is a compact connected torsion-free group of dimension one, so its dual $K^\wedge$ is a discrete divisible torsion-free group of rank one. Therefore, $K^\wedge \cong \mathbb{Q}$ and $K \cong_t \mathbb{Q}^\wedge$.

We also obtain a "canonical resolution", where the arbitrary $\Delta \in \mathcal{D}(G)$ is replaced by the canonical subgroup $\mathbf{\Delta}(G)$.

Theorem 15 (Canonical Resolution Theorem)**.** *Let G be a compact abelian group and $\mathbf{\Delta}(G) = \bigcup \mathcal{D}(G)$. Then*

(1) the map $\varphi : \mathbf{\Delta}(G) \times \mathfrak{L}(G) \to G : \varphi((d, \chi)) = d + \exp(\chi) = d + \chi(1)$ is surjective, continuous, and open;

(2) $\Gamma := \mathrm{Ker}(\varphi) = \{(\exp(\chi), -\chi) \mid \chi \in \exp^{-1}[\mathbf{\Delta}(G)]\} \cong_t \exp^{-1}[\mathbf{\Delta}(G)] \subset \mathfrak{L}(G)$ is torsion-free and φ induces an isomorphism $(\mathbf{\Delta}(G) \times \mathfrak{L}(G))/\Gamma \cong_t G$;

(3) If G is connected of finite dimension $\dim(G) = n$, then $\Gamma \cong_t \mathbb{Q}^n$.

(4) $\exp[\exp^{-1}[\mathbf{\Delta}(G)]] = \mathrm{a}(G) \cap \mathbf{\Delta}(G)$ is dense in G.

Proof. (1) The map φ is clearly homomorphic, continuous and surjective. To show that it is open, let W be an open set in $\mathbf{\Delta}(G) \times \mathfrak{L}(G)$. We can assume without loss of generality that it is a basic open set, i.e., $W = U \times U'$, where U is open in $\mathbf{\Delta}(G)$ and U' is open in $\mathfrak{L}(G)$. Then $\forall\, \Delta \in \mathcal{D}(G) : \Delta \cap U$ is an open set of Δ, so $(\Delta \cap U) \times U'$ is an open set of $\Delta \times \mathfrak{L}(G)$. By the ordinary Resolution Theorem $O_\Delta := \varphi[(\Delta \cap U) \times U']$ is open in G. Hence, so is

$$\varphi[U \times U'] = \varphi\left[\left(\bigcup_{\Delta\in\mathcal{D}(G)} (\Delta \cap U)\right) \times U'\right] = \varphi\left[\left(\bigcup_{\Delta\in\mathcal{D}(G)} (\Delta \cap U) \times U'\right)\right] = \bigcup_{\Delta\in\mathcal{D}(G)} O_\Delta.$$

(2) The map $\Gamma \to \exp^{-1}[\mathbf{\Delta}] : (\exp(\chi), -\chi) \to \chi$ clearly is bijective, homomorphic, continuous and open. Being isomorphic to a subgroup of $\mathfrak{L}(G)$, the group Γ is torsion-free. The last assertion is obvious.

(3) Fix arbitrarily $\Delta \in \mathcal{D}(G)$ and let

$$\varphi_\Delta : \Delta \times \mathfrak{L}(G) \to G, \text{ defined by } \varphi_\Delta(d, \chi) = d + \exp(\chi), \text{ and } \Gamma_\Delta = \mathrm{Ker}(\varphi_\Delta).$$

By (2) Γ is torsion-free. We will show that Γ is divisible and Γ/Γ_Δ is a torsion group. This says that Γ is the usual algebraic divisible hull of $\Gamma_\Delta \cong_t \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \subset \mathfrak{L}(G)$ (see the proof of item (4) of Theorem 14). Hence, $\exp^{-1}[\mathbf{\Delta}(G)] \cong_t \Gamma$ is the divisible hull of $\exp^{-1}[\Delta]$ and is $\mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_n \subset \mathfrak{L}(G)$ with the subspace topology. This shows that $\Gamma \cong_t \mathbb{Q}^n$.

To show that $\exp^{-1}[\mathbf{\Delta}]$, and hence Γ, is divisible, suppose that $x \in \mathfrak{L}(G)$ and $\exp(x) \in \mathbf{\Delta}(G)$. As $\mathfrak{L}(G)$ is divisible, given $m \in \mathbb{N}$, there is $y \in \mathfrak{L}(G)$ such that $my = x$. Hence, $m\exp(y) = \exp(x)$ and as $\mathbf{\Delta}(G)$ is divisible (Proposition 7(4)) there is $d \in \mathbf{\Delta}(G)$ such that

$m\exp(y) = md$. It follows that $\exp(y) - d \in \operatorname{tor}(G) \subset \mathbf{\Delta}(G)$, hence $\exp(y) \in \mathbf{\Delta}(G)$ and $y \in \exp^{-1}(\mathbf{\Delta})$ which establishes the claim.

Finally, to show that Γ/Γ_Δ is a torsion group let $\chi \in \exp^{-1}[\mathbf{\Delta}(G)]$, i.e., $\exp(\chi) \in \mathbf{\Delta}(G)$. By Proposition 7, there is $m \in \mathbb{N}$ such that $m\chi \in \Delta$. It follows that $m(\exp(\chi), -\chi) = (\exp(m\chi), -m\chi) \in \Gamma_\Delta$.

(4) Write $\mathbf{\Delta}(G) = \bigcup_{\Delta \in \mathcal{D}(G)} \Delta$ and use the fact that $\mathrm{a}(G) \cap \Delta$ is dense in Δ for every $\Delta \in \mathcal{D}(G)$, by Theorem 14(5). Then

$$\overline{\mathrm{a}(G) \cap \mathbf{\Delta}(G)} = \overline{\mathrm{a}(G) \cap \bigcup_{\Delta \in \mathcal{D}(G)} \Delta} = \overline{\bigcup_{\Delta \in \mathcal{D}(G)} \mathrm{a}(G) \cap \Delta} \supseteq \bigcup_{\Delta \in \mathcal{D}(G)} \overline{\mathrm{a}(G) \cap \Delta} \supseteq \bigcup_{\Delta \in \mathcal{D}(G)} \Delta = \mathbf{\Delta}(G).$$

Since $\mathbf{\Delta}(G)$ is dense in G, this proves that $\mathrm{a}(G) \cap \mathbf{\Delta}(G)$ is dense in G. □

In the next example, we apply the canonical resolution theorem 15 to two solenoids. The first one is $\mathbb{T} = \mathbb{Z}^\wedge$ and its canonical resolution adds nothing essentially new.

Example 2. *(a) For the solenoid, $\mathbb{T} = \mathbb{Z}^\wedge$ there is an isomorphism $\rho : \mathbb{R} \to \mathfrak{L}(G)$ and $\exp(\rho(r)) = r + \mathbb{Z}$, where $r \in \mathbb{R}$, by Definition 3(iii). Since $\mathbf{\Delta}(T) = \operatorname{tor}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$, we obtain the canonical resolution $\varphi : \mathbb{Q}/\mathbb{Z} \times \mathbb{R} \to \mathbb{T} : \varphi((a + \mathbb{Z}, r)) = (a + \mathbb{Z}) + (r + \mathbb{Z}) = a + r + \mathbb{Z}$ with $\Gamma = \{(r + \mathbb{Z}, -r) \mid r \in \mathbb{Q}\}$ and evidently $\Gamma \cong_t \mathbb{Q}$.*

(b) For the solenoid $\Sigma_2 = \mathbb{Q}^\wedge$ from Example 1 (2) $\Delta(\Sigma_2)$ is the divisible hull of its delta subgroup $\Delta = \prod_p \widehat{\mathbb{Z}}_p$. Moreover, $\mathfrak{L}(\Sigma_2) \cong_t \mathbb{R}$ (see Remark 5(c)). Theorem 15 gives the canonical resolution $\varphi : \Delta(\Sigma_2) \times \mathbb{R} \to \Sigma_2$ with $\Gamma = \ker\varphi \cong_t \mathbb{Q}$ as in (a) and $\mathrm{a}(\Sigma_2) \cap \mathbf{\Delta}(\Sigma_2) \cong \mathbb{Q}$ dense in Σ_2.
Denote by $\widetilde{\mathbb{Q}}$ the group $\Delta(\Sigma_2)$ equipped with the finer topology obtained by taking Δ as an open topological subgroup of $\widetilde{\mathbb{Q}}$. Then $\widetilde{\mathbb{Q}}$ is a locally compact ring and $\mathbb{A} := \widetilde{\mathbb{Q}} \times \mathbb{R}$ is the adele ring of $\mathbb{Q}$. Composing φ with the identity $\mathbb{A} \to \Delta(\Sigma_2) \times \mathbb{R}$ we obtain a continuous surjective homomorphism $\varphi : \mathbb{A} \to \Sigma_2$ which is again open by the Open Mapping Theorem (as $\mathbb{A}$ is σ-compact). Hence, Σ_2 is a quotient of $\mathbb{A}$.

7. Fat Delta Through the Looking Glass of Quasi-Torsion Elements

Fat Delta existed previously in the literature in a rather different form and in greater generality. In Section 7.1 we recall the definition of quasi-torsion element and the subgroup $\operatorname{td}(G)$ of quasi-torsion elements, showing that $\operatorname{td}(G) = \mathbf{\Delta}(G)$ for compact groups (Proposition 9).

7.1. Quasi-Torsion Elements

Definition 4 (([3], p. 127), [4])**.** *Let G be a Hausdorff abelian topological group. Define $\operatorname{td}(G)$ to be the set of all quasi-torsion elements of G, where $x \in G$ is **quasi-torsion** if $\langle x \rangle$ is either finite or its subspace topology is non-discrete and linear.*

This definition was given by [4] for arbitrary, not necessarily abelian, topological groups. Then $\operatorname{td}(G)$ need not be a subgroup of G, as the following example shows.

Example 3. *Take the compact group $G = SL_3(\mathbb{R})$ of rotations of $\mathbb{R}^3$. Then $\operatorname{td}(G) = \operatorname{tor}(G)$ is the set of all torsion elements of G, while the subgroup $\langle \operatorname{td}(G) \rangle$ generated by $\operatorname{td}(G)$ is the whole G since $\operatorname{td}(G)$ is invariant under conjugations and G is a simple group. A geometric proof of the equality $\langle \operatorname{td}(G) \rangle = G$ is based on the well-known fact that every rotation can be presented as a composition of two symmetries (known to have order 2).*

Remark 6. *If every convergent sequence is eventually constant in a topological abelian group G, then $\operatorname{td}(G) = \operatorname{tor}(G)$ (the assumption $\operatorname{td}(G) \neq \operatorname{tor}(G)$ leads to a contradiction: if $x \in \operatorname{td}(G) \setminus \operatorname{tor}(G)$, then the group $\langle x \rangle$ is non-discrete and metrizable, so $\langle x \rangle$ has convergent sequences that are not eventually constant).*

Infinite compact groups always have convergent sequences that are not eventually constant (since they contain copies of the Cantor set $\{0,1\}^\omega$). An example of an infinite precompact abelian group where every convergent sequence is eventually constant can be obtained as follows. For a TAG*–group (G,τ) the Bohr topology of (G,τ) is the initial topology τ^+ of all $\chi \in (G,\tau)^\wedge$ (that can be obtained by the diagonal embedding $G \to \mathbb{T}^{G^\wedge}$). For the sake of brevity we also write G^+ for (G,τ^+). In case τ is discrete, G^+ is usually denoted by $G^\#$. It is a well-known fact that in $G^\#$ every convergent sequence is eventually constant ([3]), so* $\mathrm{td}(G^\#) = \mathrm{tor}(G^\#)$.

Proposition 9. *Let G be a topological abelian group.*

1. *If $x \in G$, then $x \in \mathrm{td}(G)$ if and only if there exists a continuous homomorphism $f : (\mathbb{Z}, \nu_{\mathbb{Z}}) \to G$ with $f(1) = x$;*
2. $\mathrm{td}(G)$ *is a subgroup of G containing every compact totally disconnected subgroup of G;*
3. *If G is complete (in particular, locally compact), then* $\mathrm{td}(G)$ *coincides with the union of all compact, totally disconnected subgroups of G.*

Proof. (1) Assume that $x \in \mathrm{td}(G)$. If $\langle x\rangle$ is finite, then $\langle x\rangle$ is isomorphic to a quotient group of $(\mathbb{Z}, \nu_{\mathbb{Z}})$, so the desired homomorphism f is easy to obtain. If $\langle x\rangle$ is infinite and carries a non-discrete linear topology, then the homomorphism $f : (\mathbb{Z}, \nu_{\mathbb{Z}}) \to G$ with $f(1) = x$ is obviously continuous. On the other hand, if there exists a continuous homomorphism $f : (\mathbb{Z}, \nu_{\mathbb{Z}}) \to G$ with $f(1) = x$, then the subgroup $\langle x\rangle$ is either finite or has linear precompact topology, so $x \in \mathrm{td}(G)$.

(2) If $x, y \in \mathrm{td}(G)$, then by (1) there exist continuous homomorphisms $f, g : (\mathbb{Z}, \nu_{\mathbb{Z}}) \to G$ with $f(1) = x$ and $g(1) = y$. This gives a continuous homomorphism $h = f \oplus g : (\mathbb{Z}, \nu_{\mathbb{Z}}) \times (\mathbb{Z}, \nu_{\mathbb{Z}}) \to G$ defined by $h(n,m) = nx + my$. The restriction $h \restriction_{\Delta_{\mathbb{Z}}} : \Delta_{\mathbb{Z}} \to G$ satisfies $h(1,1) = x + y$ and since $\Delta_{\mathbb{Z}} \cong (\mathbb{Z}, \nu_{\mathbb{Z}})$, witnesses $x + y \in \mathrm{td}(G)$ by (1).

If N is a compact, totally disconnected subgroup of G, then N has a linear topology. Therefore, for every $x \in N$, the subgroup $\langle x\rangle$ is either finite or its subspace topology is linear and non-discrete (as otherwise $\langle x\rangle$ it would be a closed (so compact) discrete subgroup of N, a contradiction). Therefore, $x \in \mathrm{td}(G)$.

(3) Assume now that G complete and $x \in \mathrm{td}(G)$. Then x is quasi-torsion and $\langle x\rangle$ is either finite or its subspace topology is non-discrete and linear. Hence, its closure $\overline{\langle x\rangle}$ is the completion of $\langle x\rangle$, and thus, compact and totally disconnected. □

For a compact group $G = A^\wedge$, by Proposition 5 and Proposition 9(2), we have $\mathrm{td}(G) = \mathbf{\Delta}(G)$, and by Theorem 10 $\mathbf{\Delta}(G) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. We summarize:

Theorem 16. *Let $G = A^\wedge$ where $A \in$* AG*. Then*

$$\mathbf{\Delta}(G) = \mathrm{td}(G) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}).$$

We quote from previous papers reconfirming foregoing results.

Proposition 10. *(1) ([3], Theorem 4.1.7(a)) If $f : G \to H$ is a continuous homomorphism of topological abelian groups, then $f[\mathrm{td}(G)] \subseteq \mathrm{td}(H)$, i.e.,* td *is a functorial subgroup; in particular* $\mathrm{td}(G)$ *is fully invariant in G.*

(2) ([27], Theorem 11) If G and H in (1) are compact and f is surjective, then $f[\mathrm{td}(G)] = \mathrm{td}(H)$.

(3) (([4], Proposition 1.3(a)) and ([3], Theorem 4.1.7(b))) If G is a topological abelian group and H is a subgroup of G, then $\mathrm{td}(H) = H \cap \mathrm{td}(G)$;

(4) (([4], Proposition 1.4(a)) and ([3], Theorem 4.1.7(e))) Let $\{G_i : i \in I\}$ be a family of topological abelian groups. Then $\mathrm{td}(\prod_{i\in I} G_i) = \prod_{i\in I} \mathrm{td}(G_i)$.

Remark 7. *Comments on the various items of Proposition 10.*

(a) Items (1), (3) and (4) follow from Proposition 9 and reinforce Corollary 3(1) by showing that td *is a functorial subgroup in the* **larger** *category* TAG.

(b) *In (2) "compact" cannot be replaced by "locally compact" (take $G = \mathbb{R}$, $H = \mathbb{T}$ and f the canonical quotient map, then* $\mathrm{td}(\mathbb{R}) = \{0\}$*, while* $\mathrm{td}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z} \neq \{0\}$*).*
(c) *Item (4) reinforces Theorem 10(3) showing that it remains valid in the* **larger** *category* TAG.

Now we use item (1) from Proposition 10 to show that the subgroup $\mathrm{td}(G)$ is zero-dimensional when G is precompact, i.e., a subgroup of a compact group. We shall see in [16] that this remains true under the weaker assumption that G is locally precompact, i.e., a subgroup of a locally compact group.

Proposition 11. *Let G be a precompact abelian group. Then every subgroup H of G with $[H : (H \cap \mathrm{td}(G))] < \mathfrak{c}$ is zero-dimensional. In particular,* $\mathrm{td}(G)$ *is zero-dimensional.*

Proof. The following folklore fact will be needed in the sequel:

Claim 1. *Every proper subgroup H of $\mathbb{T}$ is zero-dimensional.*

Proof. H is either finite of dense. If H is finite then it is clearly zero-dimensional. If H is dense, then for any fixed $a \in \mathbb{T} \setminus H$ also $a + H$ is dense and disjoint from H. Hence, $\{\Gamma_{b,c} \cap H \colon b, c \in a + H\}$, where $\Gamma_{b,c}$ is an open arc in $\mathbb{T}$ with ends b and c, is a base of the induced topology on H consisting of clopen sets of H. □

First, we show that $\chi[\mathrm{td}(G)] \subseteq \mathbb{Q}/\mathbb{Z}$ for any $\chi \in G^\wedge$. Assume that $x \in \mathrm{td}(G)$, to check that $\chi(x) \in \mathbb{Q}/\mathbb{Z}$ pick an arbitrary $\chi \in G^\wedge$. Then $\chi(x) \in \mathrm{td}(\mathbb{T})$, by Proposition 10(1). By Example 1(3), $\mathrm{td}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$, so $\chi(x) \in \mathbb{Q}/\mathbb{Z}$.

Since $H/(H \cap \mathrm{td}(G)) \cong (H + \mathrm{td}(G))/\mathrm{td}(G)$, our hypothesis implies that $[(H + \mathrm{td}(G)) : \mathrm{td}(G)] < \mathfrak{c}$. Hense, for every $\chi \in G^\wedge$ the subgroup $\chi[H + \mathrm{td}(G)]$ contains the countable subgroup $\chi[\mathrm{td}(G)] \subseteq \mathbb{Q}/\mathbb{Z}$ as a subgroup of index $< \mathfrak{c}$, so $|\chi[H]| \leq |\chi[H + \mathrm{td}(G)]| < \mathfrak{c}$ too. Consequently $\chi[H] \neq \mathbb{T}$, so $\chi[H]$ is zero-dimensional for every $\chi \in G^\wedge$. Since zero-dimensionality is preserved under taking direct products, $\prod_{\chi \in G^\wedge} \chi[H]$ is zero-dimensional, by Claim 1. Since H is precompact (as a subgroup of G), H isomorphic to a subgroup of $\prod_{\chi \in G^\wedge} \chi[H]$ by ([3], Theorem 2.3.2). Since zero-dimensionality is preserved under taking subgroups, we deduce that H is zero-dimensional. □

7.2. The Subgroup td(G) *of Compact Groups and Minimality*

The Open Mapping Theorem can be reached in two steps:

Definition 5. *A Hausdorff topological group G is:*
(a) ***minimal*** *if every continuous isomorphism $f : G \to H$ onto a Hausdorff topological group H is open.*
(b) ***totally minimal*** *if G satisfies the (full) Open Mapping Theorem, i.e., every continuous homomorphism $f : G \to H$ onto a Hausdorff topological group H is open.*

Compact groups are well-known to be totally minimal. On the other hand, a Hausdorff topological group G is totally minimal if and only if all Hausdorff quotients of G are minimal.

The first supply of non-compact (totally) minimal groups was obtained by means of the following notions of "strong" density:

Definition 6 ([28])**.** *A subgroup H of a topological abelian group G is* ***totally dense*** *if $\overline{N \cap H} = N$ for every closed subgroup N of G.*

Clearly, totally dense subgroups are dense (while $\mathbb{Z}(p^\infty)$ is dense in $\mathbb{T}$, but not totally dense). Obviously, the totally dense subgroups have the following weaker property:

Definition 7 ([5,11,15])**.** *A subgroup H of a topological abelian group G is **topologically essential** if $N \cap H \neq \{0\}$ for every non-trivial closed subgroup N of G.*

The term used for this property in [5,11,15] and in the remaining literature on the Open Mapping Theorem is "essential", but we prefer the more precise term "topologically essential" to avoid possible confusion.

Theorem 17. *Let H be a dense subgroup of a compact abelian group G.*

(a) ([11,15]) *H is minimal if and only if H is topologically essential in G.*
(b) ([28]) *H is totally minimal if and only if H is totally dense in G.*

Banaschewski [5] found the following general criterion: *if H is a dense subgroup of a topological abelian group G, then H is minimal if and only if G is minimal and H is topologically essential in G.* These criteria match perfectly the following remarkable result of Prodanov and Stoyanov [14] proved at a later stage, but conjectured by Prodanov in 1972 (see [13] for an earlier partial result in the totally minimal case):

Theorem 18 (Prodanov–Stoyanov Theorem)**.** *Minimal abelian groups are precompact.*

This theorem allows one to use exclusively the form of the criteria given in Theorem 17, so to reduce the study of the (totally) minimal abelian groups to that of the dense topologically essential (resp., totally dense) subgroups of the compact abelian groups. This explains the interest in topologically essential or totally dense subgroups of the compact abelian groups.

Proposition 12 ([11])**.** *The minimal topologies on $\mathbb{Z}$ are precisely the p-adic topologies.*

It was proved in [9] that the 2–adic topology of $\mathbb{Z}$ is minimal.

Proof. Assume that τ is a minimal topology on $\mathbb{Z}$ and let K be the completion of $(\mathbb{Z}, \tau)$. By the Prodanov–Stoyanov Theorem the group K is compact. By Theorem 17(a), $\mathbb{Z}$ is essential in K, hence K is torsion-free. Therefore, the dual of K is a discrete divisible group [1,3,23,29], hence a direct sums of copies of $\mathbb{Q}$ and of $\mathbb{Z}(p^\infty)$, $p \in \mathbb{P}$. Therefore, $K = (\mathbb{Q}^\wedge)^\alpha \times \prod_p \widehat{\mathbb{Z}}_p^{\beta_p}$. Again by Theorem 17(a), $\mathbb{Z}$ must be essential in this product, hence only one of these cardinals α, β_p can be non-zero, and it must be equal to 1. Since $\mathbb{Q}^\wedge$ has a Delta subgroup isomorphic to $\prod_p \widehat{\mathbb{Z}}_p$, again Theorem 17(a) implies that $\alpha = 0$. In other words, $K \cong \widehat{\mathbb{Z}}_p$ for some prime p, therefore, τ coincides with the p-adic topology on $\mathbb{Z}$. To conclude, the minimality of the p-adic topology follows from Theorem 17(a), since $\mathbb{Z}$ is essential in $K = \widehat{\mathbb{Z}}_p$, as all non-trivial closed subgroups of K are open. □

A similar argument shows that $\mathbb{Q}^n$ *admits no minimal topologies for* $0 < n < \infty$.

The functorial subgroup $\mathrm{td}(G)$ of a compact abelian group G is not only dense in G (Theorem 6(2)), but it is totally dense in G, as the next proposition shows.

Proposition 13. *Let G be a compact abelian group. Then* $\mathrm{td}(G)$ *is totally dense in G.*

Proof. Let N be a closed subgroup of G. Then $N \cap \mathrm{td}(G) = \mathrm{td}(N)$ by Proposition 10. Therefore, it suffices to check that $\mathrm{td}(G)$ is dense in G for every compact group G. This follows from Theorem 6, but we prefer to give an independent proof here.

Let $N := \overline{\mathrm{td}(G)}$. Applying to the closed subgroup N of G the exactness of td in the sense of Proposition 10(2), we deduce that $\mathrm{td}(G/N) = \{0\}$. To see that this implies $G/N = \{0\}$ and so $N = G$, consider the discrete dual $X = (G/N)^\wedge$ and assume by way of contradiction that $X \neq \{0\}$. Then there exists a subgroup Y of X such that $X/Y \neq \{0\}$ is torsion. Then $Y^\perp \cong (X/Y)^\wedge$ is a non-trivial compact totally disconnected subgroup of G/N, so $\mathrm{td}(G/N) \neq \{0\}$, a contradiction. □

We obtain the following theorem which, among other things, reconfirms that $\mathbf{\Delta}(G)$ is dense in G when G is compact.

Theorem 19. *Let G be a compact abelian group. Then* $\mathrm{td}(G)$ *is a dense totally minimal zero-dimensional subgroup of G.*

Proof. Proposition 13 ensures the total density (hence, density as well) of $\mathrm{td}(G)$. Total minimality of $\mathrm{td}(G)$ is then an immediate consequence of Theorem 17. To prove that $\mathrm{td}(G)$ is zero-dimensional, apply Proposition 11. □

Since $\mathrm{td}(G) \neq G$ when G is not totally disconnected, this theorem provides a universal example of a non-compact totally minimal (and zero-dimensional) abelian group. This explains why it is not surprising that most of the first known examples of non-compact totally minimal groups known in the seventies were just $\mathbb{Q}/\mathbb{Z} = \mathrm{td}(\mathbb{T})$ ([15]), $(\mathbb{Q}/\mathbb{Z})^n = \mathrm{td}(\mathbb{T}^n)$ ([9]), $(\mathbb{Q}/\mathbb{Z})^{\mathbb{N}} = \mathrm{td}(\mathbb{T}^{\mathbb{N}})$ ([10]), and $(\mathbb{Q}/\mathbb{Z})^{\alpha} = \mathrm{td}(\mathbb{T}^{\alpha})$ ([27,30]).

Corollary 4. *Let G be a compact abelian group and H be a closed subgroup of G. Then* $\mathrm{td}(H) \overset{\mathrm{ins}}{\rightarrowtail} \mathrm{td}(G) \twoheadrightarrow \mathrm{td}(G/H)$ *is a proper short exact sequence in* TAG.

Proof. By Proposition 10 $\mathrm{td}(H) = \mathrm{td}(G) \cap H$ and $q[\mathrm{td}(G)] = \mathrm{td}(G/H)$ for the quotient homomorphism $q : G \to G/H$. This proves the exactness of the short exact sequence $\mathrm{td}(H) \overset{\mathrm{ins}}{\rightarrowtail} \mathrm{td}(G) \overset{f}{\twoheadrightarrow} \mathrm{td}(G/H)$, where $f = q \restriction_{\mathrm{td}(G)}$. The openness of f follows from the fact that $\mathrm{td}(G)$ is totally minimal, in view of Theorem 19. □

7.3. Sylow Subgroups of $\mathrm{td}(G)$ *for* $G \in$ TAG

The characterization in Theorem 9 of the compact abelian groups G with $\mathrm{td}(G) = \mathrm{tor}(G)$ gives a very narrow class (practically rather close to the class of Lie groups). This shows that the restraint $\mathrm{td}(G) = \mathrm{tor}(G)$ is too stringent, or from another point of view, the subgroup $\mathrm{td}(G)$ is too large to be useful in certain circumstances. This is why here we recall a smaller subgroup of $\mathrm{td}(G)$ containing $\mathrm{tor}(G)$ that still keeps the advantages of $\mathrm{td}(G)$, but it is closer to $\mathrm{tor}(G)$. This subgroup is simply the subgroup generated by all topologically p-Sylow subgroups $\mathrm{td}_p(G)$ of $\mathrm{td}(G)$ defined as follows:

Definition 8 ([3,31])**.** *An element x of a topological abelian group G is **topologically p-torsion** if $p^n x \to 0$. Let*

$$G_p := \{x \in G \mid x \text{ is topologically } p\text{-torsion}\}$$

and let $\mathrm{td}_p(G) := (\mathrm{td}(G))_p$.

Then G_p is a subgroup of G. In case G is a profinite group, G_p is usually called the ***topological p-Sylow subgroup*** of G. We shall also keep this terminology when G is not necessarily profinite. Clearly, $H_p = G_p \cap H$ for a subgroup H of G.

Obviously, $\mathrm{tor}_p(G) \leq \mathrm{td}_p(G) \leq G_p$ for every G.

The notation $\mathrm{td}_p(G)$ used in Definition 8 is borrowed from [4,27], where $\mathrm{td}_p(G)$ denotes the subgroup of all elements $x \in G$ (called ***quasi-p-torsion*** in [4]) such that $\langle x \rangle$ is either a cyclic p-group, or $\langle x \rangle$ is isomorphic to $\mathbb{Z}$ equipped with the p-adic topology.

The equivalence of both definitions follows from: if $\langle x \rangle \cong \mathbb{Z}$ is equipped with a Hausdorff linear topology such that $p^n x \to 0$, then this linear topology necessarily coincides with the p-adic topology.

The sum $\sum_p \mathrm{td}_p(G)$ is direct ([4]). Following [4], we write $\mathrm{wtd}(G) = \bigoplus_{p \in \mathbb{P}} \mathrm{td}_p(G)$ in the sequel. Clearly,

$$\mathrm{tor}(G) \leq \mathrm{wtd}(G) \leq \mathrm{td}(G),$$

but these subgroups need not coincide in general. It is proved in [4] that, when G is compact, even the smaller subgroup $\mathrm{wtd}(G)$ is still totally dense in G. Since both total density and topological essentiality are transitive properties, a dense subgroup G of a compact abelian group K is totally dense (resp., topologically essential) in K if and only if $\mathrm{td}(G) = G \cap \mathrm{td}(K)$ is totally dense (resp., topologically essential) in $\mathrm{td}(K)$ if and only if $\mathrm{wtd}(G)$ is totally dense (resp., topologically essential) in $\mathrm{wtd}(K)$.

The next theorem from [29] shows that one can characterize the totally disconnected compact abelian groups in the class of all compact abelian groups G by specifying whether the subgroups $\mathrm{td}_p(G)$ of G are closed (compact) or not:

Theorem 20 ([3,29])**.** *For a compact abelian group G and every prime p the subgroup $\mathrm{td}_p(\mathrm{c}(G))$ is dense in $\mathrm{c}(G)$. In particular, the following conditions are equivalent:*

(1) *G is totally disconnected;*
(2) $\mathrm{td}(G) = G$, *i.e.,* $\mathrm{td}(G)$ *is compact;*
(3) $\mathrm{td}_p(G)$ *is compact for every prime p;*
(4) $\mathrm{td}_p(G)$ *is compact for some prime p;*
(5) $\mathrm{td}_p(G)$ *is closed in G for some prime p (equivalently, for all primes p);*
(6) *the topology induced from G on* $\mathrm{wtd}(G) = \bigoplus_{p\in\mathbb{P}} \mathrm{td}_p(G)$ *coincides with the topology induced by the product topology of* $\prod_{p\in\mathbb{P}} \mathrm{td}_p(G)$.

In case these conditions hold, then $G \cong_t \prod_{p\in\mathbb{P}} \mathrm{td}_p(G)$.

In Theorem 9, we determined the compact groups for which $\mathbf{\Delta}(G) = \mathrm{tor}(G)$. Using the smaller subgroup $\mathrm{wtd}(G)$ instead of $\mathbf{\Delta}(G)$, we impose the condition $\mathrm{wtd}(G) = \mathrm{tor}(G)$ instead of collapsing the whole chain $\mathrm{tor}(G) \le \mathrm{wtd}(G) \le \mathrm{td}(G)$. This leads to a concept introduced in [32]:

Definition 9 ([32])**.** *A compact abelian groups G is an **exotic torus**, if* $\mathrm{wtd}(G) = \mathrm{tor}(G)$.

Clearly, the usual tori are also exotic tori, but the solenoid Σ_1 defined in Example 1(1) is an exotic torus that is not a torus. The next theorem from [32] giving eleven equivalent characterizations of exotic tori (of those (2) was used in [32] as the original definition) provides further examples of exotic tori (see also Example 4 (3), (4)).

Theorem 21 ([32])**.** *For a compact abelian group $G = A^\wedge$ the following are equivalent:*

(1) $\mathrm{wtd}(G)$ *is torsion;*
(2) $\mathrm{Soc}(G)$ *is topologically essential;*
(3) *G contains copies of the p-adic integers $\widehat{\mathbb{Z}}_p$ for no prime p;*
(4) *$n = \dim(G) < \infty$ and for every continuous surjective homomorphism $f : G \to \mathbb{T}^n$ we have* $\mathrm{Ker}\, f = \prod_p B_p$*, where each B_p is a (bounded) compact p-group;*
(5) *$n = \dim(G) < \infty$ and there exists a homomorphism $f : G \to \mathbb{T}^n$ as in (3);*
(6) $\mathrm{wtd}(G) \cong (\mathbb{Q}/\mathbb{Z})^n \times \bigoplus_{p\in\mathbb{P}} B_p$ *algebraically, where each B_p is a (bounded) compact p-group;*
(7) *A is **strongly non-divisible**, i.e., all non-trivial quotients of A are non-divisible;*
(8) *every proper subgroup of A is contained in some maximal subgroup of A;*
(9) *A admits a surjective homomorphism $A \to \mathbb{Z}(p^\infty)$ for no prime p;*
(10) *$n = \mathrm{rk}(A) < \infty$ and $A/F \cong \bigoplus T_p$, where each T_p is a bounded p-group, for every $F \in \mathcal{F}(A)$;*
(11) *$n = \mathrm{rk}(A) < \infty$ and there exists $F \in \mathcal{F}(A)$ as in (10).*

Corollary 5. *If G is a non-trivial connected exotic torus, then $n = \dim(G) < \infty$ and* $\mathrm{wtd}(G) = \mathrm{tor}(G) \cong (\mathbb{Q}/\mathbb{Z})^n$*, i.e., all p-ranks of G coincide and equal* $\dim(G)$.

It was deduced from this corollary that the only divisible torsion abelian group that may carry minimal topologies are the groups $(\mathbb{Q}/\mathbb{Z})^n$, $n \in \mathbb{N}$ ([32]).

Following [12], call a compact group ***almost countable*** if it is the completion of countable minimal abelian group. This class of compact groups was described by Prodanov [12] as follows: a compact abelian group G is almost countable if and only if $n = \dim(G) < \infty$ and there exists a homomorphism $f : G \to \mathbb{T}^n$ such that $\operatorname{Ker} f = \prod_p (\widehat{\mathbb{Z}}_p^{e_p} \times F_p)$, where F_p is a finite p group and $e_p \in \{0, 1\}$ for every prime p. These are the compact abelian groups G such that $\operatorname{td}(G)$ has a countable essential subgroup.

The larger class $\mathcal{K}$ of compact abelian groups, that contain copies of the group $\widehat{\mathbb{Z}}_p^{\mathbb{N}}$ for no prime p was studied in [6]. It is stable under extension and contains all almost countable compact groups, as well as all exotic tori. Its subclass of compact groups G that contain copies of the group $\widehat{\mathbb{Z}}_p^2$ for no prime p coincides with the completions of minimal abelian groups of countable rank, or equivalently, these are the compact abelian groups G such that $\operatorname{td}(G)$ has an essential subgroup of countable rank (see [3] or [6]).

Example 4. *Let $G = A^\wedge$ where $\mathbb{Z} \le A \le \mathbb{Q}$, be a solenoid, as in Lemma 11 and Theorem 13. It follows from $\mathbb{Z} \rightarrowtail A \twoheadrightarrow A/\mathbb{Z} \cong \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^{h_p})$ that*

$$(A/\mathbb{Z})^\wedge = \operatorname{Hom}(A/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to G = A^\wedge \overset{\phi}{\twoheadrightarrow} \mathbb{T}$$

is exact and $(A/\mathbb{Z})^\wedge \to \mathbf{\Delta}(G) = \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}) \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is exact, with $(A/\mathbb{Z})^\wedge \cong_{\mathrm{t}} \prod_{p\in\mathbb{P}} \Delta_p$ where $\Delta_p \cong_{\mathrm{t}} \mathbb{Z}(p^{h_p})^\wedge$, so $\Delta_p \cong_{\mathrm{t}} \widehat{\mathbb{Z}}_p$ when $p \in \mathbb{P}_\infty$ and Δ_p is a cyclic p-group otherwise.

(1) G is an exotic torus if and only if $\mathbb{P}_\infty = \emptyset$ (i.e., $\operatorname{tp}(A)$ has no entries ∞).

(2) It follows from (1) that there are $\mathfrak{c}$ many pairwise non-isomorphic connected one-dimensional exotic tori G; they all have $\operatorname{wtd}(G) \cong \mathbb{Q}/\mathbb{Z}$, according to Corollary 5. Nevertheless, for these exotic tori G the subgroups $\operatorname{wtd}(G)$ remain pairwise non isomorphic (since, similarly to Theorem 12, if $\operatorname{wtd}(G) \cong_t \operatorname{wtd}(H)$, then $G \cong_t H$ for every pair of compact abelian groups G, H).

(3) According to Theorem 13, if G is an exotic torus, then $\operatorname{Soc}(G)$ is dense in G if and only if $\mathbb{P}_0$ is infinite (see ([32], Proposition 2.5) for a more general result in the case of connected exotic tori of arbitrary dimension). According to Theorem 21, in this case, $\operatorname{Soc}(G)$ is the smallest dense topologically essential subgroups of G.

(4) The second assertion in (3) is related to the following more general fact proved in ([33], Theorem 5.1) justifying the interest in dense socles: a connected compact abelian group G contains a smallest dense topologically essential (i.e., smallest dense minimal) subgroup of G if and only if G is an exotic torus with dense $\operatorname{Soc}(G)$.

8. Final Comments and Open Problems

One can deduce from Lemma 11(2) that for a solenoid Σ all delta subgroups Δ of Σ have the property that all subgroups of finite index of Δ are open.

Problem 1. *Classify the compact abelian groups whose delta subgroups have the property that all their subgroups of finite index are open.*

If $G = A^\wedge$ is a finite-dimensional compact connected abelian group, one can easily extend the argument in the proof of Theorem 13 and prove that $\operatorname{Soc}(G)$ is dense in G if $\mathbb{P}_0(G)$ is infinite, where $\mathbb{P}_0(G)$ is defined in this more general case as follows (a different proof in case G is an exotic torus can be found in ([32], Proposition 2.5)). Let $n = \dim G$, then there exists a short exact sequence $\mathbb{Z}^n \rightarrowtail A \twoheadrightarrow A/\mathbb{Z}^n$, where $A/\mathbb{Z}^n$ is torsion (actually, isomorphic to a subgroup of $(\mathbb{Q}/\mathbb{Z})^n$). In this notation, $\mathbb{P}_0(G) = \{p \in \mathbb{P} : \operatorname{rk}_p(A/\mathbb{Z}^n) = 0\}$. Obviously, $\mathbb{P}_0(G) = \mathbb{P}_0$, as defined in Theorem 13, when $n = 1$. The following example shows that when $\dim G > 1$, infinity of $\mathbb{P}_0(G)$ is not a necessary condition for the density of $\operatorname{Soc}(G)$.

Example 5. *Split $\mathbb{P} = \pi_1 \sqcup \pi_2$ in two disjoint infinite subsets π_1, π_2 (e.g., take π_1 to be the set of all primes of the form $4k + 1$). For $i = 1, 2$ define the rational group $A_i = \langle 1/p : p \in \pi_i \rangle$ and the* *solenoid* $\Sigma_i = A_i^{\wedge}$*. Then both* Σ_1 *and* Σ_2 *have dense socles, by Theorem 13, so* $G = \Sigma_1 \times \Sigma_2$ *has dense socle as well. Nevertheless,* $\mathbb{P}_0(G) = \emptyset$*.*

Problem 2. *Find a criterion for density of* $\mathrm{Soc}(G)$ *for a finite-dimensional compact connected abelian group G.*

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References

1. Hofmann, K.H.; Morris, S.A. *The Structure of Compact Groups, a Primer for the Student—A Handbook for the Expert*, 4th ed.; De Gruyter: Berlin, Germany, 2020.
2. Lewis, W.; Loth, P.; Mader, A. Free subgroups with torsion quotients and profinite subgroups with torus quotients. *Rend. Sem. Mat. Padova* **2020**, *144*, 177–195. [CrossRef]
3. Dikranjan, D.; Prodanov, I.; Stoyanov, L. *Topological Groups: Dualities and Minimal Group Topologies*; Pure and Applied Mathematics; Marcel Dekker Inc.: New York, NY, USA; Basel, Switzerland, 1989; Volume 130.
4. Stojanov, L. Weak periodicity and minimality of topological groups. *Annu. Univ. Sofia Fac. Math. Méc.* **1979**, *73*, 155–167.
5. Banaschewski, B. Minimal topological algebras. *Math. Ann.* **1974**, *211*, 107–114.
6. Dikranjan, D. On a class of finite-dimensional compact abelian groups. In *Topology, Theory and Applications (Eger, 1983), Colloquia Mathematica Societatis János Bolyai;* North-Holland Publishing Company: Amsterdam, The Netherlands, 1985; Volume 41, pp. 215–231.
7. Dikranjan, D. Recent advances in minimal topological groups. *Topol. Appl.* **1998**, *85*, 53–91. [CrossRef]
8. Dikranjan, D.; Megrelishvili, M. *Minimality Conditions in Topological Groups*; Recent Progress in General Topology. III; Atlantis Press: Paris, France, 2014; pp. 229–327.
9. Doïtchinov, D. Produits des groupes topologiques minimaux. *Bull. Sci. Math.* **1972**, *97*, 59–64.
10. Eberhard, V.; Schwanengel, U. $(\mathbb{Q}/\mathbb{Z})^{\omega}$ est un groupe topologique minimal. *Rev. Roum. Math. Pures Appl.* **1982**, *27*, 957–964.
11. Prodanov, I. Precompact minimal group topologies and p–adic numbers. *Annu. Univ. Sofia Fac. Math. Méc.* **1971**, *66*, 249–266.
12. Prodanov, I. Minimal topologies on countable abelian groups. *Annu. Univ. Sofia Fac. Math. Méc.* **1975**, *70*, 107–118.
13. Prodanov, I. Some minimal group topologies are precompact. *Math. Ann.* **1977**, *227*, 117–125. [CrossRef]
14. Prodanov, I.; Stoyanov, L. Every minimal abelian group is precompact. *C. R. Acad. Bulg. Sci.* **1984**, *37*, 23–26.
15. Stephenson, R.M., Jr. Minimal topological groups. *Math. Ann.* **1971**, *192*, 193–195. [CrossRef]
16. Dikranjan, D.; Lewis, W.; Loth, P.; Mader, A. Fat Delta 2: Functorial subgroups of topological abelian groups. *Topol. Proc.* 2022, *submitted.*
17. Fuchs, L. *Abelian Groups*; Springer International Publishing: Cham, Switzerland, 2015.
18. Moskowitz, M. Homological algebra in locally compact abelian groups. *Trans. Am. Math. Soc.* **1967**, *127*, 182–212. [CrossRef]
19. Mader, A. *Almost Completely Decomposable Groups (Algebra, Logic and Applications)*; Gordon and Breach Science Publishers: Amsterdam, The Netherlands, 2000; Volume 13.
20. Reid, J. A note on torsion–free abelian groups of infinite rank. *Proc. Am. Math. Soc.* **1962**, 13, 222–225. [CrossRef]
21. Jacoby, C.; Loth, P. *Abelian Groups: Structures and Classifications*; De Gruyter: Berlin, Germany, 2019.
22. Lewis, W.; Loth, P.; Mader, A. The main decomposition of finite–dimensional protori. *J. Group Theory* **2021**, *24*, 109–127. [CrossRef]
23. Hewitt, E.; Ross, K.A. *Abstract Harmonic Analysis*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1963; Volume I.
24. Stroppel, M. *Locally Compact Groups*; European Mathematical Society: Zürich, Switzerland, 2006.
25. Chigogidze, A.; Hofmann, K.H.; Martin, J.R. Compact Groups and Fixed Point Sets. *Trans. Am. Math. Soc.* **1997**, *349*, 4537–4554. [CrossRef]
26. Ribes, L.; Zalesskii, P. *Profinite Groups*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2010.
27. Dikranjan, D.; Stoyanov, L. A–classes of minimal abelian groups. *Annu. Univ. Sofia Fac. Math. Méc.* **1976**, *71 Pt 2*, 53–62.
28. Dikranjan, D.; Prodanov, I. Totally minimal groups. *Annu. Univ. Sofia Fac. Math. Méc.* **1974**, *69*, 5–11.
29. Außenhofer, L.; Dikranjan, D.; Bruno, A.G. *Topological Groups and the Pontryagin–van Kampen Duality. An Introduction*; De Gruyter Studies in Mathematics; Walter de Gruyter GmbH: Berlin, Germany; Boston, MA, USA, 2022; Volume 83.
30. Grant, D.L. Arbitrary powers of the roots of unity are minimal Hausdorff topological groups. *Topol. Proc.* **1979**, *4*, 103–108.
31. Armacost, D.L. *The Structure of Locally Compact Abelian Groups*; Monographs and Textbooks in Pure and Applied Mathematics; Marcel Dekker, Inc.: New York, NY, USA, 1981; Volume 68.

32. Dikranjan, D.; Prodanov, I. A class of compact abelian groups. *Annu. Univ. Sofia Fac. Math. Méc.* **1975**, *70*, 191–206.
33. Comfort, W.; Dikranjan, D. On the poset of totally dense subgroups of compact groups. *Topol. Proc.* **1999**, *24*, 103–128.

axioms

Article

On Self-Aggregations of Min-Subgroups

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Abstract: Preservation of structures under aggregation functions is an active area of research with applications in many fields. Among such structures, min-subgroups play an important role, for instance, in mathematical morphology, where they can be used to model translation invariance. Aggregation of min-subgroups has only been studied for binary aggregation functions. However, results concerning preservation of the min-subgroup structure under binary aggregations do not generalize to aggregation functions with arbitrary input size since they are not associative. In this article, we prove that arbitrary self-aggregation functions preserve the min-subgroup structure. Moreover, we show that whenever the aggregation function is strictly increasing on its diagonal, a min-subgroup and its self-aggregation have the same level sets.

Keywords: aggregation function; T-subgroup; strictly monotone function

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1. Introduction

Aggregation operators have become an important research topic in the last two decades. The motivation to use such functions comes from the need to summarize different pieces of information into a single object, which is a particularly challenging task when the incoming information is heterogeneous, imprecise, or incomplete. These operators are nowadays a fundamental tool of computer sciences with applications in classification, databases, control, decision making, or image processing among others. Recent monographs on this topic are [1–3].

An aggregation operator is a non-decreasing function $A : [0,1]^n \to [0,1]$ satisfying certain boundary conditions (see Definition 1). This construction allows one to aggregate not only numerical values but also any functions, or structures on a set that have output in the unit interval.

Min-subgroups were introduced by Rosenfeld in ([4]) as a fuzzy set μ whose domain is a group G such that $\mu(x) = \mu(x^{-1})$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all x, y in G. Note that from the definition, we immediately obtain $\mu(e) \geq \mu(x)$ for all x in G, and hence the normalization condition $\mu(e) = 1$ is often added to the definition of fuzzy subgroup. Das studied min-subgroups thoroughly in [5], introducing a characterization in terms of level sets in which the level sets of μ correspond to crisp subgroups of G. Das also introduced an equivalence relation between fuzzy groups concerning level sets. Anthony and Sherwood (see [6]) extended Rosenfeld's definition using an arbitrary t-norm T instead of the minimum. These groups are called T-subgroups. Formato and Gerla constructed a correspondence between T-indistinguishability operators on a set (relations that are reflexive, symmetric, and T-transitive) and T-subgroups related with the permutation group of the set further motivating the study of T-subgroups (see [7]).

Min-subgroups can be identified as indistinguishability operators that are invariant by translations (see [8]). This type of indistinguishability operator plays a fundamental role in some applications, notably in mathematical morphology (see [8–11]).

When the set of inputs of an aggregation function share a structure (i.e., they are all indistinguishability operators, min-subgroups, or other fuzzy relations with additional properties), the main problem is the preservation of that structure. In other words, the problem is determining conditions guarantee that the output has the same structure. Preservation of structures under aggregation has been widely studied in recent decades (see [12–21]).

In particular, preservation of the min-subgroup structure under binary aggregations was studied in [12]. However, these results cannot be immediately translated into n-ary aggregation functions since these operators are not necessarily associative. In this article, we obtain the first results concerning the preservation of the min-subgroup structure for aggregation of more than two min-subgroups. More concretely, we focus on the preservation of the min-subgroup structure under self-aggregation motivated by the central role they play in the binary case. Note that the minimum t-norm is the only t-norm that is idempotent, and it is characterized by its level-sets, which makes it very useful in certain contexts ([22]).

The remainder of the article is organized as follows. In Section 2, we introduce the relevant definitions and known facts. Section 3 contains our first new results. We show that the aggregations of an arbitrary number of min-subgroups are also min-subgroups. We also study the behavior of the fuzzy subgroup obtained from conjunctive, averaging, disjunctive, and mixed aggregation functions. Section 4 is devoted to investigate self-aggregations with respect to the equivalence classes of fuzzy subgroups given by its level sets. Our main result states that, for aggregation functions that are strictly increasing on their diagonal, the self-aggregation of a min-subgroup has the same level sets that the original min-subgroup. The article ends with some concluding remarks and future lines of research.

2. Preliminary Facts

Definition 1 ([1])**.** *An operation $A : [0,1]^n \longrightarrow [0,1]$ is called an aggregation function if it satisfies the following axioms:*

$(A1)$ *Monotonicity. If $x_i \leq y_i$ for each $i \in \{1,\dots,n\}$, then $A(x_1,\dots,x_n) \leq A(y_1,\dots,y_n)$.*
$(A2)$ *Boundary conditions. $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$.*

Moreover, A is called jointly strictly monotone if whenever $x_i < y_i$ for all $i \in \{1,\dots,n\}$, then $A(x_1,\dots,x_n) < A(y_1,\dots,y_n)$.

Among the most relevant aggregation functions, we find the arithmetic mean, the geometric mean, the harmonic mean, and the quadratic mean (see [1,3]). Aggregation functions are classified into four broad classes: conjunctive, averaging, disjunctive, and mixed functions.

1. A conjunctive aggregation function A is an aggregation function such that $A(r_1,\dots,r_n) \leq \min\{r_1,\dots,r_n\}$ for all $(r_1,\dots,r_n) \in [0,1]^n$. A prototypical example is any t-norm.
2. An averaging aggregation function A is an aggregation function such that $\min\{r_1,\dots, r_n\} \leq A(r_1,\dots,r_n) \leq \max\{r_1,\dots,r_n\}$ for all $(r_1,\dots,r_n) \in [0,1]^n$. Ordered weighted averaging operators belong to this category.
3. A disjunctive aggregation function A is an aggregation function such that $\max\{r_1,\dots, r_n\} \leq A(r_1,\dots,r_n)$ for all $(r_1,\dots,r_n) \in [0,1]^n$. One example is any t-conorm.
4. An aggregation function A is called mixed if A is not conjunctive, averaging, nor disjunctive. Uninorms belong to this type of aggregation functions.

Note that the averaging class is frequently called idempotent class since every averaging aggregation function A satisfies $A(r,\dots,r) = r$ for all $(r,\dots,r) \in [0,1]^n$. Extensive information about aggregation functions can be found in [3].

Definition 2 ([4])**.** *Let $(G,\cdot)$ be a group. We say that $\mu : G \longrightarrow [0,1]$ is a min-subgroup of G if:*
$(G1)$ *For all $x \in G$, $\mu(x) \geq \mu(x^{-1})$.*

$(G2)$ *For all* $x, y \in G$, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

Note that $G1$ is equivalent to $\mu(x) = \mu(x^{-1})$ for all $x \in G$. In the paper, e denotes the neutral element of the group G.

Definition 3 ([23]). *Let μ be a fuzzy subset of a given universe X. For each $t \in [0,1]$, the level set μ_t and strict level set μ^t are defined as follows:*

$$\mu_t = \{x \in X \mid \mu(x) \geq t\} \qquad \mu^t = \{x \in X \mid \mu(x) > t\}$$

The support of μ is defined by supp $\mu = \mu^0$.

Level sets (or α-cuts) have been studied extensively in fuzzy subgroups (see for instance [24,25]). P. Das used level sets to characterize the notion of min-subgroup ([5]).

Proposition 1 ([5]). *Let G be a group and μ a fuzzy set of G; then μ is a min-subgroup of G if and only if all its non-empty level sets are subgroups of G.*

3. Self-Aggregation

Given an aggregation function A and n fuzzy subsets $\mu_1, \dots, \mu_n$ of a group G, we consider the fuzzy set $A(\mu_1, \dots, \mu_n)$ on G defined by

$$A(\mu_1, \dots, \mu_n)(x) = A(\mu_1(x), \dots, \mu_n(x))$$

for each $x \in G$. We say that $A(\mu_1, \dots, \mu_n)$ is the aggregation of $\mu_1, \dots, \mu_n$ through A.

In this section, we will study the aggregation of $A(\mu, \dots, \mu)$ whenever μ is a min-subgroup of a group G, i.e., the self-aggregation of μ through A.

Anthony and Sherwood (see [6]) introduced T-subgroups as an extension of min-subgroups using an arbitrary t-norm T instead of the minimum.

The following theorem underlines the relevance of min-subgroups within T-subgroups since the minimum is the only t-norm that guarantees preservation of the T-subgroup structure for any binary self-aggregation process.

Theorem 1 ([12]). *Let G be a group with at least four elements and T a t-norm satisfying $T \neq T_D$, where T_D is the drastic t-norm. The following assertions are equivalent:*

1. $T = \min$.
2. *For each T-subgroup μ and each aggregation function A, $A(\mu, \mu)$ is a T-subgroup.*

Due to this result, given any aggregation function and any min-subgroup μ, $A(\mu, \mu)$ is also a min-subgroup. However, since A is not necessarily associative, the previous result does not guarantee that $A(\mu, \mu, \dots, \mu)$ is also a min-subgroup. We establish that this is the case for arbitrarily sized aggregations.

Proposition 2. *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function and μ a min-subgroup of a group G. Then, $A(\mu, \dots, \mu)$ is also a min-subgroup of G.*

Proof. Take $x \in G$; we have that

$$A(\mu, \dots, \mu)(x) = A(\mu(x), \dots, \mu(x)) = A(\mu(x^{-1}), \dots, \mu(x^{-1})) = A(\mu, \dots, \mu)(x^{-1}).$$

Take $x, y \in G$. Without loss of generality, let us assume that $\mu(x) \leq \mu(y)$. Under this premise, using the fact that A is a non-decreasing function, we have that

$$A(\mu, \dots, \mu)(x) = \min\Big\{A(\mu, \dots, \mu)(x), A(\mu, \dots, \mu)(y)\Big\}. \tag{1}$$

Therefore,

$$A(\mu,\ldots,\mu)(xy) = A(\mu(xy),\ldots,\mu(xy)) \geq A\big(\min\{\mu(x),\mu(y)\},\ldots,\min\{\mu(x),\mu(y)\}\big).$$

Since $\mu(x) \leq \mu(y)$ and the monotonicity of A,

$$A\big(\min\{\mu(x),\mu(y)\},\ldots,\min\{\mu(x),\mu(y)\}\big) = A(\mu(x),\ldots,\mu(x)) = A(\mu,\ldots,\mu)(x).$$

Taking into account (1), the proof is completed. □

We proceed to study the comparison between $A(\mu,\ldots,\mu)$ and μ with respect to the usual order of fuzzy sets, that is, if $A(\mu,\ldots,\mu) \leq \mu$ or $A(\mu,\ldots,\mu) \geq \mu$. The following result shows sufficient conditions on A in order to compare both of them.

Proposition 3. *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function and μ a min-subgroup of a group G.*

1. *If A is a conjunctive aggregation function, then $A(\mu,\ldots,\mu) \leq \mu$.*
2. *If A is an averaging aggregation function, then $A(\mu,\ldots,\mu) = \mu$.*
3. *If A is a disjunctive aggregation function, then $A(\mu,\ldots,\mu) \geq \mu$.*

Proof. Let us consider $x \in G$.

1. $A(\mu,\ldots,\mu)(x) = A(\mu(x),\ldots,\mu(x)) \leq \min\{\mu(x),\ldots,\mu(x)\} = \mu(x)$.
2. On the one hand, $\mu(x) = \min\{\mu(x),\ldots,\mu(x)\} \leq A(\mu(x),\ldots,\mu(x)) = A(\mu,\ldots,\mu)(x)$. On the other hand, $A(\mu,\ldots,\mu)(x) = A(\mu(x),\ldots,\mu(x)) \leq \max\{\mu(x),\ldots,\mu(x)\} = \mu(x)$.
3. $A(\mu,\ldots,\mu)(x) = A(\mu(x),\ldots,\mu(x)) \leq \max\{\mu(x),\ldots,\mu(x)\} = \mu(x)$.

□

However, if A is mixed, it is possible that $A(\mu,\ldots,\mu)$ is not comparable to μ, and when it is, all the above inequalities can appear, as the following example shows.

Example 1. *Consider the group $G = (\mathbb{Z}_6,+)$ and the fuzzy sets μ,η,ν,σ defined in the table below.*

G	0	1	2	3	4	5
μ	0.9	0.5	0.5	0.9	0.5	0.5
η	1	0.2	0.8	0.2	0.8	0.2
ν	0.4	0.3	0.3	0.4	0.3	0.3
σ	1	0	0	0.5	0	0

Clearly, they are min-subgroups of G because their level sets are crisp subgroups of G. Let us consider the following binary aggregation function A, where $e = 0.5$ is the neutral element:

$$A(x,y) = \begin{cases} y & \text{if} \quad x = e, \\ x & \text{if} \quad y = e, \\ 0 & \text{if} \quad x < e, y < e, \\ 1 & \text{if} \quad x > e, y > e, \\ e & \quad\text{otherwise.} \end{cases}$$

It is easy to check that A is a mixed aggregation function. The self-aggregations of the previous min-subgroups are:

G	0	1	2	3	4	5
$A(\mu,\mu)$	1	0.5	0.5	1	0.5	0.5
$A(\eta,\eta)$	1	0	1	0	1	0
$A(\nu,\nu)$	0	0	0	0	0	0
$A(\sigma,\sigma)$	1	0	0	0.5	0	0

We can conclude that

$$A(\mu,\mu) \geq \mu,$$
$$A(\nu,\nu) \leq \nu,$$
$$A(\sigma,\sigma) = \sigma,$$

but $A(\eta,\eta)$ is not comparable to η.

4. Self-Aggregation on the Equivalence Class

There are infinitely many min-subgroups that generate the same chain of subgroups. In order attempt any classification, it is natural to relate two such min-subgroups. P. Das introduced in [5] the following relation between min-subgroups of a group.

Definition 4. *Let G be a group and μ,η two min-subgroups of G. We say that μ is equivalent to η, written $\mu \sim \eta$, if $\{\mu_t\}_{t\in\mu(G)} = \{\eta_s\}_{s\in\eta(G)}$ where $\mu(G)$ and $\eta(G)$ are the ranges of μ and η, respectively. The class of an element μ will be denoted by $[\mu]$.*

There are other significant equivalences on min-subgroups [26–28]. A study on their connections has been recently presented in [29]. Our paper focuses only on the given one by P. Das, which is the most relevant in the literature. Many results can be transferred easily taking into account the implications diagram from [29]. A. Jain characterized the equivalence relation $\sim$ as follows.

Proposition 4 ([30])**.** *Let G be a group and μ,η two min-subgroups of G. The following assertions are equivalent:*
1. $\mu(x) > \mu(y)$ *if and only if* $\eta(x) > \eta(y)$.
2. $\mu(x) \geq \mu(y)$ *if and only if* $\eta(x) \geq \eta(y)$.
3. $\{\mu_t\}_{t\in\mu(G)} = \{\eta_s\}_{s\in\eta(G)}$.
4. $\{\mu^t\}_{t\in\mu(G)} = \{\eta^s\}_{s\in\eta(G)}$.

We introduce the following example showing equivalence classes according to $\sim$ in order to illustrate how self-aggregation acts on the equivalence class.

Example 2. *Consider the min-subgroups μ,η,ν,σ and the aggregation A presented in Example 1. We have:*

$$[\sigma] \neq [\mu] = [\nu] \neq [\eta] \text{ and } [\sigma] \neq [\eta].$$

Moreover, self-aggregating each of these min-subgroups through A provides:

$$[A(\mu,\mu)] = [\mu]$$
$$[A(\eta,\eta)] \neq [\eta]$$
$$[A(\nu,\nu)] \neq [\nu]$$
$$[A(\sigma,\sigma)] = [\sigma]$$

The example shows that self-aggregation does not preserve equivalence classes in general. We dedicate the last part of the section to finding conditions on an aggregation function A, which ensures that a min-subgroup and its self-aggregation by A belong to the same equivalence class.

The following result is a straightforward consequence of Proposition 3.

Proposition 5. *If A is an averaging aggregation function and μ a min-subgroup of a group G, then $[A(\mu,\ldots,\mu)] = [\mu]$.*

The next proposition shows the relevance of jointly strictly monotone aggregation functions.

Proposition 6. *Let G be a group and $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. If A is jointly strictly monotone, then $[A(\mu,\ldots,\mu)] = [\mu]$ for each min-subgroup μ of G.*

Proof. We need to prove that $A(\mu,\ldots,\mu)$ and μ induce the same level sets. We will use the characterization of the Proposition 4. Let us take $x, y \in G$. Firstly, assume that $\mu(x) \geq \mu(y)$; by monotonicity of A, we have that $A(\mu,\ldots,\mu)(x) \geq A(\mu,\ldots,\mu)(y)$.

Conversely, assume that $A(\mu,\ldots,\mu)(x) \geq A(\mu,\ldots,\mu)(y)$. We must check that $\mu(x) \geq \mu(y)$. By contradiction, $\mu(x) < \mu(y)$. Since A is jointly strictly monotone, we conclude that $A(\mu(x),\ldots,\mu(x)) < A(\mu(y),\ldots,\mu(y))$; equivalently, $A(\mu,\ldots,\mu)(x) < A(\mu,\ldots,\mu)(y)$, obtaining the desired contradiction. □

We proceed with the main result of the article. Let us recall that an aggregation function A is strictly increasing on its diagonal if for each $x, y \in [0,1]$, satisfying $x < y$; then

$$A(x,\ldots,x) < A(y,\ldots,y).$$

Theorem 2. *Let G be a group and $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. The following assertions are equivalent:*

1. *A is a strictly increasing function on its diagonal.*
2. *$A(\mu,\ldots,\mu)$ and μ induce the same level sets.*

Proof. 1 $\implies$ 2. We will use the characterization of the Proposition 4. Let us take $x, y \in G$. Assume that $\mu(x) \geq \mu(y)$; by monotonicity of A, we have that $A(\mu,\ldots,\mu)(x) \geq A(\mu,\ldots,\mu)(y)$.

Conversely, assume that $A(\mu,\ldots,\mu)(x) \geq A(\mu,\ldots,\mu)(y)$. We must check that $\mu(x) \geq \mu(y)$. By contradiction, suppose that $\mu(x) < \mu(y)$. Since A is a strict increasing function on its diagonal, we conclude that $A(\mu(x),\ldots,\mu(x)) < A(\mu(y),\ldots,\mu(y))$, and equivalently, $A(\mu,\ldots,\mu)(x) < A(\mu,\ldots,\mu)(y)$, which is a contradiction.

2 $\implies$ 1. We prove that if A is not strictly increasing on its diagonal, then there is a min-subgroup $\mu \in G$ such that $A(\mu,\ldots,\mu)$ and μ do not have the same level sets. Under this premise, there are $a, b \in [0,1]$ such that

$$a < b \text{ and } A(a,\ldots,a) \geq A(b,\ldots,b).$$

By monotonicity, we have that $A(a,\ldots,a) = A(b,\ldots,b)$. Let us create the fuzzy set $\mu : G \longrightarrow [0,1]$, satisfying $\mu(e) = b$ and $\mu(x) = a$ whenever $x \neq e$. (We remember that e denotes the neutral element of G.) Clearly, μ is a min-subgroup of G according to Proposition 1. Therefore, considering an element $x \neq e$, we conclude that

$$A(\mu(x),\ldots,\mu(x)) = A(a,\ldots,a) = A(b,\ldots,b) = A(\mu(e),\ldots,\mu(e)).$$

Since $\mu(x) < \mu(e)$, they induce different level sets. □

As a direct consequence of the previous theorem, we have obtained the desired characterization.

Corollary 1. *Let μ be a min-subgroup of a group G. If A is a strict t-norm or a strict t-conorm, then $A(\mu,\ldots,\mu)$ belongs to the same equivalence class as μ.*

5. Concluding Remarks

Let A be a generic aggregation function, G a group, μ a min-subgroup of G, and $[\mu]$ the Das class of μ.

Firstly we have shown that the structure of min-subgroup is preserved by arbitrary self-aggregation functions—i.e., $A(\mu, \ldots, \mu)$ is a min-subgroup—and we have studied when $A(\mu, \ldots, \mu)$ is comparable to μ.

Secondly, we have shown an example of an aggregation function A and a fuzzy subgroup μ satisfying $[A(\mu, \ldots, \mu)] \neq [\mu]$. Hence, the Das class of a min-subgroup is not necessarily preserved by an arbitrary aggregation function. We have shown that this class is preserved if A is an averaging or a jointly strictly monotonous aggregation function.

Thirdly, our main results states that $A(\mu, \ldots, \mu)$ and μ induce the same level sets if and only if A is a strictly increasing function on its diagonal. This result implies that if A is a strict t-norm or a strict t-conorm, $A(\mu, \ldots, \mu)$ belong to the same equivalence class as μ.

Future research could examine under what conditions the Lukasiewicz and product subgroup structures are preserved by arbitrary self-aggregation functions and explore the implications of the migrativity property ([31]) for the preservation of these subgroup structures under self-aggregation functions.

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References

1. Beliakov, G.; Pradera, A.; Calvo, T. *Aggregation Functions: A Guide for Practitioners*; Springer: Berlin/Heidelberg, Germany, 2007.
2. Beliakov, G.; Sola, H.B.; Sánchez, T.C. *A Practical Guide to Averaging Functions*; Springer: Heilderberg, Germany; New York, NY, USA; Dordrech, the Netherlands; London, UK, 2016.
3. Calvo, T.; Kolesárová, A.; Komorníková, M.; Mesiar, R. *Aggregation Operators*; Physica-Verlag: Heidelberg, Germany, 2002; pp. 3–104.
4. Rosenfeld, A. Fuzzy groups. *J. Math. Anal. Appl.* **1971**, *35*, 512–517. [CrossRef]
5. Das, P. Fuzzy groups and level subgroups. *J. Math. Anal. Appl.* **1981**, *84*, 264–269. [CrossRef]
6. Anthony, J.; Sherwood, H. Fuzzy groups redefined. *J. Math. Anal. Appl.* **1979**, *69*, 124–130. [CrossRef]
7. Formato, F.; Gerla, G.; Scarpati, L. Fuzzy subgroups and similarities. *Soft Comput.* **1999**, *3*, 1–6. [CrossRef]
8. Boixader, D.; Recasens, J. On the relationship between fuzzy subgroups and indistinguishability operators. *Fuzzy Sets Syst.* **2019**, *373*, 149–163. [CrossRef]
9. Bloch, I.; Maître, H. Fuzzy mathematical morphologies: A comparative study. *Pattern Recognit.* **1995**, *28*, 1341–1387. [CrossRef]
10. Elorza, J.; Fuentes-González, R.; Bragard, J.; Burillo, P. On the relation between fuzzy closing morphological operators, fuzzy consequence operators induced by fuzzy preorders and fuzzy closure and co-closure systems. *Fuzzy Sets Syst.* **2013**, *218*, 73–89. [CrossRef]
11. Soille, P. *Morphological Image Analysis: Principles and Applications*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2013.
12. Bejines, C.; Chasco, M.; Elorza, J. Aggregation of fuzzy subgroups. *Fuzzy Sets Syst.* **2021**, *418*, 170–184. [CrossRef]
13. Bejines, C.; Ardanza-Trevijano, S.; Chasco, M.; Elorza, J. Aggregation of indistinguishability operators. *Fuzzy Sets Syst.* **2021**. [CrossRef]
14. Drewniak, J.; Dudziak, U. Preservation of properties of fuzzy relations during aggregation processes. *Kybernetika* **2007**, *43*, 115–132.
15. Dudziak, U. Preservation of t-norm and t-conorm based properties of fuzzy relations during aggregation process. In Proceedings of the EUSFLAT Conference, Milano, Italy, 11–13 September 2013.

16. Fodor, J.C.; Roubens, M. *Fuzzy Preference Modelling and Multicriteria Decision Support*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 1994; Volume 14.
17. He, X.; Li, Y.; Qin, K.; Meng, D. On the TL-transitivity of fuzzy similarity measures. *Fuzzy Sets Syst.* **2017**, *322*, 54–69. [CrossRef]
18. Jacas, J.; Recasens, J. Aggregation of T-transitive relations. *Int. J. Intell. Syst.* **2003**, *18*, 1193–1214. [CrossRef]
19. Mayor, G.; Recasens, J. Preserving T-Transitivity. In *Artificial Intelligence Research and Development*; Nebot, A., Binefa, X., de Mantaras, L., Eds.; IOS Press: Amsterdam, The Netherlands; Berlin, Germany; Washington, DC, USA, 2016; pp. 79–87.
20. Pedraza, T.; Rodríguez-López, J.; Valero, Ó. Aggregation of fuzzy quasi-metrics. *Inf. Sci.* **2020**. [CrossRef]
21. Saminger, S.; Mesiar, R.; Bodenhofer, U. Domination of aggregation operators and preservation of transitivity. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.* **2002**, *10*, 11–35. [CrossRef]
22. Klement, E.P.; Mesiar, R.; Pap, E. *Triangular Norms*; Springer Science & Business Media: Dordrecht, The Netherlands, 2013; Volume 8.
23. Zadeh, L. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
24. Dixit, V.; Kumar, R.; Ajmal, N. Level subgroups and union of fuzzy subgroups. *Fuzzy Sets Syst.* **1990**, *37*, 359–371. [CrossRef]
25. Murali, V.; Makamba, B. On an equivalence of fuzzy subgroups I. *Fuzzy Sets Syst.* **2001**, *123*, 259–264. [CrossRef]
26. Murali, V.; Makamba, B. On an equivalence of fuzzy subgroups III. *Int. J. Math. Math. Sci.* **2003**, *2003*, 2303–2313. [CrossRef]
27. Ray, S. Isomorphic fuzzy groups. *Fuzzy Sets Syst.* **1992**, *50*, 201–207. [CrossRef]
28. Zhang, Y. Some properties on fuzzy subgroups. *Fuzzy Sets Syst.* **2001**, *119*, 427–438. [CrossRef]
29. Bejines, C.; Chasco, M.J.; Elorza, J.; Montes, S. Equivalence relations on fuzzy subgroups. In Proceedings of the Conference of the Spanish Association for Artificial Intelligence, Granada, Spain, 23–26 October 2018; pp. 143–153.
30. Jain, A. Fuzzy subgroups and certain equivalence relations. *Iran. J. Fuzzy Syst.* **2006**, *3*, 75–91.
31. Bustince, H.; De Baets, B.; Fernández, J.; Mesiar, R.; Montero, J. A generalization of the migrativity property of aggregation functions. *Inf. Sci.* **2012**, *191*, 76–85. [CrossRef]

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Bounded Sets in Topological Spaces

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‡ Dedicated to Professor María Jesús Chasco.

Abstract: Let G be a monoid that acts on a topological space X by homeomorphisms such that there is a point $x_0 \in X$ with $GU = X$ for each neighbourhood U of x_0. A subset A of X is said to be G-bounded if for each neighbourhood U of x_0 there is a finite subset F of G with $A \subseteq FU$. We prove that for a metrizable and separable G-space X, the bounded subsets of X are completely determined by the bounded subsets of any dense subspace. We also obtain sufficient conditions for a G-space X to be locally G-bounded, which apply to topological groups. Thereby, we extend some previous results accomplished for locally convex spaces and topological groups.

Keywords: bounded set; group action; G-space; barrelled space

MSC: 22A05; 22D35; 22B05; 54H11; 54A25

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1. Introduction and Basic Facts

The notion of a bounded subset is ubiquitous in many parts of mathematics, particularly in functional analysis and topological groups. Here, we approach this concept from a broader viewpoint. Namely, we consider the action of a monoid G on a topological space X and associate it with a canonical family of G-bounded subsets. This provides a very general notion of boundedness that includes both the bounded subsets considered in functional analysis and in topological groups. In this paper, we initiate the study of this new notion of G-bounded subset. Among other results, it is proved that for a metrizable and separable G-space X, the bounded subsets of X are completely determined by the bounded subsets of any dense subspace, extending results obtained by Grothendieck for metrizable separable locally convex spaces [1], generalized subsequently by Burke and Todorčević and, separately, Saxon and Sánchez-Ruiz for metrizable locally convex spaces [2,3] and by Chis, Ferrer, Hernández and Tsaban for metrizable groups [4,5]. We also obtain sufficient conditions for a G-space X to be locally G-bounded, which applies to topological groups. This also provides the frame for extending to this setting some results by Burke and Todorčević and, separately, Saxon and Sáchez-Ruiz (loc. cit.) for metrizable locally convex spaces. A different approach to the notion of the bounded set was given by Hejcman [6] and Hu [7], who studied this concept in the realm of uniform and even topological spaces. Vilenkin [8] applied this general approach in the realm of topological groups.

2. G-Spaces

Let X be a topological space and let G be a *monoid*, i.e, a semigroup with a neutral element, which will be denoted by e. A *left action* of G on X is a map $\pi\colon G \times X \to X$ satisfying that $ex = x$ and $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in G$ and $x \in X$, where as usual, we write gx instead of $\pi(g, x) = \pi_g(x) = \pi_x(g)$. A topological space X is said to be a (left) *G-space* if all translations $\pi_g\colon X \to X$ are homeomorphisms. We sometimes denote the G-space X by the pair (G, X). Let $G \times X \to X$ and $G \times Y \to Y$ be two actions. A map $f \colon X \to Y$ between G-spaces is a *G-map* if $f(gx) = gf(x)$ for every $(g, x) \in G \times X$. Given $x \in X$, its *orbit* is the set $Gx = \{gx : g \in G\}$. Given $A \subseteq X$, we define $GA = \cup\{Gx : x \in A\}$.

A *right G-space* (X, G) can be defined analogously. If G^{op} is the opposite semigroup of G with the same topology then (X, G) can be treated as a left G-space (G^{op}, X) (and vice versa).

We say that a point $x \in X$ *topologically generates* a G-space X if for each neighborhood U of x we have $GU = X$. The set of generating points is denoted by X_{tg}. We say that X is *point-generated* when $X_{tg} \neq \emptyset$. We refer to [9] for unexplained topological definitions.

2.1. G-Boundedness

Let (G, X) be a point-generated G-space and let us fix a point $x_0 \in X_{tg}$. We say that a set $A \subseteq X$ is (G, x_0)*-bounded* (or *G-bounded* for short when there is no possible confusion) if for every neighborhood U of x_0, there is a finite set $F \subseteq G$ such that $A \subseteq FU$. The set $\mathfrak{B}(G, X, x_0)$ (or $\mathfrak{B}(G, X)$ for short) of all G-bounded sets in X is called the canonical (G, x_0)*-boundedness* on X. The G-space (G, X) is said to be *homogeneous* if for every pair of points x, y in X, there is a homeomorphism $f_{xy} \colon X \to X$ such that $f_{xy}(x) = y$ and $f_{xy}(A)$ is G-bounded for every G-bounded subset $A \subseteq X$. The proof of the following proposition is straightforward.

Proposition 1. *Let (G, X) be a G-space with a generating point $x_0 \in X_{tg}$. The following assertions hold true:*

1. *$A \subseteq X$ is (G, x_0)-bounded if and only if A is (G, x_1)-bounded for any other point $x_1 \in X_{tg}$.*
2. *Subsets of G-bounded sets are G-bounded.*
3. *If A and B are G-bounded so is $A \cup B$.*
4. *Finite sets are G-bounded.*
5. *If A is G-bounded so is gA for all $g \in G$.*
6. *Relatively compact subsets are G-bounded.*
7. *Every topological vector space E is an $\mathbb{R}^*$-space with the action $(r, v) \mapsto rv$, $r \in \mathbb{R}^*$ and $v \in E$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. The usual family of bounded subsets of E coincides with the canonical $\mathbb{R}^*$-boundedness, with $0 \in E$ as the point that topologically generates E.*
8. *If H is a topological group, K is a closed subgroup and G is a dense submonoid of H then the coset space H/K defined by the quotient map $p \colon H \to H/K$ is canonically a G-space by the action $ghK := p(gh)$. We say that a set $A \subseteq H/K$ is G-bounded if for every neighborhood U of K (seen as an element of H/K) there is a finite set $F \subseteq G$ such that $A \subseteq FU$. This defines the canonical G-boundedness on H/K, where K is the point that topologically generates H/K. Here, the family of G-bounded subsets coincide with the family of all precompact subsets for the left uniformity on H/K.*

Definition 1. *A point-generated G-space X is said to be* locally G-bounded *if for every point $x \in X$ there is a G-bounded open subset U containing it.*

The proof of the following proposition is straightforward.

Proposition 2. *Let X be a point-generated G-space. If there is a point $x \in X_{tg}$ and a neighborhood U of x that is G-bounded, then X is locally G-bounded.*

Remark 1. *From the above proposition, it follows that if a point-generated G-space X is not locally G-bounded then no neighborhood of a point $x \in X_{tg}$ can be G-bounded.*

2.2. Infinite Cardinals

In what follows, we shall use the notation ZFC for Zermelo-Fraenkel set theory including the axiom of choice, CH for the continuum hypothesis ($\mathcal{C} = \aleph_1$) and GCH for the generalized continuum hypothesis ($2^{\aleph_l} = \aleph_{l+1}$ for each cardinal $\aleph_l$). If CH is false, then there are cardinals strictly between $\aleph_0$ and $\mathcal{C}$.

Following [10], consider the set of functions $\mathbb{N}^{\mathbb{N}}$ from $\mathbb{N}$ into $\mathbb{N}$ endowed with the quasi-order $\leq^*$ defined by

$$f \leq^* g \text{ if } \{n \in \mathbb{N} : f(n) > g(n)\} \text{ is finite.}$$

A subset C of $\mathbb{N}^{\mathbb{N}}$ is said to be *cofinal* if for each $f \in \mathbb{N}^{\mathbb{N}}$ there is some $g \in C$ with $f \leq^* g$. A subset of $\mathbb{N}^{\mathbb{N}}$ is said to be *unbounded* if it is unbounded in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. One defines

$$\mathfrak{b} = min\{|B| : B \text{ is an unbounded subset of } \mathbb{N}^{\mathbb{N}}\}$$

and

$$\mathfrak{d} = min\{|D| : D \text{ is a cofinal subset of } \mathbb{N}^{\mathbb{N}}\},$$

yielding $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

If instead of $f \leq^* g$ we consider $f \leq g$, that is $f(n) \leq g(n)$ for all $n \in \mathbb{N}$, the value of $\mathfrak{b}$ would be $\aleph_0$. As for $\mathfrak{d}$, it would not change its value. Indeed, let D be a $\mathfrak{d}$-sized cofinal subset of $\mathbb{N}^{\mathbb{N}}$. Thus, given any $f \in \mathbb{N}^{\mathbb{N}}$, there exists $g \in D$ with $f(n) \leq g(n)$ for almost all $n \in \mathbb{N}$. Now the set $\mathbb{D} = \{mg : m \in \mathbb{N} \text{ and } g \in D\}$ still has size $\aleph_0 \cdot \mathfrak{d} = \mathfrak{d}$.

3. Dense Subspaces

In [1], Grothendieck proved that, when E is a metrizable and separable locally convex space, the bounded subsets of E are completely determined by the bounded subsets of any dense subspace. This result has been extended by Burke and Todorčević [2] and, separately, Saxon and Sánchez-Ruiz [3] for some nonseparable spaces. Subsequenly, Chis, Ferrer, Hernández and Tsaban [5] extended these results for metrizable groups. As we show next, the same assertion holds for point-generated G-spaces if G is a countable monoid. First, we need the following lemma, which is analogous to ([4], Lemma 2.2.10) (resp. [5], Th. 3.6). We include its proof here for the reader's sake.

Lemma 1. *Let $G = \{g_i : i \in \mathbb{N}\}$ be a countable monoid and let X be a non locally G-bounded G-space with a generating point $x_0 \in X_{tg}$ that has a countable neighborhood basis. Then there are two order preserving maps*

$$\Phi_{\mathcal{V}} : \mathfrak{B}(G, X) \to \mathbb{N}^{\mathbb{N}} \qquad \Psi : \mathbb{N}^{\mathbb{N}} \to \mathfrak{B}(G, X)$$

such that $\Phi_{\mathcal{V}}(\mathfrak{B}(G, X))$ is cofinal in $\mathbb{N}^{\mathbb{N}}$ and $\Psi(\mathbb{N}^{\mathbb{N}})$ is cofinal in $\mathfrak{B}(G, X)$.

Proof. The map $\Phi_{\mathcal{V}}$ is defined in a similar way as in ([4], Section 2.2.4) (resp. [5], Def. 3.5). Indeed, let $\mathcal{U} = \{U_m\}_{m<\omega}$ be a countable neighborhood basis at x_0. By Proposition 1, no neighborhood of x_0 is G-bounded. Therefore, there is $U_{m_0} \in \mathcal{U}$ such that $U_1 \not\subseteq \bigcup_{i \leq n} g_i U_{m_0}$, $\forall n < \omega$. Analogously there is $U_{m_1} \in \mathcal{U}$ such that $V_1 := U_1 \cap U_{m_0} \not\subseteq \bigcup_{i \leq n} g_i U_{m_1}$, $\forall n < \omega$. Repeating this procedure, we obtain a decreasing neighborhood base $\mathcal{V} = \{V_m\}_{m<\omega}$ at x_0 by $V_{m+1} := V_m \cap U_{n+1} \cap U_{m_n} \not\subseteq \bigcup_{i \leq n} g_i U_{m_{n+1}}$, $\forall n < \omega$. □

Define

$$\Phi_{\mathcal{V}} : \mathfrak{B}(G, X) \to \mathbb{N}^{\mathbb{N}}$$

by the rule

$$\Phi_{\mathcal{V}}(K)(m) := min\left\{ n \,:\, K \subseteq \bigcup_{i \leq n} g_i V_m \right\}.$$

Obviously,

$$\Phi_{\mathcal{V}}(K) := \{\Phi_{\mathcal{V}}(K)(m)\}_{m\leq\omega}.$$

This map is order preserving and relates the confinality of $\mathfrak{B}(G,X)$ and $\mathbb{N}^{\mathbb{N}}$. Indeed, take $\alpha \in \mathbb{N}^{\mathbb{N}}$. Set $V_0 := U_1$ and take $x_m \in V_{m-1} \setminus \bigcup_{i=1}^{\alpha(m)} g_i V_m$. The sequence $K := \{x_m\}_{m<\omega}$ converges to x_0. Thus $K \cup \{x_0\}$ is G-bounded and $\Phi_{\mathcal{V}}(K)(m) = min\{n : K \subseteq \bigcup_{i\leq n} g_i V_m\}$. It follows that $\alpha \leq \Phi_{\mathcal{V}}(K)$.

As for the map Ψ, set

$$\Psi : \mathbb{N}^{\mathbb{N}} \to \mathfrak{B}(G,X)$$

by

$$\Psi(\alpha)(n) := \bigcap_{m<\omega} \bigcup_{i=1}^{\alpha(n)} g_i V_m.$$

Obviously this map is order preserving. Moreover, $\Psi(\mathbb{N}^{\mathbb{N}})$ is cofinal in $\mathfrak{B}(G,X)$. To see this, take an arbitrary G-bounded subset K, then for every $n < \omega$ there is a finite subset $F_n \subseteq \mathbb{N}$ such that $K \subseteq \bigcup_{i\in F_m} g_i V_m$. Set $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha(m) := \max\{i : i \in F_m\}$ for every $m < \omega$. Then $K \subseteq \Psi(\alpha)$.

Theorem 1. *Let $G = \{g_n : n \in \mathbb{N}\}$ be a countable monoid and let X be a first countable G-space with a generating point $x_0 \in X_{tg}$. If Y is a dense subset of X, then for each G-bounded $K \subseteq X$ whose density is less than $\mathfrak{b}$, there is a G-bounded $P \subseteq Y$ such that $\overline{P} \supseteq K$.*

Proof. Suppose first that X is locally G-bounded and let U be a G-bounded neighborhood of x_0. Let F be a finite subset of G such that $K \subseteq FU$. Since G acts on X by homeomorphisms and Y is dense in X, it follows that $\overline{F(U\cap Y)}^X \supseteq FU$. Therefore, it suffices to take $P = F(U \cap Y)$. □

Assume without loss of generality that X is not locally G-bounded and set $D \subseteq K$ such that $|D| < \mathfrak{b}$ and $\overline{D}^K = K$. Since K is G-bounded, we take the map $\Phi_{\mathcal{V}}$ defined in Lemma 1 above, where $\mathcal{V} = \{V_m\}_{m<\omega}$ is a decreasing basis at x_0. We have

$$K \subseteq \bigcup_{n=1}^{\Phi_{\mathcal{V}}(K)(m)} g_n V_m$$

for all $m < \omega$. On the other hand, since Y is dense in X, for all $d \in D \subseteq K$, there is a sequence $S_d \subseteq Y$ which converges to d. Therefore, since $S_d \bigcup\{d\}$ is compact, we have

$$\overline{S_d} = S_d \bigcup\{d\} \subseteq \bigcup_{n=1}^{\Phi_{\mathcal{V}}(\overline{S_d})(m)} g_n V_m$$

for all $m < \omega$. So, we have a family $\{\Phi_{\mathcal{V}}(\overline{S_d})\}_{d\in D} \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality less than $\mathfrak{b}$, then it is bounded in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. Therefore, there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\Phi_{\mathcal{V}}(\overline{S_d}) \leq^* \alpha \quad \forall d \in D$. That is, if $d \in D$, then there is $m_d < \omega$ with $\Phi_{\mathcal{V}}(\overline{S_d})(m) \leq \alpha(m) \quad \forall m \geq m_d$. We also assume that $\Phi_{\mathcal{V}}(K)(m) \leq \alpha(m) \quad \forall m < \omega$. Pick now a fixed element $d \in D$. If $m < m_d$, we have

$$K \subseteq \bigcup_{n=1}^{\Phi_{\mathcal{V}}(K)(m)} g_n V_m \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V_m.$$

Therefore,

$$K \subseteq \bigcap_{m=1}^{m_d-1} \left(\bigcup_{n=1}^{\alpha(m)} g_n V_m \right) = A_d$$

that is an open set. Since this open set contains the element $d \in D$ and the sequence S_d converges to d, there is $S'_d = S_d \setminus \{\text{a finite subset}\}$ such that $S'_d \subseteq A_d$. Consider now

$$P := \bigcup_{d \in D} S'_d \subseteq Y$$

and let us verify that P is G-bounded. Take an open set V of X such that $x_0 \in V$, then there is $V_m \in \mathcal{V}$ such that $V_m \subseteq V$. For each $d \in D$ we have one of the following two options:

(1) $m < m_d$, which implies $S'_d \subseteq A_d \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V_m$.

(2) $m \geq m_d$, then $S'_d \subseteq S_d \subseteq \bigcup_{n=1}^{\Phi_{\mathcal{V}}(\overline{S_d})(m)} g_n V_m \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V_m$.

In both cases, $S'_d \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V_m \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V$.

Therefore, $P = \bigcup_{d \in D} S'_d \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V$, and since V is arbitrary this means that P is G-bounded.

It is readily seen that $\overline{P} \supseteq K$.

A consequence of this theorem is the following.

Corollary 1. *Let G be a countable monoid and let X be a point-generated, metrizable, G-space. If X contains a dense subset of cardinality less than $\mathfrak{b}$, and D is an arbitrary dense subset of X, then for each G-bounded $K \subseteq X$, there is a G-bounded $P \subseteq D$ such that $\overline{P} \supseteq K$.*

Proof. Since X is metrizable, it is first countable and the generating point x_0 has a countable neighborhood basis and K contains a dense subset of cardinality less than $\mathfrak{b}$. □

The following result improves Corollary 2.3.3 in [4] (resp. Corollary 3.19 in [5]).

Corollary 2. *Let H be a topological group, K a closed subgroup of H such that H/K is metrizable and let L be a dense subgroup of H. If $P \subseteq H/K$ is precompact, then there is a precompact subset $Q \subseteq L/K$ such that $P \subseteq \overline{Q}$.*

Proof. Let $p \colon H \to H/K$ denote the canonical quotient map. Observe that P is separable because it is metrizable and precompact. Let D be a countable dense subset of P. For every $d \in D$, there is a sequence $S_d \subseteq L$ such that $p(S_d)$ converges to d. Consider the countable subset $E = D \bigcup \left(\bigcup_{d \in D} p(S_d) \right) = \bigcup_{d \in D} \overline{p(S_d)} = \{y_i\}_{i=1}^{\infty}$ and the set $H_E = \overline{< E >}$ with the topology inherited from H/K. We have that $P \subseteq H_E$, and H_E is separable and metrizable. Let G be a countable subgroup of $p^{-1}(H_E)$ such that $p(G) = \langle \{y_i\}_{i=1}^{\infty} \rangle$, which is dense in H_E. Then H/K is a point generated G-space according to Proposition 1(viii), where the family of G-bounded subsets coincides with the family of precompact subsets of the left uniformity of H/K. On the other hand, $L \bigcap H_E$ is countable and dense in H_E and P is G-bounded. Accordingly, we apply Theorem 1 to deduce that there is $Q \subseteq L \bigcap H_E$, which is G-bounded (therefore, precompact) and $P \subseteq \overline{Q}^{H_E} \subseteq \overline{Q}$. It is readily seen that Q is precompact in L. □

The metrizability condition in the previous theorem is essential even for the special case of topological groups ([4], Example 2.3.5) (resp. [5], [Remark 3.21]).

4. G-Barrelled Groups

In this section, we have a countable monoid $G = \{g_i : i \in \mathbb{N}\}$ and a metrizable G-space X. We assume WLOG that $g_1 = e_G$ is the neutral element of G.

Definition 2. *Given a G-space X, we say that $A \subseteq X$ is* G-absorbent *(or simply A is* absorbent *for short) when $GA = X$. A G-space X is said to be* barrelled *when for every closed absorbent subset Q there is an index $i \in \mathbb{N}$ such that $g_i Q$ has a nonempty interior.*

Theorem 2. *Suppose that $G = \{g_i : i \in \mathbb{N}\}$ is a countable monoid and X is a homogeneous, barrelled G-space with a generating point $x_0 \in X_{tg}$ that has a countable neighborhood basis at x_0. If X can be covered by less than $\mathfrak{b}$ bounded subsets, then X is locally bounded.*

Proof. Let $\mathcal{V} = \{V_m\}_{m<\omega}$ be a decreasing neighborhood base at x_0 defined as in Lemma 1 and let $\pi : G \times X \to X$ denote the action of G on X. For every $g_m \in G$ we define the map

$$p_m : X \to \mathbb{N} \quad \text{by} \quad p_m(x) = min\{n : x \in \bigcup_{j \le n} g_j V_m\}.$$

As a consequence, every element $x \in X$ defines a sequence $\{p_m(x)\}_{m<\omega}$ and, therefore, we have defined the map $p : X \to \mathbb{N}^{\mathbb{N}}$ as $p(x) = \{p_m(x)\}_{m<\omega}$ so that $p(x)[m] = p_m(x)$. Suppose there is a collection of G-bounded sets $\mathfrak{B}$ such that $|\mathfrak{B}| < \mathfrak{b}$ and $X = \bigcup_{P \in \mathfrak{B}} P$. Every $P \in \mathfrak{B}$ is associated with a map $\Phi_{\mathcal{V}}(P) \in \mathbb{N}^{\mathbb{N}}$ defined previously; that is

$$\Phi_{\mathcal{V}}(P)(m) = min\{n : P \subseteq \bigcup_{j \le n} g_j V_m\}.$$

Take $x \in X$. Then, there is $P \subseteq \mathfrak{B}$ such that $x \in P$. Therefore $p(x) \le \Phi_{\mathcal{V}}(P)$. Since $|\mathfrak{B}| < \mathfrak{b}$ it follows that $\Phi_{\mathcal{V}}(\mathfrak{B}) = \{\Phi_{\mathcal{V}}(P) : P \in \mathfrak{B}\}$ is bounded in $(\mathbb{N}^{\mathbb{N}}, \le^*)$. Thus, there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\Phi_{\mathcal{V}}(P) \le^* \alpha$ and, since $p(x) \le \Phi_{\mathcal{V}}(P)$, we have $p(x) \le^* \alpha$ for all $x \in X$. So, for every $x \in X$, there is $m_x < \omega$ such that $p_m(x) \le \alpha(m)$ for all $m \ge m_x$.

Define

$$Q_\alpha = \{x \in X : p_m(x) \le \alpha(m) \quad \forall m < \omega\} = \bigcap_{m<\omega} \left(\bigcup_{j \le \alpha(m)} g_j V_m \right).$$

Clearly, the set Q_α is bounded. Let us verify that Q_α is also absorbent. Take $x \in X$. Then, since $p_m(x) \le \alpha(m) \quad \forall m \ge m_x$, we have

$$x \in \bigcap_{m \ge m_x} \left(\bigcup_{j \le \alpha(m)} g_j V_m \right).$$

Thus,

$$x \in Q_\alpha \bigcup \left(\bigcap_{m<m_x} \left(\bigcup_{j \le p_m(x)} g_j V_m \right) \right).$$

Set

$$F_x = \{i \in \mathbb{N} : i \le p_m(x), m < m_x\}.$$

We claim that

$$x \in \bigcup_{i \in F_x} g_i Q_\alpha.$$

Indeed, since each map π_g, is a bijection and g_1 is the neutral element of G, we have

$$\bigcup_{i \in F_x} g_i Q_\alpha = \bigcup_{i \in F_x} g_i \left(\bigcap_{m<\omega} \bigcup_{j \le \alpha(m)} g_j V_m \right) = \bigcup_{i \in F_x} \left(\bigcap_{m<\omega} g_i \bigcup_{j \le \alpha(m)} g_j V_m \right)$$

$$= \bigcup_{i \in F_x} \left(\bigcap_{m<\omega} \bigcup_{j \leq \alpha(m)} g_i g_j V_m \right) \supseteq \left(\bigcap_{m \geq m_x} \bigcup_{j \leq \alpha(m)} g_j V_m \right) \bigcap \left(\bigcap_{m < m_x} \bigcup_{i \in F_x} g_i V_m \right) \ni x.$$

This proves that Q_α is absorbent. Therefore $\overline{Q_\alpha}$ is absorbent too and, since X is G-barrelled, there is $g \in G$ such that $g\overline{Q_\alpha}$ has nonempty interior. Thus, $g\overline{Q_\alpha}$ is a G-bounded subset containing an open, G-bounded, subset U. Take any point $u \in U$. Since X is homogeneous, there is a homeomorphism $f_{ux_0} \colon X \to X$ such that $f_{ux_0}(u) = x_0$ and $f_{ux_0}(U)$ is an open, bounded subset containing x_0. By Proposition 2, it follows that X is locally G-bounded. □

As a consequence, we next obtain results that contain the previous results obtained by locally convex spaces [2] and topological groups [5].

Let G be a topological group, we say that a subset $A \subseteq G$ is *absorbent* when for every dense subgroup H of G it holds that $HA = G$. The group G is said to be *barrelled* when every closed absorbent subset Q has a nonempty interior. Remark that every separable Baire group is barrelled.

Corollary 3. *Let G be either a metrizable, barrelled, locally convex space or a separable, metrizable, barrelled group. If G is covered by less than $\mathfrak{b}$ bounded (resp. precompact) subsets. Then G is normable (resp.locally precompact).*

Proof. In both cases, G is homogeneous and the homeomorphisms preserving bounded subsets are translations. If G is a metrizable, barrelled, locally convex space, applying Theorem 2, we obtain that G has a neighborhood basis of zero consisting of bounded subsets, which implies that G is normable. If G is a topological group, take any countable dense subgroup H of G and consider the canonical action of H on G that makes G an H-space. By Proposition 1, a subset A of G is H-bounded if and only if it is precompact. Again, it suffices now to apply Theorem 2. □

5. Discussion

We have considered the action of a monoid G on a topological space X and associated it with a canonical family of G-bounded subsets. This provides a very general notion of boundedness that include both the bounded subsets considered in functional analysis and in topological groups. In this paper, we have initiated the study of this new notion of a G-bounded subset. Among other results, it is proved that for a metrizable and separable G-space X, the bounded subsets of X are completely determined by the bounded subsets of any dense subspace, extending results obtained by Grothendieck for metrizable separable locally convex spaces [1], generalized subsequently by Burke and Todorčević and, separately, Saxon and Sánchez-Ruiz for metrizable locally convex spaces [2,3] and by Chis, Ferrer, Hernández and Tsaban for metrizable groups [4,5]. We have also obtained sufficient conditions for a G-space X to be locally G-bounded, which applies to topological groups. This also provides the frame for extending to this setting some results by Burke and Todorčević and, separately, Saxon and Sáchez-Ruiz (loc. cit.) for metrizable locally convex spaces.

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References

1. Grothendieck, A. Critères de compacité dans les espaces fonctionnels généraux. *Am. J. Math.* **1952**, *74*, 168–186. [CrossRef]
2. Burke, M.R.; Todorćevixcx, S. Bounded sets in topological vector spaces. *Math. Ann.* **1996**, *305*, 103–125. [CrossRef]
3. Saxon, S.A.; Sánchez-Ruiz, L. Optimal cardinals for metrizable barrelled spaces. *J. Lond. Math.* **1995**, *51*, 137–147. [CrossRef]
4. Chis, C. Bounded Sets in Topological Groups. Ph.D. Thesis, Universitat Jaume I, Castelló, Spain, 2010.
5. Chis, C.; Ferrer, M.; Hernández, S.; Tsaban, B. The character of topological groups, via bounded systems, Pontryagin-van Kampen duality and pcf theory. *J. Algebra* **2014**, *420*, 86–119. [CrossRef]
6. Hejcman, J. Boundedness in uniform spaces and topological groups (Russian summary). *Czechoslovak Math. J.* **1959**, *9*, 544–563. [CrossRef]
7. Hu, S. Boundedness in a topological space. *J. Math. Pures Appl.* **1949**, *28*, 287–320.
8. Vilenkin, N.Y. The theory of characters of topological Abelian groups with boundedness given (Russian). *Izvestiya Akad. Nauk SSSR. Ser. Mat.* **1951**, *15*, 439–462.
9. Engelking, R. *General Topology*; PWN-Polish Scientific Publishers: Warsaw, Poland, 1977.
10. van Douwen, E. The integers and topology. In *Handbook of Set Theoric Topology*; Kunen, K., Vaughan, J., Eds.; North-Holland: Amsterdam, The Netherlands, 1984; pp. 111–167.

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Article

Distinguished Property in Tensor Products and Weak* Dual Spaces

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Abstract: A local convex space E is said to be distinguished if its strong dual E'_β has the topology $\beta(E', (E'_\beta)')$, i.e., if E'_β is barrelled. The distinguished property of the local convex space $C_p(X)$ of real-valued functions on a Tychonoff space X, equipped with the pointwise topology on X, has recently aroused great interest among analysts and C_p-theorists, obtaining very interesting properties and nice characterizations. For instance, it has recently been obtained that a space $C_p(X)$ is distinguished if and only if any function $f \in \mathbb{R}^X$ belongs to the pointwise closure of a pointwise bounded set in $C(X)$. The extensively studied distinguished properties in the injective tensor products $C_p(X) \otimes_\varepsilon E$ and in $C_p(X, E)$ contrasts with the few distinguished properties of injective tensor products related to the dual space $L_p(X)$ of $C_p(X)$ endowed with the weak* topology, as well as to the weak* dual of $C_p(X, E)$. To partially fill this gap, some distinguished properties in the injective tensor product space $L_p(X) \otimes_\varepsilon E$ are presented and a characterization of the distinguished property of the weak* dual of $C_p(X, E)$ for wide classes of spaces X and E is provided.

Keywords: distinguished space; injective and projective tensor product; vector-valued continuous function; Fréchet space; nuclear space

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1. Introduction

In this paper, X is an infinite Tychonoff space and $C(X)$ is the linear space of all real-valued continuous functions over X. $C_p(X)$ and $C_k(X)$ denote the space $C(X)$ equipped with the pointwise and compact-open topology, respectively. $L_p(X)$ represents the weak* dual of $C_p(X)$, i.e., the topological dual $L(X)$ of $C_p(X)$ endowed with the weak topology $\sigma(L(X), C(X))$ of the dual pair $\langle L(X), C(X)\rangle$, i.e., $L_p(X)$ has the topology of pointwise convergence on $C(X)$.

Moreover, all local convex spaces are assumed to be real and Hausdorff and the symbol '$\simeq$' indicates some canonical algebraic isomorphism or linear homeomorphism. The strong dual E'_β of a local convex space E is the topological dual E' of E equipped with the strong topology $\beta(E', E)$, which is the topology of uniform convergence on the bounded subsets of E. $\langle E, E'\rangle$ is a dual pair. For a subset A of E the *polar* A^0 of A with respect to a dual pair $\langle E, F\rangle$ is

$$A^0 = \{x \in F : |\langle a, x\rangle| \leq 1, \forall a \in A\}.$$

A local convex space E is barrelled if for each pointwise bounded subset M of E' there exists a neighborhood of the origin U in E such that M is uniformly bounded on U. Hence E is barrelled if and only if its topology is the topology $\beta(E, E')$, i.e., $[E'(\text{weak}^*)]'_\beta = E$.

Roughly speaking, E is barrelled if it verifies the local convex version of the Banach–Steinhaus uniform boundedness theorem.

The local convex space E is called *distinguished* if E'_β is barrelled. In [1–7] the distinguished property of the space $C_p(X)$ has been extensively studied. Furthermore, [8] [Proposition 6.4] is connected with distinguished $C_p(X)$ spaces. It is observed in [3] [Theorem 10] that $C_p(X)$ is distinguished if and only if $C_p(X)$ is a large subspace of $\mathbb{R}^X$, i.e., if each bounded set in $\mathbb{R}^X$ is contained in the closure in $\mathbb{R}^X$ of a bounded set of $C_p(X)$, or, equivalently, if the strong bidual of $C_p(X)$ is $\mathbb{R}^X$ [5]. In [7], [Theorem 2.1] it is shown that $C_p(X)$ is distinguished if and only if X is a Δ-space in the sense of Knight [9], and several applications of this fact are given. Equivalently, $C_p(X)$ is distinguished if for each countable partition $\{X_k : k \in \mathbb{N}\}$ of X into nonempty pairwise disjoint sets, there are open sets $\{U_k : k \in \mathbb{N}\}$ with $X_k \subseteq U_k$, for each $k \in \mathbb{N}$, such that each point $x \in X$ belongs to U_n for only finitely many $n \in \mathbb{N}$, [5] [Theorem 5].

If E and F are local convex spaces, $E \otimes_\varepsilon F$ and $E \otimes_\pi F$ represent the *injective* and *projective tensor product* of E and F, respectively. A basis of neighborhoods of the origin in $E'_\beta \otimes_\varepsilon F'_\beta$ is determined by the sets $\varepsilon(A, B) := \left(A^{00} \otimes B^{00}\right)^0$, where A is a bounded set in E, B is a bounded set in F, $A^0 \subseteq E'$, $A^{00} \subseteq E''$, $B^0 \subseteq F'$, $B^{00} \subseteq F''$ and $\left(A^{00} \otimes B^{00}\right)^0 \subseteq E' \otimes F'$. Analogously, a basis of neighborhoods of the origin in the tensor product space $E'_\beta \otimes_\pi F'_\beta$ is formed by the sets $\pi(A, B) := \mathbf{acx}\left(A^0 \otimes B^0\right)$, where A is a bounded set in E, B is a bounded set in F and $\mathbf{acx}\left(A^0 \otimes B^0\right)$ denotes the absolutely convex cover of the tensor product $A^0 \otimes B^0$. Recall that if E carries the weak topology, then $(E \otimes_\varepsilon F)' \simeq (E \otimes_\pi F)' \simeq E' \otimes F'$, [10] [41.3 (9) and 45.1 (2)]. A local convex space E is called *nuclear* if $E \otimes_\varepsilon F = E \otimes_\pi F$ for every local convex space F, [11] [21.2].

The distinguished property of $C_p(X)$ under the formation of some tensor products is examined in [2]. Among other results it is showed in [2] [Corollary 6] that for a local convex space E the injective tensor product $C_p(X) \otimes_\varepsilon E$ is distinguished if both $C_p(X)$ is distinguished and $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta$ is barrelled, where $\mathbb{R}^{(X)}$ the local convex direct sum of $|X|$ real lines.

If E is a local convex space $C_p(X, E)$ and $C_k(X, E)$ will denote the linear space of all E-valued continuous functions defined on X equipped with the pointwise topology and compact-open topology, respectively. It is also proved in [2] [Corolary 21] that, for any Tychonoff space X and any normed space E, the vector-valued function space $C_p(X, E)$ is distinguished if and only if $C_p(X) \otimes_\varepsilon E$ is distinguished. In particular, if X is a countable Tychonoff space and E a normed space, then $C_p(X, E)$ is distinguished. Indeed, if X is countable, on the one hand $C_p(X)$ is distinguished by [5] [Corollary 6] and on the other hand $\mathbb{R}^{(X)}$ is both barrelled and nuclear (the latter because [11] [21.2.3 Corollary]), so that $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta = \mathbb{R}^{(X)} \otimes_\pi E'_\beta$ is barrelled by [12] [Theorem 1.6.6]. Thus, $C_p(X) \otimes_\varepsilon E$ is distinguished by the already mentioned [2] [Corolary 6] and, since E is normed, $C_p(X, E)$ is distinguished too by [2] [Corollary 21]. A corresponding result for the compact-open topology, due to Díaz and Domański [13] [Corolary 2.5], states that the space $C_k(K, E)$ of continuous functions defined on a compact Hausdorff space K and with values in a reflexive Fréchet space E is also distinguished, being its strong dual naturally isomorphic to $L_1(\mu)\hat{\otimes}_\pi E'_\beta$.

According to [1] [Theorem 3.9], the strong dual $L_\beta(X)$ of $C_p(X)$ is always distinguished. The distinguished property of the weak* dual $L_p(X)$ of $C_p(X)$ is investigated in [5], where the following theorem is proved.

Theorem 1 ([5] [Theorem 27])**.** *If X is a μ-space, then the weak* dual $L_p(X)$ of $C_p(X)$ is distinguished.*

Recall that a Tychonoff space X is called a μ-space if each functionally bounded set is relatively compact.

The extensively studied distinguished properties in the injective tensor products $C_p(X) \otimes_\varepsilon E$ and in $C_p(X, E)$ contrasts with the few distinguished properties related with the injective tensor products $L_p(X) \otimes_\varepsilon E$ and with the weak* dual of $C_p(X, E)$. Theorem 1 and the fact that $L_p(X)$ spaces are studied so extensively as $C_p(X)$ spaces motivated us to fill partially this gap in this paper obtaining distinguished properties of injective tensor products $L_p(X) \otimes_\varepsilon E$ and providing a characterization of the distinguished property of the weak* dual of $C_p(X, E)$ for wide classes of spaces X and E. To reach these goals we require [2] [Theorem 5] and [2] [Proposition 19], which we include here for convenience.

Theorem 2 ([2] [Theorem 5])**.** *Let E and F be local convex spaces, where E carries the weak topology. If τ_ε and τ_π denote the injective and projective topologies of $E'_\beta \otimes F'_\beta$, the following properties hold*

1. *If $E'_\beta \otimes_\varepsilon F'_\beta$ is barrelled, then $\tau_\varepsilon = \beta(E' \otimes F', E \otimes F)$ and $E \otimes_\varepsilon F$ is distinguished.*
2. *If $E'_\beta \otimes_\pi F'_\beta$ is barrelled then $\tau_\varepsilon \leq \beta(E' \otimes F', E \otimes F) \leq \tau_\pi$.*

Theorem 3 ([2] [Proposition 19])**.** *For any local convex space E, the dual of the space $C_p(X, E)$ is algebraically isomorphic to $L(X) \otimes E'$, i.e., $C_p(X, E)' \simeq L(X) \otimes E'$.*

It should be noted that if $\Sigma_{i=1}^n f_i \otimes u_i$ is a representation of $\varphi \in C(X) \otimes E$ then Theorem 3 is due to the fact that the canonical map $T : C_p(X) \otimes_\varepsilon E \to C_p(X, E)$ given by

$$(T\varphi)(x) = \Sigma_{i=1}^n f_i(x) u_i,$$

is a linear homeomorphism from $C_p(X) \otimes_\varepsilon E$ into a dense linear subspace of $C_p(X, E)$. Furthermore, $\left(C_p(X) \otimes_\varepsilon E\right)' \simeq L(X) \otimes E'$, because $C_p(X)$ carries the weak topology, so one has $C_p(X, E)' \simeq L(X) \otimes E'$, as stated.

2. Distinguished Tensor Products of $L_p(X)$ Spaces

This section deals mainly with the injective tensor product of $L_p(X)$ and a nuclear metrizable space E. It should be noted that the class of nuclear metrizable spaces is large. Recall that the space s of all rapidly decreasing sequences, as well as the test space of distributions $\mathcal{D}(\Omega)$, where Ω is an open set in $\mathbb{R}^n$, with their usual local convex inductive topologies, are examples of nuclear Fréchet spaces [11] [Section 21.6]. The strong dual of $\mathcal{D}(\Omega)$ is the space of distributions on Ω and it is denoted by $\mathcal{D}'(\Omega)$.

Theorem 4. *Assume that X is a μ-space and let E be a nuclear metrizable local convex space. If every countable union of compact subsets of X is relatively compact, then $L_p(X) \otimes_\varepsilon E$ is distinguished.*

Proof. The space X is a μ-space if and only if $C_k(X)$ is barrelled, by the Nachbin-Shirota theorem [14] [Proposition 2.15]. On the other hand, as every countable union of compact subsets of X is assumed to be relatively compact, the space $C_k(X)$ is also a (DF)-space [15] [Theorem 12]. In addition, the strong dual E'_β of a metrizable local convex space E it is a complete (DF)-space by [16] [see 29.3 -in "By 2(1)"-]. Moreover, nuclearity of E implies that E'_β is nuclear too by [11] [21.5.3 Theorem]. As E'_β is a nuclear (DF)-space, one has that E'_β is a quasi-barrelled space [11] [21.5.4 Corollary]. Finally, the completeness of the quasi-barrelled space E'_β implies that E'_β is barrelled [16] [27.1.(1)], so E is distinguished.

The projective tensor product $C_k(X) \otimes_\pi E'_\beta$ is barrelled by [11] [15.6.8 Proposition]. Thus, taking into consideration E'_β nuclearity, it can be obtained that $C_k(X) \otimes_\varepsilon E'_\beta$ is also barrelled. On the other hand, since X is a μ-space it follows from [5] [Theorem 27] that $C_k(X)$ coincides with the strong dual of $L_p(X)$, i.e., $\left(L_p(X)\right)'_\beta = C_k(X)$, hence

$$\left(L_p(X)\right)'_\beta \otimes_\varepsilon E'_\beta$$

is barrelled. Finally, as $L_p(X)$ carries the weak topology, the first statement of Theorem 2, ensures that the space $L_p(X) \otimes_\varepsilon E$ is distinguished. □

Example 1. *In particular, for each compact topological space X and for each nuclear metrizable local convex space E it follows that* $L_p(X) \otimes_\varepsilon E$ *is distinguished.*

Hence, if X is the Cantor space K or the interval $[0,1]$, and if E is one of the local convex spaces $\mathcal{D}(\Omega)$ or s, then the injective tensor products $L_p(K) \otimes_\varepsilon \mathcal{D}(\Omega)$, $L_p(K) \otimes_\varepsilon s$, $L_p([0,1]) \otimes_\varepsilon \mathcal{D}(\Omega)$ and $L_p([0,1]) \otimes_\varepsilon s$ are distinguished.

Corollary 1. *If X is a compact space and Y is a countable Tychonoff space, then the space* $L_p(X) \otimes_\varepsilon C_p(Y)$ *is distinguished.*

Proof. Clearly, $C_p(Y)$ is metrizable (hence distinguished [1] [Theorem 3.3]) and nuclear (by [11] [21.2.3 Corollary]), so the statement follows from the previous theorem. □

If we apply this Corollary with X equal to the Stone-Čech compactification $\beta\mathbb{N}$ of the topological space $\mathbb{N}$ formed by the natural numbers endowed with the discrete topology and Y equal to the space $\mathbb{Q}$ of rational numbers endowed with the usual metrizable topology then we get that $L_p(\beta\mathbb{N}) \otimes_\varepsilon C_p(\mathbb{Q})$ is a distinguished space.

If the factor E of $L_p(X) \otimes_\varepsilon E$ is a normed space, the following theorem holds true.

Theorem 5. *If X is a μ-space with finite compact sets (equivalently, if every functionally bounded subset of X is finite) and E is a normed space, then* $L_p(X) \otimes_\varepsilon E$ *is distinguished.*

Proof. If X is a μ-space with finite compact sets, the space $C_k(X) = C_p(X)$ is barrelled and nuclear. As E'_β is a Banach space, [12] [Corollary 1.6.6] assures that $C_k(X) \otimes_\pi E'_\beta$ is a barrelled space, and $C_k(X)$ nuclearity yields that $C_k(X) \otimes_\varepsilon E'_\beta$ is also a barrelled space. Bearing in mind that $\left(L_p(X)\right)'_\beta = C_k(X)$, as a consequence of the fact that X is a μ-space (*cf.* [5] [Theorem 27]), Theorem 2 ensures that $L_p(X) \otimes_\varepsilon E$ is distinguished. □

A P-space in the sense of Gillman–Henriksen is a topological space in which every countable intersection of open sets is open.

Corollary 2. *If X is a P-space and E is a normed space, then* $L_p(X) \otimes_\varepsilon E$ *is distinguished.*

Proof. Every P-space is a μ-space with finite compact sets (*cf.* [17] [Problem 4K]). □

Example 2. *If* $L(\mathfrak{m})$ *denotes the Lindelöfication of the discrete space of cardinal* $\mathfrak{m} \geq \aleph_1$*, the space* $L_p(L(\mathfrak{m})) \otimes_\varepsilon C_k([0,1])$ *is distinguished.* In this case $L(\mathfrak{m})$ is a Lindelöf P-space.

Theorem 6. *If X is a μ-space with finite compact sets and E is normed space, then* $L_p(X) \otimes_\varepsilon E'(\text{weak}^*)$ *is distinguished.*

Proof. By [12] [Theorem 1.6.6] the projective tensor product $C_k(X) \otimes_\pi E$ is a barrelled space, hence $C_k(X)$ nuclearity yields that $C_k(X) \otimes_\varepsilon E$ is barrelled. So, the conclusion follows from the first statement of Theorem 2. □

Example 3. *The space* $L_p(L(\mathfrak{m})) \otimes_\varepsilon \ell_p(\text{weak}^*)$ *is distinguished for* $1 \leq p < \infty$.

A topological space is said to be *hemicompact* if it has a sequence of compact subsets such that every compact subset of the space lies inside some compact set in the sequence.

Theorem 7. *If X is a hemicompact space and E is a nuclear metrizable barrelled space (for instance a nuclear Fréchet space), then* $L_p(X) \otimes_\varepsilon E'(\text{weak}^*)$ *is distinguished.*

Proof. Clearly X is a Lindelöf space, hence it is a μ-space, and then both $C_k(X)$ and E are metrizable and barrelled spaces. Then [12] [Corollary 1.6.4] ensures that $C_k(X) \otimes_\pi E$ is also a (metrizable) barrelled space. This property and the E nuclearity imply that $C_k(X) \otimes_\varepsilon E$ is a barrelled space. Consequently, using that $\left(L_p(X)\right)'_\beta = C_k(X)$ and $E'(\text{weak}^*)'_\beta = E$, we get

$$L_p(X)'_\beta \otimes_\varepsilon E'(\text{weak}^*)'_\beta = C_k(X) \otimes_\varepsilon E.$$

So, Theorem 2 applies to guarantee that $L_p(X) \otimes_\varepsilon E'(\text{weak}^*)$ is distinguished. □

By Theorem 7 the injective tensor product $L_p(\mathbb{R}) \otimes_\varepsilon \mathcal{D}'(\Omega)(\text{weak}^*)$ is distinguished since $\mathbb{R}$ is hemicompact and $\mathcal{D}'(\Omega)(\text{weak}^*)$ is a nuclear Fréchet space. Theorem 7 is also applied in the next Example 4.

Example 4. *If $\mathbb{N}$ is equipped with the discrete topology, $p \in \beta\mathbb{N} \setminus \mathbb{N}$ and $Z = \mathbb{N} \cup \{p\}$ has the topology induced by $\beta\mathbb{N}$, then $L_p(Z) \otimes_\varepsilon L_p(Z)$ is distinguished.*

Proof. The subspace $Z = \mathbb{N} \cup \{p\}$ of $\beta\mathbb{N}$ is countable and has finite compact sets, so that it is hemicompact. Since Z is countable, $C_p(Z)$ is metrizable and, on the other hand, as a subspace of the nuclear space $\mathbb{R}^X$, the space $C_p(Z)$ is nuclear. In addition, since Z is a μ-space with finite compact sets, the space $C_p(Z)$ is barrelled [18]. So, according to the previous theorem, $L_p(Z) \otimes_\varepsilon L_p(Z)$ is distinguished. □

3. Distinguished Property of the Weak* Dual of $C_p(X) \otimes_\varepsilon E$

The preceding theorems are going to be applied to examine the distinguished property of the weak* dual of the injective tensor product $C_p(X) \otimes_\varepsilon E$. To get this property we need the following lemma.

Lemma 1. *The injective topology of the tensor product $L_p(X) \otimes E'(\text{weak}^*)$ coincides with the weak topology $\sigma(L(X) \otimes E', C(X) \otimes E)$.*

Proof. Since $L_p(X)$ carries the weak topology

$$\left(L_p(X) \otimes_\varepsilon E'(\text{weak}^*)\right)' \simeq C(X) \otimes E.$$

Hence, the injective topology τ_ε of $L_p(X) \otimes E'(\text{weak}^*)$ is stronger than the weak topology $\sigma(L(X) \otimes E', C(X) \otimes E)$. We prove that both topologies are the same. Indeed, if U is a closed absolutely convex neighborhood of the origin in $L_p(X)$ and V is a closed absolutely convex neighborhood of the origin in $E'(\text{weak}^*)$, there are finite sets Φ in $C(X)$ and Δ in E such that $\Phi^0 = U$ and $\Delta^0 = V$. Setting $\Lambda = \Phi \otimes \Delta$, then Λ is a finite set in $C(X) \otimes E$ such that

$$\varepsilon_{U,V}(\psi) = \sup_{f \in U^0, u \in V^0} \left| \sum_{i=1}^n f(x_i) \langle v_i, u \rangle \right| = \sup_{f \in U^0, u \in V^0} \left| \left\langle \sum_{i=1}^n \delta_{x_i} \otimes v_i, f \otimes u \right\rangle \right| \leq \sup_{F \in \Lambda} |\langle \psi, F \rangle|$$

for any $\psi = \sum_{i=1}^n \delta_{x_i} \otimes v_i \in L(X) \otimes_\varepsilon E'$, since

$$U^0 \otimes V^0 = \Phi^{00} \otimes \Delta^{00} = \mathbf{acx}(\Phi) \otimes \mathbf{acx}(\Delta) \subseteq \mathbf{acx}(\Lambda).$$

Therefore $\tau_\varepsilon = \sigma(L(X) \otimes E', C(X) \otimes E)$. □

Corollary 3. *If X is a hemicompact space and E is a nuclear Fréchet space, the weak* dual of $C_p(X) \otimes_\varepsilon E$ is distinguished.*

Proof. According to Lemma 1 the weak* dual of $C_p(X) \otimes_\varepsilon E$ is linearly homeomorphic to $L_p(X) \otimes_\varepsilon E'(\text{weak}^*)$, so Theorem 7 applies. □

The space Z considered in Example 4 is hemicompact, hence from Corollary 3 we have that the weak* duals of $C_p(\mathbb{R}) \otimes_\varepsilon C_p(Z)$ and $C_p(Z) \otimes_\varepsilon \mathcal{D}(\Omega)$ are distinguished.

Corollary 4. *If X is a μ-space with finite compact sets and E is a normed space, the weak* dual of $C_p(X) \otimes_\varepsilon E$ is distinguished.*

Proof. The proof is analogous to the proof of Corollary 3, with the difference of using Theorem 6 instead of Theorem 7. □

Example 5. *The weak* dual of $C_p(L(\mathfrak{m})) \otimes_\varepsilon C_k([0,1])$ is distinguished.*

4. A Characterization of the Distinguished Weak* Dual of $C_p(X,E)$

Let E be a local convex space. We will denote by $L_p(X,E')$ the weak* dual of $C_p(X,E)$. Since by Theorem 3 the dual space $C_p(X,E)'$ is algebraically isomorphic to $L(X) \otimes E'$, one has

$$L_p(X,E') \simeq \big(L(X) \otimes E', \sigma\big(L(X) \otimes E', C(X,E)\big)\big).$$

A completely regular topological space X is a $k_\mathbb{R}$-space if every real function f defined on X whose restriction to every compact subset K of X is continuous, is continuous on X.

Theorem 8. *Let X be a hemicompact $k_\mathbb{R}$-space and let E be a nuclear Fréchet space. The space $L_p(X,E')$ is distinguished if and only if the strong dual of $L_p(X,E')$ coincides with $C_k(X,E)$.*

Proof. We will denote by $C_\beta(X,E)$ the linear space $C(X,E)$ equipped with the strong topology $\beta(C(X,E), L(X) \otimes E')$, i.e., the strong dual of $L_p(X,E')$. Since X is a $k_\mathbb{R}$-space and E is complete, [11] [16.6.3 Corollary] ensures that

$$C_k(X,E) \simeq C_k(X) \,\widehat{\otimes}_\varepsilon\, E. \tag{1}$$

So, as both $C_k(X)$ and E are metrizable, $C_k(X,E)$ is a Fréchet space. Consequently, if $C_\beta(X,E) = C_k(X,E)$ then $C_\beta(X,E)$ is barrelled and $L_p(X,E')$ is distinguished.

Assume, conversely, that $L_p(X,E')$ is distinguished. From $C_k(X,E) \simeq C_k(X) \,\widehat{\otimes}_\varepsilon\, E$ it follows that $C_k(X,E)' = (C_k(X) \otimes_\varepsilon E)'$. Since $L(X) \otimes E'$ is algebraically isomorphic to a subspace of $(C_k(X) \otimes_\varepsilon E)'$, it follows that the compact-open topology of $C(X,E)$ is stronger than $\beta(C(X,E), L(X) \otimes E')$. Hence, the identity map $J : C_k(X,E) \to C_\beta(X,E)$ is continuous.

Since X is a hemicompact, $C_k(X)$ is metrizable. As a consequence of E nuclearity, $C_k(X) \otimes_\varepsilon E = C_k(X) \otimes_\pi E$ is a metrizable space. Hence, by (1) $C_k(X,E)$ is a Fréchet space. If $L_p(X,E')$ is distinguished, then $C_\beta(X,E)$ is barrelled. So J is a linear homeomorphism by the closed graph theorem. Thus, $C_\beta(X,E) = C_k(X,E)$. □

5. Conclusions and Two Open Problems

This paper has been motivated by the contrast between the extensively distinguished properties obtained recently in the injective tensor products $C_p(X) \otimes_\varepsilon E$ and in the spaces $C_p(X,E)$ with the few distinguished properties of injective tensor products related to the dual space $L_p(X)$ of $C_p(X)$ endowed with the weak* topology, as well as to the weak* dual of $C_p(X,E)$. In Section 2, distinguished properties in the injective tensor product space $L_p(X) \otimes_\varepsilon E$ are provided and in Sections 3 and 4, the distinguished property of the weak* dual of $C_p(X) \otimes_\varepsilon E$ and a characterization of the distinguished property of the weak* dual of $C_p(X,E)$ for wide classes of spaces X and E are provided.

We do not know the answer for the following two problems when the Tychonoff space X is uncountable. It is easy to prove that the answer of these two problems is positive if X is countable.

Problem 1. *Is it true that if X is an uncountable P-space and E is a Fréchet space, then $L_p(X) \otimes_\varepsilon E'(\text{weak}^*)$ is distinguished?*

Problem 2. *Is it true that if X is an uncountable P-space and E is a Fréchet space, then the weak* dual of $C_p(X) \otimes_\varepsilon E$ is distinguished?*

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References

1. Ferrando, J.C.; Kąkol, J. Metrizable bounded sets in $C(X)$ spaces and distinguished $C_p(X)$ spaces. *J. Convex. Anal.* **2019**, *26*, 1337–1346. Available online: https://www.heldermann.de/JCA/JCA26/JCA264/jca26070.htm (accessed on 7 July 2021).
2. Ferrando, J.C.; Kąkol, J. Distinguished metrizable spaces E and injective tensor products $C_p(X) \otimes_\varepsilon E$. **2021**, submitted
3. Ferrando, J.C.; Kąkol, J.; Leiderman, A.; Saxon, S.A. Distinguished $C_p(X)$ spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2021**, *115*, 1–18. [CrossRef]
4. Ferrando, J.C.; Kąkol, J.; Saxon, S.A. Examples of Nondistinguished Function Spaces $C_p(X)$. *J. Convex. Anal.* **2019**, *26*, 1347–1348. Available online: https://www.heldermann.de/JCA/JCA26/JCA264/jca26071.htm (accessed on 7 July 2021).
5. Ferrando, J.C.; Saxon, S.A. If not distinguished, is $C_p(X)$ even close? *Proc. Am. Math. Soc.* **2021**, *149*, 2583–2596. [CrossRef]
6. Ferrando, J.C.; Saxon, S.A. Distinguished $C_p(X)$ spaces and the strongest locally convex topology. **2021**, submitted
7. Kąkol, J.; Leiderman, A. A characterization of X for which spaces $C_p(X)$ are distinguished and its applications. *Proc. Am. Math. Soc. Ser. B* **2021**, *8*, 86–99. [CrossRef]
8. Banakh, T.; Kąkol, J.; Schürz, J.P. ω^ω-base and infinite-dimensional compact sets in locally convex spaces. *Rev. Mat. Complut.* **2021**. [CrossRef]
9. Knight, R.W. Δ-Sets. *Trans. Amer. Math. Soc. Ser. B* **1993** , *339*, 45–60. Available online: https://www.ams.org/journals/tran/1993-339-01/home.html (accessed on 7 July 2021). [CrossRef]
10. Köthe, G. *Topological Vector Spaces II*; Grundlehren der Mathematischen Wissenschaften 237; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 1979. Available online: https://link.springer.com/book/10.1007/978-1-4684-9409-9 (accessed on 7 July 2021).
11. Jarchow, H. *Locally Convex Spaces*; Mathematical Textbooks; B. G. Teubner: Stuttgart, Germany, 1981. [CrossRef]
12. Ferrando, J.C.; López-Pellicer, M.; Sánchez Ruiz, L.M. *Metrizable Barrelled Spaces*; Pitman Research Notes in Mathematics Series 332; Longman: Harlow, UK, 1995.
13. Díaz, J.C.; Domański, P. On the injective tensor product of distinguished Fréchet spaces. *Math. Nachr.* **1999**, *198*, 41–50. [CrossRef]
14. Kąkol, J.; Kubiś, W.; López-Pellicer, M. *Descriptive Topology in Selected Topics of Functional Analysis*; Developments in Mathematics 24; Springer: New York, NY, USA, 2011. [CrossRef]
15. Warner, S. The topology of compact convergence on continuous function spaces. *Duke Math. J.* **1958**, *25*, 265–282. [CrossRef]
16. Köthe, G. *Topological Vector Spaces I*; Die Grundlehren der Mathematischen Wissenschaften Band 159; Springer: New York, NY, USA, 1969. Available online: https://link.springer.com/book/10.1007%2F978-3-642-64988-2 (accessed on 7 July 2021).
17. Gillman, L.; Jerison, M. *Rings of Continuous Functions*, 1960 ed.; Graduate Texts in Mathematics No. 43; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 1976. Available online: https://link.springer.com/book/10.1007/978-1-4615-7819-2 (accessed on 7 July 2021).
18. Buchwalter, H.; Schmets, J. Sur quelques propiértés de l'espace $C_s(T)$. *J. Math. Pures Appl. IX Sér.* **1973**, *52*, 337–352.

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Aspects of Differential Calculus Related to Infinite-Dimensional Vector Bundles and Poisson Vector Spaces

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Abstract: We prove various results in infinite-dimensional differential calculus that relate the differentiability properties of functions and associated operator-valued functions (e.g., differentials). The results are applied in two areas: (1) in the theory of infinite-dimensional vector bundles, to construct new bundles from given ones, such as dual bundles, topological tensor products, infinite direct sums, and completions (under suitable hypotheses); (2) in the theory of locally convex Poisson vector spaces, to prove continuity of the Poisson bracket and continuity of passage from a function to the associated Hamiltonian vector field. Topological properties of topological vector spaces are essential for the studies, which allow the hypocontinuity of bilinear mappings to be exploited. Notably, we encounter $k_{\mathbb{R}}$-spaces and locally convex spaces E such that $E \times E$ is a $k_{\mathbb{R}}$-space.

Keywords: vector bundle; dual bundle; direct sum; completion; tensor product; cocycle; smoothness; analyticity; hypocontinuity; k-space; compactly generated space; infinite-dimensional Lie group; Poisson vector space; Poisson bracket; Hamiltonian vector field; group action; multilinear map

MSC: 26E15 (primary); 17B63; 22E65; 26E20; 46G20; 54B10; 54D50; 55R25; 58B10

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1. Introduction

We study questions of infinite-dimensional differential calculus in the setting of Keller's C^k_c-theory [1] (going back to [2]). Applications to infinite-dimensional vector bundles are given, and also applications in the theory of locally convex Poisson vector spaces.

Differentiability properties of operator-valued maps. Our results are centred around the following basic problem: Consider locally convex spaces X, E and F, an open set $U \subseteq X$ and a map $f\colon U \to L(E,F)_b$ to the space of continuous linear maps, endowed with the topology of uniform convergence on bounded sets. How are the differentiability properties of the operator-valued map f related to those of

$$f^\wedge\colon U \times E \to F\,, \quad f^\wedge(x,v) := f(x)(v)\,?$$

We show that if $f^\wedge$ is smooth, then also f is smooth (Proposition 1). Conversely, exploiting the hypocontinuity of the bilinear evaluation map

$$L(E,F)_b \times E \to F\,, \quad (\alpha, v) \mapsto \alpha(v)\,,$$

we find natural hypotheses on E and F ensuring that smoothness of f entails smoothness of $f^\wedge$ (Proposition 2; likewise for compact sets in place of bounded sets). Without extra hypotheses on E and F, this conclusion becomes false, e.g., if $U = X$ is a non-normable, real, locally convex space with dual space $X' := L(X,\mathbb{R})$. Then, $f := \mathrm{id}_{X'}\colon X'_b \to X'_b$ is continuous linear and thus smooth, but $f^\wedge\colon X'_b \times X \to \mathbb{R}$ is the bilinear evaluation map taking (λ, x) to $\lambda(x)$, which is discontinuous for non-normable X (see [3] (p. 2)) and hence not smooth in the sense of Keller's C^∞_c-theory. We also obtain results concerning finite-order differentiability properties, as well as real and complex analyticity. Furthermore, $L(E,F)$ can be replaced with the space $L^k(E_1, \ldots, E_k, F)$ of continuous k-linear maps $E_1 \times \cdots \times E_k \to F$, if $E_1, \ldots, E_k$ are locally convex spaces. (Related questions also play a role in the comparative study of differential calculi [1].) As a very special case of our studies, the differential

$$f' \colon U \to L(E, F)_b$$

is C^{r-2}, for each $r \in \mathbb{N} \cup \{\infty\}$ with $r \geq 2$, locally convex spaces E and F, and C^r-map $f \colon U \to F$ on an open set $U \subseteq E$ (see Corollary 1).

Applications to infinite-dimensional vector bundles. Apparently, mappings of the specific form just described play a vital role in the theory of vector bundles: If F is a locally convex space, M a (not necessarily finite-dimensional) smooth manifold and $(U_i)_{i \in I}$ an open cover of M, then the smooth vector bundles $E \to M$, with fibre F, which are trivial over the sets U_i, can be described by cocycles $g_{ij} \colon U_i \cap U_j \to \mathrm{GL}(F)$ such that $G_{ij} := g_{ij}^{\wedge} \colon (U_i \cap U_j) \times F \to F$, $(x, v) \mapsto g_{ij}(x)(v)$ is smooth (Proposition 3, Remark 7). Then, g_{ij} is smooth as a mapping to the space $L(F)_b := L(F, F)_b$ (see Proposition 1). In various contexts—for example, when trying to construct dual bundles—we are in the opposite situation: we know that each g_{ij} is smooth, and would like to conclude that also the mappings G_{ij} are smooth. Although this is not possible in general (as examples show), our results provide additional conditions ensuring that the conclusion is correct in the specific situation at hand. Notably, we obtain conditions ensuring the existence of a canonical dual bundle (Proposition 13). Without extra conditions, a canonical dual bundle need not exist (Example 2).

Besides dual bundles, we discuss a variety of construction principles of new vector bundles from given ones, including topological tensor products, completions, and finite or infinite direct sums. More generally, given a (finite- or infinite-dimensional) Lie group acting on the base manifold M, we discuss the construction of new equivariant vector bundles from given ones. Most of the constructions require specific hypotheses on the base manifold, the fibre of the bundle, and the Lie group.

As to completions, complementary topics were considered in the literature: Given an infinite-dimensional smooth manifold M, completions of the tangent bundle with respect to a weak Riemannian metric occur in [4] (p. 549), in hypotheses for a so-called *robust* Riemannian manifold.

We mention that multilinear algebra and vector bundle constructions can be performed much more easily in an inequivalent setting of infinite-dimensional calculus, the convenient differential calculus [3]. However, a weak notion of vector bundles is used there, which need not be topological vector bundles. Our discussion of vector bundles intends to pinpoint additional conditions ensuring that the natural construction principles lead to vector bundles in a stronger sense (which are, in particular, topological vector bundles).

The work [5] was particularly important for our studies. For an open subset U of a Fréchet space E, smoothness of $f^{\wedge} \colon U \times E^k \to \mathbb{R}$ is deduced from smoothness of $f \colon U \to \Lambda^k(E')_b$ in the proof of [5] (Proposition IV.6). A typical hypocontinuity argument already appears in the proof of [5] (Lemma IV.7). In contrast to the local calculations in charts, the global structure on a dual bundle (and bundles of k-forms) asserted in the first remark of [5] (p. 339) is problematic if Keller's C_c^{∞}-theory is used, without further hypotheses.

Applications related to locally convex Poisson vector spaces. In the wake of works by Odzijewicz and Ratiu on Banach–Poisson vector spaces and Banach–Poisson manifolds [6,7], certain locally convex Poisson vector spaces were introduced [8], which generalise the Lie–Poisson structure on the dual space of a finite-dimensional Lie algebra going back to Kirillov, Kostant and Souriau. By now, the latter spaces can be embedded in a general theory of locally convex Poisson manifolds (see [9]; for generalisations of finite-dimensional Poisson geometry with a different thrust, cf. [10]). Recall that many important examples of bilinear mappings between locally convex topological vector spaces are not continuous, but at least hypocontinuous (cf. [11] for this classical concept). In Sections 12 and 13, we provide the proofs for two fundamental results in the theory of locally convex Poisson vector spaces which are related to hypocontinuity. (These proofs were stated in the preprint version of [8], but not included in the actual publication.) We show that the Poisson bracket associated with a continuous Lie bracket is always continuous (Theorem 1) and that the linear map $C^\infty(E,\mathbb{R}) \to C^\infty(E,E)$ taking a smooth function to the associated Hamiltonian vector field is continuous (Theorem 2). Ideas from [8] and the current article were also taken further in [12] (Section 13).

2. Preliminaries and Notation

We describe our setting of differential calculus and compile useful facts. Either references to the literature are given or a proof; the proofs can be looked up in Appendix A.

Infinite-dimensional calculus. We work in the framework of infinite-dimensional differential calculus known as Keller's C^k_c-theory [1]. Our main reference is [13] (see also [14–17]). If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we let $\mathbb{D} := \{t \in \mathbb{K} \colon |t| \leq 1\}$ and $\mathbb{D}_\varepsilon := \{t \in \mathbb{K} \colon |t| \leq \varepsilon\}$ for $\varepsilon > 0$. We write $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. All topological vector spaces considered in the article are assumed Hausdorff, unless the contrary is stated. For brevity, Hausdorff locally convex topological vector spaces will be called locally convex spaces. As usual, a subset M of a $\mathbb{K}$-vector space is called *balanced* if $tx \in M$ for all $x \in M$ and $t \in \mathbb{D}$. The subset M is called *absolutely convex* if it is both convex and balanced. If $q \colon E \to [0,\infty]$ is a seminorm on a $\mathbb{K}$-vector space E, we write $B^q_\varepsilon(x) := \{y \in E \colon q(y-x) < \varepsilon\}$ for $x \in E$ and $\varepsilon > 0$. We also write $\|x\|_q$ in place of $q(x)$. If E is a locally convex $\mathbb{K}$-vector space, we let E' be the dual space of continuous $\mathbb{K}$-linear functionals $\lambda \colon E \to \mathbb{K}$. We write $M^\circ := \{\lambda \in E' \colon \lambda(M) \subseteq \mathbb{D}\}$ for the polar of a subset $M \subseteq E$. If $\alpha \colon E \to F$ is a continuous $\mathbb{K}$-linear map between locally convex $\mathbb{K}$-vector spaces, we let $\alpha' \colon F' \to E'$, $\lambda \mapsto \lambda \circ \alpha$ be the dual linear map. We say that a mapping $f \colon X \to Y$ between topological spaces is a *topological embedding* if it is a homeomorphism onto its image. We recall:

Definition 1. *Let E and F be locally convex $\mathbb{K}$-vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $U \subseteq E$ be an open subset. A map $f \colon U \to F$ is called $C^0_\mathbb{K}$ if it is continuous, in which case we set $d^0 f := f$. Given $x \in U$ and $y \in E$, we define*

$$df(x,y) := (D_y f)(x) := \lim_{t \to 0} \frac{f(x+ty) - f(x)}{t}$$

if the limit exists (using $t \in \mathbb{K}^\times$ such that $x + ty \in U$). Let $r \in \mathbb{N} \cup \{\infty\}$. We say that a continuous map $f \colon U \to F$ is a $C^r_\mathbb{K}$-map *if the iterated directional derivative*

$$d^k f(x, y_1, \ldots, y_k) := (D_{y_k} \cdots D_{y_1})(f)(x)$$

exists for all $k \in \mathbb{N}$ such that $k \leq r$ and all $(x, y_1, \ldots, y_k) \in U \times E^k$, and if the mappings $d^k f \colon U \times E^k \to F$ so obtained are continuous. Thus, $d^1 f = df$. If $\mathbb{K}$ is understood, we write C^r instead of $C^r_\mathbb{K}$. As usual, C^∞-maps are also called smooth.

Remark 1. *For $k \in \mathbb{N}$, it is known that a map $f \colon U \to F$ as before is $C^k_\mathbb{K}$ if and only if f is $C^1_\mathbb{K}$ and $df \colon U \times E \to F$ is $C^{k-1}_\mathbb{K}$ (cf. [13] (Proposition 1.3.10)).*

Remark 2. *If $\mathbb{K} = \mathbb{C}$, it is known that a map $f \colon E \supseteq U \to F$ as before is $C^\infty_\mathbb{C}$ if and only if it is* complex analytic *in the sense of [18] (Definition 5.6): f is continuous and for each $x \in U$, there exists a* 0*-neighbourhood $Y \subseteq E$ such that $x + Y \subseteq U$ and $f(x+y) = \sum_{n=0}^\infty \beta_n(y)$ for all $y \in Y$ as a pointwise limit, where $\beta_n \colon E \to F$ is a continuous homogeneous polynomial over $\mathbb{C}$ of degree n, for each $n \in \mathbb{N}_0$ [13] (Theorem 2.1.12). Furthermore, f is complex analytic if and only if f is $C^\infty_\mathbb{R}$ and $df(x,\cdot) \colon E \to F$ is complex linear for all $x \in U$ (see [13] (Theorem 2.1.12)). Complex analytic maps will also be called* $\mathbb{C}$-analytic *or $C^\omega_\mathbb{C}$.*

Definition 2. *If $\mathbb{K} = \mathbb{R}$, then a map $f \colon U \to F$ as before is called* real analytic *(or* $\mathbb{R}$-analytic, *or $C^\omega_\mathbb{R}$) if it extends to a complex analytic mapping $\widetilde{U} \to F_\mathbb{C}$ on some open neighbourhood $\widetilde{U}$ of U in the complexification $E_\mathbb{C}$ of E.*

In the following, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, unless the contrary is stated. We use the conventions $\infty + k := \infty - k := \infty$ and $\omega + k := \omega - k := \omega$, for each $k \in \mathbb{N}$. Furthermore, we extend the order on $\mathbb{N}_0$ to an order on $\mathbb{N}_0 \cup \{\infty, \omega\}$ by declaring $n < \infty < \omega$ for each $n \in \mathbb{N}_0$.

Remark 3. *Compositions of composable $C^r_{\mathbb{K}}$-mappings are $C^r_{\mathbb{K}}$-mappings (see Proposition 1.3.4, Remark 2.1.13, and Proposition 2.2.4 in [13]). Thus, $C^r_{\mathbb{K}}$-manifolds modelled on locally convex $\mathbb{K}$-vector spaces can be defined in the usual way (see [13] (Chapter 3) for a detailed exposition). In this article, the word "manifold" (resp., "Lie group") always refers to a manifold (resp., Lie group) modelled on a locally convex space.*

The following basic fact will be used repeatedly.

Lemma 1. *For $k \in \mathbb{N}$, let X, $E_1, \ldots, E_k$, and F be locally convex $\mathbb{K}$-vector spaces, $U \subseteq X$ be an open subset and*

$$f \colon U \times E_1 \times \cdots \times E_k \to F$$

be a $C^1_{\mathbb{K}}$-map such that $f^\vee(x) := f(x, \cdot) \colon E_1 \times \cdots \times E_k \to F$ is k-linear, for each $x \in U$. Let $x \in U$ and q be a continuous seminorm on F. Then, there exists a continuous seminorm p on X with $B^p_1(x) \subseteq U$, and continuous seminorms p_j on E_j for $j \in \{1, \ldots, k\}$ such that

$$\|f(y, v_1, \ldots, v_k)\|_q \leq \|v_1\|_{p_1} \cdots \|v_k\|_{p_k} \quad \text{and} \tag{1}$$

$$\|f(y, v_1, \ldots, v_k) - f(x, v_1, \ldots, v_k)\|_q \leq \|y - x\|_p \|v_1\|_{p_1} \cdots \|v_k\|_{p_k} \tag{2}$$

for all $y \in B^p_1(x)$ and $(v_1, \ldots, v_k) \in E_1 \times \cdots \times E_k$.

We shall also use the following fact:

Lemma 2. *Let E and F be locally convex $\mathbb{K}$-vector spaces, $k \geq 2$ be an integer and $f \colon U \times E^k \to F$ be a mapping such that $f(x, \cdot) \colon E^k \to F$ is k-linear and symmetric for each $x \in U$. Let $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$. If*

$$h \colon U \times E \to F, \quad (x, y) \mapsto f(x, y, \ldots, y)$$

is $C^r_{\mathbb{K}}$, then also f is $C^r_{\mathbb{K}}$. Notably, f is continuous if h is continuous.

***k*-spaces, $k_{\mathbb{R}}$-spaces, k^∞-spaces, and k_ω-spaces.** Recall that a topological space X is said to be *completely regular* if it is Hausdorff and its topology is initial with respect to the set $C(X, \mathbb{R})$ of all continuous real-valued functions on X. Every locally convex space is completely regular, as with every Hausdorff topological group (cf. [19] (Theorem 8.2)). Compare [20,21] for the following.

A topological space X is called a *k-space* if it is Hausdorff and a subset $A \subseteq X$ is closed if and only if $A \cap K$ is closed in K for each compact subset $K \subseteq X$. Every metrisable topological space is a k-space, and every locally compact Hausdorff space. A Hausdorff space X is a k-space if and only if, for each topological space Y, a map $f \colon X \to Y$ is continuous if and only if f is *k-continuous* in the sense that $f|_K$ is continuous for each compact subset $K \subseteq X$. If X is a k-space, then also every subset $M \subseteq X$ which is open or closed in X, when the induced topology is used on M.

A topological space X is called a *$k_{\mathbb{R}}$-space* if it is Hausdorff and a function $f \colon X \to \mathbb{R}$ is continuous if and only if f is k-continuous. Then also a map $f \colon X \to Y$ to a completely regular topological space Y is continuous if and only if it is k-continuous (as the latter condition implies continuity of $g \circ f$ for each $g \in C(Y, \mathbb{R})$). For more information, cf. [22].

Every k-space is a $k_{\mathbb{R}}$-space. The converse is not true: $\mathbb{R}^I$ is known to be a $k_{\mathbb{R}}$-space for each set I (see [22]). If I has cardinality $\geq 2^{\aleph_0}$, then $\mathbb{R}^I$ is not a k-space. (If $\mathbb{R}^I$ was a k-space, then a certain non-discrete subgroup G of $(\mathbb{R}^{\mathbb{R}}, +)$ constructed in [23] would be discrete, which is a contradiction (see [13] (Remark A.6.16 (a)) for more details). Compare also [22].)

The following facts are well known (cf. [22]):

Lemma 3. (a) *If a $k_{\mathbb{R}}$-space X is a direct product $X_1 \times X_2$ of Hausdorff spaces and $X_1 \neq \emptyset$, then X_2 is a $k_{\mathbb{R}}$-space.*
(b) *Every open subset U of a completely regular $k_{\mathbb{R}}$-space X is a $k_{\mathbb{R}}$-space in the induced topology.*

Notably, U is a $k_{\mathbb{R}}$-space for each open subset U of a locally convex space E which is a $k_{\mathbb{R}}$-space. If $E \times E$ is a $k_{\mathbb{R}}$-space, then also E.

Following [8], a topological space X is called a *k^∞-space* if the Cartesian power X^n is a k-space for each $n \in \mathbb{N}$, using the product topology. A Hausdorff space X is called *hemicompact* if $X = \bigcup_{n \in \mathbb{N}} K_n$ for a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets $K_n \subseteq X$ such that each compact subset of X is a subset of some K_n. Hemicompact k-spaces are also called *k_ω-spaces*. If X and Y are k_ω-spaces, then the product topology makes $X \times Y$ a k_ω-space. Notably, every k_ω-space is a k^∞-space. See [24,25] for further information. Finite products of metrisable spaces being metrisable, every metrisable topological space is a k^∞-space. Recall that a locally convex space E is said to be a *Silva space* or (DFS)-space if it is the locally convex inductive limit of a sequence $E_1 \subseteq E_2 \subseteq \cdots$ of Banach spaces such that each inclusion map $E_n \to E_{n+1}$ is a compact operator. Every Silva space is a k_ω-space (see, e.g., [13] (Proposition B13.13(g))).

Spaces of multilinear maps. Given $k \in \mathbb{N}$, locally convex $\mathbb{K}$-vector spaces $E_1, \ldots, E_k$ and F, and a set $\mathcal{S}$ of bounded subsets of $E_1 \times \cdots \times E_k$, we write $L^k(E_1, \ldots, E_k, F)_{\mathcal{S}}$ or $L^k_{\mathbb{K}}(E_1, \ldots, E_k, F)_{\mathcal{S}}$ for the space of continuous k-linear maps $E_1 \times \cdots \times E_k \to F$, endowed with the topology $\mathcal{O}_{\mathcal{S}}$ of uniform convergence on the sets $B \in \mathcal{S}$. Recall that finite intersections of sets of the form

$$\lfloor B, U \rfloor \;:=\; \{\beta \in L^k(E_1, \ldots, E_k, F) \colon \beta(B) \subseteq U\}$$

yield a basis of 0-neighbourhoods for this (not necessarily Hausdorff) locally convex vector topology, for U ranging through the 0-neighbourhoods in F and B through $\mathcal{S}$. If $\bigcup_{B \in \mathcal{S}} B = E_1 \times \cdots \times E_k$, then $\mathcal{O}_{\mathcal{S}}$ is Hausdorff. If $E_1 = \cdots = E_k$, we abbreviate $L^k(E, F)_{\mathcal{S}} := L^k(E, \ldots, E, F)_{\mathcal{S}}$. If $k = 1$ and $E := E_1$, we abbreviate $L(E, F)_{\mathcal{S}} := L^1(E, F)_{\mathcal{S}}$, $L_{\mathbb{K}}(E, F)_{\mathcal{S}} := L^1_{\mathbb{K}}(E, F)_{\mathcal{S}}$ and $L(E)_{\mathcal{S}} := L(E, E)_{\mathcal{S}}$. We write $\mathrm{GL}(E) = L(E)^\times$ for the group of all automorphisms of the locally convex $\mathbb{K}$-vector space E. If $\mathcal{S}$ is the set of all bounded, compact, and finite subsets of $E_1 \times \cdots \times E_k$, respectively, we shall usually write "b," "c," and "p" in place of $\mathcal{S}$. For example, we shall write $L^k(E_1, \ldots, E_k, F)_b$, $L^k(E_1, \ldots, E_k, F)_c$, and $L^k(E_1, \ldots, E_k, F)_p$.

Remark 4. *Let $E_1, \ldots, E_k$ and F be complex locally convex spaces and $f \colon U \to L^k_{\mathbb{C}}(E_1, \ldots, E_k, F)$ be a map, defined on an open subset U of a real locally convex space. Let $\mathcal{S} := b$ or $\mathcal{S} := c$. Since $L^k_{\mathbb{C}}(E_1, \ldots, E_k, F)_{\mathcal{S}}$ is a closed real vector subspace of $L^k_{\mathbb{R}}(E_1, \ldots, E_k, F)_{\mathcal{S}}$, the map f is $C^r_{\mathbb{R}}$ as a map to $L^k_{\mathbb{C}}(E_1, \ldots, E_k, F)_{\mathcal{S}}$ if and only if f is $C^r_{\mathbb{R}}$ as a map to $L^k_{\mathbb{R}}(E_1, \ldots, E_k, F)_{\mathcal{S}}$ (see [13] (Lemma 1.3.19 and Exercise 2.2.4)).*

Given a $C^r_{\mathbb{K}}$-map $f \colon E \supseteq U \to F$ as in Definition 1, we define $f^{(0)} := f$ and

$$f^{(j)} \colon U \to L^j_{\mathbb{K}}(E, F), \quad f^{(j)}(x) := (d^j f)^\vee(x) = d^j f(x, \cdot)$$

for $j \in \mathbb{N}$ such that $j \leq r$.

Hypocontinuous multilinear maps. Beyond normed spaces, typical multilinear maps are not continuous, but merely hypocontinuous. Hypocontinuous *bi*linear maps are discussed in many textbooks. An analogous notion of hypocontinuity for multilinear maps (to be described presently) is useful to us. It can be discussed similarly to the bilinear case.

Lemma 4. *For an integer $k \geq 2$, let $\beta \colon E_1 \times \cdots \times E_k \to F$ be a separately continuous k-linear mapping and $j \in \{2, \ldots, k\}$ such that, for each $x \in E_1 \times \cdots \times E_{j-1}$, the map*

$$\beta^\vee(x) := \beta(x, \cdot) \colon E_j \times \cdots \times E_k \to F$$

is continuous. Let $\mathcal{S}$ be a set of bounded subsets of $E_j \times \cdots \times E_k$. Consider the conditions:

(a) *For each $M \in \mathcal{S}$ and each 0-neighbourhood $W \subseteq F$, there exists a 0-neighbourhood $V \subseteq E_1 \times \cdots \times E_{j-1}$ such that $\beta(V \times M) \subseteq W$.*

(b) *The $(j-1)$-linear map $\beta^\vee \colon E_1 \times \cdots \times E_{j-1} \to L^{k-j+1}(E_j, \ldots, E_k, F)_\mathcal{S}$ is continuous.*

(c) *$\beta|_{E_1 \times \cdots \times E_{j-1} \times M} \colon E_1 \times \cdots \times E_{j-1} \times M \to F$ is continuous, for each $M \in \mathcal{S}$.*

Then (a) *and* (b) *are equivalent, and* (b) *implies* (c)*. If*

$$(\forall M \in \mathcal{S})\ (\exists N \in \mathcal{S})\quad \mathbb{D}M \subseteq N, \tag{3}$$

then (a), (b)*, and* (c) *are equivalent.*

Definition 3. *A k-linear map β which satisfies the hypotheses and Condition* (a) *of Lemma 4 is called $\mathcal{S}$-hypocontinuous in its arguments $(j, \ldots, k)$. If $j = k$, we also say that β is $\mathcal{S}$-hypocontinuous in the k-th argument. Analogously, we define $\mathcal{S}$-hypocontinuity of β in the j-th argument, if $j \in \{1, \ldots, k\}$ and a set $\mathcal{S}$ of bounded subsets of E_j are given.*

We are mainly interested in *b*-, *c*-, and *p*-hypocontinuity, viz., in $\mathcal{S}$-hypocontinuity with respect to the set $\mathcal{S}$ of all bounded subsets of $E_j \times \cdots \times E_k$, the set $\mathcal{S}$ of all compact subsets, and the set $\mathcal{S}$ of all finite subsets, respectively. If $\mathcal{S}$ and $\mathcal{T}$ are sets of bounded subsets of $E_j \times \cdots \times E_k$ such that $\mathcal{S} \subseteq \mathcal{T}$ and β is $\mathcal{T}$-hypocontinuous in its variables $(j, \ldots, k)$, then β is also $\mathcal{S}$-hypocontinuous in the latter. The following is obvious from Lemma 4 (c) (as the elements of a convergent sequence, together with its limit, form a compact set):

Lemma 5. *If $\beta \colon E_1 \times \cdots \times E_k \to F$ is c-hypocontinuous in some argument, or in its arguments $(j, \ldots, k)$ for some $j \in \{2, \ldots, k\}$, then β is sequentially continuous.*

In many cases, separately continuous bilinear maps are automatically hypocontinuous. Recall that a subset B of a locally convex space E is a *barrel* if it is closed, absolutely convex, and absorbing. The space E is called *barrelled* if every barrel is a 0-neighbourhood. See Proposition 6 in [11] (Chapter III, §5, no. 3) for the following fact.

Lemma 6. *If $\beta \colon E_1 \times E_2 \to F$ is a separately continuous bilinear map and E_1 is barrelled, then β is $\mathcal{S}$-hypocontinuous in its second argument, with respect to any set $\mathcal{S}$ of bounded subsets of E_2.*

Evaluation maps are paradigmatic examples of hypocontinuous multilinear maps.

Lemma 7. *Let $E_1, \ldots, E_k$ and F be locally convex $\mathbb{K}$-vector spaces and $\mathcal{S}$ be a set of bounded subsets of $E := E_1 \times \cdots \times E_k$ with $\bigcup_{M \in \mathcal{S}} M = E$. Then, the $(k+1)$-linear map*

$$\varepsilon \colon L^k(E_1, \ldots, E_k, F)_\mathcal{S} \times E_1 \times \cdots \times E_k \to F\,, \quad (\beta, x) \mapsto \beta(x)$$

is $\mathcal{S}$-hypocontinuous in its arguments $(2, \ldots, k+1)$. If $k = 1$ and $E = E_1$ is barrelled, then $\varepsilon \colon L(E, F) \times E \to F$ is also hypocontinuous in the first argument, with respect to any locally convex topology $\mathcal{O}$ on $L(E, F)$ which is finer than the topology of pointwise convergence, and any set $\mathcal{T}$ of bounded subsets of $(L(E, F), \mathcal{O})$.

Lemma 8. *Consider locally convex spaces $E_1, \ldots, E_k$ and F with $k \geq 2$ and a k-linear map $\beta \colon E_1 \times \cdots \times E_k \to F$.*

(a) *If β is sequentially continuous, then the composition $\beta \circ f$ is continuous for each continuous function $f \colon X \to E_1 \times \cdots \times E_k$ on a topological space X which is metrisable or satisfies the first axiom of countability.*

(b) *If β is c-hypocontinuous in its arguments $(j, \ldots, k)$ for some $j \in \{2, \ldots, k\}$ and X is a $k_\mathbb{R}$-space, then $\beta \circ f$ is continuous for each continuous function $f \colon X \to E_1 \times \cdots \times E_k$.*

Lipschitz differentiable maps. In Section 7, it will be useful to work with certain Lipschitz differentiable maps, instead of C^r-maps. We briefly recall concepts and facts.

Definition 4. *Let E and F be locally convex $\mathbb{K}$-vector spaces, $U \subseteq E$ be open and $f\colon U \to F$ be a map. We say that f is* locally Lipschitz continuous *or $LC^0_{\mathbb{K}}$ if it has the following property: For each $x \in U$ and continuous seminorm q on F, there exists a continuous seminorm p on E such that $B^p_1(x) \subseteq U$ and*

$$q(f(z) - f(y)) \leq p(z - y) \quad \text{for all } y, z \in B^p_1(x).$$

Given $r \in \mathbb{N}_0 \cup \{\infty\}$, we say that f is $LC^r_{\mathbb{K}}$ if f is $C^r_{\mathbb{K}}$ and $d^k f\colon U \times E^k \to F$ is $LC^0_{\mathbb{K}}$ for each $k \in \mathbb{N}_0$ such that $k \leq r$.

Every C^1-map is $LC^0_{\mathbb{K}}$ (see, for example, [13] (Exercise 1.5.4)). As a consequence, for each $r \in \mathbb{N} \cup \{\infty\}$, every $C^r_{\mathbb{K}}$-map is $LC^{r-1}_{\mathbb{K}}$. Notably, every smooth map is $LC^\infty_{\mathbb{K}}$. Moreover, a $C^r_{\mathbb{K}}$-map with finite r is $LC^r_{\mathbb{K}}$ if and only if $d^r f$ is $LC^0_{\mathbb{K}}$. The following facts are known, or part of the folklore.

Lemma 9. *For locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $r \in \mathbb{N}_0 \cup \{\infty\}$, we have:*

(a) *A map $f\colon E \supseteq U \to \prod_{j \in J} F_j$ to a direct product of locally convex spaces is $LC^r_{\mathbb{K}}$ if and only each component is $LC^r_{\mathbb{K}}$;*
(b) *Compositions of composable $LC^r_{\mathbb{K}}$-maps are $LC^r_{\mathbb{K}}$;*
(c) *Let F be a locally convex space and $F_0 \subseteq F$ be a vector subspace which is closed in F, or sequentially closed. Then, a map $f\colon E \supseteq U \to F_0$ is $FC^r_{\mathbb{K}}$ if and only if it is $FC^r_{\mathbb{K}}$ as a map to F.*
(d) *A map $E \supseteq U \to P$ to a projective limit $P = \varprojlim F_j$ of locally convex spaces is $LC^r_{\mathbb{K}}$ if and only if $p_j \circ f\colon U \to F_j$ is $LC^r_{\mathbb{K}}$ for all $j \in J$, where $p_j\colon P \to F_j$ is the limit map.*

Our concept of local Lipschitz continuity is weaker than the one in [13] (Definition 1.5.4).

The compact-open C^r-topology. If E and F are locally convex $\mathbb{K}$-vector spaces, $U \subseteq E$ is an open set and $r \in \mathbb{N}_0 \cup \{\infty\}$, then the vector space $C^r_{\mathbb{K}}(U, F)$ of all $C^r_{\mathbb{K}}$-maps $U \to F$ carries a natural topology (the "compact-open C^r-topology"), namely the initial topology with respect to the mappings

$$C^r_{\mathbb{K}}(U, F) \to C(U \times E^j, F)_{c.o.} \quad f \mapsto d^j f$$

for $j \in \mathbb{N}_0$ such that $j \leq r$, where the right-hand side is endowed with the compact-open topology. Then, $C^r_{\mathbb{K}}(U, F)$ is a locally convex $\mathbb{K}$-vector space. If F is a complex locally convex space, then also $C^r_{\mathbb{K}}(U, F)$. See, e.g., [13] (§1.7) for further information, or [26].

3. Differentiability Properties of Operator-Valued Maps

Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$. In this section, we establish the following proposition.

Proposition 1. *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, $E_1, \ldots, E_k$ and F be locally convex $\mathbb{L}$-vector spaces, X be a locally convex $\mathbb{K}$-vector space, and $U \subseteq X$ be an open subset. Let $f\colon U \to L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)$ be a map such that*

$$f^\wedge\colon U \times E_1 \times \cdots \times E_k \to F, \quad f^\wedge(x, v) := f(x)(v) \quad \text{for } x \in U, v \in E_1 \times \cdots \times E_k$$

is $C^r_{\mathbb{K}}$. Then, the following holds:

(a) *f is $C^r_{\mathbb{K}}$ as a map to $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c$.*
(b) *If $r \geq 1$, then f is $C^{r-1}_{\mathbb{K}}$ as a map to $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_b$.*

Furthermore,

$$d^j f(x,y_1,\ldots,y_j)(v) = d^j(f^\wedge)((x,v),(y_1,0),\ldots,(y_j,0)) \tag{4}$$

for all $j \in \mathbb{N}$ with $j \leq r$ (resp., $j \leq r-1$, in (b)*), all $x \in U$, $v \in E_1 \times \cdots \times E_k$, and $y_1,\ldots,y_j \in X$.*

Corollary 1. *Let E and F be locally convex $\mathbb{K}$-vector spaces and $f\colon U \to F$ be a $C^r_{\mathbb{K}}$-map on an open subset $U \subseteq E$, where $r \in \mathbb{N} \cup \{\infty, \omega\}$. Then, the following holds:*

(a) *The map $f^{(k)}\colon U \to L^k_{\mathbb{K}}(E,F)_c$, $x \mapsto f^{(k)}(x) = d^k f(x,\cdot)$ is $C^{r-k}_{\mathbb{K}}$, for each $k \in \mathbb{N}$ such that $k \leq r$.*

(b) *The map $f^{(k)}\colon U \to L^k_{\mathbb{K}}(E,F)_b$ is $C^{r-k-1}_{\mathbb{K}}$, for each $k \in \mathbb{N}$ such that $k \leq r-1$.*

Furthermore, $d^j(f^{(k)})(x,y_1,\ldots,y_j) = d^{j+k} f(x,\cdot,y_1,\ldots,y_j)$, for all $j \in \mathbb{N}$ with $j+k \leq r$ (resp., $j+k \leq r-1$), all $x \in U$, and $y_1,\ldots,y_j \in E$.

Proof. For each $k \in \mathbb{N}$ such that $k \leq r$, the map $d^k f\colon U \times E^k \to F$ is $C^{r-k}_{\mathbb{K}}$ (see [13] (Remark 1.3.13 and Exercise 2.2.7)), and $f^{(k)}(x) = d^k f(x,\cdot)$ is k-linear for each $x \in U$, by [13] (Proposition 1.3.17). Moreover, $(f^{(k)})^\wedge = d^k f$. Thus, Proposition 1 applies with $f^{(k)}$ in place of f and $r-k$ in place of r. □

Given a topological space X and locally convex space F, we endow the space $C(X,F)$ of continuous F-valued functions on X with the compact-open topology. It is known that this topology coincides with the topology of uniform convergence on compact sets. The next lemma will be useful when we discuss mappings to $L^k(E,F)_c$.

Lemma 10. *Let X, E, and F be locally convex $\mathbb{K}$-vector spaces, $U \subseteq X$ and $W \subseteq E$ be open subsets, and $f\colon U \times W \to F$ be a $C^r_{\mathbb{K}}$-map, with $r \in \mathbb{N}_0 \cup \{\infty\}$. Then, also the map*

$$f^\vee\colon U \to C(W,F), \quad x \mapsto f(x,\cdot)$$

is $C^r_{\mathbb{K}}$. If $\mathbb{K} = \mathbb{R}$ and f admits a complex analytic extension $h\colon \widetilde{U} \times \widetilde{W} \to F_{\mathbb{C}}$ for suitable open neighbourhoods $\widetilde{U}$ of U in $X_{\mathbb{C}}$ and $\widetilde{W}$ of W in $E_{\mathbb{C}}$, then $f^\vee$ is real analytic.

Proof. We first assume that $r \in \mathbb{N}_0$, and proceed by induction. For $r = 0$, the assertion is well known (see, e.g., [13] (Proposition A.6.17)). Now assume that $r \in \mathbb{N}$. Given $x \in U$ and $y \in X$, there exists $\varepsilon > 0$ such that $x + \mathbb{D}^0_\varepsilon y \subseteq U$, where $\mathbb{D}^0_r := \{t \in \mathbb{K}\colon |t| < \varepsilon\}$. Consider

$$g\colon \mathbb{D}^0_\varepsilon \times W \to F, \quad (t,w) \mapsto \begin{cases} \frac{f(x+ty,w)-f(x,w)}{t} & \text{if } t \neq 0; \\ df((x,w),(y,0)) & \text{if } t = 0. \end{cases}$$

Then, $g(t,w) = \int_0^1 df((x+sty,w),(y,0))\,ds$, by the Mean Value Theorem. The integrand being continuous, also g is continuous (by the Theorem on Parameter-Dependent Integrals, [13] (Lemma 1.1.11)). Hence, $g^\vee\colon V \to C(W,F)$ is continuous, by induction, and hence

$$\frac{f^\vee(x+ty) - f^\vee(x)}{t} \;=\; g^\vee(t) \;\to\; g^\vee(0)$$

as $t \to 0$, where $g^\vee(0) = df((x,\cdot),(y,0)) = k^\vee(x,y)$ with

$$k\colon (U \times E) \times W \to F, \quad (x,y,w) \mapsto df((x,w),(y,0)).$$

Since k is $C^{r-1}_{\mathbb{K}}$, the map $d(f^\vee) = k^\vee$ is $C^{r-1}_{\mathbb{K}}$, by the inductive hypothesis. Notably, $d(f^\vee)$ is continuous and hence $f^\vee$ is $C^1_{\mathbb{K}}$. Now, $f^\vee$ being $C^1_{\mathbb{K}}$ with $d(f^\vee)$ a $C^{r-1}_{\mathbb{K}}$-map, $f^\vee$ is $C^r_{\mathbb{K}}$.

The case $r = \infty$. If f is $C^\infty_{\mathbb{K}}$, then f is $C^k_{\mathbb{K}}$ for each $k \in \mathbb{N}_0$. Hence, $f^\vee$ is $C^k_{\mathbb{K}}$ for each $k \in \mathbb{N}_0$ (by the case already treated), and thus $f^\vee$ is $C^\infty_{\mathbb{K}}$.

Final assertion. By the $C^\infty_{\mathbb{C}}$-case already treated, the map

$$h^\vee\colon \widetilde{U} \to C(\widetilde{W}, F_{\mathbb{C}})$$

is $C^\infty_{\mathbb{C}}$. The restriction map

$$\rho\colon C(\widetilde{W}, F_{\mathbb{C}}) \to C(W, F_{\mathbb{C}})\,, \quad \gamma \mapsto \gamma|_W$$

being continuous $\mathbb{C}$-linear and thus $C^\infty_{\mathbb{C}}$, it follows that the composition

$$\rho \circ h^\vee\colon \widetilde{U} \to C(W, F_{\mathbb{C}}) = C(W, F)_{\mathbb{C}}$$

is $C^\infty_{\mathbb{C}}$ and thus complex analytic. Since $\rho \circ h^\vee$ extends $f^\vee$, we see that $f^\vee$ is real analytic. □

Proof of Proposition 1. (a) Abbreviate $E := E_1 \times \cdots \times E_k$. Because $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c$ is a closed $\mathbb{K}$-vector subspace of $C(E, F)$ and carries the induced topology, f will be $C^r_{\mathbb{K}}$ as a map to $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c$ if we can show that f is $C^r_{\mathbb{K}}$ as a map to $C(E, F)$ (see [13] (Lemma 1.3.19 and Exercise 2.2.4)). Since $f^\wedge$ is $C^r_{\mathbb{K}}$ and $f = (f^\wedge)^\vee$, the latter follows from Lemma 10. This is obvious unless $\mathbb{K} = \mathbb{R}$ and $r = \omega$. In this case, the map $f^\wedge$ admits a $\mathbb{C}$-analytic extension $p\colon Q \to F_{\mathbb{C}}$ to an open neighbourhood Q of $U \times E$ in $X_{\mathbb{C}} \times E_{\mathbb{C}}$. For each $x \in U$, there exists an open, connected neighbourhood U_x of x in $X_{\mathbb{C}}$ and a balanced, open 0-neighbourhood $W_x \subseteq E_{\mathbb{C}}$ such that $U_x \times W_x \subseteq Q$ and $U_x \cap X \subseteq U$. Let $D := \{z \in \mathbb{C}\colon |z| < 1\}$. Then,

$$q\colon U_x \times W_x \times D \to F_{\mathbb{C}}\,, \quad (y, w, z) \mapsto p(y, zw) - z^k p(y, w)$$

is a $\mathbb{C}$-analytic map which vanishes on $(U_x \times W_x \times D) \cap (X \times E \times \mathbb{R})$. Hence, $q = 0$, by the Identity Theorem (see [13] (Theorem 2.1.16 (c))). Then, $p(y, zw) = z^k p(y, w)$ for all $z \in \mathbb{C}$ such that $|z| \leq 1$, by continuity. This implies that the map

$$g\colon U_x \times E_{\mathbb{C}} \to F_{\mathbb{C}}\,, \quad (y, w) \mapsto z^k p(y, z^{-1}w) \quad \text{for some } z \in \mathbb{C}^\times \text{ with } z^{-1}w \in W_x$$

is well defined. Since g is $\mathbb{C}$-analytic, the final statement of Lemma 10 applies.

(b) We prove the assertion for $r \in \mathbb{N}$ first; then, also the case $r = \infty$ follows. If $r = 1$, let $x \in U$. Given an open 0-neighbourhood $W \subseteq F$ and bounded subset $B \subseteq E := E_1 \times \cdots \times E_k$, let q be a continuous seminorm on F such that $B^q_1(0) \subseteq W$. By Lemma 1, there exist continuous seminorms p on X and p_j on E_j for $j \in \{1, \ldots, k\}$ such that $B^p_1(x) \subseteq U$ and

$$\|f^\wedge(y, v) - f^\wedge(x, v)\|_q \leq \|y - x\|_p \|v_1\|_{p_1} \cdots \|v_k\|_{p_k}$$

for all $y \in B^p_1(x)$ and all $v = (v_1, \ldots, v_k) \in E_1 \times \cdots \times E_k$. Since B is bounded, we have

$$C := \sup\{\|v_1\|_{p_1} \cdots \|v_k\|_{p_k}\colon v = (v_1, \ldots, v_k) \in B\} \;<\; \infty\,.$$

Choose $\delta \in\,]0, 1]$ such that $\delta C \leq 1$. For each $y \in B^p_\delta(x)$, we get $\|f^\wedge(y, v) - f^\wedge(x, v)\|_q < \delta C \leq 1$ for each $v \in B$ and thus $f^\wedge(y, v) - f^\wedge(x, v) \in B^q_1(0) \subseteq W$. Hence,

$$f(y) - f(x) \in \lfloor B, W \rfloor \quad \text{for each } y \in B^p_1(x),$$

entailing that f is continuous.

Induction step: Now, assume that $r \geq 2$. Given $x \in U$ and $y \in X$, there exists $\varepsilon > 0$ such that $x + \mathbb{D}^0_\varepsilon y \subseteq U$, where $\mathbb{D}^0_r := \{t \in \mathbb{K}\colon |t| < \varepsilon\}$. Consider

$$g\colon \mathbb{D}^0_\varepsilon \times E^k \to F\,, \quad (t, v) \mapsto \begin{cases} \frac{f^\wedge(x+ty, v) - f^\wedge(x, v)}{t} & \text{if } t \neq 0; \\ d(f^\wedge)((x, v), (y, 0)) & \text{if } t = 0. \end{cases}$$

Then, g is $C^{r-1}_{\mathbb{K}}$ and hence $C^1_{\mathbb{K}}$, as a consequence of [27] (Propositions 7.4 and 7.7). Since $g(t,v)$ is k-linear in v, it follows that $g^\vee\colon U\to L^k(E,F)_b$ is continuous, by induction. As a consequence,

$$\frac{f(x+ty)-f(x)}{t} \;=\; g^\vee(t)\;\to\; g^\vee(0)$$

as $t\to 0$, where $g^\vee(0)=d(f^\wedge)((x,\cdot),(y,0))=h^\vee(x,y)$ with

$$h\colon (U\times E^k)\times W\to F\,,\quad h((x,y),v):=d(f^\wedge)((x,v),(y,0))\,.$$

Since h is $C^{r-1}_{\mathbb{K}}$ and $h((x,y),v)$ is k-linear in v, the map $df=h^\vee$ is $C^{r-2}_{\mathbb{K}}$, by induction. Hence, df is continuous and thus f is $C^1_{\mathbb{K}}$. Now, f being $C^1_{\mathbb{K}}$ with df a $C^{r-2}_{\mathbb{K}}$-map, f is $C^{r-1}_{\mathbb{K}}$.

The case $\mathbb{K}=\mathbb{R}$, $r=\omega$. By Remark 4, we may assume that $\mathbb{L}=\mathbb{R}$ (the case $\mathbb{L}=\mathbb{C}$ then follows). Given $x\in U$, let $g\colon U_x\times E_{\mathbb{C}}\to F_{\mathbb{C}}$ be as in the proof of (a). Identifying $E_{\mathbb{C}}$ with $(E_1)_{\mathbb{C}}\times\cdots\times(E_k)_{\mathbb{C}}$, the mapping g is complex k-linear in the second variable. Hence $g^\vee\colon U_x\to L^k_{\mathbb{C}}((E_1)_{\mathbb{C}},\ldots,(E_k)_{\mathbb{C}},F_{\mathbb{C}})_b$ is $\mathbb{C}$-analytic, by the $C^\infty_{\mathbb{C}}$-case already discussed. Because the map $\rho\colon L^k_{\mathbb{C}}((E_1)_{\mathbb{C}},\ldots,(E_k)_{\mathbb{C}},F_{\mathbb{C}})_b\to L^k_{\mathbb{R}}(E_1,\ldots,E_k,F_{\mathbb{C}})_b=(L^k_{\mathbb{R}}(E_1,\ldots,E_k,F)_b)_{\mathbb{C}}$, $\alpha\mapsto\alpha|_E$ is continuous $\mathbb{C}$-linear, the composition $\rho\circ g^\vee$ is $\mathbb{C}$-analytic. However, this mapping extends $f|_{U_x\cap X}$. Hence, $f|_{U_x\cap X}$ is real analytic and hence so is f, using that the open sets $U_x\cap X$ form an open cover of U.

Formula for the differentials: Let $j\in\mathbb{N}$ with $j\leq r$, $x\in U$, $v\in E_1\times\cdots\times E_k$ and $y_1,\ldots,y_j\in X$. Exploiting that $\mathrm{ev}_v\colon L^k_{\mathbb{L}}(E_1,\ldots,E_k,F)_c\to F$, $\beta\mapsto\beta(v)$ is continuous and linear, we deduce that

$$\begin{aligned}\mathrm{ev}_v(d^jf(x,y_1,\ldots,y_j)) &= d^j(\mathrm{ev}_v\circ f)(x,y_1,\ldots,y_j)\;=\;d^j(f^\wedge(\cdot,v))(x,y_1,\ldots,y_j)\\ &= d^j(f^\wedge)((x,v),(y_1,0),\ldots,(y_j,0))\end{aligned}$$

for f as a map to $L^k_{\mathbb{L}}(E_1,\ldots,E_k,F)_c$. If $j\leq r-1$, the same calculation applies to f as a mapping to $L^k_{\mathbb{L}}(E_1,\ldots,E_k,F)_b$. □

For the special case of (a) when $r=0$ and X as well as $E_1=\cdots=E_k$ are metrisable, see already [1] (Lemma 0.1.2).

4. Compositions with Hypocontinuous k-Linear Maps

We study the differentiability properties of compositions of the form $\beta\circ f$, where β is a k-linear map which need not be continuous.

Lemma 11. *Let $k\geq 2$ be an integer, $E_1,\ldots,E_k$, X, and F be locally convex $\mathbb{K}$-vector spaces, $\beta\colon E_1\times\cdots\times E_k\to F$ be a k-linear map, $r\in\mathbb{N}_0\cup\{\infty,\omega\}$ and $f\colon U\to E_1\times\cdots\times E_k=:E$ be a $C^r_{\mathbb{K}}$-map on an open subset $U\subseteq X$. Assume that*

(a) *β is sequentially continuous and X is metrisable; or*
(b) *For some $j\in\{2,\ldots,k\}$, the k-linear map β is c-hypocontinous in its variables $(j,\ldots,k)$. Moreover, $X\times X$ is a $k_{\mathbb{R}}$-space, or $r=0$ and X is a $k_{\mathbb{R}}$-space, or $(r,\mathbb{K})=(\infty,\mathbb{C})$ and X is a $k_{\mathbb{R}}$-space.*

Then, $\beta\circ f\colon U\to F$ is a $C^r_{\mathbb{K}}$-map.

Proof. The case $r=0$ was treated in Lemma 8. We first assume that $r\in\mathbb{N}$.

(a) Assuming (a), let $x\in U$, $y\in X$, and $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}\setminus\{0\}$ such that $t_n\to 0$ as $n\to\infty$ and $x+t_ny\in U$ for all $n\in\mathbb{N}$. Using the components of $f=(f_1,\ldots,f_k)$, we can write the difference quotient $\frac{1}{t_n}(\beta(f(x+t_ny))-\beta(f(x)))$ as the telescopic sum

$$\sum_{\nu=1}^{k}\beta\Big(f_1(x+t_ny),\ldots,f_{\nu-1}(x+t_ny),\frac{f_\nu(x+t_ny)-f_\nu(x)}{t_n},f_{\nu+1}(x),\ldots,f_k(x)\Big),$$

which converges to

$$\sum_{\nu=1}^{k} \beta(f_1(x), \ldots, f_{\nu-1}(x), df_\nu(x,y), f_{\nu+1}(x), \ldots, f_k(x)) = d(\beta \circ f)(x,y) \tag{5}$$

as $n \to \infty$, using the sequential continuity of β. By Lemma 8, $d(\beta \circ f)$ is continuous, whence $\beta \circ f$ is $C^1_{\mathbb{K}}$. If $r \geq 2$, then

$$g_\nu \colon U \times X \to E, \quad (x,y) \mapsto (f_1(x), \ldots, f_{\nu-1}(x), df_\nu(x,y), f_{\nu+1}(x), \ldots, f_k(x))$$

is a $C^{r-1}_{\mathbb{K}}$-map and $d(\beta \circ f) = \sum_{\nu=1}^{k} \beta \circ g_\nu$ is $C^{r-1}_{\mathbb{K}}$ by induction; thus $\beta \circ f$ is $C^r_{\mathbb{K}}$. If $r = \infty$, the preceding shows that $\beta \circ f$ is $C^s_{\mathbb{K}}$ for each $s \in \mathbb{N}_0$, whence $\beta \circ f$ is $C^r_{\mathbb{K}}$.

(b) If $X \times X$ is a $k_{\mathbb{R}}$-space, then $U \times X$ and U are $k_{\mathbb{R}}$-spaces. By Lemma 5, β is sequentially continuous. The argument from (a) shows that $d(\beta \circ f)(x,y)$ exists for all $(x,y) \in U \times X$ and is given by (5). Thus $d(\beta \circ f)$ is continuous, by Lemma 8, and thus $\beta \circ f$ is $C^1_{\mathbb{K}}$. Let f be $C^{r+1}_{\mathbb{K}}$ now and assume $\beta \circ f$ is $C^r_{\mathbb{K}}$ with rth differential of the form

$$d^r(\beta \circ f)(x, y_1, \ldots, y_r) = \sum_{(I_1, \ldots, I_r)} \beta(d^{|I_1|} f_1(x, y_{I_1}), \ldots, d^{|I_k|} f_k(x, y_{I_k})) \tag{6}$$

for $x \in U$ and $y_1, \ldots, y_r \in X$, where $(I_1, \ldots, I_k)$ ranges through k-tuples of (possibly empty) disjoint sets $I_1, \ldots, I_k$ with $I_1 \cup \cdots \cup I_k = \{1, \ldots, r\}$, and the following notation is used: For $\nu \in \{1, \ldots, k\}$, we let $|I_\nu| \in \mathbb{N}_0$ be the cardinality of I_ν and define $y_{I_\nu} := (y_{i_1}, \ldots, y_{i_m}) \in X^m$ if $i_1 < i_2 < \cdots < i_m$ are the elements of I_ν, abbreviating $m := |I_j|$ (if I_ν is empty, the symbol $y_\emptyset$ is to be ignored). Holding $y_1, \ldots, y_r$ fixed, we can apply the case $r = 1$ to the function $d^r f(\cdot, y_1, \ldots, y_r)$ and find that, for each $x \in U$ and $y_{r+1} \in X$, the directional derivative at x in the direction y_{r+1} exists and is given by

$$\begin{aligned} d^{r+1}(\beta \circ f)(x, y_1, \ldots, y_{r+1}) \;=\; & \sum_{(I_1, \ldots, I_r)} \sum_{\nu=1}^{k} \beta\big(d^{|I_1|} f_1(x, y_{I_1}), \ldots, d^{|I_{\nu-1}|} f_{\nu-1}(x, y_{I_{\nu-1}}), \\ & d^{|I_\nu|+1} f_\nu(x, y_{I_\nu}, y_{r+1}), d^{|I_{\nu+1}|} f_{\nu+1}(x, y_{I_{\nu+1}}), \ldots, d^{|I_k|} f_k(x, y_{I_k})\big). \end{aligned}$$

Thus, also $d^{r+1}(\beta \circ f)$ is of the form (6), with $r+1$ in place of r. Using Lemma 8, we deduce from the preceding formula that the map

$$U \times E \to F, \quad (x,y) \mapsto d^{r+1}(\beta \circ f)(x, y, \ldots, y)$$

is continuous. Thus, $d^{r+1}(\beta \circ f)$ is continuous, by Lemma 2, and thus $\beta \circ f$ is $C^{r+1}_{\mathbb{K}}$.

If $(r, \mathbb{K}) = (\infty, \mathbb{R})$, then $\beta \circ f$ is $C^s_{\mathbb{R}}$ for each $s \in \mathbb{N}_0$ and hence $C^\infty_{\mathbb{R}}$ (still assuming (b)).

If $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and X is only assumed $k_{\mathbb{R}}$, then $\beta \circ f$ is continuous by the case $r = 0$. Moreover, the restriction $\beta \circ f|_{U \cap Y}$ is $C^\infty_{\mathbb{C}}$ for each finite-dimensional vector subspace $Y \subseteq X$, by case (a). Hence, f is $C^\omega_{\mathbb{C}}$ (and thus $C^\infty_{\mathbb{C}}$) as a mapping to a completion of F (see [18] (Theorem 6.2)). Then, f is also $C^\infty_{\mathbb{C}}$ as a map to F, as all of its iterated directional derivatives are in F.

Both in (a) and (b), it remains to consider the case $(r, \mathbb{K}) = (\omega, \mathbb{R})$. Then, f admits a $\mathbb{C}$-analytic extension $\widetilde{f} \colon \widetilde{U} \to (E_1)_{\mathbb{C}} \times \cdots \times (E_k)_{\mathbb{C}}$, defined on an open neighbourhood $\widetilde{U}$ of U in $X_{\mathbb{C}}$. The complex k-linear extension $\beta_{\mathbb{C}} \colon (E_1)_{\mathbb{C}} \times \cdots \times (E_k)_{\mathbb{C}} \to F_{\mathbb{C}}$ of β is given by

$$z \mapsto \sum_{a_1, \ldots, a_k = 0}^{1} i^{a_1 + \cdots + a_k} \beta(x_{1,a_1}, \ldots, x_{k,a_k})$$

for $z = (x_{1,0} + i x_{1,1}, \ldots, x_{k,0} + i x_{k,1})$ with $x_{\nu,0} \in E_\nu$ and $x_{\nu,1} \in E_\nu$ for $\nu \in \{1, \ldots, k\}$. By the latter formula, $\beta_{\mathbb{C}}$ is sequentially continuous in the situation of (a), and c-hypocontinuous in its arguments $(j, \ldots, k)$ in the situation of (b). The case $(\infty, \mathbb{C})$ shows that $\beta_{\mathbb{C}} \circ \widetilde{f}$ is complex analytic. As this mapping extends $\beta \circ f$, the latter map is real analytic. In case (b), we used here that $X_{\mathbb{C}} \cong X \times X$ is a $k_{\mathbb{R}}$-space. $\square$

Moreover, the following variant will be useful.

Lemma 12. *Let X_1, X_2, E_1, E_2 and F be locally convex $\mathbb{K}$-vector spaces, and $U_1 \subseteq X_1$, $U_2 \subseteq X_2$ be open subsets. Let $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ and $\beta\colon E_1 \times E_2 \to F$ be a $\mathbb{K}$-bilinear map. Assume that X_1 is finite-dimensional and β is c-hypocontinuous in its first variable. Then, for all $C^r_{\mathbb{K}}$-maps $f_1\colon U_1 \to E_1$ and $f_2\colon U_1 \times U_2 \to E_2$, also the following map is $C^r_{\mathbb{K}}$:*

$$g\colon U_1 \times U_2 \to F\,, \quad (x_1, x_2) \mapsto \beta(f_1(x_1), f_2(x_1, x_2))\,.$$

Proof. We first prove the assertion for $r \in \mathbb{N}_0$ (from which the case $r = \infty$ follows). If $r = 0$, we have to show that g is continuous. If $(x_1, x_2) \in U_1 \times U_2$, then x_1 has a compact neighbourhood $W = W_{x_1}$ in U_1. Then, $f_1(W)$ is compact, and thus $\beta|_{f_1(W)\times E_2}$ is continuous, by c-hypocontinuity. Hence, $g|_{W\times U_2} = \beta|_{f_1(W)\times E_2} \circ (f_1 \circ \pi_W, f_2)$ is continuous, where $\pi_W\colon W \times U_2 \to W$ is the projection onto the first factor. Since $(W^0_{x_1} \times U_2)_{x_1\in U_1}$ is an open cover of $U_1 \times U_2$, the map g is continuous.

Since β is sequentially continuous by Lemma 5, we see as in the preceding proof that the directional derivative $dg(x, y)$ exists for all $x = (x_1, x_2) \in U_1 \times U_2$ and $y = (y_1, y_2) \in X_1 \times X_2$, and is given by

$$dg(x, y) = \beta(df_1(x_1, y_1), f_2(x)) + \beta(f_1(x_1), df_2(x, y))\,. \tag{7}$$

Note that $(x_1, y_1) \mapsto f_1(x_1)$ and df_1 are $C^{r-1}_{\mathbb{K}}$-mappings $U_1 \times X_1 \to E_1$. Moreover, $((x_1, y_1),(x_2, y_2)) \mapsto f_2(x_1, x_2)$ and $((x_1, y_1),(x_2, y_2)) \mapsto df_2((x_1, x_2),(y_1, y_2))$ are $C^{r-1}_{\mathbb{K}}$-maps $(U_1 \times X_1) \times (U_2 \times X_2) \to E_2$ (cf. Remark 1). By induction, the right-hand side of (7) is a $C^{r-1}_{\mathbb{K}}$-map. Hence, g is $C^r_{\mathbb{K}}$.

The case $(r, \mathbb{K}) = (\omega, \mathbb{R})$ follows from the case $(\infty, \mathbb{C})$ as in the preceding proof. $\square$

Remark 5. *In a setting of differential calculus in which continuity on products is replaced with k-continuity (as championed by E. G. F. Thomas), every bilinear map β which is c-hypocontinuous in the second factor is smooth (see [28] (Theorem 4.1)); smoothness of $\beta \circ f$ for a smooth map f then follows from the Chain Rule (cf. also [29]). Likewise, β is smooth in the sense of convenient differential calculus.*

5. Differentiability Properties of $f^\wedge$

For $k = 1$, the following result is essential for our constructions of vector bundles.

Proposition 2. *Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $k \in \mathbb{N}$, $E_1, \ldots, E_k$ and F be locally convex $\mathbb{L}$-vector spaces, X be a locally convex $\mathbb{K}$-vector space, and $U \subseteq X$ be an open subset. Then, the following holds.*

(a) *If $(X \times E_1 \times \cdots \times E_k) \times (X \times E_1 \times \cdots \times E_k)$ is a $k_{\mathbb{R}}$-space, or $r = 0$ and $X \times E_1 \times \cdots \times E_k$ is a $k_{\mathbb{R}}$-space, or $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and $X \times E_1 \times \cdots \times E_k$ is a $k_{\mathbb{R}}$-space, or all of the vector spaces $E_1, \ldots, E_k$ are finite dimensional, then*

$$f^\wedge\colon U \times E_1 \times \cdots \times E_k \to F\,, \quad (x, y_1, \ldots, y_k) \mapsto f(x)(y_1, \ldots, y_k)$$

is $C^r_{\mathbb{K}}$ for each $C^r_{\mathbb{K}}$-map $f\colon U \to L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c$.

(b) *If $E := E_1 = E_2 = \cdots = E_k$ holds and, moreover, $(X \times E) \times (X \times E)$ is a $k_{\mathbb{R}}$-space or $r = 0$ and $X \times E$ is a $k_{\mathbb{R}}$-space, or $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and $X \times E$ is a $k_{\mathbb{R}}$-space, then*

$$f^\wedge\colon U \times E^k \to F\,, \quad (x, y_1, \ldots, y_k) \mapsto f(x)(y_1, \ldots, y_k)$$

is $C^r_{\mathbb{K}}$ for each $C^r_{\mathbb{K}}$-map $f\colon U \to L^k_{\mathbb{L}}(E, F)_c$ such that $f(x)$ is a symmetric k-linear map for each $x \in U$.

(c) *If X is finite-dimensional, $k = 1$, and $E := E_1$ is barrelled, then $f^\wedge\colon U \times E \to F$, $(x, y) \mapsto f(x)(y)$ is $C^r_{\mathbb{K}}$ for each $C^r_{\mathbb{K}}$-map $f\colon U \to L_{\mathbb{L}}(E, F)_c$.*

(d) *If all of the spaces $E_1, \ldots, E_k$ are normable, then $f^\wedge\colon U \times E_1 \times \cdots \times E_k \to F$ is $C^r_{\mathbb{K}}$ for each $C^r_{\mathbb{K}}$-map $f\colon U \to L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_b$.*

Proof. Let $\mathrm{ev}\colon L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c \times E_1 \times \cdots \times E_k \to F$ be the evaluation map, which is c-hypocontinuous in its arguments $(2, \ldots, k+1)$ by Lemma 7.

(a) Assuming the respective $k_{\mathbb{R}}$-property, the map $f^\wedge = \mathrm{ev} \circ (f \times \mathrm{id}_{E_1 \times \cdots \times E_k})$ is $C^r_{\mathbb{K}}$, by Lemma 11 (b). If $E_1, \ldots, E_k$ are finite-dimensional, then $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c$ equals $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_b$, whence the conclusion of (a) is a special case of (d).

(b) By Lemma 11 (b), the map

$$g\colon U \times E \to F, \quad (x, y) \mapsto f^\wedge(x, y, \ldots, y)$$

is $C^r_{\mathbb{K}}$, as $g = \mathrm{ev} \circ (f \times \delta)$ with $\delta\colon E \to E^k$, $y \mapsto (y, \ldots, y)$, which is continuous $\mathbb{K}$-linear. Then, also $f^\wedge$ is $C^r_{\mathbb{K}}$, by Lemma 2.

(c) The bilinear map $\mathrm{ev}\colon L_{\mathbb{K}}(E, F)_c \times E \to F$ is c-hypocontinuous in its first argument, by Lemma 7. Hence, $f^\wedge = \mathrm{ev} \circ (f \times \mathrm{id}_E)$ is $C^r_{\mathbb{K}}$, by Lemma 12.

(d) If $E_1, \ldots, E_k$ are normable, then the evaluation map

$$\varepsilon\colon L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_b \times E_1 \times \cdots \times E_k \to F$$

is continuous $(k+1)$-linear and hence $C^r_{\mathbb{K}}$, whence also $f^\wedge = \varepsilon \circ (f \times \mathrm{id}_{E_1 \times \cdots \times E_k})$ is $C^r_{\mathbb{K}}$. □

Remark 6. *If X and all of $E_1, \ldots, E_k$ are metrisable, then the topological space $(X \times E_1 \times \cdots \times E_k) \times (X \times E_1 \times \cdots \times E_k)$ is metrisable and hence a k-space. If X and all of $E_1, \ldots, E_k$ are k_ω-spaces, then also $(X \times E_1 \times \cdots \times E_k) \times (X \times E_1 \times \cdots \times E_k)$ is a k_ω-space and hence a k-space. In either case, we are in the situation of (a).*

6. Infinite-Dimensional Vector Bundles

In this section, we provide foundational material concerning vector bundles modelled on locally convex spaces (cf. also [13] (Chapter 3)). Notably, we discuss the description of vector bundles via cocycles, and define equivariant vector bundles.

Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$. The word "manifold" always refers to a manifold modelled on a locally convex space. Likewise, the Lie groups that we consider need not have finite dimension.

Definition 5. *Let M be a $C^r_{\mathbb{K}}$-manifold and F be a locally convex $\mathbb{L}$-vector space. An* $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M, with typical fibre F, *is a $C^r_{\mathbb{K}}$-manifold E, together with a surjective $C^r_{\mathbb{K}}$-map $\pi\colon E \to M$ and endowed with an $\mathbb{L}$-vector space structure on each fibre $E_x := \pi^{-1}(\{x\})$, such that, for each $x \in M$, there exists an open neighbourhood $U \subseteq M$ of x and a $C^r_{\mathbb{K}}$-diffeomorphism*

$$\psi\colon \pi^{-1}(U) \to U \times F$$

(called a "local trivialisation") such that $\psi(E_y) = \{y\} \times F$ for each $y \in U$ and the map $\mathrm{pr}_F \circ \psi|_{E_y}\colon E_y \to F$ is $\mathbb{L}$-linear (and hence an isomorphism of topological vector spaces, if we give E_y the topology induced by E), where $\mathrm{pr}_F\colon U \times F \to F$ is the projection.

In the situation of Definition 5, let $(\psi_i)_{i \in I}$ be an atlas of local trivialisations for E, *i.e.*, a family of local trivialisations

$$\psi_i\colon \pi^{-1}(U_i) \to U_i \times F$$

of E whose domains U_i cover M. Then, given $i, j \in I$, we have

$$\psi_i(\psi_j^{-1}(x, v)) = (x, g_{ij}(x)(v))$$

for $x \in U_i \cap U_j$, $v \in F$, for some function

$$g_{ij}\colon U_i \cap U_j \to \mathrm{GL}(F) \subseteq L(F).$$

Here,

$$G_{ij}\colon (U_i \cap U_j) \times F \to F, \quad (x, v) \mapsto g_{ij}(x)(v)$$

is $C^r_{\mathbb{K}}$, as $\psi_i(\psi_j^{-1}(x, v)) = (x, G_{ij}(x, v))$ is $C^r_{\mathbb{K}}$ in $(x, v) \in (U_i \cap U_j) \times F$. By Proposition 1, $g_{ij}\colon U_i \cap U_j \to L(F)_c$ is a $C^r_{\mathbb{K}}$-map, and as a map to $L(F)_b$, it is at least $C^{r-1}_{\mathbb{K}}$ (if $r \geq 1$). Note that the "transition maps" g_{ij} satisfy the "cocycle conditions"

$$\begin{cases} (\forall i \in I)\,(\forall x \in U_i) & g_{ii}(x) = \mathrm{id}_F \quad \text{and} \\ (\forall i, j, k \in I)\,(\forall x \in U_i \cap U_j \cap U_k) & g_{ij}(x) \circ g_{jk}(x) = g_{ik}(x)\,. \end{cases}$$

Proposition 3. *Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$. Assume that*

(a) *M is a $C^r_{\mathbb{K}}$-manifold modelled on a locally convex $\mathbb{K}$-vector space Z;*
(b) *E is a set and $\pi\colon E \to M$ a surjective map;*
(c) *F is a locally convex $\mathbb{L}$-vector space;*
(d) *$(U_i)_{i \in I}$ is an open cover of M;*
(e) *$(\psi_i)_{i \in I}$ is a family of bijections $\pi^{-1}(U_i) \to U_i \times F$ such that $\psi_i(\pi^{-1}(\{x\})) = \{x\} \times F$ for all $x \in U_i$;*
(f) *$g_{ij}(x)(v) := \mathrm{pr}_F(\psi_i(\psi_j^{-1}(x, v)))$ depends $\mathbb{L}$-linearly on $v \in F$, for all $i, j \in I$, $x \in U_i \cap U_j$;*
(g) *$G_{ij}\colon (U_i \cap U_j) \times F \to F$, $G_{ij}(x, v) := g_{ij}(x)(v)$ is a $C^r_{\mathbb{K}}$-map.*

Then, there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_{\mathbb{K}}$ on E making ψ_i a local trivialisation for each $i \in I$.

Proof. For $i, j \in I$, let $\mathrm{pr}_{ij}\colon (U_i \cap U_j) \times F \to U_i \cap U_j$ be the projection onto the first component. As the maps

$$\psi_i \circ \psi_j^{-1}|_{(U_i \cap U_j) \times F} = (\mathrm{pr}_{ij}, G_{ij})$$

are $C^r_{\mathbb{K}}$, there is a uniquely determined $C^r_{\mathbb{K}}$-manifold structure on E making ψ_i a $C^r_{\mathbb{K}}$-diffeomorphism for each $i \in I$. Given $x \in M$, we pick $i \in I$ with $x \in U_i$; we give $E_x := \pi^{-1}(\{x\})$ the unique $\mathbb{L}$-vector space structure making the bijection $\mathrm{pr}_F \circ \psi_i|_{E_x}\colon E_x \to F$ an isomorphism of vector spaces. It is easy to see that the vector space structure on E_x is independent of the choice of ψ_i, and it is easily verified that we have turned E into an $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ with the asserted properties. □

Remark 7. *Let M be a $C^r_{\mathbb{K}}$-manifold, F be a locally convex $\mathbb{L}$-vector space, $(U_i)_{i \in I}$ be an open cover of M, and $(g_{ij})_{i,j \in I}$ be a family of maps $g_{ij}\colon U_i \cap U_j \to \mathrm{GL}(F)$ satisfying the cocycle conditions and such that*

$$G_{ij}\colon (U_i \cap U_j) \times F \to F\,, \quad (x, v) \mapsto g_{ij}(x)(v)$$

is $C^r_{\mathbb{K}}$, for all $i, j \in I$. Using Proposition 3, the usual construction familiar from the finite-dimensional case provides an $\mathbb{L}$-vector bundle $\pi\colon E \to M$ of class $C^r_{\mathbb{K}}$, with typical fibre F, and a family $(\psi_i)_{i \in I}$ of local trivialisations $\pi^{-1}(U_i) \to U_i \times F$, whose associated transition maps are the given g_{ij}'s. The bundle E is unique up to canonical isomorphism.

Combining Proposition 3 and Proposition 2, we obtain:

Corollary 2. *Retaining the hypotheses* (a)–(f) *from Proposition* 3 *but omitting* (g)*, consider the following conditions:*

(g)′ *$g_{ij}(x) \in L(F)$ for all $i, j \in I$, $x \in U_i \cap U_j$, and $g_{ij}\colon U_i \cap U_j \to L(F)_c$ is $C^r_{\mathbb{K}}$;*
(g)″ *$g_{ij}(x) \in L(F)$ for all $i, j \in I$, $x \in U_i \cap U_j$, and $g_{ij}\colon U_i \cap U_j \to L(F)_b$ is $C^r_{\mathbb{K}}$;*
(i) *$(Z \times F) \times (Z \times F)$ is a $k_{\mathbb{R}}$-space, or $r = 0$ and $Z \times F$ is a $k_{\mathbb{R}}$-space, or $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and $Z \times F$ is a $k_{\mathbb{R}}$-space;*

(ii) $\dim(M) < \infty$ *and F is barrelled;*
(iii) *F is normable.*

If (g)$'$ *holds as well as* (i) *or* (ii), *then the conclusions of Proposition* 3 *remain valid. They also remain valid if* (g)$''$ *and* (iii) *hold.*

Example 2 below shows that Conditions (a)–(f) and (g)$'$ alone are not sufficient for the conclusion of Proposition 3, without extra conditions on Z and F. Note that (i) is satisfied if both Z and F are metrisable, or both Z and F are k_ω-spaces.

Equivariant vector bundles. Beyond vector bundles, we shall discuss *equivariant* vector bundles in the following, i.e., vector bundles together with an action of a (finite- or infinite-dimensional) Lie group G. Choosing $G = \{e\}$ as a trivial group, we obtain results about ordinary vector bundles (without a group action), as a special case.

For the remainder of this section, and also in Section 7, let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $s \in \{\infty, \omega\}$, and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ with $r \leq s$. Let G be a $C^s_\mathbb{K}$-Lie group (modelled on a locally convex $\mathbb{K}$-vector space Y) and M be a $C^r_\mathbb{K}$-manifold. We assume that a $C^r_\mathbb{K}$-action

$$\alpha\colon G \times M \to M$$

is given. Then, (M, α) is called a *G-manifold of class* $C^r_\mathbb{K}$.

Definition 6. *An* equivariant $\mathbb{L}$-vector bundle of class $C^r_\mathbb{K}$ *over a G-manifold* (M, α) *of class* $C^r_\mathbb{K}$ *is an* $\mathbb{L}$*-vector bundle* $\pi\colon E \to M$ *of class* $C^r_\mathbb{K}$*, together with a* $C^r_\mathbb{K}$*-action*

$$\beta\colon G \times E \to E$$

such that $\beta(g, E_x) \subseteq E_{\alpha(g,x)}$ *for all* $(g, x) \in G \times M$*, and* $\beta(g, \cdot)|_{E_x}\colon E_x \to E_{\alpha(g,x)}$ *is* $\mathbb{L}$*-linear.*

In other words, $\beta(g, \cdot)$ takes fibres linearly to fibres and coincides with $\alpha(g, \cdot)$ on the zero section. The mapping π is then equivariant in the sense that $\alpha \circ (\mathrm{id}_G \times \pi) = \pi \circ \beta$.

Example 1. *If M is a G-manifold of class* $C^r_\mathbb{K}$*, with* $r \geq 1$*, then the tangent bundle TM is an equivariant* $\mathbb{L}$*-vector bundle of class* $C^{r-1}_\mathbb{K}$ *in a natural way, with* $\mathbb{L} := \mathbb{K}$*. In fact, the action* $\alpha\colon G \times M \to M$ *has a tangent map* $T\alpha\colon T(G \times M) \to TM$*, which is* $C^{r-1}_\mathbb{K}$*. Let* $0_G\colon G \to TG$ *be the* 0*-section. Identifying* $T(G \times M)$ *with* $TG \times TM$ *in the usual way, we obtain a* $C^{r-1}_\mathbb{K}$*-map* $\beta\colon G \times TM \to TM$ *via*

$$\beta := (T\alpha) \circ (0_G \times \mathrm{id}_{TM})\,.$$

It is easy to see that $\beta(g, v) = T_x(\alpha(g, \cdot))(v) \in T_{\alpha(g,x)}M$ *for* $g \in G$ *and* $v \in T_xM$*, whence* $\beta(g, T_xM) \subseteq T_{\alpha(g,x)}M$ *and* $\beta(g, \cdot)|_{T_xM} = T_x(\alpha(g, \cdot))$*. Clearly, β is an action making TM an equivariant* $\mathbb{K}$*-vector bundle of class* $C^{r-1}_\mathbb{K}$ *over the G-manifold M.*

Induced action on an invariant subbundle. Given an $\mathbb{L}$-vector bundle $\pi : E \to M$ of class $C^r_\mathbb{K}$, with typical fibre F, we call a subset $E_0 \subseteq E$ a *subbundle* if there exists a sequentially closed $\mathbb{L}$-vector subspace $F_0 \subseteq F$ such that for each $x \in M$ there exists a local trivialisation $\psi : \pi^{-1}(U) \to U \times F$ of E such that $\psi(E_0 \cap \pi^{-1}(U)) = U \times F_0$. It readily follows from [13] (Lemma 1.3.19 and Exercise 2.2.4) that there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_\mathbb{K}$ on $\pi|_{E_0} : E_0 \to M$ making $\psi|_{\pi^{-1}(U) \cap E_0} : \pi^{-1}(U) \cap E_0 \to U \times F_0$ a local trivialisation of E_0, for each local trivialisation ψ as before. Then, the inclusion map $E_0 \to E$ is $C^r_\mathbb{K}$, and a mapping $N \to E$ from a $C^r_\mathbb{K}$-manifold N to E with image in E_0 is $C^r_\mathbb{K}$ as a mapping to E if and only if its co-restriction to E_0 is $C^r_\mathbb{K}$, by the facts just cited. In the preceding situation, suppose that a $C^s_\mathbb{K}$-Lie group G acts $C^s_\mathbb{K}$ on M and E is an equivariant vector bundle of class $C^r_\mathbb{K}$ with respect to the action $\beta\colon G \times E \to E$. If E_0 is invariant under the G-action, i.e., if $\beta(G \times E_0) \subseteq E_0$, as a special case of the preceding observations, we deduce from the $C^r_\mathbb{K}$-property of β that $\beta|_{G \times E_0}$ and thus also $\beta|_{G \times E_0}\colon G \times E_0 \to E_0$ is $C^r_\mathbb{K}$. We can summarise as follows.

Proposition 4. *If E is an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over a G-manifold M, then the action induced on any G-invariant subbundle E_0 is $C^r_{\mathbb{K}}$ and thus makes the latter an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$.*

7. Completions of Vector Bundles

Let $\pi\colon E \to M$ be an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$, as in Definition 6, with typical fibre F and G-actions $\alpha\colon G \times M \to M$ and $\beta\colon G \times E \to E$. Assume that $r \geq 1$. Our goal is to complete the fibre of the bundle, i.e., to find a G-equivariant vector bundle $\widetilde{E}$ whose typical fibre is a completion of the locally convex space F, and which contains E as a dense subset.

Let $\widetilde{F}$ be a completion of F such that $F \subseteq \widetilde{F}$ and, for each $x \in M$, let $\widetilde{E}_x$ be a completion of E_x such that $E_x \subseteq \widetilde{E}_x$. We may assume that the sets $\widetilde{E}_x$ are pairwise disjoint for $x \in M$. Consider the (disjoint) union

$$\widetilde{E} := \bigcup_{x \in M} \widetilde{E}_x\,. \tag{8}$$

We shall turn $\widetilde{E}$ into an equivariant vector bundle. Consider the map $\widetilde{\beta}\colon G \times \widetilde{E} \to \widetilde{E}$, defined using the continuous extension $(\beta(g,\cdot)|_{E_x})^{\sim}\colon \widetilde{E}_x \to \widetilde{E}_{\alpha(g,x)}$ of the linear map $\beta(g,\cdot)|_{E_x}\colon E_x \to E_{\alpha(g,x)}$ via

$$\widetilde{\beta}(g,v) := (\beta(g,\cdot)|_{E_x})^{\sim}(v)$$

for $g \in G$, $x \in M$, and $v \in \widetilde{E}_x$. It is clear that $\widetilde{\beta}$ makes $\widetilde{E}$ a G-set. Let

$$\widetilde{\pi}\colon \widetilde{E} \to M \tag{9}$$

be the map taking elements from $\widetilde{E}_x$ to x. Then, $\widetilde{\pi}$ is G-equivariant. If $\psi\colon \pi^{-1}(U) \to U \times F$ is a local trivialisation of E and $\mathrm{pr}_F\colon U \times F \to F$, $(x,y) \mapsto y$, we define

$$\widetilde{\psi}\colon \widetilde{\pi}^{-1}(U) \to U \times \widetilde{F}\,, \quad \widetilde{E}_x \ni v \mapsto (x, (\mathrm{pr}_F \circ \psi|_{E_x})^{\sim}(v))\,. \tag{10}$$

Then, the following holds:

Proposition 5. *$(\widetilde{E}, \widetilde{\beta})$ can be made an equivariant $\mathbb{L}$-vector bundle of class $C^{r-1}_{\mathbb{K}}$ over the G-manifold M, such that $\widetilde{\psi}$ is a local trivialisation of $\widetilde{E}$ for each local trivialisation ψ of E.*

Remark 8. *Omitting the hypothesis that $r \geq 1$, assume instead that E is an equivariant $\mathbb{L}$-vector bundle* of class *$LC^r_{\mathbb{K}}$. That is, both E and M are $LC^r_{\mathbb{K}}$-manifolds (each admitting an atlas with transition maps of class $LC^r_{\mathbb{K}}$), a family of local trivialisations can be chosen with $LC^r_{\mathbb{K}}$-transition maps, and the G-actions on E and M are $LC^r_{\mathbb{K}}$. Then, also $\widetilde{E}$ is an equivariant vector bundle of class $LC^r_{\mathbb{K}}$ (and hence of class $C^r_{\mathbb{K}}$).*

Extension of differentiable maps to subsets of the completions. To enable the proof of Proposition 5, we need to discuss conditions ensuring that a C^r-map $f\colon E \supseteq U \to F$ (with locally convex spaces E and F) can be extended to a C^r-map $\widetilde{U} \to \widetilde{F}$ on an open subset of the completion $\widetilde{E}$ of E, or at least to a C^{r-1}-map. Although this is not possible in general, it is possible if F is normed and r is finite. This will be sufficient for our purposes. The natural framework for the discussion of the problem is not C^r-maps, but Lipschitz differentiable maps, as in Definition 4.

Proposition 6. *Let E be a locally convex $\mathbb{K}$-vector space, $(F, \|\cdot\|)$ be a Banach space over $\mathbb{K}$, $U \subseteq E$ be open and $f\colon U \to F$ be an $LC^r_{\mathbb{K}}$-map, where $r \in \mathbb{N}_0$. Let $\widetilde{E}$ be a completion of E such that $E \subseteq \widetilde{E}$. Then, f extends to an $LC^r_{\mathbb{K}}$-map $\widetilde{f}\colon \widetilde{U} \to F$ on an open subset $\widetilde{U} \subseteq \widetilde{E}$ which contains U as a dense subset.*

The following lemma enables an inductive proof of Proposition 6.

Lemma 13. *Let $k \in \mathbb{N}$, X be a locally convex $\mathbb{K}$-vector space, and $E_1, \ldots, E_k, F$ be locally convex $\mathbb{L}$-vector spaces, with completions $\widetilde{X}$, $\widetilde{E}_1, \ldots, \widetilde{E}_k$ and $\widetilde{F}$, respectively. Let $U \subseteq X$ be open and $f\colon U \times E_1 \times \cdots \times E_k \to F$ be a map such that $f^\vee(x) := f(x, \cdot)\colon E_1 \times \cdots \times E_k \to F$ is k-linear over $\mathbb{L}$ for each $x \in U$. Assume that there exists an $LC^r_{\mathbb{K}}$-map $h\colon W \to \widetilde{F}$ which extends f, defined on an open set $W \subseteq \widetilde{X} \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$ in which $U \times E_1 \times \cdots \times E_k$ is dense. Then, there exists an $LC^r_{\mathbb{K}}$-map*

$$\widetilde{f}\colon \widetilde{U} \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k \to \widetilde{F} \tag{11}$$

which extends f, for some open subset $\widetilde{U} \subseteq \widetilde{E}$ in which U is dense. The maps $(\widetilde{f})^\vee(x) := \widetilde{f}(x, \cdot)\colon \widetilde{E}_1 \times \cdots \times \widetilde{E}_k \to \widetilde{F}$ are k-linear over $\mathbb{L}$, for each $x \in \widetilde{U}$.

Proof. For each $x \in U$, there exists an open neighbourhood V_x of x in $\widetilde{X}$ and a balanced, open 0-neighbourhood $Q_x \subseteq \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$ such that $V_x \times Q_x \subseteq W$. After shrinking V_x, we may assume that $X \cap V_x = U$, whence $U \cap V_x = X \cap V_x$ is dense in V_x. Given $z \in \mathbb{L}$ such that $|z| \leq 1$, consider the map

$$V_x \times Q_x \to \widetilde{F}, \quad (y, v) \mapsto h(y, zv) - z^k h(y, v).$$

This map vanishes, because it is continuous and vanishes on the dense subset $(V_x \cap X) \times (Q_x \cap (E_1 \times \cdots \times E_k))$. As a consequence, we obtain a well-defined map

$$f_x\colon V_x \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k \to \widetilde{F}, \quad (y, v) \mapsto z^{-k} h(y, zv)$$

for $y \in V_x$, $v \in \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$ and $z \in \mathbb{L} \setminus \{0\}$ with $zv \in Q_x$. As $f_x(y, v) = z^{-k} h(y, zv)$ is $LC^r_{\mathbb{K}}$ in $(y, v) \in V_x \times z^{-1} Q_x$ and these sets form an open cover of $V_x \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$, we see that f_x is $LC^r_{\mathbb{K}}$. Given $x, y \in U$, the set $U \cap V_x \cap V_y = X \cap V_x \cap V_y$ is dense in the open set $V_x \cap V_y \subseteq \widetilde{X}$. Since f_x, f_y, and f coincide on the set $(U \cap V_x \cap V_y) \times E_1 \times \cdots \times E_k$, it follows that the continuous maps f_x and f_y coincide on the set $(V_x \cap V_y) \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$ in which the former set is dense. Hence, setting $\widetilde{U} := \bigcup_{x \in U} V_x$, a well-defined map $\widetilde{f}$ as in (11) is obtained if we set

$$\widetilde{f}(y, v) := f_x(y, v) \quad \text{if } x \in U, y \in V_x \text{ and } v \in \widetilde{E}_1 \times \cdots \times \widetilde{E}_k.$$

The final assertion follows by continuity from the k-linearity of the mappings $f^\vee(x)$ for $x \in U$. □

Proof of Proposition 6. We proceed by induction on $r \in \mathbb{N}_0$.

The case $r = 0$. Given $x \in U$, there exists a continuous seminorm q on E such that $B^q_1(x) \subseteq U$ and

$$\|f(z) - f(y)\| \leq q(z - y) \quad \text{for all } y, z \in B^q_1(x). \tag{12}$$

Then, $N_q := \{y \in E\colon q(y) = 0\}$ is a closed vector subspace of E and $\|y + N_q\|_q := q(y)$ for $y \in E$ defines a norm on $E_q := E/N_q$ making the map $\alpha_q\colon E \to E_q$, $y \mapsto y + N_q$ continuous linear. By (12), we have $\|f(z) - f(y)\| = 0$ for all $y, z \in B^q_1(x)$ such that $y - z \in N_q$. Hence,

$$h\colon \alpha_q(B^q_1(x)) \to F, \quad y + N_q \mapsto f(y)$$

is a well-defined map. Note that $\alpha_q(B^q_1(x))$ is the open ball $B := \{y \in E_q\colon \|y - \alpha_q(x)\|_q < 1\}$ in E_q. Let $\widetilde{E}_q$ be the completion of the normed space E_q; the extended norm will again be denoted by $\|.\|_q$. Applying (12) to representatives, we see that

$$\|h(z) - h(y)\| \leq \|z - y\|_q \quad \text{for all } y, z \in B.$$

Hence, h satisfies a global Lipschitz condition (with Lipschitz constant 1), and hence h is uniformly continuous, entailing that h extends uniquely to a uniformly continuous map

$$\widetilde{h}\colon \widetilde{B} \to F$$

on the corresponding open ball $\widetilde{B}$ in $\widetilde{E}_q$. Then, $\|\widetilde{h}(z)-\widetilde{h}(y)\| \leq \|z-y\|_q$ for all $y,z \in \widetilde{B}$, by continuity. Let $\widetilde{\alpha}_q\colon \widetilde{E} \to \widetilde{E}_q$ be the continuous extension of the continuous linear map α_q. Then, $V_x := (\widetilde{\alpha}_q)^{-1}(\widetilde{B})$ is an open neighbourhood of x in $\widetilde{E}$ such that $V_x \cap E = B_1^q(x) \subseteq U$. Moreover, $f_x := \widetilde{h} \circ \widetilde{\alpha}_q|_{V_x}$ is a continuous map extending $f|_{V_x \cap E}$, which furthermore satisfies

$$\|f_x(z)-f_x(y)\| \leq \widetilde{q}(z-y) \quad \text{for all } y,z \in V_x, \tag{13}$$

where we use the continuous seminorm $\widetilde{q} := \|.\|_q \circ \widetilde{\alpha}_q\colon \widetilde{E} \to [0,\infty[$ extending q. Then

$$\widetilde{U} := \bigcup_{x \in U} V_x$$

is an open subset of $\widetilde{E}$ and $E \cap \widetilde{U} = U$ is dense in $\widetilde{U}$. Given $x,y \in U$, the set $U \cap V_x \cap V_y = E \cap V_x \cap V_y$ is dense in the open set $V_x \cap V_y \subseteq \widetilde{E}$. Since

$$f_x|_{U \cap V_x \cap V_y} = f|_{U \cap V_x \cap V_y} = f_y|_{U \cap V_x \cap V_y},$$

it follows that $f_x|_{V_x \cap V_y} = f_y|_{V_x \cap V_y}$. Hence

$$\widetilde{f}\colon \widetilde{U} \to F, \quad z \mapsto f_x(z) \quad \text{for } x \in U \text{ such that } z \in V_x$$

is a well-defined map. Since $\widetilde{f}|_{V_x} = f_x$ is $LC^0_{\mathbb{K}}$ for each $x \in U$ (by (13)), the map $\widetilde{f}$ is $LC^0_{\mathbb{K}}$. Furthermore, $\widetilde{f}$ extends f by construction.

Induction step. If f is $LC^{r+1}_{\mathbb{K}}$, then f extends to an $LC^0_{\mathbb{K}}$-map $\widetilde{f}\colon \widetilde{U} \to F$ on an open subset $\widetilde{U} \subseteq \widetilde{E}$ such that $\widetilde{U} \cap E = U$, and $df\colon U \times E \to F$ extends to an $LC^r_{\mathbb{K}}$-map $h\colon W \to F$ on an open subset W of $\widetilde{E} \times \widetilde{E}$, by induction. Using Lemma 13, we find an open neighbourhood V of U in $\widetilde{E}$ and an $LC^r_{\mathbb{K}}$-map $g\colon V \times \widetilde{E} \to F$ which extends df. After replacing $\widetilde{U}$ and V with their intersection, we may assume that $\widetilde{U} = V$. If $x_0 \in \widetilde{U}$ and $y_0 \in \widetilde{E}$, there exist open neighbourhoods Q of x_0 and P of y_0 in $\widetilde{E}$, and $\varepsilon > 0$ such that $Q + \mathbb{D}_\varepsilon P \subseteq \widetilde{U}$. Then, the map

$$\ell\colon Q \times P \times \mathbb{D}_\varepsilon \to F, \quad (x,y,t) \mapsto \int_0^1 g(x+sty,y)\,ds$$

is continuous, being given by a parameter-dependent weak integral with continuous integrand. For (x,y,t) in the dense subset $(Q \cap E) \times (P \cap E) \times (\mathbb{D}_\varepsilon \setminus \{0\})$ of the set $Q \times P \times (\mathbb{D}_\varepsilon \setminus \{0\})$, the Mean Value Theorem implies that

$$\ell(x,y,t) = \frac{f(x+ty)-f(x)}{t} = \frac{\widetilde{f}(x+ty)-\widetilde{f}(x)}{t}.$$

Then, $\ell(x,y,t) = \frac{\widetilde{f}(x+ty)-\widetilde{f}(x)}{t}$ for all $(x,y,t) \in Q \times P \times (\mathbb{D}_\varepsilon \setminus \{0\})$, by continuity. Thus,

$$\frac{f(x_0+ty_0)-f(x_0)}{t} = \ell(x_0,y_0,t) \to \ell(x_0,y_0,0) = g(x_0,y_0)$$

as $t \to 0$. Hence, $d\widetilde{f}(x_0,y_0) = g(x_0,y_0)$. Since g is $LC^r_{\mathbb{K}}$, it follows that $\widetilde{f}$ is $LC^{r+1}_{\mathbb{K}}$. □

The conclusion of Proposition 6 becomes false in general if the Banach space F is replaced by a complete locally convex space. In fact, there exists a smooth map $E \to (\ell^1)^\Omega$ from a proper, dense vector subspace E of ℓ^1 to a suitable power of ℓ^1, which has no continuous extension to $E \cup \{x\}$ for any $x \in \ell^1 \setminus E$ (see Appendix B). Nonetheless, we have the following result.

Proposition 7. *Let $k \in \mathbb{N}$, X be a locally convex $\mathbb{K}$-vector space, and $E_1,\ldots,E_k,F$ be locally convex $\mathbb{L}$-vector spaces, with completions $\widetilde{X}$, $\widetilde{E}_1,\ldots,\widetilde{E}_k$ and $\widetilde{F}$, respectively. Let $U \subseteq X$ be open and $f\colon U \times E_1 \times \cdots \times E_k \to F$ be a mapping such that $f^\vee(x) := f(x,\cdot)\colon E_1 \times \cdots \times E_k \to F$*

is k-linear over $\mathbb{L}$ for each $x \in U$. If f is $LC^r_{\mathbb{K}}$ for some $r \in \mathbb{N}_0 \cup \{\infty\}$ (resp., $C^r_{\mathbb{K}}$ for some $r \in \mathbb{N} \cup \{\infty, \omega\}$), then there exists a unique map

$$\widetilde{f} \colon U \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k \to \widetilde{F} \tag{14}$$

which is $LC^r_{\mathbb{K}}$ (resp., $C^{r-1}_{\mathbb{K}}$) and extends f. The maps $\widetilde{f}^\vee(x) := \widetilde{f}(x, \cdot) \colon \widetilde{E}_1 \times \cdots \times \widetilde{E}_k \to \widetilde{F}$ are k-linear over $\mathbb{L}$, for each $x \in U$.

Proof. Abbreviate $E := E_1 \times \cdots \times E_k$ and $\widetilde{E} := \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$. Assume first that $r \neq \omega$. Since $LC^r_{\mathbb{K}}$-maps are continuous and $U \times E$ is dense in $U \times \widetilde{E}$, there is at most one map $\widetilde{f}$ with the asserted properties. We may therefore assume that $r \in \mathbb{N}_0$. We may also assume that F is complete. Then, $F = \varprojlim F_j$ for some projective system $((F_j)_{j \in J}, (p_{ij})_{i \leq j})$ of Banach spaces F_j and continuous linear maps $p_{ij} \colon F_j \to F_i$, with limit maps $p_j \colon F \to F_j$. We claim that $p_j \circ f \colon U \times E \to F_j$ has an $LC^r_{\mathbb{K}}$-extension $g_j := (p_j \circ f)^\sim \colon U \times \widetilde{E} \to F_j$, for each $j \in J$. If this is true, then $p_{ij} \circ g_j = g_i$ for $i \leq j$, by uniqueness of continuous extensions. Hence, by the universal property of the projective limit, there exists a unique map $\widetilde{f} \colon U \times \widetilde{E} \to F$ such that $p_j \circ \widetilde{f} = g_j$. Then, $p_j \circ \widetilde{f}|_{U \times E} = g_j|_{U \times E} = p_j \circ f$ and hence $\widetilde{f}|_{U \times E} = f$. Furthermore, $\widetilde{f}$ is $LC^r_{\mathbb{K}}$, by Lemma 9 (d). To prove the claim, note that Proposition 6 yields an $LC^r_{\mathbb{K}}$-extension $h_j \colon W_j \to F_j$ of $p_j \circ f$ to an open subset $W_j \subseteq \widetilde{X} \times \widetilde{E}$, which contains $U \times E$ as a dense subset. Now, Lemma 13 yields an open subset $U_j \subseteq \widetilde{X}$ in which U is dense, and an $LC^r_{\mathbb{K}}$-extension $e_j \colon U_j \times \widetilde{E} \to F_j$ of $p_j \circ f$. Then, $g_j := e_j|_{U \times \widetilde{E}}$ is as desired.

We now consider the case $(r, \mathbb{K}) = (\omega, \mathbb{R})$. If $\mathbb{L} = \mathbb{C}$, by the density of $U \times E$ in $U \times \widetilde{E}$, for any real analytic extension $\widetilde{f} \colon U \times \widetilde{E} \to \widetilde{F}$ and $x \in U$, the map $\widetilde{f}(x, \cdot)$ will be k-linear over $\mathbb{L}$. We may therefore assume that $\mathbb{L} = \mathbb{R}$. Let $h \colon W \to F_{\mathbb{C}}$ be a $\mathbb{C}$-analytic extension of f, defined on an open subset $W \subseteq X_{\mathbb{C}} \times E_{\mathbb{C}}$ such that $U \times E \subseteq W$. For each $x \in U$, there exist an open x-neighbourhood $U_x \subseteq U$ and balanced open 0-neighbourhoods $V_x \subseteq X$ and $W_x \subseteq E_{\mathbb{C}}$ such that $(U_x + iV_x) \times W_x \subseteq W$. We claim that there exists a $\mathbb{C}$-analytic map $g_x \colon (U_x + iV_x) \times E_{\mathbb{C}} \to F_{\mathbb{C}}$ such that $g_x|_{U_x \times E} = f|_{U_x \times E}$. For $x, y \in U$, the intersection $((U_x + iV_x) \times E_{\mathbb{C}}) \cap ((U_y + iV_y) \times E_{\mathbb{C}}) = ((U_x \cap U_y) + i(V_x \cap V_y)) \times E_{\mathbb{C}}$ is connected and meets $U \times E$ whenever it is non-empty. Hence, by the Identity Theorem, g_x and g_y coincide on the intersection of their domains. We therefore obtain a well-defined $\mathbb{C}$-analytic map $g \colon Q \times E_{\mathbb{C}} \to F_{\mathbb{C}}$ such that $g|_{(U_x + iV_x) \times E_{\mathbb{C}}} = g_x$ for each $x \in U$, using the open subset $Q := \bigcup_{x \in U}(U_j + iV_j)$ of $X_{\mathbb{C}}$. For each $x \in U$, the map $g(x, \cdot)|_E = g_x(x, \cdot)|_E = f(x, \cdot)$ is k-linear over $\mathbb{R}$. Using the Identity Theorem, we see that $g(x, \cdot)$ is k-linear over $\mathbb{C}$ for each $x \in U$, and hence for each $x \in Q$ by the Identity Theorem. By the case $(\infty, \mathbb{C})$, g has a $\mathbb{C}$-analytic extension $\widetilde{g} \colon Q \times \widetilde{E}_{\mathbb{C}} \to \widetilde{F}_{\mathbb{C}}$. Since $g(U \times E) = f(U \times E) \subseteq F \subseteq \widetilde{F}$ and $U \times E$ is dense in $U \times \widetilde{E}$, we deduce that $\widetilde{g}(U \times \widetilde{E}) \subseteq \widetilde{F}$; we therefore obtain a map

$$\widetilde{f} \colon U \times \widetilde{E} \to \widetilde{F}, \quad (x, y) \mapsto \widetilde{g}(x, y)$$

for $x \in U$, $y \in \widetilde{E}$. Since $\widetilde{g}$ is a $\mathbb{C}$-analytic extension for $\widetilde{f}$, the function $\widetilde{f}$ is $\mathbb{R}$-analytic. To prove the claim, consider for $x \in U$ and $n \in \mathbb{N}$ the $\mathbb{C}$-analytic map

$$g_{x,n} \colon (U_x + iV_x) \times nW_x \to F_{\mathbb{C}}, \quad (z, y) \mapsto n^k h(z, (1/n)y).$$

If $n \leq m$ and $y \in nW_x \cap E$, we have for all $z \in U_x$

$$g_{x,m}(z, y) = m^k h(z, (1/m)y) = m^k f(z, (1/m)y) = f(z, y) = n^k f(t, (1/n)y) = g_{x,n}(z, y),$$

whence $g_{x,m}(z, y) = g_{x,n}(z, y)$ for all $z \in U_x + iV_x$ and $y \in nW_x$, by the Identity Theorem. Thus, $g_x \colon (U_x + iV_x) \times E_{\mathbb{C}} \to F_{\mathbb{C}}$, $(z, y) \mapsto g_{x,n}(z, y)$ if $y \in nW_x$ is a well-defined $\mathbb{C}$-analytic extension of $f|_{U_x \times E}$. □

Proof of Proposition 5. It suffices to prove the strengthening described in Remark 8. Let $(\psi_i)_{i \in I}$ be a family of local trivialisations $\psi_i \colon \pi^{-1}(U_i) \to U_i \times F$ of an $LC^r_{\mathbb{K}}$-vector bundle E

such that each local trivialisation is some ψ_i. Let $(g_{ij})_{i,j\in I}$ be the corresponding cocycle and G_{ij} be the $LC^r_{\mathbb{K}}$-map $g_{ij}^{\wedge}\colon (U_i \cap U_j) \times F \to F$, which is $\mathbb{L}$-linear in the second argument. By Proposition 7, there is a unique $LC^r_{\mathbb{K}}$-map $\widetilde{G}_{ij}\colon U \times \widetilde{F} \to \widetilde{F}$ which extends G_{ij}, and $\widetilde{G}_{ij}$ is $\mathbb{L}$-linear in the second argument. Thus, we obtain a map

$$\widetilde{g}_{ij}\colon U_i \cap U_j \to L_{\mathbb{L}}(\widetilde{F}), \quad x \mapsto \widetilde{G}_{ij}(x,\cdot).$$

By continuity and density, for all $i \in I$, we have $\widetilde{G}_{ii}(x,y) = y$ for all $(x,y) \in U_i \times \widetilde{F}$. Thus, $\widetilde{g}_{ii}(x) = \mathrm{id}_{\widetilde{F}}$ for all $x \in U$. For all $i,j,k \in I$, we have

$$\widetilde{G}_{ij}(x, \widetilde{G}_{jk}(x,y)) = \widetilde{G}_{ik}(x,y) \quad \text{for all } (x,y) \in (U_i \cap U_j \cap U_k) \times \widetilde{F},$$

as both sides are continuous in (x,y) and equality holds for y in the dense subset F of $\widetilde{F}$; thus, $\widetilde{g}_{ij}(x) \circ \widetilde{g}_{jk}(x) = \widetilde{g}_{ik}(x)$. Notably, $\widetilde{g}_{ij}(x) \circ \widetilde{g}_{ji}(x) = \widetilde{g}_{ii}(x) = \mathrm{id}_{\widetilde{F}}$ for all $x \in U$ and $i,j \in I$, entailing that $\widetilde{g}_{ij}(x) \in \mathrm{GL}(\widetilde{F})$. By the preceding, the $\widetilde{g}_{ij}$ satisfy the cocycle conditions. Let $\widetilde{E}$ and $\widetilde{\pi}$ be as in (8) and (9); define $\widetilde{\psi}_i\colon \widetilde{\pi}^{-1}(U_i) \to U_i \times \widetilde{F}$ as in (10), replacing ψ with ψ_i. For all $i,j \in I$ and $x \in U$, we then have that

$$\widetilde{\psi}_i(\widetilde{\psi}_j^{-1}(x,y)) = (x, \widetilde{G}_{ij}(x,y))$$

holds for all $y \in \widetilde{F}$, as equality holds for all $y \in F$. As an analogue of Proposition 3 holds with $LC^r_{\mathbb{K}}$-maps in place of $C^r_{\mathbb{K}}$-maps, we get a unique $\mathbb{L}$-vector bundle structure of class $LC^r_{\mathbb{K}}$ on $\widetilde{E}$ making $\widetilde{\psi}_i$ a local trivialisation for each $i \in I$.

It is apparent that $\widetilde{\beta}\colon G \times \widetilde{E} \to \widetilde{E}$ is an action, and $\widetilde{E}_x$ is taken $\mathbb{L}$-linearly to $\widetilde{E}_{\alpha(g,x)}$ by $\widetilde{\beta}(g,\cdot)$, for each $g \in G$ and $x \in M$. It only remains to show that $\widetilde{\beta}$ is $LC^r_{\mathbb{K}}$. To this end, let $g_0 \in G$ and $x_0 \in M$; we show that $\widetilde{\beta}$ is $LC^r_{\mathbb{K}}$ on $U \times \widetilde{\pi}^{-1}(V)$ for some open neighbourhood U of g_0 in G and an open neighbourhood V of x_0 in M. Indeed, there exists a local trivialisation $\psi\colon \pi^{-1}(W) \to W \times F$ of E over an open neighbourhood W of $\alpha(g_0, x_0)$ in M. The action α being continuous, we find an open neighbourhood U of g_0 in G and an open neighbourhood V of x_0 in M over which E is trivial, such that $\alpha(U \times V) \subseteq W$. Let $\phi\colon \pi^{-1}(V) \to V \times F$ be a local trivialisation of E over V. Then,

$$\phi(\beta(g^{-1}, \psi^{-1}(\alpha(g,x),v))) = (x, A(g,x,v)) \quad \text{for all } g \in U,\, x \in V, \text{ and } v \in F,$$

for an $LC^r_{\mathbb{K}}$-map $A\colon U \times V \times F \to F$, which is $\mathbb{L}$-linear in the third argument. By Proposition 7, there is a unique extension of A to an $LC^r_{\mathbb{K}}$-map

$$\widetilde{A}\colon U \times V \times \widetilde{F} \to \widetilde{F},$$

and the latter is $\mathbb{L}$-linear in its third argument. For all $g \in U$ and $x \in V$, we then have

$$\widetilde{\phi}(\widetilde{\beta}(g^{-1}, \widetilde{\psi}^{-1}(\alpha(g,x),v))) = (x, \widetilde{A}(g,x,v))$$

for all $v \in \widetilde{F}$, as equality holds for all $v \in F$. Thus, $\widetilde{\beta}$ is $LC^r_{\mathbb{K}}$. □

8. Tensor Products of Vector Bundles

Throughout this section, let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $s \in \{\infty, \omega\}$, and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ such that $r \leq s$. Let G be a $C^s_{\mathbb{K}}$-Lie group modelled on a locally convex $\mathbb{K}$-vector space Y, M be a $C^r_{\mathbb{K}}$-manifold modelled on a locally convex $\mathbb{K}$-vector space Z, and $\alpha\colon G \times M \to M$ be a $C^r_{\mathbb{K}}$-action. For $k \in \{1,2\}$, let $\pi_k\colon E_k \to M$ be an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M, whose typical fibre is a locally convex $\mathbb{L}$-vector space F_k. Let $\beta_k\colon G \times E_k \to E_k$ be the G-action of class $C^r_{\mathbb{K}}$. Consider the set $\mathcal{A}$ of all pairs of local trivialisations of E_1 and E_2 trivialising these over the same open subset of M. Using an index set I, we have

$\mathcal{A} = \{(\psi_i^1, \psi_i^2) : i \in I\}$, where $\psi_i^k : \pi_k^{-1}(U_i) \to U_i \times F_k$ is a local trivialisation of E_k for $k \in \{1,2\}$, for each $i \in I$. Apparently, $(U_i)_{i \in I}$ is an open cover of M.

For our first result concerning tensor products, Proposition 8, we assume that F_1 is finite-dimensional. Then, fixing a basis $e_1, \ldots, e_n$ for F_1, the map $\theta\colon (F_2)^n \to F_1 \otimes_{\mathbb{L}} F_2$, $(y_1, \ldots, y_n) \mapsto \sum_{\tau=1}^n e_\tau \otimes y_\tau$ is an isomorphism of $\mathbb{L}$-vector spaces. We give $F_1 \otimes_{\mathbb{L}} F_2$ the topology $\mathcal{T}$, making θ a homeomorphism. This topology makes $F_1 \otimes_{\mathbb{L}} F_2$ a locally convex $\mathbb{L}$-vector space and θ an isomorphism of topological $\mathbb{L}$-vector spaces. It is easy to check (and well known) that the topology $\mathcal{T}$ is independent of the chosen basis. Let $e_1^*, \ldots, e_n^* \in F_1'$ be the basis dual to $e_1, \ldots, e_n$. Our goal is to make the union

$$E_1 \otimes E_2 := \bigcup_{x \in M} (E_1)_x \otimes_{\mathbb{L}} (E_2)_x$$

an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M, with typical fibre $F_1 \otimes_{\mathbb{L}} F_2$; the tensor products $(E_1)_x \otimes_{\mathbb{L}} (E_2)_x$ are chosen pairwise disjoint here for $x \in M$. Let $\pi\colon E_1 \otimes E_2 \to M$ be the mapping which takes $v \in (E_1)_x \otimes_{\mathbb{L}} (E_2)_x$ to x.

We define $\psi_i\colon \pi^{-1}(U_i) \to U_i \times (F_1 \otimes_{\mathbb{L}} F_2)$ via

$$\psi_i(v) := (x, ((\mathrm{pr}_{F_1} \circ \psi_i^1|_{(E_1)_x}) \otimes (\mathrm{pr}_{F_2} \circ \psi_i^2|_{(E_2)_x}))(v))$$

for $x \in U_i$ and $v \in (E_1)_x \otimes_{\mathbb{L}} (E_2)_x$, where $\mathrm{pr}_{F_k}\colon M \times F_k \to F_k$ is the projection.

Given $i, j \in I$ and $x \in U_i \cap U_j$, we have $\psi_i^k((\psi_j^k)^{-1}(x,v)) = (x, G_{ij}^k(x,v))$ for all $k \in \{1,2\}$ and $v \in F_k$, where $G_{ij}^k\colon (U_i \cap U_j) \times F_k \to F_k$ is $C^r_{\mathbb{K}}$ and $g_{ij}^k(x) := G_{ij}^k(x, \cdot)$ an $\mathbb{L}$-linear mapping. Then, $c_{\sigma,\tau}\colon U_i \cap U_j \to \mathbb{K}$, $x \mapsto e_\sigma^*(G_{ij}^1(x, e_\tau))$ is $C^r_{\mathbb{K}}$, and $\psi_i((\psi_j)^{-1}(x,v)) = (x, G_{ij}(x,v))$ for $x \in U_i \cap U_j$ and $v = \sum_{\tau=1}^n e_\tau \otimes v_\tau \in F_1 \otimes_{\mathbb{L}} F_2$, where

$$\begin{aligned} G_{ij}(x,v) &= (g_{ij}^1(x) \otimes g_{ij}^2(x))(v) = \sum_{\tau=1}^n (g_{ij}^1(x)e_\tau) \otimes (g_{ij}^2(x)v_\tau) \\ &= \sum_{\sigma,\tau=1}^n e_\sigma \otimes (c_{\sigma,\tau}(x) g_{ij}^2(x) v_\tau) = \theta\left(\left(\sum_{\tau=1}^n c_{\sigma,\tau}(x) G_{ij}^2(x, v_\tau)\right)_{\sigma=1}^n\right). \end{aligned}$$

As $F_1 \otimes_{\mathbb{L}} F_2 \to F_2$, $v \mapsto v_\tau = \mathrm{pr}_\tau(\theta^{-1}(v))$ is a continuous linear map (where $\mathrm{pr}_\tau :$ $(F_2)^n \to F_2$ is the projection onto the τ-component), in view of the preceding formula G_{ij} is $C^r_{\mathbb{K}}$. Thus, by Proposition 3, there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_{\mathbb{K}}$ on $E_1 \otimes E_2$ making each ψ_i a local trivialisation.

Note that $\beta\colon G \times (E_1 \otimes E_2) \to E_1 \otimes E_2$, $(g, v) \mapsto (\beta_1(g, \cdot)|_{(E_1)_x}^{E_{\alpha(g,x)}} \otimes \beta_2(g, \cdot)|_{(E_2)_x}^{(E_2)_{\alpha(g,x)}})(v)$ for $g \in G$, $x \in M$, $v \in (E_1 \otimes E_2)_x$ defines an action of G on $E_1 \otimes E_2$ by $\mathbb{L}$-linear mappings, which makes $\pi\colon E_1 \otimes E_2 \to M$ an equivariant mapping and such that $\beta(g, \cdot)$ is $\mathbb{L}$-linear on $(E_1)_x \otimes_{\mathbb{L}} (E_2)_x$ for all $g \in G$ and $x \in M$.

To show that β is $C^r_{\mathbb{K}}$, let $g_0 \in G$ and $x_0 \in M$. We pick $i \in I$ such that $\alpha(g_0, x_0) \in U_i$. The mapping α being continuous, we find open neighbourhoods U of g_0 in G and V of x_0 in M such that $\alpha(U \times V) \subseteq U_i$. There is $j \in I$ such that $x_0 \in U_j \subseteq V$. For $k \in \{1,2\}$, $g \in U$, $x \in U_j$ and $v \in F_k$, we have

$$\psi_i^k(\beta_k(g, (\psi_j^k)^{-1}(x,v))) = (\alpha(g,x), a_k(g,x,v))$$

for some $C^r_{\mathbb{K}}$-map $a_k\colon U \times U_j \times F_k \to F_k$, which is $\mathbb{L}$-linear in the final argument. Define $b_{\sigma,\tau} : U \times U_j \to \mathbb{L}$, $(g,x) \mapsto e_\sigma^*(a_1(g,x,e_\tau))$; then, $b_{\sigma,\tau}$ is $C^r_{\mathbb{K}}$. If $g \in U$, $x \in U_j$ and $v = \sum_{\tau=1}^n e_\tau \otimes v_\tau \in F_1 \otimes_{\mathbb{L}} F_2$, then $\psi_i(\beta(g, \psi_j^{-1}(x,v)))$ equals

$$\left(\alpha(g,x), \sum_{\tau=1}^n a_1(g,x,e_\tau) \otimes a_2(g,x,v_\tau)\right) = \left(\alpha(g,x), \theta\left(\left(\sum_{\tau=1}^n b_{\sigma,\tau}(g,x) a_2(g,x,v_\tau)\right)_{\sigma=1}^n\right)\right),$$

which is a $C^r_{\mathbb{K}}$-function of (g, x, v). As a consequence, $\beta|_{U\times\pi^{-1}(U_j)}$ is $C^r_{\mathbb{K}}$ and thus β, being $C^r_{\mathbb{K}}$ locally, is $C^r_{\mathbb{K}}$. We summarise as follows.

Proposition 8. *Let G be a $C^s_{\mathbb{K}}$-Lie group and M be a G-manifold of class $C^r_{\mathbb{K}}$. Let E_1 and E_2 be equivariant $\mathbb{L}$-vector bundles of class $C^r_{\mathbb{K}}$ over M. If the typical fibre of E_1 is finite-dimensional, then $E_1 \otimes E_2$, as defined above, is an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M.*

Instead of $\dim(F_1) < \infty$ (as before) assume that F_1 and F_2 are Fréchet spaces and the modelling spaces of G and M are metrisable. The completed projective tensor product

$$F := F_1 \widehat{\otimes}_\pi F_2$$

over $\mathbb{L}$ then is a Fréchet space (cf. [30] (p. 438, lines after Definitions 43.4)). We define

$$E := E_1 \widehat{\otimes}_\pi E_2 := \bigcup_{x\in M} (E_1)_x \widehat{\otimes}_\pi (E_2)_x,$$

where the $(E_1)_x \widehat{\otimes}_\pi (E_2)_x$ for $x \in M$ are chosen pairwise disjoint. Let $\pi\colon E \to M$ be the map taking $v \in E_x := (E_1)_x \widehat{\otimes}_\pi (E_2)_x$ to x. Define $\psi_i\colon \pi^{-1}(U_i) \to U_i \times (F_1 \widehat{\otimes}_\pi F_2)$ via

$$\psi_i(v) := \big(x, ((\mathrm{pr}_{F_1} \circ \psi^1_i|_{(E_1)_x}) \widehat{\otimes}_\pi (\mathrm{pr}_{F_2} \circ \psi^2_i|_{(E_2)_x}))(v)\big)$$

for $x \in U_i$ and $v \in (E_1)_x \widehat{\otimes}_\pi (E_2)_x$, where $\mathrm{pr}_{F_k}\colon M \times F_k \to F_k$ is the projection. Note that $\beta\colon G \times E \to E$, $(g, v) \mapsto (\beta_1(g, \cdot)|_{(E_1)_x} \widehat{\otimes}_\pi \beta_2(g, \cdot)|_{(E_2)_x})(v)$ for $g \in G$, $x \in M$, $v \in E_x$ defines an action of G on E which makes $\pi\colon E \to M$ an equivariant mapping. We show:

Proposition 9. *$\pi\colon E_1 \widehat{\otimes}_\pi E_2 \to M$ admits a unique structure of equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M such that ψ_i is a local trivialisation for each $i \in I$.*

Proof. The uniqueness for prescribed local trivialisations is clear. Let us show the existence of the structure. Given $i, j \in I$ and $x \in U_i \cap U_j$, we have $\psi^k_i((\psi^k_j)^{-1}(x, v)) = (x, G^k_{ij}(x, v))$ for all $k \in \{1, 2\}$ and $v \in F_k$, where $G^k_{ij}\colon (U_i \cap U_j) \times F_k \to F_k$ is $C^r_{\mathbb{K}}$ and $g^k_{ij}(x) := G^k_{ij}(x, \cdot)$ an $\mathbb{L}$-linear mapping. By Proposition 1 (a), the map $g^k_{ij}\colon U_i \cap U_j \to L(F_k)_c$ is $C^r_{\mathbb{K}}$. Now,

$$L_{\mathbb{L}}(F_1)_c \times L_{\mathbb{L}}(F_2) \to L_{\mathbb{L}}(F_1 \widehat{\otimes}_\pi F_2)_c, \quad (S, T) \mapsto S \widehat{\otimes}_\pi T$$

being continuous $\mathbb{L}$-bilinear (as recalled in Lemma 14), we deduce that

$$g_{ij}\colon U_i \cap U_j \to L_{\mathbb{L}}(F_1 \widehat{\otimes}_\pi F_2)_c, \quad x \mapsto g^1_{ij}(x) \widehat{\otimes}_\pi g^2_{ij}(x)$$

is $C^r_{\mathbb{K}}$. Hence, $G_{ij} := g^\wedge_{ij}\colon (U_i \cap U_j) \times (F_1 \widehat{\otimes}_\pi F_2) \to F_1 \widehat{\otimes}_\pi F_2$, $(x, v) \mapsto g_{ij}(x)(v)$ is $C^r_{\mathbb{K}}$, by Proposition 2 (a). We easily check that $\psi_i((\psi_j)^{-1}(x, v)) = (x, G_{ij}(x, v))$ holds for G_{ij} as just defined, for all $x \in U_i \cap U_j$ and $v \in F_1 \widehat{\otimes}_\pi F_2$. Hence, $E_1 \widehat{\otimes}_\pi E_2$ can be made an $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ in such a way that each ψ_i is a local trivialisation, by Proposition 3. Note that $\beta(g, \cdot)$ is $\mathbb{L}$-linear on E_x for all $g \in G$ and $x \in M$. To show that β is $C^r_{\mathbb{K}}$, let g_0, x_0, i, U, V, j and the $C^r_{\mathbb{K}}$-map a_k be as in the proof of Proposition 8. By Proposition 1 (a), $a^\vee_k\colon U \times U_j \to L(F_k)_c$, $(g, x) \mapsto a_k(g, x, \cdot)$ is $C^r_{\mathbb{K}}$. Hence,

$$a\colon U \times U_j \to L(F_1 \widehat{\otimes}_\pi F_2)_c, \quad (g, x) \mapsto a^\vee_1(g, x) \widehat{\otimes}_\pi a^\vee_2(g, x)$$

is $C^r_{\mathbb{K}}$, by the Chain Rule and Lemma 14. Using Proposition 2 (a), we find that the map $a^\wedge\colon U \times U_j \times (F_1 \widehat{\otimes}_\pi F) \to F_1 \widehat{\otimes}_\pi F_2$, $(g, x, v) \mapsto a(g, x)(v)$ is $C^r_{\mathbb{K}}$. We easily verify that $\psi_i(\beta(g, (\psi_j)^{-1}(x, v))) = (\alpha(g, x), a^\wedge(g, x, v))$ for all $(g, x, v) \in U \times U_j \times (F_1 \widehat{\otimes}_\pi F_2)$. Thus, $\psi_i(\beta(g, (\psi_j)^{-1}(x, v)))$ is $C^r_{\mathbb{K}}$ in (g, x, v), which completes the proof. □

We used the following fact:

Lemma 14. *Let E_1, E_2, F_1, and F_2 be Fréchet spaces over $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$. Then, the following bilinear map is continuous:*

$$\Xi\colon L_{\mathbb{L}}(E_1, F_1)_c \times L_{\mathbb{L}}(E_2, F_2)_c \to L_{\mathbb{L}}((E_1 \widehat{\otimes}_\pi E_2), (F_1 \widehat{\otimes}_\pi F_2))_c, \quad (S_1, S_2) \mapsto S_1 \widehat{\otimes}_\pi S_2.$$

Proof. Let $K \subseteq E_1 \widehat{\otimes} E_2$ be compact, q be a continuous seminorm on $F_1 \widehat{\otimes}_\pi F_2$, and $\varepsilon > 0$. After increasing q, we may assume that $q = q_1 \otimes q_2$ for continuous seminorms q_k on F_k for $k \in \{1, 2\}$. By [30] (p. 465, Corollary 2 to Theorem 45.2), K is contained in the closed, absolutely convex hull of $K_1 \otimes K_2$ for certain compact subsets $K_k \subseteq E_k$ for $k \in \{1, 2\}$. For all $S_k \in L(E_k, F_k)$ such that $\sup q_k(S_k(K_k)) \leq \sqrt{\varepsilon}$, we have

$$\sup q((S_1 \widehat{\otimes}_\pi S_2)(K)) \leq \sup q((S_1 \widehat{\otimes}_\pi S_2)(K_1 \otimes K_2)) = \sup q_1(S_1(K_1)) q_2(S_2(K_2)) \leq \sqrt{\varepsilon}^2 = \varepsilon,$$

using [30] (Proposition 43.1). The assertion follows. □

Remark 9. *If E_1 and E_2 are Hilbert spaces over $\mathbb{L}$ with Hilbert space tensor product $E_1 \widehat{\otimes} E_2$, and also F_1 and F_2 are Hilbert spaces over $\mathbb{L}$, then the bilinear map*

$$\Xi\colon L(E_1, F_1)_b \times L(E_2, F_2)_b \to L((E_1 \widehat{\otimes} E_2), (F_1 \widehat{\otimes} F_2))_b$$

is continuous, as $\|S_1 \widehat{\otimes} S_2\|_{\mathrm{op}} \leq \|S_1\|_{\mathrm{op}} \|S_2\|_{\mathrm{op}}$.

Replace the hypotheses in Proposition 9 with the requirements that G and M are modelled on metrisable locally convex spaces, $r \geq 1$ and F_1, F_2 are Hilbert spaces. We now use Remark 9 instead of Lemma 14, replace $F_1 \widehat{\otimes}_\pi F_2$ with the Hilbert space $F_1 \widehat{\otimes} F_2$, Proposition 1 (a) with Proposition 1 (b) (so that operator-valued maps are only $C^{r-1}_{\mathbb{K}}$) and use Proposition 2 (b) with $r - 1$ in place of r. Repeating the proof of Proposition 9, we get:

Proposition 10. *On $E_1 \widehat{\otimes} E_2 = \bigcup_{x \in M} (E_1)_x \widehat{\otimes} (E_2)_x$, there is a unique equivariant $\mathbb{L}$-vector bundle structure of class $C^{r-1}_{\mathbb{K}}$ over M whose typical fibre is the Hilbert space $F_1 \widehat{\otimes} F_2$, such that $\psi_i\colon \pi^{-1}(U_i) \to U_i \times (F_1 \widehat{\otimes} F_2)$ is a local trivialisation for each $i \in I$.*

Remark 10. *If $r \geq 1$, G and M are modelled on metrisable spaces and both F_1 and F_2 are pre-Hilbert spaces with Hilbert space completions $\widetilde{F}_1$ and $\widetilde{F}_2$, we can use the non-completed tensor product $F_1 \otimes_{\mathbb{L}} F_2 \subseteq \widetilde{F}_1 \widehat{\otimes} \widetilde{F}_2$ with the induced topology as the fibre and get an equivariant $\mathbb{L}$-vector bundle structure over M of class $C^{r-1}_{\mathbb{K}}$ over M on $E_1 \otimes E_2 = \bigcup_{x \in M} (E_1)_x \otimes_{\mathbb{L}} (E_2)_x$, exploiting that the $\mathbb{L}$-bilinear map $L_{\mathbb{L}}(F_1)_b \times L_{\mathbb{L}}(F_2)_b \to L_{\mathbb{L}}(F_1 \otimes_{\mathbb{L}} F_2)_b$, $(S_1, S_2) \mapsto S_1 \otimes S_2$ is continuous.*

9. Locally Convex Direct Sums of Vector Bundles

Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $s \in \{\infty, \omega\}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ such that $r \leq s$, G be a $C^s_{\mathbb{K}}$-Lie group modelled on a locally convex space Y, and M be a $C^r_{\mathbb{K}}$-manifold modelled on a locally convex $\mathbb{K}$-vector space Z, together with a $C^r_{\mathbb{K}}$-action $\alpha\colon G \times M \to M$.

Let $n \in \mathbb{N}$ and $\pi_k\colon E_k \to M$ be an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M for $k \in \{1, \ldots, n\}$, with typical fibre a locally convex $\mathbb{L}$-vector space F_k; let $\beta_k\colon G \times E_k \to E_k$ be the G-action and $\mathrm{pr}_{F_k}\colon M \times F_k \to F_k$ be the projection onto the second component. We easily check that there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_{\mathbb{K}}$ on the "Whitney sum"

$$E := E_1 \oplus \cdots \oplus E_n := \bigcup_{x \in M} (E_1)_x \times \cdots \times (E_n)_x,$$

with the apparent map $\pi : E \to M$, such that $\psi : \pi^{-1}(U) \to U \times F_1 \times \cdots \times F_n$, $v = (v_1, \ldots, v_n) \mapsto (\pi(v), \mathrm{pr}_{F_1}(\psi_1(v_1)), \ldots, \mathrm{pr}_{F_n}(\psi_n(v_n)))$ is a local trivialisation of E, for all families $(\psi_k)_{k=1}^n$ of local trivialisations $\psi_k\colon (\pi_k)^{-1}(U) \to U \times F_k$, which trivialise the E_ks over a joint open subset U of M. Then, $\beta(g,v) := (\beta_1(g,v_1),\ldots,\beta_n(g,v_n))$ for $g \in G$, $v = (v_1,\ldots,v_n) \in E$ yields an action of G on E. It is straightforward that β is $C^r_{\mathbb{K}}$. Thus,

Proposition 11. *If $E_1,\ldots,E_n$ are equivariant $\mathbb{L}$-vector bundles of class $C^r_{\mathbb{K}}$ over a G-manifold M of class $C^r_{\mathbb{K}}$, then also $E_1 \oplus \cdots \oplus E_n$ is an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M.*

The following lemma allows infinite direct sums to be tackled.

Lemma 15. *Let $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$ be families of locally convex spaces over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, with locally convex direct sums $E := \bigoplus_{i\in I} E_i$ and $F := \bigoplus_{i\in I} F_i$, respectively. Let V be an open subset of a locally convex $\mathbb{K}$-vector space Z. Let $r \in \mathbb{N}_0 \cup \{\infty\}$, and assume that $f_i \colon V \times E_i \to F_i$ is a map which is linear in the second argument, for each $i \in I$. Moreover, assume that* (a) *or* (b) *holds:*

(a) *Z is finite-dimensional; or*
(b) *Z and each E_i is a k_ω-space and I is countable.*

If f_i is of class $C^r_{\mathbb{K}}$ for each $i \in I$, then also the following map is $C^r_{\mathbb{K}}$:

$$f\colon V \times E \to F\,, \quad (x,(v_i)_{i\in I}) \mapsto (f_i(x,v_i))_{i\in I}.$$

Proof. If (b) holds, we may assume that I is countably infinite, excluding a trivial case. Thus, assume that $I = \mathbb{N}$. For each $n \in \mathbb{N}$, identify $E_1 \times \cdots \times E_n$ with a vector subspace of E, identifying $x \in E_1 \times \cdots \times E_n$ with $(x,0)$. For each $n \in \mathbb{N}$, we then have

$$Z \times E = \bigcup_{n\in\mathbb{N}} (Z \times E_1 \times \cdots \times E_n) \quad \text{and} \quad V \times E = \bigcup_{n\in\mathbb{N}} (V \times E_1 \times \cdots \times E_n),$$

where $Z \times E_1 \times \cdots \times E_n$ is a k_ω-space in the product topology. The inclusion map

$$\lambda_n \colon F_1 \times \cdots \times F_n \to \bigoplus_{i\in\mathbb{N}} F_i, \;\; v \mapsto (v,0)$$

is continuous and $\mathbb{K}$-linear. Moreover,

$$g_n \colon V \times E_1 \times \cdots \times E_n \to F_1 \times \cdots \times F_n, \;\; (x,v_1,\ldots,v_n) \mapsto (f_1(x,v_1),\ldots,f_n(x,v_n))$$

is a $C^r_{\mathbb{K}}$-map and so is $f|_{V\times E_1\times\cdots\times E_n} = \lambda_n \circ g_n$, for each $n \in \mathbb{N}$. Hence, f is $C^r_{\mathbb{K}}$ on the open subset $V \times E$ of $Z \times E$, considered as the locally convex direct limit $\varinjlim(Z \times E_1 \times \cdots \times E_n)$, by [31] (Proposition 4.5 (a)). This locally convex space equals $Z \times \varinjlim(E_1 \times \cdots \times E_n) = Z \times E$ with the product topology (see [32] (Theorem 3.4)).

If (a) holds, it suffices to prove the assertion for $r \in \mathbb{N}_0$. We proceed by induction. *The case $r = 0$.* Let $(x,v) = (x,(v_i)_{i\in I}) \in V \times E$; we show that f is continuous at (x,v). To this end, let Q be an absolutely convex, open 0-neighbourhood in F. There is a finite subset $J \subseteq I$ such that $v_i = 0$ for all $i \in I \setminus J$. Let $N := |J| + 1$. For each $i \in I$, the intersection $Q_i := (\frac{1}{N}Q) \cap F_i$ is an absolutely convex, open 0-neighbourhood in F_i. For the absolutely convex hull, we get $\operatorname{absconv}(\bigcup_{i\in I} Q_i) \subseteq \frac{1}{N}Q$. Since f_i is continuous for each $i \in J$ and J is finite, we find a compact neighbourhood K of x in V such that $f_i(y,v_i) - f_i(x,v_i) \in Q_i$ for all $y \in K$ and $i \in J$. Since $f_i(K \times \{0\}) = \{0\}$, where K is compact and f_i is continuous, for each $i \in I$, there is an absolutely convex, open 0-neighbourhood P_i in E_i such that $f_i(K \times P_i) \subseteq Q_i$. Then, $W := v + \operatorname{absconv}(\bigcup_{i\in I} P_i)$ is an open neighbourhood of v in E. Let $y \in K$ and $w \in W$ be given, say $w = (w_i)_{i\in I} = v + (t_i p_i)_{i\in I}$, where $p_i \in P_i$ and $(t_i)_{i\in I} \in \bigoplus_{i\in I}\mathbb{R}$ such that $t_i \in [0,1]$ and $\sum_{i\in I} t_i = 1$. Then, for each $i \in I \setminus J$, since $v_i = 0$, we obtain

$$f_i(y,w_i) - f(x,v_i) = f_i(y,t_i p_i) = t_i f_i(y,p_i) \in t_i Q_i\,.$$

For $i \in J$, on the other hand, we have

$$\begin{aligned} f_i(y,w_i) - f(x,v_i) &= f_i(y,w_i - v_i) + (f_i(y,v_i) - f_i(x,v_i)) \\ &= t_i f_i(y,p_i) + (f_i(y,v_i) - f_i(x,v_i)) \in t_i Q_i + Q_i. \end{aligned}$$

As a consequence, $f(y,w) - f(x,v) \in (\prod_{i\in I} t_i Q_i) + \sum_{i\in J} Q_i \subseteq \frac{1}{N} Q + \sum_{i\in J} \frac{1}{N} Q = Q$, using the convexity of Q. We have shown that f is continuous at (x,v).

Induction step. Let $r \geq 1$ and assume the assertion is true for all numbers $< r$. Given $u,v \in E$, $x \in V$, and $z \in Z$, we have $u,v \in \bigoplus_{i\in J} E_i = \prod_{i\in J} E_i$ for some finite subset $J \subseteq I$. The map $f_J \colon V \times \prod_{i\in J} E_i \to \prod_{i\in J} F_i$, $(x,(v_i)_{i\in J}) \mapsto (f_i(x,v_i))_{i\in J}$ is $C^1_{\mathbb{K}}$, whence

$$\begin{aligned} df_J((x,u),(z,v)) &= \lim_{t\to 0} t^{-1}(f_J((x,u)+t(z,v)) - f_J(x,u)) \\ &= \lim_{t\to 0} t^{-1}(f((x,u)+t(z,v)) - f(x,u)) = df((x,u),(z,v)) \end{aligned}$$

exists in $\prod_{i\in J} F_i$ and thus in F; its ith component is

$$df_i((x,u_i),(z,v_i)) = d_1 f_i(x,u_i,z) + d_2 f_i(x,u_i,v_i)$$

in terms of partial differentials. Note that the mappings $g_i \colon (V\times Z)\times(E_i\times E_i) \to F_i$, $(x,z,u_i,v_i) \mapsto d_1 f_i(x,u_i,z)$ and $h_i \colon (V\times Z)\times(E_i\times E_i) \to F_i$, $(x,z,u_i,v_i) \mapsto d_2 f_i(x,u_i,v_i) = f_i(x,v_i)$ are $C^{r-1}_{\mathbb{K}}$ and linear in (u_i,v_i). By induction, the mappings

$$g \colon (V\times Z)\times(E\times E) \to F, \quad (x,z,(u_i)_{i\in I},(v_i)_{i\in I}) \mapsto (g_i(x,z,u_i,v_i))_{i\in I} \quad \text{and}$$

$$h \colon (V\times Z)\times(E\times E) \to F, \quad (x,z,(u_i)_{i\in I},(v_i)_{i\in I}) \mapsto (h_i(x,z,u_i,v_i))_{i\in I}$$

are $C^{r-1}_{\mathbb{K}}$, using that $E\times E \cong \bigoplus_{i\in I}(E_i\times E_i)$. Hence, also $df \colon (V\times E)\times(Z\times E) \to F$ is $C^{r-1}_{\mathbb{K}}$, as $df((x,u),(z,v)) = g(x,z,u,v) + h(x,z,u,v)$. Since df exists and is $C^{r-1}_{\mathbb{K}}$, the continuous map f is $C^r_{\mathbb{K}}$. $\square$

Remark 11. *The conclusion of Lemma 15 does not hold for $(r,\mathbb{K}) = (\omega,\mathbb{R})$ in the example $I = \mathbb{N}$, $V = Z = \mathbb{R}$, $E_k = \mathbb{R}$, $f_k(r,t) := \frac{t}{1+kr^2}$, using that the Taylor series of $f_k(\cdot,t)$ around 0 has radius of convergence $\frac{1}{\sqrt{k}}$ for all $t \in \mathbb{R}\setminus\{0\}$.*

Assuming now $r \neq \omega$, consider a family $(E_i)_{i\in I}$ of equivariant $\mathbb{L}$-vector bundles $\pi_i \colon E_i \to M$ of class $C^r_{\mathbb{K}}$ with typical fibre F_i and G-action $\beta_i \colon G\times E_i \to E_i$. We assume that (a) or (b) is satisfied:

(a) G and M are finite-dimensional; or
(b) I is countable and each F_i as well as the modelling spaces of G and M are k_ω-spaces.

Moreover, we assume:

(c) For each $x \in M$, there exists an open neighbourhood U of x in M, such that, for each $i \in I$, the vector bundle E_i admits a local trivialisation $\psi_i \colon (\pi_i)^{-1}(U) \to U \times F_i$.

Thus, the $C^r_{\mathbb{K}}$-vector bundle $E_i|_U$ is trivialisable for each $i \in I$. Define $E := \bigcup_{x\in M} \bigoplus_{i\in I}(E_i)_x$ with pairwise disjoint direct sums and $\pi \colon E \to M$, $\bigoplus_{i\in I}(E_i)_x \ni v \mapsto x$. Then

$$\beta \colon G\times E \to E, \quad (g,(v_i)_{i\in I}) \mapsto (\beta_i(g,v_i))_{i\in I}$$

is a G-action such that $\beta(g,\cdot)|_{E_x} \colon E_x \to E_{\alpha(g,x)}$ is $\mathbb{L}$-linear for all $(g,x) \in G\times M$, where $E_x := \pi^{-1}(\{x\})$. We readily deduce from Proposition 3 and Proposition 15 that there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_{\mathbb{K}}$ on E such that

$$\pi^{-1}(U) \to U \times \bigoplus_{i\in I} F_i, \quad E_x \ni (v_i)_{i\in I} \mapsto (x,(\mathrm{pr}_{F_i}(\psi_i(v_i)))_{i\in I})$$

is a local trivialisation for E, for each family $(\psi_i)_{i\in I}$ of local trivialisations as above. The latter makes E an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$. In fact, the $C^r_{\mathbb{K}}$-property of β can be checked using pairs of local trivialisations, as in the proofs of Propositions 5, 8, and 9. Then, apply Proposition 15, with F_i in place of E_i and $Y \times Z$ in place of Z. Thus,

Proposition 12. *In the preceding situation, $\bigoplus_{i\in I} E_i$ is an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M.*

Remark 12. *If M is a $C^r_{\mathbb{R}}$-manifold, then every $x \in M$ has an open neighbourhood U which is $C^r_{\mathbb{R}}$-diffeomorphic to a convex open subset W in the modelling space Z of M. If W can be chosen $C^r_{\mathbb{R}}$-paracompact, then every $C^r_{\mathbb{R}}$-vector bundle over U is trivialisable (see [12] (Corollary 15.10)). The latter condition is satisfied, for example, if Z is finite-dimensional, a Hilbert space, or a countable direct limit of finite-dimensional vector spaces (and hence a nuclear Silva space), cf. [3] (Theorem 16.10 and Corollary 16.16). If $(r,\mathbb{K}) = (\infty,\mathbb{C})$ and Z has finite dimension, then each finite-dimensional holomorphic vector bundle over a, say, polycylinder in Z is $C^\infty_{\mathbb{C}}$-trivialisable (cf. [33]). Under suitable hypotheses, holomorphic Banach vector bundles over contractible bases are $C^\infty_{\mathbb{C}}$-trivialisable as well [34].*

10. Dual Bundles and Cotangent Bundles

In this section, we discuss conditions ensuring that a vector bundle has a canonical dual bundle. Let $\mathbb{L} \in \{\mathbb{R},\mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R},\mathbb{L}\}$, $r \in \mathbb{N}_0 \cup \{\infty,\omega\}$, and M be a $C^r_{\mathbb{K}}$-manifold modeled on a locally convex space Z.

Definition 7. *Let $\pi\colon E \to M$ be an $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$, with typical fibre F. Consider the disjoint union*

$$E' := \bigcup_{x\in M} (E_x)';$$

let $p\colon E' \to M$ be the map taking $\lambda \in (E_x)'$ to x, for each $x \in M$. Given $t \in \mathbb{N}_0 \cup \{\infty,\omega\}$ such that $t \le r$, we say that E has a canonical dual bundle *of class $C^t_{\mathbb{K}}$ with respect to $\mathcal{S} \in \{b,c\}$ if E' can be made an $\mathbb{L}$-vector bundle of class $C^t_{\mathbb{K}}$ over M, with typical fibre $F'_{\mathcal{S}}$ and bundle projection p, such that*

$$\widetilde{\psi}\colon p^{-1}(U) \to U \times F'_{\mathcal{S}}, \quad (E')_x = (E_x)' \ni \lambda \mapsto (x, ((\mathrm{pr}_F \circ \psi|_{E_x})^{-1})'(\lambda)) \qquad (15)$$

is a local trivialisation of E', for each local trivialisation $\psi\colon \pi^{-1}(U) \to U \times F$ of E.

To pinpoint situations where the dual bundle exists, we recall a fact concerning the formation of dual linear maps (see [8] (Proposition 16.30)):

Lemma 16. *Let E and F be locally convex spaces, and $\mathcal{S} \in \{b,c\}$. If the evaluation homomorphism $\eta_{F,\mathcal{S}}\colon F \to (F'_{\mathcal{S}})'_{\mathcal{S}}$, $\eta_{F,\mathcal{S}}(x)(\lambda) := \lambda(x)$ is continuous, then*

$$\Theta\colon L(E,F)_{\mathcal{S}} \to L(F'_{\mathcal{S}}, E'_{\mathcal{S}})_{\mathcal{S}}, \quad \alpha \mapsto \alpha'$$

is a continuous linear map.

Remark 13. *Let F be a locally convex $\mathbb{K}$-vector space over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. It is known that $\eta_{F,b}$ is continuous if and only if F is quasi-barrelled, i.e., every bornivorous barrel in F is a 0-neighbourhood [35] (Proposition 2 in Section 11). In particular, $\eta_{F,b}$ is continuous if F is bornological or barrelled. It is also known that $\eta_{F,c}$ is continuous (and actually a topological embedding) if F is a $k_{\mathbb{R}}$-space. If $\mathbb{K} = \mathbb{R}$, this follows from [36] (Theorem 2.3) and [37] (Lemma 14.3) (cf. also [37] (Propositions 2.3 and 2.4)). If $\mathbb{K} = \mathbb{C}$ and F is a $k_{\mathbb{R}}$-space, then $\eta_{F_{\mathbb{R}},c}$ is a topological embedding for the real topological vector space $F_{\mathbb{R}}$ underlying F. Now, $(F'_c)_{\mathbb{R}} \cong (F_{\mathbb{R}})'$ as a real topological vector space, using that a continuous $\mathbb{C}$-linear functional $\lambda\colon F \to \mathbb{C}$ is determined by its real part. Transporting the complex vector space structure from F'_c to $(F_{\mathbb{R}})'$, the latter becomes a complex locally convex space. Thus,*

$((F'_c)'_c)_{\mathbb{R}}$ can be identified with $((F_{\mathbb{R}})'_c)'_c$, and it is easy to verify that $\eta_{F,c}$ corresponds to $\eta_{F_{\mathbb{R}},c}$ if we make the latter identification.

Proposition 13. *Let $\pi\colon E \to M$ be an $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$, with typical fibre F. Let $\mathcal{S} \in \{b, c\}$. If $\mathcal{S} = c$, let $r_- := r$; if $\mathcal{S} = b$, assume $r \geq 1$ and set $r_- := r - 1$. Consider the following conditions:*

- (α) *The modelling space Z of M is finite-dimensional, $\eta_{F,\mathcal{S}}$ is continuous, and $F'_{\mathcal{S}}$ is barrelled.*
- (β) *$\eta_{F,\mathcal{S}}$ is continuous and, moreover, $(Z \times F'_{\mathcal{S}}) \times (Z \times F'_{\mathcal{S}})$ is a $k_{\mathbb{R}}$-space, or $r_- = 0$ and $Z \times F'_{\mathcal{S}}$ is a $k_{\mathbb{R}}$-space, or $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and $Z \times F'_{\mathcal{S}}$ is a $k_{\mathbb{R}}$-space.*
- (γ) *F is normable.*

If (α) or (β) is satisfied with $\mathcal{S} = c$, then E has a canonical dual bundle of class $C^r_{\mathbb{K}}$ with respect to $\mathcal{S} = c$. If (α), (β), or (γ) is satisfied with $\mathcal{S} = b$, then E has a canonical dual bundle of class $C^{r-1}_{\mathbb{K}}$ with respect to $\mathcal{S} = b$.

For $\mathcal{S} = b$, condition (α) of Proposition 13 is satisfied, for example, if F is a reflexive locally convex space (then $\eta_{F,b}$ is continuous and F'_b is barrelled, being reflexive.)

Proof. Let E' be the disjoint union $\bigcup_{x \in M}(E_x)'$, and $p\colon E' \to M$ be as in Definition 7. Let $(\psi_i)_{i \in I}$ be a family such that the $\psi_i\colon \pi^{-1}(U_i) \to U_i \times F$ form the set of all local trivialisations of E. Let $(g_{ij})_{i,j \in I}$ be the associated cocycle (as explained before Proposition 3). Then, $G_{ij} := g_{ij}^{\wedge}$ is $C^r_{\mathbb{K}}$ and hence $g_{ij} = (G_{ij})^{\vee}$ is $C^{r_-}_{\mathbb{K}}$, by Proposition 1. Given $i \in I$, we define $\widetilde{\psi}_i\colon p^{-1}(U_i) \to U_i \times F'_{\mathcal{S}}$ as in (15), using ψ_i instead of ψ. Then,

$$\begin{aligned}\widetilde{\psi}_i(\widetilde{\psi}_j^{-1}(x,\lambda)) &= (x, ((\mathrm{pr}_F \circ \psi_i|_{E_x})^{-1})' \circ (\mathrm{pr}_F \circ \psi_j|_{E_x})'(\lambda)) \\ &= (x, (\mathrm{pr}_F \circ \psi_j|_{E_x} \circ (\mathrm{pr}_F \circ \psi_i|_{E_x})^{-1})'(\lambda)) \;=\; (x, g_{ji}(x)'(\lambda))\end{aligned}$$

for all $x \in U_i \cap U_j$ and $\lambda \in F'$ shows that

$$(\widetilde{\psi}_i \circ \widetilde{\psi}_j^{-1})(x,\lambda) = (x, h_{ij}(x)(\lambda)),$$

where $h_{ij}(x) := g_{ji}(x)' \in \mathrm{GL}(F'_{\mathcal{S}})$. If ($\alpha$) or ($\beta$) holds, then $\eta_{F,\mathcal{S}}\colon F \to (F'_{\mathcal{S}})'_{\mathcal{S}}$ is continuous by hypothesis. If $\mathcal{S} = b$ and (γ) holds, then $\eta_{F,b}$ is an isometric embedding (as is well known) and hence continuous. Thus, $\Theta\colon L(F)_{\mathcal{S}} \to L(F'_{\mathcal{S}})_{\mathcal{S}}$, $\alpha \mapsto \alpha'$ is a continuous $\mathbb{L}$-linear map (Lemma 16). Since $g_{ji}\colon U_i \cap U_j \to L(F)_{\mathcal{S}}$ is $C^{r_-}_{\mathbb{K}}$, we deduce that $h_{ij} = \Theta \circ g_{ji}\colon U_i \cap U_j \to L(F'_{\mathcal{S}})_{\mathcal{S}}$ is $C^{r_-}_{\mathbb{K}}$. Thus Condition (g)$'$ of Corollary 2 is satisfied, with r_- in place of r. Conditions (a)–(f) being apparent, the cited corollary provides an $\mathbb{L}$-vector bundle structure of class $C^{r_-}_{\mathbb{K}}$ on E'. □

Without specific hypotheses, a canonical dual bundle need not exist.

Example 2. *Let A be a unital, associative, locally convex topological $\mathbb{K}$-algebra whose group of units $A^\times$ is open in A, and such that the inversion map $\iota\colon A^\times \to A^\times$ is continuous. Then, ι is smooth (and indeed $\mathbb{K}$-analytic); see, e.g., [13] (Propositions 10.1.12 and 10.1.13). We assume that the locally convex space underlying A is a non-normable Fréchet–Schwartz space and hence Montel, ensuring that $L(A)_b = L(A)_c$. For example, we might take $A := C^\infty(K, \mathbb{K})$, where K is a connected, compact, smooth manifold of positive dimension (cf. [13] (Lemma 10.2.2 (c))). Let $r, t \in \mathbb{N}_0 \cup \{\infty, \omega\}$ with $t \leq r$ and $\mathcal{S} \in \{b, c\}$. We consider the trivial vector bundle*

$$\mathrm{pr}_1\colon E := A^\times \times A \to A^\times .$$

(Thus, $E \cong TA^\times$, the tangent bundle). Then, E is a $\mathbb{K}$-vector bundle of class $C^r_{\mathbb{K}}$ over the base $A^\times$, with typical fibre A. Both $\psi_1 := \mathrm{id}\colon A^\times \times A \to A^\times \times A$ and $\psi_2\colon A^\times \times A \to A^\times \times A$, $(a, v) \mapsto (a, av)$ are global trivialisations of E. Identifying $E' := \bigcup_{a \in A^\times}(E_a)'$ with the set $A^\times \times A'$, we consider the associated bijections $\widetilde{\psi}_i\colon E' = A^\times \times A' \to A^\times \times A'$ for $i \in \{1, 2\}$

(cf. (15)). Thus, $\widetilde{\psi}_1 = \mathrm{id}$*, and* $\widetilde{\psi}_2(a,\lambda) = (a, \lambda(a^{-1}\cdot))$ *for* $a \in A^\times$*,* $\lambda \in A'$*. The map* $G_{ij}\colon A^\times \times A \to A$*,* $(a,v) \mapsto \mathrm{pr}_2(\psi_i(\psi_j^{-1}(a,v)))$ *is* $C^r_{\mathbb{K}}$ *for* $i,j \in \{1,2\}$*, where* $\mathrm{pr}_2\colon A^\times \times A \to A$ *is the projection onto the second factor. Then, also* $g_{ij}\colon A^\times \to L(A)_c = L(A)_b$*,* $a \mapsto G_{ij}(a,\cdot)$ *is* $C^r_{\mathbb{K}}$*, by Proposition 1 (a). Now, A being Fréchet and thus barrelled, the evaluation homomorphism* $\eta_{A,b}$ *is continuous; since A is metrisable and hence a k-space, also* $\eta_{A,c}$ *is continuous (see Remark 13). Since* g_{ij} *is* $C^r_{\mathbb{K}}$*, we deduce with Lemma 16 that also* $h_{ij}\colon A^\times \to L(A'_{\mathcal{S}})_{\mathcal{S}}$*,* $a \mapsto (g_{ji}(a))'$ *is* $C^r_{\mathbb{K}}$*. Define*

$$H_{ij}\colon A^\times \times A'_{\mathcal{S}} \to A'_{\mathcal{S}} \quad (a,\lambda) \mapsto h_{ij}(a)(\lambda)$$

for $i,j \in \{1,2\}$*. Then,* H_{12} *is discontinuous. To see this, we compose* H_{12} *with the map* $\mathrm{ev}_1\colon A'_b \to \mathbb{K}$*,* $\lambda \mapsto \lambda(1)$*, which evaluates functionals at the identity element* $1 \in A$*, and recall that* ev_1 *is continuous. Then,* $\mathrm{ev}_1(H_{12}(a,\lambda)) = \lambda(g_{21}(a)(1)) = \lambda(a)$ *for* $a \in A^\times$ *and* $\lambda \in A'$*. However, A being a non-normable locally convex space, the bilinear, separately continuous evaluation map* $\varepsilon\colon A \times A'_b \to \mathbb{K}$*,* $(a,\lambda) \mapsto \lambda(a)$ *is discontinuous, and hence so is its restriction* $\varepsilon|_{A^\times \times A'_b} = \mathrm{ev}_1 \circ H_{12}$ *to the non-empty open subset* $A^\times \times A'_b$*, as is readily verified. Now,* $\mathrm{ev}_1 \circ H_{12}$ *being discontinuous, also* H_{12} *is discontinuous (and therefore not* $C^t_{\mathbb{K}}$*). As a consequence, also* $\widetilde{\psi}_1 \circ \widetilde{\psi}_2^{-1} = (\mathrm{pr}_1, H_{12})$ *is discontinuous. Summing up:*

There is no canonical vector bundle structure of class $C^t_{\mathbb{K}}$ on E' because the two vector bundle structures on E' making $\widetilde{\psi}_1$ (resp., $\widetilde{\psi}_2$) a global trivialisation do not coincide.

Remark 14. *In the preceding situation, set* $M := A^\times$*,* $F := A'_b$*,* $I := \{1,2\}$*,* $U_i := M$ *for* $i \in I$*, and* $\pi := \mathrm{pr}_1\colon M \times F \to M$*. If we let* $M \times A'_b$ *play the role of E in Proposition 3 and* $\widetilde{\psi}_i\colon \pi^{-1}(U_i) \to U_i \times F$ *the role of* ψ_i *in Proposition 3 (e), then all of Conditions (a)–(f) of Proposition 3 and Condition (g)′ of Corollary 2 are satisfied for* $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ *(with* $\mathbb{L} := \mathbb{K}$*). However, there is no* $C^r_{\mathbb{K}}$*-vector bundle structure on* $M \times F$ *making each* $\widetilde{\psi}_i$ *a trivialisation, as just observed, i.e., the conclusion of Corollary 2 becomes false.*

Remark 15. *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$*,* $r \in \mathbb{N} \cup \{\infty, \omega\}$*,* $t \in \mathbb{N}_0 \cup \{\infty, \omega\}$ *with* $t \le r$ *and M be a* $C^r_{\mathbb{K}}$*-manifold modelled on a locally convex space Z. Then, the tangent bundle TM is a* $\mathbb{K}$*-vector bundle of class* $C^{r-1}_{\mathbb{K}}$ *over M, with typical fibre Z. Pick a locally convex vector topology* $\mathcal{T}$ *on* Z'*. Let* $\mathcal{A}$ *be the set of all maps* $\widetilde{\psi}$ *as in (15), with* $(Z', \mathcal{T})$ *in place of* $F'_{\mathcal{S}}$*, for* ψ *ranging through the set of all local trivialisations of TM (alternatively, only those of the form* $(\pi_{TU}, d\phi)$ *for charts* $\phi\colon U \to V \subseteq Z$ *of M, using the bundle projection* $\pi_{TU}\colon TU \to U$*). Let us say that M has a* canonical cotangent bundle *of class* $C^t_{\mathbb{K}}$ *with respect to* $\mathcal{T}$ *if* $T'M := \bigcup_{x \in M} (T_x M)'$ *admits a* $\mathbb{K}$*-vector bundle structure of class* $C^t_{\mathbb{K}}$ *over M with typical fibre* $(Z', \mathcal{T})$*, which makes each* $\widetilde{\psi}\colon p^{-1}(U) \to U \times (Z', \mathcal{T})$ *a local trivialisation (with* $p\colon T'M \to M$*,* $(T_xM)' \ni \lambda \mapsto x$*). Then, the evaluation map*

$$\varepsilon\colon (Z', \mathcal{T}) \times Z \to \mathbb{K}, \quad (\lambda, x) \mapsto \lambda(x)$$

must be continuous and hence Z normable. For $\mathbb{K} = \mathbb{R}$*, this is explained in [17] (Remark 1.3.9) (written after Example 2 was found) if* $r = \infty$*. This implies the case* $r \in \mathbb{N}$*. As the diffeomorphism f employed as a change of charts is real analytic, the case* $(\omega, \mathbb{R})$ *follows and also the complex case, using a* $\mathbb{C}$*-analytic extension of f. When* $\mathcal{T}$ *is the compact-open topology, existence of a canonical cotangent bundle for M even implies that Z is finite-dimensional. (If* ε *is continuous on* $Z'_c \times Z$*, then there exists a compact subset* $K \subseteq Z$ *and a 0-neighbourhood* $W \subseteq Z$ *such that* $\varepsilon((K^\circ) \times W) \subseteq \mathbb{D}$*. Hence,* $K^\circ \subseteq W^\circ$*. Since* K° *is a 0-neighbourhood in* Z'_c *and* W° *compact (by Ascoli's Theorem),* Z'_c *is locally compact and hence finite-dimensional. As* Z'_c *separates points on Z, also Z must be finite-dimensional.)*

Cotangent bundles are not needed to define 1-forms on an infinite-dimensional manifold M. Following [38], these can be considered as smooth maps on TM which are linear on the fibres (and a similar remark applies to differential forms of higher order).

Differentiability properties of the *G*-action on the dual bundle. Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $s \in \{\infty, \omega\}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ with $r \le s$, and G be a $C^s_{\mathbb{K}}$-Lie group modelled on a locally convex $\mathbb{K}$-vector space Y. Let M be a $C^r_{\mathbb{K}}$-manifold modelled on a locally convex $\mathbb{K}$-vector space Z and $\alpha\colon G \times M \to M$ be a G-action of class $C^r_{\mathbb{K}}$.

Proposition 14. *Let $\pi\colon E \to M$ be an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$, with typical fibre F and G-action $\beta\colon G \times E \to E$ of class $C^r_{\mathbb{K}}$. Let $\mathcal{S} \in \{b, c\}$. If $\mathcal{S} = c$, set $r_- := r$; if $\mathcal{S} = b$, assume $r \geq 1$ and set $r_- := r - 1$. Consider the following conditions:*

(a) *$\eta_{F,\mathcal{S}}$ is continuous, and, moreover, $(Y \times Z \times F'_{\mathcal{S}}) \times (Y \times Z \times F'_{\mathcal{S}})$ is a $k_{\mathbb{R}}$-space, or $r_- = 0$ and $Y \times Z \times F'_{\mathcal{S}}$ is a $k_{\mathbb{R}}$-space, or $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and $Y \times Z \times F'_{\mathcal{S}}$ is a $k_{\mathbb{R}}$-space;*
(b) *M and G are finite-dimensional, $\eta_{F,\mathcal{S}}$ is continuous, and $F'_{\mathcal{S}}$ is barrelled; or*
(c) *F is normable.*

If $\mathcal{S} = c$ and (a) *or* (b) *holds, then E has a canonical dual bundle E' of class $C^{r_-}_{\mathbb{K}}$ with respect to $\mathcal{S}$, and the map $\beta^*\colon G \times E' \to E'$, defined using adjoint linear maps via*

$$\beta^*(g, \lambda) := (\beta(g^{-1}, \cdot)|^{E_x}_{E_{\alpha(g,x)}})'(\lambda)$$

for $g \in G$, $\lambda \in (E_x)'$, turns E' into an equivariant $\mathbb{L}$-vector bundle of class $C^{r_-}_{\mathbb{K}}$ over the G-manifold M. If $\mathcal{S} = b$ and (a), (b), *or* (c) *is satisfied, then the same conclusion holds.*

Proof. In view of Proposition 13, the hypotheses imply that E has a canonical dual bundle $p\colon E' \to M$ of class $C^{r_-}_{\mathbb{K}}$. It is apparent that $\beta^*\colon G \times E' \to E'$ is an action, and E'_x is taken $\mathbb{L}$-linearly to $E'_{\alpha(g,x)}$ by $\beta^*(g, \cdot)$, for each $g \in G$ and $x \in M$. It therefore only remains to show that β^* is $C^{r_-}_{\mathbb{K}}$. To this end, let $g_0 \in G$ and $x_0 \in M$; we show that β^* is $C^{r_-}_{\mathbb{K}}$ on $U \times p^{-1}(V)$, for some open neighbourhood U of g_0 in G and an open neighbourhood V of x_0 in M. Indeed, there exists a local trivialisation $\psi\colon \pi^{-1}(W) \to W \times F$ of E over an open neighbourhood W of $\alpha(g_0, x_0)$ in M. The action α being continuous, we find an open neighbourhood U of g_0 in G and an open neighbourhood V of x_0 in M over which E is trivial, such that $\alpha(U \times V) \subseteq W$. Let $\phi\colon \pi^{-1}(V) \to V \times F$ be a local trivialisation of E over V. Then

$$\phi(\beta(g^{-1}, \psi^{-1}(\alpha(g, x), v))) = (x, A(g, x, v)) \quad \text{for all } g \in U, x \in V, \text{and } v \in F,$$

for a $C^r_{\mathbb{K}}$-map $A\colon U \times V \times F \to F$, which is $\mathbb{L}$-linear in the third argument. By Corollary 1, the map $a\colon U \times V \to L(F)_{\mathcal{S}}$, $(g, x) \mapsto A(g, x, \cdot)$ is $C^{r_-}_{\mathbb{K}}$. In view of the hypotheses, Lemmas 16 and 13 entail that also $a^*\colon U \times V \to L(F'_{\mathcal{S}})_{\mathcal{S}}$, $(g, x) \mapsto (a(g, x))'$ is $C^{r_-}_{\mathbb{K}}$-map. Now, again using the specific hypotheses, Proposition 2 shows that also the mapping $A^*\colon U \times V \times F'_{\mathcal{S}} \to F'_{\mathcal{S}}$, $(g, x, \lambda) \mapsto a^*(g, x)(\lambda)$ is $C^{r_-}_{\mathbb{K}}$. However, for $g \in U$, $x \in V$, and $\lambda \in F'$, we calculate

$$\begin{aligned}
\widetilde{\psi}(\beta^*(g, \widetilde{\phi}^{-1}(x, \lambda))) &= \Big(\alpha(g, x), \Big(\mathrm{pr}_F \circ \phi|_{E_x} \circ \beta(g^{-1}, \cdot)|^{E_x}_{E_{\alpha(g,x)}} \circ (\mathrm{pr}_F \circ \psi|_{E_{\alpha(g,x)}})^{-1}\Big)'(\lambda)\Big) \\
&= (\alpha(g, x), A^*(g, x, \lambda)),
\end{aligned}$$

using the notation as in (15). We conclude that $\beta^*|_{U \times p^{-1}(V)}$ is $C^{r_-}_{\mathbb{K}}$. □

Example 3. *For elementary examples, recall that the group* $\mathrm{Diff}(M)$ *of all smooth diffeomorphisms of a σ-compact, finite-dimensional smooth manifold M can be made a smooth Lie group, modelled on the (LF)-space $\Gamma_c(TM)$ of compactly supported smooth vector fields on M (see [13,15]). The natural action* $\mathrm{Diff}(M) \times M \to M$ *is smooth [13]. In view of Example 1, Proposition 14 (b), Proposition 8 and Proposition 4, we readily deduce that also the natural action of* $\mathrm{Diff}(M)$ *on TM is smooth, as well as the natural actions on $T^*M := (TM)'$, $TM^{\otimes n} \otimes (T^*M)^{\otimes m}$ for all $n, m \in \mathbb{N}_0$, and the natural action on the subbundles $S^n(T^*M)$ and $\bigwedge^n T^*M$ of $(T^*M)^{\otimes n}$ given by symmetric and exterior powers, respectively.*

11. Locally Convex Poisson Vector Spaces

We discuss a slight generalisation of the concept of a locally convex Poisson vector space introduced in [8]. Fix $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

A bounded set-functor $\mathcal{S}$ associates with each locally convex $\mathbb{K}$-vector space E a set $\mathcal{S}(E)$ of bounded subsets of E, such that $\{\lambda(M) \colon M \in \mathcal{S}(E)\} \subseteq \mathcal{S}(F)$ for each continuous $\mathbb{K}$-linear map $\lambda \colon E \to F$ between locally convex $\mathbb{K}$-vector spaces (cf. [8] (Definition 16.15)). Given locally convex $\mathbb{K}$-vector spaces E and F, we shall write $L(E,F)_{\mathcal{S}}$ as a shorthand for $L_{\mathbb{K}}(E,F)_{\mathcal{S}(E)}$. We write $E'_{\mathcal{S}} := L_{\mathbb{K}}(E,\mathbb{K})_{\mathcal{S}}$.

Throughout this section, we let $\mathcal{S}$ be a bounded set-functor such that, for each locally convex space E, we have

$$\{K \subseteq E \colon K \text{ is compact}\} \subseteq \mathcal{S}(E). \tag{16}$$

Then, $\{x\} \in \mathcal{S}(E)$ for each $x \in E$, and we get a continuous linear point evaluation

$$\eta_{E,\mathcal{S}}(x) \colon E'_{\mathcal{S}} \to \mathbb{K}, \quad \lambda \mapsto \lambda(x).$$

Definition 8. *A* locally convex Poisson vector space *with respect to $\mathcal{S}$ is a locally convex $\mathbb{K}$-vector space E such that $E \times E$ is a $k_{\mathbb{R}}$-space and*

$$\eta_{E,\mathcal{S}} \colon E \to (E'_{\mathcal{S}})'_{\mathcal{S}}, \quad x \mapsto \eta_{E,\mathcal{S}}(x)$$

a topological embedding, together with a bilinear map $[.,.] \colon E'_{\mathcal{S}} \times E'_{\mathcal{S}} \to E'_{\mathcal{S}}$, $(\lambda,\eta) \mapsto [\lambda,\eta]$, which makes $E'_{\mathcal{S}}$ a Lie algebra, is $\mathcal{S}(E'_{\mathcal{S}})$-hypocontinuous in its second argument, and satisfies

$$\eta_{E,\mathcal{S}}(x) \circ \mathrm{ad}_\lambda \;\in\; \eta_{E,\mathcal{S}}(E) \quad \textit{for all } x \in E \textit{ and } \lambda \in E', \tag{17}$$

writing $\mathrm{ad}_\lambda := \mathrm{ad}(\lambda) := [\lambda,.] \colon E' \to E'$.

Remark 16. *(a) Definition 16.35 in [8] was more restrictive; E was assumed to be a k^∞-space there.*
(b) In [8] (16.31 (b)), the following additional condition was imposed: For each $M \in \mathcal{S}(E'_{\mathcal{S}})$ and $N \in \mathcal{S}(E)$, the set $\varepsilon(M \times N)$ is bounded in $\mathbb{K}$, where $\varepsilon \colon E' \times E \to \mathbb{K}$ is the evaluation map. As we assume (16), the latter condition is automatically satisfied, by [8] (Proposition 16.11 (a) and Proposition 16.14).
*(c) Let us say that a locally convex space E is $\mathcal{S}$-*reflexive *if $\eta_{E,\mathcal{S}} \colon E \to (E'_{\mathcal{S}})'_{\mathcal{S}}$ is an isomorphism of topological vector spaces.*
(d) Of course, we are mostly interested in the case where $[.,.]$ is continuous, but only hypocontinuity is required for the basic theory.

Definition 9. *Let $(E,[.,.])$ be a locally convex Poisson vector space with respect to $\mathcal{S}$, and $U \subseteq E$ be open. Given $f,g \in C^\infty_{\mathbb{K}}(U,\mathbb{K})$, we define a function $\{f,g\} \colon U \to \mathbb{K}$ via*

$$\{f,g\}(x) \;:=\; \langle [f'(x),g'(x)],x\rangle \quad \textit{for } x \in U, \tag{18}$$

where $\langle .,.\rangle \colon E' \times E \to \mathbb{K}$, $\langle \lambda,x\rangle := \lambda(x)$ is the evaluation map and $f'(x) = df(x,.)$.

Condition (17) in Definition 8 enables us to define a map $X_f \colon U \to E$ via

$$X_f(x) \;:=\; \eta^{-1}_{E,\mathcal{S}}\big(\eta_{E,\mathcal{S}}(x) \circ \mathrm{ad}(f'(x))\big) \quad \textit{for } x \in U. \tag{19}$$

Using Lemma 11 instead of [8] (Theorem 16.26), we see as in the proof of [8] (Theorem 16.40 (a)) that the function $\{f,g\} \colon U \to \mathbb{K}$ is $C^\infty_{\mathbb{K}}$. The $C^\infty_{\mathbb{K}}$-function $\{f,g\}$ is called the *Poisson bracket* of f and g. Using Lemma 11 instead of [8] (Theorem 16.26), we see as in the proof of [8] (Theorem 16.40 (b)) that $X_f \colon U \to E$ is a $C^\infty_{\mathbb{K}}$-map; it is called the *Hamiltonian vector field* associated with f. As in [8] (Remark 16.43), we see that the Poisson bracket just defined makes $C^\infty_{\mathbb{K}}(U,\mathbb{K})$ a Poisson algebra.

We shall write "b" and "c" in place of $\mathcal{S}$ if $\mathcal{S}$ is the bounded set functor, taking a locally convex space E to the set $\mathcal{S}(E)$ of all bounded subsets and compact subsets of E, respectively. Both of these satisfy the hypothesis (16).

In the following, we describe new results for locally convex Poisson vector spaces over $\mathcal{S} = c$. We mention that the embedding property of $\eta_{E,c}$ is automatic in this case, as $E \times E$ is a $k_{\mathbb{R}}$-space in Definition 9; thus, E is a $k_{\mathbb{R}}$-space and Remark 13 applies.

Example 4. *Let $(\mathfrak{g}_j)_{j\in J}$ be a family of finite-dimensional real Lie algebras $\mathfrak{g}_j$. Endow $\mathfrak{g} := \bigoplus_{j\in J} \mathfrak{g}_j$ with the locally convex direct sum topology, which coincides with the finest locally convex vector topology. Then, $\mathfrak{g}$ is c-reflexive, as with every vector space with its finest locally convex vector topology (see [39] (Theorem 7.30 (a))). As a consequence, also $\mathfrak{g}'_c$ is c-reflexive (cf. [39] (Proposition 7.9 (iii))). Using [40] (Proposition 7.1), we see that the component-wise Lie bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is continuous on the locally convex space $\mathfrak{g} \times \mathfrak{g}$, which is naturally isomorphic to the locally convex direct sum $\bigoplus_{j\in J}(\mathfrak{g}_j \times \mathfrak{g}_j)$. We set $E := \mathfrak{g}'_c$ and give E'_c the continuous Lie bracket $[.,.]$ making $\eta_{\mathfrak{g},c} \colon \mathfrak{g} \to (\mathfrak{g}'_c)'_c = E'_c$ an isomorphism of topological Lie algebras. Then*

$$E = \mathfrak{g}'_c \cong \prod_{j\in J}(\mathfrak{g}_j)'_c$$

and $E \times E$ are $k_{\mathbb{R}}$-spaces, being Cartesian products of locally compact spaces (see [22]). Thus, $(E, [.,.])$ is a locally convex Poisson vector space over $\mathcal{S} = c$, in the sense of Definition 8. If J has cardinality $\geq 2^{\aleph_0}$ and $\mathfrak{g}_j \neq \{0\}$ for all $j \in J$ (e.g., if we take an abelian 1-dimensional Lie algebra $\mathfrak{g}_j$ for each $j \in J$), then $E \cong \mathbb{R}^J$ is not a k-space. Hence, E is not a k^∞-space, and hence it is not a Poisson vector space in the more restrictive sense of [8].

12. Continuity Properties of the Poisson Bracket

If E and F are locally convex $\mathbb{K}$-vector spaces and $U \subseteq E$ an open subset, we endow $C^\infty(U, F)$ with the compact-open C^∞-topology. Our goal is the following result:

Theorem 1. *Let $(E, [.,.])$ be a locally convex Poisson vector space with respect to $\mathcal{S} = c$. Let $U \subseteq E$ be open. Then, the Poisson bracket*

$$\{.,.\} \colon C^\infty_{\mathbb{K}}(U, \mathbb{K}) \times C^\infty_{\mathbb{K}}(U, \mathbb{K}) \to C^\infty_{\mathbb{K}}(U, \mathbb{K})$$

is c-hypocontinuous in its second variable. If $[.,.] \colon E'_c \times E'_c \to E'_c$ is continuous, then also the Poisson bracket is continuous.

Various auxiliary results are needed to prove Theorem 1. With little risk of confusion with subsets of spaces of operators, given a 0-neighbourhood $W \subseteq F$ and a compact set $K \subseteq U$, we shall write $\lfloor K, W \rfloor := \{f \in C(U, F) \colon f(K) \subseteq W\}$.

Lemma 17. *Let E, F be locally convex spaces and $U \subseteq E$ be open. Then, the linear map*

$$D \colon C^\infty_{\mathbb{K}}(U, F) \to C^\infty_{\mathbb{K}}(U, L(E, F)_c), \quad f \mapsto f'$$

is continuous.

Proof. By Corollary 1, $f' \in C^\infty_{\mathbb{K}}(U, L(E, F)_c)$ for each $f \in C^\infty_{\mathbb{K}}(U, F)$. As D is linear and also $C^\infty(U, L(E, F)_c) \to C(U \times E^k, L(E, F)_c)$, $f \mapsto d^k f$ is linear for each $k \in \mathbb{N}_0$,

$$d^k \circ D \colon C^\infty(U, F) \to C(U \times E^k, L(E, F)_c)_{c.o.} \tag{20}$$

is linear, whence it will be continuous if it is continuous at 0. We pick a typical 0-neighbourhood in $C(U \times E^k, L(E, F)_c)_{c.o.}$, say $\lfloor K, V \rfloor$ with a compact subset $K \subseteq U \times E^k$ and a 0-neighbourhood $V \subseteq L(E, F)_c$. After shrinking V, we may assume that $V = \lfloor A, W \rfloor$ for some compact set $A \subseteq E$ and 0-neighbourhood $W \subseteq F$.

We now recall that for $f \in C^\infty_{\mathbb{K}}(U,F)$, we have

$$d^k(f')(x,y_1,\ldots,y_k) \;=\; d^{k+1}f(x,y_1,\ldots,y_k,\cdot)\colon E \to F \tag{21}$$

for all $k \in \mathbb{N}_0$, $x \in U$ and $y_1,\ldots,y_k \in E$ (cf. Corollary 1). Since $\lfloor K \times A, W\rfloor$ is an open 0-neighbourhood in $C(U \times E^{k+1}, F)$ and the map $C^\infty(U,F) \to C(U \times E^{k+1}, F)_{c.o.}$, $f \mapsto d^{k+1}f$ is continuous, we see that the set Ω of all $f \in C^\infty(U,F)$ such that $d^{k+1}f \in \lfloor K \times A, W\rfloor$ is a 0-neighbourhood in $C^\infty(U,F)$. In view of (21), we have $d^k(f') \in \lfloor K, \lfloor A, W\rfloor\rfloor$ for each $f \in \Omega$. Hence, $d^k \circ D$ from (20) is continuous at 0, as required. □

Lemma 18. *Let X be a Hausdorff topological space, F be a locally convex space, $K \subseteq X$ be compact and $M \subseteq C(X,F)_{c.o.}$ be compact. Let $\mathrm{ev}\colon C(X,F) \times X \to F$, $(f,x) \mapsto f(x)$ be the evaluation map. Then, $\mathrm{ev}(M \times K)$ is compact.*

Proof. The map $\rho\colon C(X,F)_{c.o.} \to C(K,F)_{c.o.}$, $f \mapsto f|_K$ is continuous by [20] (§3.2 (2)). Thus, $\rho(M)$ is compact in $C(K,F)_{c.o.}$. The map $\varepsilon\colon C(K,F) \times K \to F$, $(f,x) \mapsto f(x)$ is continuous by [20] (Theorem 3.4.2). Hence, $\mathrm{ev}(M \times K) = \varepsilon(\rho(M) \times K)$ is compact. □

Lemma 19. *Let E, F_1, F_2, and G be locally convex $\mathbb{K}$-vector spaces and $\beta\colon F_1 \times F_2 \to G$ be a bilinear map which is c-hypocontinuous in its second argument. Let $U \subseteq E$ be an open subset and $r \in \mathbb{N}_0 \cup \{\infty\}$. Assume that $E \times E$ is a $k_{\mathbb{R}}$-space, or $r = 0$ and E is a $k_{\mathbb{R}}$-space, or $(r,\mathbb{K}) = (\infty,\mathbb{C})$ and E is a $k_{\mathbb{R}}$-space. Then, the following holds:*

(a) *We have $\beta \circ (f,g) \in C^r_{\mathbb{K}}(U,G)$ for all $(f,g) \in C^r_{\mathbb{K}}(U,F_1) \times C^r_{\mathbb{K}}(U,F_2)$. The map*

$$C^r_{\mathbb{K}}(U,\beta)\colon C^r_{\mathbb{K}}(U,F_1) \times C^r_{\mathbb{K}}(U,F_2) \to C^r_{\mathbb{K}}(U,G),\;\; (f,g) \mapsto \beta \circ (f,g)$$

is bilinear. For each compact subset $M \subseteq C^r_{\mathbb{K}}(U,F_2)$ and 0-neighbourhood $W \subseteq C^r_{\mathbb{K}}(U,G)$, there is a 0-neighbourhood $V \subseteq C^r_{\mathbb{K}}(U,F_1)$ such that $C^r_{\mathbb{K}}(U,\beta)(V \times M) \subseteq W$.

(b) *For each $g \in C^r_{\mathbb{K}}(U,F_2)$, the map $C^r_{\mathbb{K}}(U,F_1) \to C^r_{\mathbb{K}}(U,G)$, $f \mapsto \beta \circ (f,g)$ is continuous and linear.*

(c) *If β is also c-hypocontinuous in its first argument, then $C^r_{\mathbb{K}}(U,\beta)$ is c-hypocontinuous in its second argument and c-hypocontinuous in its first argument.*

(d) *If β is continuous, then $C^r_{\mathbb{K}}(U,\beta)$ is continuous.*

Proof. (a) By Lemma 11, $\beta \circ (f,g) \in C^r_{\mathbb{K}}(U,G)$. The bilinearity of $C^r(U,\beta)$ is clear. It suffices to prove the remaining assertion for each $r \in \mathbb{N}_0$. To see this, let $M \subseteq C^\infty_{\mathbb{K}}(U,F_2)$ be a compact subset and $W \subseteq C^\infty_{\mathbb{K}}(U,G)$ be a 0-neighbourhood. Since the topology on $C^\infty_{\mathbb{K}}(U,G)$ is initial with respect to the family of inclusion maps $C^\infty_{\mathbb{K}}(U,G) \to C^r_{\mathbb{K}}(U,G)$ for $r \in \mathbb{N}_0$, there exists $r \in \mathbb{N}_0$ and a 0-neighbourhood Q in $C^r_{\mathbb{K}}(U,G)$ such that $C^\infty_{\mathbb{K}}(U,G) \cap Q \subseteq W$. If the assertion holds for r, we find a 0-neighbourhood $P \subseteq C^r_{\mathbb{K}}(U,F_1)$ such that $C^r_{\mathbb{K}}(U,\beta)(P \times M) \subseteq Q$. Then, $V := C^\infty_{\mathbb{K}}(U,F_1) \cap P$ is a 0-neighbourhood in $C^\infty_{\mathbb{K}}(U,F_1)$ and $C^\infty_{\mathbb{K}}(U,\beta)(V \times M) \subseteq C^\infty_{\mathbb{K}}(U,G) \cap C^r_{\mathbb{K}}(U,\beta)(P \times M) \subseteq C^\infty_{\mathbb{K}}(U,G) \cap Q \subseteq W$.

The case $r = 0$. Let $M \subseteq C(U,F_2)$ be compact and $W \subseteq C(U,G)$ be a 0-neighbourhood. Then, $\lfloor K, Q\rfloor \subseteq W$ for some compact subset $K \subseteq U$ and some 0-neighbourhood $Q \subseteq G$. By Lemma 18, the set $N := \mathrm{ev}(M \times K) \subseteq F_2$ is compact, where $\mathrm{ev}\colon C(U,F_2) \times U \to F_2$ is the evaluation map. Since β is c-hypocontinuous in its second argument, there exists a 0-neighbourhood $P \subseteq F_1$ with $\beta(P \times N) \subseteq Q$. Then, $\beta \circ (\lfloor K, P\rfloor \times M) \subseteq \lfloor K, Q\rfloor \subseteq W$.

Induction step. Let $M \subseteq C^r_{\mathbb{K}}(U,F_2)$ be a compact subset and $W \subseteq C^r_{\mathbb{K}}(U,G)$ be a 0-neighbourhood. The topology on $C^r(U,G)$ is initial with respect to the linear maps $\lambda_1\colon C^r_{\mathbb{K}}(U,G) \to C(U,G)_{c.o.}$, $f \mapsto f$ and $\lambda_2\colon C^r_{\mathbb{K}}(U,G) \to C^{r-1}_{\mathbb{K}}(U \times E, G)$, $f \mapsto df$ (by [26] (Lemma A.1 (d))). Note that the ordinary C^r-topology is used there, by [26] (Proposition 4.19 (d) and Lemma A2). After shrinking W, we may therefore assume that

$$W = (\lambda_1)^{-1}(W_1) \cap (\lambda_2)^{-1}(W_2)$$

with absolutely convex 0-neighbourhoods $W_1 \subseteq C(U,G)$ and $W_2 \subseteq C^{r-1}_{\mathbb{K}}(U \times E, G)$. Applying the case $r = 0$ to $C(U,\beta)$, we find a 0-neighbourhood $V_1 \subseteq C(U,F_1)$ such that $C(U,\beta)(V_1 \times M) \subseteq W_1$. The map $\delta_j\colon C^r_{\mathbb{K}}(U,F_j) \to C^{r-1}_{\mathbb{K}}(U \times E, F_j)$, $f \mapsto df$ is continuous linear and $\pi\colon U \times E \to U$, $(x,y) \mapsto x$ is smooth, whence $\rho_j\colon C^r_{\mathbb{K}}(U,F_j) \to C^{r-1}_{\mathbb{K}}(U \times E, F_j)$, $f \mapsto f \circ \pi$ is continuous linear (cf. [26] (Lemma 4.4) or [13] (Proposition 1.7.11)). By (5),

$$\lambda_2 \circ C^r_{\mathbb{K}}(U,\beta) \;=\; C^{r-1}_{\mathbb{K}}(U \times E,\beta) \circ (\delta_1 \times \rho_2) \;+\; C^{r-1}_{\mathbb{K}}(U \times E,\beta) \circ (\rho_1 \times \delta_2)\,. \tag{22}$$

The subsets $\rho_2(M) \subseteq C^{r-1}_{\mathbb{K}}(U \times E, F_2)$ and $\delta_2(M) \subseteq C^{r-1}_{\mathbb{K}}(U \times E, F_2)$ are compact. Using the case $r-1$ (with $U \times E$ in place of U), which holds as the inductive hypothesis, we find 0-neighbourhoods $V_2, V_3 \subseteq C^{r-1}_{\mathbb{K}}(U \times E, F_1)$ such that $C^{r-1}_{\mathbb{K}}(U,\beta)(V_2 \times \rho_2(M)) \subseteq (1/2)W_2$ and $C^{r-1}_{\mathbb{K}}(U,\beta)(V_3 \times \delta_2(M)) \subseteq (1/2)W_2$. Then, $Q := (\delta_1)^{-1}(V_2) \cap (\rho_1)^{-1}(V_3)$ is an open 0-neighbourhood in $C^r_{\mathbb{K}}(U,F_1)$. Since $(1/2)W_2 + (1/2)W_2 = W_2$, we deduce from (22) that

$$\lambda_2(C^r_{\mathbb{K}}(U,\beta)(Q \times M)) \subseteq C^{r-1}_{\mathbb{K}}(U \times E,\beta)(V_2 \times \rho_2(M)) + C^{r-1}_{\mathbb{K}}(U \times E,\beta)(V_3 \times \delta_2(M)) \subseteq W_2.$$

Thus, $C^r_{\mathbb{K}}(U,\beta)(Q \times M) \subseteq (\lambda_2)^{-1}(W_2)$. Now, $V := V_1 \cap Q$ is a 0-neighbourhood in $C^r_{\mathbb{K}}(U,F_1)$ such that $C^r_{\mathbb{K}}(U,\beta)(V \times M) \subseteq (\lambda_1)^{-1}(W_1) \cap (\lambda_2)^{-1}(W_2) = W$.

(b) Since $C^r_{\mathbb{K}}(U,\beta)$ is bilinear, the map $f \mapsto \beta \circ (f,g)$ is linear. Its continuity follows from (a), applied with the singleton $M := \{g\}$.

(c) By (a) just established, the condition in Lemma 4 (a) is satisfied. By (b), the map $C^r_{\mathbb{K}}(U,\beta)$ is continuous in its first argument. Interchanging the roles of F_1 and F_2, we see that $C^r_{\mathbb{K}}(M,\beta)$ is also continuous in its second argument and hence c-hypocontinuous in its second argument. Likewise, $C^r_{\mathbb{K}}(U,\beta)$ is c-hypoocontinuous in its first argument.

(d) If β is continuous and hence smooth, then $C^r(U,\beta)$ is smooth and hence continuous, as a very special case of [26] (Proposition 4.16) or [13] (Corollary 1.7.13). □

Proof of Theorem 1. By Lemma 17, the mapping $D\colon C^\infty(U,\mathbb{K}) \to C^\infty(U,E'_c)$, $f \mapsto f'$ is continuous and linear. By Lemma 19 (c), the bilinear map

$$C^\infty(U,[.,.])\colon C^\infty(U,E') \times C^\infty(U,E') \to C^\infty(U,E')\,, \quad (f,g) \mapsto (x \mapsto [f(x),g(x)])$$

is c-hypocontinuous in its second argument; if $[.,.]$ is continuous, then also $C^\infty(U,[.,.])$, by Lemma 19 (d). The evaluation map $\beta\colon E \times E'_c \to \mathbb{K}$, $(x,\lambda) \mapsto \lambda(x)$ is c-hypocontinuous in its first argument, by Proposition 7. As a consequence, $\beta_*\colon C^\infty(U,E'_c) \to C^\infty(U,\mathbb{K})$, $f \mapsto \beta \circ (\mathrm{id}_U, f)$ is continuous linear by Lemma 19 (b). Since

$$\{.,.\} \;=\; \beta_* \circ C^\infty(U,[.,.]) \circ (D \times D)$$

by definition, we see that $\{.,.\}$ is a composition of continuous maps if $[.,.]$ is continuous, and hence continuous. In the general case, $\{.,.\}$ is a composition of a bilinear map which is c-hypocontinuous in its second argument and continuous linear maps, whence $\{.,.\}$ is c-hypocontinuous in its second arguemnt. □

13. Continuity of the Map Taking f to the Hamiltonian Vector Field X_f

In this section, we show the continuity of the mapping which takes a smooth function to the corresponding Hamiltonian vector field, in the case $\mathcal{S} = c$.

Theorem 2. *Let $(E,[.,.])$ be a locally convex Poisson vector space with respect to $\mathcal{S} = c$. Let $U \subseteq E$ be an open subset. Then, the map*

$$\Psi\colon C^\infty_{\mathbb{K}}(U,\mathbb{K}) \to C^\infty_{\mathbb{K}}(U,E)\,, \quad f \mapsto X_f \tag{23}$$

is continuous and linear.

Proof. Let $\eta_E\colon E \to (E'_c)'_c$ be the evaluation homomorphism and $V := \{A \in L(E'_c, E'_c)\colon (\forall x \in E)\ \eta_E(x) \circ A \in \eta_E(E)\}$. Then, V is a vector subspace of $L(E'_c, E'_c)$ and $\mathrm{ad}(E') \subseteq V$. The composition map $\Gamma\colon (E'_c)'_c \times L(E'_c, E'_c)_c \to (E'_c)'_c$, $(\alpha, A) \mapsto \alpha \circ A$ is hypocontinuous with respect to equicontinuous subsets of $(E'_c)'_c$, by Proposition 9 in [11] (Chapter III, §5, no. 5). If $K \subseteq E$ is compact, then the polar K° is a 0-neighbourhood in E'_c, entailing that $(K^\circ)^\circ \subseteq (E'_c)'$ is equicontinuous. Hence, η_E takes compact subsets of E to equicontinuous subsets of $(E'_c)'$, and hence

$$\beta\colon E \times V \to E\,, \quad (x, A) \mapsto \eta_E^{-1}(\Gamma(\eta_E(x), A))$$

is c-hypocontinuous in its first argument. By Lemma 19 (c), $\beta_*\colon C^\infty(U, V) \to C^\infty(U, E)$, $f \mapsto \beta \circ (\mathrm{id}_U, f)$ is continuous linear. Moreover, the map $D\colon C^\infty(U, \mathbb{K}) \to C^\infty(U, E'_c)$, $f \mapsto f'$ is continuous linear by Lemma 17. Furthermore, $\mathrm{ad} = [.,.]^\vee\colon E'_c \to L(E'_c, E'_c)_c$ is continuous linear since $[.,.]$ is c-hypocontinuous in its second argument (see Lemma 4 (b)), whence

$$C^\infty(U, \mathrm{ad})\colon C^\infty(U, E'_c) \to C^\infty(U, L(E'_c, E'_c)_c)\,, \quad f \mapsto \mathrm{ad} \circ f$$

is continuous linear (see, e.g., [26] (Lemma 4.13), or [13] (Corollary 1.7.13)). Hence, $\Psi = \beta_* \circ C^\infty(U, \mathrm{ad}) \circ D$ is continuous and linear. □

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Appendix A. Proofs for Some Basic Facts

We give proofs for various facts stated in Section 2.

Proof of Lemma 1. Let $E := E_1 \times \cdots \times E_k$. Since $df\colon U \times E \times X \times E \to F$ is continuous and $df(x, 0, 0, 0) = 0$, given q, there exists a continuous seminorm p on X such that $B_1^p(x) \subseteq U$, and continuous seminorms p_j on E_j for $j \in \{1, \ldots, k\}$ such that

$$\|df(y, v_1, \ldots, v_k, z, w_1, \ldots, w_k)\|_q \leq 1 \tag{A1}$$

for all $v_j, w_j \in B_1^{p_j}(0)$, $y \in B_1^p(x)$, and $z \in B_1^p(0)$. For $y \in B_1^p(x)$ and $(v_1, \ldots, v_k) \in B_1^{p_1}(0) \times \cdots \times B_1^{p_k}(0)$, the Mean Value Theorem (see [13] (Proposition 1.2.6)) shows that

$$f(y, v_1, \ldots, v_k) = \int_0^1 df(y, tv_1, \ldots, tv_k, 0, v_1, \ldots, v_k)\, dt\,.$$

Since $\|df(y, tv_1, \ldots, tv_k, 0, v_1, \ldots, v_k)\|_q \leq 1$ for each t, it follows that $\|f(y, v_1, \ldots, v_k)\|_q \leq 1$ in the preceding situation. Because $f(y, \cdot)$ is k-linear, we deduce that (1) holds. To prove (2), we first note that (A1) implies that

$$\|df(y, v_1, \ldots, v_k, z, 0, \ldots, 0)\|_q \leq \|z\|_p \tag{A2}$$

for all $y \in B_1^p(x)$, $(v_1, \ldots, v_k) \in B_1^{p_1}(0) \times \cdots \times B_1^{p_k}(0)$ and $z \in X$, exploiting the linearity of $df(y, v_1, \ldots, v_k, z, 0, \ldots, 0)$ in z. We now use the Mean Value Theorem to write

$$f(y, v_1, \ldots, v_k) - f(x, v_1, \ldots, v_k) = \int_0^1 df(x + t(y - x), v_1, \ldots, v_k, y - x, 0, \ldots, 0)\, dt$$

for $y \in B_1^p(x)$ and $(v_1, \dots, v_k) \in B_1^{p_1}(0) \times \cdots \times B_1^{p_k}(0)$. By (A2), we have

$$\|df(x + t(y - x), v_1, \dots, v_k, y - x, 0, \dots, 0)\|_q \leq \|y - x\|_p$$

and hence $\|f(y, v_1, \dots, v_k) - f(x, v_1, \dots, v_k)\|_q \leq \|y - x\|_p$. Now, (2) follows, using the k-linearity of the map $f(y, \cdot) - f(x, \cdot)\colon E_1 \times \cdots \times E_k \to F$. □

Proof of Lemma 2. By the Polarisation Formula for symmetric k-linear maps (see, e.g., ([13], Proposition 1.6.19)), we have

$$f(x, y_1, \dots, y_k) = \frac{1}{k!2^k} \sum_{\varepsilon_1, \dots, \varepsilon_k \in \{-1,1\}} \varepsilon_1 \cdots \varepsilon_k\, h(x, \varepsilon_1 y_1 + \cdots + \varepsilon_k y_k)$$

for all $x \in U$ and $y_1, \dots, y_k \in E$. Thus, f is $C^r_{\mathbb{K}}$ if h is so. □

Proof of Lemma 3. (a) Let $\mathrm{pr}_2\colon X_1 \times X_2 \to X_2$, $(x, y) \mapsto y$ be the projection onto the second component and pick $x_0 \in X_1$. Since pr_2 is continuous, every k-continuous function $f\colon X_2 \to \mathbb{R}$ yields a k-continuous function $f \circ \mathrm{pr}_2$ on X. Then, $f \circ \mathrm{pr}_2$ is continuous and hence also $f = (f \circ \mathrm{pr}_2)(x_0, \cdot)$.

(b) Let $f\colon U \to \mathbb{R}$ be k-continuous and $x \in U$. As X is completely regular, we find a continuous function $g\colon X \to \mathbb{R}$ with $g(x) \neq 0$ and support $\mathrm{supp}(g) \subseteq U$. Define $h\colon X \to \mathbb{R}$ via $h(y) := f(y)g(y)$ if $y \in U$, $h(y) := 0$ if $y \in X \setminus \mathrm{supp}(g)$. If $K \subseteq X$ is a compact subset, then each $x \in K$ has a compact neighbourhood K_x in K which is contained in U or in $X \setminus \mathrm{supp}(g)$. In the first case, $h|_{K_x} = f|_{K_x} g|_{K_x}$ is continuous by k-continuity of f. In the second case, $h|_{K_x} = 0$ is continuous as well. Thus, $h|_K$ is continuous. Since X is a $k_{\mathbb{R}}$-space, continuity of h follows. Thus, f is continuous on the open x-neighbourhood $g^{-1}(\mathbb{R} \setminus \{0\})$. □

A simple fact will be useful (see, e.g., [8] (Lemma 1.13)).

Lemma A1. *Let X be a topological space, F be a locally convex space, and $BC(X, F)$ be the space of bounded F-valued continuous functions on X, endowed with the topology of uniform convergence. Then, $\mu\colon BC(X, F) \times X \to F$, $(f, x) \mapsto f(x)$ is continuous.*

Proof of Lemma 4. (If $k = 2$, see Proposition 3 and 4 in [11] (Chapter III, §5, no. 3) for the equivalence (a)⇔(b) and the implication (b)⇒(c); (c)⇒(a) can be found in [8] (Proposition 1.8).)

(a)⇔(b): $\beta(V \times M) \subseteq W$ is equivalent to $\beta^\vee(V) \in \lfloor M, W \rfloor$. Hence, (a) is equivalent to continuity of $\beta^\vee$ in 0 and hence to its continuity (see Proposition 5 in [11] (Chapter I, §1, no. 6)).

(b)⇒(c): If $M \in \mathcal{S}$, then $\varepsilon\colon L^{k-j+1}(E_j, \dots, E_k, F)_{\mathcal{S}} \times M \to F$, $\varepsilon(\alpha, x) := \alpha(x)$ is continuous as a consequence of Lemma A1. Hence, $\beta|_{E_1 \times \cdots \times E_{j-1} \times M} = \varepsilon \circ (\beta^\vee \times \mathrm{id}_M)$ is continuous.

(c)⇒(a) if (3) holds: Given $M \in \mathcal{S}$ and a 0-neighbourhood $W \subseteq F$, by hypothesis, we can find $N \in \mathcal{S}$ such that $\mathbb{D}M \subseteq N$. By continuity of $\beta|_{E_1 \times \cdots \times E_{j-1} \times N}$, there exist 0-neighbourhoods $V_i \subseteq E_i$ for $i \in \{1, \dots, k\}$ such that $\beta(V \times (N \cap U)) \subseteq W$, where $V := V_1 \times \cdots \times V_{j-1}$ and $U := V_j \times \cdots \times V_k$. Set $a := \frac{j-1}{k-j+1}$. Since M is bounded, $M \subseteq n^a U$ for some $n \in \mathbb{N}$. Then, $\frac{1}{n^a} M \subseteq N \cap U$. Using that β is k-linear, we obtain $\beta((\frac{1}{n} V) \times M) = \beta(V \times (\frac{1}{n^a} M)) \subseteq \beta(V \times (N \cap U)) \subseteq W$. □

Proof of Lemma 7. Given $\alpha \in L^k(E_1, \dots, E_k, F)$, we have $\varepsilon^\vee(\alpha) = \varepsilon(\alpha, \cdot) = \alpha$, which is a continuous k-linear map. The map ε is also continuous in its first argument, as the topology on $L^k(E_1, \dots, E_k, F)_{\mathcal{S}}$ is finer than the topology of pointwise convergence, by the hypothesis on $\mathcal{S}$. The linear map $\varepsilon^\vee\colon L^k(E_1, \dots, E_k)_{\mathcal{S}} \to L^k(E_1, \dots, E_k)_{\mathcal{S}}$, $\alpha \mapsto \alpha$ being continuous, condition (b) of Lemma 4 is satisfied by ε in place of β and hence also the equivalent condition (a), whence ε is $\mathcal{S}$-hypocontinuous in its arguments $(2, \dots, k+1)$.

Now, assume that $k = 1$. Since $\mathcal{O}$ is finer than the topology of pointwise convergence, the map ε remains separately continuous in the situation described at the end of the lemma. Hence, if E is barrelled, Lemma 6 ensures hypocontinuity with respect to $\mathcal{T}$. □

Proof of Lemma 8. (a) The composition $\beta \circ f$ is sequentially continuous and hence continuous, its domain X being first countable.

(b) Write $f = (f_1, \ldots, f_k)$ with components $f_j\colon X \to E_\nu$ for $\nu \in \{1, \ldots, k\}$. If K is a compact subset of X, then $M := (f_j, \ldots, f_k)(K)$ is a compact subset of $E_j \times \cdots \times E_k$. Since $\beta|_{E_1 \times \cdots \times E_{j-1} \times M}$ is continuous by Lemma 4 (c), the composition

$$\beta \circ f|_K = \beta|_{E_1 \times \cdots \times E_{j-1} \times M} \circ f|_K$$

is continuous. Thus, $\beta \circ f$ is k-continuous and hence continuous, as X is a $k_{\mathbb{R}}$-space and F is completely regular. □

Proof of Lemma 9. (a) The case $r = 0$: Let q be a continuous seminorm on $F := \prod_{j \in J} F_j$, and $x \in U$. After increasing q, we may assume that

$$q(y) = \max\{q_j(y_j)\colon j \in \Phi\} \quad \text{for all } y = (y_j)_{j \in J} \in F, \tag{A3}$$

for some non-empty, finite subset $\Phi \subseteq J$ and continuous seminorms q_j on F_j for $j \in \Phi$. If each f_j is $LC^0_{\mathbb{K}}$, then we find a continuous seminorm p_j on E for each $j \in \Phi$ such that $B^{p_j}_1(x) \subseteq U$ and $q_j(f_j(z) - f_j(y)) \leq p_j(z - y)$ for all $z, y \in B^{p_j}_1(x)$. Then

$$p\colon E \to [0, \infty[, \quad y \mapsto \max\{p_j(y)\colon j \in \Phi\}$$

is a continuous seminorm on E such that $B^p_1(x) \subseteq U$ and $q(f(z) - f(y)) \leq p(z - y)$ for all $z, y \in B^p_1(x)$. If f is $LC^0_{\mathbb{K}}$, let us show that f_j is $LC^0_{\mathbb{K}}$ for each $j \in J$. Let q be a continuous seminorm on F_j and $x \in U$. Let $\mathrm{pr}_j\colon F \to F_j$, $(y_i)_{i \in J} \mapsto y_j$ be the continuous linear projection onto the jth component. Then, $q \circ \mathrm{pr}_j$ is a continuous seminorm on F, whence we find a continuous seminorm p on E such that $B^p_1(x) \subseteq U$ and $(q \circ \mathrm{pr}_j)(f(z) - f(y)) \leq p(z - y)$ for all $z, y \in B^p_1(x)$. Since $(q \circ \mathrm{pr}_j)(f(z) - f(y)) = q(f_j(z) - f_j(y))$, we see that f_j is $LC^0_{\mathbb{K}}$.

If $r \in \mathbb{N} \cup \{\infty\}$, then f is $C^r_{\mathbb{K}}$ if and only if each f_j is $C^r_{\mathbb{K}}$, and $d^k f = (d^k f_j)_{j \in J}$ in this case for all $k \in \mathbb{N}_0$ such that $k \leq r$ (see [13] (Lemma 1.3.3)). By the case $r = 0$, the map $d^k f$ is $LC^0_{\mathbb{K}}$ if and only if $d^k(f_j)$ is $LC^0_{\mathbb{K}}$ for all $j \in J$. The assertion follows.

(b) Let E, F, and Y be locally convex $\mathbb{K}$-vector spaces and $f\colon U \to F$ as well as $g\colon V \to Y$ be $LC^r_{\mathbb{K}}$-maps on open subsets $U \subseteq E$ and $V \subseteq F$, such that $f(U) \subseteq V$.

If $r = 0$, let $x \in U$ and q be a continuous seminorm on Y. There exists a continuous seminorm p on F such that $B^p_1(f(x)) \subseteq V$ and $q(g(b) - g(a)) \leq p(b - a)$ for all $a, b \in B^p_1(f(x))$. There exists a continuous seminorm P on E with $B^P_1(x) \subseteq U$ and $p(f(z) - f(y)) \leq P(z - y)$ for all $z, y \in B^P_1(x)$. Then, $f(B^P_1(x)) \subseteq B^p_1(f(x))$ and hence

$$q(g(f(z)) - g(f(y))) \leq p(f(z) - f(y)) \leq P(z - y)$$

for all $y, z \in B^P_1(x)$. Thus, $g \circ f\colon U \to Y$ is $LC^0_{\mathbb{K}}$.

If $r \in \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}$ such that $k \leq r$, we can use Faà di Bruno's Formula

$$d^k(g \circ f)(x, y) = \sum_{j=1}^{k} \sum_{P \in P_{k,j}} d^j g(f(x), d^{|I_1|}(x, y_{I_1}), \ldots, d^{|I_j|}(x, y_{I_j})) \tag{A4}$$

for $x \in U$ and $y = (y_1, \ldots, y_k) \in E^k$, as in [13] (Theorem 1.3.18). Here, $P_{k,j}$ is the set of all partitions $P = \{I_1, \ldots, I_j\}$ of $\{1, \ldots, k\}$ into j disjoint, non-empty subsets $I_1, \ldots, I_j \subseteq \{1, \ldots, k\}$. For a non-empty subset $J \subseteq \{1, \ldots, k\}$ with elements $j_1 < \cdots < j_m$, let $y_J := (y_{j_1}, \ldots, y_{j_m})$. Using (a) and the case $r = 0$, we deduce from (A4) that $d^k(g \circ f)$ is $LC^0_{\mathbb{K}}$.

(c) For each continuous seminorm q on F, the restriction $q|_{F_0}$ is a continuous seminorm on F_0, and each continuous seminorm Q on F_0 arises in this way. In fact, we find an open, absolutely convex 0-neighbourhood $V \subseteq F$ such that $V \cap F_0 \subseteq B_1^Q(0)$. Then, the absolutely convex hull W of $V \cup B_1^Q(0)$ is a 0-neighbourhood in F with $W \cap F_0 = B_1^Q(0)$, whence $q|_{F_0} = Q$ holds for the Minkowski funtional q of W. The case $r = 0$ follows.

If $r \in \mathbb{N} \cup \{\infty\}$, let $\iota \colon F_0 \to F$ be the inclusion map and $f \colon U \to F_0$ be a map on an open subset $U \subseteq E$. Then, f is $C^r_{\mathbb{K}}$ if and only if $\iota \circ f$ is $C^r_{\mathbb{K}}$, and $d^k(\iota \circ f) = \iota \circ (d^k f)$ for all $k \in \mathbb{N}_0$ such that $k \leq r$ (see [13] (Lemma 1.3.19)). By the case $r = 0$, each of the maps $d^k f$ is $LC^0_{\mathbb{K}}$ if and only if $\iota \circ (d^k f)$ is so, from which the assertion follows.

(d) is immediate from (a) and (c). □

Appendix B. Smooth Maps Need Not Extend to the Completion

Let $E := \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \colon (\exists N \in \mathbb{N})(\forall n \geq N)\, x_n = 0\}$ be the space of finite sequences, endowed with the topology induced by the real Banach space ℓ^1 of absolutely summable real sequences. Then, E is a dense proper vector subspace of ℓ^1, and ℓ^1 is a completion of E. In this appendix, we provide a smooth map with the following pathological properties.

Proposition A1. *There exists a smooth map $f \colon E \to F$ to a complete locally convex space F which does not admit a continuous extension to $E \cup \{z\}$ for any $z \in \ell^1 \setminus E$.*

Proof. Given $z = (z_n)_{n\in\mathbb{N}} \in \ell^1 \setminus E$, the set $S := \{n \in \mathbb{N} \colon z_n \neq 0\}$ is infinite. For each $n \in \mathbb{N}$, we pick a smooth map $h_n \colon \mathbb{R} \to \mathbb{R}$ such that $h_n(z_n) = 1$; if $n \in S$, we also require that h_n vanishes on some 0-neighbourhood. Endow $\mathbb{R}^{\mathbb{N}}$ with the product topology. Then

$$g \colon \ell^1 \to \mathbb{R}^{\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}} \mapsto (h_1(x_1) \cdots h_n(x_n))_{n\in\mathbb{N}}$$

is a smooth map, as its components $g_n \colon \ell^1 \to \mathbb{R}$, $x \mapsto h_1(x_1) \cdots h_n(x_n)$ are smooth. If $x = (x_n)_{n\in\mathbb{N}} \in E$, then there is $N \in S$ such that $x_n = 0$ for all $n \geq N$. Thus, $g_n(x) = 0$ for all $n \geq N$ and hence $g(x) \in E$. Notably, $g(x) \in \ell^1$. It therefore makes sense to define

$$f_z \colon E \to \ell^1, \quad x \mapsto g(x).$$

We now show: $f_z \colon E \to \ell^1$ *is a smooth map to* ℓ^1 *which does not admit a continuous extension to* $E \cup \{z\}$.

In fact, for x and N as above, there exists $\varepsilon > 0$ such that $h_N(t) = 0$ for each $t \in]-\varepsilon, \varepsilon[$. Identify $\mathbb{R}^N$ with the closed vector subspace $\mathbb{R}^N \times \{0\}$ of E and $\mathbb{R}^{\mathbb{N}}$. Then,

$$U := \{y = (y_n)_{n\in\mathbb{N}} \in E \colon |y_N| < \varepsilon\}$$

is an open neighbourhood of x in E such that $f_z(U) \subseteq \mathbb{R}^N$. Thus, $f_z|_U$ is smooth as a map to $\mathbb{R}^N$ and hence also as a map to ℓ^1. As a consequence, $f_z \colon E \to \ell^1$ is smooth.

Now, suppose that $p = (p_n)_{n\in\mathbb{N}} \colon E \cup \{z\} \to \ell^1$ was a continuous extension of f_z; we shall derive a contradiction. To this end, set $y_k := (z_1, \ldots, z_k, 0, 0, \ldots) \in E$ for $k \in \mathbb{N}$. Then, $y_k \to z$ in E as $k \to \infty$. The inclusion map $\ell^1 \to \mathbb{R}^{\mathbb{N}}$ being continuous, we deduce that

$$p_n(y_k) \to p_n(z) \quad \text{as } k \to \infty,$$

for each $n \in \mathbb{N}$. Since $p_n(y_k) = g_n(y_k) = h_1(z_1) \cdots h_n(z_n) = 1$ for all $k \geq n$, it follows that $p_n(z) = 1$ for all $n \in \mathbb{N}$ and thus $(1, 1, \ldots) = p(z) \in \ell^1$, which is absurd. Therefore, f_z has all of the asserted properties.

We now define $\Omega := \ell^1 \setminus E$ and endow $F := (\ell^1)^{\Omega}$ with the product topology. We let $f := (f_z)_{z\in\Omega} \colon E \to F$ be the map with components f_z as defined before. By construction, f has the properties described in Proposition A1. □

References

1. Keller, H.H. *Differential Calculus in Locally Convex Spaces*; Springer: Berlin, Germany, 1974.
2. Bastiani, A. Applications différentiables et variétés différentiables de dimension infinie. *J. Anal. Math.* **1964**, *13*, 1–114. [CrossRef]
3. Kriegl, A.; Michor, P.W. *The Convenient Setting of Global Analysis*; American Mathematical Society: Providence, RI, USA, 1997.
4. Micheli, M.; Michor, P.W.; Mumford, D. Sobolev metrics on diffeomorphism groups and the derived geometry of spaces of submanifolds. *Izv. Math.* **2013**, *77*, 541–570. [CrossRef]
5. Wurzbacher, T. Fermionic second quantization and the geometry of the restricted Grassmannian. In *Infinite-Dimensional Kähler Manifolds*; Huckleberry, A.T., Wurzbacher, T., Eds.; Birkhäuser: Basel, Switzerland, 2001; pp. 287–375.
6. Odzijewicz, A.; Ratiu, T.S. Banach Lie-Poisson spaces and reduction. *Commun. Math. Phys.* **2003**, *243*, 1–54. [CrossRef]
7. Odzijewicz, A.; Ratiu, T.S. Extensions of Banach Lie-Poisson spaces. *J. Funct. Anal.* **2004**, *217*, 103–125. [CrossRef]
8. Glöckner, H. Applications of hypocontinuous bilinear maps in infinite-dimensional differential calculus. In *Generalized Lie Theory in Mathematics, Physics and Beyond*; Silvestrov, S., Paal, E., Abramov, V., Stolin, A., Eds.; Springer, Berlin, Germany, 2008; pp. 171–186.
9. Neeb, K.-H.; Sahlmann, H.; Thiemann, T. Weak Poisson structures on infinite dimensional manifolds and Hamiltonian actions. In *Lie Theory and Its Applications in Physics*; Dobrev, V., Ed.; Springer: Tokyo, Japan, 2014; pp. 105–135.
10. Beltiţă, D.; Goliński, T.; Tumpach, A.B. Queer Poisson brackets. *J. Geom. Phys.* **2018**, *132*, 358–362. [CrossRef]
11. Bourbaki, N. *Topological Vector Spaces*; Springer: Berlin, Germany, 1987; Chapters 1–5.
12. Glöckner, H. Smoothing operators for vector-valued functions and extension operators. *arXiv* **2020**, arXiv:2006.00254.
13. Glöckner, H.; Neeb, K.-H. Infinite Dimensional Lie Groups. Universität Paderborn, Paderborn, Germany. 2022, *book in preparation*.
14. Hamilton, R. The inverse function theorem of Nash and Moser. *Bull. Am. Math. Soc.* **1982**, *7*, 65–222. [CrossRef]
15. Michor, P.W. *Manifolds of Differentiable Mappings*; Shiva: Orpington, UK, 1980.
16. Milnor, J. Remarks on infinite-dimensional Lie groups. In *Relativity, Groups and Topology II*; DeWitt, B., Stora, R., Eds.; North Holland: Amsterdam, The Netherlands, 1984; pp. 1008–1057.
17. Neeb, K.-H. Towards a Lie theory of locally convex groups. *Jpn. J. Math.* **2006**, *1*, 291–468. [CrossRef]
18. Bochnak, J.; Siciak, J. Analytic functions in topological vector spaces. *Studia Math.* **1971**, *39*, 77–112. [CrossRef]
19. Hewitt, E.; Ross, K.A. *Abstract Harmonic Analysis I*; Springer: New York, NY, USA, 1979.
20. Engelking, R. *General Topology*; Heldermann: Berlin, Germany, 1989.
21. Kelley, J.L. *General Topology*; Springer: New York, NY, USA, 1975.
22. Noble, N. The continuity of functions on cartesian products. *Trans. Am. Math. Soc.* **1970**, *149*, 187–198. [CrossRef]
23. Ferrer, M.V.; Hernández, S.; Shakhmatov, D. A countable free closed non-reflexive subgroup of $\mathbb{Z}^c$. *Proc. Am. Math. Soc.* **2017**, *145*, 3599–3605. [CrossRef]
24. Ardanza-Trevijano, S.; Chasco, M.J. The Pontryagin duality of sequential limits of topological Abelian groups. *J. Pure Appl. Algebra* **2005**, *202*, 11–21. [CrossRef]
25. Franklin, S.P.; Smith Thomas, B.V. A survey of k_ω-spaces. *Topol. Proc.* **1978**, *2*, 111–124.
26. Glöckner, H. Lie groups over non-discrete topological fields. *arXiv* **2004**, arXiv:math/0408008.
27. Bertram, W.; Glöckner, H.; Neeb, K.H. Differential calculus over general base fields and rings. *Expo. Math.* **2004**, *22*, 213–282. [CrossRef]
28. Thomas, E.G.F. *Calculus on Locally Convex Spaces*; University of Groningen: Groningen, The Netherlands, 1996.
29. Seip, U. *Kompakt erzeugte Vektorräume und Analysis*; Springer: Berlin, Germany, 1972.
30. Treves, F. *Topological Vector Spaces, Distributions and Kernels*; Academic Press: New York, NY, USA, 1967.
31. Glöckner, H. Direct limits of infinite-dimensional Lie groups. In *Developments and Trends in Infinite-Dimensional Lie Theory*; Neeb, K.-H., Pianzola, A., Eds.; Birkhäuser: Basel, Switzerland, 2011; pp. 243–280.
32. Hirai, T.; Shimomura, H.; Tatsuuma, N.; Hirai, E. Inductive limits of topologies, their direct products, and problems related to algebraic structures. *J. Math. Kyoto Univ.* **2001**, *41*, 475–505. [CrossRef]
33. Grauert, H. Analytische Faserungen über holomorph-vollständigen Räumen. *Math. Ann.* **1958**, *135*, 263–273. [CrossRef]
34. Patyi, I. On holomorphic Banach vector bundles over Banach spaces. *Math. Ann.* **2008**, *341*, 455–482. [CrossRef]
35. Jarchow, H. *Locally Convex Spaces*, B. G. Teubner: Stuttgart, Germany, 1981.
36. Noble, N. *k*-groups and duality. *Trans. Am. Math. Soc.* **1970**, *151*, 551–561.
37. Banaszczyk, W. *Additive Subgroups of Topological Vector Spaces*; Springer: Berlin, Germany, 1991.
38. Beggs, E. De Rham's theorem for infinite-dimensional manifolds. *Q. J. Math.* **1987**, *38*, 131–154. [CrossRef]
39. Hofmann, K.H.; Morris, S.A. *The Structure of Compact Groups*; de Gruyter: Berlin, Germany, 1998.
40. Glöckner, H. Lie groups of measurable mappings. *Can. J. Math.* **2003**, *55*, 969–999. [CrossRef]

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Review

Advances in the Theory of Compact Groups and Pro-Lie Groups in the Last Quarter Century †

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Abstract: This article surveys the development of the theory of compact groups and pro-Lie groups, contextualizing the major achievements over 125 years and focusing on some progress in the last quarter century. It begins with developments in the 18th and 19th centuries. Next is from Hilbert's Fifth Problem in 1900 to its solution in 1952 by Montgomery, Zippin, and Gleason and Yamabe's important structure theorem on almost connected locally compact groups. This half century included profound contributions by Weyl and Peter, Haar, Pontryagin, van Kampen, Weil, and Iwasawa. The focus in the last quarter century has been structure theory, largely resulting from extending Lie Theory to compact groups and then to pro-Lie groups, which are projective limits of finite-dimensional Lie groups. The category of pro-Lie groups is the smallest complete category containing Lie groups and includes all compact groups, locally compact abelian groups, and connected locally compact groups. Amongst the structure theorems is that each almost connected pro-Lie group G is homeomorphic to $\mathbb{R}^I \times C$ for a suitable set I and some compact subgroup C. Finally, there is a perfect generalization to compact groups G of the age-old natural duality of the group algebra $\mathbb{R}[G]$ of a finite group G to its representation algebra $R(G, \mathbb{R})$, via the natural duality of the topological vector space $\mathbb{R}^I$ to the vector space $\mathbb{R}^{(I)}$, for *any* set I, thus opening a new approach to the Hochschild-Tannaka duality of compact groups.

Keywords: topological group; Lie group; compact group; pro-Lie group; Lie algebra; duality; Tannaka duality; Pontryagin duality; LCA group

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1. Introduction

Certain areas of mathematical research draw their particular fascination from the fact that they are based between two principal domains of mathematics, such as algebra and topology. Between these two, we find algebraic topology and topological algebra. An observer looking at mathematics from a distance may wonder if these two fields differ much. The language itself points out the difference: Topological algebra is a specialty located in algebra, the art of calculating—adding and multiplying, while using the tools of geometry, and manipulating the concept of continuity adds an extra attraction.

Groups emerged in 1770 in the work on permutation groups of Joseph-Louis Lagrange (1736–1813) and in 1799 in the context of solving quintic equations in the work of Paolo Ruffini (1765–1822). Groups in their abstract form can to be traced back to Augustin-Louis Cauchy (1789–1857), Niels Henrik Abel (1802–1829), and Évariste Galois (1811–1832), when groups were formative in the development of abstract algebra. Galois was, in fact, the first to use the word group (*groupe* in French). The beginnings of *Topology* reach back hundreds of years; however, as August Ferdinan Möbius (1790–1868) said, it was Jules Henri Poincaré (1854–1912) who "gave topology wings" in several articles, the first of which appeared in 1895. (Johann Benedict Listing (1808–1882) introduced the (German) term *Topologie* in 1847.) *Topology* as an independent area had not yet crystallized, though *Geometry* was quite present, when Felix Klein (1849–1925) and Sophus Lie (1842–1899) (and followers, such as Friedrich Engel (1861–1941) and Wilhelm Karl Joseph Killing (1847–1923)) founded the area of what later became named *Lie groups*. Algebra, geometry, and analysis were thoroughly mixed into the genesis of Lie group theory.

2. Hilbert's Fifth Problem and Locally Compact Groups

In 1900, David Hilbert (1862–1943) gave his famous address to the International Congress of Mathematicians in Paris. In an apparently unforgettable fashion, it foreshadowed crucial developments of mathematical research in the 20th century. Hilbert formulated 23 open problems leading to groundbreaking research in the 20th century. By that time, *topology* was present in the minds of mathematicians, although it may not have reached the heights it would attain in the course of the century. Yet, enough was available to Hilbert for him to formulate, for instance, his famous Fifth Problem:

If a group is defined on a euclidean manifold in such a way that multiplication and inversion are continuous functions, can it be given the structure of a **differentiable** *manifold so that the continuous group operations are in fact differentiable?*

This would make it a group of the kind that Lie had created in a visionary way. In modern parlance, Hilbert posed the question:

Is a locally euclidean topological group a Lie group?

He envisioned a positive answer. However, it would take a little over half a century to confirm his vision.

Yet, this half century advanced the research of topological groups enormously. The most consequential steps were:

(i) the discovery of fundamental properties of *compact groups* by Hermann Weyl (1885–1955) and his doctoral student Fritz Peter (1899–1949) in 1927;

(ii) the discovery that every *locally compact group* has a (left) invariant measure by Alfréd Haar (1885–1933) in 1932; and

(iii) the discovery in 1934 of the duality between the *category of (discrete) abelian groups* and the *category of compact abelian groups* by Lev Semyonovich Pontryagin (1908–1988), rounded off in 1935 with the extension to arbitrary *locally compact abelian groups* by Egbert van Kampen (1908–1942), and by André Weil (1906–1998) in 1938, who also established that *a complete topological group with a Haar measure has to be* locally compact. (See References [1–3]. For a discussion of Pontryagin Duality outside the class of locally compact abelian groups, see Reference [4] and its references. For a category theory proof of Pontryagin Duality, see Reference [5].)

Inasmuch as these milestones were set up close to Lie groups, they are naturally linked to topological groups whose underlying topological spaces (for the most part) are connected. It was recognized early on that, in a topological group G, the connected component G_0 of the identity is a closed normal subgroup which is mapped into itself by any continuous endomorphism of G. (We recall that such a subgroup is called *fully characteristic*.) Obviously, therefore, it is very special. Indeed, in any Lie group (real or complex), the benefit drawn from the presence of the Lie algebra of G invented by Sophus Lie reaches as far as G_0, and not the tiniest bit beyond. One is tempted to remark that G_0 *supports all the (traditional) geometry of group theory*.

The observation that real Lie algebras are attached to topological groups in a more general sense was first anticipated by Richard Lashof (1922–2010) in 1957 for locally compact groups. More recently, as explained and illustrated in our book (Reference [6]), this was extended to a much wider class of topological groups. It is natural to ask how much of the structure of a topological group G is supported by the elementary concept of

$\mathrm{Hom}(\mathbb{R}, G)$, the space of morphisms of topological groups $\mathbb{R} \to G$, also called *one-parameter subgroups* of G.

That half-a-century of developments of topological groups went alongside an astounding unfolding of topology. However, there was a second impact on the domain of topological groups. This advance emerged from algebra itself, more specifically, from GALOIS THEORY. As a typical example, the algebraic completion A of a field F is the directed colimit of all finite extensions $(K{:}F)$. Inevitably, the Galois group $G(A{:}F)$ is the projective limit of the finite Galois groups $G(K{:}F)$. A projective limit G of a directed inverse system of finite groups automatically carries a group topology making it a compact totally disconnected topological group. Here, *totally disconnected* means exactly that G_0 is a singleton subgroup. This example clearly illustrates the fact that this group theory, belonging to the home of pure classical algebra, uncontroversially leads to a class of topological groups located opposite to the type of connected topological groups which have arisen, historically, out of Lie theory. Yet, the link between the two disparate classes of topological groups was, from the very beginning, the fact that:
every topological group G gives rise to a connected topological subgroup G_0, its identity component, *and, by contrast, the totally disconnected quotient group $G_t = G/G_0$.*

The complete solution of Hilbert's Fifth Problem arrived in 1952 (9 years after the death of Hilbert), when Andrew Mattei Gleason (1921–2008), Deane Montgomery (1909–1992), and Leo Zippin (1905–1995) settled it with a positive answer. This effort was crowned by the fundamental discovery in 1953 by Hidehiko Yamabe (1923–1960) that:
in a topological group G whose component factor group G_t is compact, any compact identity neighborhood of G contains a closed normal subgroup N, such that the factor group G/N is a Lie group, indeed, precisely one of those Lie groups, which had so fascinated Hilbert in 1900. The compactness of the factor group $G_t = G/G_0$ led to the standard terminology that a topological group having this property is called *almost connected*.

Yamabe's major contribution to the solution of Hilbert's Fifth Problem was soon followed by an immensely influential paper [7], by Kenkichi Iwasawa (1917–1998), on the structure of locally compact groups.

One way of expressing the theorem of Yamabe was to say that:
every almost connected group is a projective limit of Lie groups.
(Projective limits are discussed and explained, e.g., in References [1,6].)

This fact caused much of the work on *locally compact groups* in the second half of last century to be focused on *projective limits of Lie groups*. In this endeavor, it is truly very helpful that the projective limit presentation of an *almost connected locally compact group G*, in terms of its Lie group quotients G/N, has limit maps $G \to G/N$ that are particularly well behaved because each of them is a *proper* morphism, i.e., it is a closed continuous map such that the inverse image of each compact set is compact.

A substantial step in a general structure theory of locally compact totally disconnected groups occurred in the 1990s, when George A. Willis innovated the theory by introducing concepts, such as *tidy subgroups* and *scaling functions* [8,9].

The second half of the twentieth century saw a substantial amount of research on what has become known as *Abstract Harmonic Analysis*. This subject, outside the scope of this survey, was built on the realization by André Weil that, using Haar measure, Fourier series and Fourier integrals are a special case of a construction on locally compact groups. The standard references are References [2,10,11], but also see Reference [12].

3. Pro-Lie Groups: From Connected to Almost Connected Ones

One should be aware of the fact that not every locally compact group is a projective limit of Lie groups, as $\mathrm{SL}(2, \mathbb{Q}_p)$, the group of p-adic 2 by 2 matrices of determinant 1, illustrates for any prime p.

However, within topological group theory, in this immense activity of the 20th century on the projective limit representation of locally compact (and, in particular, compact) groups, it was almost overlooked that a topological group G which has a filter basis $\mathcal{N}$ of closed normal subgroups N such that, firstly,

(1) *G/N is a Lie group for all $N \in \mathcal{N}$*

and, secondly,

(2) *the natural map $G \to \lim_{N\in\mathcal{N}} G/N$ is an isomorphism*

certainly does not force G to be locally compact. In fact, under these circumstances, G is locally compact if and only if $\mathcal{N}$ contains a compact member. The simplest examples failing to be locally compact are groups, such as $\mathbb{Z}^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{N}}$, with their product topologies. Indeed, the second of these examples illustrates the fact that we are touching a subject that is understood with the concepts of basic linear algebra over the real or complex field (or, indeed, any locally compact field). Given the Axiom of Choice, we know that every vector space V over $\mathbb{R}$ has a basis, equivalently, that it is a direct sum $\bigoplus_{j\in J}\mathbb{R}_j$ of some family of copies $\mathbb{R}_j \cong \mathbb{R}$ of $\mathbb{R}$, denoted by $\mathbb{R}^{(J)}$. The vector space $\mathrm{Hom}(V,\mathbb{R})$ of all linear forms of V is (naturally isomorphic to) the Cartesian product $\mathrm{Hom}(\bigoplus_{j\in J}\mathbb{R}_j,\mathbb{R}) \cong \prod_{j\in J}\mathrm{Hom}(\mathbb{R}_j,\mathbb{R}) \cong \mathbb{R}^J$ of copies of $\mathbb{R}$. Since $\mathbb{R}$ has a natural topology, this is true for $\mathbb{R}^J$ with its *Tychonov* topology or product topology—and that is locally compact if and only if J is a finite set. So, with $J = \mathbb{N}$, the topological vector space $\mathbb{R}^{\mathbb{N}}$ is the first one to break this barrier. Topological vector spaces which are isomorphic to $\mathbb{R}^J$ for some set J are called *weakly complete vector spaces*. There is no problem in extending this terminology to vector spaces over the complex ground field $\mathbb{C}$.

It has become customary to call a topological group satisfying (1) and (2) above a *pro-Lie group*. Their systematic study coincides neatly with the beginning of the twenty-first century. The simplest examples are the weakly complete vectors spaces themselves. They are even closer to elementary vector spaces than one spontaneously thinks. Indeed, if $W \cong \mathbb{R}^J$ is a weakly complete vector space, then the vector space $\mathrm{Hom}_{\mathrm{continuous}}(W,\mathbb{R})$ of all continuous linear forms on W is isomorphic to $\mathbb{R}^{(J)}$, and a slightly more detailed consideration shows that this is the background of a perfect *duality between the category of real vector spaces and that of weakly complete vector spaces*. This rather elementary duality is discussed in detail in the first edition of Reference [1] in 1998 and in the first monograph of Reference [6] to have a systematic study of pro-Lie groups in 2007.

Here, the natural question arises how the concepts of a pro-Lie group and that of the historically fundamental one of a manifold based Lie group differ. The concept of a manifold had developed at that time vastly, being now based on locally convex topological vector spaces. Accordingly, the concept of a Lie group had developed deeply into the domain of infinite dimensional manifolds [13]. Nevertheless, from Reference [14], we know precisely how the two concepts are related:

Theorem 1. *A pro-Lie group is a Lie group if and only if it is locally contractible.*

Here, a topological group G is called *locally contractible*, if some identity neighborhood U can be homotopically contracted to a point in G, and it is called 1-*connected* if $\pi_1(G)$ is singleton. In the spirit of Lie theory from any viewpoint, it is fascinating that local contractibility of a 1-connected pro-Lie group can be detected purely from the Lie algebra $\mathfrak{g}$ of G: Every pro-finite dimensional Lie algebra $\mathfrak{g}$ has a maximal (semi-)direct summand $\mathfrak{s}$ being a product of some collection of simple finite dimensional Lie algebras. Indeed,
a 1-connected pro-Lie group G is locally contractible iff, apart from a finite number of these factors, each of the factors is isomorphic to the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ (the group of 2 by 2 real matrices of determiant 1).

The weakly complete real vector spaces provide an exemplarily simple class of pro-Lie groups beyond traditional Lie groups. In Reference [6], the authors proved the fairly deep theorem, saying that:
a connected pro-Lie group G contains a closed subspace E and a compact subgroup C such that E is homeomorphic to some weakly complete real vector space and the function

$$(e,c) \mapsto ec : E \times C \to G \text{ is a homeomorphism.}$$

We might say, so far so good, for connected pro-Lie groups. However, the free abelian group $\mathbb{Z}^{(\mathbb{N})}$ of countably infinitely many generators supports a nondiscrete pro-Lie topology with rather bizarre properties. (This is described in Proposition 2 in Chapter 5 on abelian pro-Lie groups in Reference [6].) So, one ventures outside *connected* pro-Lie groups with trepidation. Even basic issues are settled only very partially, exemplified by the question: when is a quotient of a pro-Lie group a pro-Lie group? (See, e.g., Reference [6], Chapter 4, Theorem 4.28.)

It is, therefore, astonishing how much positive information has been gathered on pro-Lie groups, even if they fail to be connected.

Our monograph [6] presents a reasonably comprehensive theory of connected pro-Lie groups. While classical Lie theory is used intensively, the technical difficulties to bring them to bear on the general situation are often painfully complex on the technical level.

At the opposite end, we face totally disconnected pro-Lie groups. By definition, such a group G is a projective limit of Lie group quotients G/N. The pro-Lie algebra map $\mathfrak{L}(G) \to \mathfrak{L}(G/N)$ induced by the quotient morphism is surjective. (See Reference [6], 4.21.) However, $\mathfrak{L}(G) = \{0\}$, since G is totally disconnected. So, the Lie algebra of the Lie group G/N vanishes. Therefore, it is discrete. Accordingly, G is a projective limit of discrete quotients. Therefore, it is called *prodiscrete*. In the domain of locally compact groups, *prodiscrete groups* are generally considered still tractable. This applies certainly to the realm of compact groups where they are traditionally known as *profinite groups* and are treated extensively in the monograph literature. (See, e.g., Reference [15].) By contrast, one would have to admit, however, that no coherent structure or representation theory exists for prodiscrete groups, in general, outside the locally compact domain.

So, there arise obvious questions which link connectivity and prodiscreteness.

Problem 1. *Let G be a pro-Lie group. Is there a neighborhood of G_0 whose structure is reasonably well understood, at least topologically?*

Perhaps more explicitly (and optimistically):

Problem 2. *Let G be a pro-Lie group. Is there a closed totally disconnected subgroup H of G such that the subgroup G_0H is open?*

The consequences of such pieces of information would be far reaching. In the case of a locally compact group G, indeed, there exists a totally disconnected compact subgroup D such that G_0D is open. So, the answers for both Problem 1 and Problem 2 are affirmative if G is locally compact. Conclusive answers are not available if G fails to be locally compact, but partial answers to these questions were provided after the appearance of Reference [6] by the authors in Reference [16], and in a survey in Reference [17], including the following result:

Theorem 2. *Let G be an almost connected pro-Lie group. Then, every compact subgroup is contained in a maximal one and all of these are conjugate. There is a closed subspace homeomorphic to a weakly complete vector space E in G such that, for each maximal compact subgroup C, the function*

$$(e, c) \mapsto ec : E \times C \to G$$

is a homeomorphism.

The proof in Reference [16], in 2011 (after the appearance of Reference [6]), provides additional information on the way that E is constructed. A shorter, but perhaps more easily recalled, formulation is the following:

Corollary 1. *Any almost connected pro-Lie group is homeomorphic to $\mathbb{R}^J \times C$, for some set J and a compact subgroup C of G.*

It should be emphasized that this theorem gives a definitive insight into the *topological* structure of an almost connected pro-Lie group modulo the known structure of a compact group, as detailed in Reference [1]. Indeed,
a compact group C is homeomorphic to $C_0 \times C/C_0$, where C/C_0 is either finite or is homeomorphic to a power $\{0,1\}^J$ of the two element space for a suitable set J. (See Reference [1], 10.40.)
The compact connected group C_0 itself is a semidirect product of the closed commutator group C_0' and a compact connected abelian subgroup $A \cong C/C_0'$. (See Reference [1], 9.39.)

The compact semisimple commutator subgroup is described explicitly in Reference [1], 9.19, where it is argued that it is not too far from a product of a possibly large family of compact connected simple Lie groups.

For the pro-Lie group-theoretical understanding of the abelian connected compact group A, we also have explicit knowledge, namely the *Resolution Theorem* (Reference [1], 8.20), which specifies a profinite subgroup Δ of A and a continuous open surjective homomorphism $\Delta \times \mathfrak{L}(A) \to A$ for the Lie algebra $\mathfrak{L}(A)$ of A. Here, the Lie algebra $\mathfrak{L}(A)$ is none other than a weakly complete real vector space. In particular, these pieces of information together with Theorem 2 above yield the following rather complete information of the topology of an almost connected pro-Lie group:

Theorem 3. The Topology of Almost Connected Pro-Lie Groups: *Any infinite almost connected pro-Lie group is homeomorphic to a pro-Lie group of the form*

$$\mathbb{R}^I \times S \times A \times F,$$

where F is either finite or $\mathbb{Z}(2)^J$; here, I and J are sets, $\mathbb{Z}(2)$ is the two-element group, where S is a compact connected group that agrees with its commutator subgroup S' and is, modulo a central profinite subgroup, a Cartesian product of compact connected simple Lie groups, and, where, finally, A is a compact connected abelian group.

It may be helpful here to recall a consequence of Pontryagin Duality, namely that
the category of all compact connected abelian groups is dual to the (vast) category of all torsion-free abelian groups.

The history of locally compact groups has illustrated that an insight into the structure of *abelian* locally compact groups preceded the solution of Hilbert's 5th Problem. In this spirit, we have had some success in getting the basics of a structure theory of *abelian* pro-Lie groups formulated. (See Reference [6], 5.20).) Indeed, we proved the following result.

Theorem 4. Main Structure Theorem of Abelian Pro-Lie Groups: *Any abelian pro-Lie group G is the direct sum $E \oplus H$ of closed subgroups, where E is isomorphic to $\mathbb{R}^J$, for a set J, and H has the following properties:*
- *(i) H_0 is compact and is the unique largest compact connected subgroup;*
- *(ii) every compact subgroup of G is contained in H;*
- *(iii) the totally disconnected quotient groups $G_t = G/G_0$ and $H_t = H/H_0$ are isomorphic; and*
- *(iv) The union* comp(G) *of all compact subgroups of G is a fully characteristic closed subgroup of G that is contained in H, and*

$$G_0 + \mathrm{comp}(G) = E \oplus \mathrm{comp}(G)$$

is a fully characteristic closed subgroup G_1 of G such that every monothetic subgroup of G/G_1 is isomorphic to the discrete group $\mathbb{Z}$.

The factor group $G/\,\mathrm{comp}(G)$ does not contain any nonsingleton compact subgroup, and the Main Structure Theorem implies immediately that its identity component is a weakly complete real vector space isomorphic to $\mathbb{R}^J$ and is a direct summand.

The factor group G/G_1 is a totally disconnected abelian pro-Lie group without any nontrivial compact subgroup whose structure remains largely uncharted and mysterious.

Indeed, *A. Weil's Lemma* on the Classification of Monothetic Subgroups of Locally Compact Groups (Reference [1], 7.43) was extended by the authors to pro-Lie groups in the following fashion:

Theorem 5. Weil's Lemma for Pro-Lie Groups: *Let $E = \mathbb{Z}$ or $E = \mathbb{R}$ and $X\colon E \to G$ a morphism of topological groups into a pro-Lie group. Then, exactly one of the following statement holds:*

(i) $r \mapsto X(r) : E \to X(E)$ *is an isomorphism of topological groups;*

(ii) $\overline{X(E)}$ *is compact.*

As a consequence, *if a pro-Lie group G has no nontrivial compact subgroups, then every monothetic subgroup is isomorphic to $\mathbb{Z}$ as a topological group.*

In all of topological group theory, the subclass of commutative topological groups is usually considered a test class which is representative of the status of information provided by current research. This is exemplified by information provided for locally compact abelian groups (often called *LCA-groups*) and, similarly, by all the information on real topological vector spaces made available by functional analysis.

It was, therefore, natural to raise the issue of duality for abelian pro-Lie groups in Reference [6], pp. 237ff.

Notably, satisfactory results emerged for *almost connected* abelian pro-Lie groups, and some interesting general additional aspects were pointed out in Reference [6] (5.36, 5.40, 5.41). In particular, it was observed in Reference [6] (Comments to 14.15) that an abelian pro-Lie group G may fail to be reflexive. (Here, a topological abelian group is called *reflexive*, if the natural morphism $G \to \widehat{\widehat{G}}$ is an isomorphism of topological groups.) Overall, one might consider the structure theory of abelian pro-Lie groups still as regrettably incomplete. Some aspects that we do know are collected in the following theorem.

Theorem 6. The Structure of Almost Connected Abelian Pro-Lie Groups: *Let G be an almost connected abelian pro-Lie group. Then,* $\mathrm{comp}(G)$ *is a compact subgroup, and*

(i) $G \cong \mathbb{R}^J \times \mathrm{comp}(G)$. *In particular, each weakly complete* real *vector space is reflexive.*

(ii) *The annihilator of G_0 in $\widehat{G}$ is* $\mathrm{comp}\,\widehat{G}$.

Now, assume that G is an abelian pro-Lie group which is algebraically generated by a compact subset. Let $G_1 = G_0 + \mathrm{comp}(G)$ be the fully characteristic subgroup of G introduced in Theorem 3(iv). Then, G_1 is locally compact, and G/G_1 is a Polish space (i.e., it is completely metrizable and separable) if and only if $G \cong \mathbb{R}^m \times \mathrm{comp}(G) \times \mathbb{Z}^n$, for nonnegative integers m and n.

More details can be found in Reference [6], including a version of a universal covering theorem which, in Reference [1], was called a ''Resolution Theorem''.

Wayne Lewis noted recently that the Resolution Theorem suggests introducing into the study of LCA groups a more systematic use of the *adele ring*

$$\prod_{p\,\mathrm{prime}}^{\mathrm{local}} (\mathbb{Q}_p, \mathbb{Z}_p) \times \mathbb{R}\,,$$

thus relating the structure theory of LCA groups to algebraic number theory. (The term *idele* was introduced by Claude Chevalley (1909–1984) and is an abbreviation of "ideal element". The term *adele* stands for "additive idele".)

Theorem 5 confirms the impression that we can regard the condition of being *almost connected* in the theory of pro-Lie groups as very satisfactory, but that we do not have a comprehensive theory of totally disconnected abelian pro-Lie groups, in general. The recent study of Reference [18] on locally compact totally disconnected abelian groups G satisfying $G = \mathrm{comp}(G)$ deals with this subject, as well as illustrates the fact that not even the presence of a wealth of compact open subgroups provides for structural simplicity.

While we noted that each locally compact group G having a pro-Lie group as identity component G_0 is largely determined by a profinite dimensional Lie algebra $\mathfrak{L}(G)$, nevertheless, we observed that $\mathfrak{L}(G)$ has no effect on the totally disconnected portion $G_t = G/G_0$ of G.

One recent branch in the research on locally compact groups provides noteworthy connections between locally compact groups and topology without having such restrictions. Indeed, the set of all closed subgroups of any locally compact group G always supports a compact topology making that set into a compact Hausdorff space $\mathfrak{Ch}(G)$, called the *Chabauty space* of G. (The names of Leopold Vietoris (1891–2002) or James Michael Gardner Fell (1923–2016) would have been just as appropriate as that of Claude Chabauty (1910–1990).) The example of the circle group $G = \mathbb{T}$ shows that the Chabauty space may have pathological aspects even in the compact connected case. On the other hand, this tool appears to come in handy for totally disconnected locally compact groups G, as the following example shows:
For any locally compact group G, the function $g \mapsto \overline{\langle g\rangle} : G \to \mathfrak{Ch}(G)$ *is continuous iff G is totally disconnected.*

In this sense, the operators $\mathfrak{L}$ and $\mathfrak{Ch}$ are opposite in their prospect as tools for the structure theory of G. (See Reference [19].)

4. Linear Algebra Meets Pro-Lie Group Theory

In Reference [20], in 1939, Tadeo Tannaka (1908–1986) formalized the process of reconstructing a compact group from the systematically structured class of finite dimensional linear representations. This approach he proved to be a way of generalizing Pontryagin's duality of the categories of *abelian* compact, respectively, discrete groups to a noncommutative situation. This led to vast generalizations in the abstract world of category theory. (See Reference [21].) On the other hand, at a very early point in his book [22], Gerhard Paul Hochschild (1915–2010) formalized very concretely the idea that the real vector space $R(G,\mathbb{R})$ of coefficient functions of finite dimensional linear representations of a compact group G is not only a commutative algebra, but also a coalgebra and, indeed, a symmetric *Hopf algebra*. He specified the conditions under which the spectrum of a symmetric Hopf algebra is a compact group G whose Hopf algebra $R(G,\mathbb{R})$ is isomorphic to the given one. This produces a duality between the category of compact groups and a category of *certain* symmetric Hopf algebras. The connection between $R(G,\mathbb{R})$ and the linear representations indicates an existing equivalence of Hochschild's duality with Tannaka's.

We have proposed a topological group algebra $\mathbb{R}[G]$ of any compact group G. This allows us to produce a certain category of *topological* symmetric Hopf algebra which is *equivalent* to the category of compact groups via $G \mapsto \mathbb{R}[G]$. This links us with Hochschild-Tannaka duality through the fact that $R(G,\mathbb{R})$ and $\mathbb{R}[G]$ are natural duals of each other as symmetric Hopf algebras in their respective domains of plain vector spaces and topological vector spaces.

From the very beginning of the study of pro-Lie groups, it was clear that one would have to consider pro-Lie algebras. One of the difficult problems with which Sophus Lie found himself confronted was the question of whether, for each Lie algebra $\mathfrak{g}$, one could find a Lie group G whose Lie algebra was (isomorphic to) $\mathfrak{g}$. A satisfactory answer became known in the history of Lie theory as Lie's *Third Fundamental Theorem*. In the development of the theory of pro-Lie groups, it seemed conceptually fitting to find a response to a more comprehensive question. At that point in the history of topological groups, one had a good hold of category theory, and it was understood that the Lie algebra functor $\mathfrak{L}$ from the category $\mathcal{G}$ of pro-Lie groups to the category of profinite-dimensional Lie algebras $\mathcal{L}$ has a right adjoint Γ. Thus, for every morphism $f\colon \mathfrak{g} \to \mathfrak{L}(G)$, there is a unique morphism $f'\colon \Gamma(\mathfrak{g}) \to G$, producing a natural isomorphism

$$f \mapsto f'\colon \mathcal{L}(\mathfrak{g}, \mathfrak{L}(G)) \to \mathcal{G}(\Gamma(\mathfrak{g}), G)).$$

In particular, the right adjoint functor $\mathfrak{L}$ preserves all limits, so, if G is a projective limit of finite dimensional Lie groups, then $\mathfrak{L}(G)$ is a projective limit of finite dimensional Lie algebras, i.e., a *profinite dimensional Lie algebra*. (See Reference [23].) As an immediate elementary consequence, the real topological vector space underlying $\mathfrak{L}(G)$ is a projective limit of finite dimensional vector spaces. This returns us to the fact that one had to discuss at a comparatively early stage in References [1,6] that the projective limit property, indeed, characterized a real or complex topological vector space to be weakly complete. The insight that the category of $\mathbb{K}$-vector spaces, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, is dual to the category of weakly complete topological $\mathbb{K}$-vector spaces was explicitly elucidated both in References [1,6]. We note here with some circumspection that, for $\mathbb{K} = \mathbb{R}$, the duality between real vector spaces, on the one hand, and weakly complete topological real vector spaces, on the other, may be regarded a special case for abelian pro-Lie groups of Pontryagin duality (also see Reference [1], A7.10).

It is clear that pro-Lie group theory and elementary linear algebra are tied together from the beginning. However, when the first systematic study of pro-Lie groups [6] was compiled, another avenue leading from ‘‘elementary linear algebra'' directly to pro-Lie groups had not yet been observed, even though its mathematical underpinning would have been available. This avenue leads from weakly complete topological vector spaces to weakly complete associative topological algebras. That associative unital algebras would appear in the vicinity of groups and their linear representation theory is perhaps not surprising, given the history of representation theory and module theory. It is perhaps astonishing that the concept of weakly complete algebras appeared so late.

Indeed, a *weakly complete unital algebra A* is an associative algebra whose addition and scalar multiplication are that of a weakly complete vector space and whose multiplication is associative and continuous and has an identity. Let us denote the multiplicative group of invertible elements by A^{-1}. At first glance, and in light of the numerous types of associative unital algebras that functional analysis deals with in the representation theory of topological groups, the following fact may come as a surprise:

Theorem 7. *Every weakly complete unital algebra A is a projective limit of finite dimensional unital quotient algebras.*

In other words, a weakly complete associative algebra is automatically *profinite dimensional*.

The essence of the above result was first observed by Bogfiellmo, Dahmen, and Schmeding [24]. For more on this theorem, see Reference [1], A7.32–A7.43.

These facts require absolutely no additional hypothesis apart from the fact that the algebra topology is the weakly complete one. We recorded that the categories $\mathcal{V}$ of vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and the category $\mathcal{W}$ of weakly complete topological vector spaces are dual. This suggests that Theorem 7 is just one step away from a purely algebraic result. Indeed, let us reconsider the categories $\mathcal{V}$ and $\mathcal{W}$ and, for each of the two, the occasionally tricky concept of its tensor product $\otimes_{\mathcal{V}}$, respectively, $\otimes_{\mathcal{W}}$. (The basic properties of $\otimes_{\mathcal{W}}$ were first studied in the Master's thesis (Diplomarbeit) in 2007 of Raphael Dahmen.) The most significant property of this pair of tensor products is its compatibility with duality:

$$(V_1 \otimes_{\mathcal{V}} V_2)^* \cong V_1^* \otimes_{\mathcal{W}} V_2^* \text{ and } (W_1 \otimes_{\mathcal{W}} W_2)' \cong W_1' \otimes_{\mathcal{V}} W_2'.$$

With the aid of the tensor product, the multiplication of a weakly complete algebra A may now be expressed as a $\mathcal{W}$-morphism $m\colon A \otimes_{\mathcal{W}} A \to A$ subject to the commutativity of a diagram expressing associativity (Reference [1], Definition A3.63a), and the identity element 1 of the algebra may be expressed by a morphism $u\colon \mathbb{K} \to A$, $u(t) = t \cdot 1$, subject to a commutative diagram (cf. loc. cit.). Now, the dual object $m'\colon A' \to A' \otimes_{\mathcal{V}} A'$, together with $u'\colon A' \to \mathbb{K}$, represents a coassociative *coalgebra* with *coidentity*. So, all such coalgebras, being purely algebraic objects in the category $\mathcal{V}$, are *locally finite* in the sense that every element is contained in a finite dimensional subcoalgebra. In other words, each associative counital coalgebra is a directed colimit of finite dimensional subcoalgebras, or, once again reformulated, each counital coassociative coalgebra in $\mathcal{V}$ is a projective colimit of finite dimensional subalgebras. This result is referred to as the "CARTIER Lemma", and also as "The Fundamental Theorem of Coalgebras". (See Michaelis, in Reference [25].) Now, we see that Theorem 7 is the dual of the Cartier Lemma.

An almost immediate consequence of Theorem 7 is the following.

Theorem 8. Fundamental Theorem of Weakly Complete Algebras: *Let A be a weakly complete unital algebra. The group of units (that is, multiplicatively invertible elements), A^{-1}, is an almost connected pro-Lie group. It is dense in A, and the exponential function* $\exp\colon A \to A^{-1}$ *converges everywhere and defines the exponential function of the pro-Lie group A^{-1} if A is considered as a Lie algebra with respect to the bracket $[a,b] = ab - ba$.*

The Fundamental Theorem of Weakly Complete Algebras yields an assignment $A \mapsto A^{-1}$, which is clearly functorial, mapping the category $\mathcal{WA}$ of weakly complete unital algebras into the category $\mathcal{L}$ of pro-Lie groups. Its left adjoint functor $G \to \mathbb{K}[G] : \mathcal{L} \to \mathcal{WA}$ assigns to a pro-Lie group G its *group algebra* (over the groundfield $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). In the case of $\mathbb{K} = \mathbb{R}$ the duality yields an isomorphism $\mathbb{R}[G]' \cong R(G,\mathbb{R})$ with the topological dual $\mathbb{R}[G]'$ of the weakly complete group algebra $\mathbb{R}[G]$ and the ring of representative functions $R(G,\mathbb{R}) \subseteq C(G,\mathbb{R})$, familiar notably in the representation theory of compact groups. (See Reference [1], Chapter 3, Definition 3.3.) The group algebra $\mathbb{K}[G]$ was discussed in detail in References [26,27] and in the book of Reference [6]. In a natural way, $\mathbb{K}[G]$ is, in fact, a *symmetric Hopf algebra*. Here, a Hopf algebra is simultaneously an associative unital algebra and an associative counital coalgebra linked in a compatible fashion. It is a *symmetric* Hopf algebra if it further includes a "symmetry", an involutory self-map, acting in a similar way as "inversion" makes a semigroup into a group.

For compact groups, this concept, the equivalence of the category of compact groups with a certain category of weakly complete symmetric Hopf algebras, via duality, eventually leads us to the conclusive form of the *Hochschild-Tannaka Duality* of the category of compact groups and a certain subcategory of the category of purely algebraic symmetric Hopf algebras. (The interested reader will find this discussed in Reference [1], Chapter 3: Part 3, pp. 90–12, and in Appendix 3 on Category Theory: Section on Commutative Monoidal Categories and their Monoids, Part 5: Symmetric Hopf Algebras over $\mathbb{R}$ and C, pp. 856–862, and, finally, in Appendix 7: Weakly Complete Topological Vector Spaces, Subsection on: Weakly Complete Unital Algebra, pp. 936–941.)

It must be noted here that, for a Hopf algebra A with multiplication $m\colon A \otimes A \to A$ and identity $u\colon \mathbb{K} \to A$, comultiplication $c\colon A \to A \otimes A$, and coidentity $k\colon A \to \mathbb{K}$, we call an element $a \in A$ *group-like* if $c(a) = a \otimes a$ and $k(a) = 1$, and *primitive* if $c(a) = a \otimes 1 + 1 \otimes a$. Then, an additional general structural feature is to be added to Theorem 8:

Theorem 9. Fundamental Theorem of Weakly Complete Hopf Algebras: *If A is a weakly complete symmetric Hopf algebra, then the set $\mathbb{G}(A)$ of group-like elements is a closed subgroup of A^{-1} and, thus, is a pro-Lie subgroup of A^{-1}.*

The set $\mathbb{P}(A)$ of primitive elements is a closed Lie subalgebra of $A_{\rm Lie}$ and is the Lie algebra of $\mathbb{G}(A)$, and its exponential function $\exp_{\mathbb{G}(A)}\colon \mathbb{P}(A) \to \mathbb{G}(A)$ *is induced by the exponential function of A.*

This applies, in particular, to the group algebra $\mathbb{R}[G]$ of each compact group G, where we have:

Theorem 10. The Group Algebra Theorem for Compact Groups: *A compact group G may be identified with the subgroup $\mathbb{G}(\mathbb{R}[G])$ of group-like elements in the group algebra $\mathbb{R}[G]$, and its Lie algebra $\mathfrak{L}(G)$ may be identified with the Lie subalgebra of $\mathbb{P}(\mathbb{R}[G])$ of primitive elements of $\mathbb{R}[G]$, and, finally, its exponential function* $\exp\colon \mathfrak{L}(G) \to G$ *is then induced by the exponential* *function of the weakly complete algebra $\mathbb{R}[G]$. The cocommutative weakly complete symmetric Hopf algebra $\mathbb{R}[G]$ is dual to the commutative symmetric Hopf algebra $R(G, \mathbb{R})$.* (See Reference [26].)

The essential feature of a Lie group G is its Lie algebra $\mathfrak{g}$, which is at the heart of its algebraic structure. In analogy to the way that leads from groups to group algebras, there is a traditional path leading from Lie algebras to associative algebras. It has been observed recently that the functor from the category of weakly complete unital algebras to the category of profinite dimensional Lie algebras which associates with a weakly complete unital algebra A the Lie algebra A_{Lie} whose underlying vector space is that underlying A with the Lie bracket $[a, b] = ab - ba$. Since A is profinite dimensional, so is the weakly complete Lie algebra A_{Lie}. The assignment $A \mapsto A_{\mathrm{Lie}}$ is a functor from the category $\mathcal{WA}$ of weakly complete associative unital algebras to the category $\mathcal{FL}$ of profinite dimensional Lie algebras. The left adjoint $\mathbf{U}\colon \mathcal{FL} \to \mathcal{WA}$ yields for a profinite dimensional Lie algebra $\mathfrak{g}$ the weakly complete unital associative algebra $\mathbf{U}(\mathfrak{g})$. (See References [27,28].)

Theorem 11. The Enveloping Algebra Theorem: *Let $\mathfrak{g}$ be a profinite dimensional Lie algebra and $U(\mathfrak{g})$ its traditional enveloping algebra over $\mathbb{K}$. Then, $\mathbf{U}(\mathfrak{g})$ is a weakly complete unital associative symmetric Hopf algebra containing the classical enveloping algebra $U(\mathfrak{g})$ as a dense sub-Hopf algebra. The weakly complete algebra $\mathbf{U}(\mathfrak{g})$ has an exponential function* $\exp\colon \mathbf{U}(\mathfrak{g})_{\mathrm{Lie}} \to \mathbf{U}(\mathfrak{g})^{-1}$.

The Lie subalgebra $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ of primitive elements contains naturally a copy of $\mathfrak{g}$ which generates $\mathbf{U}(\mathfrak{g})$ algebraically and topologically as an algebra. $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ is the Lie algebra of the pro-Lie group of group-like elements $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$.

While the classical enveloping algebra does not contain any nonidentity group-like elements, the weakly complete enveloping algebra $\mathbf{U}(\mathfrak{g})$ contains within the pro-Lie group $\mathbf{U}(\mathfrak{g})^{-1}$ the group $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ of group-like elements, which, in turn, contains the group $\Gamma(\mathfrak{g}) = \overline{\langle \exp \mathfrak{g} \rangle}$ that is attached to $\mathfrak{g}$ by Lie's Third Theorem, and the exponential function $\exp\colon \mathfrak{g} \to \Gamma(\mathfrak{g})$ is induced by the exponential of the weakly complete algebra $\mathbf{U}(\mathfrak{g})$.

5. Postscript

After a brief review of 100 years of history of Lie groups and locally compact groups, we have tried to emphasize the widening of the horizon from the landscape of classical Lie group and locally compact group theory to pro-Lie groups. Apart from an emphasis to include functorial thinking into the study of topological groups, this enlargement of scope is strengthened by the viewpoint that Lie group theory deals in essence with topological groups G having a Lie algebra $\mathfrak{L}(G)$ and an exponential function $\exp : \mathfrak{L}(G) \to G$ that crucially determines the structure of G via the Lie algebra structure of $\mathfrak{L}(G)$. Functorial thinking tells us how far we have to go from the long standing classical field of finite dimensional Lie algebras and connected (or at least almost connected!) Lie groups, and, so, we shall unquestionably arrive at pro-Lie groups.

The prime testing ground for pro-Lie group theory remains the field of compact groups. At the beginning of their history, decades ago, it was detected that they were pro-Lie groups automatically by their representation theory. Now, they tell us how far we can go with a clear structure theory of pro-Lie groups past the boundaries imposed by connectivity. In that process, we redetect the significance of "almost connected" groups, namely those G whose space $G_t = G/G_0$ of connected components is compact. In the realm of locally compact groups, Hidehiko Yamabe had justly drawn attention to almost connected locally compact groups for which one could demonstrate that they were pro-Lie groups.

A second testing ground for any theory of topological groups is the class of commutative ones. As far as pro-Lie groups are concerned, this territory is largely uncharted. Yet, once more, the subterritory of almost connected abelian pro-Lie groups is crystal clear: it comprises all groups which are direct products $E \times C$, where E is (the additive group of) a so-called "weakly complete" real topological vector space. These topological vector spaces are also the ones that are underlying the Lie algebras of all pro-Lie groups. So, they play a significant role in pro-Lie theory on both the group and the algebra level. How complicated are they?

The answer was simple since the beginning of their presence a quarter of a century ago: They are simply the duals of ordinary real vector spaces V, together with the topology that these inherit from their nature as function spaces $E = \mathrm{Hom}(V, \mathbb{R}) \stackrel{\mathrm{def}}{=} V^* \subseteq \mathbb{R}^V$ in the form of the topology of pointwise convergence, or, equivalently expressed, the topology induced by the Tychonov product topology of $\mathbb{R}^V$. Traditionally, this topology on E is called the "weak-* topology", which led to the terminology of *weakly complete vector spaces*. Their truly basic nature is emphasized by the fact that the topological dual $E' = \mathrm{Hom}_{\mathrm{continuous}}(E, \mathbb{R}) \subseteq C(E, \mathbb{R})$ is naturally isomorphic to V, that is $V \cong V^{*\prime}$, and that, likewise, $E \cong E'^*$. Compactly phrased, the categories $\mathcal{W}$ of *weakly complete vector spaces* and the category $\mathcal{V}$ of (ordinary) real vector spaces are dual. This interplay pertains, therefore, to *elementary linear algebra*. Moreover, the quotient map $\mathbb{R} \mapsto \mathbb{T} \stackrel{\mathrm{def}}{=} \mathbb{R}/\mathbb{Z}$ induces an isomorphism:

$$V^* = \mathrm{Hom}(V, \mathbb{R}) \cong \mathrm{Hom}(V, \mathbb{T}) = \widehat{V} = \textit{Pontryagin Dual of } V.$$

Thus, a closer appropriate inspection shows that the duality between $\mathcal{V}$ and $\mathcal{W}$ is just another manifestation of Pontryagin Duality expressed as $V \cong \widehat{\widehat{V}}$ and $E \cong \widehat{\widehat{E}}$ (where an *ordinary vector space* V is equipped with its unique smallest locally convex topology).

The category $\mathcal{W}$ allows an immediate natural access from elementary linear algebra to the category $\mathcal{W}A$ of all weakly complete unital associative algebras. It is astonishing that each such algebra A provides an immediate connection to the world of pro-Lie groups insofar as A is a projective limit of finite dimensional algebras and as the group of units A^{-1} is a pro-Lie group whose Lie algebra $\mathfrak{L}(A^{-1})$ is the Lie algebra A_{Lie} defined on A by the Lie bracket, while their exponential function is the ordinary exponential function $\exp A_{\mathrm{Lie}} \to A^{-1}$, $\exp a = 1 + a + \frac{1}{2!} \cdot a^2 + \cdots$ defined on all of A. This opens up the general definition of a weakly complete group algebra $\mathbb{R}[G]$ of a pro-Lie group G and a weakly complete universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of a profinite-dimensional Lie algebra $\mathfrak{g}$. Here, pro-Lie group theory meets weakly complete algebras in the form of appropriate group algebras and appropriate weakly complete enveloping algebras on the basis of a weakly complete symmetric Hopf algebra theory which we have described. Yet, even for compact groups, this opens up previously unnoticed connections to the classical Tannaka-Hochschild duality theory.

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References

1. Hofmann, K.H.; Morris, S.A. *The Structure Theory of Compact Groups–A Primer for the Student–A Handbook for the Expert*, 4th ed.; De Gruyter Studies in Mathematics: Berlin, Germany; Boston, MA, USA, 2020; p. xxvi+1006.
2. Hewitt, E.; Ross, K.A. *Abstract Harmonic Analysis I*; Springer: Berlin, Germany, 1963.
3. Halmos, P. *Measure Theory*; Springer: New York, NY, USA, 1978.
4. Chasco, M.J. Pontryagin Duality for Metrizable Groups. *Arch. Math.* **1998**, *70*, 22–28. [CrossRef]
5. Morris, S.A. *Pontryagin Duality and the Structure of Locally Compact Abelian Groups*; Cambridge University Press: Cambridge, UK, 1977.
6. Hofmann, K.H.; Morris, S.A. *The Lie Theory of Connected Pro-Lie Groups–A Structure Theory for Pro-Lie Algebras, Pro-Lie Groups, and Connected Locally Compact Groups*; European Mathematical Society Publishing House: Zürich, Switzerland, 2008.

7. Iwasawa, K. On some types of topological groups. *Ann. Math.* **1949**, *50*, 507–558. [CrossRef]
8. Willis, G. The structure of totally disconnected locally compact groups. *Math. Ann.* **1994**, *300*, 341–363. [CrossRef]
9. Willis, G. Totally disconnected groups and proofs of conjectures of Hofmann and Mukherjea. *Bull. Austral. Math. Soc.* **1995**, *51*, 489–494. [CrossRef]
10. Hewitt, E.; Ross, K.A. *Abstract Harmonic Analysis II*; Springer: Berlin, Germany, 1970.
11. Rudin, W. *Fourier Analysis on Groups*; Interscience Publ.: New York, NY, USA, 1967.
12. Folland, G.B. *A Course in Abstract Harmonic Analysis*; CRC Press: Boca Raton, FL, USA; London, UK, 1995.
13. Glöckner, H.; Neeb, K.-H. *Infinite-Dimensional Lie Groups and Main Examples, Reprint*; Spinger: New York, NY, USA, 2017.
14. Hofmann, K.H.; Neeb, K.-H. Pro-Lie Groups which are Infinite Dimensional Lie Groups. *Math. Proc. Cambridge Phil. Soc.* **2009**, *146*, 351–378. [CrossRef]
15. Ribes, L.; Zalesskii, P. *Profinite Groups*, 2nd ed.; Springer: Berlin, Germany, 2010.
16. Hofmann, K.H.; Morris, S.A. The Structure of Almost Connected Pro-Lie Groups. *J. Lie Theory* **2011**, *21*, 347–383.
17. Hofmann, K.H.; Morris, S.A. Pro-Lie Groups: A Survey with Open Problems. *Axioms* **2015**, *4*, 294–312. [CrossRef]
18. Herfort, W.; Hofmann, K.H.; Russo, F.G. *Periodic Locally Compact Groups*; De Gruyter Studies in Mathematics: Berlin, Germany; Boston, MA, USA, 2018.
19. Hofmann, K.H.; Willis, G.A. Continuity Characterizing Totally Disconnected Locally Compact Groups. *J. Lie Theory* **2015**, *25*, 1–7.
20. Tannaka, T. Über den Dualitätssatz der nichtkommutativen topologischen Gruppen. *Tohoku Math. J.* **1939**, *45*, 1–12.
21. Joyal, A.; Street, R. An Introduction to Tannaka Duality and Quantum Groups. In Proceedings of the Category Theory: International Conference, Como, Italy, 22–28 July 1990; Carboni, A., Peddicio, M.C., Rossolini, G., Eds.; Springer: Berlin, Germany, 1991; pp. 413–492.
22. Hochschild, G.P. *The Structure of Lie Groups*; Holden-Day Inc.: San Francisco, CA, USA, 1965.
23. Hofmann, K.H.; Morris, S.A. Sophus Lie's Third Fundamental Theorem and the Adjoint Functor Theorem. *J. Group Theory* **2005**, *8*, 115–133. [CrossRef]
24. Bogfiellmo, G.; Dahmen, R.; Schmeding, A. Character groups of Hopf algebras as infinite dimensional Lie groups. *Ann. Inst. Fourier* **2016**, *66*, 2101–2155. [CrossRef]
25. Michaelis, W. Coassociative coalgebras. In *Handbook of Algebra*; Hazewinkel, M., Ed.; Elsevier: Amsterdam, The Netherlands, 2003; pp. 587–788.
26. Dahmen, R.; Hofmann, K.H. The Pro-Lie Group Aspect of Weakly Complete Algebras and Weakly Complete Group Hopf Algebras. *J. Lie Theory* **2019**, *29*, 413–455.
27. Hofmann, K.H.; Kramer, L. On Weakly Complete Group Algebras of Compact Groups. *J. Lie Theory* **2020**, *30*, 407–424.
28. Hofmann, K.H.; Kramer, L. *On Weakly Complete Enveloping Algebras: A Poincaré–Birkhoff–Witt Theorem*; In preparation.

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Normed Spaces Which Are Not Mackey Groups

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Abstract: It is well known that every normed (even quasibarrelled) space is a Mackey space. However, in the more general realm of locally quasi-convex abelian groups an analogous result does not hold. We give the first examples of normed spaces which are not Mackey groups.

Keywords: normed space; Mackey group; locally quasi-convex; compatible group topology

MSC: 46A8; 46E10; 54H11

1. Introduction

Let (E,τ) be a locally convex space (lcs for short). A locally convex vector topology ν on E is called *compatible with* τ if the spaces (E,τ) and (E,ν) have the same topological dual space. The famous Mackey–Arens Theorem states that there is a finest locally convex vector space topology μ on E compatible with τ. The topology μ is called the *Mackey topology* on E associated with τ, and if $\mu=\tau$, the space E is called a *Mackey space*. The most important class of Mackey spaces is the class of quasibarrelled spaces. This class is sufficiently rich and contains all metrizable locally convex spaces. In particular, every normed space is a Mackey space.

For an abelian topological group (G,τ) we denote by $\widehat{G}$ the group of all continuous characters of (G,τ). Two topologies μ and ν on an abelian group G are said to be *compatible* if $\widehat{(G,\mu)}=\widehat{(G,\nu)}$. Being motivated by the concept of Mackey spaces, the following notion was implicitly introduced and studied in [1], and explicitly defined in [2] (for all relevant definitions see the next section): A locally quasi-convex abelian group (G,μ) is called a *Mackey group* if for every locally quasi-convex group topology ν on G compatible with μ it follows that $\nu\le\mu$.

Every lcs considered as an abelian topological group is locally quasi-convex. So, it is natural to ask whether every Mackey space is also a Mackey group. Surprisingly, the answer to this question is negative. Indeed, answering a question posed in [2], we show in [3] that there is even a metrizable lcs which is not a Mackey group. Recall that for every Tychonoff space X, the space $C_p(X)$ of all continuous functions on X endowed with the pointwise topology is quasibarrelled, and hence it is a Mackey space. However, in [4] we proved that the space $C_p(X)$ is a Mackey group if and only if it is barrelled. In particular, the metrizable space $C_p(\mathbb{Q})$ is *not* a Mackey group. These results motivate the following question. For $1\le p\le\infty$, denote with $\mathfrak{T}_{\ell_p}$ the topology on the direct sum $\mathbb{R}^{(\mathbb{N})}:=\bigcup_{n\in\mathbb{N}}\mathbb{R}^n$ induced from ℓ_p.

Problem 1 ([3]). *Does there exist a* normed *space E which is not a Mackey group? What about* $(\mathbb{R}^{(\mathbb{N})},\mathfrak{T}_{\ell_p})$*?*

The main goal of this note is to answer Problem 1 in the affirmative. More precisely, we show that the normed spaces $c_{00}:=(\mathbb{R}^{(\mathbb{N})},\mathfrak{T}_{\ell_\infty})$ and $\ell_{00}^1:=(\mathbb{R}^{(\mathbb{N})},\mathfrak{T}_{\ell_1})$ are not Mackey groups.

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2. Main Result

Set $\mathbb{N} := \{1, 2, \dots\}$. Denote by $\mathbb{S}$ the unit circle group and set $\mathbb{S}_+ := \{z \in \mathbb{S} : \operatorname{Re}(z) \geq 0\}$.

Let G be an abelian topological group. A character $\chi \in \widehat{G}$ is a continuous homomorphism from G into $\mathbb{S}$. A subset A of G is called *quasi-convex* if for every $g \in G \setminus A$ there exists $\chi \in \widehat{G}$ such that $\chi(g) \notin \mathbb{S}_+$ and $\chi(A) \subseteq \mathbb{S}_+$. An abelian topological group is called *locally quasi-convex* if it admits a neighborhood base at the neutral element 0 consisting of quasi-convex sets. It is well known that the class of locally quasi-convex abelian groups is closed under taking products and subgroups.

The following group plays an essential role in the proof of our main results, Theorems 1 and 2. Set

$$c_0(\mathbb{S}) := \{(z_n) \in \mathbb{S}^{\mathbb{N}} : z_n \to 1\},$$

and denote by $\mathfrak{F}_0(\mathbb{S})$ the group $c_0(\mathbb{S})$ endowed with the metric $d\big((z_n^1), (z_n^2)\big) = \sup\{|z_n^1 - z_n^2|, n \in \mathbb{N}\}$. Then $\mathfrak{F}_0(\mathbb{S})$ is a Polish group, and the sets of the form $V^{\mathbb{N}} \cap c_0(\mathbb{S})$, where V is a neighborhood at the unit $1 \in \mathbb{S}$, form a base at the identity $1 = (1_n) \in \mathfrak{F}_0(\mathbb{S})$. In [5] (Theorem 1), we proved that the group $\mathfrak{F}_0(\mathbb{S})$ is reflexive and hence locally quasi-convex.

A proof of the next important result can be found in [6] [Proposition 2.3].

Fact 1. *Let E be a real lcs. Then the map $\psi : E' \to \widehat{E}, \psi(\chi) := e^{2\pi i \chi}$, is an algebraic isomorphism.*

We use the next standard notations. Let $\{e_n\}_{n\in\mathbb{N}}$ be the standard basis of the Banach space $(c_0, \|\cdot\|_\infty)$, and let $\{e_n^*\}_{n\in\mathbb{N}}$ be the canonical basis in the dual Banach space $(c_0)' = \ell_1$, i.e.,

$$e_n = (0, \dots, 0, 1, 0, \dots) \quad \text{and} \quad e_n^* = (0, \dots, 0, 1, 0, \dots),$$

where 1 is placed in position n. Then $c_{00} = (\mathbb{R}^{(\mathbb{N})}, \mathfrak{T}_{\ell_\infty})$ is a dense subspace of c_0 consisting of all vectors with finite support.

Theorem 1. *The normed space c_{00} is not a Mackey group.*

Proof. For simplicity and clearness of notations we set $E := c_{00}$ and $\tau := \mathfrak{T}_{\ell_\infty}$. For every $n \in \mathbb{N}$, set $\chi_n := n e_n^*$. It is clear that $\chi_n \to 0$ in the weak* topology on E' and hence in $\sigma(\widehat{E}, E)$. Therefore we can define the linear injective operator $F : E \to E \times c_0$ and the monomorphism $p : E \to E \times \mathfrak{F}_0(\mathbb{S})$ setting (for all $x = (x_n) \in E$)

$$F(x) := \big(x, R(x)\big), \text{ where } R(x) := \big(\chi_n(x)\big) = (n x_n) \in c_0,$$
$$p(x) := \big(x, R_0(x)\big), \text{ where } R_0(x) := Q \circ R(x) = \big(\exp\{2\pi i \chi_n(x)\}\big) = \big(\exp\{2\pi i n x_n\}\big) \in \mathfrak{F}_0(\mathbb{S}).$$

Denote with $\mathfrak{T}$ and $\mathfrak{T}_0$ the topologies on E induced from $E \times c_0$ and $E \times \mathfrak{F}_0(\mathbb{S})$, respectively. So $\mathfrak{T}$ is a locally convex vector topology on E and $\mathfrak{T}_0$ is a locally quasi-convex group topology on E. By construction, $\tau \leq \mathfrak{T}_0 \leq \mathfrak{T}$, so taking into account Fact 1 and the Hahn–Banach extension theorem, we obtain

$$\psi(E') = \psi\big(\ell_1\big) \subseteq \widehat{(E, \mathfrak{T}_0)} \subseteq \psi\big((E, \mathfrak{T})'\big) \subseteq \psi\big(\ell_1 \times \ell_1\big). \tag{1}$$

Step 1: The topologies τ and $\mathfrak{T}_0$ are compatible. By (1), it is sufficient to show that each continuous character of $\big(E, \mathfrak{T}_0\big)$ belongs to $\psi\big(\ell_1\big)$. Fix $\chi \in \widehat{\big(E, \mathfrak{T}_0\big)}$. Then (1) implies that $\chi = \psi(\eta) = \exp\{2\pi i \eta\}$ for some

$$\eta = \big(\nu, (c_n)\big) \in \ell_1 \times \ell_1, \text{ where } \nu \in \ell_1 \text{ and } (c_n) \in \ell_1,$$

and

$$\eta(x) = \nu(x) + \sum_{n\in\mathbb{N}} c_n \chi_n(x) = \nu(x) + \sum_{n\in\mathbb{N}} c_n \cdot n x_n \quad (x = (x_n) \in E).$$

To prove that $\chi \in \psi\big(\ell_1\big)$ it is sufficient (and also necessary) to show that $\big(c_n n\big)_n \in \ell_1$. Replacing, if needed, η by $\eta - \nu$, we assume that $\nu = 0$.

Suppose for a contradiction that $\sum_n |c_n|n = \infty$. Since χ is continuous, Fact 1 shows that, for every $\varepsilon < 0.01$, there is a $\delta < \varepsilon$ such that

$$\eta(x) = \sum_{n\in\mathbb{N}} nc_n x_n \in (-\varepsilon, \varepsilon) + \mathbb{Z}, \text{ for every } x \in U_\delta, \tag{2}$$

where U_δ is a canonical $\mathfrak{T}_0$-neighborhood of zero

$$U_\delta := \{x = (x_n) \in E : |x_n| \le \delta \text{ and } nx_n \in [-\delta, \delta] + \mathbb{Z} \text{ for every } n \in \mathbb{N}\}. \tag{3}$$

In what follows ε and δ are fixed as above. We distinguish between three cases.

Case 1: There is a subsequence $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $|c_{n_k}|n_k \to \infty$ as $k \to \infty$. As $|c_{n_k}|n_k \to \infty$ and $c_n \to 0$, there is $k \in \mathbb{N}$ such that

$$\frac{1}{8|c_{n_k}|} > 1 \quad \text{and} \quad \frac{3}{8|c_{n_k}|n_k} < \delta. \tag{4}$$

The first inequality in (4) implies that there is

$$m_k \in \left(\frac{1}{8|c_{n_k}|}, \frac{3}{8|c_{n_k}|} \right) \cap \mathbb{N}. \tag{5}$$

Set $x = (x_n) := (0, \dots, 0, \mathsf{sign}(c_{n_k})\frac{m_k}{n_k}, 0, \dots)$, where the nonzero element is placed in position n_k. Then $nx_n \in \mathbb{Z}$ for every $n \in \mathbb{N}$, and the second inequality of (4) and (5) imply

$$\|x\|_\infty = |x_{n_k}| = \frac{m_k}{n_k} < \frac{3}{8|c_{n_k}|n_k} < \delta.$$

Therefore $x \in U_\delta$. On the other hand, (5) implies

$$\frac{1}{8} < \eta(x) = \sum_{n\in\mathbb{N}} c_n n x_n = |c_{n_k}|n_k \frac{m_k}{n_k} = |c_{n_k}|m_k < \frac{3}{8}.$$

Hence $\eta(x) \notin (-\varepsilon, \varepsilon) + \mathbb{Z}$ since $\varepsilon < 0.01$. However, this contradicts (2).

Case 2: There is a subsequence $\{n_k\}_{k\in\mathbb{N}}$ and a number $a > 0$ such that $|c_{n_k}|n_k \to a$ as $k \to \infty$. Choose $N \in \mathbb{N}$ such that

$$\frac{a}{2} < |c_{n_k}|n_k < \frac{3a}{2} \quad \text{for all} \quad k \ge N. \tag{6}$$

Choose a finite subset F of $\{n_k\}_{k\ge N}$ and, for every $n \in F$, a natural number i_n such that the following two conditions are satisfied:

$$i_n \in \{1, 2, \dots, \lfloor \delta n \rfloor\} \quad \text{for every} \quad n \in F, \tag{7}$$

and

$$\frac{10}{72a} < \sum_{n\in F} \frac{i_n}{n} < \frac{30}{72a} \tag{8}$$

(this is possible because $\frac{1}{n} \le \frac{i_n}{n} \le \frac{\lfloor \delta n \rfloor}{n} \approx \delta$ and $n \to \infty$: so, if $\frac{10}{72a} < \delta$ the set F can be chosen to have only one element, and if $\delta \le \frac{10}{72a}$, the set F also can be easily chosen to be finite). Now we define $x = (x_n) \in E$ by

$$x_n = \mathsf{sign}(c_n) \cdot \frac{i_n}{n} \text{ if } n \in F, \text{ and } x_n = 0 \text{ if } n \in \mathbb{N} \setminus F.$$

Then $nx_n \in \mathbb{Z}$ for every $n \in \mathbb{Z}$, and, by (7), $\|x\|_\infty = \max\left\{\frac{i_n}{n} : n \in F\right\} \le \delta$. Therefore $x \in U_\delta$. On the other hand, (6) and (8) imply

$$\frac{5}{24} < \frac{a}{2} \sum_{n\in F} \frac{i_n}{n} < \eta(x) = \sum_{n\in\mathbb{N}} c_n n x_n = \sum_{n\in F} |c_n| n \cdot \frac{i_n}{n} < \frac{3a}{2} \sum_{n\in F} \frac{i_n}{n} < \frac{5}{8}.$$

Hence $\eta(x) \notin (-\varepsilon, \varepsilon) + \mathbb{Z}$ which contradicts (2).

Case 3: $\lim_n |c_n|n = 0$. Choose $N \in \mathbb{N}$ such that (recall that $(c_n) \in \ell_1$)

$$\sum_{n \geq N} |c_n| < \tfrac{\delta}{100} \quad \text{and} \quad \sup\left\{|c_n|n : n \geq N\right\} < \tfrac{\delta}{100}. \tag{9}$$

Since $\sum_n |c_n|n = \infty$, choose a finite subset $F \subseteq \{N, N+1, \dots\}$ such that

$$\sum_{n \in F} |c_n|n > \tfrac{2}{\delta}. \tag{10}$$

Define $x = (x_n) \in E$ by

$$x_n := \Delta_n \cdot \mathsf{sign}(c_n) \cdot \tfrac{\lfloor \delta n \rfloor}{n} \;\text{ if }\; n \in F, \text{ and }\; x_n := 0 \;\text{ if }\; n \in \mathbb{N} \backslash F,$$

where $\Delta_n \in \{0, 1\}$ will be chosen afterwards. Then, for all $n \in \mathbb{N}$ and arbitrary Δ_ns, we have $x_n \cdot n \in \mathbb{Z}$ and $|x_n| \leq \delta$. Therefore $x \in U_\delta$. On the other hand, we have

$$0 < \eta(x) = \sum_{n \in \mathbb{N}} c_n n x_n = \sum_{n \in F} |c_n| n \Delta_n \cdot \tfrac{\lfloor \delta n \rfloor}{n} \leq \sum_{n \in F} |c_n| n \delta + \sum_{n \in F} |c_n| n \tfrac{\lfloor \delta n \rfloor - \delta n}{n}, \tag{11}$$

(to obtain the last inequality we put $\Delta_n = 1$ for all $n \in F$) and (9) and (10) imply

$$\sum_{n \in F} |c_n| n \delta + \sum_{n \in F} |c_n| n \tfrac{\lfloor \delta n \rfloor - \delta n}{n} > 2 - \tfrac{\delta}{100} > 1. \tag{12}$$

From the second inequality in (9), we have

$$c_n n x_n = |c_n| n \Delta_n \leq |c_n| n < \tfrac{\delta}{100} < \tfrac{1}{100} \quad \text{for every} \quad n \in F.$$

Using this inequality and (11) and (12), one can easily find a family $\{\Delta_n : n \in F\}$ such that

$$\tfrac{1}{4} < \eta(x) = \sum_{n \in \mathbb{N}} c_n n x_n < \tfrac{3}{4},$$

and hence $\eta(x) \notin (-\varepsilon, \varepsilon) + \mathbb{Z}$ which contradicts (2).

Cases 1–3 show that the assumption $\sum_n |c_n|n = \infty$ is wrong. Thus the topologies τ and $\mathfrak{T}_0$ are compatible.

Step 2. The topology $\mathfrak{T}_0$ is strictly finer than the original topology τ. Thus, E is not a Mackey group. Indeed, it is clear that $\frac{1}{2k} e_k \to 0$ in the norm topology τ on E. On the other hand, since

$$R_0\left(\tfrac{1}{2k} e_k\right) = \Big(\exp\Big\{2\pi i \cdot \chi_n\left(\tfrac{1}{2k} e_k\right)\Big\}\Big)_{n \in \mathbb{N}} = (1, \dots, 1, -1, 1, \dots,) \quad \text{for every} \quad k \in \mathbb{N},$$

where -1 is placed in position k, we obtain that $\frac{1}{2k} e_k \not\to 0$ in the topology $\mathfrak{T}_0$. Since, by construction, $\tau \leq \mathfrak{T}_0$ we obtain $\tau \subsetneq \mathfrak{T}_0$ as desired. □

Analogously we prove that the normed space $\ell^1_{00} = (\mathbb{R}^{(\mathbb{N})}, \mathfrak{T}_{\ell_1})$ is not a Mackey group. To this end, let $\{e_n\}_{n \in \mathbb{N}}$ be the standard basis of the Banach space $(\ell_1, \|\cdot\|_1)$, and let $\{e_n^*\}_{n \in \mathbb{N}}$ be the canonical dual sequence in the dual Banach space $(\ell_1)' = \ell_\infty$, i.e.,

$$e_n = (0, \dots, 0, 1, 0, \dots) \quad \text{and} \quad e_n^* = (0, \dots, 0, 1, 0, \dots),$$

where 1 is placed in position n. Then ℓ^1_{00} is a dense subspace of ℓ_1 consisting of all vectors with finite support.

Theorem 2. *The normed space ℓ^1_{00} is not a Mackey group.*

Proof. For simplicity and clearness of notations we set $E := \ell^1_{00}$ and $\tau := \mathfrak{T}_{\ell_1}$. For every $n \in \mathbb{N}$, set $\chi_n := ne^*_n$. It is clear that $\chi_n \to 0$ in the weak* topology on E' and hence in $\sigma(\widehat{E}, E)$. Therefore we can define the linear injective operator $F : E \to E \times c_0$ and the monomorphism $p : E \to E \times \mathfrak{F}_0(\mathbb{S})$ setting (for all $x = (x_n) \in E$)

$$F(x) := \big(x, R(x)\big), \text{ where } R(x) := \big(\chi_n(x)\big) = (nx_n) \in c_0,$$
$$p(x) := \big(x, R_0(x)\big), \text{ where } R_0(x) := Q \circ R(x) = \big(\exp\{2\pi i \chi_n(x)\}\big) = \big(\exp\{2\pi i n x_n\}\big) \in \mathfrak{F}_0(\mathbb{S}).$$

Denote with $\mathfrak{T}$ and $\mathfrak{T}_0$ the topologies on E induced from $E \times c_0$ and $E \times \mathfrak{F}_0(\mathbb{S})$, respectively. So $\mathfrak{T}$ is a locally convex vector topology on E and $\mathfrak{T}_0$ is a locally quasi-convex group topology on E. By construction, $\tau \leq \mathfrak{T}_0 \leq \mathfrak{T}$, so taking into account Fact 1 and the Hahn–Banach extension theorem we obtain

$$\psi(E') = \psi\big(\ell_\infty\big) \subseteq \widehat{(E, \mathfrak{T}_0)} \subseteq \psi\big((E, \mathfrak{T})'\big) \subseteq \psi\big(\ell_\infty \times \ell_1\big). \tag{13}$$

Step 1: The topologies τ and $\mathfrak{T}_0$ are compatible. By (13), it is sufficient to show that each continuous character of $\big(E, \mathfrak{T}_0\big)$ belongs to $\psi\big(\ell_\infty\big)$. Fix $\chi \in \widehat{(E, \mathfrak{T}_0)}$. Then (1) implies that $\chi = \psi(\eta) = \exp\{2\pi i \eta\}$ for some

$$\eta = \big(\nu, (c_n)\big) \in \ell_\infty \times \ell_1, \ \text{ where } \nu \in \ell_\infty \ \text{ and } \ (c_n) \in \ell_1,$$

and

$$\eta(x) = \nu(x) + \sum_{n\in\mathbb{N}} c_n \chi_n(x) = \nu(x) + \sum_{n\in\mathbb{N}} c_n \cdot n x_n \quad \big(x = (x_n) \in E\big).$$

To prove that $\chi \in \psi\big(\ell_\infty\big)$ it is sufficient (and also necessary) to show that $\big(c_n n\big)_n \in \ell_\infty$. Replacing if needed η by $\eta - \nu$, we assume that $\nu = 0$.

Suppose for a contradiction that $\big(|c_n| n\big)_n$ is unbounded. Then there is a subsequence $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $|c_{n_k}| n_k \to \infty$ as $k \to \infty$. Since χ is continuous, Fact 1 shows that, for every $\varepsilon < 0.01$, there is a $\delta < \varepsilon$ such that

$$\eta(x) = \sum_{n\in\mathbb{N}} n c_n x_n \in (-\varepsilon, \varepsilon) + \mathbb{Z}, \ \text{ for every } \ x \in U_\delta, \tag{14}$$

where U_δ is a canonical $\mathfrak{T}_0$-neighborhood of zero

$$U_\delta := \big\{x = (x_n) \in E : \|x\|_1 \leq \delta \ \text{ and } \ n x_n \in [-\delta, \delta] + \mathbb{Z} \ \text{ for every } \ n \in \mathbb{N}\big\}. \tag{15}$$

As $|c_{n_k}| n_k \to \infty$ and $c_n \to 0$, there is $k \in \mathbb{N}$ such that

$$\tfrac{1}{8|c_{n_k}|} > 1 \quad \text{and} \quad \tfrac{3}{8|c_{n_k}| n_k} < \delta. \tag{16}$$

The first inequality in (16) implies that there is

$$m_k \in \left(\tfrac{1}{8|c_{n_k}|}, \tfrac{3}{8|c_{n_k}|}\right) \cap \mathbb{N}. \tag{17}$$

Set $x = (x_n) := (0, \dots, 0, \mathrm{sign}(c_{n_k}) \frac{m_k}{n_k}, 0, \dots)$, where the nonzero element is placed in position n_k. Then $n x_n \in \mathbb{Z}$ for every $n \in \mathbb{N}$, and the second inequality of (16) and (17) imply

$$\|x\|_1 = |x_{n_k}| = \tfrac{m_k}{n_k} < \tfrac{3}{8|c_{n_k}| n_k} < \delta.$$

Therefore $x \in U_\delta$. On the other hand, (17) implies

$$\tfrac{1}{8} < \eta(x) = \sum_{n\in\mathbb{N}} c_n n x_n = |c_{n_k}| n_k \tfrac{m_k}{n_k} = |c_{n_k}| m_k < \tfrac{3}{8}.$$

Hence $\eta(x) \notin (-\varepsilon, \varepsilon) + \mathbb{Z}$ since $\varepsilon < 0.01$. However, this contradicts (14).

Step 2. The topology $\mathfrak{T}_0$ is strictly finer than the original topology τ. Thus E is not a Mackey group. Indeed, it is clear that $\frac{1}{2k}e_k \to 0$ in the norm topology τ on E. On the other hand, since

$$R_0\big(\tfrac{1}{2k}e_k\big) = \Big(\exp\Big\{2\pi i\cdot\chi_n\big(\tfrac{1}{2k}e_k\big)\Big\}\Big)_{n\in\mathbb{N}} = (1,\dots,1,-1,1,\dots,) \quad \text{for every} \quad k\in\mathbb{N},$$

where -1 is placed in position k, we obtain that $\frac{1}{2k}e_k \not\to 0$ in the topology $\mathfrak{T}_0$. Since, by construction, $\tau \le \mathfrak{T}_0$ we obtain $\tau \subsetneq \mathfrak{T}_0$ as desired. □

We finish this note with the following problem.

Problem 2. *Let E be a real normed (metrizable, bornological or quasibarrelled) locally convex space. Is it true that E is a Mackey group if and only if it is barrelled?*

Note that every barrelled lcs is a Mackey group, see [1].

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References

1. Chasco, M.J.; Martín-Peinador, E.; Tarieladze, V. On Mackey topology for groups. *Studia Math.* **1999**, *132*, 257–284.
2. Dikranjan, D.; Martín-Peinador, E.; Tarieladze, V. Group valued null sequences and metrizable non-Mackey groups. *Forum Math.* **2014**, *26*, 723–757. [CrossRef]
3. Gabriyelyan, S. On the Mackey topology for abelian topological groups and locally convex spaces. *Topol. Appl.* **2016**, *211*, 11–23. [CrossRef]
4. Gabriyelyan, S. A characterization of barrelledness of $C_p(X)$. *J. Math. Anal. Appl.* **2016**, *439*, 364–369. [CrossRef]
5. Gabriyelyan, S. Groups of quasi-invariance and the Pontryagin duality. *Topol. Appl.* **2010**, *157*, 2786–2802. [CrossRef]
6. Banaszczyk, W. *Additive Subgroups of Topological Vector Spaces*; LNM 1466; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1991.

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Article

Krein's Theorem in the Context of Topological Abelian Groups †

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† Dedicated to our colleague and good friend M. J. Chasco on the occassion of her 65th birthday.

Abstract: A topological abelian group G is said to have the quasi-convex compactness property (briefly, qcp) if the quasi-convex hull of every compact subset of G is again compact. In this paper we prove that there exist locally quasi-convex metrizable complete groups G which endowed with the weak topology associated to their character groups $G^\wedge$, do not have the qcp. Thus, Krein's Theorem, a well known result in the framework of locally convex spaces, cannot be fully extended to locally quasi-convex groups. Some features of the qcp are also studied.

Keywords: quasi-convex subset; determining subgroup; quasi-convex compactness property; Krein's Theorem

MSC: 54H11; 54D50; 46A20

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1. Introduction

Krein's Theorem is a key result in the classical theory of topological vector spaces. It admits different formulations with varying degrees of generality; for instance the one presented in [1] (5.3, Theorem 4) reads as follows:

Theorem 1. *Let E be a locally convex space and let A be a weakly compact subset of E. Then the closed convex hull of A is weakly compact if and only if it is complete for the given topology.*

From this result it immediately follows that the weak topology of a complete locally convex space has the convex compactness property (we recall the relevant notions below). In this paper we deal with the extension of Krein's Theorem to topological abelian groups, in the natural form, which consists of replacing classical objects of the theory of topological vector spaces by corresponding objects of the theory of topological abelian groups. For instance, the natural substitution of continuous linear forms is realized by continuous characters, the notion of convexity by that of quasi-convexity etc. This process may result in natural counterparts of important theorems of Functional Analysis for the class of topological abelian groups. Our main result Theorem 6 confirms that this is not always the case.

We first deal with the quasi-convex compactness property (qcp), a convenient notion which mimics the convex compactness property as defined in [2]. In Sections 2 and 3 some aspects of the qcp are outlined, including the description of classes of abelian topological groups which have the qcp, the hereditary behaviour and some obstructions to the qcp. In Section 4 we study the relation between the qcp and the convex compactness property in topological vector spaces. With these instruments at hand, in Section 5 we prove that a counterpart of Krein's Theorem holds for the class of locally convex spaces considered as topological abelian groups. However, it cannot be extended to the bigger class of locally quasi-convex groups. We provide a family of locally quasi-convex metrizable groups G which endowed with their weak topology $\sigma(G, G^\wedge)$ do not have the qcp. The groups in this family can be considered as counterexamples to the version of Krein's Theorem for topological groups.

In Section 6 we study some interactions between the qcp and other properties like being g-barrelled, or the Glicksberg property.

The many ways in which completeness-like properties relate to convexity in topological vector spaces have been studied for a long time and Krein's theorem is just an important milestone in this ongoing exploration. As suggested by one of the referees of this paper, it would make sense to look for relevant topological group counterparts of other concepts and results which have arisen in connection with this topic, e.g., those concerning the metric convex compactness property, or Mackey completeness.

Preliminaries

Define $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$ and denote by $\pi : \mathbb{R} \to \mathbb{T}$ the canonical projection. On $\mathbb{T}$ we consider the group norm $|\pi(r)| = d_{\mathbb{R}}(r + \mathbb{Z}, 0)$. Note that $|\pi(r)| \leq \min\{|r|, 1/2\}$ for every $r \in \mathbb{R}$, and $|\pi(r)| = |r|$ whenever $|r| \leq 1/2$. We put $\mathbb{T}_+ = \pi([-1/4, 1/4])$.

A *character* of an abelian group X is a homomorphism $\chi : X \to \mathbb{T}$. We denote by $\mathrm{Hom}(X, \mathbb{T})$ the group of all characters of X with pointwise sum.

If X is a topological abelian group, we write $X^\wedge$ for the subgroup of $\mathrm{Hom}(X, \mathbb{T})$ whose elements are the continuous characters of X. A topological abelian group X is said to be MAP (an abbreviation of "maximally almost periodic") if for every $x \neq 0$ in X there exists $\chi \in X^\wedge$ with $\chi(x) \neq 0$. Two group topologies on an abelian group are called *compatible* if they give rise to the same family of continuous characters.

If U is a subset of the topological abelian group X, the set $U^\triangleright := \{\chi \in X^\wedge : \chi(U) \subseteq \mathbb{T}_+\}$ is called the *polar* of U. Note that a subset of $X^\wedge$ is equicontinuous if and only if it is a subset of the polar of a neighborhood of zero in X. If B is a subset of $X^\wedge$, where X is clear from the context, we will sometimes denote by $B^\triangleleft$ the set $\{x \in X : B(x) \subseteq \mathbb{T}_+\}$. Note that $U^{\triangleright\triangleleft} = \bigcap_{\chi \in U^\triangleright} \chi^{-1}(\mathbb{T}_+)$ for every $U \subseteq X$. We will say that U is *quasi-convex* if $U = U^{\triangleright\triangleleft}$. For an arbitrary $U \subseteq X$, the set $U^{\triangleright\triangleleft}$ is the smallest quasi-convex subset of X which contains U; we call it the quasi-convex hull of U in X and denote it usually by $\mathrm{qc}_X(U)$.

A topological abelian group is called *locally quasi-convex* if it has a basis of neighborhoods of zero formed by quasi-convex sets. Given a topological abelian group X with topology τ, the quasi-convex neighborhoods of zero in X form a basis of neighborhoods of zero for the finest topology among the group topologies on X which are coarser than τ and locally quasi-convex. We call X_{lqc} the group X endowed with this locally quasi-convex modification of its original topology. The groups X and X_{lqc} have the same continuous characters. Moreover, X_{lqc} is Hausdorff if and only if X is MAP. See ([3], Proposition 6.18) for details on this construction.

Given an abelian group X and a subgroup H of $\mathrm{Hom}(X, \mathbb{T})$, we denote by $\sigma(X, H)$ the initial topology on X with respect to the characters in H. The topological group $(X, \sigma(X, H))$ is precompact and its dual group is exactly H. In what follows X will often carry a group topology and H will be taken as $X^\wedge$; also, for a MAP topological group X we will sometimes consider the weak topology $\sigma(X^\wedge, X)$ where X is regarded as a subgroup of $\mathrm{Hom}(X^\wedge, \mathbb{T})$ in the natural way.

A topological abelian group (X, τ) is said to *satisfy the Glicksberg property* if every $\sigma(X, (X, \tau)^\wedge)-$compact subset of X is $\tau-$compact. The classical Glicksberg Theorem establishes that all locally compact abelian groups have the Glicksberg property.

If X is a topological abelian group, we denote by $\mathcal{T}_c$ the topology on $X^\wedge$ of uniform convergence on compact subsets of X. We will often write $X_c^\wedge$ as a shorthand for $(X^\wedge, \mathcal{T}_c)$. The topology $\mathcal{T}_c$ admits as a basis of neighborhoods of zero the family of all sets of the form $K^\triangleright$ where K runs over all compact subsets of X.

For any topological abelian group X we define the homomorphism $\alpha_X : X \to (X_c^\wedge)_c^\wedge$ by $\alpha_X(x)(\chi) = \chi(x)$. We say that X is *semi-reflexive* if α_X is onto. We say that X is *reflexive* if α_X is a topological isomorphism. The classical Pontryagin-van Kampen theorem asserts that every locally compact abelian group is reflexive.

Let H be a subgroup of a topological abelian group X, and let $\imath : H \to X$ be the inclusion mapping. We say that H is *dually embedded* in X if every continuous character of H can be extended to X, i.e., if the restriction mapping $\imath^\wedge : X^\wedge \to H^\wedge$ is onto. It is clear that $\imath^\wedge : X_c^\wedge \to H_c^\wedge$ is always continuous; we say that H is *strongly dually embedded* in X if it is actually a quotient mapping (i.e., it is onto and open). Every open subgroup is strongly dually embedded ([4], Lemma 2.2). Assume that H is dense in X; then clearly H is strongly dually embedded in X if and only if $\imath^\wedge : X_c^\wedge \to H_c^\wedge$ is a topological isomorphism. We say in this case that H *determines* X. A metrizable group is determined by all of its dense subgroups (see ([3], Theorem 4.3) and ([5], Theorem 2)).

The $k-$*refinement* or *k-modification* of a topological space (X, t) is the topological space (X, kt) where kt is the family formed by those $U \subseteq X$ with $U \cap K \in t\restriction_K$ for every $t-$compact set $K \subseteq X$. The topology kt is the finest topology on X among those which induce $t\restriction_K$ on every $t-$compact $K \subseteq X$. If t is a Hausdorff topology, kt admits the following characterization: a subset $C \subseteq X$ is $kt-$closed if and only if $C \cap K$ is $t-$compact for every $t-$compact subset $K \subseteq X$. The space (X, t) is a $k-$space if $kt = t$. Metrizable spaces and locally compact spaces are $k-$spaces. If an abelian topological group X is a $k-$space, then all compact subsets of $X_c^\wedge$ are equicontinuous, i.e., the homomorphism $\alpha_X : X \to (X_c^\wedge)_c^\wedge$ is continuous.

We say that the topological abelian group X is *locally precompact* if it admits a nonempty precompact open subset or, equivalently if it is a subgroup of a locally compact group.

A topological group is *almost metrizable* if it contains a compact subset of countable character. As proved in ([3], 2.20), an abelian Hausdorff topological group X is almost metrizable if and only if it has a compact subgroup K such that X/K is metrizable.

We only consider vector spaces over $\mathbb{R}$. A subset A of a topological vector space E is said to be *balanced* if $[-1, 1]A \subseteq A$. A subset of E is *absolutely convex* if it is both convex and balanced. For any subset $A \subseteq E$, we denote by $\mathrm{acc}_E(A)$ the closure of the absolutely convex hull of A. A topological vector space E is said to have the *convex compactness property* (*ccp* in what follows) if $\mathrm{acc}_E(A)$ is compact for every compact subset $A \subseteq E$.

If E is a topological vector space, we denote by E^* the dual space of E, i.e., the space of all linear continuous functionals defined on E. We say that E is dually separated if for every $x \neq 0$ in E there exists $x^* \in E^*$ with $x^*(x) \neq 0$. We denote by $\omega(E, E^*)$ the initial vector space topology on E with respect to all linear functionals in E^*.

The symbol E_c^* stands for the space E^* endowed with the topology of uniform convergence on compact subsets of E. For any topological abelian group E the mapping

$$T_E : E_c^* \longrightarrow E_c^\wedge, \quad T_E(f) = \pi \circ f \tag{1}$$

is an isomorphism of topological abelian groups (see [6] or ([7], Proposition 2.3)).

If U is a subset of a topological vector space E, we put $U^\circ := \{f \in E^* : f(U) \subseteq [-1, 1]\}$. We call the set $\bigcap_{f \in U^\circ} f^{-1}([-1, 1])$ the *bipolar* of U and denote it by $U^{\circ\circ}$. The well-known Bipolar Theorem asserts ([8], II.4, Corollary 1) that $U^{\circ\circ} = \mathrm{acc}_E(U)$ for any subset U of a locally convex space E.

Every topological vector space has a topological abelian group structure which arises from its internal operation and its topology. The topological vector spaces whose underlying topological groups are MAP (resp. locally quasi-convex) are exactly the dually separated (resp. locally convex) ones ([7], Proposition 2.4). Other aspects of topological vector spaces considered as abelian groups are studied in the book [9].

2. Generalities on the Quasi-Convex Compactness Property

We start by formulating the natural group counterpart of the convex compactness property:

Definition 1. *Let X be a topological abelian group. We say that X satisfies the quasi-convex compactness property (qcp) if* $\mathrm{qc}_X(K)$ *is a compact subset of X for any compact subset $K \subseteq X$.*

The qcp was defined for the first time in [10]. In the next Proposition we collect some (mostly known) information regarding this property.

Proposition 1.

(a) *Every semi-reflexive locally quasi-convex group has the qcp.*
(b) *Every complete locally quasi-convex group has the qcp.*
(c) *A locally quasi-convex group with the qcp can fail to be semi-reflexive. Actually there exists a complete, metrizable, locally quasi-convex group which is not semi-reflexive.*
(d) *A locally quasi-convex group with the qcp can fail to be complete.*
(e) *A metrizable, locally quasi-convex group with the qcp is necessarily complete.*
(f) *If X is a topological abelian group such that $\alpha_X : X \to (X_c^\wedge)_c^\wedge$ is continuous, then $X_c^\wedge$ has the qcp.*
(g) *If σ and τ are compatible locally quasi-convex group topologies on an abelian group X where $\sigma \leq \tau$, and (X, σ) has the qcp, then (X, τ) has the qcp too.*

Proof. *(a)* and *(b)* are proved in ([11], Proposition 3.1).
(c) Such an example can be found in ([3], Corollary 11.15). Note that this group has the qcp by *(a)*.
(d) Let G be any locally compact, noncompact abelian group. Put $X = (G, \sigma(G, G^\wedge))$. By Glicksberg's Theorem, $X_c^\wedge = G_c^\wedge$. This implies, on the one hand, that $(X_c^\wedge)^\wedge = G$, that is, X is semi-reflexive and by *(a)* has the qcp. On the other hand, X is not complete since otherwise it would be compact and in particular $(X_c^\wedge)_c^\wedge = (G_c^\wedge)_c^\wedge \cong G$ would be compact as well, a contradiction.
(e) *(f)* and *(g)* are Theorem 3.6, Proposition 3.4 and Proposition 3.3 in [11], respectively. A different proof of *(e)* can be found in ([12], Theorem 2).
□

It makes sense to ask whether local quasi-convexity can be relaxed to the MAP property in Proposition 1 *(a)*, *(b)* and *(e)*. The answer is negative in the case of *(b)* (see Example 3 below) and positive in the case of *(e)*; actually within the class of MAP metrizable groups the qcp already implies local quasi-convexity (see Theorem 2).

Lemma 1. *Let X be a metrizable, MAP topological abelian group. The identity mapping $(X_{lqc})_c^\wedge \to X_c^\wedge$ is a topological isomorphism.*

Proof. See ([3], Proposition 6.18). □

The following result is included in the preprint [13] as Theorem I.34. We provide a different, very natural proof.

Theorem 2. *Let X be a metrizable, MAP topological abelian group with the qcp. Then X is locally quasi-convex and complete.*

Proof. Call τ the given metrizable topology on X and τ_{lqc} its locally quasi-convex modification. We are going to show that $\tau = \tau_{lqc}$. Since $\tau_{lqc} \leq \tau$ and τ_{lqc} (being metrizable) is a $k-$space topology, it is enough to show that every $\tau_{lqc}-$compact set is $\tau-$compact. Fix a $\tau_{lqc}-$compact set K. The set $K^\triangleright$ is a neighborhood of zero in $(X_{lqc})_c^\wedge$. By Lemma 1 $K^\triangleright$ is also a neighborhood of zero in $X_c^\wedge$. Hence there exists a $\tau-$compact set C with $C^\triangleright \subseteq K^\triangleright$. This implies $K \subseteq K^{\triangleright\triangleleft} \subseteq C^{\triangleright\triangleleft}$. Since $C^{\triangleright\triangleleft}$ is $\tau-$compact by hypothesis, and K is $\tau_{lqc}-$closed, we deduce that K is actually $\tau-$compact.

This implies that X is locally quasi-convex. The fact that it is also complete follows from Proposition 1*(e)*. $\square$

Problem 1. *Is it possible to extend Theorem 2 to the class $\mathcal{K}$ of MAP topological groups which are k-spaces? Is the dual group of any group X in $\mathcal{K}$ a k-space?*

Note that if X is a metrizable topological vector space, we can even remove the restriction of being MAP from Theorem 2 (see Proposition 7 below). A non-metrizable topological vector space with the qcp does not need to be locally (quasi-)convex; see ([14], Example 2) and Section 4 below.

In order to give a characterization of the qcp in terms of topologies of uniform convergence on the dual group, we need the following result:

Lemma 2. *Let X be an abelian group. Let τ_1 and τ_2 be group topologies on X such that $\tau_1 < \tau_2$ and τ_2 has a basis of neighborhoods of zero formed by τ_1-closed sets. If $L \subset X$ is τ_1-complete, then it is τ_2-complete, too.*

Proof. Assume that L is τ_1-complete. Let $\{x_\alpha\}_\alpha \subseteq L$ be a Cauchy net in τ_2. The inequality $\tau_1 < \tau_2$ implies that $\{x_\alpha\}_\alpha$ is a Cauchy net in τ_1, thus we can find $x \in L$ such that $x_\alpha \xrightarrow{\tau_1} x \in L$.

Let V be a zero neighborhood in τ_2, which is τ_1-closed. Since $\{x_\alpha\}_\alpha$ is a Cauchy net in τ_2 there is an index α_0 such that $x_\alpha - x_\beta \in V$ for all $\alpha, \beta > \alpha_0$. For a fixed α, $x_\alpha - x_\beta \xrightarrow{\tau_1} x_\alpha - x$ and $x_\alpha - x \in V$ since V is τ_1-closed. This is true for all $\alpha > \alpha_0$, thus $x_\alpha \in x + V$ for all $\alpha > \alpha_0$. In other words, $x_\alpha \xrightarrow{\tau_2} x$. $\square$

Recall that for any topological abelian group X we denote by $\mathcal{T}_c$ the topology on $X^\wedge$ of uniform convergence on compact subsets of X. In what follows we also denote by $\mathcal{T}_{\sigma qc}$ the topology on $X^\wedge$ of uniform convergence on $\sigma(X, X^\wedge)-$compact, quasi-convex subsets of X.

Proposition 2. *For a Hausdorff locally quasi-convex topological group (X, τ) the following statements are equivalent:*

(a) *X has the qcp.*
(b) *$\mathcal{T}_c \leq \mathcal{T}_{\sigma qc}$.*

Proof. $(a) \Rightarrow (b)$ A basic $\mathcal{T}_c$-neighborhood of zero has the form $K^\triangleright$ for some compact $K \subset X$. Fix such a subset K. By *(a)* the quasi-convex hull $\mathrm{qc}_X K$ of K is compact in τ, and hence also in the weaker topology $\sigma(X, X^\wedge)$. Now $(\mathrm{qc}_X K)^\triangleright = K^\triangleright$ is a neighborhood of zero in $\mathcal{T}_{\sigma qc}$.

$(b) \Rightarrow (a)$ Let $K \subset X$ be τ-compact. By *b)* we can find a $\sigma(X, X^\wedge)$-compact, quasi-convex $C \subset X$ such that $C^\triangleright \subseteq K^\triangleright$. This implies $\mathrm{qc}_X K \subseteq \mathrm{qc}_X C = C$. Since $\mathrm{qc}_X K$ is a $\sigma(X, X^\wedge)$-closed subset of C, it is $\sigma(X, X^\wedge)$-compact, therefore complete with respect to $\sigma(X, X^\wedge)$. By Lemma 2, $\mathrm{qc}_X K$ is also complete with respect to τ (note that quasi-convex subsets of X are $\sigma(X, X^\wedge)-$closed). On the other hand it is τ- precompact, being the quasi-convex hull of a τ-compact set ([3], Theorem 7.12). Thus $\mathrm{qc}_X K$ is τ-compact and (X, τ) has the qcp. $\square$

3. The qcp on Subgroups

In this section we analyze the hereditary behavior of the quasi-convex compactness property. Clearly, the qcp is not preserved by proper dense subgroups in general. Actually a noncomplete metrizable group cannot have the qcp (Proposition 1*(e)*). This can be generalized to proper dense, determining subgroups of groups that are $k-$spaces (Corollary 2).

The following result gives a quite general condition under which a subgroup inherits the qcp from its ambient group.

Proposition 3. *Let X be a topological abelian group with the qcp. Let H be a subgroup of X which is closed in the $k-$modification of X. Then H also has the qcp.*

Proof. Since H is closed in the k-modification of X, the set $C \cap H$ is compact for any compact subset C in X. By hypothesis for every compact subset K of H the set $\text{qc}_X K$ is compact. Hence $(\text{qc}_X K) \cap H$ is compact as well. Since $\text{qc}_H K$ is closed in H and is clearly a subset of $(\text{qc}_X K) \cap H$, the result follows. □

Corollary 1. *Let X be a topological abelian group with the qcp. Let H be a closed subgroup of X. Then H has the qcp.*

The following result is a partial converse of Proposition 3. Note that the qcp of the group X is not required.

Theorem 3. *Let X be a Hausdorff topological abelian group and let H be a strongly dually embedded subgroup of X. If H has the qcp, then H is closed in the $k-$modification of X.*

Proof. Fix a compact subset C in X. We need to show that $C \cap H$ is compact. Since $C \cap H$ is closed in H, it suffices to find a compact subset of H which contains it. Let us denote by $r : X_c^\wedge \to H_c^\wedge$ the restriction mapping given by $r(\chi) = \chi\restriction_H$; this is an open mapping by hypothesis, so there exists a compact subset K of H such that $K^\blacktriangleright \subseteq r(C^\triangleright)$. (The symbol $\blacktriangleright$ denotes a polar set computed in the dual pair $\langle H, H^\wedge\rangle$.) Since clearly $r(K^\triangleright) = K^\blacktriangleright$, we deduce that $K^\triangleright \subseteq C^\triangleright + H^\perp$, where $H^\perp$ denotes the subgroup of $X^\wedge$ formed by those characters which are identically zero on H. This implies $\text{qc}_X K \supseteq (C^\triangleright + H^\perp)^\triangleleft$ and consequently

$$C \cap H \subseteq (C^\triangleright + H^\perp)^\triangleleft \cap H \subseteq (\text{qc}_X K) \cap H = \text{qc}_H K$$

which is compact by hypothesis. (The identity $(\text{qc}_X K) \cap H = \text{qc}_H K$ follows easily from the fact that H is dually embedded in X.) □

As expressed in Section 1, the notion of strongly dually embedded subgroup directly leads to that of determining subgroup. Thus, Theorem 3 yields the following results:

Corollary 2. *Let X be a Hausdorff topological abelian group which is a $k-$space. If H is a proper dense subgroup of X which determines X, then H fails to have the qcp.*

Note that Corollary 2 implies *(e)* in Proposition 1, since every metrizable group determines its completion. An analogous consequence in a different context follows:

Corollary 3. *If a topological abelian group H is (locally) precompact, has the qcp and determines its completion, then it is actually (locally) compact.*

It is known ([3], Theorem 7.11) that the quasi-convex hull of a finite subset of a MAP group is again finite. This gives the following

Corollary 4. *If H is a precompact non-compact topological group whose compact subsets are finite, then H does not determine its completion.*

We prove below (Theorem 6) that the locally precompact groups X which determine their completions can be characterized by means of the joint continuity of the evaluation mapping $e_X : X_c^\wedge \times X \to \mathbb{T}$, defined by $e_X(\phi, x) = \phi(x)$. Previously we establish a few results related with the weaker condition of continuity of the associated mapping α_X.

The following result has a straightforward proof.

Lemma 3. *Let X be a topological abelian group. If $e_X : X_c^\wedge \times X \to \mathbb{T}$ is continuous then $\alpha_X : X \to (X_c^\wedge)_c^\wedge$ is continuous.*

Lemma 4. *Let H be a dense subgroup of a topological abelian group X, $r : X^\wedge \to H^\wedge$ the restriction mapping, and $L \subset H^\wedge$ equicontinuous with respect to H. Then $r^{-1}(L)$ is equicontinuous with respect to X.*

Proof. Let V be an open zero neighborhood in H with $\phi(V) \subseteq \mathbb{T}_+$ for every $\phi \in L$. Let W be an open zero neighborhood in X such that $V = W \cap H$. We next check that $r^{-1}(L) \subset W^\triangleright$. If $\phi \in L$ and $\widetilde{\phi} = r^{-1}(\phi)$ is its unique extension to a continuous character on X, we have $\widetilde{\phi}(W \cap H) \subset \mathbb{T}_+$. Since $\widetilde{\phi}$ is continuous, also $\widetilde{\phi}(\overline{W \cap H}) \subset \mathbb{T}_+$ where the closure is taken in X. The density of H implies $\overline{W \cap H} = \overline{W}$. Thus $\widetilde{\phi} \in \overline{W}^\triangleright = W^\triangleright$. □

Proposition 4. *Let H be a dense subgroup of a topological abelian group X and let $r : X^\wedge \to H^\wedge$ be the restriction mapping. If α_H is continuous then the inverse image $r^{-1}(K)$ of any compact subset $K \subset H_c^\wedge$ is compact in $X_c^\wedge$.*

Proof. This is Theorem I.19(b) in the preprint [13]. We provide the reader with a proof anyway. Pick $K \subset H_c^\wedge$ compact. Since α_H is continuous, K is equicontinuous with respect to H and by Lemma 4, $r^{-1}(K)$ is equicontinuous with respect to X. On the other hand, $r^{-1}(K)$ is closed in $X_c^\wedge$ by the continuity of r. By Ascoli's Theorem ([15], Theorem 9), $r^{-1}(K)$ is compact in $X_c^\wedge$. □

Proposition 5. *Let H be a dense subgroup of a topological abelian group X. If α_H is continuous (in particular, if H is a k-space) and $H_c^\wedge$ is a $k-$space, then H determines X.*

Proof. The restriction mapping $r : X_c^\wedge \to H_c^\wedge$ is a continuous isomorphism whenever H is a dense subgroup. Thus it is only left to prove that r is open, equivalently closed. Pick a closed $C \subset X_c^\wedge$. We must prove that $r(C) \cap K$ is compact in $H_c^\wedge$ for every compact $K \subset H_c^\wedge$. Since $r^{-1}(r(C) \cap K) = C \cap r^{-1}(K)$ and $r^{-1}(K)$ is compact by Proposition 4, we obtain that $r^{-1}(r(C) \cap K)$ is compact. Now r continuous implies that $r(C) \cap K$ is compact. □

The continuity of α_H in Proposition 5 cannot be removed as the following example shows.

Example 1. *Let L be a locally compact, non-compact abelian group and let $H := (L, \sigma(L, L^\wedge))$. By Glicksberg's theorem $H_c^\wedge = L_c^\wedge$, therefore $H_c^\wedge$ is even locally compact. However H does not determine its completion X: clearly, $X_c^\wedge$ is discrete whereas $H_c^\wedge$ is non-discrete. Observe further that H has the qcp (see the proof of Proposition 1(d)).*

Under the more restrictive assumption that $e_H : H_c^\wedge \times H \to \mathbb{T}$ is continuous, it is easily obtained that $H_c^\wedge$ is locally compact, ([16], Proposition 1.2). We claim the following:

Theorem 4. *Let H be a locally quasi-convex, Hausdorff group. The following conditions are equivalent:*

(i) $e_H : H_c^\wedge \times H \to \mathbb{T}$ *is continuous.*
(ii) *H is locally precompact and determines its completion.*

Proof. $(ii) \Rightarrow (i)$: Let X be the completion of H and $\imath : H \to X$ be the inclusion mapping. By hypothesis X is locally compact and the restriction mapping $r : X_c^\wedge \to H_c^\wedge$ is a topological isomorphism. The diagram

$$\begin{array}{ccc} H_c^\wedge \times H & & \\ {\scriptstyle r^{-1}\times\imath}\big\downarrow & \searrow^{e_H} & \\ X_c^\wedge \times X & \xrightarrow[e_X]{} & \mathbb{T} \end{array}$$

is commutative and e_X is clearly continuous. The assertion follows.

$(i) \Rightarrow (ii)$: By Prop. 1.2 in [16], $H_c^\wedge$ is locally compact, and in particular a $k-$space. The homomorphism $\alpha_H : H \to (H_c^\wedge)_c^\wedge$ is an embedding: take into account that H is Hausdorff, locally quasi-convex and apply Lemma 3 and ([7], Lemma 14.3). Since $(H_c^\wedge)_c^\wedge$ is locally compact, we deduce that H is locally precompact.

From Proposition 5 we obtain that H determines X. □

Corollary 5. *Let X be a locally compact abelian group and let H be a dense subgroup of X. Then H determines X if and only if $e_H : H_c^\wedge \times H \to \mathbb{T}$ is continuous.*

Remark 1. *In the class of reflexive groups, continuity of e_X implies local compactness of X, as proved in [17]. On the other hand, a reflexive noncomplete group does not determine its completion ([18], Theorem 5.2). For a general topological group X, continuity of e_X is equivalent to continuity of α_X plus local compactness of $X_c^\wedge$ [19].*

Example 1 shows that a topological group which is a k-space may have dense subgroups which are not k-spaces. In particular, H does not determine X in the mentioned example. The following are natural questions:

Problem 2.

(i) *If a topological group X contains a dense subgroup which is a k-space and determines X, must X be a k-space?*
(ii) *If a topological group X contains a dense subgroup H which is a k-space, does H determine X?*

Corollary 6. *Let H be a dense subgroup of X. If H is almost metrizable, then H determines X.*

Proof. If H is almost metrizable, then H is a k-space ([3], Proposition 1.24) therefore α_H is continuous and $H_c^\wedge$ is a k-space (Proposition 5.20 in the same reference). Thus H satisfies the hypothesis of Proposition 5. □

Remark 2. *If H is a dense subgroup of X and H is almost metrizable, then X is almost metrizable too. In fact, if K is a compact subgroup of H such that H/K is metrizable, clearly K is also a compact subgroup of X. On the other hand H/K is dense in X/K, therefore H/K metrizable implies X/K metrizable. Thus, X is almost metrizable.*

In the preceding Remark "almost metrizable" cannot be replaced by "$k-$space", as the following example shows:

Example 2. *Let $X := \mathbb{R}^\beta$, where β is any uncountable ordinal, and let H be the corresponding Σ-product (i.e., the subgroup formed by those $x \in \mathbb{R}^\beta$ with countable support). If X is endowed with its usual product topology, H is Fréchet-Urysohn ([20], Theorem 2.1), therefore it is a k-space ([21], 1.6.14, 3.3.20). Clearly H is dense in X. However X is not a k-space. ([22], Chapter 7, Ex. J(b)).*

In ([23], Theorem 4.8) it is proved that any compact abelian group X contains an almost metrizable proper dense subgroup which determines X. Our Corollary 6 shows that the fact that H determines X is a consequence of the remaining hypothesis. The following question arises naturally:

Problem 3. *Does every almost metrizable (resp. k-space) X contain an almost metrizable (resp. k-space) proper dense subgroup which determines X?*

4. The qcp in Topological Vector Spaces

In this section we study the relationship between the ccp and the qcp on a topological vector space.

The next result is a slight improvement of Proposition 4.5 in [11] (see also ([7], Proposition 15.1)). In its proof we will need the following fact: If E is any dually separated topological vector space, $\omega(E,E^*)-$compact subsets and $\sigma(E,E^\wedge)-$compact subsets coincide. This result is proved in ([24], Lemma 1.2) in the locally convex case but it can be easily generalized since it only involves weak topologies.

Proposition 6. *Let E be a topological vector space. Consider the following properties:*

(a) *For every compact subset K of E, the set* $\mathrm{qc}_E(K)$ *is compact (i.e., E has the qcp).*
(a′) *For every compact subset K of E, the set* $\mathrm{qc}_E(K)$ *is* $\sigma(E,E^\wedge)-$*compact.*
(b) *For every compact subset K of E, the set* $K^{\circ\circ}$ *is compact.*
(b′) *For every compact subset K of E, the set* $K^{\circ\circ}$ *is* $\omega(E,E^*)-$*compact.*
(c) *For every compact subset K of E, the set* $\mathrm{acc}_E(K)$ *is compact (i.e., E has the ccp).*
(d) *The natural mapping* $\alpha_E : E \to (E_c^\wedge)^\wedge$ *defined by* $\alpha_E(x)(\chi) = \chi(x)$ *is onto (i.e., E is semi-reflexive as a topological abelian group).*
(e) *The natural mapping* $\gamma_E : E \to (E_c^*)^*$ *defined by* $\gamma_E(x)(f) = f(x)$ *is onto.*

Then the following implications hold:

$$\begin{array}{ccccccc} (a) & \Longleftrightarrow & (b) & \Longrightarrow & (c) \qquad & (d) & \Longleftrightarrow\ (e) \\ \Downarrow & & \Downarrow & & & & \\ (a') & \Longleftrightarrow & (b') & & & & \end{array}$$

If E is locally convex then all these properties are equivalent.

Proof. $(a) \Leftrightarrow (b)$: Note that if $B \subseteq E$ is balanced and nonempty then $\mathrm{qc}_E B = B^{\circ\circ}$ ([25], Prop. 1.11(c)). Since $[-1,1]K$ is compact for every compact $K \subseteq E$, and quasi-convex hulls are closed sets, it is clear that *(a)* holds if and only if $\mathrm{qc}_E([-1,1]K)$ is compact for every compact set $K \subseteq E$. Now $\mathrm{qc}_E([-1,1]K) = ([-1,1]K)^{\circ\circ} = K^{\circ\circ}$ and the equivalence is proved.

$(a') \Leftrightarrow (b')$: Again, since $[-1,1]K$ is compact for every compact $K \subseteq E$, and quasi-convex hulls are $\sigma(E,E^\wedge)-$closed sets, it is clear that *(a′)* holds if and only if $\mathrm{qc}_E([-1,1]K) = K^{\circ\circ}$ is $\sigma(E,E^\wedge)-$compact for every compact set $K \subseteq E$. It only remains to apply that $\omega(E,E^*)-$compact subsets and $\sigma(E,E^\wedge)-$compact subsets coincide.

$(a) \Rightarrow (a')$ and $(b) \Rightarrow (b')$ are trivial.

$(b) \Rightarrow (c)$: Let K be a compact subset of E. The set $\mathrm{acc}_E K$ is closed and a subset of $K^{\circ\circ}$, which is compact by hypothesis. Hence it is compact, too.

$(d) \Leftrightarrow (e)$: As we have mentioned (1), the mapping

$$T_E : E_c^* \longrightarrow E_c^\wedge, \quad T_E(f) = \pi \circ f$$

is an isomorphism of topological abelian groups. Hence its adjoint mapping

$$T_E^\wedge : (E_c^\wedge)^\wedge \longrightarrow (E_c^*)^\wedge, \quad T_E^\wedge(\kappa) = \kappa \circ T_E$$

is an isomorphism of abelian groups.

Analogously, the mapping

$$T_{E^*} : (E_c^*)^* \longrightarrow (E_c^*)^\wedge, \quad T_{E^*}(\lambda) = \pi \circ \lambda$$

is an isomorphism of abelian groups.

It is easy to check that $T_{E^*}^{-1} \circ T_E^\wedge \circ \alpha_E = \gamma_E$. This shows that *(d)* and *(e)* are equivalent.

$(c) \Rightarrow (b)$ if E is locally convex: This is an immediate consequence of the Bipolar Theorem.

$(c) \Leftrightarrow (e)$ if E is locally convex: This is known ([26], Theorem 9.2.12).

$(a') \Rightarrow (a)$ if E is locally convex: Fix a compact subset $K \subseteq E$. By hypothesis the set $\mathrm{qc}_E(K)$ is $\sigma(E, E^\wedge)-$compact. Since E is a locally quasi-convex group, $\mathrm{qc}_E(K)$ is both complete (Lemma 2) and precompact ([3], 7.12), hence compact. □

The equivalence between qcp and ccp holds for metrizable spaces, and actually these properties characterize Fréchet spaces within this class:

Proposition 7. *Let E be a metrizable topological vector space. The following properties are equivalent:*

(a) *E has the qcp.*
(b) *E has the ccp.*
(c) *E is locally convex and complete.*

Proof. $(a) \Rightarrow (b)$ follows from Proposition 6.

$(b) \Rightarrow (c)$: If E has the ccp then it is locally convex by ([27], 1.642). Hence it is also complete ([2], Theorem 2.3).

$(c) \Rightarrow (a)$: This is true for locally quasi-convex complete groups ([11], Proposition 3.1). □

Local convexity in (c) plays an essential role. Below we present an example of a metrizable complete topological vector space which does not have the qcp.

Example 3. *Consider the space ℓ_p (with $0 < p < 1$) endowed with the p-norm $\|x\|_p = \sum_{k=1}^{\infty} |x_k|^p$. It is known that ℓ_p is a non locally convex, complete metric linear space. Its dual space is ℓ_∞ (in the usual sense for sequence spaces), and in particular ℓ_p is a MAP group. (Details can be found for instance in Chapter 2.3 of [28].) The fact that ℓ_p does not have the ccp follows from $(b) \Rightarrow (c)$ in Proposition 7. Let us give a concrete example of a compact subset of this space whose absolutely convex closure is not compact. Define the sequence $\{x_n\} \in \ell_p$ by*

$$x_n(n) = n^{p-1} \text{ and } x_n(m) = 0 \text{ if } n \neq m.$$

The sequence $\{x_n\}$ converges to 0 in the space ℓ_p, since

$$\|x_n\|_p = x_n(n)^p = n^{(p-1)p} \to 0.$$

However, the convex hull of the compact subset $K := \{0\} \cup \{x_n : n \in \mathbb{N}\}$ is unbounded with respect to the p-norm $\| \cdot \|_p$. Indeed, define

$$y_N = \frac{x_1 + \cdots + x_N}{N} = (\frac{1}{N}, \frac{2^{p-1}}{N}, \ldots, \frac{N^{p-1}}{N}, 0, 0, \ldots)$$

for each $N \in \mathbb{N}$. This sequence is clearly contained in the convex hull of K. We have

$$\|y_N\|_p = \sum_{k=1}^{N} \frac{k^{(p-1)p}}{N^p} \geq \sum_{k=1}^{N} \frac{N^{(p-1)p}}{N^p} = N^{(p-1)^2}$$

which goes to infinity as $N \to \infty$. Thus $\{y_N\}$ is unbounded in ℓ_p and consequently, the closed convex hull of K is not compact.

Let us now analyze the presence of the qcp in weak vector space topologies.

Proposition 8. *Let E be a dually separated topological vector space. The following properties are equivalent:*

(a) *The group $(E, \sigma(E, E^\wedge))$ has the qcp.*
(b) *The space $(E, \omega(E, E^*))$ has the ccp.*
(c) *$(E, \sigma(E, E^\wedge))$ is a semi-reflexive group.*

Proof. It is known that for any dually separated topological vector space E, the dual space of $(E, \omega(E, E^*))$ is E^* (see for instance ([29], Chapter IV, 1.2)). Moreover, as we have mentioned above, given any topological vector space F the natural group homomorphism $F^* \to F^\wedge$ given by $f \mapsto \pi \circ f$ is actually an isomorphism. These facts clearly imply that

$$(E, \omega(E, E^*))^\wedge = E^\wedge = (E, \sigma(E, E^\wedge))^\wedge \tag{2}$$

From (2) and the above mentioned fact that $\omega(E, E^*)-$compact subsets and $\sigma(E, E^\wedge)-$compact subsets coincide, we deduce on the one hand that *(a)* is equivalent to

(a') $(E, \omega(E, E^*))$ has the qcp and on the other hand that

$$(E, \omega(E, E^*))^\wedge_c = (E, \sigma(E, E^\wedge))^\wedge_c \tag{3}$$

By $(a) \Leftrightarrow (c)$ in Proposition 6 applied to the locally convex space $(E, \omega(E, E^*))$, *(b)* is equivalent to *(a')*. Since *(a')* is equivalent to *(a)*, we have proved $(a) \Leftrightarrow (b)$.

By $(c) \Leftrightarrow (d)$ in Proposition 6 applied to the locally convex space $(E, \omega(E, E^*))$, *(b)* is equivalent to

(b') $(E, \omega(E, E^*))$ is semi-reflexive as a topological abelian group.

From (3) we deduce that *(b')* is equivalent to *(c)*. This shows $(b) \Leftrightarrow (c)$. □

5. The Krein Property for Topological Abelian Groups

In the sequel we will call Krein's Theorem the following statement which is an immediate consequence of Theorem 1:

Theorem 5. *If E is a complete locally convex space, then the space $(E, \omega(E, E^*))$ has the ccp.*

We will see below that Krein's Theorem cannot be totally extended to the class of locally quasi-convex groups, but some approach is possible and we first study positive results in this line. For convenience we introduce the *Krein property*:

Definition 2. *Let X be a MAP topological abelian group. We say that X has the Krein property if $(X, \sigma(X, X^\wedge))$ has the qcp.*

By Proposition 1*(g)*, any locally quasi-convex group with the Krein property has the qcp.

Denote by $\mathcal{T}_{\sigma c}$ the topology on $X^\wedge$ of uniform convergence on $\sigma(X, X^\wedge)-$compact subsets of a topological abelian group X. Proposition 2 yields the following:

Proposition 9. *Let X be a MAP topological abelian group. The following conditions are equivalent:*

(a) *X has the Krein property.*

(b) *The topologies $\mathcal{T}_{\sigma c}$ and $\mathcal{T}_{\sigma qc}$ coincide on $X^\wedge$.*

We denote by bX the completion of the group $(X, \sigma(X, X^\wedge))$, where X is a MAP topological abelian group. The compact group bX can be realized as $\mathrm{Hom}(X^\wedge, \mathbb{T})$ with the topology it carries as a subgroup of $\mathbb{T}^{X^\wedge}$. The following result is an immediate consequence of Corollary 3:

Proposition 10. *Let X be a MAP topological abelian group. If X has the Krein property then either $(X, \sigma(X, X^\wedge)) = bX$ or $(X, \sigma(X, X^\wedge))$ does not determine bX.*

The following result follows at once from Proposition 8. It shows that the Krein property, as we have just defined it, generalizes its natural vector-space counterpart in a satisfactory way.

Proposition 11. *Let (X, τ) be a dually separated topological vector space. Then its underlying group has the Krein property if and only if $(X, \omega(X, X^*))$ has the ccp.*

Hence Krein's (Theorem 5 above) can be restated as the fact that all complete locally convex spaces have the Krein property as groups. In order to show that Krein's Theorem does **not** remain true for complete locally quasi-convex groups, we present a family of counterexamples considered in [30] with a different purpose.

We follow the notation of the mentioned paper, and outline the parts that allow us to reach our conclusion.

Notation 1. For a Hausdorff topological abelian group X, let u and p be, respectively, the uniform and the product topology on $X^{\mathbb{N}}$. A basis of zero neighborhoods for u is given by the family $\{U^{\mathbb{N}}, U \in \mathcal{N}(X)\}$ where $\mathcal{N}(X)$ stands for a zero-neighborhood basis at X. Denote by $c_0(X)$ the subgroup of $(X^{\mathbb{N}}, u)$ formed by the null sequences of X, by u_0 the topology induced by u in $c_0(X)$ and by p_0 the topology induced by p in $c_0(X)$.

Theorem 6. *Let X be an infinite compact, connected metrizable topological abelian group, and let $G := (c_0(X), u_0)$. Then G is a complete metrizable group. However, G does not have the Krein property.*

Proof. Straightforward calculations show that $c_0(X)$ is closed in $(X^{\mathbb{N}}, u)$. Therefore G is complete and metrizable.

The important fact is that its dual group $G^\wedge$ is countable, and this is obtained in [30], after several steps that include the definition of a subclass of the locally generated abelian groups. A steady reasoning leads to the fact that $(c_0(X), u_0)^\wedge = (c_0(X), p_0)^\wedge$, whenever X is a nontrivial compact connected metrizable group ([30], Theorem 7.3).

Now it is easy to prove that $(c_0(X), p_0)^\wedge$ is countable. Taking into account that $c_0(X)$ is dense in $(X^{\mathbb{N}}, p)$, $(c_0(X), p_0)^\wedge$ can be identified with the dual group of $(X^{\mathbb{N}}, p)$ which is the direct sum of countably many copies of $X^\wedge$, say $(X^\wedge)^{(\mathbb{N})}$. Since X is a compact metrizable group, $X^\wedge$ is countable. Thus, $G^\wedge = (X^\wedge)^{(\mathbb{N})}$ is also countable.

The topology $\sigma(G, G^\wedge)$ coincides with p_0, so we have that $(G, \sigma(G, G^\wedge))$ is metrizable. The fact that G does not have the Krein property follows by contradiction: had $(G, \sigma(G, G^\wedge))$ the qcp, by Proposition 1*(e)*, it would be complete. But this is not the case since $(G, \sigma(G, G^\wedge)) = (c_0(X), p_0)$ and $c_0(X)$ is a proper dense subgroup of the complete group $(X^{\mathbb{N}}, p)$. □

6. The Krein and the Glicksberg Properties in the Context of Duality

There is some interaction between these properties as we present below.

Proposition 12. *Let X be a locally quasi-convex topological group with the Krein property. The following statements are equivalent:*

(a) *X has the Glicksberg property.*
(b) *$\mathcal{T}_c = \mathcal{T}_{\sigma c} = \mathcal{T}_{\sigma qc}$.*

Proof. $(a) \Rightarrow (b)$ derives from the equality $\mathcal{T}_c = \mathcal{T}_{\sigma c}$ and from Proposition 9.

$(b) \Rightarrow (a)$: Let $K \subset X$ be $\sigma(X, X^\wedge)$-compact. Thus $K^\triangleright$ is a $\mathcal{T}_{\sigma c}$-neighborhood of zero, and since $\mathcal{T}_c = \mathcal{T}_{\sigma c}$ we can find a compact subset C of (X, τ) such that $C^\triangleright \subset K^\triangleright$. This implies $K \subset \mathrm{qc}_X C$. Since, by Proposition 1 (g), (X, τ) also has the qcp, we obtain that $\mathrm{qc}_X C$ is compact in τ. Consequently, K is τ-compact. □

We recall that a topological abelian group (X, τ) is *g-barrelled* if every $\sigma((X, \tau)^\wedge, X)-$compact subset of $X^\wedge$ is $\tau-$equicontinuous. For reflexive groups, the "Glicksberg property" and "being g-barrelled" are dual to each other as shown below (Corollary 7).

Proposition 13. *Let (X, τ) be a Hausdorff locally quasi-convex group. Consider the assertions:*

(a) *$X_c^\wedge$ is g-barrelled.*

(b) X has the Glicksberg property.

Then $(a) \Rightarrow (b)$. If (X,τ) is further semi-reflexive, then $(b) \Rightarrow (a)$.

Proof. $(a) \Rightarrow (b)$ Let $S \subset X$ be $\sigma(X, X^{\wedge})$-compact. Through the natural embedding

$$\beta : (X, \sigma(X, X^{\wedge})) \hookrightarrow ((X_c^{\wedge})^{\wedge}, \sigma((X_c^{\wedge})^{\wedge}, X^{\wedge}))$$

we obtain that $\beta(S)$ is a $\sigma((X_c^{\wedge})^{\wedge}, X^{\wedge})$-compact subset of $(X_c^{\wedge})^{\wedge}$. By (a) there is a zero-neighborhood V in $X_c^{\wedge}$ such that $\beta(S) \subset V^{\triangleright}$. Since $V^{\triangleright}$ is a compact subset of $(X_c^{\wedge})_c^{\wedge}$ and $\beta(S)$ is closed, we obtain that $\beta(S)$ is also compact in $(X_c^{\wedge})_c^{\wedge}$. From the assumption that X is locally quasi-convex, we have that $\alpha : (X,\tau) \to (X_c^{\wedge})_c^{\wedge}$ is relatively open, thus $\alpha^{-1}(\beta(S)) = S$ is compact in τ, which ends the proof.

Assume now that (X,τ) is semi-reflexive. In order to prove $(b) \Rightarrow (a)$ observe that $\beta : (X, \sigma(X, X^{\wedge})) \to ((X_c^{\wedge})^{\wedge}, \sigma((X_c^{\wedge})^{\wedge}, X^{\wedge}))$ is a topological isomorphism. Thus, if $K \subset (X_c^{\wedge})^{\wedge}$ is $\sigma((X_c^{\wedge})^{\wedge}, X^{\wedge})$-compact, $\beta^{-1}(K)$ is a $\sigma(X, X^{\wedge})$-compact subset of X. By (b) it is also τ-compact, therefore $(\beta^{-1}(K))^{\triangleright} = K^{\triangleleft}$ is a neighborhood of zero in $X_c^{\wedge}$ such that $K \subset (\beta^{-1}(K))^{\triangleright\triangleright}$. Thus, K is equicontinuous. Consequently, $X_c^{\wedge}$ is g-barrelled. □

Corollary 7. *Let (X,τ) be a reflexive group. The following two properties are equivalent:*

(a) $X_c^{\wedge}$ is g-barrelled.

(b) X has the Glicksberg property.

Remark 3. *$(a) \Rightarrow (b)$ of Corollary 7 is a generalization of ([25], Proposition 1.7). In [31] it was wrongly stated that every reflexive group satisfies (b). A counterexample can be seen in [24]. Thus, we conclude that the dual of a reflexive group is not necessarily g-barreled.*

Corollary 7 can also be obtained from Proposition 5.3 of [32], where several notions of barreledness for groups are considered.

Melting Proposition 12 and Corollary 7, we obtain:

Corollary 8. *Let X be a reflexive group with the Krein property. The following statements are equivalent:*

(i) $X_c^{\wedge}$ is g-barrelled.
(ii) X has the Glicksberg property.
(iii) The topologies $\mathcal{T}_c$ and $\mathcal{T}_{\sigma qc}$ coincide on $X^{\wedge}$.

Example 4.

(i) Banach spaces provide examples of reflexive topological groups with the Krein property. Just take into account that a Banach space is a reflexive topological group ([6]), and Theorem 5 and Proposition 11 of the present paper.
(ii) A reflexive group (G,τ) with the Krein property, such that $G_c^{\wedge}$ is not g-barrelled: Let G be an infinite dimensional, reflexive Banach space (in the ordinary sense of reflexivity for Banach spaces). It does not have the Glicksberg property: in fact, the unit ball B is $\omega(G, G^*)$-compact and by [24] also $\sigma(G, G^{\wedge})$-compact. Clearly B is not compact in the norm topology of G. Thus, Corollary 8 applies to obtain that $G_c^{\wedge}$ is not g-barrelled.
(iii) A non reflexive group with Krein and Glicksberg properties such that $G_c^{\wedge}$ is g-barrelled: Let $G := (E, \omega(E, E^*))$ where E is an infinite dimensional Banach space and $\omega(E, E^*)$ is its weak topology. The group G is locally quasi-convex nonreflexive (α_G is not continuous) and by (i) it has the Krein property. Since the $\omega(E, E^*)$-compact subsets of E coincide with the $\sigma(E, E^{\wedge})$-compact subsets ([24], Lemma 1.2), G has the Glicksberg property. By Proposition 12, the compact-open topology on $G^{\wedge}$ coincides with $\mathcal{T}_{\sigma qc}$.
By Proposition 8, G is semi-reflexive and Proposition 13 proves that $G_c^{\wedge}$ is g-barrelled. Observe also that G itself is not g-barrelled.

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References

1. Grothendieck, A. *Topological Vector Spaces*; Gordon and Breach: New York, NY, USA; London, UK; Paris, France, 1973.
2. Ostling, E.G.; Wilansky, A. Locally convex topologies and the convex compactness property. *Proc. Camb. Philos. Soc.* **1974**, *75*, 45–50. [CrossRef]
3. Aussenhofer, L. *Contributions to the Duality Theory of Abelian Topological Groups and to the Theory of Nuclear Groups*; Dissertationes Mathematicae; PWN: Warszawa, Poland, 1999; Volume 384.
4. Banaszczyk, W.; Chasco, M.J.; Martín Peinador, E. Open subgroups and Pontryagin duality. *Math. Z.* **1994**, *215*, 195–204. [CrossRef]
5. Chasco, M.J. Pontryagin duality for metrizable groups. *Arch. Math.* **1998**, *70*, 22–28. [CrossRef]
6. Smith, M.F. The Pontrjagin Duality Theorem in Linear Spaces. *Ann. Math.* **1952**, *56*, 248–253. [CrossRef]
7. Banaszczyk, W. Additive subgroups of topological vector spaces. In *Lecture Notes in Mathematics*; Springer: Berlin/Heidelberg, Germany, 1991; Volume 1466.
8. Robertson, A.P.; Robertson, W. *Topological Vector Spaces*, 2nd ed.; Cambridge University Press: London, UK; New York, NY, USA, 1973.
9. Hofmann, K.H.; Morris, S.A. *The Structure of Compact Groups*, 4th ed.; Studies in Mathematics; De Gruyter: Berlin, Germany; Boston, MA, USA, 2020; Volume 25.
10. Bruguera, M. Grupos Topológicos y Grupos de Convergencia: Estudio de la Dualidad de Pontryagin. Ph.D. Thesis, University of Barcelona, Barcelona, Spain, 1999.
11. Bruguera, M.; Martín-Peinador, E. Banach-Dieudonné theorem revisited. *J. Aust. Math. Soc.* **2003**, *75*, 1–15. [CrossRef]
12. Hernández, S. Pontryagin duality for topological abelian groups. *Math. Z.* **2001**, *238* 493–503. [CrossRef]
13. Lukács, G. Notes on duality theories of abelian groups. *arXiv* **2006**, arXiv:math/0605149.
14. Araki, T. A characterization of non-local convexity in some class of topological vector spaces. *Math. Japon.* **1995**, *41*, 573–577.
15. Morris, S.A. *Pontryagin Duality and the Structure of Locally Compact Abelian Groups*; London Mathematical Society Lecture Notes Series; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA; Melbourne, Australia, 1977; Volume 29.
16. Martín-Peinador, E.; Tarieladze, V. A property of Dunford-Pettis type in topological groups. *Proc. Amer. Math. Soc.* **2003**, *132*, 1827–1837. [CrossRef]
17. Martín-Peinador, E. A reflexive admissible topological group must be locally compact. *Proc. Am. Math. Soc.* **1995**, *123*, 3563–3566. [CrossRef]
18. Comfort, W.W.; Raczkowki, S.U.; Trigos-Arrieta, F.J. The dual group of a dense subgroup. *Czechoslovak Math. J.* **2004**, *54*, 509–533. [CrossRef]
19. Turnwald, G. On the continuity of the evaluation mapping associated with a group and its character group. *Proc. Amer. Math. Soc.* **1998**, *126*, 3413–3415. [CrossRef]
20. Noble, N. The continuity of functions on Cartesian products. *Trans. Amer. Math. Soc.* **1970**, *149*, 187–198. [CrossRef]
21. Engelking, R. *General Topology*; Sigma Series in Pure Mathematics; Heldermann Verlag: Berlin, Germany, 1989; Volume 6.
22. Kelley, J.L. *General Topology*; University Series in Higher Mathematics; D. Van Nostrand: Toronto, ON, Canada; New York, NY, USA; London, UK, 1955.
23. Bruguera, M.; Tkachenko, M. Pontryagin duality in the class of precompact Abelian groups and the Baire property. *J. Pure Appl. Algebra* **2012**, *216*, 2636–2647. [CrossRef]
24. Remus, D.; Trigos-Arrieta, F.J. Abelian groups which satisfy Pontryagin duality need not respect compactness. *Proc. Amer. Math. Soc.* **1993**, *117*, 1195–1200. [CrossRef]
25. Chasco, M.J.; Martín-Peinador, E.; Tarieladze, V. On Mackey topology for groups. *Studia Math.* **1999**, *132*, 257–284. [CrossRef]
26. Wilansky, A. *Modern methods in Topological Vector Spaces*; McGraw-Hill International Book Co.: New York, NY, USA, 1978.
27. Mazur, S.; Orlicz, W. Sur les espaces métriques linéaires (I). *Stud. Math.* **1948**, *10*, 184–208. [CrossRef]
28. Kalton, N.J.; Peck, N.T.; Roberts, J.W. *An F-Space Sampler*; London Mathematical Society Lecture Note Series; Cambridge University Press: Cambridge, UK; New York, NY, USA; Melbourne, Australia, 1984; Volume 89.

29. Schaefer, H.H. *Topological Vector Spaces*; Graduate Texts in Mathematics; Springer: New York, NY, USA; Heidelberg/Berlin, Germany, 1971; Volume 3.
30. Dikranjan, D.; Martín-Peinador, E.; Tarieladze, V. Group valued null sequences and metrizable non-Mackey groups. *Forum Math.* **2014**, *26*, 723–757. [CrossRef]
31. Venkataraman, R. Compactness in abelian topological groups. *Pac. J. Math.* **1975**, *57*, 591–595. [CrossRef]
32. Gabriyelyan, S. Maximally almost periodic groups and respecting properties. In *Descriptive Topology and Functional Analysis II*; Ferrando, J.C., Ed.; Springer Proceedings in Mathematics & Statistics; Springer: Cham, Switzerland, 2019; Volume 286, pp. 103–106.

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Article

On the Group of Absolutely Summable Sequences †

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Abstract: For an abelian topological group G, the sequence group $\ell^1(G)$ of all absolutely summable sequences in G is studied. It is shown that $\ell^1(G)$ is a Pontryagin reflexive group in case G is a reflexive metrizable group or an LCA group. Further, $\ell^1(G)$ has the Schur property if and only if G has it and $\ell^1(G)$ is a Schwartz group if and only if G is linearly topologized.

Keywords: summable sequence; absolutely summable sequence; locally quasi-convex group; Schwartz group; nuclear group; Pontryagin reflexive group; Schur property; LCA groups

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1. Preliminaries

1.1. Introduction

It is a well-known result in the theory of locally convex vector spaces that for a metrizable locally convex space (E, τ), the underlying topology τ is the finest locally convex topology giving rise to the dual space $(E, \tau)'$ in all continuous linear forms ([1], p. 263). The idea of a finest compatible topology was generalized in [2] to locally quasi-convex groups. More precisely, for a locally quasi-convex group (G, τ), the topology τ is called the Mackey topology (see [2] for details) if it is the finest among all locally quasi-convex group topologies giving rise to the character group $(G, \tau)^\wedge$. For several years, it was an open question as to whether every metrizable locally quasi-convex group topology is a Mackey topology. The first example giving a negative answer to this question was the group of all null-sequences in the torus $c_0(\mathbb{T}) = \{(z_n) \in \mathbb{T}^{\mathbb{N}} : z_n \to 0\}$ endowed with the topology of uniform convergence. The important observation was that the dual group of $c_0(\mathbb{T})$ is isomorphic to $\mathbb{Z}^{(\mathbb{N})}$; in particular, it is countable. This implies that the weak topology $\sigma(c_0(\mathbb{T}), c_0(\mathbb{T})^\wedge)$ is metrizable and precompact. Because this topology is strictly weaker than the topology of uniform convergence on $c_0(\mathbb{T})$, the metrizable weak topology cannot be the Mackey topology. In [3], this was generalized to $c_0(G)$ where G is a compact connected abelian metrizable group. The main idea was to show that the character group of such a group has a countable dual group. In [4] (Theorem 3.4), an alternative proof for this was given, the structure of the character group of $c_0(G)$ was described, and many properties of these groups have been studied since then (cf. [4–7]).

In [7] (Theorem 1.3), Gabriyelyan proves that for an LCA group G, the following assertions are equivalent: G is totally disconnected iff $c_0(G)$ is a nuclear group iff $c_0(G)$ is a Schwartz group iff $c_0(G)$ respects compactness. Further, in [4] (Theorem 1.2), he generalized the results from [5] and shows that $c_0(G)$ is a reflexive group.

In [5], groups of the form $\ell^p(\mathbb{T}) = \{(z_n) \in \mathbb{T}^{\mathbb{N}} : \sum_{n=1}^{\infty} |1 - z_n|^p < \infty\}$ were investigated and it was shown that for $0 < p < \infty$, $\ell^p(\mathbb{T})$ is a monothetic Polish group which is topologically isomorphic to $\ell^p/\mathbb{Z}^{(\mathbb{N})}$ ([5] Proposition 5/Theorem 1) and $\ell^1(\mathbb{T})$ is reflexive.

Because in the theory of Banach spaces, the sequence space c_0 of (real or complex) null-sequences, the space ℓ^1 of all absolutely summable sequences, and the space ℓ^∞ of bounded sequences play an important role, it is natural to generalize them to the corresponding sequence groups for abelian Hausdorff groups G. This was performed in the case $c_0(G)$ by Gabriyelyan and will now be carried out for the groups $\ell^1(G)$ of absolutely summable sequences (Definition 3).

Alternatively, unconditionally Cauchy sequences and absolutely summable sequences (suitably defined) were studied in the realm of topological vector spaces in order to characterize nuclear vector spaces (cf. ([8], 21.2.1) and ([9], p.73)). This idea was picked up by Domínguez Pérez and Tarieladze in [10,11] in order to characterize nuclear groups (see below).

Our main interest is to find sufficiency conditions for a group G such that $\ell^1(G)$ is reflexive. We prove that a metrizable group G is reflexive if and only if the sequence group $\ell^1(G)$ is reflexive (Corollary 6). Moreover, for every LCA group G, the group $\ell^1(G)$ is reflexive (Theorem 4).

A normed vector space has the Schur property if every sequence which converges in the weak vector space topology is also convergent with respect to the norm. As the vector space ℓ^1 has the Schur property ([12], 27.13), it is natural to ask whether $\ell^1(G)$ also has a similar property. It turns out that for a locally quasi-convex group G, $\ell^1(G)$ has the (analogue of the) Schur property for groups if and only if G has this property (Theorem 6).

In [13], Banaszczyk introduced nuclear groups, a Hausdorff variety of groups which contains all locally convex nuclear vector spaces and all LCA groups. In [14], Schwartz groups were defined, examples were given, and first properties were shown. Because no infinite-dimensional normed space is neither a Schwartz space nor a nuclear vector space, it is not surprising that the hypotheses on a group G such that $\ell^1(G)$ is a Schwartz group or a nuclear group must be rather restrictive. Indeed, we show that for a locally quasi-convex group G, the group $\ell^1(G)$ is a Schwartz group iff $\ell^1(G)$ is a nuclear group iff G is linearly topologized (Theorem 8). This is an analogue of Gabriyelyan's result for $c_0(G)$ as every totally disconnected LCA group is linearly topologized.

The paper is organized as follows:

In Section 1.2, we gather material concerning reflexive groups, and in Section 1.3, we study properties of the Minkowski functional for groups. Section 2 is dedicated to the study of the sequence group $\ell^1(G)$, the focus of the paper. We start in Section 2.1 with the definition and basic properties of the topological group $\ell^1(G)$. We show that, on the one hand, G can be embedded in $\ell^1(G)$ and, on the other, G is a quotient group of $\ell^1(G)$ (Lemma 1). Thus, it is not surprising that G and $\ell^1(G)$ have many properties in common in the sense that G satisfies property P iff $\ell^1(G)$ satisfies P. For example, this holds for cardinal invariants, separation axioms, completeness, and local quasi-convexity. The mapping $G \to \ell^1(G)$ is a covariant functor from the category of abelian topological groups into itself (Lemma 6). Further, the compact subsets of $\ell^1(G)$ are characterized (Proposition 8). In Section 2.2, the dual group of $\ell^1(G)$ is described and it is shown that G is a locally quasi-convex group if and only if $\ell^1(G)$ has this property. Further, sufficiency conditions are established for the continuity of $\alpha_{\ell^1(G)}$, the canonical mapping in the bidual group $G^{\wedge\wedge}$ (see Section 1.2 for a precise definition). In Theorem 2, it is shown that $\alpha_{\ell^1(G)}$ is continuous if G is reflexive and $G^\wedge$ is complete with a countable point-separating subgroup. In Section 2.3, the second character group is studied. It is shown that under mild conditions on the group G (e.g., if G is reflexive), $\ell^1(G)^{\wedge\wedge}$ can be canonically identified with $\ell^1(G^{\wedge\wedge})$, from which it follows that $\ell^1(G)$ is reflexive if G is a metrizable reflexive group or an LCA group.

In Section 2.4, we recall first the Schur property for groups (Definition 4) and prove for G locally quasi-convex that $\ell^1(G)$ has the Schur property if and only if G does. In Section 2.5 of this chapter, we recall the definition of Schwartz groups, properties of nuclear groups, and classify locally quasi-convex groups for which $\ell^1(G)$ is a Schwartz group, respectively, a nuclear group.

Finally, in Section 3, we present some open questions related to this article.

1.2. Notation and Preliminaries

Let $\mathbb{N} = \{1, 2, \ldots\}$ denote the natural numbers. For $m \in \mathbb{N}$, put $\underline{m} := \{1, \ldots, m\}$ and denote by $\aleph_0$ the cardinality of $\mathbb{N}$. As usual, $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}$ denotes the set of integers.

For a topological group G, let $\mathcal{N}_G(0)$ denote the set of all **symmetric** neighborhoods of 0. If the group G is clear from context, the index G will be omitted.

The compact torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is isomorphic to the complex numbers of modulus one. For technical reasons, we prefer the additive notation.

Let G be an abelian Hausdorff group. The set of all continuous characters (i.e., continuous homomorphisms from G into the torus $\mathbb{T}$) is called the **character group** of G, denoted $G^\wedge$. With pointwise addition, $G^\wedge$ is an abelian group; endowed with the compact-open topology, it is an abelian Hausdorff group, allowing us to form the second character group $(G^\wedge)^\wedge =: G^{\wedge\wedge}$. An abelian Hausdorff group G is called **(Pontryagin) reflexive** if the evaluation homomorphism

$$\alpha_G : G \to G^{\wedge\wedge},\ x \mapsto (\alpha_G(x) : \chi \mapsto \chi(x))$$

is a topological isomorphism. The famous Pontryagin–van Kampen duality theorem states that every locally compact abelian group (abbreviated LCA group) is Pontryagin reflexive. It was shown by Smith [15] that every reflexive topological vector space and every Banach space are Pontryagin reflexive groups. The latter result depends deeply on the fact that, in the character group (which can be algebraically identified with the dual space), the compact-open and strong topologies do not agree in general. However, this implies that the real or complex vector spaces c_0, ℓ^1, and ℓ^∞, well-known to be non-reflexive topological vector spaces, are Pontryagin reflexive groups. All other notation and terminology not recalled here can be found in [16] or [17].

Let $\mathbb{T}_+ = \{x + \mathbb{Z} \in \mathbb{T} : |x| \le \frac{1}{4}\}$. For a subset A of G, we call the set $A^\triangleright = \{\chi \in G^\wedge : \chi(A) \subseteq \mathbb{T}_+\}$ the **polar** of A, and for a subset $B \subseteq G^\wedge$, we consider $B^\triangleleft = \{x \in G : \chi(x) \in \mathbb{T}_+ \ \forall \chi \in B\}$, the **prepolar** of B. A subset A of an abelian topological group G is called **quasi-convex** if, for every $x \in G \setminus A$, there exists a continuous character $\chi \in A^\triangleright$ such that $\chi(x) \notin \mathbb{T}_+$. An abelian topological group G is named **locally quasi-convex** (abbreviated lqc) if there is a neighborhood base at 0 consisting of quasi-convex sets. According to ([13], 2.4), a topological vector space is lqc (as an abelian topological group) if and only if it is locally convex.

A subset B of the character group $G^\wedge$ is called **equicontinuous** if $B \subseteq U^\triangleright$ for a suitable neighborhood $U \in \mathcal{N}_G(0)$. It is well known that the polar of each neighborhood U is a compact subset of $G^\wedge$. The canonical mapping α_G is continuous if and only if every compact subset of $G^\wedge$ is equicontinuous. By a result of Kye ([18]), α_G restricted to every compact subset of G is continuous ([17], 13.4.1). In particular, if G is metrizable (more generally, a k-space), then α_G is continuous.

If G is reflexive, then the sets $\alpha_G^{-1}(U^{\triangleright\triangleright}) = U^{\triangleright\triangleleft}$ form a neighborhood base at 0. Hence, every reflexive group is lqc. The set $U^{\triangleright\triangleleft} =: \mathrm{qc}(U)$ is called the **quasi-convex hull** of U. It is the smallest quasi-convex set containing U.

If a group G is lqc and Hausdorff, then the characters of G separate points; in other words, α_G is injective or, equivalently, G is a maximally almost periodic group (abbreviated MAP group). Further, it is straightforward to prove that if G is an lqc Hausdorff group, then the mapping $\alpha_G^{-1} : \alpha_G(G) \to G,\ \alpha_G(x) \mapsto x$ is continuous.

Thus, in order to prove that G is reflexive, one has to verify that:

- G is an lqc Hausdorff group;
- Every compact subset of $G^\wedge$ is equicontinuous;
- α_G is surjective.

Next, we collect some elementary properties applied later.

Proposition 1. *If G is a second countable Hausdorff group, then $G^\wedge$ is separable.*

Proof. Because G is a second countable regular space, it is separable and metrizable ([16], 4.2.9), in particular, first countable. Thus, $G^\wedge = \bigcup_{n\in\mathbb{N}} U_n^\triangleright$ where (U_n) is a countable neighborhood base at 0. It suffices, therefore, to prove that every $U_n^\triangleright$ is separable. However, on the compact set $U_n^\triangleright$, the compact-open topology coincides with the point-separating topology $\sigma(G^\wedge, D)$ for D, a countable dense subset of G. Thus, each polar $U_n^\triangleright$, whence G is separable. □

Note that the character group of a separable group need not be separable, as $\mathbb{T}^\mathbb{R}$ shows. It is separable by the Pondiczery theorem ([16], 2.3.16), but its discrete character group is uncountable.

Proposition 2. *Let G be an abelian MAP group. If $G^\wedge$ endowed with the compact-open topology is separable, then $G^\wedge$ has a countable point-separating subgroup.*

Proof. The weak topology $\sigma(G^\wedge, G)$, induced by the mapping $G^\wedge \to \mathbb{T}^G$, $\chi \mapsto (\chi(x))_{x\in G}$, is coarser than the compact-open topology on $G^\wedge$ and hence also separable. Let $D \leq G^\wedge$ be a countable dense subgroup and let $H = \bigcap_{\chi\in D} \ker(\chi)$. We have to show that $H = \{0\}$ is the trivial subgroup of G. Thus, assume there exists $0 \neq x \in H$. Because G is a MAP group, there exists $\chi \in G^\wedge$ which satisfies $\chi(x) \neq 0_\mathbb{T}$. Because D is dense in $(G^\wedge, \sigma(G^\wedge, G))$, there exists a net $(\chi_\alpha)_{\alpha\in A}$ in D such that $(\chi_\alpha(x))$ converges to $\chi(x)$. Hence, $\chi_\alpha(x) \neq 0_\mathbb{T}$ for some $\alpha \in A$, which shows that D separates the points of G. □

Definition 1 ([19])**.** *A subset A of a topological group G is called* qc-**precompact** *if for every $U \in \mathcal{N}(0)$ there exists a finite subset F of G such that $A \subseteq \mathrm{qc}(F + U)$.*

Proposition 3 ([19], Corollary 3.7)**.** *If G is a locally quasi-convex group, then every* qc*-precompact subset of G is precompact.*

Remark 1 ([20], 6.3.10)**.** *Let C be a compact subset of a reflexive group G, then also* $\mathrm{qc}(C)$ *is compact.*

Indeed, $\mathrm{qc}(C) = C^{\triangleright\triangleleft} = \alpha_G^{-1}(C^{\triangleright\triangleright})$ holds. Because $C^\triangleright$ is a neighborhood of 0 in $G^\wedge$, its polar $C^{\triangleright\triangleright}$ is a compact subset of $G^{\wedge\wedge}$. Because α_G is a topological isomorphism, $\mathrm{qc}(C) = \alpha_G^{-1}(C^{\triangleright\triangleright})$ is a compact subset of G.

1.3. The Minkowski Functional for Groups

We define an analogue of the Minkowski functional for groups:

Definition 2 ([13], p.8)**.** *Let G be an abelian group and let $S \subseteq G$ be a symmetric subset containing 0. Set*

$$\kappa_S : G \to \mathbb{R},\ x \mapsto \begin{cases} 2 & : \quad x \notin S \\ \inf\{\frac{1}{m} :\ kx \in S\ \forall 1 \leq k \leq m\} & : \quad x \in S. \end{cases}$$

We omit an index indicating the group, because κ_S depends only on $S \subseteq G$ and not on the group containing S.

In [13], κ_S was only defined for elements of S. Kaplan defined a generalization of the Minkowski functional slightly differently in [21].

For $n \in \mathbb{N}$, we define $\mathbb{T}_n = \{x + \mathbb{Z} :\ -\frac{1}{4n} \leq x \leq \frac{1}{4n}\}$ and we put $\mathbb{T}_1 =: \mathbb{T}_+$.

Fact 1. *For $w \in \mathbb{T}$ and $n \in \mathbb{N}$ the following assertions are equivalent:*

(a) $w \in \mathbb{T}_n$;

(b) $kw \in \mathbb{T}_+$ *for all* $1 \leq k \leq n$.

Thus, Fact 1 can be reformulated as follows: $\kappa_{\mathbb{T}_+}(w) \leq \frac{1}{n}$ for some $w \in \mathbb{T}$ is equivalent to $w \in \mathbb{T}_n$.

Lemma 1.

(a) *If $A \subseteq B$ are symmetric sets containing 0, then $\kappa_B \leq \kappa_A$.*
(b) *Let A and B be symmetric subsets of G and $k \in \mathbb{N}$ such that $0 \in A$ and $\underbrace{A + \ldots + A}_{k \text{ summands}} \subseteq B$. Then, $\kappa_B(x) \leq \frac{1}{k}\kappa_A(x)$ holds for all $x \in A$.*
(c) *If A is quasi-convex, then $\kappa_A(x) \leq \frac{1}{m}$ for some $m \in \mathbb{N}$ if and only if $\chi(x) \in \mathbb{T}_m$ for all $\chi \in A^{\triangleright}$.*
(d) *If A is a subgroup of G, then $\kappa_A(x) = 0$ if $x \in A$ and $\kappa_A(x) = 2$ for $x \notin A$.*
(e) *If H is a subgroup of G and $A \subseteq G$ is a symmetric set containing $\{0\}$, then $\kappa_A(x) = \kappa_{A\cap H}(x)$ holds for all $x \in H$.*
(f) *If $A_1 \subseteq G_1$ and $A_2 \subseteq G_2$ are symmetric subsets containing the respective neutral elements, then $\kappa_{A_1\times A_2}(x_1, x_2) = \max\{\kappa_{A_1}(x_1), \kappa_{A_2}(x_2)\}$ for all $(x_1, x_2) \in G_1 \times G_2$.*

Proof. The proofs of (a) and (b) are straightforward.

(c) Fix $m \in \mathbb{N}$ and $x \in G$ with $\kappa_A(x) \leq \frac{1}{m}$. This means, $kx \in A$ for all $1 \leq k \leq m$. Because A is quasi-convex, $y \in A$ if and only if $\chi(y) \in \mathbb{T}_+$ for all $\chi \in A^{\triangleright}$. Thus, we obtain $k\chi(x) = \chi(kx) \in \mathbb{T}_+$ for all $1 \leq k \leq m$ and all $\chi \in A^{\triangleright}$. By Fact 1, this is equivalent to $\chi(x) \in \mathbb{T}_m$.

(d) and (e) are trivial.

(f) Fix $m \in \mathbb{N}$. Assume that $\kappa_{A_1\times A_2}(x_1, x_2) \leq \frac{1}{m}$. This is equivalent to $kx_1 \in A_1$ and $kx_2 \in A_2$ for all $1 \leq k \leq m$. Thus, $\kappa_{A_1}(x_1) \leq \frac{1}{m}$ and $\kappa_{A_2}(x_2) \leq \frac{1}{m}$. This shows that $\kappa_{A_1\times A_2}(x_1, x_2) \geq \max\{\kappa_{A_1}(x_1), \kappa_{A_2}(x_2)\}$. Conversely, if $\max\{\kappa_{A_1}(x_1), \kappa_{A_2}(x_2)\} \leq \frac{1}{m}$, then $k(x_1, x_2) \in A_1 \times A_2$ for all $1 \leq k \leq m$ and consequently $\kappa_{A_1\times A_2}(x_1, x_2) \leq \frac{1}{m}$. This implies $\kappa_{A_1\times A_2}(x_1, x_2) \leq \max\{\kappa_{A_1}(x_1), \kappa_{A_2}(x_2)\}$. □

κ_S does, in general, not satisfy the triangle inequality, as the following example shows: Let $A = [-1, 1] \subseteq \mathbb{R}$;
$\kappa_A(\frac{3}{2}) = 2 > 1 + \frac{1}{2} = \kappa_A(1) + \kappa_A(\frac{1}{2})$.

However, we have:

Proposition 4. *If $0 \in A \subseteq G$ is symmetric, then $\kappa_{A+A}(x + y) \leq \max\{\kappa_A(x), \kappa_A(y)\} \leq \kappa_A(x) + \kappa_A(y)$.*

Proof. It is sufficient to prove the first inequality. If $x \notin A$ or $y \notin A$, the assertion trivially holds. Thus, let us assume that $x, y \in A$. Fix $m \in \mathbb{N}$. If $\kappa_A(x), \kappa_A(y) \leq \frac{1}{m}$, then $kx, ky \in A$ for all $1 \leq k \leq m$ and hence $k(x + y) \in A + A$. This implies $\kappa_{A+A}(x + y) \leq \frac{1}{m}$. □

Lemma 2. *If $A \subseteq G$ is quasi-convex, $m \in \mathbb{N}$, and $x, y \in G$ satisfy $\kappa_A(x), \kappa_A(y) \leq \frac{1}{2m}$, then $\kappa_A(x + y) \leq \frac{1}{m}$.*

Proof. By Lemma 1 (c), $\kappa_A(x), \kappa_A(y) \leq \frac{1}{2m}$ is equivalent to $\chi(\{x, y\}) \subseteq \mathbb{T}_{2m}$ for all $\chi \in A^{\triangleright}$. Thus, $\chi(x + y) \in \mathbb{T}_m$ for all $\chi \in A^{\triangleright}$, which is equivalent to $\kappa_A(x + y) \leq \frac{1}{m}$. □

Lemma 3. *For $A \subseteq G$ and $\chi \in G^{\wedge}$ and $m \in \mathbb{N}$, the following holds:*

(a) *$\kappa_{A^{\triangleright}}(\chi) = \frac{1}{m}$ if and only if $\chi(A) \subseteq \mathbb{T}_m$ but $\chi(A) \nsubseteq \mathbb{T}_{m+1}$;*
(b) *$\kappa_{A^{\triangleright}}(\chi) = 0$ if and only if $\chi(A) = \{0\}$.*

Proof.

(a) $\kappa_{A^\triangleright}(\chi) = \frac{1}{m}$ is equivalent to $k\chi \in A^\triangleright$ for all $1 \leq k \leq m$ and $(m+1)\chi \notin A^\triangleright$. This means that $k\chi(a) \in \mathbb{T}_+$ for all $1 \leq k \leq m$ and all $a \in A$ and there exists $a_0 \in A$ such that $(m+1)\chi(a_0) \notin \mathbb{T}_+$. The first assertion is equivalent to $\chi(A) \subseteq \mathbb{T}_m$, the second (combined with the first) is equivalent to $\chi(A) \nsubseteq \mathbb{T}_{m+1}$.

(b) The assertions $\kappa_{A^\triangleright}(\chi) = 0$ are equivalent to $k\chi \in A^\triangleright$ and to $k\chi(a) \in \mathbb{T}_+$ for all $a \in A$ and $k \in \mathbb{N}$. The latter is equivalent to $\chi(A) = \{0\}$.

□

Lemma 4. *Let $\varphi : G \to H$ be a homomorphism. Assume that $0 \in A \subseteq G$ and $0 \in B \subseteq H$ are symmetric subsets such that $\varphi(A) \subseteq B$ holds. Then, $\kappa_B \circ \varphi \leq \kappa_A$ follows.*

Proof. Let $x \in G$. WLOG, we may assume that $x \in A$. Assume that $\kappa_A(x) \leq \frac{1}{m}$ for some $m \in \mathbb{N}$. Hence, $kx \in A$ for all $1 \leq k \leq m$ and hence $k\varphi(x) \in B$, which implies $\kappa_B(\varphi(x)) \leq \frac{1}{m}$. □

Lemma 5. *Let G be an abelian topological group and $A \subseteq G$ a symmetric and closed set containing 0. Then, κ_A is lower semicontinuous (i.e., $\kappa_A^{-1}(]y,\infty])$ is open for all $y \in \mathbb{R}$ or, equivalently, $\kappa_A^{-1}([0,y])$ is closed for all $y \geq 0$).*

For any sequence (A_n) of closed symmetric subsets of G containing 0, the mapping $G \to [0,\infty]$, $x \mapsto \sum_{n\in\mathbb{N}} \kappa_{A_n}(x)$ is lower semicontinuous as well.

Proof. For $y < 0$, $\kappa_A^{-1}(]y,\infty[) = G$. Fix $y \geq 0$ and let $x_0 \in G$ satisfy $\kappa_A(x_0) > y$. If $\kappa_A(x_0) = 2$, then $G \setminus A$ is an open neighborhood of x_0 contained in $\kappa_A^{-1}(]y,\infty[)$. Otherwise, $\kappa_A(x_0) = \frac{1}{m}$ for some $m \in \mathbb{N}$. Thus, $(m+1)x_0 \notin A$. For a suitable open neighborhood W of x_0, we have $(m+1)x \notin A$ for all $x \in W$. This implies $\kappa_A(x) \geq \frac{1}{m} > y$ for all $x \in W$ and hence $x_0 \in W \subseteq \kappa_A^{-1}(]y,\infty[)$.

Assume now that (A_n) is a sequence of closed and symmetric sets containing 0. Put $\kappa := \sum_{n\in\mathbb{N}} \kappa_{A_n}$. Fix $y \in \mathbb{R}$. As above, $\kappa^{-1}(]y,\infty[) = G$ in case $y < 0$. Thus, assume now that $y \geq 0$ and let $x_0 \in G$ satisfy $\kappa(x_0) > y$. Then, there is $N \in \mathbb{N}$ such that $\sum_{n=1}^N \kappa_{A_n}(x_0) > y$. Let $y_n := \kappa_{A_n}(x_0)$ and $\varepsilon := \frac{1}{N}\Big((\sum_{n=1}^N y_n) - y\Big)$. By what was shown above, there exists an open neighborhood W of x_0 such that $\kappa_{A_n}(x) > y_n - \varepsilon$ for all $1 \leq n \leq N$ and all $x \in W$. Then, $\kappa(x) \geq \sum_{n=1}^N \kappa_{A_n}(x) > \sum_{n=1}^N (y_n - \varepsilon) = \sum_{n=1}^N y_n - N\varepsilon = y$. This shows that κ is lower semicontinuous. □

2. The Group of Absolutely Summable Sequences $\ell^1(G)$

2.1. Basic Properties of $\ell^1(G)$

Definition 3. *Let (G,τ) be an abelian topological group. Denote by*

$$\ell^1(G) = \ell^1(G,\tau) = \{(x_n) \in G^{\mathbb{N}} : \ \forall U \in \mathcal{N}_G(0) : \sum_{n\in\mathbb{N}} \kappa_U(x_n) < \infty\}.$$

The set $\ell^1(G)$ is a group under pointwise addition.

(Indeed, let $(x_n),\ (y_n) \in \ell^1(G)$. For $U \in \mathcal{N}(0)$, there exists $W \in \mathcal{N}(0)$ such that $W + W \subseteq U$. Then, by Lemma 1 (a) and Proposition 4, $\sum_{n\in\mathbb{N}} \kappa_U(x_n + y_n) \leq \sum_{n\in\mathbb{N}} \kappa_{W+W}(x_n + y_n) \leq \sum_{n\in\mathbb{N}} \kappa_W(x_n) + \sum_{n\in\mathbb{N}} \kappa_W(y_n) < \infty$ holds.)

The group $\ell^1(G)$ is the **group of all absolutely summable sequences** *in G. The family of sets $(S_U)_{U\in\mathcal{N}(0)}$ where*

$$S_U = \{(x_n) \in \ell^1(G) : \sum_{n\in\mathbb{N}} \kappa_U(x_n) \leq 1\}$$

forms a neighborhood base at 0 of a group topology on $\ell^1(G)$.

(Indeed, fix a symmetric neighborhood $U \in \mathcal{N}(0)$ and let $(x_n), (y_n) \in S_U$. Then,

$$\sum_{n\in\mathbb{N}} \kappa_{U+U+U+U}(x_n + y_n) \leq \sum_{n\in\mathbb{N}} \kappa_{U+U}(x_n) + \sum_{n\in\mathbb{N}} \kappa_{U+U}(y_n)$$

$\leq \frac{1}{2}(\sum_{n\in\mathbb{N}} \kappa_U(x_n) + \sum_{n\in\mathbb{N}} \kappa_U(y_n)) \leq 1$ by Proposition 4 and Lemma 1 (b).
Thus, the symmetric set S_U satisfies $S_U + S_U \subseteq S_{U+U+U+U}$).

This topology will be denoted $\Sigma_{\ell^1(G)}$.

Further, for $N \in \mathbb{N}$ and $U \in \mathcal{N}(0)$, let

$$S_{N,U} := \{(x_n) \in \ell^1(G) : \sum_{n\geq N} \kappa_U(x_n) \leq 1\}.$$

Thus, $S_U = S_{1,U}$ for all $U \in \mathcal{N}(0)$.

Remark 2. *The direct sum $G^{(\mathbb{N})}$ is contained in $\ell^1(G)$, while the latter group is a subgroup of $c_0(G)$, the group of all null sequences in G.* (The first assertion is trivial. In order to prove the second one, fix $(x_n) \in \ell^1(G)$ and $U \in \mathcal{N}(0)$. Because $\sum_{n\in\mathbb{N}} \kappa_U(x_n) < \infty$, there exists $n_0 \in \mathbb{N}$ such that $\kappa_U(x_n) \leq 1$ for all $n \geq n_0$. However, this means that $x_n \in U$ for all $n \geq n_0$. Hence, $x_n \to 0$.)

In case G does not admit any non-trivial convergent sequences, $G^{(\mathbb{N})} = \ell^1(G) = c_0(G)$ holds algebraically. Hrušák, van Mill, Ramos-García, and Shelah [22] proved (under ZFC) that there exists an infinite countably compact group G of exponent 2 which has no non-trivial convergent sequences, whence $\ell^1(G) = G^{(\mathbb{N})}$.

Lemma 6. *If $\varphi : G \to H$ is a continuous homomorphism of topological groups, then $\varphi_\# : \ell^1(G) \to \ell^1(H)$, $(x_n) \mapsto (\varphi(x_n))$ is a well-defined continuous homomorphism. More precisely, if $\varphi(U) \subseteq V$ holds for symmetric neighborhoods $U \in \mathcal{N}_G(0)$ and $V \in \mathcal{N}_H(0)$, then $\varphi_\#(S_U) \subseteq S_V$.*

*Thus, $\mathfrak{F}_1 :$ **ATOP** $\to$ **ATOP**, $G \mapsto \ell^1(G)$ and $\varphi \mapsto \varphi_\#$ defines a covariant functor from the category of all abelian topological groups into itself. In particular, if φ is a topological isomorphism, then so is $\varphi_\#$.*

Proof. For $V \in \mathcal{N}_H(0)$, there exists $U \in \mathcal{N}_G(0)$ such that $\varphi(U) \subseteq V$. By Lemma 4, $\kappa_V(\varphi(x)) \leq \kappa_U(x)$ holds for all $x \in G$. Thus, for $(x_n) \in \ell^1(G)$ this gives $\sum_{n\in\mathbb{N}} \kappa_V(\varphi(x_n)) \leq \sum_{n\in\mathbb{N}} \kappa_U(x_n) < \infty$. This yields that $\varphi_\#$ is well-defined and obviously a homomorphism which satisfies $\varphi_\#(S_U) \subseteq S_V$. Thus, in particular, $\varphi_\#$ is continuous. It is straightforward to check that $(\varphi \circ \psi)_\# = \varphi_\# \circ \psi_\#$ for an appropriate continuous homomorphism $\psi : G_0 \to G$. Now, the assertion follows easily. □

Corollary 1. *Let G be a non-necessarily Hausdorff abelian group and denote by $N = \overline{\{0\}}$ the core of G and by $\pi : G \to G/N$ the canonical projection. Then, $\pi_\# : \ell^1(G) \to \ell^1(G/N)$ is a projection.*

Proof. By Lemma 6, $\pi_\#$ is continuous, and for a symmetric neighborhood $U \in \mathcal{N}_G(0)$, we have $\pi_\#(S_U) \subseteq S_{\pi(U)}$. Conversely, we are going to show that $\pi_\#(S_{U+U}) \supseteq S_{\pi(U)}$ holds. Therefore, we verify first that $\kappa_{U+U}(x) \leq \kappa_{\pi(U)}(\pi(x))$ holds for all $x \in G$. Thus, assume that $\kappa_{\pi(U)}(\pi(x)) \leq \frac{1}{m}$ for some $m \in \mathbb{N}$. This implies that $k\pi(x) \in \pi(U)$ for all $1 \leq k \leq m$ and hence $kx \in U + N \subseteq U + U$ for all $1 \leq k \leq m$. Thus, $\kappa_{U+U}(x) \leq \frac{1}{m}$. Next, fix $(\pi(x_n)) \in S_{\pi(U)}$. Then, $\sum_{n\in\mathbb{N}} \kappa_{U+U}(x_n) \leq \sum_{n\in\mathbb{N}} \kappa_{\pi(U)}(\pi(x_n)) \leq 1$ follows. Thus, $(x_n) \in S_{U+U}$ and hence $(\pi(x_n)) \in \pi_\#(S_{U+U})$. □

Proposition 5. *Let G be an abelian topological group and F a finite subset of $\mathbb{N}$. Then:*

(a) $\mu_F : G^F \to \ell^1(G)$, $(x_n)_{n\in F} \mapsto (x_n)_{n\in\mathbb{N}}$, *where $x_n = 0$ for all $n \in \mathbb{N} \setminus F$, is an embedding.*

(b) *$p_F : \ell^1(G) \to G^F,\ (x_n)_{n\in\mathbb{N}} \mapsto (x_n)_{n\in F}$ is a projection.*
(c) *G is Hausdorff if and only if $\ell^1(G)$ is Hausdorff.*
(d) *G is linearly topologized if and only if $\ell^1(G)$ has this property.*

For $F = \{n\}$, we write μ_n and p_n instead of $\mu_{\{n\}}$ and $p_{\{n\}}$.

Proof. We start with the following observation:

For every $U \in \mathcal{N}_G(0)$ and $W \in \mathcal{N}_G(0)$ such that $W \underbrace{+\ldots+}_{|F| \text{ times}} W \subseteq U$ one has

$$\mu_F(W^F) = W^F \times \{0\}^{\mathbb{N}\setminus F} \subseteq S_U.$$

Proof of observation: For $x \in W$, one has $\kappa_U(x) \le \frac{1}{|F|}\kappa_W(x)$ by Lemma 1 (b); hence, $\mu_F(W \times \ldots \times W) \subseteq S_U$, because $\sum_{n\in F} \kappa_U(x_n) \le \sum_{n\in F} \frac{1}{|F|} \cdot \kappa_W(x_n) \le 1$ for all $(x_n)_{n\in F} \in W^F$, as desired.

(a) The observation above implies that μ_F is continuous. In order to show that μ_F is an embedding, observe the following: $\mu_F(G^F) \cap S_W \subseteq \mu_F(W^F)$, because $\mu_F((x_n)_{n\in F}) \in S_W$ if and only if $\sum_{n\in F} \kappa_W(x_n) \le 1$, which implies $x_n \in W$ for all $n \in F$.
(b) Because $p_F(S_U) \subseteq U^F$ for all $U \in \mathcal{N}_G(0)$, the mapping p_F is continuous. In order to show that p_F is open, let U and W be as in the observation. Then, $p_F(S_U) \supseteq p_F(W^F \times \{0\}^{\mathbb{N}\setminus F}) \supseteq W^F$. This shows that p_F is open.
(c) Assume that G is Hausdorff. It is straightforward to prove that $\bigcap_{U\in\mathcal{N}} S_U = \{0\}$. Thus, $\ell^1(G)$ is also a Hausdorff group. Conversely, because $\mu_1 : G \to \ell^1(G)$ is an embedding by item (a), G is Hausdorff provided $\ell^1(G)$ has this property.
(d) Assume that G is linearly topologized. If U is an open subgroup of G, then $\kappa_U = 2 \cdot 1_{G\setminus U}$ where $1_{G\setminus U}$ denotes the indicator function (by Lemma 1 (d)). Thus, $S_U = \{(x_n) \in \ell^1(G) :\ x_n \in U\ \forall n \in \mathbb{N}\} = U^{\mathbb{N}} \cap \ell^1(G)$ is a subgroup. Hence, $\ell^1(G)$ is also linearly topologized.

The converse implication is a consequence of item (a). □

A consequence of item (b) is the continuity of the canonical projections p_n, which immediately implies the following.

Corollary 2. *The canonical mapping $(\ell^1(G), \Sigma_{\ell^1(G)}) \to (G^{\mathbb{N}}, \tau_p)$, where τ_p denotes the product topology, is continuous.*

Proposition 6.

(a) *If H is a subgroup of G and $\iota : H \to G$ denotes the embedding , then $\iota_\# : \ell^1(H) \to \ell^1(G)$ is an embedding. Furthermore, if H is an open, respectively, closed subgroup of G, then $\iota_\#(\ell^1(H))$ is an open, respectively, closed subgroup of $\ell^1(G)$.*
(b) *For abelian topological groups G_1 and G_2, the sequence space $\ell^1(G_1 \times G_2)$ is canonically topologically isomorphic to $\ell^1(G_1) \times \ell^1(G_2)$.*

Proof.

(a) Because for every symmetric neighborhood $U \in \mathcal{N}_G(0)$ the equation $\iota_\#(S_{U\cap H}) = S_U \cap \iota_\#(\ell^1(H))$ holds by Lemma 1 (e), this yields that $\iota_\#$ is an embedding. Further, if H is open, U can be chosen to be contained in H and then $S_U \subseteq \iota_\#(\ell^1(H))$, so $\iota_\#(\ell^1(H)$ is an open subgroup of $\ell^1(G)$. Now, let H be a closed subgroup of G and let $p_n : \ell^1(G) \to G$ denote the projection on the n-th coordinate. Then, $\iota_\#(\ell^1(H)) = \bigcap_{n\in\mathbb{N}} p_n^{-1}(H)$ is closed in $\ell^1(G)$ by Proposition 5 (b).
(b) For $i \in \{1, 2\}$, let $\pi_i : G_1 \times G_2 \to G_i$ be the canonical projection and consider the canonical mapping

$\psi = ((\pi_1)_\# \times (\pi_2)_\#) : \ell^1(G_1 \times G_2) \to \ell^1(G_1) \times \ell^1(G_2),\ ((x_n, y_n)) \mapsto ((x_n), (y_n))$, which is a continuous monomorphism by Lemma 6.

By Lemma 1 (f), we have for $U_i \in \mathcal{N}_{G_i}(0)$ and $x \in G_1, y \in G_2$ $\kappa_{U_1 \times U_2}(x, y) \le \kappa_{U_1}(x) + \kappa_{U_2}(y)$. This implies that ψ is surjective. In order to prove that ψ is open, we are going to show that $\psi(S_{(U_1+U_1)\times(U_2+U_2)}) \supseteq S_{U_1} \times S_{U_2}$ for $U_i \in \mathcal{N}_{G_i}(0)$. Thus, fix $(x_n) \in S_{U_1}$ and $(y_n) \in S_{U_2}$. Then, by Lemma 1 (b),

$$\sum_{n\in\mathbb{N}} \kappa_{(U_1+U_1)\times(U_2+U_2)}((x_n, y_n)) = \sum_{n\in\mathbb{N}} \max\{\kappa_{U_1+U_1}(x_n), \kappa_{U_2+U_2}(y_n)\} \le$$

$$\le \sum_{n\in\mathbb{N}} \kappa_{U_1+U_1}(x_n) + \sum_{n\in\mathbb{N}} \kappa_{U_2+U_2}(y_n) \le \sum_{n\in\mathbb{N}} \frac{1}{2}\kappa_{U_1}(x_n) + \sum_{n\in\mathbb{N}} \frac{1}{2}\kappa_{U_2}(y_n) \le 1.$$

This shows that ψ is open and completes the proof. □

Lemma 7. *Let (C_n) be a sequence of complete subsets of a Hausdorff abelian group G and let $((x_n^{(\alpha)})_n)_{\alpha\in A}$ be a Cauchy net in $\ell^1(G)$. Assume that $\{x_n^{(\alpha)} : \alpha \in A\} \subseteq C_n$ for every $n \in \mathbb{N}$. Then, $((x_n^{(\alpha)})_n)_{\alpha\in A}$ is convergent.*

Proof. By Proposition 5 (b), all p_n are continuous, so for every $n \in \mathbb{N}$, the net $(x_n^{(\alpha)})_{\alpha\in A}$ is a Cauchy net in G contained in C_n. Because C_n was assumed to be complete, $x_n = \lim_{\alpha\in A} x_n^{(\alpha)}$ exists for all $n \in \mathbb{N}$. □

Claim: For every $U \in \mathcal{N}(0)$, there exists $\alpha_0 \in A$ such that $\sum_{n\in\mathbb{N}} \kappa_U(x_n^{(\alpha)} - x_n) \le 1$ for all $\alpha \ge \alpha_0$.

Proof. Fix $U \in \mathcal{N}(0)$. We choose a closed and symmetric neighborhood $W \in \mathcal{N}(0)$ such that $W + W + W + W \subseteq U$. By assumption, there exists $\alpha_W \in A$ such that $\sum_{n\in\mathbb{N}} \kappa_W(x_n^{(\alpha)} - x_n^{(\beta)}) \le 1$ holds for all $\alpha, \beta \ge \alpha_W$. Because W is closed and $x_n^{(\alpha)} - x_n^{(\beta)} \in W$ for all $\alpha, \beta \ge \alpha_W$, we obtain $x_n^{(\alpha)} - x_n \in W$ for all $\alpha \ge \alpha_W$. Now, fix $\alpha \ge \alpha_W$ and assume that $\sum_{n\in\mathbb{N}} \kappa_U(x_n^{(\alpha)} - x_n) > 1$. Choose a finite subset $F \subseteq \mathbb{N}$ such that $\kappa_U(x_n^{(\alpha)} - x_n) > 0$ for all $n \in F$ and $\sum_{n\in F} \kappa_U(x_n^{(\alpha)} - x_n) > 1$. For $n \in F$, we have $0 < \kappa_U(x_n^{(\alpha)} - x_n) \le \kappa_W(x_n^{(\alpha)} - x_n) \le 1$. Thus, choose $m_n \in \mathbb{N}$ such that $\frac{1}{m_n} = \kappa_W(x_n^{(\alpha)} - x_n)$. Because $(x_n^{(\beta)})_{\beta\in A}$ converges to x_n, there exists $\beta \ge \alpha_W$ such that $\kappa_W(x_n^{(\beta)} - x_n) \le \frac{1}{|F|}$ for all $n \in F$. We obtain

$$\begin{aligned}
1 &< \sum_{n\in F} \kappa_U(x_n^{(\alpha)} - x_n) \\
&\le \sum_{n\in F} \kappa_{W+W+W+W}(x_n^{(\alpha)} - x_n^{(\beta)} + x_n^{(\beta)} - x_n) \\
&\overset{Proposition\ 4}{\le} \sum_{n\in F} \kappa_{W+W}(x_n^{(\alpha)} - x_n^{(\beta)}) + \sum_{n\in F} \kappa_{W+W}(x_n^{(\beta)} - x_n) \\
&\overset{Lemma\ 1(b)}{\le} \frac{1}{2}\sum_{n\in F} \kappa_W(x_n^{(\alpha)} - x_n^{(\beta)}) + \frac{1}{2}\sum_{n\in F} \kappa_W(x_n^{(\beta)} - x_n) \\
&\le \frac{1}{2}\sum_{n\in\mathbb{N}} \kappa_W(x_n^{(\alpha)} - x_n^{(\beta)}) + \frac{1}{2}\sum_{n\in F} \frac{1}{|F|} \le \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

This contradiction proves the Claim with $\alpha_0 = \alpha_W$.

We now show that $(x_n) \in \ell^1(G)$. Fix a symmetric closed neighborhood $U \in \mathcal{N}(0)$. Choose α_0 as in the Claim. We obtain $\sum_{n\in\mathbb{N}} \kappa_{U+U}(x_n) \le \sum_{n\in\mathbb{N}} \kappa_U(x_n - x_n^{(\alpha_0)}) + \sum_{n\in\mathbb{N}} \kappa_U(x_n^{(\alpha_0)}) \le 1 + \sum_{n\in\mathbb{N}} \kappa_U(x_n^{(\alpha_0)}) < \infty$. Thus, $(x_n) \in \ell^1(G)$. It follows from the Claim that $((x_n^{(\alpha)})_n)_{\alpha\in A}$ converges to $(x_n)_n$. □

Corollary 3. *If G is a Hausdorff complete abelian group, then so is $\ell^1(G)$.*

Proof. Apply Lemma 7 to $C_n = G$ for all $n \in \mathbb{N}$. □

Proposition 7. *$G^{(\mathbb{N})}$ is dense in $\ell^1(G)$.*

Proof. Fix $(x_n) \in \ell^1(G)$ and $U \in \mathcal{N}(0)$. Because $\sum_{n\in\mathbb{N}} \kappa_U(x_n) < \infty$, there exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0+1}^{\infty} \kappa_U(x_n) \leq 1$. This shows that $(x_n) - \mu_{\underline{n_0}}(x_1, \ldots, x_{n_0}) \in S_U$. □

Proposition 8. *Let G be an abelian Hausdorff group. A subset K of $\ell^1(G)$ is compact if and only if the following three conditions hold:*

(a) *K is closed;*
(b) *$p_n(K)$ is compact for every $n \in \mathbb{N}$;*
(c) *For every $U \in \mathcal{N}(0)$, there exists $N_U \in \mathbb{N}$ such that $K \subseteq S_{N_U,U}$.*

Proof. Assume that $K \subseteq \ell^1(G)$ is compact. Then, obviously, conditions (a) and (b) are satisfied. In order to prove (c), fix $U \in \mathcal{N}(0)$. Because K is totally bounded and $G^{(\mathbb{N})}$ is dense in $\ell^1(G)$ by Proposition 7, there exists a finite subset $F \subseteq G^{(\mathbb{N})}$ such that $K \subseteq F + S_U$. Fix $N_U \in \mathbb{N}$ such that $p_k((y_n)) = 0$ for all $k \geq N_U$ and all $(y_n) \in F$. Fix $(x_n) \in K$. There exists $(y_n) \in F$ such that $(x_n - y_n) \in S_U$. Hence, $\sum_{n\geq N_U} \kappa_U(x_n) = \sum_{n\geq N_U} \kappa_U(x_n - y_n) \leq \sum_{n\geq 1} \kappa_U(x_n - y_n) \leq 1$. This shows that $K \subseteq S_{N_U,U}$.

Conversely, assume that $K \subseteq \ell^1(G)$ satisfies the conditions (a), (b), and (c). By Lemma 7 (with $C_n = p_n(K)$), we conclude that K is complete. In order to prove that K is totally bounded, we fix $U \in \mathcal{N}(0)$. By item (c), there exists $N_U \in \mathbb{N}$ such that $\sum_{n\geq N_U} \kappa_U(x_n) \leq 1$ for all $(x_n) \in K$. Because $K \subseteq \mu_{\underline{N_U}}(\prod_{n=1}^{N_U} p_n(K)) + S_U$ and $\mu_{\underline{N_U}}(\prod_{n=1}^{N_U} p_n(K))$ is compact, K is totally bounded. □

Corollary 4. *Let G be an abelian Hausdorff group. For the density (the minimal cardinality of a dense subset), the following holds: $d(\ell^1(G)) = \max\{\aleph_0, d(G)\}$ in case $d(G) > 1$.*

Proof. Let $D \subseteq G$ be a dense subset of cardinality $d(G)$. Because $\mu_{\underline{n}}$ is an embedding for every $n \in \mathbb{N}$, the closure of $D^{(\mathbb{N})}$ contains the dense set $G^{(\mathbb{N})}$. This shows that $D^{(\mathbb{N})}$ is dense in $\ell^1(G)$ and hence $d(\ell^1(G)) \leq \max\{\aleph_0, d(G)\}$. In case $d(G)$ is infinite, $d(G) = d(\ell^1(G))$, because p_1 maps a dense subset of $\ell^1(G)$ onto a dense subset of G.

Assume now that $1 < d(G) < \infty$. Then, G is a finite discrete group and hence $\ell^1(G) = G^{(\mathbb{N})}$ is a countably infinite discrete group. Hence, $d(\ell^1(G)) = \aleph_0$ in this case. □

Proposition 9. *Let G be an abelian Hausdorff group. For the character χ (the minimal cardinality of a neighborhood base at 0) and the weight w (the minimal cardinality of a base), the following holds:*

(a) *$\chi(G) = \chi(\ell^1(G))$.*
(b) *$w(G) = w(\ell^1(G))$ if $w(G)$ is infinite.*

Proof.

(a) is trivial.
(b) Recall that for every topological group H, one has $w(H) = \chi(H) \cdot d(H)$ (Lemma 5.1.7 in [17]). If $d(G)$ were finite, then G would be a finite discrete group and hence $w(G)$ had to be finite in contradiction to the assumption. Thus, $d(G)$ is infinite. Applying item (a) and Corollary 4, we obtain

$$w(\ell^1(G)) = \chi(\ell^1(G)) \cdot d(\ell^1(G)) = \chi(G) \cdot \max\{\aleph_0, d(G)\} = \chi(G) \cdot d(G) = w(G).$$

□

2.2. The Character Group of $\ell^1(G)$

Proposition 10. *The mapping*

$$(\mu_n^\wedge) : \ell^1(G)^\wedge \longrightarrow G^{\wedge\mathbb{N}},\ \chi \longmapsto (\mu_n^\wedge(\chi)) = (\chi \circ \mu_n)$$

is a continuous injective homomorphism. Thus, algebraically, $\ell^1(G)^\wedge$ can be identified with a subgroup of $G^{\wedge\mathbb{N}}$.

Proof. Because μ_n is continuous for every $n \in \mathbb{N}$ by Proposition 5 (a), so is $(\mu_n^\wedge)$. We are going to show now that $(\mu_n^\wedge)$ is injective: Let $\chi \in \ell^1(G)^\wedge$ and assume that $\mu_n^\wedge(\chi) = \chi \circ \mu_n$ is the trivial character for every $n \in \mathbb{N}$. This implies that χ restricted to the subgroup $G^{(\mathbb{N})}$ is trivial. By Proposition 7, $G^{(\mathbb{N})}$ is dense in $\ell^1(G)$; hence, χ is trivial. □

This result allows us to identify a character $\chi \in \ell^1(G)^\wedge$ with the sequence $(\chi_n) = (\mu_n^\wedge(\chi))_{n\in\mathbb{N}}$.

Next, we are going to describe the structure of the dual group of $\ell^1(G)$.

Proposition 11. *For an abelian topological group, the following assertions hold:*

$$\ell^1(G)^\wedge \quad = \quad \bigcup_{U\in\mathcal{N}(0)} (U^\triangleright)^{\mathbb{N}}.$$

and

$$(S_U)^\triangleright = (U^\triangleright)^{\mathbb{N}}.$$

Proof. A homomorphism $\chi : \ell^1(G) \to \mathbb{T}$ is continuous if and only if χ maps a suitable neighborhood of 0 in $\ell^1(G)$ into $\mathbb{T}_+$ or, equivalently, if χ belongs to the polar of a neighborhood of 0. Hence, $\ell^1(G)^\wedge = \bigcup_{U\in\mathcal{N}_G(0)} (S_U)^\triangleright$.

Next, we are going to describe such a polar $(S_U)^\triangleright$: Fix $\chi = (\chi_n) \in (S_U)^\triangleright$. Because $\mu_n(U) \subseteq S_U$ for all $n \in \mathbb{N}$, we obtain $\chi_n = \mu_n^\wedge(\chi) = \chi \circ \mu_n \in U^\triangleright$. This shows that $(S_U)^\triangleright \subseteq (U^\triangleright)^{\mathbb{N}}$.

Conversely, assume that $\chi = (\chi_n) \in (U^\triangleright)^{\mathbb{N}}$ and fix $(x_n) \in S_U$. Recall that for $\psi \in U^\triangleright$ and $x \in U$ with $\kappa_U(x) \le \frac{1}{m}$, one has $k\psi(x) \in \mathbb{T}_+$ for all $1 \le k \le m$ and hence $\psi(x) \in \mathbb{T}_m$ (Fact 1). We obtain $\chi(x_n) = \sum_{n\in\mathbb{N}} \chi_n(x_n) \in \mathbb{T}_+$, so $\chi \in (S_U)^\triangleright$. □

Proposition 12. *A topological group G is lqc if and only if $\ell^1(G)$ is lqc.*

Proof. Because $\mu_1 : G \to \ell^1(G)$ is an embedding and because subgroups of lqc groups are again lqc, the condition is necessary. Conversely, assume that G is lqc. Fix a quasi-convex neighborhood $U \in \mathcal{N}_G(0)$ and choose $W \in \mathcal{N}_G(0)$ quasi-convex such that $W + W + W \subseteq U$. We are going to prove that $\mathrm{qc}(S_W) \subseteq S_U$. Thus, let $(x_n) \notin S_U$, i.e., $\sum_{n\in\mathbb{N}} \kappa_U(x_n) > 1$. We have to find $\chi = (\chi_n) \in (S_W)^\triangleright = (W^\triangleright)^{\mathbb{N}}$ such that $\chi(x_n) \notin \mathbb{T}_+$. In case there is $n \in \mathbb{N}$ such that $x_n \notin W$, there exists $\chi_n \in W^\triangleright$ such that $\chi_n(x_n) \notin \mathbb{T}_+$. Then, $\chi = p_n^\wedge(\chi_n)$ has the desired property. Assume now that $x_n \in W$ for all $n \in \mathbb{N}$. This implies $\kappa_U(x_n) \le \frac{1}{3}$. Let $N \in \mathbb{N}$ be minimal with the property that $\sum_{n=1}^{N} \kappa_U(x_n) > 1$, $F = \{n :\ 1 \le n \le N,\ \kappa_U(x_n) > 0\}$, and put $\kappa_U(x_n) = \frac{1}{m_n}$ for $n \in F$, where $m_n \ge 3$ must hold. By the minimality condition, $N \in F$. For $n \in F$, we choose $\chi_n \in U^\triangleright$ such that $\chi_n(x_n) = t_n + \mathbb{Z}$ where $\dfrac{1}{4(m_n+1)} < t_n \le \dfrac{1}{4m_n}$ (cf. Lemma 1 (c)). Because $W + W \subseteq U$, we obtain $U^\triangleright + U^\triangleright \subseteq W^\triangleright$ (Fact 1). Thus, $\chi = p_F^\wedge((2\chi_n)_{n\in F}) \in (W^\triangleright)^{\mathbb{N}}$. We obtain: $\chi(x_n) = \sum_{n\in F} 2\chi_n(x_n) = \sum_{n\in F} 2t_n + \mathbb{Z}$ where

$$\frac{1}{4} < \frac{1}{4}\sum_{n\in F} \kappa_U(x_n) = \sum_{n\in F} \frac{1}{4m_n} \le \sum_{n\in F} \frac{1}{2m_n+2} < \sum_{n\in F} 2t_n \le$$

$$\le \sum_{n\in F} \frac{1}{2m_n} = \sum_{n\in F\setminus\{N\}} \frac{1}{2m_n} + \frac{1}{2m_N} \le \frac{1}{2} + \frac{1}{2m_N} \le \frac{2}{3} < \frac{3}{4}$$

because N was chosen to be minimal and hence $\sum_{n\in F\setminus\{N\}} \frac{1}{m_n} \leq 1$; further, because $x_N \in W$, we have $\frac{1}{m_N} = \kappa_U(x_N) \leq \frac{1}{3}$. This shows that $\chi = p_F^\wedge((2\chi_n)_{n\in F})$ has the desired properties. □

Proposition 13. *Let G be a Hausdorff abelian group. Then,* $(G^\wedge)^{(\mathbb{N})}$ *is dense in* $\ell^1(G)^\wedge$.

Proof. Fix $\chi = (\chi_n) \in \ell^1(G)^\wedge$ and a compact subset K of $\ell^1(G)$. Because χ is continuous, there exists $U \in \mathcal{N}_G(0)$ such that $\chi \in (S_U)^\triangleright = (U^\triangleright)^\mathbb{N}$. Choose $N_U \in \mathbb{N}$ such that $K \subseteq S_{N_U,U}$ (cf. Proposition 8).

For $(x_n) \in S_{N_U,U}$ we have
$(\chi - p_{\underline{N_U}}^\wedge(\chi_1, \ldots, \chi_{N_U}))(x_n) = \sum_{n>N_U} \chi_n(x_n) \in \mathbb{T}_+$ because $\sum_{n>N_U} \kappa_U(x_n) \leq 1$ and $\chi_n \in U^\triangleright$ for all $n \in \mathbb{N}$, so $\chi - p_{\underline{N_U}}^\wedge(\chi_1, \ldots, \chi_{N_U}) \in (S_{N_U,U})^\triangleright \subseteq K^\triangleright$, as desired. □

Next, we are going to study the continuity of $\alpha_{\ell^1(G)}$ and start with the following obvious

Proposition 14. *Let G be a metrizable group. Then,* $\alpha_{\ell^1(G)}$ *is continuous.*

Proof. Because G is first countable, so is $\ell^1(G)$ by Proposition 9 (a). Hence, $\alpha_{\ell^1(G)}$ is continuous. □

Lemma 8. *Let G be an abelian Hausdorff group. Then,* $\alpha_{\ell^1(G)}$ *is continuous if and only if for every compact subset* $K \subseteq \ell^1(G)^\wedge$ *the set* $T_K := \overline{\bigcup_{m\in\mathbb{N}} \mu_m^\wedge(K)} \subseteq G^\wedge$ *is equicontinuous.*

Proof. Recall that for an abelian topological group G, the canonical homomorphism α_G is continuous if and only if every compact subset of $G^\wedge$ is equicontinuous. Thus, $\alpha_{\ell^1(G)}$ is continuous if and only if for every compact subset K of $\ell^1(G)^\wedge$ there exists a neighborhood $U \in \mathcal{N}_G(0)$ such that $K \subseteq (S_U)^\triangleright = (U^\triangleright)^\mathbb{N}$. This implies $\mu_m^\wedge(K) \subseteq U^\triangleright$ for all $m \in \mathbb{N}$ and hence $T_K \subseteq U^\triangleright$.

Conversely, assume that for every compact subset $K \subseteq \ell^1(G)^\wedge$ there exists $U \in \mathcal{N}_G(0)$ such that $T_K \subseteq U^\triangleright$. Then, $K \subseteq \prod_{m\in\mathbb{N}} \mu_m^\wedge(K) \subseteq (U^\triangleright)^\mathbb{N} = (S_U)^\triangleright$. This shows that K is equicontinuous and hence $\alpha_{\ell^1(G)}$ is continuous. □

For a continuous homomorphism $\psi : H \to G$ between Hausdorff groups, the homomorphism $\psi_\# : \ell^1(H) \to \ell^1(G)$ is continuous and so is its dual homomorphism $(\psi_\#)^\wedge : \ell^1(G)^\wedge \to \ell^1(H)^\wedge$.

Lemma 9. *Let* $\psi : H \to G$ *be a continuous homomorphism between abelian Hausdorff groups. Then,* $(\psi_\#)^\wedge(\chi_n) = (\psi^\wedge(\chi_n))$ *holds for all* $(\chi_n) \in \ell^1(G)^\wedge$.

If $K \subseteq \ell^1(G)^\wedge$ *is compact and* $T_K = \overline{\bigcup_{m\in\mathbb{N}} \mu_m^\wedge(K)}$ *and* $T_{(\psi_\#)^\wedge(K)}$ *is the analogous subset of* $H^\wedge$ *corresponding to the compact set* $(\psi_\#)^\wedge(K)$, *then* $\psi^\wedge(T_K) \subseteq T_{(\psi_\#)^\wedge(K)}$.

Proof. By Lemma 6, the mapping $\psi_\# : \ell^1(H) \to \ell^1(G)$ is a continuous homomorphism. Hence, $(\psi_\#)^\wedge : \ell^1(G)^\wedge \to \ell^1(H)^\wedge$ is a well-defined continuous homomorphism. Fix $(\chi_n) \in \ell^1(G)^\wedge$ and $(h_n) \in \ell^1(H)$. Then, we have $(\psi_\#)^\wedge((\chi_n))(h_n) = (\chi_n)(\psi_\#(h_n)) = (\chi_n)(\psi(h_n)) = \sum_{n\in\mathbb{N}} \chi_n(\psi(h_n)) = \sum_{n\in\mathbb{N}} \psi^\wedge(\chi_n)(h_n) = (\psi^\wedge(\chi_n))(h_n)$. Now, the first assertion follows. This yields

$$\psi^\wedge(T_K) \subseteq \overline{\{\psi^\wedge(\mu_m^\wedge(\chi)) :\ \chi \in K,\ m \in \mathbb{N}\}}$$

$$= \overline{\{\psi^\wedge(\chi_m) :\ \chi = (\chi_n) \in K,\ m \in \mathbb{N}\}} = T_{(\psi_\#)^\wedge(K)}.$$

□

Theorem 1. *For every compact abelian group G, the mapping* $\alpha_{\ell^1(G)}$ *is continuous.*

Proof. Let $K \subseteq \ell^1(G)^\wedge$ be compact and let T_K be as in Lemma 8. Assume that T_K is an infinite subset of $G^\wedge$. Let D be the divisible hull of the discrete group $G^\wedge$ and consider the embedding $G^\wedge \to D$. Let D_0 be a divisible countably infinite subgroup of D such that $T_K \cap D_0$ is infinite. Because D_0 splits, there is a continuous homomorphism $\gamma : G^\wedge \to D_0$ such that $\gamma(T_K)$ is infinite. Because G and D_0 are reflexive groups, we may consider $\gamma = \psi^\wedge$ for a suitable homomorphism $\psi : D_0^\wedge \to G$ (after identifying D_0 with its second dual group). Indeed, let $\gamma^\wedge : D_0^\wedge \to G^{\wedge\wedge}$ be the dual homomorphism and let $\psi = \alpha_G^{-1} \circ \gamma^\wedge : D_0^\wedge \to G$ be the composition of $\gamma^\wedge$ with the topological isomorphism α_G^{-1}. (Observe that G is compact and hence reflexive.) Then, $\psi^\wedge = \gamma^{\wedge\wedge} \circ (\alpha_G^{-1})^\wedge = \gamma^{\wedge\wedge} \circ \alpha_{G^\wedge} : G^\wedge \to D_0^{\wedge\wedge}$ holds, because $(\alpha_G^{-1})^\wedge = \alpha_{G^\wedge}$. Finally, because the discrete group D_0 is reflexive, $\alpha_{D_0}^{-1} \circ \psi^\wedge = \alpha_{D_0}^{-1} \circ \gamma^{\wedge\wedge} \circ \alpha_{G^\wedge} = \gamma$. We are going to identify $D_0^{\wedge\wedge}$ with D_0 via the topological isomorphism $\alpha_{D_0}^{-1}$ and obtain that $\psi^\wedge = \gamma$. Thus, $\psi^\wedge(T_K)$ is an infinite subset of D_0.

By Lemma 9, $\psi^\wedge(T_K)$ is contained in the set $T_{(\psi_\#)^\wedge(K)}$. Because $D_0^\wedge$ is metrizable, $\alpha_{\ell^1(D_0^\wedge)}$ is continuous by Proposition 14. As $(\psi_\#)^\wedge(K)$ is a compact subset of $\ell^1(D_0^\wedge)^\wedge$, the set $T_{(\psi_\#)^\wedge(K)}$ is an equicontinuous and hence compact subset of the discrete group $D_0^{\wedge\wedge}$ by Lemma 8, and hence finite. This contradiction proves that T_K must be finite and hence equicontinuous. Thus, again by Lemma 8, $\alpha_{\ell^1(G)}$ is continuous. □

Lemma 10. *Let (G, τ) be a reflexive group such that $G^\wedge$ has a countable point-separating subgroup. For a compact subset K of $\ell^1(G)^\wedge$ and $m \in \mathbb{N}$, put $T_m = \mu_m^\wedge(K)$ and $T = \bigcup_{m \in \mathbb{N}} T_m$. Then, T is totally bounded.*

Recall that the hypothesis that $G^\wedge$ has a countable point-separating subgroup is fulfilled in case G is second countable or $G^\wedge$ is separable by Propositions 1 and 2.

Proof. Let $D = \{\psi_k : k \in \mathbb{N}\}$ be a countable point-separating subgroup of $G^\wedge$. Because the topology $\sigma(G, D)$ induced by the mapping $G \to \mathbb{T}^D$, $x \mapsto (\psi(x))_{\psi \in D}$ is Hausdorff, we obtain that on every τ-compact subset C of G, the subspace topologies induced by τ and by $\sigma(G, D)$ coincide. Denote by $\mathcal{F}$ the set of all finite subsets of $G^\wedge$ containing 0.

Assume that T is not precompact. By Proposition 3, T is not qc-precompact either, because $G^\wedge$ is lqc. Thus, there exists a compact subset $0 \in C \subseteq G$ such that for every $F \in \mathcal{F}$ we have $T \nsubseteq \mathrm{qc}(F + C^\triangleright)$. Because $\mathrm{qc}(F \cup C^\triangleright) \subseteq \mathrm{qc}(F + C^\triangleright)$, we even have $T \nsubseteq \mathrm{qc}(F \cup C^\triangleright)$ for all $F \in \mathcal{F}$. This is equivalent to

$$\mathrm{qc}(T) \nsubseteq \mathrm{qc}(F \cup C^\triangleright).$$

As $C^\triangleright = \mathrm{qc}(C)^\triangleright$ and because $\mathrm{qc}(C)$ is compact according to Remark 1, we may assume that C is quasi-convex. Hence, we have

$$T^{\triangleleft\triangleright} \overset{(*)}{=} T^{\triangleright\triangleleft} = \mathrm{qc}(T) \nsubseteq \mathrm{qc}(F \cup C^\triangleright) = (F \cup C^\triangleright)^{\triangleright\triangleleft} \overset{(*)}{=} (F \cup C^\triangleright)^{\triangleleft\triangleright} = (F^\triangleleft \cap \underbrace{C^{\triangleright\triangleleft}}_{=\mathrm{qc}(C)=C})^\triangleright = (F^\triangleleft \cap C)^\triangleright$$

The equations marked by $(*)$ hold because G is reflexive, so α_G is surjective. Hence, we have

$$\forall F \in \mathcal{F} \quad T^\triangleleft \nsupseteq F^\triangleleft \cap C. \tag{1}$$

We inductively construct:

(a) A sequence $(\chi^{(n)})_{n \in \mathbb{N}_0}$ in K where $\chi^{(n)} = (\chi_k^{(n)})_{k \in \mathbb{N}}$;
(b) A strictly increasing sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers;
(c) An increasing sequence $(F_n)_{n \in \mathbb{N}}$ in $\mathcal{F}$ such that $\psi_n \in F_n$ for all $n \in \mathbb{N}$;
(d) A sequence $(x_n)_{n \in \mathbb{N}}$ in C such that for all $n \in \mathbb{N}$
  (i) $x_n \in C \cap F_n^\triangleleft$
  (ii) $\chi_{m_n}^{(n)}(x_n) \notin \mathbb{T}_+$;

(iii) $\chi_m^{(j)}(C \cap F_n^{\triangleleft}) \subseteq \mathbb{T}_2$ for all $m \in \mathbb{N}$ and $0 \le j < n$.

Choose $\chi^{(0)} \in K$ arbitrarily.

Assume now that for some $n \in \mathbb{N}_0$ $(\chi^{(0)}, \chi^{(1)}, \ldots, \chi^{(n)})$, $m_1 < \ldots < m_n$, $F_1 \subseteq \ldots \subseteq F_n$ and $(x_1, \ldots, x_n)$ have been constructed, satisfying the above-listed properties.

Because $\chi^{(0)}, \chi^{(1)}, \ldots, \chi^{(n)}$ are continuous, there exists $U \in \mathcal{N}_G(0)$ such that $\chi^{(j)} \in (S_{U+U})^{\triangleright}$ for all $0 \le j \le n$, which implies

$$\chi_m^{(j)}(U) \subseteq \mathbb{T}_2 \quad \text{for all } m \in \mathbb{N} \text{ and } 0 \le j \le n. \tag{2}$$

As a finite union of compact sets, $\bigcup_{k=1}^{m_n} \mu_k^{\wedge}(K)$ is compact. (In case $n = 0$, this set is empty and hence compact.) Because α_G is assumed to be continuous, $\bigcup_{k=1}^{m_n} \mu_k^{\wedge}(K)$ is equicontinuous. Thus, for a suitable neighborhood $W \in \mathcal{N}_G(0)$, we have

$$\psi(x) \in \mathbb{T}_+ \quad \text{for all } x \in W \text{ and } \psi \in \bigcup_{k=1}^{m_n} \mu_k^{\wedge}(K). \tag{3}$$

Because C is compact, the original topology τ coincides with the weak topology $\sigma(G, G^{\wedge})$ on C; hence, there exists a finite subset $F_{n+1} \in \mathcal{F}$ such that

$$0 \in C \cap F_{n+1}^{\triangleleft} \subseteq C \cap U \cap W. \tag{4}$$

WLOG, we may assume that $F_n \cup \{\psi_{n+1}\} \subseteq F_{n+1}$, so that item (c) is fulfilled. Thus, for all $0 \le j \le n$ and $m \in \mathbb{N}$, we have $\chi_m^{(j)}(C \cap F_{n+1}^{\triangleleft}) \subseteq \chi_m^{(j)}(U) \subseteq \mathbb{T}_2$ by Equations (2) and (4) (i.e., (d)(iii) is satisfied).

Because by Equation (1) $T^{\triangleleft} \not\supseteq C \cap F_{n+1}^{\triangleleft}$, there exists $x_{n+1} \in C \cap F_{n+1}^{\triangleleft} \setminus T^{\triangleleft}$. This means that there exist $\chi^{(n+1)} \in K$ and $m_{n+1} \in \mathbb{N}$ such that $\mu_{m_{n+1}}^{\wedge}(\chi^{(n+1)})(x_{n+1}) = \chi_{m_{n+1}}^{(n+1)}(x_{n+1}) \notin \mathbb{T}_+$. As $x_{n+1} \in C \cap F_{n+1}^{\triangleleft} \subseteq W$ by Equation (4), the index m_{n+1} must be strictly larger than m_n, because otherwise $\chi_{m_{n+1}}^{(n+1)}(x_{n+1}) \in \mathbb{T}_+$ would follow from Equation (3). Thus, $\chi^{(n+1)}$, x_{n+1} and m_{n+1} satisfy the properties stated in (a), (b), (d)(i), and (d)(ii). This completes the inductive step.

Let $S := \{0\} \cup \{\mu_{m_n}(x_n) : n \in \mathbb{N}\}$. Applying Proposition 8, we are going to show first that S is a compact subset of $\ell^1(G)$. Of course, $p_m(S)$ consists of at most 2 points, because the sequence $(m_n)_{n \in \mathbb{N}}$ is strictly increasing. It can be easily checked that S is closed in the product topology and by Corollary 2 also in the topology $\Sigma_{\ell^1(G)}$.

Fix $U \in \mathcal{N}_G(0)$. We have to show that there exists $N_U \in \mathbb{N}$ such that for all $(y_n) \in S$ $\sum_{n \ge N_U} \kappa_U(y_n) \le 1$ holds. By the special form of the elements of S, this is equivalent to $\kappa_U(x_n) \le 1$ for all n such that $m_n \ge N_U$. Because C is compact, there exists a finite subset F of D such that $F^{\triangleleft} \cap C \subseteq U \cap C$. By item (c), there exists $n_0 \in \mathbb{N}$ such that $F \subseteq F_{n_0} \subseteq F_n$ for all $n \ge n_0$. Thus, for $n \ge n_0$, we have $x_n \in F_n^{\triangleleft} \cap C \subseteq F_{n_0}^{\triangleleft} \cap C \subseteq U \cap C \subseteq U$ by item (d)(i) and hence $\kappa_U(x_n) \le 1$ for all $n \ge n_0$. Now, choose $N_U := 1 + m_{n_0}$. For n, such that $m_n \ge N_U$, we have (because (m_n) is strictly increasing) $n > n_0$ and hence $\kappa_U(x_n) \le 1$. This shows that S is compact.

Let us prove that

$$\forall k_1, k_2 \in \mathbb{N} \quad k_1 \ne k_2 \implies \chi^{(k_2)} - \chi^{(k_1)} \notin (S+S)^{\triangleright} : \tag{5}$$

WLOG, we may assume that $k_1 < k_2$. Because $\chi_m^{(k_1)}(F_{k_2}^{\triangleleft} \cap C) \subseteq \mathbb{T}_2$ for all $m \in \mathbb{N}$ by item (d)(iii) and $x_{k_2} \in F_{k_2}^{\triangleleft} \cap C$ by item (d)(i) and $\chi_{m_{k_2}}^{(k_2)}(x_{k_2}) \notin \mathbb{T}_+$ by item (d)(ii), this implies

$$(\chi^{(k_2)} - \chi^{(k_1)})(\mu_{m_{k_2}}(x_{k_2})) = \underbrace{\chi_{m_{k_2}}^{(k_2)}(x_{k_2})}_{\notin \mathbb{T}_+} - \underbrace{\chi_{m_{k_2}}^{(k_1)}(x_{k_2})}_{\in \mathbb{T}_2} \notin \mathbb{T}_2. \tag{6}$$

Because $\psi \in (S+S)^{\triangleright}$ if and only if $\psi(S) \subseteq \mathbb{T}_2$, Equation (5) is an immediate consequence of Equation (6).

Because by item (a) $\chi^{(n)} \in K$ for all $n \in \mathbb{N}$, Equation (5) implies that K is not totally bounded. This contradiction implies that T is precompact, whence totally bounded. □

Theorem 2. *Let G be a reflexive group which has the following additional properties:*

1. *$G^\wedge$ has a countable point-separating subgroup.*
2. *$G^\wedge$ is complete.*

Then, $\alpha_{\ell^1(G)}$ is continuous.

Proof. Let K be a compact subset of $\ell^1(G)^\wedge$. By Lemma 10, $T = \bigcup_{m\in\mathbb{N}} \mu_m^\wedge(K)$ is totally bounded. Hence, its closure $T_K = \overline{T}$ is also totally bounded and complete by the assumption that $G^\wedge$ is complete. Thus, T_K is a compact subset of $G^\wedge$. Because α_G is continuous, the compact subset T_K of $G^\wedge$ is equicontinuous. By Lemma 8, the canonical homomorphism $\alpha_{\ell^1(G)}$ is continuous. □

2.3. The Second Character Group of $\ell^1(G)$

In this section, we study the second character group of $\ell^1(G)$ and show that each element $\eta \in \ell^1(G)^{\wedge\wedge}$ can be identified with a sequence (η_n) in $G^{\wedge\wedge}$. Next, we study necessary and sufficient conditions for G such that (η_n) belongs to $\ell^1(G^{\wedge\wedge})$. As a consequence, it is possible to prove the main theorems of this paper, asserting that $\ell^1(G)$ is reflexive if G is metrizable and reflexive or an LCA group.

Proposition 15. *For every abelian topological group G, the mapping*

$$\Psi = (p_n^{\wedge\wedge}) : \ell^1(G)^{\wedge\wedge} \to (G^{\wedge\wedge})^{\mathbb{N}},\ \eta \mapsto (p_n^{\wedge\wedge}(\eta))_n$$

is a continuous monomorphism. For all $(x_n) \in \ell^1(G)$, $\quad \Psi \circ \alpha_{\ell^1(G)}(x_n) = (\alpha_G(x_n))$ holds. If α_G is continuous, then

$$\Psi \circ \alpha_{\ell^1(G)} = (\alpha_G)_\#.$$

Proof. It is clear that Ψ is a continuous homomorphism. Fix $\eta \in \ell^1(G)^{\wedge\wedge}$ with $p_n^{\wedge\wedge}(\eta) = 0$ for all $n \in \mathbb{N}$. Then, $\eta \circ p_n^\wedge$ is trivial for all $n \in \mathbb{N}$. Hence, η vanishes on the subgroup $G^{\wedge(\mathbb{N})}$ of $\ell^1(G)^\wedge$, which is dense by Proposition 13. This implies that η is trivial. Because Ψ is a homomorphism, we conclude that Ψ is injective.

Observe that for $(x_n) \in \ell^1(G)$, we have $\Psi(\alpha_{\ell^1(G)}((x_n))) = (p_m^{\wedge\wedge}(\alpha_{\ell^1(G)}((x_n))))_{m\in\mathbb{N}} = (\alpha_{\ell^1(G)}((x_n)) \circ p_m^\wedge)_{m\in\mathbb{N}}$. Further, for $\chi \in G^\wedge$, we have

$$\alpha_{\ell^1(G)}((x_n))(p_m^\wedge(\chi)) = \alpha_{\ell^1(G)}((x_n))(\chi \circ p_m) = (\chi \circ p_m)((x_n)) = \chi(x_m) = \alpha_G(x_m)(\chi).$$

Combining these observations yields $\Psi(\alpha_{\ell^1(G)}((x_n))) = (\alpha_G(x_n))$. If α_G is continuous (and hence $(\alpha_G)_\#$ is well-defined), then $\Psi \circ \alpha_{\ell^1(G)} = (\alpha_G)_\#$. □

For the remainder, we identify an element $\eta \in \ell^1(G)^{\wedge\wedge}$ with the sequence $\Psi(\eta) = (\eta_n)$ where $\eta_n = p_n^{\wedge\wedge}(\eta)$.

Proposition 16. *Let G be an abelian Hausdorff group and let Ψ be as in Proposition 15.*

(a) *If α_G is continuous, then $\Psi(\ell^1(G)^{\wedge\wedge}) \subseteq \ell^1(G^{\wedge\wedge})$.*
(b) *If α_G is surjective and G is lqc, then $\Psi(\ell^1(G)^{\wedge\wedge}) \supseteq \ell^1(G^{\wedge\wedge})$.*

In particular, if G is reflexive, then $\Psi(\ell^1(G)^{\wedge\wedge}) = \ell^1(G^{\wedge\wedge})$.

Proof. (a) Assume first that α_G is continuous. Fix $\eta \in \ell^1(G)^{\wedge\wedge}$ and let $\eta_n := p_n^{\wedge\wedge}(\eta)$. Because η is a continuous character of $\ell^1(G)^\wedge$, there exists—by definition of the compact-open topology—a compact subset $K \subseteq \ell^1(G)$ such that $\eta \in K^{\triangleright\triangleright}$. In order to show that $\Psi(\eta) = (\eta_n) \in \ell^1(G^{\wedge\wedge})$, we fix a compact subset C of $G^\wedge$ and wish to prove that $\sum_{n\in\mathbb{N}} \kappa_{C^\triangleright}(\eta_n) < \infty$. Because, by assumption, α_G is continuous, there exists a neighborhood $U \in \mathcal{N}_G(0)$ such that $C \subseteq U^\triangleright$, whence $C^\triangleright \supseteq U^{\triangleright\triangleright}$. Because $K \subseteq \ell^1(G)$ is compact, there exists by Proposition 8 $N_U \in \mathbb{N}$ such that $K \subseteq S_{N_U,U}$. Hence, $\eta \in K^{\triangleright\triangleright} \subseteq (S_{N_U,U})^{\triangleright\triangleright} = \left(\{0\}^{\{1,\dots,N_U-1\}} \times (U^\triangleright)^{\mathbb{N}\setminus\{1,\dots,N_U-1\}}\right)^\triangleright$.

Because $p_n^\wedge(U^\triangleright) \subseteq \{0\}^{\{1,\dots,N_U-1\}} \times (U^\triangleright)^{\mathbb{N}\setminus\{1,\dots,N_U-1\}} =: M$ for all $n \geq N_U$, we obtain $\eta_n(U^\triangleright) = p_n^{\wedge\wedge}(\eta)(U^\triangleright) = \eta(p_n^\wedge(U^\triangleright)) \subseteq \mathbb{T}_+$, which implies that $\eta_n \in U^{\triangleright\triangleright}$ for all $n \geq N_U$. We want to show that

$$\sum_{n\geq N_U} \kappa_{U^{\triangleright\triangleright}}(\eta_n) < 2.$$

Assume that this does not hold and let $\nu \geq N_U$ be minimal with $\sum_{n=N_U}^{\nu} \kappa_{U^{\triangleright\triangleright}}(\eta_n) \geq 2$.

Let $N = \{n \in \mathbb{N} : \ N_U \leq n \leq \nu \text{ and } \kappa_{U^{\triangleright\triangleright}}(\eta_n) > 0\}$. For $n \in N$, we have $\kappa_{U^{\triangleright\triangleright}}(\eta_n) = \frac{1}{m_n}$ for a suitable natural number m_n, because $\eta_n \in U^{\triangleright\triangleright}$. Next, for $n \in N$, choose $\chi_n \in U^\triangleright$ such that $\eta_n(\chi_n) = t_n + \mathbb{Z}$ where $\dfrac{1}{4(m_n+1)} < t_n \leq \dfrac{1}{4m_n}$. For $k \in \mathbb{N} \setminus N$, put $\chi_k = 0$. Then, $\chi = (\chi_n)_{n\in\mathbb{N}} \in M$ and hence $\eta(\chi) = (\sum_{n\in N} t_n) + \mathbb{Z} \in \mathbb{T}_+$. Further,

$$\frac{1}{4} \leq \sum_{n\in N} \frac{1}{8m_n} \leq \sum_{n\in N} \frac{1}{4(m_n+1)} < \sum_{n\in N} t_n \leq \sum_{n\in N} \frac{1}{4m_n}$$

holds. Because ν was chosen minimal, we conclude that $\displaystyle\sum_{n\in N} \frac{1}{4m_n} = \sum_{n\in N, n\neq\nu} \frac{1}{4m_n} + \frac{1}{4m_\nu} < \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. This yields $\eta(\chi) = \sum_{n\in N} t_n + \mathbb{Z} \notin \mathbb{T}_+$ and gives the desired contradiction.

Now, it easily follows that $\sum_{n\in\mathbb{N}} \kappa_{C^\triangleright}(\eta_n) \leq \sum_{n\in\mathbb{N}} \kappa_{U^{\triangleright\triangleright}}(\eta_n) < \infty$.

(b) Assume now that α_G is surjective and G is lqc. Then, $\alpha_G^{-1} : G^{\wedge\wedge} \to G$ is continuous. We show that the composition

$$\ell^1(G^{\wedge\wedge}) \overset{(\alpha_G^{-1})_\#}{\longrightarrow} \ell^1(G) \overset{\alpha_{\ell^1(G)}}{\longrightarrow} \ell^1(G)^{\wedge\wedge} \overset{\Psi}{\longrightarrow} (G^{\wedge\wedge})^{\mathbb{N}}$$

is the identity on $\ell^1(G^{\wedge\wedge})$. Fix $(\eta_n) \in \ell^1(G^{\wedge\wedge})$. Because α_G is surjective, there is a sequence $(x_n) \in G^{\mathbb{N}}$ such that $\alpha_G(x_n) = \eta_n$ for all $n \in \mathbb{N}$. Applying Proposition 15, we obtain

$$\Psi \circ \alpha_{\ell^1(G)} \circ (\alpha_G^{-1})_\#(\eta_n) = \Psi \circ \alpha_{\ell^1(G)}(x_n) = (\alpha_G(x_n)) = (\eta_n)$$

This shows $\ell^1(G^{\wedge\wedge}) \subseteq \Psi(\ell^1(G)^{\wedge\wedge})$. The final statement is an immediate consequence of (a) and (b). □

Corollary 5. *Let G be a reflexive group. Then,* $\alpha_{\ell^1(G)}$ *is surjective.*

Proof. By Proposition 15, $\Psi \circ \alpha_{\ell^1(G)} = (\alpha_G)_\#$ holds. Because α_G is a topological isomorphism, Proposition 6 implies that $(\alpha_G)_\# : \ell^1(G) \to \ell^1(G^{\wedge\wedge})$ is a topological isomorphism. Because $\Psi : \ell^1(G)^{\wedge\wedge} \to \ell^1(G^{\wedge\wedge})$ is an isomorphism by Proposition 16, we obtain that $\alpha_{\ell^1(G)} = \Psi^{-1} \circ (\alpha_G)_\#$ is an isomorphism, whence surjective. □

Theorem 3. *Let G be a reflexive group. Then,* $\alpha_{\ell^1(G)}$ *is an open isomorphism. If*

(a) *G is metrizable or*

(b) $G^\wedge$ *is complete and has a countable point-separating subgroup,*

then $\ell^1(G)$ *is reflexive.*

Proof. If G is reflexive, then G is an lqc Hausdorff group. According to Propositions 5 (c) and 12, $\ell^1(G)$ is lqc and Hausdorff as well and hence $\alpha_{\ell^1(G)}$ is an open isomorphism by Corollary 5.

It remains to show that $\alpha_{\ell^1(G)}$ is continuous if (a) or (b) holds. In case (a), it is a consequence of Proposition 14. In case (b), it is a consequence of Theorem 2. □

Theorem 4. *For every LCA group G, the group $\ell^1(G)$ is reflexive.*

Proof. Let G be an LCA group. By the structure theorem for LCA groups, G has an open subgroup H topologically isomorphic to $\mathbb{R}^n \times K$ where $n \in \mathbb{N}_0$ and K is a compact abelian group. By Proposition 6 (a), $\ell^1(H)$ can be considered to be an open subgroup of $\ell^1(G)$. Because by Theorem (2.3), in [23], a group is reflexive if and only if it has an open reflexive subgroup, it is sufficient to show that $\ell^1(H)$ is reflexive, or, by Proposition 6 (b), that $\ell^1(\mathbb{R})^n \times \ell^1(K)$ is reflexive. The group $\ell^1(\mathbb{R})$ is reflexive by Theorem 3.

By Corollary 5, $\alpha_{\ell^1(K)}$ is surjective, and by Theorem 1, $\alpha_{\ell^1(K)}$ is continuous. As the group $\ell^1(K)$ is lqc and Hausdorff by Propositions 12 and 5 (c), the assertion follows. □

Recall that a subgroup H of an abelian topological group G is **dually closed** if for every $x \in G \setminus H$ there exists a continuous character $\chi \in G^\wedge$ such that $\chi(H) = \{0\}$ and $\chi(x) \neq 0$. The subgroup H is **dually embedded** if every continuous character of H can be extended to a continuous character of G; in other words, the dual homomorphism of the canonical embedding $\iota : H \to G$ is surjective.

It is straightforward to check that $\mu_1(G)$ is a dually closed and dually embedded subgroup of $\ell^1(G)$ provided that G is an MAP group. We are going to apply the following result of Noble:

Proposition 17 ([24], Theorem 3.1)**.** *Let G be an abelian Hausdorff group such that α_G is an open isomorphism. If H is a dually closed and dually embedded subgroup of G, then also α_H is an open isomorphism.*

Theorem 5. *If $\ell^1(G)$ is Pontryagin reflexive, then so is G.*

Proof. Assume that $\ell^1(G)$ is reflexive. Because μ_1 is an embedding and $\mu_1(G)$ is a dually closed and dually embedded subgroup of $\ell^1(G)$, we obtain from Proposition 17 that α_G is an open isomorphism. Because $p_1 : \ell^1(G) \to G$ is a projection (Proposition 5 (b), Lemma (14.7) in [13]) implies α_G is continuous. □

Corollary 6. *Let G be a metrizable group. Then, G is reflexive if and only if $\ell^1(G)$ is reflexive.*

Proof. If G is a metrizable reflexive group, then $\ell^1(G)$ is reflexive by Theorem 3 (a). If $\ell^1(G)$ is reflexive, then G is reflexive by Theorem 5. □

2.4. The Schur Property of $\ell^1(G)$

A normed space V is said to have the **Schur property** if a sequence (x_n) converges to 0 provided that $(f(x_n))$ converges to 0 for every continuous linear form. In this section, we first recall the definition of the Schur property for MAP groups; afterward, having in mind that $\ell^1(\mathbb{R})$ has the Schur property, we prove that $\ell^1(G)$ has the Schur property for groups if and only if G has this property (Theorem 6).

Definition 4. *For a topological group (G, τ), denote by τ^+ the topology on G induced by $G \to \mathbb{T}^{G^\wedge}$, $x \mapsto (\chi(x))_{\chi \in G^\wedge}$. The topology τ^+ is called **weak topology**.*

The weak topology τ^+ is Hausdorff if and only if the characters of G separate the points. A subset C of G is called **weakly compact** if it is compact with respect to τ^+.

Definition 5. *A MAP group (G, τ) is said to have the* **Schur property** *if every τ^+-convergent sequence converges in τ.*

Theorem 6. *Let G be an lqc Hausdorff group. Then, G has the Schur property if and only if $\ell^1(G)$ has the* **Schur property**.

Proof. Assume first that G has the Schur property. Let $(x^{(m)})_{m\in\mathbb{N}}$ be a weakly convergent sequence in $\ell^1(G)$, where $x^{(m)} = (x_n^{(m)})_{n\in\mathbb{N}}$. WLOG, we may assume that $(x^{(m)})_{m\in\mathbb{N}}$ converges to 0. For $\chi \in G^\wedge$ and $n \in \mathbb{N}$, the sequence $(p_n^\wedge(\chi)(x^{(m)}))_{m\in\mathbb{N}} = (\chi(x_n^{(m)}))_{m\in\mathbb{N}}$ converges to 0 in $\mathbb{T}$. The assumption that G has the Schur property implies that $(x_n^{(m)})_{m\in\mathbb{N}}$ converges in the original topology of G to 0 for every $n \in \mathbb{N}$.

Assume that the sequence $(x^{(m)})_{m\in\mathbb{N}}$ is not convergent in the original topology. This means that there exists a quasi-convex neighborhood $U \in \mathcal{N}_G(0)$ such that for infinitely many $m \in \mathbb{N}$, $x^{(m)} \notin S_U$. After passing to a subsequence, we may assume that $x^{(m)} \notin S_U$ for all $m \in \mathbb{N}$. In order to obtain a contradiction, we are going to inductively construct strictly increasing sequences (m_k), (n_k) and (N_k) of natural numbers and a sequence $(\chi_k) \in (U^\triangleright)^{\mathbb{N}}$ such that

(a) $N_k \le n_k < N_{k+1}$ for all $k \in \mathbb{N}$;

(b) $\sum_{n=1}^{n_k} \chi_n(x_n^{(m_k)}) \notin \mathbb{T}_3$, and $\sum_{n>n_k} \kappa_U(x_n^{(m_k)}) < \frac{1}{8}$ for all $k \in \mathbb{N}$.

Let $m_1 = 1$. Because $x^{(1)} \notin S_U$, there exists $N_1 \in \mathbb{N}$ minimal such that $\sum_{n=1}^{N_1} \kappa_U(x_n^{(1)}) > 1$. Further, there is $n_1 \ge N_1$ such that $\sum_{n>n_1} \kappa_U(x_n^{(1)}) < \frac{1}{8}$. If for some $1 \le n \le N_1$ the element $x_n^{(1)} \notin U$, then we choose $\chi_n \in U^\triangleright$ such that $\chi_n(x_n^{(1)}) \notin \mathbb{T}_+$ and for $j \in \{1 \ldots, n_1\} \setminus \{n\}$ we put $\chi_j = 0$. Otherwise, let $F = \{n :\ 1 \le n \le N_1,\ \kappa_U(x_n^{(1)}) > 0\}$. Fix $n \in F$. Because $x_n^{(1)} \in U$, we have $0 < \kappa_U(x_n^{(1)}) \le 1$; hence, there exists $l_n \in \mathbb{N}$ such that $\frac{1}{l_n} = \kappa_U(x_n^{(1)})$ for some $l_n \in \mathbb{N}$. The minimality of N_1 implies that $N_1 \in F$ and $\sum_{n\in F} \kappa_U(x_n^{(1)}) = \sum_{n=1}^{N_1} \kappa_U(x_n^{(1)}) \le 2$. For every $n \in F$, choose $\chi_n \in U^\triangleright$ such that $\chi_n(x_n^{(1)}) = t_n + \mathbb{Z}$ for some $t_n \in]\frac{1}{4(l_n+1)}, \frac{1}{4l_n}]$. Because

$$\frac{1}{8} < \frac{1}{4}\sum_{n\in F}\frac{1}{2l_n} \le \frac{1}{4}\sum_{n\in F}\frac{1}{l_n+1} < \sum_{n\in F} t_n \le \frac{1}{4}\sum_{n\in F}\frac{1}{l_n} \le \frac{1}{2},$$

we obtain $\sum_{n\in F} \chi_n(x_n^{(1)}) \notin \mathbb{T}_3$. For $n \in \{1, \ldots, n_1\} \setminus F$, we put $\chi_j = 0$. Then, conditions (a) and (b) are satisfied for $k = 1$.

Assume now that for some $k \in \mathbb{N}$, $m_1, \ldots, m_k$, $N_1, \ldots, N_k$, $n_1, \ldots, n_k$ and $\chi_1, \ldots, \chi_{n_k} \in U^\triangleright$ have been constructed such that (a) and (b) hold. By the initial observation, $(\kappa_U(x_n^{(m)}))_{m\in\mathbb{N}}$ converges to 0 for every $n \in \mathbb{N}$; hence, there exists $m_{k+1} > m_k$ such that $\sum_{n=1}^{n_k} \kappa_U(x_n^{(m_{k+1})}) < \frac{1}{8}$. Because $x^{(m_{k+1})} \notin S_U$, there exists a minimal $N_{k+1} > n_k$ such that $\sum_{n=n_k+1}^{N_{k+1}} \kappa_U(x_n^{(m_{k+1})}) > \frac{7}{8}$. We choose $n_{k+1} \ge N_{k+1}$ such that $\sum_{n>n_{k+1}} \kappa_U(x_n^{(m_{k+1})}) < \frac{1}{8}$.

Fix $s \in [-\frac{1}{4}, \frac{1}{4}]$ such that $\sum_{n=1}^{n_k} \chi_n(x_n^{(m_{k+1})}) = s + \mathbb{Z}$. If $s + \mathbb{Z} \notin \mathbb{T}_3$, we define $\chi_j = 0$ for all $n_k < j \le n_{k+1}$.

Assume now that $s + \mathbb{Z} \in \mathbb{T}_3$ and that for some $n_k < j \le N_{k+1}$, the element $x_j^{(m_{k+1})} \notin U$. Then, we choose $\chi_j \in U^\triangleright$ such that $\chi_j(x_j^{(m_{k+1})}) \notin \mathbb{T}_+$ and then $\sum_{n=1}^{n_k} \chi_n(x_n^{(m_{k+1})}) + \chi_j(x_j^{(m_{k+1})}) \notin \mathbb{T}_3$. Further, for all $n \in \{n_k + 1, \ldots, n_{k+1}\} \setminus \{j\}$, we put $\chi_n = 0$.

Finally, assume that $s + \mathbb{Z} \in \mathbb{T}_3$ and $x_j^{(m_{k+1})} \in U$ for all $n_k < j \le N_{k+1}$. The minimality of N_{k+1} implies $\sum_{n=n_k+1}^{N_{k+1}} \kappa_U(x_n^{(m_{k+1})}) \le \frac{15}{8}$. Let $F = \{n \in \mathbb{N} :\ n_k < n \le N_{k+1}$ and $\kappa_U(x_n^{(m_{k+1})}) > 0\}$. For $n \in F$, $\kappa_U(x_n^{(m_{k+1})}) = \frac{1}{l_n}$ for suitable $l_n \in \mathbb{N}$. Hence, there exist $\chi_n \in U^{\triangleright}$ such that $\chi_n(x_n^{(m_{k+1})}) = t_n + \mathbb{Z}$ where $|t_n| \in [\frac{1}{4(l_n+1)}, \frac{1}{4l_n}]$. Because $\sum_{n\in F} \frac{1}{l_n} = \sum_{n=n_k+1}^{N_{k+1}} \kappa_U(x_n^{(m_{k+1})}) \in [\frac{7}{8}, \frac{15}{8}]$, we obtain

$$\frac{1}{12} < \frac{1}{4} \cdot \frac{7}{16} < \frac{1}{4} \sum_{n\in F} \frac{1}{2l_n} \le \frac{1}{4} \sum_{n\in F} \frac{1}{l_n+1} < \sum_{n\in F} |t_n| \le \frac{1}{4} \sum_{n\in F} \frac{1}{l_n} \le \frac{15}{32} < \frac{1}{2}.$$

For $n \in \{n_k+1, \dots, n_{k+1}\} \setminus F$, we put $\chi_n = 0$.
If $s \in [0, \frac{1}{12}]$, then $\sum_{n=1}^{n_{k+1}} \chi_n(x^{(m_{k+1})}) \notin \mathbb{T}_3$.
If $s \in [-\frac{1}{12}, 0]$, replace χ_n by $-\chi_n$ for $n \in F$ such that $\sum_{n=1}^{n_{k+1}} \chi_n(x^{(m_{k+1})}) \notin \mathbb{T}_3$ holds.

We have constructed the subsequence $(x^{(m_k)})_k$ of $(x^{(m)})_{m\in\mathbb{N}}$ and the character $\chi = (\chi_n) \in (U^{\triangleright})^{\mathbb{N}} \subseteq \ell^1(G)^{\wedge}$. We obtain

$$\chi(x^{(m_k)}) = \sum_{n=1}^{\infty} \chi_n(x_n^{(m_k)}) = \underbrace{\sum_{n=1}^{n_k} \chi_n(x_n^{(m_k)})}_{\notin \mathbb{T}_3} + \underbrace{\sum_{n>n_k} \chi_n(x_n^{(m_k)})}_{\in \mathbb{T}_8 \subseteq \mathbb{T}_6} \notin \mathbb{T}_6,$$

because $\sum_{n>n_k} \kappa_U(x_n^{(m_k)}) < \frac{1}{8}$ and $\chi_n \in U^{\triangleright}$ for all $n \in \mathbb{N}$ (cf. Lemma 1 (c)). This shows $(\chi(x^{(m_k)}))$ does not converge to 0 and gives the desired contradiction.

Because $\mu_1 : G \to \ell^1(G)$ is an embedding, and the class of groups having the Schur property is closed under taking subgroups, the result follows. □

An MAP group (G, τ) is said to have the **Glicksberg property** if every weakly compact subset is compact.

Every group which has the Glicksberg property also has the Schur property. This definition honors Glicksberg who proved that every LCA group has this property. Because then many other examples of groups having the Glicksberg property were established, for example, it is a consequence of the Eberlein–Šmulian theorem [25] and the Schur theorem [12] that $\ell^1(\mathbb{R})$ has the Glicksberg property. Further, the class of groups having the Glicksberg property is stable under taking subgroups and products. In particular, if for an MAP group G, the sequence group $\ell^1(G)$ has the Glicksberg property, then also G has the Glicksberg property (as G can be embedded in $\ell^1(G)$). However, the converse implication is not clear, see Question 6.

2.5. Schwartz Groups

In this final section, we show that only under very restrictive conditions is the sequence group $\ell^1(G)$ a Schwartz group, a class of groups introduced in [14] generalizing Schwartz topological vector spaces (see ([8], p. 201) for the definition).

Notation 1. *Let G be an abelian group. For a symmetric subset U of G containing* 0*, one defines*

$$(1/n)U := \{x \in G : \; jx \in U \; \forall \, 1 \le j \le n\} = \kappa_U^{-1}([0, \frac{1}{n}]).$$

Observe that if U is a symmetric neighborhood of 0 in a topological group, then also $(1/n)U$ is a neighborhood of 0 for every $n \in \mathbb{N}$.

Definition 6 ([14])**.** *An abelian topological group G is called a* **Schwartz group** *if for every symmetric neighborhood $U \in \mathcal{N}_G(0)$ there exists a symmetric neighborhood $V \in \mathcal{N}_G(0)$ and a sequence (F_n) of finite subsets of G such that $V \subseteq (1/n)U + F_n$ for all $n \in \mathbb{N}$.*

The class of Schwartz groups is closed under taking subgroups, arbitrary products, and Hausdorff quotients ([14], 3.6). Every lqc Schwartz group has the Glicksberg property, in particular, the Schur property ([19]). A topological vector space is a Schwartz space if and only if the additive group is a Schwartz group ([14], 4.2).

Definition 7 (Tarieladze)**.** *A symmetric subset U containing 0 of an abelian group G is called a* **GTG-set** *(***G***roup* **T***opology* **G***enerating set), if the sets $((1/n)U)_{n\in\mathbb{N}}$ form a neighborhood base at 0 of a not-necessarily Hausdorff group topology. An abelian topological group G is called a* **locally GTG-group** *if it has a neighborhood base at 0 consisting of GTG-sets.*

The following two statements follow straightforward from the definitions. For a GTG-set U in G, the intersection $U_\infty := \bigcap_{n\in\mathbb{N}}(1/n)U$ is a subgroup of G. It is a direct consequence of Lemma 1 (c) that every lqc group is locally GTG.

By ([8], 10.4.3), every bounded subset of a Schwartz space is precompact. Hence, a normed space is a Schwartz space if and only if it is finite-dimensional. Thus, $\ell^1(\mathbb{R})$ is not a Schwartz space and hence no Schwartz group either. Conversely, if $\ell^1(G)$ is a Schwartz group, then necessarily G is a Schwartz group, because G embeds in $\ell^1(G)$. However, the example $\mathbb{R} \to \ell^1(\mathbb{R})$ shows that this property is not sufficient.

Theorem 7. *For a locally GTG-group G, the following assertions are equivalent:*

(a) *$\ell^1(G)$ is a Schwartz group.*
(b) *G is linearly topologized.*

Proof. (a) $\Longrightarrow$ (b) Let $\mathcal{N}_0$ be a neighborhood base at $0 \in G$ consisting of GTG-sets. Fix a neighborhood $U \in \mathcal{N}_0$. There is neighborhood $W \in \mathcal{N}_0$ such that $W + W + W + W \subseteq U$. Because $\ell^1(G)$ is a Schwartz group by assumption, there is a sequence $(\widetilde{F}_n)$ of finite subsets in $\ell^1(G)$ and a neighborhood $V \in \mathcal{N}_0$ such that $S_V \subseteq \widetilde{F}_n + (1/n)S_W$. Because $G^{(\mathbb{N})}$ is dense in $\ell^1(G)$ (Proposition 7), there exists for every $n \in \mathbb{N}$ a finite subset $F_n \subseteq G^{(\mathbb{N})}$ such that $\widetilde{F}_n \subseteq F_n + (1/n)S_W$. Thus, we have $S_V \subseteq F_n + (1/n)S_W + (1/n)S_W$. We are going to show that $(1/n)S_W + (1/n)S_W \subseteq (1/n)S_U$. Therefore, we fix $(x_n), (y_n) \in (1/n)S_W$. For $1 \leq j \leq n$, we obtain by Lemma 1 (a) and (b) and Proposition 4

$$\textstyle\sum_{n\in\mathbb{N}} \kappa_U(j(x_n + y_n)) \leq \sum_{n\in\mathbb{N}} \kappa_{W+W+W+W}(jx_n + jy_n) \leq$$

$$\leq \textstyle\sum_{n\in\mathbb{N}} \kappa_{W+W}(jx_n) + \sum_{n\in\mathbb{N}} \kappa_{W+W}(jy_n) \leq \frac{1}{2}\sum_{n\in\mathbb{N}} \kappa_W(jx_n) + \frac{1}{2}\sum_{n\in\mathbb{N}} \kappa_W(jy_n) \leq 1.$$

It follows that

$$S_V \subseteq F_n + (1/n)S_U$$

for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Because F_n is a finite subset of $G^{(\mathbb{N})}$, we can choose $N_n \in \mathbb{N}$ such that $p_m(F_n) = \{0\}$ for all $m \geq N_n$. For $m \geq N_n$, $\mu_m(V) \subseteq S_V \subseteq F_n + (1/n)S_U$, or equivalently because $p_m(F_n) = \{0\}$, $\mu_m(V) \subseteq (1/n)S_U$. Thus, for all $x \in V$ and $1 \leq j \leq n$, we have $j\mu_m(x) \in S_U$ which is equivalent to $\kappa_U(jx) \leq 1$ for all $1 \leq j \leq n$. Thus, $jx \in U$ for all $1 \leq j \leq n$, which means that $x \in (1/n)U$. We have shown that $V \subseteq (1/n)U$ for all $n \in \mathbb{N}$. This yields $V \subseteq \bigcap_{n\in\mathbb{N}}(1/n)U = U_\infty$. As a consequence, U_∞ is an open subgroup of G. Hence, $(U_\infty)_{U\in\mathcal{N}_0}$ is a neighborhood base at 0 for G consisting of open subgroups. This means that G is linearly topologized.

(b) $\Longrightarrow$ (a) If G is linearly topologized, so is $\ell^1(G)$ by Proposition 5. It is obvious that every linearly topologized group is a Schwartz group, so the assertion follows. □

The class of nuclear groups was introduced by Banaszczyk in [13]. Nuclear groups include all Schwartz groups ([14], 4.3), all LCA groups ([13], 7.10), and all nuclear locally convex vector spaces ([13], 7.4). This class of groups is closed under taking products, subgroups, and Hausdorff quotient groups ([13], 7.5 and 7.6), and every nuclear group is lqc ([13], 8.5). Because every Hausdorff linearly topologized group can be embedded into a product of discrete groups, every linearly topologized Hausdorff group is nuclear.

Theorem 8. *For an lqc Hausdorff group G, the following are equivalent:*

(a) *$\ell^1(G)$ is a nuclear group;*
(b) *$\ell^1(G)$ is a Schwartz group;*
(c) *G is linearly topologized.*

Proof.

(a)$\Longrightarrow$ (b) holds, because every nuclear group is a Schwartz group ([14], 4.3).
(b)$\Longrightarrow$ (c) is a consequence of Theorem 7.
(c)$\Longrightarrow$ (a) If G is linearly topologized, so is $\ell^1(G)$ by Proposition 5. Hence, $\ell^1(G)$ is a nuclear group.
□

3. Open Questions

In this final chapter, we gather some open questions concerning sequence groups.

Question 1. *Characterize those abelian Hausdorff groups for which $\ell^1(G) = G^{(\mathbb{N})}$ holds. In particular, is it possible that $c_0(G) \neq G^{(\mathbb{N})} = \ell^1(G)$?*

A dense subgroup H of an abelian Hausdorff group G is said to **determine** G if the dual homomorphism $\iota^\wedge : G^\wedge \to H^\wedge$ of the natural embedding ι is a topological isomorphism. It was shown in [26] and in ([27], 4.10) that every metrizable abelian group determines its completion.

Question 2. *Assume that H is a dense subgroup of the abelian topological group G which determines G. Does $\ell^1(H)$ determine $\ell^1(G)$?*

It was shown (Theorem 5) that if $\ell^1(G)$ is reflexive, then G must be reflexive. Conversely, if G is reflexive, then $\alpha_{\ell^1(G)}$ is an open isomorphism (Theorem 3). However, we do not know if $\alpha_{\ell^1(G)}$ is continuous.

Question 3. *Let G be an abelian Hausdorff group such that α_G is continuous. Is it true that $\alpha_{\ell^1(G)}$ is continuous?*

Or, a bit weaker,

Question 4. *Let G be a reflexive group. Is it true that $\alpha_{\ell^1(G)}$ is continuous (and hence $\ell^1(G)$ is reflexive)?*

It was shown in [4] that for every LCA group G, the group of null-sequences $c_0(G)$ is reflexive.

Question 5. *Is it true that G is a reflexive group if and only if $c_0(G)$ is reflexive?*

Question 6. *Assume that G has the Glicksberg property. Does $\ell^1(G)$ have the Glicksberg property?*

Question 7. *What can be said about the groups*

$$\ell^p(G) = \{(x_n) \in G^{\mathbb{N}} : \sum_{n\in\mathbb{N}} (\kappa_U(x_n))^p < \infty \ \forall U \in \mathcal{N}_G(0)\}$$

for $1 \leq p < \infty$? In particular, what are the properties of $\ell^2(G)$?

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References

1. Koethe, G. *Topological Vector Spaces I*; Springer: Berlin/Heidelberg, Germany, 1983.
2. Chasco, M.J.; Martín-Peinador, E.; Tarieladze, V. On Mackey Topology for groups. *Stud. Math.* **1999**, *132*, 257–284. [CrossRef]
3. Dikranjan, D.; Martín-Peinador, E.; Tarieladze, V. A class of metrizable locally quasi-convex groups which are not Mackey. *Forum Math.* **2019**, *26*, 723–757.
4. Gabriyelyan, S. On reflexivity of the group of null sequences valued in an Abelian topological group. *J. Pure Appl. Algebra* **2015**, *219*, 2989–3008. [CrossRef]
5. Gabriyelyan, S. Groups of quasi-invariance and the Pontryagin duality. *Topol. Appl.* **2010**, *157*, 2786–2802. [CrossRef]
6. Gabriyelyan, S. On characterized subroups of Abelian topological groups X and the group of X-vaued null sequences. *Comment. Math. Univ. Carol.* **2014**, *55*, 73–99.
7. Gabriyelyan, S. Topological properties of the group of the null sequences valued in an Abelian topological group. *Topol. Appl.* **2016**, *207*, 136–155. [CrossRef]
8. Jarchow, H. *Locally Convex Spaces*; B.G.Teubner Verlag: Stuttgart, Germany, 1981.
9. Pietsch, A. *Nuclear Locally Convex Spaces*; Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 66; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 1972.
10. Domínguez, X.; Tarieladze, V. GP-Nuclear Groups. Nuclear groups and Lie Groups (Madrid, 1999). *Res. Exp. Math.* **2001**, *24*, 127–161.
11. Domínguez, X.; Tarieladze, V. Nuclear and GP-nuclear groups. *Acta Math. Hungar.* **2000**, *88*, 301–322. [CrossRef]
12. Jameson, G.J.O. *Topology and Normed Spaces*; Chapman and Hall: London, UK, 1974.
13. Banaszczyk, W. *Additive Subgroups of Topological Vector Spaces*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1991; p. 1466.
14. Außenhofer, L.; Chasco, M.J.; Domínguez, X.; Tarieladze, V. On Schwartz groups. *Studia Math.* **2007**, *181*, 199–210. [CrossRef]
15. Smith, M.F. The Pontrjagin duality theorem in linear spaces. *Ann. Math.* **1952**, *56*, 248–253. [CrossRef]
16. Engelking, R. *General Topology*; Sigma Series in Pure Mathematics; Heldermann Verlag: Berlin, Germany, 1989; Volume 6.
17. Außenhofer, L.; Dikranjan, D.; Giordano Bruno, A. *Topological Groups and the Pontryagin-van Kampen Duality*; Studies in Mathematics; De Gruyter: Berlin, Germany, 2021; p. 83.
18. Kye, S.-H. Pontryagin duality in real linear topological spaces. *Chin. J. Math.* **1984**, *12*, 129–136.
19. Außenhofer, L. On the Glicksberg Theorem for locally quasi-convex Schwartz groups. *Fundam. Math.* **2008**, *201*, 163–177. [CrossRef]
20. Bruguera Padró, M. Grupos Topológicos y Grupos de Convergencia: Estudia de la Dualidad de Pontryagin. Ph.D. Thesis, Universitat de Barcelona, Barcelona, Spain, 1999.
21. Kaplan, S. Extension of Pontrjagin Duality I: Infinite Products. *Duke Math. J.* **1948**, *15*, 649–658. [CrossRef]
22. Hrušák, M.; van Mill, J.; Ramos-García, U.; Shelah, S. Countably compact groups without non-trivial convergent sequences. *Trans. Amer. Math. Soc.* **2021**, *374*, 1277–1296. [CrossRef]
23. Banaszczyk, W.; Chasco, M.J.; Martín-Peinador, E. Open subgroups and Pontryagin duality. *Math. Z.* **1994**, *215*, 195–204. [CrossRef]
24. Noble, N. k-groups and duality. *Trans. Am. Math. Soc.* **1970**, *151*, 551–561.
25. Dunford, N.; Schwartz, J.T. *Linear Operators, Part I: General Theory*; Reprint of the 1958 Original; John Wiley and Sons, Inc.: New York, NY, USA, 1988.
26. Chasco, M.J. Pontryagin duality for metrizable groups. *Arch. Math.* **1998**, *70*, 22–28. [CrossRef]
27. Außenhofer, L. *Contributions to the Duality Theory of Abelian Topological Groups and to the Theory of Nuclear Groups*; Polska Akademia Nauk, Instytut Matematyczny: Warsaw, Poland, 1999.

axioms

Article

Permutations, Signs, and Sum Ranges †

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† Dedicated to M. J. Chasco on the occasion of her 65th birthday.

Abstract: The sum range $\mathrm{SR}[\mathbf{x}; X]$, for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a topological vector space X, is defined as the set of all elements $s \in X$ for which there exists a bijection (=permutation) $\pi : \mathbb{N} \to \mathbb{N}$, such that the sequence of partial sums $(\sum_{k=1}^{n} x_{\pi(k)})_{n\in\mathbb{N}}$ converges to s. The sum range problem consists of describing the structure of the sum ranges for certain classes of sequences. We present a survey of the results related to the sum range problem in finite- and infinite-dimensional cases. First, we provide the basic terminology. Next, we devote attention to the one-dimensional case, i.e., to the Riemann–Dini theorem. Then, we deal with spaces where the sum ranges are closed affine for all sequences, and we include some counterexamples. Next, we present a complete exposition of all the known results for general spaces, where the sum ranges are closed affine for sequences satisfying some additional conditions. Finally, we formulate two open questions.

Keywords: series; permutation; convergence; sum range

MSC: 54C35; 54E15

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1. Basic Definitions

We write $\mathbb{N}$ for the set $\{1, 2, \dots\}$ of natural numbers with its usual order, and

$$\mathbb{N}_n := \{k \in \mathbb{N} : k \leq n\},\ n = 1, 2, \dots .$$

For any set I, a bijection $\sigma : I \to I$ is called a permutation of I; we denote by $\mathbb{S}(I)$ the set of all permutations of I.

For a semigroup $(X, +)$, a natural number n, and a finite sequence $x_k \in X, k = 1, \dots, n$

- The sum $\sum_{k=1}^{n} x_k$ is defined in the usual way;
- It is known that if $(X, +)$ is Abelian, then for every permutation $\sigma \in \mathbb{S}(\mathbb{N}_n)$, the equality

$$\sum_{k=1}^{n} x_{\sigma(k)} = \sum_{k=1}^{n} x_k$$

holds.

A (formal infinite) series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of an additive semigroup $(X, +)$ is the sequence of partial sums

$$\left(\sum_{k=1}^{n} x_k\right)_{n\in\mathbb{N}}. \tag{1}$$

The 'multiplicative' counterpart of the similar concept would be: a (formal) infinite product corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a multiplicative Abelian semigroup $(X, \cdot)$ is the sequence of partial products

$$\left(\prod_{k=1}^{n} x_k\right)_{n\in\mathbb{N}} . \tag{2}$$

A topologized semigroup is a pair (X, τ), where X is a semigroup, and τ is a topology in X.

A topological semigroup is a topologized semigroup (X, τ) for which the semigroup operation is τ-continuous.

A D-convergence space is a pair $(X, \lim)$, where X is a set, and $\lim \subset X^{\mathbb{N}} \times X$ is a relation with natural properties, see [1,2].

If $(X, \lim)$ is a D-convergence space, $\mathbf{s} = (s_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, and $a \in X$, then instead of $(\mathbf{s}, a) \in \lim$, we write $\lim \mathbf{s} = a$ or $\lim_n s_n = a$ and say that the sequence $\mathbf{s} = (s_n)_{n\in\mathbb{N}}$ converges to the element a.

A D-convergence semigroup is a D-convergence space $(X, \lim)$, where X is a semigroup.

A series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a topologized semigroup $(X, +, \tau)$ or a D-convergence semigroup $(X, +, \lim)$ is said to be convergent in X, if there exists an element $s \in X$, such that the sequence

$$\left(\sum_{k=1}^{n} x_k\right)_{n\in\mathbb{N}}$$

converges in (X, τ), respectively, in $(X, \lim)$ to s.

If the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a topologized or D-convergence semigroup X converges to an element $s \in X$, then the element s is called a sum of the series, and we write

$$s = \sum_{k=1}^{\infty} x_k \qquad \text{or} \qquad \sum_{k=1}^{\infty} x_k = s\,.$$

Note that Bourbaki uses the notation $\mathrm{S}_{k=1}^{\infty} x_k$ instead of $\sum_{k=1}^{\infty} x_k$.

In connection with these notions, the following questions can be posed.

Question 1. *Let X be a Hausdorff topological Abelian group and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. If the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is convergent in X, and $\sigma \in \mathbb{S}(\mathbb{N})$ is a permutation, is the series corresponding to the sequence $\mathbf{x}_\sigma = (x_{\sigma(n)})_{n\in\mathbb{N}}$ convergent in X?*

Question 2. *Let X be a Hausdorff topological Abelian group and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. If the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is convergent in X, and $\sigma \in \mathbb{S}(\mathbb{N})$ is a permutation, such that the series corresponding to a sequence $\mathbf{x}_\sigma = (x_{\sigma(n)})_{n\in\mathbb{N}}$ is convergent in X too, is the equality*

$$\sum_{k=1}^{\infty} x_{\sigma(k)} = \sum_{k=1}^{\infty} x_k$$

true?

It seems that Augustin-Luis Cauchy (1789–1857) was the first who noticed (in 1833) that the answer to Question 1, in the case of the set $(\mathbb{R}, +)$ of real numbers with the usual notion of convergence, is negative.

Namely, Cauchy (pp. 57–58, [3]), first indicated (without giving any reference) a proof of the assertion that the series corresponding to the sequence $x_n = (-1)^{n+1}\frac{1}{n}$,

$n = 1, 2, \ldots$ converges in $\mathbb{R}$ and then describes a permutation $\sigma : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to the sequence $x_{\sigma(n)},\ n = 1, 2, \ldots$ does not converge in $\mathbb{R}$.

The second was Peter Lejeune-Dirichlet (1805–1859), who noticed in his 1837 paper (p. 3, [4]) (without any reference either) that the answers to both Questions 1 and 2 were negative. See Remark 1 below about Dirichlet's statements.

Motivated by the abovementioned negative answers to Questions 1 and 2, for any sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a topologized semigroup or a D-convergence semigroup $(X, +)$, we define the subsets

$$\mathfrak{P}[\mathbf{x}; X], \qquad \mathfrak{C}[\mathbf{x}; X]$$

of $\mathbb{S}(\mathbb{N})$ and the subsets

$$\mathrm{SR}[\mathbf{x}; X], \qquad \mathrm{LPR}[\mathbf{x}; X]$$

of X as follows:

- A permutation $\pi : \mathbb{N} \to \mathbb{N}$ is in $\mathfrak{P}[\mathbf{x}; X]$, if and only if the series corresponding to $(x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent in X.
- A permutation $\pi : \mathbb{N} \to \mathbb{N}$ is in $\mathfrak{C}[\mathbf{x}; X]$, if and only if some subsequence of the sequence $(\sum_{k=1}^{n} x_{\pi(k)})_{n\in\mathbb{N}}$ converges in X.
- An element $t \in X$ is in $\mathrm{SR}[\mathbf{x}; X]$, if and only if $\exists \pi \in \mathfrak{P}[\mathbf{x}; X]$, such that $t = \sum_{k=1}^{\infty} x_{\pi(k)}$.
- An element $t \in X$ belongs to $\mathrm{LPR}[\mathbf{x}; X]$, if and only if $\exists \pi \in \mathfrak{C}[\mathbf{x}; X]$ such that some subsequence of the sequence $(\sum_{k=1}^{n} x_{\pi(k)})_{n\in\mathbb{N}}$ converges in X to t.

The set $\mathrm{SR}[\mathbf{x}; X]$ is called the sum range for the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ (see Definition 2.1.1, [5]), and the set $\mathrm{LPR}[\mathbf{x}; X]$ is called the limit-point range of the series corresponding to the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ (see Definition 3.2.1, [5], where this set is denoted by $\mathrm{LPR}(\sum_{k=1}^{\infty} x_k)$).

In (p. 95, [6]), instead of $\mathrm{LPR}[(x_n)_{n\in\mathbb{N}}; X]$, the notation $\mathfrak{C}(\sum_n x_n; X)$ is used.

Evidently,

$$\mathrm{SR}[\mathbf{x}; X] \subset \mathrm{LPR}[\mathbf{x}; X]. \tag{3}$$

It may be that for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, the set $\mathfrak{P}[\mathbf{x}; X]$ (respectively, the set $\mathfrak{C}[\mathbf{x}; X]$) is empty, in which case, $\mathrm{SR}[\mathbf{x}; X] = \emptyset$ (respectively $\mathrm{LPR}[\mathbf{x}; X] = \emptyset$) as well.

In the multiplicative case, of course, we need to say that a permutation $\pi : \mathbb{N} \to \mathbb{N}$ belongs to $\mathfrak{P}(\mathbf{x})$, if and only if the infinite product corresponding to $(x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent in X, and we define the the product range

$$\mathrm{PR}[\mathbf{x}; X]$$

in a similar way.

The **sum range problem** can be stated as follows: *to describe the structure of the set* $\mathrm{SR}[\mathbf{x}; X]$ *for a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *of elements of a topologized semigroup* $(X, +, \tau)$ *or of a D-convergence semigroup* $(X, +, \lim)$.

Similarly, we can state the **product range problem** as follows: *to describe the structure of the set* $\mathrm{PR}[\mathbf{x}; X]$ *for a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *of elements of a topologized semigroup* $(X, \cdot, \tau)$ *or of a D-convergence semigroup* $(X, \cdot, \lim)$.

Let us first comment on the case of the set of extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with the usual order, addition, and notion of convergence.

For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, for which the set $\{n \in \mathbb{N} : x_n < 0\}$ is finite, the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is always convergent in $\overline{\mathbb{R}}$; so, the expression

$$\sum_{k=1}^{\infty} x_k$$

is always defined.

Surely the following observation was known much earlier, but it is precisely formulated in one of the first papers [7] written by Maurice Fréchet (1878–1973) in 1903.

Proposition 1. *Let $X = \overline{\mathbb{R}}_+ = \{x \in \overline{\mathbb{R}} : x \geq 0\}$ with the usual order, addition, and topology. Then, X is a compact metrizable topological Abelian monoid, which has the following properties:*

(I) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is convergent in X.*

(II) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X and for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the equality*

$$\sum_{k=1}^{\infty} x_{\sigma(k)} = \sum_{k=1}^{\infty} x_k$$

holds.

(III) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, the sum range $\mathrm{SR}[\mathbf{x}; X]$ is a singleton.*

(IV) *For every $x \in X$, there exists a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X for which $\mathrm{SR}[\mathbf{x}; X] = \{x\}$.*

The following 'multiplicative' analogue of Proposition 1 is true as well.

Proposition 2. *Let $X = [0, 1]$ with the usual multiplication, order, and topology. Then, X is a compact metrizable topological Abelian monoid, which has the following properties:*

(I) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, the infinite product corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is convergent in X.*

(II) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X and for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the equality*

$$\prod_{k=1}^{\infty} x_{\sigma(k)} = \prod_{k=1}^{\infty} x_k$$

holds.

(III) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, the product range $\mathrm{PR}[\mathbf{x}; X]$ is a singleton.*

(IV) *For every $x \in X$, there exists a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X for which $\mathrm{PR}[\mathbf{x}; X] = \{x\}$.*

We adopt the following definitions.

Definition 1. *The series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topologized semigroup $(X, +, \tau)$ or a D-convergence semigroup $(X, +, \lim)$ is called unconditionally convergent (Bourbaki says commutatively convergent [8]) in $(X, +, \tau)$, if*

$$\mathfrak{P}[\mathbf{x}; X] = \mathbb{S}(\mathbb{N})\,;$$

i.e., if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the series corresponding to $\mathbf{x}_\sigma = (x_{\sigma(n)})_{n\in\mathbb{N}}$ is convergent in $(X, +, \tau)$ or in $(X, +, \lim)$.

Definition 2. *The infinite product corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topologized semigroup $(X, \cdot, \tau)$ or a D-convergence semigroup $(X, \cdot, \lim)$ is called unconditionally convergent, if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the infinite product corresponding to $\mathbf{x}_\sigma = (x_{\sigma(n)})_{n\in\mathbb{N}}$ is convergent in $(X, \cdot, \tau)$ or in $(X, \cdot, \lim)$.*

Sometimes the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is called conditionally convergent or semi-convergent, if it converges but does not converge unconditionally. We do not use these terms.

The following statement, which in a more general setting was obtained in [9], implies that the sum range problem has an easy solution in the case of unconditional convergence.

Theorem 1. *For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a Hausdorff topologized Abelian semigroup $(X, +, \tau)$, the following statements are true.*

(a') If the series corresponding to $\mathbf{x}$ is convergent in $(X, +, \tau)$, and $\mathrm{SR}[\mathbf{x}; X]$ is not a singleton, then there is a permutation $\lambda : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to $\mathbf{x}_\lambda = (x_{\lambda(n)})_{n\in\mathbb{N}}$ is not convergent in $(X, +, \tau)$.

(a) (Commutativity theorem) *If the series corresponding to* $\mathbf{x}$ *is unconditionally convergent in* $(X, +, \tau)$, *then* $\mathrm{SR}[\mathbf{x}; X]$ *is a singleton.*

In the next section, we consider the problem in the case of $\mathbb{R}$. We see in particular that the converse to Theorem 1(a) is true for $X = \mathbb{R}$, but it fails in general, see Remark 7.

A topological group X is called protodiscrete, if every neighborhood of the neutral element of X contains an open subgroup of X.

The following assertion shows that for protodiscrete groups, the sum range problem has an easy solution too.

Proposition 3. *Let* $(X, +, \tau)$ *be a topological group and* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence of elements of* X. *Consider the statements:*

(i) *The set* $\mathrm{SR}[\mathbf{x}; X]$ *is not empty.*
(ii) *The sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *converges in* X *to the neutral element.*
(iii) *The series corresponding to* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *is unconditionally convergent in* X.

Then,

(I) $(i) \Longrightarrow (ii)$.
(II) $(ii) \Longrightarrow (iii)$, *provided* $(X, +, \tau)$ *is protodiscrete, sequentially complete, and Abelian.*
(III) (See (Ch.III, Section 5, Exercise 2) [8], $(i) \Longrightarrow (iii)$ *provided* $(X, +, \tau)$ *is protodiscrete sequentially complete and Abelian.*
(IV) *If* $(X, +, \tau)$ *is protodiscrete, sequentially complete, Hausdorff, and Abelian, then* $\mathrm{SR}[\mathbf{x}; X]$ *either is empty or is a singleton.*

Proof.

(I) This is well-known and is easy to verify.

(II) We fix a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ and set $u_n = x_{\sigma(n)}$, $s_n = \sum_{k=1}^{n} u_k$, $n = 1, 2, \ldots$. Since (ii) is satisfied, it is easy to verify that the sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ also converges in X to the neutral element. Let us deduce from this that (s_n) is a Cauchy sequence in X. Indeed, let V be an arbitrary neighborhood of zero in X. Since X is protodiscrete, there is an open subgroup H of X with $H \subset V$. Since $\lim_n u_n = 0$, there exists $N_H \in \mathbb{N}$, such that $u_n \in H$ for each $n > N_H$. We now fix arbitrarily natural numbers n and m, such that $N_H < m < n$; then, $s_n - s_m = \sum_{k=m+1}^{n} u_k \in H \subset V$, and so, (s_n) is a Cauchy sequence in X.
Since X is sequentially complete, the sequence (s_n) converges in X, i.e., the series corresponding to $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ converges in X. Since $\sigma : \mathbb{N} \to \mathbb{N}$ was an arbitrary permutation, (II) is proved.

(III) Since (i) is satisfied, by (I), for $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, condition (ii) is satisfied too. Hence, by (II), we obtain that (iii) is true.

(IV) Suppose that the set $\mathrm{SR}[\mathbf{x}; X]$ is not empty. Then, by (I), condition (ii) is satisfied, and then by (II), the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent in X. From this, according to Theorem 1(a), we can conclude that $\mathrm{SR}[\mathbf{x}; X]$ is a singleton.

□

To formulate a general result related to the sum ranges, let us fix one more notation that does not directly involve permutations, see (p. 95, [6]).

For sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a topologized Abelian semigroup $(X, +, \tau)$ and for each $m = 1, 2, \ldots$, let

$$\mathrm{A_m}[\mathbf{x}; X]$$

be the closure in (X, τ) of the set

$$\{s \in X : \exists I \subset \{m, m+1, \ldots\},\ I \text{ is finite}, \quad I \neq \emptyset, \quad s = \sum_{i\in I} x_i\},$$

and

$$\mathbf{A}[\mathbf{x}; X] = \bigcap_{m=1}^{\infty} \mathrm{A_m}[\mathbf{x}; X] \,.$$

Proposition 4. (See (pp. 95–96, [6]); see also [10]) *Let X be a metrizable topological Abelian group and $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ be a sequence of elements of X for which the set* $\mathrm{SR}[\mathbf{x}; X]$ *is not empty. Then,*

$$\mathbf{A}[\mathbf{x}; X]$$

is a closed subgroup of X.

Moreover,

$$\mathbf{A}[\mathbf{x}; X] + s = \mathrm{LPR}[\mathbf{x}; X],$$

for every $s \in \mathrm{SR}[\mathbf{x}; X]$.

It can be said that this proposition is the only result related to the sum range, which is valid for all metrizable topological Abelian groups. In the next section, we consider the classical case of real numbers.

2. Riemann–Dini Theorem

Let us reproduce a piece from (p. 3, [4]):

"... we respect the essential difference which exists between two kinds of infinite series. If we regard each value instead of each term or, it being imaginary, its module, then two cases can happen. Either it is possible to give a finite value which is greater than the sum of any of however many of these values or moduli, or this condition cannot be satisfied by any finite number. In the first case, the series always converges and has a completely defined sum regardless how the series terms are ordered, ..."

It follows that the following result was discovered by Dirichlet in 1837.

Theorem 2. *Let $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ be a sequence of real or complex numbers, such that for some finite number L, we have $\sum_{k=1}^{n} |x_k| \leq L, \; n = 1, 2, \ldots$; i.e., in modern terms, the series corresponding to $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ is absolutely convergent.*

Then, the series corresponding to $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ is unconditionally convergent, and $\mathrm{SR}[\mathbf{x}; \mathbb{R}]$ *is a singleton.*

Dirichlet continues as follows:

"... In the second case the series can converge too but convergence is essentially dependent on the kind of order of terms. Does convergence hold for a specific order then it can stop when this order is changed, or, if this does not happen, then the sum of the series might become completely different.

So, for example, of the two series made from the same terms:

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \ldots \,,$$

$$1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{4}} + \ldots \,,$$

only the first converges while of the following:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \,,$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots \,,$$

both converge, but with different sums."

Remark 1. *Let us formulate Dirichlet's statements in terms of the present article. We introduce the sequences of real numbers* $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ *and* $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ *defined for a fixed* $n \in \mathbb{N}$ *by the equalities:*

$$a_n = (-1)^{n+1}\frac{1}{\sqrt{n}}, \quad c_n = (-1)^{n+1}\frac{1}{n} \,.$$

Let $\sigma : \mathbb{N} \to \mathbb{N}$ *be a mapping defined for a fixed* $n \in \mathbb{N}$ *by the equalities:*

$$\sigma(3n-2) = 4n-3,\ \sigma(3n-1) = 4n-1,\ \sigma(3n) = 2n\,.$$

Clearly, σ *is a bijection, i.e.,* $\sigma \in \mathbb{S}(\mathbb{N})$.

We have:

(D1) The series corresponding to the sequence $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ *converges in* $\mathbb{R}$.
The convergence follows from Leibniz's alternating series theorem; we have, moreover, that

$$0 < a_1 + a_2 < \sum_{k=1}^{\infty} a_k < a_1 = 1,\ \ 0 < \sum_{k=1}^{2n} a_k < \sum_{k=1}^{\infty} a_k < \sum_{k=1}^{2n-1} a_k < 1,\ \ n = 2, 3, \ldots.$$

The exact value of $\sum_{n=1}^{\infty} a_n$ *seems to be unknown.*

(D2) The series corresponding to the sequence $(a_{\sigma(n)})_{n\in\mathbb{N}}$ *does not converge in* $\mathbb{R}$.
This needs little work; it can be shown that, in fact,

$$\lim_n \sum_{k=1}^{n} a_{\sigma(k)} = +\infty\,.$$

(D3) The series corresponding to the sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ *converges in* $\mathbb{R}$.
This follows again from the alternating series theorem. The value of $\sum_{n=1}^{\infty} c_n$ *is known; it is* $\ln 2$.

(D4) The series corresponding to the sequence $(c_{\sigma(n)})_{n\in\mathbb{N}}$ *converges in* $\mathbb{R}$ *too, and*

$$\sum_{n=1}^{\infty} c_{\sigma(n)} = \frac{3}{2}\ln 2\,.$$

This needs more work.

As we see, Dirichlet's conclusions are correct. Now, we know that Dirichlet could consider only one sequence, either $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ *or* $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$, *to obtain the same conclusions, because the following statements are true as well:*

(D2') By Riemann's Theorem 3, there exists $\sigma' \in \mathbb{S}(\mathbb{N})$, *such that the series corresponding to* $(a_{\sigma'(n)})_{n\in\mathbb{N}}$ *converges in* $\mathbb{R}$, *but*

$$\sum_{n=1}^{\infty} a_{\sigma'(n)} \neq \sum_{n=1}^{\infty} a_n\,.$$

(D4') By Dini's Theorem 4(c), there exists $\sigma'' \in \mathbb{S}(\mathbb{N})$, *such that the series corresponding to* $(c_{\sigma''(n)})_{n\in\mathbb{N}}$ *does not converge in* $\mathbb{R}$.

It is not clear in advance that an unconditionally convergent series of real numbers is absolutely convergent as well. We shall see (Proposition 6 below) that this is in fact true due to the following Riemann rearrangement theorem, which was first published in 1867:

Theorem 3. *Let* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers, such that the series corresponding to it is convergent, but it is not absolutely convergent. Then,* $\mathrm{SR}[\mathbf{x}; \mathbb{R}] = \mathbb{R}$.

Let us reproduce Riemann's (1826–1866) text:

"...Dirichlet found a way to solve this problem noting that infinite series form two essentially distinct classes: those which remain convergent if all their terms are made positive and those where this is not the case. In the first case the terms of a series can be permutated arbitrarily, while in the second case the sum of a series depends on the order of terms. In fact, let for a series from the second class the positive terms be

$$a_1, a_2, a_3, \ldots,$$

and the negatives be

$$-b_1, -b_2, -b_3, \ldots$$

Then it is clear that both of the sums $\sum a$ and $\sum b$ must be divergent; in fact, if both of them are convergent, then the given series would be convergent after making all signs of its terms the same; if only one of them is convergent, then the given series would be divergent. It is not hard to see that after appropriate permutation of terms the series may take an arbitrary given value C. In fact, let us take alternately first positive terms of the series until their sum does not exceed C, and then the negative terms until the sum will not be less than C; in this way the deviation of the sum from C will never be greater than the absolute value of the preceding term whose sign has been changed. But as the values a and b when the indices increase became infinitely small, we get that the deviation from C after sufficient continuation of the series will become arbitrarily small, and hence the series converges to the value C." (Translated from (Section 3, p. 232, [11]))

"... infinite series fall into two distinct classes, depending on whether or not they remain convergent when all the terms are made positive. In the first class the terms can be arbitrarily rearranged; in the second, on the other hand, the value is dependent on the ordering of the terms. Indeed, if we denote the positive terms of a series in the second class by

$$a_1, a_2, a_3, \ldots,$$

and the negative terms by

$$-b_1, -b_2, -b_3, \ldots$$

then it is clear that $\sum a$ as well as $\sum b$ must be infinite. For if they were both finite, the series would still be convergent after making all the signs the same. If only one were infinite, then the series would diverge. Clearly now an arbitrarily given value C can be obtained by a suitable reordering of the terms. We take alternately the positive terms of the series until the sum is greater than C, and then the negative terms until the sum is less than C. The deviation from C never amounts to more than the size of the term at the last place the signs were switched. Now, since the numbers a as well as the numbers b become infinitely small with increasing index, so do also the deviations from C. If we proceed sufficiently far in the series, the deviation becomes arbitrarily small, that is, the series converges to C." (See (pp. 226–227, [12]))

According to (p. 19, [13]) Theorem 3 "made its first appearance in the work of B. Riemann (1854). It was not until after Riemann's death that a small gap in his reasoning was discovered and closed by U. Dini (1868)." Here, B. Riemann (1854) is [14], and U. Dini (1868) is [15]. We could not find any mention of 'a small gap' either in [15] or in [16].

These two theorems amount to a complete solution of the sum range problem for $\mathbb{R}$.

Proposition 5. *Let* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers. Then,*

(I) *One of the following must be true:*

- *(a)* $\mathrm{SR}[\mathbf{x};\mathbb{R}] = \emptyset$.
- *(b)* $\mathrm{SR}[\mathbf{x};\mathbb{R}]$ *is a singleton.*
- *(c)* $\mathrm{SR}[\mathbf{x};\mathbb{R}] = \mathbb{R}$.

(II) *Case (b) takes place if and only if the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent.*

Proof.

(I) Suppose that $\mathrm{SR}[\mathbf{x};\mathbb{R}] \neq \emptyset$. We fix a permutation $\pi : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to $(x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent. We write $y_n = x_{\pi(n)}, n = 1, 2, \ldots$ and $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$. It is clear that

$$\mathrm{SR}[\mathbf{x};\mathbb{R}] = \mathrm{SR}[\mathbf{y};\mathbb{R}]. \tag{4}$$

If the series corresponding to $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ is absolutely convergent, then by Theorem 2, we have that $\mathrm{SR}[\mathbf{y};\mathbb{R}]$ is a singleton, and by equality (4), we have that the set $\mathrm{SR}[\mathbf{x};\mathbb{R}]$ is a singleton too.
If the series corresponding to $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ is not absolutely convergent, then by Theorem 3, we have $\mathrm{SR}[\mathbf{y};\mathbb{R}] = \mathbb{R}$, and by equality (4), we have that the equality $\mathrm{SR}[\mathbf{x};\mathbb{R}] = \mathbb{R}$ holds, too.

(II) It remains to prove that if the set $\mathrm{SR}[\mathbf{x};\mathbb{R}]$ is a singleton, then the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent. Since $\mathrm{SR}[\mathbf{x};\mathbb{R}] \neq \emptyset$, we can fix again a permutation $\pi : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to $(x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent; we write $y_n = x_{\pi(n)}, n = 1, 2, \ldots$, and $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$. By equality (4), we have that the set $\mathrm{SR}[\mathbf{y};\mathbb{R}]$ is a singleton too; in particular, $\mathrm{SR}[\mathbf{y};\mathbb{R}] \neq \mathbb{R}$. From this, by Theorem 3, the series corresponding to $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ is absolutely convergent. Hence, by Theorem 2, the series corresponding to $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ is unconditionally convergent; so, the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent too.

□

Proposition 6 (Riemann–Dirichlet theorem)**.** *For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of real numbers, the following statements are equivalent:*

(i) The series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent in $\mathbb{R}$.
(ii) The series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is absolutely convergent in $\mathbb{R}$.

Proof. $(i) \Longrightarrow (ii)$. By Theorem 1(a), condition (i) implies that $\mathrm{SR}[\mathbf{x};\mathbb{R}]$ is a singleton. If (i) is satisfied, but (ii) is not true, then by Theorem 3, we should have that $\mathrm{SR}[\mathbf{x};\mathbb{R}] = \mathbb{R}$, a contradiction.

$(ii) \Longrightarrow (i)$ by Theorem 2. □

In what follows, for $x \in \mathbb{R}$, we write:

$$x^+ = \max(x, 0), \;\; x^- = \max(-x, 0).$$

The following version of Theorem 3 was proved by Dini in [15] in 1868 and was included in [16] too.

Theorem 4 (Dini)**.** *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.*

(a) (Dirichlet) If $\sum_{n=1}^{\infty} x_n^+ < +\infty$, and $\sum_{n=1}^{\infty} x_n^- < +\infty$, then $\mathfrak{P}[\mathbf{x};\mathbb{R}] = \mathbb{S}(\mathbb{N})$;
(b) If $\sum_{n=1}^{\infty} x_n^+ < +\infty$, but $\sum_{n=1}^{\infty} x_n^- = +\infty$, or $\sum_{n=1}^{\infty} x_n^+ = +\infty$, but $\sum_{n=1}^{\infty} x_n^- < +\infty$, then $\mathfrak{P}[\mathbf{x};\mathbb{R}] = \emptyset$;
(c) If $x_n \to 0$, and $\sum_{n=1}^{\infty} x_n^+ = \sum_{n=1}^{\infty} x_n^- = +\infty$, then $\mathrm{SR}\left[\mathbf{x};\overline{\mathbb{R}}\right] = \overline{\mathbb{R}}$; moreover, there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$, such that

$$-\infty \le \liminf_n \sum_{k=1}^{n} x_{\pi(k)} < \limsup_n \sum_{k=1}^{n} x_{\pi(k)} \le +\infty\,,$$

where the lower and the upper limits are taken in $\overline{\mathbb{R}}$.

Remark 2. *Note that:*

(1) *In Theorem 4(c), unlike in Theorem 2, it is not required in advance that the initial series be convergent. Theorem 4(c) easily implies Theorem 2, although this is not noted in [15], where, as we have noted already, the name of Riemann is not mentioned at all.*
(2) *The conclusion of Theorem 4(a) in [15] reads as follows: the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is "convergent in whatever order its terms are taken".*
As we see, Dini did not write that reorderings do not affect the sum (however, prior to the formulation of his theorem, he did point this out).
(3) *The "moreover" part of Theorem 4(c) in [15] (up to the notation) is as follows: there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the series corresponding to $(x_{\pi(n)})_{n\in\mathbb{N}}$ "will also become indeterminate".*

The following statement is related to Theorem 4; the implication $(B) \Longrightarrow (C)$ is taken from (Ch. IV, Section 7, Ex. 15 [8]), where the names of Riemann and Dini are not mentioned in connection with this.

Theorem 5 (Riemann–Dini–Bourbaki)**.** *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Consider the following statements.*

(A) *The series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$, but $\sum_{n=1}^{\infty} |x_n| = +\infty$.*
(B) $\lim_n x_n = 0$, *and* $\sum_{n=1}^{\infty} x_n^+ = \sum_{n=1}^{\infty} x_n^- = +\infty$.
(C) *For each two elements a and b of $\overline{\mathbb{R}}$ with $a \le b$, there is a permutation σ of $\mathbb{N}$ such that:*
  (I) $\liminf_n \sum_{k=1}^{n} x_{\sigma(k)} = a$ *and,* $\limsup_n \sum_{k=1}^{n} x_{\sigma(k)} = b$ *, where* lim inf *and* lim sup *are taken in $\overline{\mathbb{R}}$,*
  and
  (II) *The set of cluster points of the sequence $(\sum_{k=1}^{n} x_{\sigma(k)})_{n\in\mathbb{N}}$ coincides with the interval $[a, b]$.*
(D) $\mathrm{SR}[\mathbf{x}; \mathbb{R}] = \mathbb{R}$.
Then, the following implications are true:

$$(A) \Longrightarrow (B) \Longrightarrow (C) \Longrightarrow (D) \Longrightarrow (B).$$

Proof. The implication $(A) \Longrightarrow (B)$ is well known.

A proof of $(B) \Longrightarrow (C, (I))$ is in fact contained in (Theorem 3.54 (p. 76), [17]). We present a proof of the implication $(B) \Longrightarrow (C, (II))$ below.

To prove the implication $(C, (I)) \Longrightarrow (D)$, we fix an arbitrary $c \in \mathbb{R}$ and apply $(C, (I))$ for $a = b = c$. We obtain a permutation σ of $\mathbb{N}$, such that $\liminf_{n\to\infty} \sum_{k=1}^{n} x_{\sigma(k)} = c$ and $\limsup_{n\to\infty} \sum_{k=1}^{n} x_{\sigma(k)} = c$. Hence, $\lim_{n\to\infty} \sum_{k=1}^{n} x_{\sigma(k)} = c$. Therefore, $c \in \mathrm{SR}[\mathbf{x}; \mathbb{R}]$.

$(D) \Longrightarrow (B)$. From (D), we can find and fix a permutation $\pi : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to $\mathbf{x}_\pi := (x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent. Clearly, we have $\mathrm{SR}[\mathbf{x}_\pi; \mathbb{R}) = \mathrm{SR}[\mathbf{x}; \mathbb{R}]$. So, we have also that $\mathrm{SR}[\mathbf{x}_\pi; \mathbb{R}] = \mathbb{R}$. From this equality and Theorem 2, we conclude that the series corresponding to $\mathbf{x}_\pi := (x_{\pi(n)})_{n\in\mathbb{N}}$ is not absolutely convergent. So, we can apply the (already proved) implication $(A) \Longrightarrow (B)$ for the sequence $\mathbf{x}_\pi := (x_{\pi(n)})_{n\in\mathbb{N}}$ and obtain that $\lim_n x_{\pi(n)} = 0$, and both of the series corresponding to $(x^+_{\pi(n)})_{n\in\mathbb{N}}$ and $(x^-_{\pi(n)})_{n\in\mathbb{N}}$ are divergent. Hence, we have also that $\lim_n x_n = 0$, and both of the series corresponding to $(x_n^+)_{n\in\mathbb{N}}$ and $(x_n^-)_{n\in\mathbb{N}}$ are divergent as well. □

The following example shows that the implication $(D) \Longrightarrow (A)$ in Theorem 5 is false in general.

Example 1. *Let*

$$a_n = \frac{1 + 2(-1)^n}{n}, \; n = 1, 2, \ldots.$$

Then,

(a) *The series corresponding to $(a_n)_{n\in\mathbb{N}}$ does not converge in $\mathbb{R}$* (in fact, $\sum_{n=1}^{\infty} a_n = +\infty$).
(b) $\lim_n a_n = 0$, and $\sum_{n=1}^{\infty} a_n^+ = \sum_{n=1}^{\infty} a_n^- = +\infty$.
(c) $\mathrm{SR}[(a_n)_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R}$.

Proof. (a) and (b) are easy to verify. (c) follows from (b) by the implication $(B) \Longrightarrow (D)$ in Theorem 5. □

From the following assertion, it becomes clear that the implication $(B) \Longrightarrow (C, (II))$ in Theorem 5 is a consequence of the implication $(B) \Longrightarrow (C, (I))$ in the same Theorem.

Theorem 6. *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Consider the statements:*

(B1) $\lim_n x_n = 0$.

(B2) *The set of cluster points in $\overline{\mathbb{R}}$ of the sequence*

$$(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$$

coincides with the interval

$$\left[\liminf_n \sum_{k=1}^{n} x_k, \limsup_n \sum_{k=1}^{n} x_k\right],$$

where the lower and the upper limits are taken in $\overline{\mathbb{R}}$.

Then, $(B1) \Longrightarrow (B2)$.

We prove Theorem 6 by means of the next two propositions, which are "series free" and may be of an independent interest.

Proposition 7. *Let (s_n) be a sequence of real numbers, such that both the sets $N_+ = \{n \in \mathbb{N} : s_n \geq 0\}$ and $N_- = \{n \in \mathbb{N} : s_n < 0\}$ are infinite. Consider the following statements:*

(1) $\lim_n (s_{n+1} - s_n) = 0$.

(2) *For every sequence (β_n) of strictly positive real numbers, the sequence (s_n) has a subsequence (s_{j_n}), such that*

$$0 \leq s_{j_n} < \beta_n, \ \ n = 1, 2, \ldots.$$

(3) *The sequence (s_n) has a subsequence (s_{j_n}) such that*

$$s_{j_n} \geq 0, \ \ n = 1, 2, \ldots, \textit{and } \lim_n s_{j_n} = 0\,.$$

Then, $(1) \Longrightarrow (2) \Longrightarrow (3)$.

Proof. $(1) \Longrightarrow (2)$.

We fix a sequence (β_n) of strictly positive real numbers. (1) implies the existence of a sequence (k_n) of natural numbers, such that

$$k \in \mathbb{N}, k \geq k_n \Longrightarrow |s_{k+1} - s_k| < \beta_n, \quad n = 1, 2, \ldots.$$

Since N_+ is an infinite set, we have that $\{i \in N_+ : i \geq k_1\} \neq \emptyset$; so, we can define

$$l_1 := \min\{i \in N_+ : i \geq k_1\}\,.$$

We have: $l_1 \geq k_1$.

Since N_- is an infinite set as well, we have that $\{i \in N_- : i > l_1\} \neq \emptyset$; so, we can define

$$m_1 := \min\{i \in N_- : i > l_1\}\,.$$

We have: $m_1 > l_1$, $k_1 \leq j_1 := m_1 - 1 \in \mathbb{N} \setminus N_- = N_+$, and

$$0 \leq s_{j_1} < s_{j_1} - s_{m_1} = |s_{j_1} - s_{m_1}| < \beta_1\,.$$

In this way, we can inductively construct a sequence (l_n) of elements of N_+ and a sequence (m_n) of elements of N_-, such that

$$k_n \le l_n = \min\{i \in N_+ : i \ge k_n\}\,,\ m_n = \min\{i \in N_- : i > l_n\},\ \text{and}\ \ n = 2, 3, \ldots.$$

Then, we have:

$$m_n > l_n,\ k_n \le j_n := m_n - 1 \in \mathbb{N} \setminus N_- = N_+,\ \ n = 1, 2, \ldots,$$

and

$$0 \le s_{j_n} < s_{j_n} - s_{m_n} = |s_{j_n} - s_{m_n}| < \beta_n,\ n = 1, 2, \ldots.$$

(2) $\Longrightarrow$ (3). From (2) applied for the sequence (β_n) with $\lim_n \beta_n = 0$, we obtain a subsequence (s_{j_n}) of (s_n) for which (3) is satisfied. □

Proposition 8. *Let (t_n) be a sequence of real numbers, such that*

$$\lim_n (t_{n+1} - t_n) = 0\,.$$

Then, the set of cluster points of the sequence $(t_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ is the interval $[a, b]$, where

$$a := \liminf t_n,\ \ b := \limsup t_n.$$

(The lower and the upper limits are taken in $\overline{\mathbb{R}}$.)

Proof. The assertion is clearly true (without the assumption that $\lim_n(t_{n+1} - t_n) = 0$), if $a = b \in \overline{\mathbb{R}}$. So, we can suppose that $a < b$.

We fix $c \in]a, b[$ and put

$$s_n = t_n - c,\ n = 1, 2, \ldots$$

We observe that

$$\liminf_n s_n = \liminf_n t_n - c = a - c < 0, \text{and}\ \limsup_n s_n = \limsup_n t_n - c = b - c > 0\,. \quad (5)$$

Clearly,

(1b) $\lim_n(s_{n+1} - s_n) = 0$, and (5) implies that

(2b) The sets $N_+ = \{n \in \mathbb{N} : s_n \ge 0\}$ and $N_- = \{n \in \mathbb{N} : s_n < 0\}$ are infinite.

So, by the implication of (1) $\Longrightarrow$ (3) in Proposition 7, we obtain that (s_n) has a subsequence (s_{j_n}), such that $\lim_n s_{j_n} = 0$. Hence,

$$\lim_n t_{j_n} = c,$$

and since $c \in]a, b[$ is arbitrary, Proposition 8 is proved. □

Proof of Theorem 6. Consider the sequence $t_n = \sum_{k=1}^n x_k,\ n = 1, 2, \ldots$ Observe that since $(B1)$ is satisfied, we have $\lim_n(t_{n+1} - t_n) = 0$. An application of Proposition 8 for (t_n) gives that $(B2)$ holds for (x_n). □

Using Theorem 5, we can prove the following rearrangement theorem:

Theorem 7. *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\mathrm{LPR}[\mathbf{x}; \mathbb{R}] \neq \emptyset$. The following statements are equivalent:*

(i) $\lim_n x_n = 0$.

(ii) $\mathrm{SR}[\mathbf{x}; \mathbb{R}] = \mathrm{LPR}[\mathbf{x}; \mathbb{R}]$.

Proof. $(i) \Longrightarrow (ii)$. Take $s \in \mathrm{LPR}[\mathbf{x}; \mathbb{R}]$. We can find and fix a permutation $\pi : \mathbb{N} \to \mathbb{N}$ and a strictly increasing sequence $(j_n)_{n\in\mathbb{N}}$ of natural numbers, such that the sequence $\left(\sum_{k=1}^{j_n} x_{\pi(k)}\right)_{n\in\mathbb{N}}$ converges to s. If the sequence $\left(\sum_{k=1}^{n} x_{\pi(k)}\right)_{n\in\mathbb{N}}$ converges to s, then $s \in \mathrm{SR}[\mathbf{x}; \mathbb{R}]$. Suppose now that the sequence $\left(\sum_{k=1}^{n} x_{\pi(k)}\right)_{n\in\mathbb{N}}$ does not converge to s. From these properties, we conclude easily that $\sum_{k=1}^{\infty} x^+_{\pi(k)} = +\infty$, and $\sum_{k=1}^{\infty} x^-_{\pi(k)} = +\infty$.

These equalities together with (i) according to implication $(B) \implies (D)$ of Theorem 5 implies that $\mathrm{SR}[\mathbf{x};\mathbb{R}] = \mathbb{R}$; in particular, $s \in \mathrm{SR}[\mathbf{x};\mathbb{R}]$.

$(ii) \implies (i)$. The equality (ii) and the condition $\mathrm{LPR}[\mathbf{x};\mathbb{R}] \neq \varnothing$ imply $\mathrm{SR}[\mathbf{x};\mathbb{R}] \neq \varnothing$. Hence, (i) holds. □

We conclude this section with the following theorem, the second item of which can be considered as a "sign analogue" of Theorem 3.

Theorem 8. *For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of real numbers, the following statements are true.*

(Ia) *If $\lim_n x_n = 0$, then there exists a sequence $t_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(t_n x_n)$ converges in $\mathbb{R}$.*

(Ib) *Chapter 6, Example 6,[18]) If $\lim_n x_n = 0$, and the series corresponding to the sequence $(|x_n|)$ does not converge in $\mathbb{R}$, then for every $s \in \mathbb{R}$, there exists a sequence $t_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(t_n x_n)$ converges in $\mathbb{R}$ and*

$$\sum_{k=1}^{\infty} t_k x_k = s\,.$$

Proof. *(Ia)* If $\sum_{n=1}^{\infty} |x_n| < +\infty$, then the assertion is true for every sequence $t_n \in \{1, -1\}$, $n = 1, 2, \ldots$.

If $\sum_{n=1}^{\infty} |x_n| = +\infty$, then the conclusion follows from (Ib).

(Ib) A proof of (Ib) can be seen in (Chapter 6, Example 6 [18]), where the requirements of (Ib) are used.

□

Remark 3. Theorem $8(Ia)$ remains true for a sequence of complex numbers [19].

In a footnote of the Russian translation of (Chapter 6, Example 6 [18]), it is noted that the following variant of Theorem $8(Ib)$ is true as well: let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers, such that the series corresponding to the sequence (x_n) converges in $\mathbb{C}$, but $\sum_{n=1}^{\infty} |x_n| = +\infty$. Then, for every $s \in \mathbb{C}$, there exists a sequence $t_n \in \mathbb{C}$, $|t_n| = 1$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(t_n x_n)$ converges in $\mathbb{C}$, and $\sum_{n=1}^{\infty} t_n x_n = s$.

3. Levy–Steinitz Theorem

The sum range problem for complex numbers was treated by Paul Pierre Lévy (1886–1971) in his first article [20] written in 1905 (which contains no separately formulated theorems). The problem for 'numbers' belonging to $\mathbb{R}^d$, $d = 2, 3, \ldots$ was investigated by Ernst Steinitz (1871–1928) in his cycle of articles [21–23]. As written in [23], this problem was proposed to Steinitz by Edmund Landau (1877–1938), who included his version and proof of the Riemann rearrangement theorem (without mentioning Riemann's name) in [24], as Theorem 217.

In this section, we treat the case of Hausdorff topological vector spaces over $\mathbb{R}$.

A general statement which includes Levy's and Steinitz's results was published by Banaszczyk in [25]. To formulate this statement, we first comment on the notion of a nuclear space (whose definition is not given in [25]).

We follow [26–28]. For a nonempty subset U of a monoid $(X, +)$, let $k_U : X \to [0, 1]$ be the functional defined at $x \in X$ by the equality

$$k_U(x) = \sup\{\frac{1}{n} : n \in \mathbb{N} \text{ and } nx \notin U\}$$

(we agree that $\sup(\varnothing) = 0$). Let us call the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topologized Abelian monoid $(X, +, \tau)$ absolutely convergent, if

$$\sum_{n=1}^{\infty} k_U(x_n) < +\infty$$

for every τ-neighborhood U of the neutral element of $(X, +, \tau)$.

It follows from [27] that the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of

- Real numbers is absolutely convergent, if and only if $\sum_{n=1}^{\infty} |x_n| < +\infty$.
- Elements of a normed space $(X, \|\cdot\|)$ are absolutely convergent, if and only if $\sum_{n=1}^{\infty} \|x_n\| < +\infty$.
- Elements of a locally convex topological vector space X are absolutely convergent, if and only if $\sum_{n=1}^{\infty} \|x_n\| < +\infty$ for every continuous semi-norm $\|\cdot\| : X \to \mathbb{R}_+$.

The notion of a nuclear locally convex space was introduced in 1953 by Grothendieck in terms of topological tensor products. We use as a definition the second item of the following consequence of Grothendieck–Pietsch's theorem (see (Ch. IV, 10.7, Corollary 2) [28] and the text following it):

Theorem 9. *For a metrizable locally convex topological vector space X over $\mathbb{R}$, the following statements are equivalent:*

(N) *X is nuclear.*

(GPS) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, such that the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent in X, we have that the same series is absolutely convergent as well.*

We note that the metrizability assumption in Theorem 9 is essential only for the validity of the implication $(GPS) \Longrightarrow (N)$.

It follows from the Riemann–Dirichlet theorem that $(\mathbb{R}, +)$ with the usual topology is nuclear. This implies that the spaces $(\mathbb{R}^d, +)$, $d = 2, 3 \ldots$ and $(\mathbb{R}^{\mathbb{N}}, +)$ with their usual topologies are nuclear as well. It follows that any finite-dimensional normed space is nuclear. For other examples and for the general definition of nuclearity, we refer the reader to [28].

In what follows, for a topological vector space X over $\mathbb{R}$,

- We write X^* for the (topological) dual space, which consists of all continuous linear functionals $x^* : X \to \mathbb{R}$;
- The set X^* is regarded as a vector space over $\mathbb{R}$ with the usual pointwise addition and multiplication by real scalars.

A topological vector space X is called dually separated or a DS-space, if X^* separates the points of X.

The Hahn–Banach theorem implies that a Hausdorff locally convex topological vector space X over $\mathbb{R}$ is a DS-space. There are also DS-spaces, which are not locally convex.

For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topological vector space X over $\mathbb{R}$, let

$$\Gamma_{\mathbf{x}} = \{x^* \in X^* : \sum_{n=1}^{\infty} |x^*(x_n)| < +\infty\},$$

(the set $\Gamma_{\mathbf{x}}$ is called in [29] the weak summability domain of $\mathbf{x}$), and let

$$^{\perp}\Gamma_{\mathbf{x}} := \{x \in X : x^*(x) = 0 \ \ \forall x^* \in \Gamma_{\mathbf{x}}\}$$

be the inverse polar of $\Gamma_{\mathbf{x}}$ in X.

For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topological vector space X over $\mathbb{R}$, the set $\Gamma_{\mathbf{x}}$ is a vector subspace of X^*, while the set $^{\perp}\Gamma_{\mathbf{x}}$ is a closed vector subspace of X.

Let us introduce also, for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topological vector space X, Steinitz's range

$$\mathrm{StR}[\mathbf{x}; X]$$

as follows:

$$\mathrm{StR}[\mathbf{x}; X] = \{s \in X : x^*(s) \in \mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] \ \forall x^* \in X^*\}.$$

For a non-empty subset A of a vector space X and an element $a \in X$, we write

$$A - a = \{x - a : x \in A\},\ A + a = \{x + a : x \in A\},\ a + A = \{a + x : x \in A\}.$$

A subset A of a vector space X over $\mathbb{R}$ or over $\mathbb{C}$ is called real affine, if

$$a_1 \in A, a_2 \in A, t \in \mathbb{R} \implies ta_1 + (1-t)a_2 \in A.$$

The empty set is affine. A non-empty subset A of a vector space X over $\mathbb{R}$ is real affine, if and only if for some $a \in A$, the set $A - a$ is a vector subspace of X.

Proposition 9. *For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topological vector space X over $\mathbb{R}$, the following statements are true.*

(1) $\mathrm{SR}[\mathbf{x}; X] \subset \mathrm{StR}[\mathbf{x}; X]$.

(2) *Ref. [30], ([Proposition 2.1(b))*

$$\mathrm{StR}[\mathbf{x}; X] \neq \emptyset,\ s \in \mathrm{StR}[\mathbf{x}; X] \implies \mathrm{StR}[\mathbf{x}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}. \tag{6}$$

In particular, $\mathrm{StR}[\mathbf{x}; X]$ *is always a closed real affine subset of X.*

(3) *Ref. [31], (Proposition 1)*

$$\mathrm{SR}[\mathbf{x}; X] \neq \emptyset,\ s \in \mathrm{SR}[\mathbf{x}; X] \implies \mathrm{StR}[\mathbf{x}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}. \tag{7}$$

Proof.

(1) This is evident.

(2) Let us show first that

$$\mathrm{StR}[\mathbf{x}; X] \subset s + {}^{\perp}\Gamma_{\mathbf{x}} \tag{8}$$

We fix $a \in \mathrm{StR}[\mathbf{x}; X]$, and we show that $a \in s + {}^{\perp}\Gamma_{\mathbf{x}}$; i.e., we need to show that $a - s \in {}^{\perp}\Gamma_{\mathbf{x}}$. To see this, we fix an arbitrary $x^* \in \Gamma_{\mathbf{x}}$, and we see that $x^*(a-s) = 0$. We can find and fix permutations $\pi : \mathbb{N} \to \mathbb{N}$ and $\sigma : \mathbb{N} \to \mathbb{N}$, such that

$$\sum_{n=1}^{\infty} x^*(x_{\pi(n)}) = x^*(s) \text{ and } \sum_{n=1}^{\infty} x^*(x_{\sigma(n)}) = x^*(a). \tag{9}$$

As $x^* \in \Gamma_{\mathbf{x}}$, the series corresponding to $(|x^*(x_n)|)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$. From this by Theorem 2, we obtain that $\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}]$ is a singleton. This and (9) imply:

$$x^*(s) = x^*(a).$$

Hence, $x^*(a-s) = 0$, and $(b1)$ is proved.

It remains to prove that

$$\mathrm{StR}[\mathbf{x}; X] \supset s + {}^{\perp}\Gamma_{\mathbf{x}} \tag{10}$$

We fix $y \in s + {}^{\perp}\Gamma_{\mathbf{x}}$ and $x^* \in X^*$. We need to verify that

$$x^*(y) \in \mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] \tag{11}$$

First, let $x^* \in \Gamma_{\mathbf{x}}$; then, (as $y \in s + {}^{\perp}\Gamma_{\mathbf{x}}$), we have $x^*(y) = x^*(s)$. As $s \in \mathrm{StR}[\mathbf{x}; X]$, for some permutation $\sigma : \mathbb{N} \to \mathbb{N}$, we have:

$$\sum_{n=1}^{\infty} x^*(x_{\sigma(n)}) = x^*(s).$$

From this, (as $x^*(y) = x^*(s)$), we obtain:

$$\sum_{n=1}^{\infty} x^*(x_{\sigma(n)}) = x^*(y).$$

So, (11) is true in this case.

Now, let $x^* \in X^* \setminus \Gamma_{\mathbf{x}}$; then, as $\mathrm{StR}[\mathbf{x}; X] \neq \emptyset$, we have that $\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] \neq \emptyset$, and the series corresponding to $(|x^*(x_n)|)_{n\in\mathbb{N}}$ is not convergent in $\mathbb{R}$. So by Riemann's theorem, we have $\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}};\mathbb{R}] = \mathbb{R}$; hence, $x^*(y) \in \mathbb{R} = \mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}};\mathbb{R}]$. Consequently, (11) is true in this case too.

(3) This follows from (i) and (iii).

□

Now, we are ready to formulate the result.

Theorem 10 (Wojciech Banaszczyk; Theorem 1, [25])**.** *Let X be a metrizable nuclear locally convex topological vector space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X.*

(I) *If* $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$*, then*

$$\mathrm{SR}[\mathbf{x}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}$$

for each $s \in \mathrm{SR}[\mathbf{x}; X]$.

(II) $\mathrm{SR}[\mathbf{x}; X]$ *is always a closed affine subset of* X.

Proof.

(I) This can be seen in [25]; see also [5], (Ch.8, Section 3 (pp. 110–117)).

(II) This follows from (I).

□

The following statement is one of the key points for the proof of Theorem 10(I).

Proposition 10 (Lemma 6, [25])**.** *Let X be a metrizable nuclear locally convex topological vector space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X for which* $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$*; then,*

$$\mathbf{A}[\mathbf{x}; X] = {}^{\perp}\Gamma_{\mathbf{x}}$$

The following surprising complement to Theorem 10(I) is true as well.

Theorem 11 (Wojciech Banaszczyk; [32])**.** *For a metrizable locally convex topological vector space over $\mathbb{R}$, the following statements are equivalent:*

(i) *X is nuclear.*

(ii) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X such that* $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$*, the equality*

$$\mathrm{SR}[\mathbf{x}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}$$

holds for each $s \in \mathrm{SR}[\mathbf{x}; X]$.

In connection with this theorem, the following question seems very natural.

Question 3 (Remark 3, [32])**.** Can condition (ii) in Theorem 11 be replaced by the following condition?

(ii') *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X, the range* $\mathrm{SR}[\mathbf{x}; X]$ *is always a closed affine subset of X.*

The question of whether the set $\mathrm{SR}[\mathbf{x}; X]$ is always convex for every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a Banach space X over $\mathbb{R}$ was posed by Banach, see (Problem 106, [33]), where a negative answer was included too. It is conjectured that the corresponding example for the case $X = L_2([0,1])$ is due to Marcinkiewicz (see an interesting story in (pp. 31–32, [5])). Our presentation of this example follows (p. 173, [34]).

Example 2. (Marcinkiewicz's Example, 1936) *For a natural number m, find the nonnegative integers m' and $m'' < 2^{m'}$, such that $m = 2^{m'} + m''$, and define the function $x_m : [0,1] \to \{0,1\}$ by the equality:*

$$x_m = 1_{\left[\frac{m''}{2^{m'}}, \frac{m''+1}{2^{m'}}\right]}.$$

Now, let $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ *be the sequence defined by*

$$y_{2n-1} = x_n,\ y_{2n} = -x_n,\ n = 1, 2, \dots.$$

This sequence has the following properties:

(a0) *The series corresponding to* $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ *converges in* $X = L_2([0,1])$ *to zero; in particular,* $0 \in \mathrm{SR}[\mathbf{y}; X]$.

(a1) $1 \in \mathrm{SR}[\mathbf{y}; X]$.

(b) *Every element of* $\mathrm{SR}[\mathbf{y}; X]$ *is an integer-valued function.*

(c) $\mathrm{SR}[\mathbf{y}; X]$ *is not a convex subset of* $X = L_2([0,1])$.

Proof. $(a0)$ follows at once from the observation that $\lim_n \|x_n\|_2 = 0$.

$(a1)$ First, we reproduce the corresponding fragment from (p. 173, [34]): since

$$y_3 + y_5 + y_2 = y_7 + y_9 + y_3 = \cdots = 0 \tag{12}$$

(it can be verified that these equalities hold almost everywhere on $[0,1]$), it follows that the series corresponding to the sequence

$$(y_1, y_3, y_5, y_2, y_7, y_9, y_3, \dots) \tag{13}$$

converges to 1.

From this, since the sequence (13) is indeed a permutation of the sequence $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$, we obtain that $1 \in \mathrm{SR}[\mathbf{y}; X]$.

(b) This is evident.

(c) From (b), we have that $\frac{1}{2} \notin \mathrm{SR}[\mathbf{y}; X]$. From this and from $(a0)$ and $(a1)$, we conclude that the set $\mathrm{SR}[\mathbf{y}; X]$ is not convex.

□

It is known that an example of the same type as Example 2 can be constructed in any infinite-dimensional Banach space X over $\mathbb{R}$ (Corollary 7.2.1 (p. 97), [5]). It follows that if the answer to Question 3 for a Banach space X over $\mathbb{R}$ is positive, then X is finite dimensional and, hence, is nuclear. In general, the answer to this question remains open.

For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in an infinite-dimensional separable Hilbert space X, the sum range $\mathrm{SR}[\mathbf{x}; X]$

- May not be closed [35] (see also (Example 3.1.3 (p. 31), [5]));
- It seems to be unknown whether it may be affine and non-closed (Remark 3.1.1 (p. 32) [5]);
- It is always an analytic subset of X; see [36] (hence, not any subset of a Hilbert space "can serve as the sum range of a series", see (p. 36, [5])). However, it seems to be unknown whether a sum range is always a Borel subset of X.

Remark 4. *Let X and* $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ *be as in Example 2. Then,*

- $\mathrm{SR}[\mathbf{y}; X] = L_2([0,1]; \mathbb{Z})$, *see (Exercise 3.1.4, [5]), where it is noted also that the inclusion* $L_2([0,1]; \mathbb{Z}) \subset \mathrm{SR}[\mathbf{y}; X]$ *was proved by Bogdan in her MSc Thesis (Zaporozhie University, Ukraine, 1992); see also (Assertion 2, [37]).*
- *Let* $X_w := (L_2([0,1]), \mathrm{weak})$. *Then,* $\mathrm{SR}[\mathbf{y}; X_w]$, *as a set, coincides with the whole space* $X = L_2([0,1]$, *see (Exercise 3.1.6, [5]).*

Remark 5. *Theorem* $10(I)$ *implies that that if X is a metrizable nuclear locally convex topological vector space over* $\mathbb{R}$, *then, for every sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *in X, such that* $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$ *and* $\Gamma_{\mathbf{x}} = \{0\}$, *we have* $\mathrm{SR}[\mathbf{x}; X] = X$. *The question of validity of a similar conclusion for the case of an infinite-dimensional separable Banach space X was posed in (Problem 3 (p.146), [29]). A negative answer for the case* $X = L_2([0,1])$ *was obtained in [38] (see also (Proposition 3.4.4 (p. 84), [39])), where it was shown that for Marcinkiewicz's sequence* $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$, *the equality* $\Gamma_{\mathbf{y}} = \{0\}$ *holds; however, by Example* $2(c)$, *we have:* $\mathrm{SR}[\mathbf{y}; X] \neq X = L_2([0,1])$.

Remark 6. *The question of whether the set* $\mathrm{SR}[\mathbf{x}; X)$ *is always convex for every sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *in a separable infinite-dimensional Hilbert space X over* $\mathbb{R}$ *was posed (independently from Banach) and was answered negatively by Hugo Hadwiger [40]. However, in [40], it was conjectured that for each sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$*, for which* $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$*, the sum range* $\mathrm{SR}[\mathbf{x}; X]$ *must be a co-set of an additive subgroup of X. We reproduce an interesting piece from (p. 32, [5]):*

> *"In this section we shall give an example of a series... whose sum range consists of two points ... This example disproves, in particular, H. Hadwiger's conjecture that the sum range of any conditionally convergent series is the coset of some additive subgroup of the space under consideration. The construction given here belongs to M. I. Kadets; its justification was first obtained independently, and about the same time, by P. A. Kornilov [37] and K. Wozniakowski (see [41]). It is interesting that similar constructions were proposed at least by two other mathematicians. A. Dvoretzky told us that many years ago he had such an example, but he, too (like M. I. Kadets), was not able to find a justification. P. Enflo constructed an example with a complete proof at about the same time [37,41] were written, but he did not publish it because I. Halperin informed him about the preprint containing the example presented below."*

Note that now we know more: for each finite subset $A \subset X$ *of an infinite-dimensional separable Banach space X over* $\mathbb{R}$*, there exists a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *in X, such that* $\mathrm{SR}[\mathbf{x}; X] = A$ *[42].*

Theorem 10 is applicable for finite-dimensional normed spaces (because they are nuclear); however, for them, more is also true.

Theorem 12 (Ernst Steinitz)**.** *Let X be a finite-dimensional normed vector space over* $\mathbb{R}$ *and* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence in X. The following statements are true.*

$(I1)$ *If* $\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] \neq \emptyset$*, for every* $x^* \in X^*$*, then* $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$*.*
$(I2)$ $\mathrm{SR}[\mathbf{x}; X] = \mathrm{StR}[\mathbf{x}; X]$.
$(I3)$ *If* $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$*, and* $s \in \mathrm{SR}[\mathbf{x}; X]$*, then* $\mathrm{SR}[\mathbf{x}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}$.
(II) $\mathrm{SR}[\mathbf{x}; X]$ *is a real affine subset of X.*

Comments on the Proof. $(I1)$ and $(I2)$ are nontrivial even when $\dim(X) = 2$. According to [43], $(I1)$ was proved in [23]; however, we did not find its proof in [44] or in [5]. Only in (Theorem II, [45]) and in [46] can one find some information about this implication (see also [47]).

$(I3)$ follows from $(I2)$ and Proposition 9.
Direct proofs of $(I3)$ can be seen in [45,48] and in (Chapter 2, Section 1 (pp. 13–20), [5]).

(II) follows from $(I3)$. (II) is formulated as Lemma 4 in [49] and proved there for the first time in the English literature. Proofs of (II) appeared in Russian for the first time in [50] and in [51]. □

Surely, the first attempt to understand and simplify Steinitz's exposition was carried out by the Austrian mathematician Wilhelm Gross (Groß) (1886–1918) in his 1917 article [52]. Later, many authors were interested in the proof of Theorem 12 (II). Among them was Kurt Gödel (1906–1978), one of the most outstanding logicians of the twentieth century [53] (see a nicely written account of this work in [54]). In [54], after commenting on [52], the following was written:

> *"Other authors, among them Abraham Wald (1933)*(=[55]) *published a proof of the theorem which was close to the proof of Gödel, and it may well be that publication of Wald's proof lessened Gödel's interest in the publication of his own manuscript."*

Let us note also that Theorem 10(II) for $X = \mathbb{R}^{\mathbb{N}}$ was derived from Theorem 12(II) earlier by Wald [56].

Theorem 12(II) coincides with (Ch. VII, Section 3, Exercise 2(ii), [57]), where a (rather complicated) hint of its proof is also given. Surely, a complete realization of Bourbaki's claim requires a separate monograph.

Let us formulate two ingredients of the proof of Theorem 12 (II), which are of independent interest. We follow the terminology of [5,45,48].

Theorem 13. (The polygonal confinement theorem.) *There exists a sequence $C_m, m = 1, 2, \ldots$ of strictly positive constants, with $C_1 = 1$ and with $C_m < m, m = 2, 3, \ldots$, for which the following statement is true.*

If X is a finite-dimensional vector space over $\mathbb{R}$ with $dim(X) = m \geq 1$ and $\|\cdot\|$ a norm (or a subadditive positively homogeneous function) on X, then for a natural number $n > 1$ and for elements $x_j \in X$, $j = 1, \ldots, n$ with

$$\sum_{j=1}^{n} x_j = 0\,,$$

there exists a permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$, such that $\sigma(1) = 1$, and

$$\|\sum_{j=1}^{k} x_{\sigma(j)}\| \leq C_m \max_{j\in\mathbb{N}_n} \|x_j\|,\ k = 1, 2, \ldots, n\,. \tag{14}$$

A proof of Theorem 13 with $C_m = 5^{\frac{m-1}{2}}$, $m = 1, 2, \ldots$ is indicated in (Ch. VII, Section 3, Exercise 1b, [57]). A version of Theorem 13 with $C_m = 2^m - 1$, $m = 1, 2, \ldots$ (without proof and with references to [23,52]) was formulated as Lemma 1 in [49].

It is remarkable that Theorem 13 holds for every norm given on a vector space X over $\mathbb{R}$ with $\dim(X) = m \geq 1$. For a fixed m (and concrete norm), the optimal value of the constant in (14), as well as an elaboration of an optimal algorithm for finding the corresponding permutation σ, plays a role in scheduling theory [58]. It is conjectured that in the case of Euclidean norms, the theorem should hold with constants $C_m, m = 1, 2, \ldots$ for which the sequence $\frac{C_m}{\sqrt{m}}, m = 1, 2, \ldots$ is bounded [59].

Using Theorem 13, it is possible to prove the following generalization of Theorem 7:

Theorem 14 (see Lemma 2, [49], and the rearrangement theorem (p. 346), [48])**.** *Let X be a finite-dimensional normed space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X, such that* $\mathrm{LPR}[\mathbf{x}; X] \neq \emptyset$. *The following statements are equivalent:*

(i) $\lim_n x_n = 0$.
(ii) $\mathrm{SR}[\mathbf{x}; X] = \mathrm{LPR}[\mathbf{x}; X]$.

It is known that if for a Banach space X over $\mathbb{R}$, an analogue (of implication $(i) \implies (ii)$) of Theorem 14 is true, then X is finite-dimensional [60].

The following assertion implies in particular that an analogue of Theorem 12$(I1)$ is not true for all nuclear spaces.

Theorem 15 (Kadets, [43])**.** *For an infinite-dimensional complete separable metrizable topological vector space X over $\mathbb{R}$, the following statements are equivalent.*

(SI) *For each sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X, such that*

$$\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] \neq \emptyset\ \forall x^* \in X^*,$$

we have that $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$.

(Is) *X is topologically isomorphic to $\mathbb{R}^{\mathbb{N}}$ endowed with the topology of coordinatewise convergence.*

The following example shows that a further improvement of Theorem 12$(I1)$ is not possible even in the two-dimensional case.

Example 3 (Kadets, [43])**.** *Let X be a two-dimensional normed vector space over $\mathbb{R}$. There is a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X, such that the set*

$$K_{\mathbf{x}} := \{x^* \in X^* : \mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] \neq \emptyset\}$$

separates points of X, but the set

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is empty.

Proof. Take $X = \mathbb{R}^2$. Fix $n \in \mathbb{N}$, put

$$a_n = \frac{1+2(-1)^n}{n},\ b_n = \frac{1-2(-1)^n}{n},\ x_n = (a_n, b_n),$$

and consider the sequences $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$, $\mathbf{b} = (b_n)_{n\in\mathbb{N}}$, and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$. Then,

(1) $\mathrm{SR}[(a_n)_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R}$, and $\mathrm{SR}[(b_n)_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R}$.
(2) $\mathbb{R}e_1^* \cup \mathbb{R}e_2^* \subset K_{\mathbf{x}}$, where e_1^* and e_2^* are the first and the second projections from $X = \mathbb{R}^2$ onto $\mathbb{R}$, respectively. In particular, $K_{\mathbf{x}}$ separates points of $X = \mathbb{R}^2$.
(3) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \emptyset$.

(1) This follows from implication $(B) \Longrightarrow (D)$ in Theorem 5.
(2) This follows from (1).
(3) Suppose (3) is not true, i.e., $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$. Then, we can find and fix a permutation $\sigma : \mathbb{N} \to \mathbb{N}$, for which the series corresponding to $(x_{\sigma(n)})_{n\in\mathbb{N}}$ converges in $X = \mathbb{R}^2$. Then, both series corresponding to $(a_{\sigma(n)})_{n\in\mathbb{N}}$ and to $(b_{\sigma(n)})_{n\in\mathbb{N}}$ will converge in $\mathbb{R}$. This would imply that the series corresponding to $(a_{\sigma(n)} + b_{\sigma(n)})_{n\in\mathbb{N}}$ will converge in $\mathbb{R}$ too; but this is impossible, as $a_{\sigma(n)} + b_{\sigma(n)} = \frac{2}{\sigma(n)},\ n = 1, 2, \ldots.$

□

4. Kadets-Type Theorems

The first result for infinite-dimensional Banach spaces was obtained in 1953 by Mikhail Iosifovich Kadets (1923–2011).

Theorem 16 (Kadets, Lemma I and Theorem II, [61]). *Let $1 < p < \infty$, (T, Σ, ν) be some σ-finite positive measure space, $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$, and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then,*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{LPR}[(x_n)_{n\in\mathbb{N}}; X],$$

and the sum range

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is a closed affine subset of X, provided the following condition is satisfied:

(KC_p) $\sum_{n=1}^{\infty} \|x_n\|_p^{\min(2,p)} < +\infty.$

At the end of [61], it was written: "It is unknown for the author whether the condition (KC_p) is necessary".

Then, the following more general result appeared:

Theorem 17. (Stanimir Troyanski, [62]) *Let X be a uniformly smooth Banach space over $\mathbb{R}$ with a modulus of smoothness ρ. Then, for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X, the sum range*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is a closed affine subset of X provided the following condition is satisfied:

$$(TC) \sum_{n=1}^{\infty} \rho(\|x_n\|) < +\infty.$$

In [62], it is noted also that it is unknown whether the condition (TC) is necessary.

Remark 7. *Ref. [62] began with the following definition: "A series of vectors of linear topological space is called conditionally convergent if two of its permutations have different sums." In view* *of this definition, one can expect that for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in Hausdorff topological vector space X, the following statement must be true:*

(HM) *If* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$ *is a singleton, then the series corresponding to* $\mathbf{x}$ *is unconditionally convergent in* X.

We have:

(a) *If X is finite-dimensional, then (HM) is true.*
This follows from Steinitz's Theorem 12$(I3)$.
(b) *If X is an infinite dimensional Hilbert space, then (HM) is not true [63].*
(c) *If X is an infinite dimensional Banach space, then (HM) is not true either [64].*
(d) *If $X = \mathbb{R}^{[0,1]}$ is endowed with the topology of point-wise convergence, then (HM) is not true, and there even exists a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X consisting of continuous functions, such that $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \{0\}$, and $\mathrm{SR}[(|x_n(t)|)_{n\in\mathbb{N}}; \mathbb{R}] = \emptyset$, for each $t \in [0,1]$ [65].*

Next was paper [66], which was the first one to be written in English dealing with infinite-dimensional Hilbert spaces (and containing a conclusion about the sum range in the line of Steinitz's Theorem 12$(I3)$).

Theorem 18 (Vladimir Drobot, Theorem 1, [66])**.** *Let (T, Σ, ν) be $[0,1]$ endowed with the sigma-algebra Σ of Lebesgue-measurable sets and the Lebesgue measure ν, $X = (L_2(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_2)$ be the Hilbert space over $\mathbb{R}$, and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X with the following properties:*

(a) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$.
(b) $\sum_{n=1}^{\infty} \|x_n\|_2 = +\infty$.
(c) $\sum_{n=1}^{\infty} \|x_n\|_2^2 < +\infty$.
(d) *The set $\Gamma_{\mathbf{x}}$ is closed in $X^* = X$.*
Then,

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}},$$

for some $s \in \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$.

Among all the previous works, ref. [66] only mentions Steinitz's 1913 paper. At the end of [66], two examples are presented.

Example 1 demonstrates that, in Theorem 18, conditions (a, b, c) may be satisfied, while condition (d) is not;

Example 2 gives a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X for which the conditions (a, b, c, d) of Theorem 18 with $\Gamma_{\mathbf{x}} = \{0\}$ are satisfied; hence, by this theorem, we have that $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = X$.

The last example, together with a version of Theorem 18 in which $\Gamma_{\mathbf{x}} = \{0\}$, is included in the monograph [67].

It is a bit strange that in [66] the result is not formulated or proved for an abstract infinite-dimensional Hilbert space over $\mathbb{R}$, while later in [68], one of the main ingredients of its proof is formulated and proved for the abstract case.

The first paper, written in Russian, to mention [66] was [69].

Theorem 19 (Vladimir Fonf, Theorem, [69])**.** *Let X be a uniformly smooth Banach space over $\mathbb{R}$ with a modulus of smoothness ρ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ a sequence in X, such that*

(a) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$.
(b) $\sum_{n=1}^{\infty} \|x_n\| = +\infty$.
(c) $\sum_{n=1}^{\infty} \rho(\|x_n\|) < +\infty$.
Then,

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}$$

for some $s \in \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$.

Note that this theorem contains and improves Theorem 18, removing condition (d) from it.

In 1971, a paper [70] by Evgenii Mikhailovich Nikishin (1945-1986) appeared, where among other results, it was shown that in Kadets' Theorem 16, condition (KC_p) is in a sense the best possible when $p \geq 2$, and the above considered Banach's question from the "Scottish book" was answered negatively without actually knowing it.

Theorem 20 (Nikishin, Corollary 4 (p. 284), [70])**.** *Let $1 \leq p < \infty$. Let (T, Σ, ν) be $[-1,1]$ endowed with the sigma-algebra Σ of Lebesgue-measurable sets and the Lebesgue measure ν $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$ Then, there exists a sequence $\varphi_n \in X,\ n = 1, 2, \ldots$, such that*

$$\sum_{n=1}^{\infty} \|\varphi_n\|_p^{2+\varepsilon} < \infty \qquad \forall \varepsilon > 0\,,$$

$\mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}; X] \neq \emptyset$*, and* $\mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}; X]$ *is not affine.*

Now, we formulate several other remarkable results contained in [70,71].

Theorem 21. *Let ν be the Lebesgue measure on $[0,1]$ and $X = L_0$ the vector space over $\mathbb{R}$ of (ν-equivalence classes of) all ν-measurable functions $\varphi : [0,1] \to \mathbb{R}$; moreover, let X_ν be the space X endowed with the topology of convergence in measure ν and $X_{\nu,a.e}$ be the space X endowed with ν-almost everywhere convergence (sequences from X). For a sequence $\varphi_n \in X,\ n = 1, 2, \ldots$, we write:*

$$\mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}] := \mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}]; X_\nu],$$

and

$$\mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}] := \mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}]; X_{\nu,a.e}]\,.$$

The following statements are valid.

(I) (Theorem 1, [71]) (see Theorem 7) If for a sequence $\psi_n \in X,\ n = 1, 2, \ldots$,

- (a) *The series corresponding to the sequence $(\psi_n^2)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ ν-almost everywhere, and*
- (b) *Some subsequence of the sequence $\sum_{k=1}^{n} \psi_k \in X,\ n = 1, 2, \ldots$ converges ν-almost everywhere to a function $\varphi \in X$,*

then $\varphi \in \mathrm{SR}_{\nu,a.e}[(\psi_n)_{n\in\mathbb{N}}]$.

(II) (Theorem 2, [71]) If for a sequence $\varphi_n \in X,\ n = 1, 2, \ldots$, the series corresponding to the sequence $(\varphi_n^2)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ ν-almost everywhere, then

(IIa) (Theorem 2, [71]) $\mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}]$ is an affine subset of X, which is closed in X_ν.

(IIb) $\mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}] = \mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}]$.

(IIc) $\mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}]$ is a closed affine subset of X_ν.

(III) Ref. [70] There exists a sequence $\varphi_n \in X,\ n = 1, 2, \ldots$, such that the series corresponding to the sequence $(|\varphi_n|^{2+\varepsilon})_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ ν-almost everywhere for every $\varepsilon > 0$, and $\mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}]$ is not an affine subset of X.

Comments on the Proof. (IIb) The inclusion $\mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}] \subset \mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}]$ is true because the convergence of sequences ν-almost everywhere implies the convergence in measure.

The proof of the inclusion $\mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}] \subset \mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}]$ is more delicate. So, we fix $\varphi \in \mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}]$ and take a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ for which the series corresponding to the sequence $(\varphi_{\sigma(n)})_{n\in\mathbb{N}}$ converges in measure to φ. Then, it is easy to see that the assumptions (a) and (b) of (I) are satisfied for the sequence $\psi_n := \varphi_{\sigma(n)},\ n = 1, 2, \ldots$, and from (I), we conclude that

$$\varphi \in \mathrm{SR}_{\nu,a.e}[(\psi_n)_{n\in\mathbb{N}}] = \mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}]\,.$$

(IIc) follows from (IIb) and (IIa). $\square$

As noted in [70], Theorem 21(III) answers negatively, in particular, a question which (according to [72]) was posed by Banach. The following assertion shows that in Kadets' Theorem 16 the condition (KC_p) is in a sense the best possible one also when $1 < p < 2$.

Theorem 22 (Kornilov, see Theorem 1, [73])**.** *Let $1 \leq p \leq 2$, (T, Σ, ν) be $[0,1]$ endowed with the sigma-algebra Σ of Lebesgue-measurable sets and the Lebesgue measure ν, $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$ and let $\omega :]0,\infty[\to]0,\infty[$ be a function, such that*

$$\lim_{t \to 0} \omega(t) = 0\,.$$

Then, there exists a sequence $\varphi_n \in X,\ n = 1, 2, \ldots$, such that

(1) $\sum_{n=1}^{\infty} \|\varphi_n\|_p^p \omega(\|\varphi_n\|_p) < \infty$,

(2) $0 \in \mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}; X]$ *and* $1 \in \mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}; X]$,
but

(3) $\lambda \in]0,1[\implies \lambda \notin \mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}; X]$.

In 1973, the following variant of Theorem 16 was announced, which can be considered the first infinite-dimensional version of Steinitz's Theorem 12($I2$):

Theorem 23. *Let $1 < p < \infty$, (T, Σ, ν) be $[0,1]$ endowed with the σ-algebra Σ of Lebesgue-measurable sets, and the Lebesgue measure ν and $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$. Let, moreover, $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which satisfies the condition (KC_p).*

Then, the following statements are valid.

(I) (Pecherskii, (Theorem 1, [74])) The equality

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

(II) Pecherskii, (Theorem 3, [74]))

$$\forall x^* \in X^* \setminus \{0\} \qquad \mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R} \implies \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = X\,.$$

We note that Theorem 23(II), in the case when $p = 2$, is included with a complete proof in (Appendix, Section 6, Theorem 1 (pp. 352–357), [75]).

The first essential improvement of Theorem 16 in case $1 < p < 2$ was the following result.

Theorem 24 (Nikishin, Theorem 1, [76])**.** *Let $1 \leq p < 2$, (T, Σ, ν) be $[0,1]$ endowed with the sigma-algebra Σ of Lebesgue-measurable sets and the Lebesgue measure ν, and $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$. Moreover, let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X that satisfies the following condition.*

$(NikC_p)$, the series corresponding to the sequence $(|x_n(t)|^2)_{n\in\mathbb{N}}$, converges in $\mathbb{R}$ for Lebesgue's almost every $t \in [0,1]$ and

$$\left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} \in X\,.$$

Then, for the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, the sum range

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is a closed affine subset of X.

In 1977, the following modification Theorem 23 appeared, which takes into account Nikishin's Theorem 24 too.

Theorem 25. *Let $1 \leq p < \infty$, (T, Σ, ν) be $[0,1]$ endowed with the σ-algebra Σ of Lebesgue-measurable sets and the Lebesgue measure ν and $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$. Moreover, let*

$\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence in X for which* $(NikC_p)$ *is satisfied when* $1 \le p < 2$, *and* (KC_2) *is satisfied, when* $2 \le p < \infty$. *The following statements are true.*

(I) *(Theorem 1, [77]) The equality*

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

(II) *(Corollary 1, [77]) If*

$$\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R} \quad \forall x^* \in X^* \setminus \{0\},$$

then the equality

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = X$$

holds too.

The following result is applicable to the non-separable Banach space $(l_\infty, \|\cdot\|_\infty)$ of all bounded real sequences.

Theorem 26 (Barany, Theorems 2 and 3, [78]). *Let* $1 \le p \le \infty$, $c(j) := 2^{3^j}$, $j = 1, 2, \ldots$, *and* $X = (l_p, \|\cdot\|_p)$. *Moreover, let* $x_n : \mathbb{N} \to \mathbb{R}$, $n = 1, 2, \ldots$ *be a sequence in X such that* $c \cdot x_n \in X$, $n = 1, 2, \ldots$.

Assume further that either

$(BaCo_\infty)$ $p = \infty$, *and* $\limsup_n \|c \cdot x_n\|_\infty = 0$,

or

$(BaCo_p)$ $1 \le p < \infty$, *and* $\sum_{n=1}^\infty \|c \cdot x_n\|_p^p < \infty$.

Then, for the sequence $(x_n)_{n\in\mathbb{N}}$, *the sum range,*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is a closed affine subset of X.

The result is new when $p = \infty$. In the case when $1 \le p \le 2$, it is a consequence of Kadets' Theorem 16, while in the case when $2 < p < \infty$, it is independent from this theorem.

From Theorem 26, unlike the previous results in the present section, it is possible to derive the following corollary.

Corollary 1 (see Proposition 5). *Let* $(t_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers. Then, the sum range*

$$\mathrm{SR}[(t_n)_{n\in\mathbb{N}}; \mathbb{R}]$$

can be either empty, a singleton, or $\mathbb{R}$.

Proof. If $\limsup_n |t_n| > 0$, then clearly $\mathrm{SR}[(t_n)_{n\in\mathbb{N}}; \mathbb{R}] = \varnothing$. So, let $\limsup_n |t_n| = 0$. Consider the sequence $x_n := t_n e_1 \in l_\infty$, $n = 1, 2, \ldots$, where $e_1 = (1, 0, 0, \ldots, 0, \ldots)$. Clearly, $c \cdot x_n = c_1 t_n e_1$, $n = 1, 2, \ldots$, and so, $\limsup_n \|c \cdot x_n\|_\infty = c_1 \limsup_n |t_n| = 0$. Hence, $(BaCo_\infty)$ is satisfied for our sequence $(x_n)_{n\in\mathbb{N}}$, and then by Theorem 26, $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$ is a closed affine subset of X. Clearly,

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(t_n e_1)_{n\in\mathbb{N}}; X] \subset \mathbb{R} \cdot e_1 = \{t e_1 : t \in \mathbb{R}\}.$$

From this relation, we conclude that

$$\mathrm{SR}[(t_n)_{n\in\mathbb{N}}; \mathbb{R}]$$

is a closed affine subset of $\mathbb{R}$. □

Of course, it would be interesting to find other sequences $c(j)$, $j = 1, 2, \ldots$ for which Theorem 26 will remain true.

The first generalizations of Nikishin's Theorem 24 appeared in [79,80]. To formulate them, we recall the needed definitions.

For a natural number n, the Rademacher function $r_n : [0,1] \to \{-1,1\}$ is defined by the equality

$$r_n(t) = (-1)^{[2^n t]}, \qquad t \in [0,1],$$

where $[x]$ stands for the integer part of $x \in \mathbb{R}$.

We say that a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a Banach space X over $\mathbb{R}$ satisfies the (RSC)-condition, if for Lebesgue's almost every $t \in [0,1]$, the series corresponding to the sequence $(r_n(t)x_n)_{n\in\mathbb{N}}$ converges in X.

For a number $q \in \mathbb{R},\ q \geq 2$, we say that a Banach space X over $\mathbb{R}$ is of cotype q, if for every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X satisfying the (RSC)-condition, the series corresponding to the sequence $(\|x_n\|^q)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$.

Theorem 27 (Theorems 8(a,b) and 9, [79])**.** *Let X be a Banach space X over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ a sequence in X, such that*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset .$$

If either

(I) $X = L_p(T,\Sigma,\nu;\mathbb{R})$, *with* $1 \leq p \leq 4$ *and with some σ-finite positive measure space* (T,Σ,ν), *and for* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, *the condition* $(NikC_p)$ *is satisfied,*
or

(II) *X is of cotype 2, and the sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *satisfies the* (RSC)*-condition,*
then

(III) *The equality*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

This theorem contains the promised first generalizations of Nikishin's Theorem 24. At the very end of [79], it was conjectured that $((II) \Longrightarrow (III))$ in Theorem 27 should be true for all Banach spaces. Soon, this conjecture was confirmed. See Theorem 30 below.

Theorem 28 (Particular cases of Theorems 1 and 2, [80])**.** *Let* $0 \leq p < \infty$, (T,Σ,ν) *be some σ-finite positive measure space,* $X = L_p(T,\Sigma,\nu;\mathbb{R})$, *and* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence in X.*

(I) *If for* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, *the condition* $(NikC_p)$ *is satisfied, then the sum range*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is a closed affine subset of X.

(II) *If* $1 \leq p < \infty$, *for* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, *the condition* $(NikC_p)$ *is satisfied, and* $\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$, *then the equality*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}$$

holds for each $s \in \mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X]$.

This theorem contains the further generalizations of Nikishin's Theorem 24. Theorem 28(II) also extends Theorem 25(I).

For a number $r \in \mathbb{R}, 1 < r \leq 2$, we say that a Banach space X over $\mathbb{R}$ is

- Of *type r*, if for every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X, for which the series corresponding to the sequence $(\|x_n\|^r)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$, the (RSC)-condition is satisfied.
- Of *infratype r*, if there exists a positive finite constant C, such that for each natural number n and elements $x_k \in X$, $k = 1,\ldots,n$, the inequality

$$\|\sum_{k=1}^{n} \theta_k x_k\| \leq C\left(\sum_{k=1}^{n} \|x_k\|^r\right)^{\frac{1}{r}}$$

holds for some choice of 'signs' $\theta_k \in \{-1, 1\}$, $k = 1, \ldots, n$.

In (Chapter 7, Section 1 (pp. 87–92) [5]), a machinery oriented to obtaining the following result is developed.

Theorem 29 (Kadets–Ostrovskii [35,81] and Theorem 7.1.2 (p. 92), [5])**.** *Let $1 < r \leq 2$, X be a Banach space over $\mathbb{R}$ having the infratype r, and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X for which*

(a) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$, *and*
(b) *The sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ satisfies the condition:*
(KC_r) The series corresponding to the sequence $(\|x_n\|^r)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$.
Then, the equality

$$\mathrm{SR}((x_n)_{n\in\mathbb{N}}) = s + {}^{\perp}\Gamma_{\mathbf{x}}$$

holds for each $s \in \mathrm{SR}((x_n)_{n\in\mathbb{N}})$.

Theorem 29 covers Theorem 16 in the case when $1 \leq p < \infty$, as if $X = \big(L_p([0,1]), \|\cdot\|_p\big)$, then it is known that X is of type $r = \min(p, 2)$.

Theorem 30 (Announced in Theorem 3, [82], and proved in Theorem 5, [31])**.** *Let X be the Banach space $\mathbb{R}$, and let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X for which*

(a) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$, *and*
(b) *The sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ satisfies the (RSC)-condition.*
Then, the equality

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

This statement was the first general result valid for all Banach spaces. It does not cover Theorem 26, when $2 < p \leq \infty$. However, it implies the following final improvement of Nikishin's Theorem 24.

Theorem 31 (Announced in Corollary 2, [82], and proved in Corollary 3, [31])**.** *Let $X = L_p(T, \Sigma, \nu; \mathbb{R})$, with $1 \leq p < +\infty$ and with some σ-finite positive measure space (T, Σ, ν), and let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X for which*

(a) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$, *and*
(b) *The series corresponding to the sequence $(|x_n(t)|^2)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ for ν-almost every $t \in T$, and*

$$\left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} \in X\,.$$

Then, the equality

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

Comments on the Proof. This follows from Theorem 30 due to the following theorem by Jorgen Hoffman-Jorgensen (1942-2017): (b) *holds if and only if the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ satisfies the (RSC)-condition* (Corollary 2(b) to Theorem 5.5.2 (pp. 323–324), [83]). □

Remark 8. *We fix $r \in]1, 2]$ and a Banach space X over $\mathbb{R}$. Let us note:*

(1) *If X is of type r, then Theorem 29 follows from Theorem 30.*
(2) *If $1 < r < 2$, and X is of infratype r, then X is of type r too [84]. From this and (1), we conclude that if $1 < r < 2$, and X is of infratype r, then again Theorem 29 follows from Theorem 30.*
(3) *Ref. [85] showed the existence of X of infratype 2, which is not of type 2. Consequently, in the case of $p = 2$, Theorem 29 does not follow from Theorem 30.*

To formulate an important generalization of Theorem 30, it would be convenient to provide a definition.

We say that a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in the Banach space (or the topological vector space) X over $\mathbb{R}$ satisfies the (PSC)-condition, if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, there exists a sequence of 'signs' $\theta_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(\theta_n x_{\sigma(n)})_{n\in\mathbb{N}}$ converges in X.

Theorem 32. *Let X be a Banach space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which satisfies the (PSC)-condition.*

Then,

(I) (Theorem 1, [86]) The equality

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

(II) (Corollary 2, [86]) The equality

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = X$$

holds, if and only if

$$\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R} \qquad \forall x^* \in X^* \setminus \{0\}\,. \tag{15}$$

In particular, if (15) *is satisfied, then* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$.

Theorem 32(I) implies Theorem 30 because, as proved in (Proposition 1, [86]) in a somewhat sophisticated way, the (RSC)-condition implies the (PSC)-condition. Note also that Theorem 32 would imply Theorem 29 in the case when $p = 2$ too, if the following conjecture is true.

Conjecture 1 (Infratype 2 conjecture; see Conjecture (p. 92), [5])**.** *Let X be a Banach space of infratype 2. Then, for every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X for which the series corresponding to the sequence $(\|x_n\|^2)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$, there exists a sequence of 'signs' $\theta_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(\theta_n x_n)_{n\in\mathbb{N}}$ converges in X.*

The following, weaker version, of Theorem 32(I) was announced (independently of [86]) in [87] and is included in [5] as "Pecherskii's theorem".

Theorem 33 (Announced in Theorem 4, [87], and proved in Theorem 2.3.1, [5])**.** *Let X be a Banach space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X for which*

(a) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$, *and*

(b) The sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ satisfies the (PSC)-condition.

Then, the equality

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

In (p. 23, [5]), the following observation is included after the formulation of Theorem 33:

(1) "This assertion provides the most general of the known sufficient conditions for linearity of the sum range of a series in an infinite-dimensional space".

(2) "In the finite-dimensional case Theorem 2.3.1 is identical to Steinitz's theorem..."

   (1) This is not completely true, as above, we state here too: Theorem 33 would imply Theorem 29 in the case when $p = 2$, if the infratype 2 conjecture were true.

   (2) This is true due to the following result.

Theorem 34 (Dvoretzky–Hanani theorem, [19] in the case when dim(X) = 2 and Theorem 2.2.1 (p. 22), [5], in general)**.** *Let X be a finite-dimensional normed space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which converges to zero in X. Then, there exists a sequence of 'signs' $\theta_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(\theta_n x_n)_{n\in\mathbb{N}}$ converges in X.*

We note that this result is presented on p. 24 of the Russian edition of [44] as Exercise 1.3.7; then, on p. 28 after Exercise 2.1.2, it is noted that it is equivalent to the following theorem.

Theorem 35 ([19] in the case when dim(X) = 2 and Lemma 2.2.1 (p. 21), [5], in general; see also [59])**.** *There exists a sequence $D_m, m = 1, 2, \ldots$ of strictly positive constants, with $D_1 = 1$ and with $D_m \le 2m - 1, m = 2, 3, \ldots$, for which the following statement is true.*

Let X be a finite-dimensional vector space over $\mathbb{R}$ with $dim(X) = m \ge 1$, and let $\|\cdot\|$ be a norm on X. Then, for a natural number $n > 1$ and for elements $x_j \in X$, $j = 1, \ldots, n$, there exist 'signs' $\theta_j \in \{-1, 1\}$, $j = 1, \ldots, n$, such that

$$\|\sum_{j=1}^{k} \theta_j x_j\| \le D_m \max_{j\in\mathbb{N}_n} \|x_j\|, \; k = 1, 2, \ldots, n\,. \tag{16}$$

The following version of Theorem 35 (without proof and with a reference to [88]) was formulated as Lemma 10 in [49].

There exists a sequence $D_m, m = 1, 2, \ldots$ of strictly positive constants for which the following statement is true.

If $X = \mathbb{R}^m$ with $m \in \mathbb{N}$ and with the maximum norm $\|\cdot\|$ on X, then for a bounded sequence of elements $x_j \in X$, $j = 1, 2, \ldots$, there exist 'signs' $\theta_j \in \{-1, 1\}$, $j = 1, 2, \ldots$, such that

$$\|\sum_{j=1}^{k} \theta_j x_j\| \le D_m \sup_{j\in\mathbb{N}} \|x_j\|, \; k = 1, 2, \ldots\,. \tag{17}$$

However, in [88] it is hard to find such a statement. It seems that in the case when $m > 2$, the first proof of Theorem 35 is Grinberg's proof, which appeared on pp. 178–179 of the Russian edition of [44] as a solution to Exercise 2.1.2.

It is conjectured (as in the case of Theorem 13) that for Euclidean norms, the theorem should hold with constants $D_m, m = 1, 2, \ldots$ for which the sequence $\frac{D_m}{\sqrt{m}}, m = 1, 2, \ldots$ is bounded [59].

The following result covers some cases of metrizable topological vector spaces, which may not be locally convex.

Theorem 36 (Giorgobiani)**.** *Let X be a metrizable topological vector space over $\mathbb{R}$, and let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which satisfies the (PSC)-condition.*

Assume further that

$(GiCo)$ the topology of X can be generated by a translation invariant metric d, such that

$$\inf\left\{ \frac{d(2x, 0)}{d(x, 0)} : x \in X \setminus \{0\} \right\} > 1\,.$$

Then, the following statements are valid.

(a) *(Theorem 1.2.1 (p. 34), [39])* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{LPR}[(x_n)_{n\in\mathbb{N}}; X]$.

(b) *(Announced in (Remark (p. 45), [87]), and proved in [89]; see also (Theorem 1.3.1 (p. 41), [39]))* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$ *is a closed affine subset of X.*

This theorem covers Theorem 28(I). Does Theorem 36 remain true for all metrizable topological vector spaces? The answer is unknown yet. The following result covers the general case of metrizable locally convex topological vector spaces.

Theorem 37 (Maria-Jesus Chasco–Sergei Chobanyan)**.** *Let X be a metrizable locally convex topological vector space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which satisfies the (PSC)-condition. Then, the following statements are valid.*

(a) *(Theorem 2, [90])* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{LPR}[(x_n)_{n\in\mathbb{N}}; X]$.

(b) *(Announced in (Theorem 5 (p. 15), [91]), also in [92], and proved in (Theorem 3, [90]); see also (Theorem 1.3.2 (p. 45), [39])*
If $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$*, then the equality*

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

The following inequality plays a key role in the proof of Theorem 37.

Proposition 11 (Lemma 1, [90])**.** *Let $n \geq 2$ be a natural number, let X be a vector space over $\mathbb{R}$, $\|\cdot\|$ be a seminorm on X, and $a_k \in X$, $k = 1, 2, \ldots, n$. Moreover, let*

(1) *$\pi : \mathbb{N}_n \to \mathbb{N}_n$ be an 'optimal' permutation in the following sense: for any permutation $\lambda : \mathbb{N}_n \to \mathbb{N}_n$, the inequality*

$$\max_{1\leq k\leq n} \|\sum_{j=1}^{k} a_{\pi(j)}\| \leq \max_{1\leq k\leq n} \|\sum_{j=1}^{k} a_{\lambda(j)}\|$$

holds, and

(2) *$\sigma : \mathbb{N}_n \to \mathbb{N}_n$ be the permutation associated with π as follows*

$$\sigma(k) = \pi(n-k+1),\ k = 1, 2, \ldots, n$$

Then,

$$\max_{1\leq k\leq n} \|\sum_{j=1}^{k} a_{\pi(j)}\| \leq \|\sum_{j=1}^{n} a_j\| + \max_{1\leq k\leq n} \|\sum_{j=1}^{k} \theta_j a_{\sigma(j)}\|$$

for every choice of 'signs' $\theta_j \in \{1, -1\}$, $j = 1, 2, \ldots, n$.

Theorem 37 would imply Banaszczyk's Theorem 10, if the following conjecture is true.

Conjecture 2 ((p. 109), [6], and (p. 615), [90])**.** *Let X be a complete metrizable nuclear locally convex topological vector space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which converges to zero in X. Then, there exists a sequence of 'signs' $\theta_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(\theta_n x_n)_{n\in\mathbb{N}}$ converges in X.*

Conjecture 2 is true when X is finite dimensional by Theorem 34, when $X = \mathbb{R}^{\mathbb{N}}$ (Theorem 2, [49]), and for some other nuclear spaces [93]. The following result, related to this conjecture, is true.

Theorem 38 (Wojciech Banaszczyk, [94]; announced in (Remark 10.15 (pp. 106–107), [6]))**.** *For a complete metrizable locally convex topological vector space X over $\mathbb{R}$, the following statements are equivalent:*

(i) *X is nuclear.*

(ii) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, which converges to zero in X, there exists a sequence of 'signs' $\theta_n \in \{1, -1\}$, $n = 1, 2, \ldots$ and a permutation $\sigma : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to the sequence $(\theta_n x_{\sigma(n)})_{n\in\mathbb{N}}$ converges in X.*

After [90], a remarkable paper by Bonet and Defant [95] and a paper by Sofi [96] appeared. The first one deals with Banaszczyk's type rearrangement theorems for (not necessarily metrizable) nuclear locally convex spaces. The second one contains Chasco–Chobanyan-type results imposing conditions on series different from the (PSC)-condition.

5. Additional Comments

During our expositions, we have indicated several problems, which would be interesting to solve. In connection with Chasco–Chobanyan's theorem, it would be interesting to answer also the following questions.

Question 4. *Is Theorem 37(b) true without the assumption that* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$*?*

Question 5. *Let X be as in Theorem 37 and* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *and* $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ *be two sequences in X, which satisfy the* (PSC)*-condition. Does their sum* $(x_n + y_n)_{n\in\mathbb{N}}$ *satisfy the* (PSC)*-condition?*

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References

1. Dudley, R.M. On sequential convergence. *Trans. Am. Math. Soc.* **1964**, *112*, 483–507. [CrossRef]
2. Dudley, R.M. Corrections to "On sequential convergence". *Trans. Am. Math. Soc.* **1970**, *148*, 623–624.
3. Cauchy, A.L *Résumés Analytiques*; De l'Imprimerie Royale: Turin, Italy, 1833.
4. Dirichlet, P. There are infinitely many prime numbers in all arithmetic progressions with first term and difference coprime. *arXiv* **2008**, arXiv:math/0808.1408; Originally published in Abhandlungen der Königlich Preussischen Akademie der Wissenschaften von 1837, 45-81. Translated by Ralf Stephan.
5. Kadets, V.M.; Kadets, M.I. *Series in Banach Spaces: Conditional and Unconditional Convergence*; Lacob, A., Translator; Operator Theory Advances and Applications; Birkhauser: Basel, Switzerland, 1997; Volume 94.
6. Banaszczyk, W. *Additive Subgroups of Topological Vector Spaces*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1991; Volume 1466.
7. Fréchet, M. Sur le résultat du changement de l'ordre des termes dans une série. *Nouv. Ann. MathéMatiques 4e Série* **1903**, *3*, 507–511.
8. Bourbaki, N. *General Topology*; Hermann-Addison Wesley: Boston, MA, USA, 1966; Part 1, Chapters I–IV.
9. Castejón, A.; Corbacho, E.; Tarieladze, V. Series with Commuting Terms in Topologized Semigroups. *Axioms* **2021**, *10*, 237. [CrossRef]
10. Wald, A. Reihen in topologischen Gruppen. In *Ergebnisse Math*; Kolloquium: Wien, Austria, 1933; Volume 5, pp. 14–16.
11. Riemann, B. On the representation of a function by a trigonometric series (1854). In *Riemann Bernhard, Works*; Translated from German to Russian and edited by V.L. Goncharov: Moscow, Russia; Royal Society of Sciences in Göttingen: Göttingen, Germany, 1948; pp. 225–261.
12. Riemann, B. On the representation of a function by a trigonometric series. In *Riemann Bernhard Collected Papers*; Translated from the 1892 German; Baker, R., Christensen, C., Orde, H., Eds.; Kendrick Press: Heber City, UT, USA, 2004; pp. 219–256.
13. Diestel, J.; Jarchow, H.; Tonge, A. *Absolutely Summing Operators*; Cambridge University Press: Cambridge, UK, 1995; No. 43.
14. Riemann, B. *Ueber die Darstellbarkeit einer Function durch eine Trigonometrische Reihe*; Habilitationsschrift Universiat: Gottingen, Germany, 1854.
15. Dini, U. Sui Prodotti Infiniti. *Annali Matematica Pura Applicata* **1868**, *2*, 28–38. [CrossRef]
16. Dini, U. *Fondamenti per la Teorica Delle Funzioni di Variable Real*; Nistri: Pisa, Italy, 1878.
17. Rudin, W. *Principles of Mathematical Analysis*, 3rd ed.; McGraw-Hill Inc.: New York, NY, USA, 1976.
18. Gelbaum, B.R.; Olmstead, J.M.H. *Counterexamples in Analysis*; Ulyanov, P.L., Ed.; Golubov, B.I., Translator; 1967; Amsterdam, The Netherlands; Mir: Moscow, Russia, 1964.
19. Dvoretzki, A.; Hanani, H. Sur les changements des signes des termes d'une série à termes complexes. *C. R. Acad. Sci. Paris* **1947**, *225*, 516–518.
20. Lévy, P. Sur les séries semi-convergentes. *Nouv. Ann. MathéMatiques* **1905**, *5*, 506–511.
21. Steinitz, E. Bedingt konvergente Reihen und konvexe Systeme. *J. Die Reine Angew. Math. Bd.* **1914**, *144*, 1–40.
22. Steinitz, E. Bedingt konvergente Reihen und konvexe Systeme. *J. Die Reine Angew. Math. Bd.* **1916**, *146*, 1–52. [CrossRef]
23. Steinitz, E. Bedingt konvergente Reihen und konvexe Systeme. *J. Die Reine Angew. Math. Bd.* **1913**, *143*, 128–176. [CrossRef]
24. Landau, E. *Einfuhrung in die Differentialrechnung und Integralrechnung*; Noordhoff: Groningen, The Netherlands, 1934.

25. Banaszczyk, W. The Steinitz theorem on rearrangement of series for nuclear spaces. *J. Reine Angew. Math.* **1990**, *403*, 187–200.
26. Domínguez, X.; Tarieladze, V. GP-nuclear groups. Nuclear Groups and Lie Groups. *Res. Exp. Math.* **2000**, *24*, 127–161.
27. Domínguez, X.; Tarieladze, V. Nuclear and GP-nuclear groups. *Acta Math. Hung.* **2000**, *88*, 301–322. [CrossRef]
28. Schaefer, H.H. *Topological Vector Spaces*, 3rd ed.; Springer: Berlin/Heidelberg, Germany, 1971.
29. Martín-Peinador, E.; Rodés, A. Sobre el dominio de sumabilidad débil de una sucesión en un espacio de Banach. In *Libro Homenaje al Profesor D. Rafael Cid*; Publicaciones de la Universidad de Zaragoza: Zaragoza, Spain, 1987; pp. 137–146.
30. Giorgobiani, G.; Tarieladze, V. On complex universal series. *Proc. A. Razmadze Math. Inst.* **2012**, *160*, 53–63.
31. Chobanyan, S. Structure of the set of sums of a conditionally convergent series in a normed space. *Math. USSR-Sb.* **1985**, *128*, 50–65. [CrossRef]
32. Banaszczyk, W. Rearrangement of series in non-nuclear spaces. *Studia Math.* **1993**, *107*, 213–222. [CrossRef]
33. Mauldin, R.D. *The Scottish Book*; Birkhäuser: Boston, MA, USA, 1981; Volume 4.
34. Maligranda, L. Józef Marcinkiewicz (1910–1940)-on the centenary of his birth. *Banach Cent. Publ.* **2011**, *95*, 133–234. [CrossRef]
35. Ostrovskii, M.I. Domains of sums of conditionally convergent series in Banach spaces. *Teor. Funkts Funkts Anal. Prilozh.* **1986**, *46*, 77–85. Ostrovskii, M.I., Translator; Set of sums of conditionally convergent series in Banach spaces. *J. Math. Sci.* **1990**, *48*, 559–566. (In Russian)
36. Tarieladze, V. *On the Sum Range Problem*; Caucasian Mathematics Conference CMC II, Book of Abstracts; Turkish Mathematical Society: Van, Turkey, 2017; pp. 126–127.
37. Kornilov, P.A. On the set of sums of a conditionally convergent series of functions. *Mat. Sb.* **1988**, *179*, 114–127; English Translation: Mathematics of the USSR-Sbornik **1990**, *65*, 119–131. [CrossRef]
38. Giorgobiani, G. Some remarks about the set of sums of a conditionally convergent series in a Banach Space. *Proc. Inst. Comp. Math.* **1988**, *33*, 38–44. (In Russian)
39. Giorgobiani, G. Rearrangements of series. *J. Math. Sci.* **2019**, *239*, 437–548. [CrossRef]
40. Hadwiger, H. Über das Umordnungsproblem im Hilbertschen Raum. *Math. Zeit.* **1940**, *46*, 70–79. [CrossRef]
41. Kadets, M.I.; Wozniakowski, K. On series whose permutations have only two sums. *Bull. Pol. Acad. Sci. Math.* **1989**, *37*, 15–21.
42. Wojtaszczyk, J.O. A series whose sum range is an arbitrary finite set. *Stud. Math.* **2005**, *171*, 261–281. [CrossRef]
43. Kadets, V.M. On a problem of the existence of convergent rearrangement. *Izv.Vyssh. Uchebn. Zaved. Mat.* **1992,** 3, 7–9.
44. Kadets, V.M.; Kadets, M.I. *Rearrangements of Series in Banach Spaces*; McFaden, H.H., Translator; Translations of Mathematical Monographs; American Mathematical Society: Providence, RI, USA, 1991; Volume 86, ISSN 0065-9282.
45. Halperin, I. Sums of a series, permitting rearrangements. *C. R. Math. Rep. Acad. Sci. Can.* **1986**, *8*, 87–102.
46. Banakh, T. A simple inductive proof of Levy-Steinitz theorem. *arXiv* **2017**, arXiv:math/1711.04136.
47. Tarieladze, V. Is "Weakly Good" Series in a Finite-Dimensional Banach Space "Good"? Lviv Scottish Book. 24 September 2017. Available online: https://mathoverflow.net/questions/281948/is-weakly-good-series-in-a-finite-dimensional-banach-space-good (accessed on 8 May 2023).
48. Rosenthal, P. The remarkable theorem of Lévy and Steinitz. *Am. Math. Mon.* **1987**, *94*, 342–351.
49. Katznelson, Y.; McGehee, O.C. Conditionally convergent series in R^{∞}. *Mich. Math. J.* **1974**, *21*, 97–106. [CrossRef]
50. Shklyarskii, D.O. Conditionally convergent series of vectors. *Uspekhi Mat. Nauk.* **1944**, *10*, 51–59.
51. Kadets, M.I. On a property of broken lines in n-dimensional space. *Uspekhi Mat. Nauk.* **1953**, *8*, 139–143.
52. Groß, W. Bedingt konvergente Reihen. *Monatshefte Math. Phys.* **1917**, *28*, 221–237. [CrossRef]
53. Gödel, K. Simplified proof of a theorem of Steinitz. In *Kurt Gödel: Collected Works: Volume III: Unpublished Essays and Lectures*; Oxford University Press on Demand: Oxford, UK, 1986; Volume 3, pp. 56–61.
54. Halperin, I. Introductory note to "Simplified proof of a theorem of Steinitz" by K. Gödel. In *Kurt Gödel: Collected Works: Volume III: Unpublished Essays and Lectures*; Oxford University Press on Demand: Oxford, UK, 1986; Volume 3, pp. 54–55.
55. Wald, A. Vereinfachter beweis des Steinitzschen satzes über vektorenreihen im R_n. In *Ergebnisse Math*; Kolloquium: Wien, Austria, 1933; Volume 5, pp. 10–13.
56. Wald, A. Bedingt konvergente Reihen von Vektoren im R_{ω}. In *Ergebnisse Math*; Kolloquium: Wien, Austria, 1933; Volume 5, pp. 13–14.
57. Bourbaki, N. *General Topology*; Hermann: Paris, France, 1966; Part 2, Chapters V–VIII.
58. Sevastyanov, S.V. Geometric Methods and Effective Algorithms in Scheduling Theory. Ph.D. Thesis, Sobolev Institute of Mathematics, Russian Academy of Sciences, Novosibirsk, Russia, 2000; 283p. (In Russian)
59. Bárány, I. On the power of linear dependencies. In *Building Bridges*; Springer: Berlin/Heidelberg, Germany, 2008; pp. 31–45.
60. Chobanyan, S.; Giorgobiani, G.; Kvaratskhelia, V.; Levental, S.; Tarieladze, V. On rearrangement theorems in Banach spaces. *Georgian Math. J.* **2014**, *21*, 157–163. [CrossRef]
61. Kadets, M.I. On conditionally convergent series in the space L^p. *Uspekhi Mat. Nauk.* **1954**, *9*, 107–109.
62. Troyanski, S. Conditionally converging series and certain F-spaces. *Teor. Funkts. Funkts. Anal. Prilozh.* **1967**, *5*, 102–107. (In Russian)
63. Hadwiger, H. Über die Konvergenzarten unendlicher Reihen im Hilbertschen Raum. *Math. Zeit.* **1941**, *47*, 325–329; [CrossRef]
64. McArthur, C.W. On relationships amongst certain spaces of sequences in an arbitrary Banach space. *Can. J. Math.* **1956**, *8*, 192–197. [CrossRef]

65. Kashin, B.S. On a property of functional series. *Mat. Zametki* **1972**, *11*, 481–490; English version: Mathematical Notes **1972**, *11*, 294–299. [CrossRef]
66. Drobot, V. Rearrangements of series of functions. *Trans. Am. Math. Soc.* **1969**, *142*, 239–248. [CrossRef]
67. Kashin, B.S.; Saakyan, A.A. *Orthogonal Series*; American Mathematical Society: Providence, RI, USA, 2005; Volume 75.
68. Drobot, V. A note on rearrangements of series. *Stud. Math.* **1970**, *35*, 177–179. [CrossRef]
69. Fonf, V.P. Conditionally convergent series in a uniformly smooth Banach space. *Matematicheskie Zametki* **1972**, *11*, 209–214. English Translation: Mathematical Notes of the Academy of Sciences of the USSR **1972**, *11*, 129–132. [CrossRef]
70. Nikishin, E.M. Rearrangements of function series. *Math. USSR-Sb.* **1971**, *127*, 272–285. English Translation: Math. USSR-Sb. **1971**, *14*, 267. (In Russian)
71. Nikishin, E.M. On the set of sums of a functional series. *Math. Notes Acad. Sci. USSR* **1970**, *7*, 243–247. [CrossRef]
72. Orlicz, W. On the independence of the order of almost everywhere convergence of function series (German). *Bull. Int. L'academie Pol. Des. Sci. Ser. A* **1927**, 117–125. Available online: https://zbmath.org/53.0243.04 (accessed on 7 May 2023)
73. Kornilov, P.A. On rearrangements of conditionally convergent series of functions. *Math. USSR-Sb.* **1980**, *113*, 598–616. [CrossRef]
74. Pecherskii, D.V. On rearrangements of terms in functional series. *Dokl. Akad. Nauk SSSR* **1973**, *209*, 1285–1287.
75. Karatsuba, A.A.; Voronin, S.M. *The Riemann Zeta-Function*; Koblitz, N., Translator; Walter de Gruyter: Berlin, Germany, 2011; Volume 5.
76. Nikishin, E.M. Rearrangements of series in L_p. *Mat. Zametki* **1973**, *14*, 31–38. English Translation: Mathematical Notes of the Academy of Sciences of the USSR **1973**, *14*, 570–574. (In Russian) [CrossRef]
77. Pecherskii, D.V. A theorem on projections of rearranged series with terms in L_p. Izv. *AN SSSR. Ser. Matem.* **1977**, *41*, 203–214. English Translation: Mathematics of the USSR-Izvestiya **1977**, *11*, 193. (In Russian)
78. Barani, I. Permutations of series in infinite-dimensional spaces. *Mat. Zametki* **1989**, *46*, 10–17. English Translation: Mathematical Notes of the Academy of Sciences of the USSR **1989**, *46*, 895–900. (In Russian)
79. Chobanyan, S. Convergence of Bernoulli series and the set of sums of a conditionally convergent functional series. *Teorya Veroyanost. Primen.* **1983**, *28*, 420–429.
80. Megrabian, R.M. On the set of sums of functional series in spaces L_ΦTeor. *Veroyatnost. Primen.* **1985**, *30*, 511–523.
81. Kadets, V.M. B-convexity and the Steinitz lemma, Izv. Severo-Kavkaz. Nauchn. *Tsentra Vyssh. Shkoly Estestv. Nauk.* **1984**, 4, 27–29. (In Russian)
82. Chobanyan, S. The structure of a set of sums of a conditionally convergent series in Banach space. *Dokl. Akad. Nauk SSSR* **1984**, *278*, 556–559.
83. Vakhania, N.N.; Tarieladze, V.; Chobanyan, S. *Probability Distributions on Banach Spaces*; Russian Edition; Nauka: Moscow, Russia, 1985; 368p. English Edition: D. Reidel Publ. C. 1987, 482p.
84. Talagrand, M. Type, infratype and the Elton-Pajor theorem. *Invent. Math.* **1992**, *107*, 41–59. [CrossRef]
85. Talagrand, M. Type and infratype in symmetric sequence spaces. *Isr. J. Math.* **2004**, *143*, 157–180. [CrossRef]
86. Pecherskii, D.V. Rearrangements of series in Banach spaces and arrangements of signs. Mat. Sb. **1988**, *135*, 24–35. English Translation: Mathematics of the USSR-Sbornik **1989**, *63*, 23–33. [CrossRef]
87. Chobanyan, S.A.; Giorgobiani, G.J. A problem of rearrangement of summands in normed space and Rademacher sums Probability Theory on Vector Spaces IV. In *Probability Theory on Vector Spaces IV, Proceedings of the Conference, Łańcut, Poland, 10–17 June 1987*; Lecture notes in Mathematics 1391; Cambanis, S., Weron, A., Eds.; Springer: Berlin/Heidelberg, Germany, 1989; pp. 33–46.
88. Calabi, E.; Dvoretzky, A. Convergence-and sum-factors for series of complex numbers. *Trans. Am. Math. Soc.* **1951**, *70*, 177–194. [CrossRef]
89. Giorgobiani, G.D. Structure of the set of sums of a conditionally converging series in a p-normed space. *Bull. Acad. Sci. Georgian SSR* **1988**, *130*, 481–484.
90. Chasco, M.J.; Chobanyan, S. Rearrangements of series in locally convex spaces. *Mich. Math. J.* **1997**, *44*, 607–617. [CrossRef]
91. Chobanyan, S. *On Some Inequalities Related to Permutations of Summands in a Normed Space*; Academy of Sciences of Georgian SSR, Muskhelishvili Institute of Computational Mathematics: Tbilisi, Georgia, 1990; 21p.
92. Chobanyan, S. Convergence a. s. of rearranged random series in Banach space and associated inequalities. In *Probability in Banach Spaces*; Birkhäuser: Boston, MA, USA, 1994; Volume 9, pp. 3–29.
93. Núñez-García, J. On Certain Varieties of Nuclear Groups (Spanish). Ph.D. Thesis, Universidad Complutense, Madrid, Spain, 2002.
94. Banaszczyk, W. Balancing vectors and convex bodies. *Stud. Math.* **1993**, *106*, 93–100. [CrossRef]
95. Bonet, J.; Defant, A. The Levy-Steinitz rearrangement theorem for duals of metrizable spaces. *Israel J. Math.* **2000**, *117*, 131–156. [CrossRef]
96. Sofi, M.A. Levy-Steinitz theorem in infinite dimension. *N. Z. J. Math.* **2008**, *38*, 63–73.

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Series with Commuting Terms in Topologized Semigroups

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Abstract: We show that the following general version of the Riemann–Dirichlet theorem is true: if every rearrangement of a series with pairwise commuting terms in a Hausdorff topologized semigroup converges, then its sum range is a singleton.

Keywords: semigroup; group; topology; permutation; convergence

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1. Introduction

In 1827, Peter Lejeune-Dirichlet was the first to notice that it is possible to rearrange the terms of certain convergent series of real numbers so that the sum changes [1]. According to [2] (Ch. 2, §2.4), In 1833, Augustin-Louis Cauchy also noticed this in his "Resumes analytiques".

Later, in 1837, Dirichlet showed that this cannot happen if the series converges absolutely: if a series formed by absolute values of a term of series of real numbers converges, then the series itself converges and every rearrangement also converges to the same sum. A series in which every rearrangement converges is called unconditionally convergent. Let us define the sum range of series as the set of all sums of all its convergent rearrangements.

It is not clear in advance that an unconditionally convergent series of real numbers is also absolutely convergent, and hence its sum range is a singleton. This is in fact true thanks to the following *Riemann rearrangement theorem*: if a convergent series of real numbers **is not** absolutely convergent, then some rearrangement **is not** convergent, and its sum range is the set of **all real numbers**.

These results depend heavily on the structure of the set of real numbers. However, the concepts of unconditional convergence and sum range make sense even in general topologized semigroups. An abelian version of the statement in the abstract appears in (unpublished) [3]. A non-abelian version for topological groups appears in [4].

Section 2 focuses on 'finite series' and Section 3 treats the general case. Section 4 contains additional comments.

2. Algebraic Part

We write $\mathbb{N}$ for the set $\{1, 2, \dots\}$ of natural numbers with its usual order and

$$\mathbb{N}_n := \{k \in \mathbb{N} : k \leq n\}, \ n = 1, 2, \dots$$

A non-empty set, X, endowed with a binary operation $+ : X \times X \to X$ is called a groupoid or a magma. For a groupoid, $(X, +)$, the value of $+$ at $(x_1, x_2) \in X \times X$ will be denoted as $x_1 + x_2$.

For a finite non-empty $I \subset \mathbb{N}$ and a family $(x_i)_{i \in I}$ of elements of a groupoid $(X, +)$, following Bourbaki, we define the (ordered) sum

$$\sum_{i \in I} x_i \in X \qquad \text{(OS)}$$

inductively as follows:

(1) If I consists of a single element, $I = \{j\}$, then $\sum_{i \in I} x_i = x_j$;

(2) If I has more than one element, j is the least element of I and $I' = I \setminus \{j\}$, then

$$\sum_{i \in I} x_i = x_j + \left(\sum_{i \in I'} x_i \right).$$

Note that:

If I consists of two elements, then $\sum_{i \in I} x_i = x_j + x_k$, where j is the least element of I and k is the last element of I;

If I consists of three elements, then $\sum_{i \in I} x_i = x_j + (x_m + x_k)$, where again, j is the least element of I, k is the last element of I and $j < m < k$.

If $I = \mathbb{N}_n$, then instead of $\sum_{i \in I} x_i$ we write also $\sum_{i=1}^{n} x_i$.

A groupoid, $(X, +)$, is a *semigroup* if its binary operation $+$ is associative, i.e., for every $(x_1, x_2, x_3) \in X \times X \times X$ we have $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$.

For a finite non-empty $I \subset \mathbb{N}$ and a family $(x_i)_{i \in I}$ of elements of a *semigroup* $(X, +)$ the above given definition of (OS) can be reformulated as follows:

(1r) if I consists of a single element, $I = \{k\}$, then $\sum_{i \in I} x_i = x_k$,

(2r) if I has more than one element, k is the last element of I and $I' = I \setminus \{k\}$, then

$$\sum_{i \in I} x_i = \left(\sum_{i \in I'} x_i \right) + x_k.$$

For a set I a bijection $\sigma : I \to I$ called a permutation of I; the set of all permutations of I is denoted by $\mathbb{S}(I)$.

For a finite non-empty $I \subset \mathbb{N}$ and a family $(x_i)_{i \in I}$ of elements of a groupoid $(X, +)$,we define its *sum range*

$$SR((x_i)_{i \in I})$$

as follows:

$$SR((x_i)_{i \in I}) := \{s \in X : \exists \sigma \in \mathbb{S}(I),\ s = \sum_{i \in I} x_{\sigma(i)}\}.$$

In a case where the multiplicative notation $\cdot$ is applied for the binary operation, it would be natural to use the word 'product' instead of 'sum'; 'ordered product' (OP) *instead of 'ordered sum'* (OS)*; 'product range'* (PR) *instead 'sum range'* (SR) *and $\prod$ instead of $\sum$.*

Two elements, x_1 and x_2, of a groupoid, $(X, +)$, are said to commute (or to be permutable) if $x_1 + x_2 = x_2 + x_1$; i.e., if $SR\big((x_i)_{i \in \mathbb{N}_2}\big)$ is a singleton.

A family $(x_i)_{i \in I}$ of elements of a groupoid $(X, +)$ is *commuting* if for each $i \in I$ and $j \in I$, the elements x_i and x_j commute.

An element a of a groupoid $(X, +)$ is *left cancellable* if the left translation mapping $x \mapsto a + x$ is injective; *right cancellable* is defined similarly. An element is *cancellable* if it is both left and right cancellable.

Theorem 1 (Commutativity theorem)**.** *For a finite non-empty $I \subset \mathbb{N}$ and a family $(x_i)_{i \in I}$ of elements of a semigroup $(X, +)$ the following statements are true.*

(a) If $(x_i)_{i \in I}$ is a **commuting** *family, then $SR((x_i)_{i \in I})$ is a singleton.*

(b) If $SR((x_i)_{i \in I})$ is a singleton and either Card(I) ≤ 2 *or for every $i \in I$ the element x_i is right (resp. left) cancellable, then $(x_i)_{i \in I}$ is a* **commuting** *family.*

Proof. (a) See [5] [Ch.1, §1.5, Theorem 2 (p. 9)].

(b) For the case Card(I) ≤ 2 the statement is evident. Now, let $n = \text{Card(I)} > 2$ and for every $i \in I$ the element x_i is right cancellable. Fix $i, j \in I$, $i \neq j$, write $I'' = I \setminus \{i, j\}$. Also write $I = \{k_1, k_2, \dots, k_n\}$, where $k_1 < k_2 < \dots < k_n$. Moreover, consider permutations σ and π of I such that $\sigma(k_1) = i$, $\sigma(k_2) = j$, $\sigma(\{k_3, \dots, k_n\}) = I''$ and $\pi(k_1) = j$, $\pi(k_2) = i$, $\pi(\{k_3, \dots, k_n\}) = I''$. As $SR((x_i)_{i \in I})$ is a singleton, we can write:

$$x_i + x_j + \left(\sum_{r \in I''} x_r \right) = \sum_{i \in I} x_{\sigma(i)} = \sum_{i \in I} x_{\pi(i)} = x_j + x_i + \left(\sum_{r \in I''} x_r \right).$$

From this equality, as $\sum_{r \in I''} x_r$ is right cancellable, we obtain $x_i + x_j = x_j + x_i$. The case where Card(I) > 2 and for every $i \in I$ the element x_i is left cancellable is considered similarly. □

Our next claim is to find an analog of Theorem 1 when $I = \mathbb{N}$.

3. Series

A (formal) *series corresponding to a sequence* $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of elements of a groupoid $(X, +)$ is the sequence

$$\left(\sum_{k \in \mathbb{N}_n} x_k \right)_{n \in \mathbb{N}}. \qquad \text{(S1)}$$

The 'multiplicative' counterpart is: a (formal) *infinite product corresponding to a sequence* $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of elements of a groupoid $(X, \cdot)$ is the sequence

$$\left(\prod_{k \in \mathbb{N}_n} x_k \right)_{n \in \mathbb{N}}. \qquad \text{(P1)}$$

We use the additive notation herein.

Let $(X, +)$ be a groupoid and τ be a topology in X; such a triplet $(X, +, \tau)$ will be called *a topologized groupoid*.

A topologized groupoid $(X, +, \tau)$ is *a topological groupoid* if its binary operation $+$ is continuous as mapping from $(X \times X, \tau \otimes \tau)$ to (X, τ) (where $\tau \otimes \tau$ stands for the product topology).

A series corresponding to a sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of elements of a topologized groupoid $(X, +, \tau)$ is said to be convergent in $(X, +, \tau)$ if the sequence (S1) converges to an element $s \in X$ in the topology τ; in such a case, we write

$$s = \sum_{k=1}^{\infty} x_k$$

and call s a sum of the series.

To a sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of elements of a topologized groupoid $(X, +, \tau)$, we associate a subset $\mathfrak{P}(\mathbf{x})$ of $\mathbb{S}(\mathbb{N})$ as follows: a permutation $\pi : \mathbb{N} \to \mathbb{N}$ belongs to $\mathfrak{P}(\mathbf{x})$ if and only if the series corresponding to $(x_{\pi(n)})_{n \in \mathbb{N}}$ is convergent in $(X, +, \tau)$ and define *the sum range* of the series corresponding to $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$

$$SR(\mathbf{x})$$

as follows (cf. [6] (Definition 2.1.1)):

$$SR(\mathbf{x}) := \{t \in X : \exists \pi \in \mathfrak{P}(\mathbf{x}),\ t = \sum_{k=1}^{\infty} x_{\pi(k)}\}.$$

It may happen that for a sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ the set $\mathfrak{P}(\mathbf{x})$ is empty; in which case, $SR(\mathbf{x}) = \emptyset$ as well.

The series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is called *unconditionally convergent* (Bourbaki says *commutatively convergent* [7]) in $(X, +, \tau)$ if

$$\mathfrak{P}(\mathbf{x}) = \mathbb{S}(\mathbb{N})\,;$$

i.e., *if for every permutation* $\sigma : \mathbb{N} \to \mathbb{N}$ *the series corresponding to* $\mathbf{x}_\sigma = (x_{\sigma(n)})_{n\in\mathbb{N}}$ *is convergent in* $(X, +, \tau)$.

We proceed to our main result, extending to topologized semigroups the results for topological groups in [4] (Theorem 2 and Theorem 1).

Theorem 2 (Commutativity Theorem 2)**.** *For a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *of elements of a Hausdorff topologized semigroup* $(X, +, \tau)$*, the following statements are true.*

(a') *If the series corresponding to* $\mathbf{x}$ *is convergent in* $(X, +, \tau)$*,* $\mathbf{x}$ *is a* **commuting** *family and* $SR(\mathbf{x})$ **is not** *a singleton, then there is a permutation* $\lambda : \mathbb{N} \to \mathbb{N}$ *such that the series corresponding to* $\mathbf{x}_\lambda = (x_{\lambda(n)})_{n\in\mathbb{N}}$ *is not convergent in* $(X, +, \tau)$.

(a) *If the series corresponding to* $\mathbf{x}$ *is* **unconditionally convergent** *in* $(X, +, \tau)$ *and* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *is a* **commuting** *family, then* $SR(\mathbf{x})$ *is a singleton.*

(b) *If* $SR(\mathbf{x})$ *is a singleton,* $(X, +)$ *is a group and for every* $n \in \mathbb{N}$ *the left translation determined by* x_n *is sequentially continuous, then* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *is a* **commuting** *family.*

Proof. (a').

To prove (a'), denote by s the limit in $(X, +, \tau)$ of the sequence (S1), i.e.,

$$(\tau)\lim_n \sum_{k\in\mathbb{N}_n} x_k = s\,. \tag{1}$$

Since $SR(\mathbf{x})$ **is not** a singleton, there is $t \in SR(\mathbf{x})$ such that $t \neq s$. Hence, there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the series corresponding to $\mathbf{x}_\pi = (x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent to t in $(X, +, \tau)$, i.e.,

$$(\tau)\lim_n \sum_{k\in\mathbb{N}_n} x_{\pi(k)} = t\,. \tag{2}$$

Construction of a permutation $\lambda : \mathbb{N} \to \mathbb{N}$.

Find and fix a **strictly increasing** sequence of natural numbers $(m_k)_{k\in\mathbb{N}}$ such that

$$1 = m_1,\ \mathbb{N}_{m_{2k-1}} \subset \pi(\mathbb{N}_{m_{2k}}) \subset \mathbb{N}_{m_{2k+1}},\ k = 1, 2, \ldots \tag{3}$$

Now, define a mapping $\lambda : \mathbb{N} \to \mathbb{N}$ as follows:

$$\lambda(1) = 1; \quad \lambda(\mathbb{N}_{m_{2k}} \setminus \mathbb{N}_{m_{2k-1}}) = \pi(\mathbb{N}_{m_{2k}}) \setminus \mathbb{N}_{m_{2k-1}};$$
$$\lambda(\mathbb{N}_{m_{2k+1}} \setminus \mathbb{N}_{m_{2k}}) = \mathbb{N}_{m_{2k+1}} \setminus \pi(\mathbb{N}_{m_{2k}}), \quad k = 1, 2, \ldots \tag{4}$$

It is easy to see that $\lambda \in \mathbb{S}(\mathbb{N})$.

From (3) and (4), we can conclude that

$$\lambda(\mathbb{N}_{m_{2k+1}}) = \mathbb{N}_{m_{2k+1}}\,, k = 1, 2, \ldots \tag{5}$$

and

$$\lambda(\mathbb{N}_{m_{2k}}) = \pi(\mathbb{N}_{m_{2k}})\,, k = 1, 2, \ldots \tag{6}$$

From (5) and (6) together with Theorem 1(a) (which is applicable because $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is a commuting family), we conclude that the following relations are true:

$$\sum_{i=1}^{m_{2k+1}} x_{\lambda(i)} = \sum_{i=1}^{m_{2k+1}} x_i,\ k = 1, 2, \ldots \tag{7}$$

and

$$\sum_{i=1}^{m_{2k}} x_{\lambda(i)} = \sum_{i=1}^{m_{2k}} x_{\pi(i)},\ k = 2, 3, \ldots \tag{8}$$

The equality (7) implies:

$$\lim_{k} \sum_{i=1}^{m_{2k+1}} x_{\lambda(i)} = s\,, \tag{9}$$

while the equality (8) implies:

$$\lim_{k} \sum_{i=1}^{m_{2k}} x_{\lambda(i)} = t\,. \tag{10}$$

From (9) and (10), since $t \neq s$ and τ is a *Hausdorff topology*, we conclude that $(\sum_{i=1}^{n} x_{\lambda(i)})_{n\in\mathbb{N}}$ is not a convergent sequence. Therefore, we found a permutation $\lambda : \mathbb{N} \to \mathbb{N}$ such that the series corresponding to $\mathbf{x}_\lambda = (x_{\lambda(n)})_{n\in\mathbb{N}}$ is not convergent in $(X, +, \tau)$ and (a') is proved.

(a) follows from (a').

(b) In view of Theorem $1(b)$, it is sufficient to show that for a fixed natural number $n > 1$ we find that $SR((x_i)_{i\in\mathbb{N}_n})$ is a singleton.

We can suppose without loss of generality that the series corresponding to $\mathbf{x}$ is convergent in $(X, +, \tau)$ to $s \in X$. This implies:

$$\lim_{m>n} \left(\sum_{i\in\mathbb{N}_n} x_i + \sum_{i\in\mathbb{N}_m\setminus\mathbb{N}_n} x_i \right) = s\,.$$

From this, since the left translations are continuous, we obtain:

$$\lim_{m>n} \sum_{i\in\mathbb{N}_m\setminus\mathbb{N}_n} x_i = -\sum_{i\in\mathbb{N}_n} x_i + s\,.$$

Now, fix an arbitrary permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi(k) = k$, $k = n+1, n+2, \ldots$ From the above equality, since the left translations are continuous, we can now write

$$\lim_{m>n} \left(\sum_{i\in\mathbb{N}_n} x_{\pi(i)} + \sum_{i\in\mathbb{N}_m\setminus\mathbb{N}_n} x_i \right) = \sum_{i\in\mathbb{N}_n} x_{\pi(i)} + (-\sum_{i\in\mathbb{N}_n} x_i + s)\,.$$

Hence, since $SR(\mathbf{x})$ is a singleton, we conclude:

$$\sum_{i\in\mathbb{N}_n} x_{\pi(i)} + (-\sum_{i\in\mathbb{N}_n} x_i + s) = s\,.$$

Therefore,

$$\sum_{i\in\mathbb{N}_n} x_{\pi(i)} = \sum_{i\in\mathbb{N}_n} x_i$$

and, as π is arbitrary, we prove that $SR((x_i)_{i\in\mathbb{N}_n})$ is a singleton. □

Remark 1. *Theorem 2(a) for a Banach space was first proved in [8], where the term "B-space" was used and it was also noticed that this term is credited to M. Frechet. In [9], where the term 'Banach space' is already used, one finds a nice discussion of equivalent characterizations of unconditional convergence.*

4. Additional Comments

4.1. On Theorem 2

The statement (b) of Theorem 2 **is not** a complete converse of statement (a) of Theorem 2; in the case of Hausdorff topological groups, such a complete converse can be formulated as follows:

If for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *of elements of a Hausdorff topological group* X *the set* $SR(\mathbf{x})$ *is a singleton, then the series corresponding to* $\mathbf{x}$ *is* **unconditionally convergent** *in* X *and* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *is a* **commuting** *family.*

Let us say that a Hausdorff topological group X *has property* (HM) if whenever for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ the set $SR(\mathbf{x})$ is a singleton, then the series corresponding to $\mathbf{x}$ is **unconditionally convergent** in X.

The Riemann rearrangement theorem implies that $X = \mathbb{R}$ *has property* (HM). In [10], it was shown that if X is an infinite-dimensional Hilbert space, then X **does not have** property (HM); a similar result was obtained in [11] for infinite-dimensional Banach spaces. From the general result of [12], we conclude that the finite-dimensional real normed spaces, as well as the countable product of real lines $\mathbb{R}^{\mathbb{N}}$, have property (HM).

4.2. On Sum Ranges

A subset A of a topological group X is *a sum range* if a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X exists such that $A = SR(\mathbf{x})$. Known results and the history of the study of the structure of sum ranges in Banach spaces are found in [6]; see also, [12–18].

A subset A of a real vector space X is called *affine* if

$$x_1 \in A,\ x_2 \in A,\ t \in \mathbb{R},\ \Longrightarrow\ tx_1 + (1-t)x_2 \in A\,.$$

It is known that:

- A subset of a finite-dimensional real Banach space is a sum range if and only if it is affine (Steinitz's theorem, see [6]);
- A subset of a real nuclear Frechet space is a sum range if and only if it is closed and affine [13];
- Every closed affine subset of a *separable* real Frechet space can be a sum range (cf. [19], where the following question is left open: *is every separable infinite-dimensional complete metrizable real topological vector space a sum range?*);
- An arbitrary *finite* subset of an *infinite-dimensional* Banach space can be a sum range [20];
- A *non-analytic* subset of an infinite-dimensional *separable* Banach space cannot be a sum range [21];
- A *non-closed* subset of an infinite-dimensional *separable* Banach space can be a sum range (see [6,22]; however, it is unknown whether a *non-closed vector subspace* of an infinite-dimensional *separable* Banach space can be a sum range [16]) .

Finally, note that it would be interesting to:

(1) Investigate, in connection with Theorem 2(a), the question of how rich the sum range $SR(\mathbf{x})$ can be for a **non-commuting** sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, the series corresponding to which is **unconditionally convergent**; may happen that $SR(\mathbf{x}) = X$?

(2) Find a "semigroup version" of Theorem 2(b).

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References

1. Galanor, S. Riemann's rearrangement Theorem. *Math. Teach.* **1987**, *80*, 675–681. [CrossRef]
2. Whittaker, E.T.; Watson, G.N. *A Course of Modern Analysis*, 4th ed.; Cambridge University Press: Cambridge, UK, 1927; [Russian translation by F. V. Shirokov, Moscow, 1962].
3. Castejón, A.; Corbacho, E.; Tarieladze, V. *Metric Monoids and Integration*; Manuscript; Vigo, Spain, 1995; 268p.
4. McArthur, C.W. Series with Sums Invariant Under Rearrangement. *Am. Math. Mon.* **1968**, *75*, 729–731. [CrossRef]
5. Bourbaki, N. *Algebra 1, Chapters 1–3*; Hermann: Paris, France, 1974.
6. Kadets, M.; Kadets, V. *Series in Banach Spaces: Conditional and Unconditional Convergence*; Birkhauser: Basel, Switzerland, 1997; Volume 94.
7. Bourbaki, N. *General Topology, Part 1, Chapters I–IV*; Hermann: Paris, France, 1966.
8. Orlicz, W. Beitrage zur Theorie derOrthogonalentwicklungen II. *Studia Math.* **1929**, *1*, 241–255. [CrossRef]
9. Hildebrandt, T.H. On Unconditional Convergence in Normed Vector Spaces. *Bull. Am. Math. Soc.* **1940**, *46*, 959–962. [CrossRef]
10. Hadwiger, H. Uber das Umordnungsproblem im Hilbertschen Raum. *Math. Z.* **1941**, *47*, 325–329. [CrossRef]
11. McArthur, C.W. On relationships amongs certain spaces of sequences in an arbitrary Banach space. *Can. J. Math.* **1956**, *8*, 192–197. [CrossRef]
12. Chasco, M.J.; Chobanyan, S. Rearrangements of series in locally convex spaces. *Mich. Math. J.* **1997**, *44*, 607–617. [CrossRef]
13. Banaszczyk, W. The Steinitz theorem on rearrangement of series for nuclear spaces. *J. Reine Angew. Math.* **1990**, *403*, 187–200.
14. Banaszczyk, W. *Additive Subgroups of Topological Vector Spaces*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1991; Volume 1466.
15. Bonet, J.; Defant, A. The Levy-Steinitz rearrangement theorem for duals of metrizable spaces. *Isr. J. Math.* **2000**, *117*, 131–156. [CrossRef]
16. Bonet, J. Reordenacion de series. El teorema de Levy-Steinitz. *Gac. RSME* **2013**, *16*, 449–464.
17. Giorgobiani, G. Rearrangements of series. *J. Math. Sci.* **2019**, *239*, 437–548. [CrossRef]
18. Sofi, M.A. Levy-Steinitz theorem in infinite dimension. *N. Z. J. Math.* **2008**, *38*, 63–73.
19. Giorgobiani, G.; Tarieladze, V. Special universal series. In *Several Problems of Applied Mathematics and Mechanics, Chapter 12*; Nova Science Publishers: Hauppauge, NY, USA, 2013; pp. 125–130.
20. Wojtaszczyk, J.O. A series whose sum range is an arbitrary finite set. *Stud. Math.* **2005**, *171*, 261–281. [CrossRef]
21. Tarieladze, V. On the sum range problem. In *Book of Abstracts, Proceedings of the Caucasian Mathematics Conference CMC II, Van, Turkey, 22–24 August 2017*; Turkish Mathematical Society: Istanbul, Turkey, 2017; pp. 126–127.
22. Ostrovskii, M.I. Domains of sums of conditionally convergent series in Banach spaces. *Teor. Funkts. Funkts. Anal. Prilozh.* **1986**, *46*, 77–85. (In Russian). [English translation: Ostrovskii, M.I. Set of sums of conditionally convergent series in Banach spaces. *J. Math. Sci.* **1990**, *48*, 559–566]. [CrossRef]

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Article

An Expository Lecture of María Jesús Chasco on Some Applications of Fubini's Theorem

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Abstract: The usefulness of Fubini's theorem as a measurement instrument is clearly understood from its multiple applications in Analysis, Convex Geometry, Statistics or Number Theory. This article is an expository paper based on a master class given by the second author at the University of Vigo and is devoted to presenting some Applications of Fubini's theorem. In the first part, we present Brunn–Minkowski's and Isoperimetric inequalities. The second part is devoted to the estimations of volumes of sections of balls in $\mathbb{R}^n$.

Keywords: Fubini's theorem; Brunn–Minkowski inequality; isoperimetric inequality; volumes of section of balls

MSC: 28A35; 52A40; 52A38

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1. Introduction

Fubini's theorem and Brunn–Minkowski's inequality are two cornerstones of analytical methods in convex geometry with important applications to probability theory, partial differential equations and combinatorics. The present paper is an expository note on the subject based on a master class given by the second author at the University of Vigo some years ago. The aim of including it in this volume is to commemorate her teaching trajectory. We have tried to maintain the original exposition, other than removing some very easy facts from the original lecture. In this introduction, we intend to show that the subject is still interesting and to provide the reader with some useful references in order to explore the evolution of the subject until the present time.

The paper starts by recalling Fubini's theorem. After that, we give a detailed proof of Brunn–Minkowski's inequality and, as a corollary of it, the classical isoperimetric inequality which states that, among bodies of a given volume in $\mathbb{R}^n$, the Euclidean balls have the least surface area. This result appears to have been known in ancient times for two dimensions. By the end of the last century, there were a number of proofs which worked arbitrarily in many dimensions. It is interesting to remark that the formulation of the reverse isoperimetric problem needs some care because even convex bodies can have a large surface area and a small volume [1]. A big part of the classical Brunn–Minkowski theory is concerned with establishing generalizations and analogues of the Brunn–Minkowski inequality for other geometric invariants. See the excellent survey article of Gardner [2] and the book of Schneider [3], which contains a comprehensive account of different aspects and consequences of Brunn–Minkowski inequality. More recent papers about Brunn–Minkowski-type inequalities include [4–7].

The second part of this paper is devoted to applying Fubini's theorem and Brunn–Minkowski's inequality to obtain estimations of volumes of sections of balls in $\mathbb{R}^n$. The study of the geometry of convex bodies based on information about sections and projections of the bodies has important applications in many areas of science. The Fourier analytic approach to sections of convex bodies is based on certain formulas expressing the volume of sections in terms of the Fourier transform of powers of the Minkowski functional of a body. This approach was extended to obtain volumes of projections of convex bodies obtaining counterparts of the results of sections (see [8,9]).

In the study of convex bodies from a geometric and analytic point of view, some other basic questions appeared. One is about the distribution of the volume of high-dimensional convex bodies [10]. Moreover, in [11] the authors established the log-concavity of the volume of central sections of dilations of the cross-polytope B_1^n. Another remarkable paper on the subject is [12], where the maximal and minimal volume of non-central sections of the cross-polytope are obtained. There are also very recent, interesting results concerning sections of other convex bodies, such as the cube (see [13]).

2. Preliminaries

We recall in this section the concepts and notations used in the rest of the article. We will not go into great detail because they are elementary and can be found in any introductory book on Functional Analysis or Measure Theory (see for instance [14]).

If $\|\|$ is a fixed norm in $\mathbb{R}^n$, the set $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is called the *unit ball*. The dual space of $\mathbb{R}^n$ is the space of continuous linear forms endowed with the norm $\|f\| = \sup_{\|x\|\leq 1} |f(x)|$ and can be identified with $\mathbb{R}^n$. For a subset B of $\mathbb{R}^n$, $\|x\| = \|x\|_B := \inf\{\lambda > 0 : \lambda^{-1}x \in B\}$ denotes the *Minkowski functional* corresponding to the set B. Whenever you have a *convex body* B in $\mathbb{R}^n$, that is, B is a *compact convex set with non-empty interior and symmetric*, its Minkowski functional $\|\|_B$ defines a norm whose unit ball is B.

The unit ball for the normed spaces $(\mathbb{R}^n, \|\|_p)$, where $1 \leq p < \infty$ and $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ for all $x \in \mathbb{R}^n$, will be denoted by $\mathbb{B}_p^n = \{x \in \mathbb{R}^n \ s.t \ \|x\|_p \leq 1\}$. In particular, when $p = 2$, $\|\|_2$ is called the *euclidean norm* and it generates the euclidean topology in $\mathbb{R}^n$.

A *measure space* $(X, \mathcal{M}, \mu)$ is a triple formed by any set X, a σ-algebra $\mathcal{M}$ defined on its subsets and a measure μ defined on $\mathcal{M}$. Members of $\mathcal{M}$ are called measurable sets. A measure space is called *sigma-finite* if there exists a countable number $\{A_n \ n \in \mathbb{N}\}$ of measurable sets in $\mathcal{M}$ such that $X = \cup_{n\in\mathbb{N}} A_n$ and $\mu(A_n) < \infty$ for any $n \in \mathbb{N}$.

A map f between two measure spaces $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ is called *measurable* if $f^{-1}(B) \in \mathfrak{M} \ \forall \ B \in \mathfrak{N}$. Given two such measure spaces you can canonically construct the measure product space $(X \times Y, \mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu)$. $\mathfrak{M} \otimes \mathfrak{N}$ is called the product σ-algebra of $\mathfrak{M}$ and $\mathfrak{N}$, and $\mu \times \nu$ the product measure of μ and ν.

We are especially interested in the case where $X = \mathbb{R}^n$, $\mathcal{M} = \mathcal{M}_n$ is the Lebesgue σ-algebra in $\mathbb{R}^n$ and $\mu = m_n$ is the Lebesgue measure on $\mathcal{M}_n$ (m_n is the completion of the product measure $m \times ..._{n \text{ times}}... \times m$, where m is the Lebesgue measure on $\mathbb{R}$). This measure space is σ-finite. $\mathcal{M}_n$ properly contains the Borel σ-algebra $\mathcal{B}_n$ (generated by the open sets of the euclidean topology in $X = \mathbb{R}^n$). Moreover, the Lebesgue measure is a *Radon measure*: that is, all compact sets K have finite measures, and it is outer and inner regular (for every Borel set, its measure is the infimum of the measures of the open sets containing it and for every open set its measure is the maximum of the measures of the compact sets contained in it, respectively). For a measurable set A, *vol(A), volume* of A, will be just $m_n(A)$.

Our measurable functions will be defined on $(\mathbb{R}^n, \mathcal{M}_n, m_n)$ and will take real values in $(\mathbb{R}, \mathcal{M}, m)$. By $\int_{\mathbb{R}^n} f \ dm_n$,we denote the Lebesgue integral of a measurable function f. We say that f is integrable if $\int_{\mathbb{R}^n} |f| \ dm_n < \infty$. The set of all integrable functions is a normed space denoted by $\mathbb{L}^1(\mathbb{R}^n)$ and $\|f\|_1 = \int_{\mathbb{R}^n} |f| \ dm_n$. In the same way that $\mathbb{L}^1(\mathbb{R}^n)$, it can be defined as the normed space $\mathbb{L}^p(\mathbb{R}^n)$ for $1 < p \in \mathbb{R}$ taking $\|\|_p$ as the norm defined by $\|f\|_p = (\int_{\mathbb{R}^n} |f|^p \ dm_n)^{\frac{1}{p}}$. We recall here the *Dominated Convergence Theorem,* which will be used later on: if $\{f_n\}$ is a sequence of measurable functions pointwisely convergent to a function f and there exists an integrable g such that $|f_n| \leq g \ \forall \ n \in \mathbb{N}$, then f is integrable and the limit of the integrals of f_n equals the integral of f.

In the computation of volumes it plays an important role in the *Euler* Γ*-function*, which is defined this way:

$$\begin{array}{rccc} \Gamma : & \mathbb{R}_+ & \to & \mathbb{R} \\ & x & \mapsto & \int_0^\infty t^{x-1}e^{-t}dt, \end{array}$$

with the following property and values:

$$\Gamma(x+1) = x\Gamma(x) \quad \forall x > 0, \qquad \Gamma(1) = 1, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

We finish this section with the statement of Fubini's theorem ([14], Theorem 8.8):

Theorem 1. *Let* $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ *be* σ*-finite measure spaces. Let* $F : X \times Y \to \mathbb{R}$ *be an* $\mathcal{M} \times \mathcal{N}$*-measurable function. Let us consider the functions:*

$$\begin{array}{rcl} \varphi^* : X & \longrightarrow & [0,\infty) \\ x & \longrightarrow & \int_Y |F(x,.)|d\nu \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi^* : Y & \longrightarrow & [0,\infty) \\ y & \longrightarrow & \int_X |F(.,y)|d\mu \end{array}, \text{ then:}$$

1. $\varphi^* \in \mathbb{L}^1(X,\mu) \Rightarrow F \in \mathbb{L}^1(X \times Y, \mu \times \nu)$.
2. $\psi^* \in \mathbb{L}^1(Y,\nu) \Rightarrow F \in \mathbb{L}^1(X \times Y, \mu \times \nu)$.

If $F \in \mathbb{L}^1(X \times Y, \mu \times \nu)$*, then:*
3. There is $E \subset X$ *with* $\mu(X \setminus E) = 0$ *such that* $F(x,.) \in \mathbb{L}^1(Y,\nu) \, \forall x \in E$ *and*

$$\begin{array}{rcl} \varphi : E & \longrightarrow & \mathbb{R} \\ x & \longrightarrow & \int_Y F(x,.)d\nu \end{array} \quad \text{is in } \mathbb{L}^1(E, \mu_E).$$

4. There is $G \subset Y$ *with* $\nu(Y \setminus G) = 0$ *such that* $F(.,y) \in \mathbb{L}^1(X,\mu) \, \forall y \in G$ *and*

$$\begin{array}{rcl} \psi : G & \longrightarrow & \mathbb{R} \\ y & \longrightarrow & \int_X F(.,y)d\mu \end{array} \quad \text{is in } \mathbb{L}^1(G, \nu_G).$$

Moreover,

$$\int_E \varphi d\mu_E = \int_{X \times Y} F d(\mu \times \nu) = \int_G \psi d\nu_G.$$

3. Brunn-Minkowski's Inequality

Next, we are going to use Fubini's theorem in the proof of Brunn–Minkowski inequality [15], which will be done by induction.

Theorem 2. *If* A, B *are compact sets in* $\mathbb{R}^n$ *with* $n \geq 1$,
(1) $\forall \lambda \in [0,1], \quad vol(\lambda A + (1-\lambda)B) \geq vol(A)^\lambda \cdot vol(B)^{1-\lambda}$
(2) $vol(A+B) \geq (vol(A)^{1/n} + vol(B)^{1/n})^n \qquad (Brunn--Minkowski)$

Proof. *First step: (2) is consequence of (1).*

In fact, taking

$$\lambda = \frac{vol(A)^{1/n}}{vol(A)^{1/n} + vol(B)^{1/n}}$$

and, considering the compact sets $A' = vol(A)^{-1/n} \cdot A$, $B' = vol(B)^{-1/n} \cdot B$, we have

$$vol(\lambda A' + (1-\lambda)B') \geq [vol(vol(A)^{-1/n}A)]^\lambda \cdot [vol(vol(B)^{-1/n}B)]^{1-\lambda}$$
$$= [vol(A)^{-1} \cdot vol(A)]^\lambda [vol(B)^{-1} \cdot vol(B)]^{1-\lambda} = 1.$$

In other words:

$$vol\left(\frac{A+B}{vol(A)^{1/n} + vol(B)^{1/n}}\right) \geq 1$$

and then

$$vol(A+B) \geq (vol(A)^{1/n} + vol(B)^{1/n})^n$$

Second step: (1) is a consequence of the following lemma

Lemma 1. *Let $f, g, \varphi : \mathbb{R}^n \to [0, \infty]$ be measurable functions, such that for some $\lambda \in (0,1)$*

$$\varphi(\lambda r + (1-\lambda)s) \geq f(r)^\lambda \cdot g(s)^{1-\lambda}, \ \forall r, s \in \mathbb{R}^n.$$

Then,

$$\int_{\mathbb{R}^n} \varphi(x) dm_n(x) \geq (\int_{\mathbb{R}^n} f(x) dm_n(x))^\lambda (\int_{\mathbb{R}^n} g(x) dm_n(x))^{1-\lambda}.$$

Indeed, taking

$$\varphi = 1_{\lambda A + (1-\lambda)B}, \ f = 1_A, \ g = 1_B$$

(1) is obtained.

Third step: it is enough to prove the lemma for $\|f\|_\infty = \|g\|_\infty = 1$.

In fact if the lemma holds for $\|f\|_\infty = \|g\|_\infty = 1$, it will also be true (by linearity of the integral) for any pair of bounded functions f, g, just applying the lemma to

$$\Phi = \frac{\varphi}{\|f\|_\infty^\lambda \|g\|_\infty^{1-\lambda}}, \ F = \frac{f}{\|f\|_\infty} \text{ and } G = \frac{g}{\|g\|_\infty}.$$

Fourth step: proof of the lemma for $\|f\|_\infty = \|g\|_\infty = 1$, $n = 1$.

For $0 \leq t < 1$, whenever $f(x) \geq t, g(y) \geq t$, we will have

$$\varphi(\lambda x + (1-\lambda)y) \geq f(x)^\lambda \cdot g(y)^{1-\lambda} \geq t.$$

So,

$$\{x \in \mathbb{R}, \varphi(x) \geq t\} \supset \lambda\{x \in \mathbb{R}, f(x) \geq t\} + (1-\lambda)\{x \in \mathbb{R}, g(x) \geq t\}$$

Now, since for non-empty compact sets A, B of $\mathbb{R}$, we have

$$\begin{aligned} & \{minA\} + B \subset A + B \text{ and } A + \{maxB\} \subset A + B \\ \Rightarrow \quad & m(A+B) \geq m[(\{minA\} + B) \cup (A + \{maxB\})] \\ = \quad & m(\{minA\} + B) + m(A + \{maxB\}) = m(B) + m(A), \end{aligned}$$

by the regularity of Lebesgue's measure in $\mathbb{R}$, for the measurable sets $A = \lambda\{x \in \mathbb{R}, f(x) \geq t\}$ and $B = (1-\lambda)\{x \in \mathbb{R}, g(x) \geq t\}$ we have

$$m\{x \in \mathbb{R}, \varphi(x) \geq t\} \geq \lambda m\{x \in \mathbb{R}, f(x) \geq t\} + (1-\lambda) m\{x \in \mathbb{R}, g(x) \geq t\}.$$

Integrating with respect to t in $\mathbb{R}^+$:

$$\begin{aligned} & \int_0^\infty m\{x \in \mathbb{R}, \varphi(x) \geq t\} dm(t) \\ & \quad \geq \lambda \int_0^\infty m\{x \in \mathbb{R}, f(x) \geq t\} dm(t) + (1-\lambda) \int_0^\infty m\{x \in \mathbb{R}, g(x) \geq t\} dm(t). \end{aligned}$$

The first integral is

$$\int_0^\infty (\int_{\{x \in \mathbb{R}: \varphi(x) \geq t\}} dm(x)) dm(t) = \int_{\mathbb{R}} (\int_0^{\varphi(x)} dm(t)) dm(x) = \int_{\mathbb{R}} \varphi(x) dm(x).$$

In the same way, the second and third integrals are:

$$\int_{\mathbb{R}} f(x)dm(x) \text{ and } \int_{\mathbb{R}} g(x)dm(x).$$

So:

$$\int_{\mathbb{R}} \varphi(x)dm(x) \geq \lambda \int_{\mathbb{R}} f(x)dm(x) + (1-\lambda) \int_{\mathbb{R}} g(x)dm(x)$$
$$\geq (\int_{\mathbb{R}} f(x)dm(x))^{\lambda} (\int_{\mathbb{R}} g(x)dm(x))^{1-\lambda}),$$

where the last inequality comes from

$$\lambda a + (1-\lambda)b \geq a^{\lambda} \cdot b^{1-\lambda}, \ \forall \, a, b > 0,$$

because $\ln(x)$ is concave.

Let $n > 1$ and suppose the result is proved for $n-1$.

Take a fixed $y \in \mathbb{R}$ and define

$$\begin{array}{rcl} \varphi_y : \mathbb{R}^{n-1} & \longrightarrow & [0, \infty) \\ t & \longrightarrow & \varphi(t, y). \end{array}$$

Define $f_y, \ g_y$ analogously.

If $y_0, y_1 \in \mathbb{R}$ are such that $y = \lambda y_1 + (1-\lambda)y_0$, then $\forall r, s \in \mathbb{R}^{n-1}$ we have:

$$\varphi_y(\lambda r + (1-\lambda)s) = \varphi(\lambda(r, y_1) + (1-\lambda)(s, y_0))$$
$$\geq (f(r, y_1))^{\lambda} \cdot (g(s, y_0)^{1-\lambda} = (f_{y_1}(r))^{\lambda} \cdot (g_{y_0}(s))^{1-\lambda}.$$

So, if we apply the induction hypothesis to $\varphi_y, f_{y_1}, g_{y_0}$, we get

$$\int_{\mathbb{R}^{n-1}} \varphi_y dm_{n-1} \geq (\int_{\mathbb{R}^{n-1}} f_{y_1} dm_{n-1})^{\lambda} \cdot (\int_{\mathbb{R}^{n-1}} g_{y_0} dm_{n-1})^{1-\lambda}$$

and applying again the result for $n = 1$,

$$\int_{\mathbb{R}^n} \varphi \, dm_n = \int_{\mathbb{R}} (\int_{\mathbb{R}^{n-1}} \varphi_y dm_{n-1}) dm(y)$$
$$\geq [\int_{\mathbb{R}} (\int_{\mathbb{R}^{n-1}} f_{y_1} dm_{n-1}) dm(y)]^{\lambda} \cdot [\int_{\mathbb{R}} (\int_{\mathbb{R}^{n-1}} g_{y_0} dm_{n-1}) dm(y)]^{1-\lambda}$$
$$= (\int_{\mathbb{R}^n} f dm_n)^{\lambda} . (\int_{\mathbb{R}^n} g dm_n)^{1-\lambda}.$$

□

4. Isoperimetric Inequality

Brunn–Minkowski's inequality allows us to easily obtain the isoperimetric inequality.

Theorem 3. *Let C be a convex body in $\mathbb{R}^n$ with $n \geq 2$, let $\partial(C)$ its border and $A(\partial(C))$ its surface area or perimeter,*

$$\mathbb{B}_2^n = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\} \text{ and } \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$$

$$A(\partial(C)) \geq (\frac{vol(C)}{vol(\mathbb{B}_2^n)})^{\frac{n-1}{n}} A(\mathbb{S}^{n-1})$$

(Among all convex bodies with fixed area, the maximum volume is attained by the spheres).

Proof. Although it is difficult to give a notion of the perimeter or surface area (area for short) of a general compact, the convex ones are well approximated by polytopes and their area can be defined by continuity. Thus, we obtain a notion of area which coincides, for differentiable manifolds of class C^1, with that corresponding to the canonical measure.

If such definition is accepted, the area is obtained from the volume by the intuitive formula [16]:

$$A(\partial(C)) = \lim_{t\to 0} \frac{vol(C + t\mathbb{B}_2^n) - vol(C)}{t}.$$

Using the Brunn–Minkowski's inequality,

$$vol(C + t\mathbb{B}_2^n) \geq (vol(C)^{\frac{1}{n}} + (t\,vol(\mathbb{B}_2^n)^{\frac{1}{n}})^n \geq vol(C) + nt\,vol(\mathbb{B}_2^n)^{\frac{1}{n}} vol(C)^{\frac{n-1}{n}} + o(t)$$

and so

$$A(\partial(C)) \geq nvol(\mathbb{B}_2^n)^{\frac{1}{n}} vol(C)^{\frac{n-1}{n}}$$
$$= nvol(\mathbb{B}_2^n) vol(C)^{\frac{n-1}{n}} vol(\mathbb{B}_2^n)^{\frac{1}{n}-1} = A(\mathbb{S}^{n-1})(\frac{vol(C)}{vol(\mathbb{B}_2^n)})^{\frac{n-1}{n}}.$$

□

The volume of convex bodies is related to the geometrical properties of the corresponding spaces. So its study is important in the local theory of Banach spaces [15]. Next, we will try to show how Fubini's theorem can be used in the estimation of volumes of sections of balls. We will see two illustrative theorems.

5. Estimations of Volumes of Sections of Balls in $\mathbb{R}^n$

In the sequel, a ball B will be a symmetric convex body in $\mathbb{R}^n$.

If $\|\|_B$ is the Minkowski's functional associated with B, $(\mathbb{R}^n, \|\|_B)$ is a Banach space whose unit ball is B. $(\mathbb{R}^n, \|\|_B)$ is a Hilbert space if and only if B is an ellipsoid.

If E is a k-dimensional subspace of $(\mathbb{R}^n, \|\|_B)$ and $E^\perp$ is the orthogonal complement of E, the section $E \cap B$ is the unit ball of the normed subspace E and the projection $P_{E^\perp}(B)$ is the unit ball of the quotient normed space $\mathbb{R}^n/E$.

Theorem 4. *[15]*

$$\binom{n}{k}^{-1} vol(E\cap B)vol(P_{E^\perp}(B)) \leq vol(B) \leq vol(E\cap B)vol(P_{E^\perp}(B))$$

Proof. *First step: $vol(B)$ can be expressed as $vol(B) = \int_{P_{E^\perp}(B)} vol((x+E)\cap B)dm_{n-k}(x)$.* By Fubini's theorem,

$$vol(B) = m_n(B) = \int_{E^\perp} m_k\{y \in E : x + y \in B\} dm_{n-k}(x)$$
$$= \int_{E^\perp} vol((x+E)\cap B)dm_{n-k}(x) = \int_{P_{E^\perp}(B)} vol((x+E)\cap B)dm_{n-k}(x),$$

because if $x \notin P_{E^\perp}(B)$, $(x+E)\cap B = \emptyset$.

Second step: We obtain the inequality on the right $vol(B) \leq vol(E\cap B)vol(P_{E^\perp}(B))$

$$E\cap B = \frac{1}{2}[((x+E)\cap B) + ((-x+E)\cap B)]$$

and

$$vol((x+E)\cap B) = vol((-x+E)\cap B)).$$

Then applying Brunn–Minkowski's inequality, it yields

$$vol(E \cap B)^{\frac{1}{k}} \geq \frac{1}{2}[vol((x+E) \cap B)^{\frac{1}{k}} + vol((-x+E) \cap B)^{\frac{1}{k}}] = vol((x+E) \cap B)^{\frac{1}{k}}$$

and hence, using First Step, we obtain $vol(B) \leq vol(E \cap B)vol(P_{E^\perp}(B))$.
Third step: We obtain the inequality on the left.

If $x \in tP_{E^\perp}(B), 0 \leq t \leq 1$, then $x = tP_{E^\perp}(b)$ being $b \in B$ and $tb \in x + E$.

By convexity $tb + (1-t)(E \cap B) \subset B$, so $tb + (1-t)(E \cap B) \subset (x+E) \cap B$ and, being Lebesgue measure translation invariant

$$vol[(1-t)(E \cap B)] \leq vol((x+E) \cap B)$$

hence

$$(1-t(x))^k vol(E \cap B) \leq vol((x+E) \cap B),$$

where $t(x)$ represents the Minkowski functional of $P_{E^\perp}(B)$. Finally,

$$\begin{aligned}
vol(B) &\geq vol(E \cap B) \int_{P_{E^\perp}(B)} (1-t(x))^k dm_{n-k}(x) \\
&= vol(E \cap B) \int_{P_{E^\perp}(B)} (\int_{t(x)}^1 k(1-t)^{k-1} dt) dm_{n-k}(x) \\
&= vol(E \cap B) \int_0^1 k(1-t)^{k-1} (\int_{tP_{E^\perp}(B)} dm_{n-k})(x) dt \\
&= vol(E \cap B) vol(P_{E^\perp}(B)) \int_0^1 k(1-t)^{k-1} t^{n-k} dt \\
&= vol(E \cap B) \cdot vol(P_{E^\perp}(B)) \cdot \binom{n}{k}^{-1}.
\end{aligned}$$

□

The following lemma gives us an expression of the volumes of sections of balls in $\mathbb{R}^n$.

Lemma 2. *Let $\{u^1, \ldots, u^{n-k}\}$ be an orthonormal basis of $E^\perp$, $\| \cdot \|$ the norm associated with the ball B and $E(\varepsilon) = \{x \in \mathbb{R}^n : |\langle x, u^j \rangle| \leq \varepsilon,\ 1 \leq j \leq n-k\}$.*

Then,

$$\Gamma(1 + \frac{k}{p}) vol(E \cap B) = \lim_{\varepsilon \to 0} (2\varepsilon)^{k-n} \int_{E(\varepsilon)} e^{-\|x\|^p} dm_n(x), \quad p > 0$$

Proof. *First step.*

$$vol(E \cap B) \geq (2\varepsilon)^{k-n} vol(E(\varepsilon) \cap B),\ \forall \varepsilon > 0$$

and

$$vol(E \cap B) = \lim_{\varepsilon \to 0} (2\varepsilon)^{k-n} vol(E(\varepsilon) \cap B).$$

By Fubini's theorem,

$$\begin{aligned}
vol(E(\varepsilon) \cap B) &= \int_{E^\perp} m_k\{y \in E : x + y \in E(\varepsilon) \cap B\} dm_{n-k}(x) \\
&= \int_{E^\perp \cap E(\varepsilon)} m_k\{y \in E : x + y \in B\} dm_{n-k}(x) \\
&= \int_{P_{E^\perp}(E(\varepsilon))} vol((x+E) \cap B) dm_{n-k}(x).
\end{aligned}$$

Then, doing the change of variable $x = \varepsilon z$,

$$(2\varepsilon)^{k-n} vol(E(\varepsilon) \cap B) = 2^{k-n} \int_{P_{E^\perp}(E(1))} vol((\varepsilon z + E) \cap B) dm_{n-k}(z)$$
$$\leq 2^{k-n} vol(E \cap B) \int_{P_{E^\perp}(E(1))} dm_{n-k}(z) = vol(E \cap B).$$

This last inequality allows us to apply the dominated convergence theorem and also obtain that

$$\lim_{\varepsilon \to 0} (2\varepsilon)^{n-k} vol(E(\varepsilon) \cap B) = vol(E \cap B)$$

Second step: Obtaining the result.

$$\begin{aligned}
& (2\varepsilon)^{k-n} \int_{E(\varepsilon)} e^{-\|x\|^p} dm_n(x) = (2\varepsilon)^{k-n} \int_{E(\varepsilon)} (\int_{\|x\|^p}^{+\infty} e^{-t} dt) dm_n(x) \\
= \quad & (2\varepsilon)^{k-n} \int_0^\infty e^{-t} (\int_{t^{\frac{1}{p}} B \cap E(\varepsilon)} dm_n(x)) dt \\
= \quad & (2\varepsilon)^{k-n} \int_0^\infty e^{-t} vol(t^{\frac{1}{p}} B \cap E(\varepsilon)) dt \\
= \quad & \int_0^\infty (2\varepsilon t^{\frac{-1}{p}})^{k-n} e^{-t} t^{\frac{k}{p}} vol(B \cap E(\varepsilon t^{\frac{-1}{p}})) dt \\
\underset{\varepsilon \to 0}{\longrightarrow} \quad & \int_0^\infty vol(B \cap E) t^{\frac{k}{p}} e^{-t} dt = vol(B \cap E) \Gamma(1 + \frac{k}{p}).
\end{aligned}$$

□

Remark 1. *If $E = \mathbb{R}^n$, we have $\Gamma(1 + \frac{n}{p}) vol(B) = \int_{\mathbb{R}^n} e^{-\|x\|^p} dm_n(x)$, which for $B = \mathbb{B}_p^n$ allows us to easily compute $vol(\mathbb{B}_p^n)$ because the integral $\int_{\mathbb{R}^n} e^{-\|x\|_p^p} dm_n(x)$ is transformed by Fubini's theorem into:*

$$\Pi_{i=1}^n \int_{\mathbb{R}} e^{-|x_i|^p} dx_i = (2 \int_0^\infty e^{-t^p} dt)^n = (2 \int_0^\infty e^{-s} s^{\frac{1}{p}} ds)^n = (2\Gamma(1 + \frac{1}{p}))^n$$

and so,

$$m_n(\mathbb{B}_p^n) = \frac{(2\Gamma(1 + \frac{1}{p}))^n}{\Gamma(1 + \frac{n}{p}).}$$

In particular,

$$m_n(\mathbb{B}_1^n) = \frac{2^n}{n!}$$

$$m_n(\mathbb{B}_2^{2k}) = \frac{\pi^k}{k!} \text{ and } m_n(\mathbb{B}_2^{2k+1}) = \frac{\pi^k}{1/2(1+1/2)\dots(k+1/2).}$$

From the above lemma, we will obtain the next Theorem. In order to do that we need two definitions:

Definition 1. *Let*

$$\begin{aligned} f : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longrightarrow e^{-\|\alpha_p x\|_p^p}, \end{aligned}$$

where $\alpha_p = 2\Gamma(1 + \frac{1}{p})$. We define the measure μ_p^n as $\mu_p^n(A) = \int_A f(x) dm_n(x)$.

So defined, μ_p^n turns out to be a probability measure with density $f(x)$ with respect to m_n, because precisely

$$\int_{\mathbb{R}^n} e^{-\|x\|_p^p} dm_n(x) = \alpha_p^n.$$

Definition 2. *Let μ, ν be Radon positive measures on $\mathbb{R}^n$. The measure μ is said to be finer than the measure ν ($\mu \succ \nu$), if for any ball $B \subset \mathbb{R}^n$, $\mu(B) \geq \nu(B)$.*

Theorem 5. *[17] If $1 \leq q \leq p < \infty$, $\frac{vol(\mathbb{B}_p^n \cap E)}{vol(\mathbb{B}_p^k)} \geq \frac{vol(\mathbb{B}_q^n \cap E)}{vol(\mathbb{B}_q^k)}$.*

Proof. Applying the former lemma to $\mathbb{B}_p^n$, we have

$$vol(E \cap \mathbb{B}_p^n) = \frac{1}{\Gamma(1+k/p)} \lim_{\varepsilon \to 0} (2\varepsilon)^{k-n} \int_{E(\varepsilon)} e^{-\|x\|_p^p} dm_n(x).$$

Changing the variables $x = \alpha_p z$

$$vol(E \cap \mathbb{B}_p^n) = \frac{\alpha_p^k}{\Gamma(1+k/p)} \lim_{\varepsilon \to 0} (\frac{2\varepsilon}{\alpha_p})^{k-n} \int_{E(\frac{\varepsilon}{\alpha_p})} e^{-\|\alpha_p z\|_p^p} dm_n(z)$$

and calling η to $\frac{\varepsilon}{\alpha_p}$

$$vol(E \cap \mathbb{B}_p^n) = vol(\mathbb{B}_p^k) \lim_{\eta \to 0} (2\eta)^{k-n} \mu_p^n(E(\eta))$$

or equivalently

$$\frac{vol(E \cap \mathbb{B}_p^n)}{vol(\mathbb{B}_p^k)} = \lim_{\eta \to 0} (2\eta)^{k-n} \mu_p^n(E(\eta))$$

and analogously

$$\frac{vol(E \cap \mathbb{B}_q^n)}{vol(\mathbb{B}_q^k)} = \lim_{\eta \to 0} (2\eta)^{k-n} \mu_q^n(E(\eta)).$$

Let us see now that for $p \geq q$, $\mu_p^1 \succ \mu_q^1$.

In fact, it is enough to see that $g(x) = \int_0^x (e^{-|\alpha_p t|^p} - e^{-|\alpha_q t|^q}) dt \geq 0$, $\forall\, x > 0$ and this is so because $g(0) = 0$, $g(\infty) = 1/2 - 1/2 = 0$, $g'(x)$ vanishes in one single point and moreover it is positive on a neighbourhood of 0.

Moreover, if $\mu_1 \succ \nu_1$ and $\mu_2 \succ \nu_2$ being μ_i, ν_i, $i = 1, 2$ Radon positive measures with concave logarithm density with respect to m_{s_i} in $\mathbb{R}^{s_i}$, for $i = 1, 2$, then $\mu_1 \times \mu_2 \succ \nu_1 \times \nu_2$ in $\mathbb{R}^{s_1+s_2}$ [13].

Hence, if $p \geq q$, $\mu_p^n \succ \mu_q^n$.

Now, being $E(\eta)$ symmetric convex with non-empty interior and the measures μ_p^n, μ_q^n regular and satisfying $\mu_p^n \succ \mu_q^n$, we have that $\mu_p^n(E(\eta)) \geq \mu_q^n(E(\eta))$ and so

$$\frac{vol(E \cap \mathbb{B}_p^n)}{vol(\mathbb{B}_p^k)} \geq \frac{vol(E \cap \mathbb{B}_q^n)}{vol(\mathbb{B}_q^k)}$$

□

We finish this note with some consequences:

Remark 2. *Taking into account that $E \cap \mathbb{B}_2^n = \mathbb{B}_2^k$, we obtain from Theorem 5:*

$$\textit{For } 2 \leq p < \infty,\ vol(E \cap \mathbb{B}_p^n) \geq vol(\mathbb{B}_p^k)$$

$$\textit{For } 1 \leq p \leq 2,\ vol(E \cap \mathbb{B}_p^n) \leq vol(\mathbb{B}_p^k).$$

On the other hand, if B, B' are balls in $\mathbb{R}^n$ such that $B \subset B'$, we obtain from Theorem 5 that:

$$\frac{vol(B' \cap E)}{vol(B')} \leq \binom{n}{k} \frac{1}{vol(P_{E^\perp}(B'))} \leq \binom{n}{k} \frac{1}{vol(P_{E^\perp}(B))} \leq \binom{n}{k} \frac{vol(B \cap E)}{vol(B)}.$$

In particular:

$$\textit{For } 2 \leq p \leq \infty,\ vol(E \cap \mathbb{B}_p^n) \leq \binom{n}{k} \frac{vol(\mathbb{B}_2^k)}{vol(\mathbb{B}_2^n)} vol(\mathbb{B}_p^n)$$

$$\textit{For } 1 \leq p \leq 2,\ vol(E \cap \mathbb{B}_p^n) \geq \binom{n}{k}^{-1} \frac{vol(\mathbb{B}_2^k)}{vol(\mathbb{B}_2^n)} vol(\mathbb{B}_p^n).$$

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References

1. Ball, K. Convex geometry and functional analysis. *Handb. Geom. Banach Spaces* **2001**, *1*, 161–194.
2. Gardner, R.J. The Brunn-Minkowski inequality. *Bull. Am. Math. Soc.* **2002**, *39*, 355–405. [CrossRef]
3. Schneider, R. *Convex Bodies: The Brunn-Minkowski Theory*; Cambridge University Press: Cambridge, UK, 2014.
4. Böröczky, K.J.; Lutwak, E.; Yang, D.; Zhang, G. The log-Brunn-Minkowski inequality. *Adv. Math.* **2012**, *231*, 1974–1997. [CrossRef]
5. Böröczky, K.J.; Lutwak, E.; Yang, D.; Zhang, G. The logarithmic Minkowski problem. *J. Am. Math. Soc.* **2013**, *26*, 831–852. [CrossRef]
6. Ma, L. A new proof of the log-Brunn-Minkowski inequality. *Geom. Dedicata* **2015**, *177*, 75–82. [CrossRef]
7. Yang, Y.; Zhang, D. The log-Brunn-Minkowski inequality in $\mathbb{R}^3$. *Proc. Am. Math. Soc.* **2019**, *147*, 4465–4475. [CrossRef]
8. Barthe, F.; Naor, A. Hyperplane projections of the unit ball of l_p. *Discret. Comput. Geom.* **2002**, *27*, 215–226. [CrossRef]
9. Koldobsky, A. *Fourier Analysis in Convex Geometry. Mathematical Surveys and Monogrhs (No. 116)*; American Mathematical Society: Providence, RI, USA, 2005.
10. Brazitikos, S.; Giannopoulos, A.; Valettas, P.; Vritsiou, B.H. *Geometry of Isotropic Convex Bodies. Mathematical Surveys and Monograhs*; American Mathematical Society: Providence, RI, USA, 2014; Volume 196.
11. Nayar, P.; Tkocz, T. On a convexity property of sections of the cross-polytope. *Proc. Am. Math. Soc.* **2020**, *148*, 1271–1278. [CrossRef]
12. Liu, R.; Tkocz, T. A note on the extremal non-central sections of the cross-polytope. *Adv. Appl. Math.* **2020**, *118*, 102031. [CrossRef]
13. König, H.; Rudelson, M. On the volume of non-central sections of a cube. *Adv. Math.* **2020**, *360*, 106929. [CrossRef]
14. Rudin, W. *Real and Complex Analysis*; McGraw-Hill Series in Higher Mathematics; McGraw-Hill: New York, NY, USA, 1987.
15. Pisier, G. *The Volume of Convex Bodies and Banach Space Geometry*; Cambridge Tracts in Mathematics; Cambridge University Press: Cambridge, UK, 1989; Volume 94.
16. Berger, M. *Geometry I, II*; Universitext; Springer: Berlin/Heidelberg, Germany, 1987.
17. Meyer, M.; Pajor, A. Sections of the unit ball of ℓ_p^n. *J. Funct. Anal.* **1988**, *80*, 109–123. [CrossRef]

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