

Article

Analytical Error Bounds for Lunar-Surface Relative Positioning Under Non-Identical Line-of-Sight Geometry

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Abstract

Lunar navigation systems are being developed to support sustained surface operations, but their early deployment phase may involve limited satellite visibility and degraded accuracy in orbit and clock determination compared with terrestrial global navigation satellite systems (GNSS). Relative positioning with a lunar-surface reference station can mitigate common error sources through differenced measurements. In conventional GNSS, this mitigation often relies on the assumption that the line-of-sight (LOS) vectors from the reference and user receivers to the same satellite are nearly identical. Because lunar navigation satellite ranges can be considerably shorter than GNSS ranges, this approximation may be insufficient for kilometer-level surface baselines. This paper analytically quantifies three error mechanisms resulting from non-identical LOS geometries: the deterministic bias of the identical-LOS-vector approximation, the residual projection of broadcast ephemeris errors into single-differenced measurements, and the satellite-position-evaluation-time error caused by receiver clock initialization. For each mechanism, a simple closed-form bound is derived as a function of the baseline length and satellite range and is verified against time-series evaluations for representative lunar orbits. For a 10 km baseline in a low lunar orbit, the approximation alone can bias positioning by several tens of meters, whereas single differencing suppresses broadcast ephemeris errors to the meter level, and the clock-initialization effect can reach the meter level or larger. These results indicate that in lunar-surface relative positioning, the single-differenced range should be evaluated from the exact receiver-dependent geometry rather than from a common LOS vector. Additionally, the derived bounds provide a practical basis for budgeting broadcast ephemeris and receiver clock synchronization requirements.

Keywords: lunar navigation; relative positioning; line-of-sight geometry; single-differenced measurements; broadcast ephemeris



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1. Introduction

Lunar positioning, navigation, and timing (PNT) systems are expected to play an essential role in future lunar exploration activities such as surface mobility, landing support, resource exploration, and operations around the lunar south pole [1–5]. LunaNet is a proposed lunar communication and navigation network designed to provide robust, interoperable, and scalable services for exploration missions [6]. Developed by an international partnership of NASA, ESA, and JAXA, LunaNet aims to establish a framework similar to the terrestrial Internet and GNSS. Representative ongoing or proposed initiatives include NASA's Lunar Communications Relay and Navigation Systems (LCRNS) [7,8],

ESA's Moonlight Lunar Communications and Navigation Services (LCNS) program [9,10], and JAXA's Lunar Navigation Satellite System (LNSS) [11]. In such systems, the relative positioning between a reference station and a user receiver is a promising technique for improving local positioning accuracy, particularly during the early deployment phase when the number of available satellites may be limited [12,13].

In conventional relative positioning for terrestrial global navigation satellite system (GNSS), single- and double-differenced measurements are widely used to reduce receiver clock, satellite clock, and common-mode satellite orbit errors [14,15]. A key assumption behind this error cancellation is that the line-of-sight (LOS) vectors from the reference and user receivers to the same satellite are nearly identical. This assumption is often well-justified for terrestrial GNSS because the satellite range is much larger than the baseline length between receivers.

In lunar-surface relative positioning, this geometric assumption may not always hold because the satellite range can be much shorter than that of terrestrial GNSS satellites. When the satellite range is not sufficiently large compared with the receiver baseline, the difference between the LOS vectors observed at the reference and user receivers can become non-negligible. Consequently, identical-LOS-vector approximation can introduce a deterministic geometric error, and satellite-related errors, typically treated as common-mode errors in terrestrial GNSS relative positioning, may remain in the differenced measurements. To date, however, these short-range effects have not been characterized analytically, and no closed-form bounds relating them to the receiver baseline and satellite geometry have been established.

This paper focuses on three related effects that result from non-identical LOS geometries in lunar-surface relative positioning. The first is the deterministic geometric error caused by an identical-LOS-vector approximation. When the single-differenced geometric range is approximated only by projecting the baseline vector onto a common LOS vector, the higher-order terms associated with the baseline component perpendicular to the LOS are neglected. The second is the residual effect of errors in the broadcast ephemeris. Although satellite position errors are common to both receivers, their projection onto the LOS vectors differs between the reference and user receivers. The third is the effect of receiver clock errors on the evaluation of the satellite position. When the direct receiver clock term is removed through double differencing, an error in the receiver time can lead to an incorrect signal transmission time, and consequently an incorrect satellite position used in the measurement model. This effect manifests as a satellite position error associated with the satellite velocity and can persist after differencing.

The objective of this paper is to analytically formulate these geometric and satellite-related residual-error mechanisms and quantitatively assess their impact on lunar-surface relative positioning. The analysis clarifies the dependency of the errors on the baseline length, satellite range, satellite geometry, magnitude of broadcast ephemeris errors, and receiver clock initialization errors. In addition, the paper compares the conventional identical-LOS-vector approximation with an iterative least-squares formulation based on the original nonlinear single-differenced observation equation. Numerical evaluations are performed for representative lunar navigation satellite orbits to illustrate the magnitude of these effects and discuss their implications for the design and operation of early lunar PNT systems.

The remainder of this paper is organized as follows. Section 2 describes the representative lunar navigation satellite orbits and evaluation conditions used for the numerical assessments. Section 3 defines the relative positioning measurement model used in this study. Section 4 examines the effects of the identical-LOS-vector approximation and presents a least-squares formulation based on the exact single-differenced range model.

Sections 5 and 6 analyze the residual effects of broadcast ephemeris errors and receiver clock initialization errors, respectively, on the differenced measurements. Section 7 discusses the implications of the analytical and numerical results for lunar-surface relative positioning and summarizes the resulting processing recommendations. Section 8 concludes the paper.

2. Lunar Navigation Satellite Orbits and Evaluation Conditions

Motivated by the ongoing and proposed LunaNet-related initiatives introduced above, this study considers three representative lunar satellite orbits for the numerical evaluation. These systems employ different orbital configurations depending on the mission objectives and coverage requirements. Therefore, rather than evaluating a specific operational constellation, this section defines the representative orbital scenarios and the receiver location that are used throughout the subsequent analyses.

Beyond these constellation-based architectures, previous studies have examined the use of differenced measurements for lunar positioning under extremely sparse visibility conditions, including cases with only one or two visible satellites [16,17]. This sparse-visibility scenario is particularly relevant to early phases of lunar PNT deployment, when a complete navigation constellation may not yet be available. For this reason, the following evaluation focuses on the measurement-level error characteristics for representative single-satellite geometries, rather than on a full constellation-level positioning performance analysis.

The first representative lunar satellite orbit is a circular low lunar orbit (LLO) at an altitude of 300 km [16]. The second is a circular orbit corresponding to JAXA's LNSS Demo orbit [18]. The third is an elliptical lunar frozen orbit (ELFO), corresponding to ESA Moonlight LCNS [18]. In all cases, the user receiver is located at the lunar south pole, which is one of the most important target areas for future lunar exploration. The same orbital and receiver geometry is used throughout all analyses.

The Keplerian orbital elements used in the evaluation are summarized in Table 1. The semi-major axis (a), eccentricity (e), inclination (i), right ascension of the ascending node (Ω), argument of periapsis (ω), and initial true anomaly (ν_0) are defined in a Moon-centered inertial frame.

Table 1. Representative lunar navigation satellite orbits used in the evaluation.

Orbit	a [km]	e	i [deg]	Ω [deg]	ω [deg]	ν_0 [deg]
LLO	2037.4	0.00	110.0	0.0	0.0	0.0
LNSS Demo	3870.0	0.00	104.428	53.563	90.0	−5.0
Moonlight ELFO	9748.1	0.70	48.04	89.49	123.60	90.0

The three representative orbits are illustrated in Figure 1. They span nearly an order of magnitude in satellite range from the lunar south pole and differ in shape and orientation: the two circular orbits are near-polar, whereas the apoapsis of the eccentric ELFO lies over the southern hemisphere, providing long dwell times over the south pole. This diversity in range, velocity, and geometry makes the three cases suitable for assessing the sensitivity of the derived error bounds to the orbital configuration.

Near-rectilinear halo orbits (NRHOs) are also important orbit candidates in the broader LunaNet architecture, particularly for the Gateway and possible lunar communication relay or hub functions [19]. However, the present study focuses on representative orbital configurations discussed for lunar PNT service constellations. Therefore, NRHO cases are not included in the numerical evaluations listed in Table 1. It should be noted, however, that the analytical formulations derived in this paper are not restricted to the three representative

orbits considered here and can also be applied to NRHO-based configurations by using the corresponding satellite position, velocity, and range geometry.

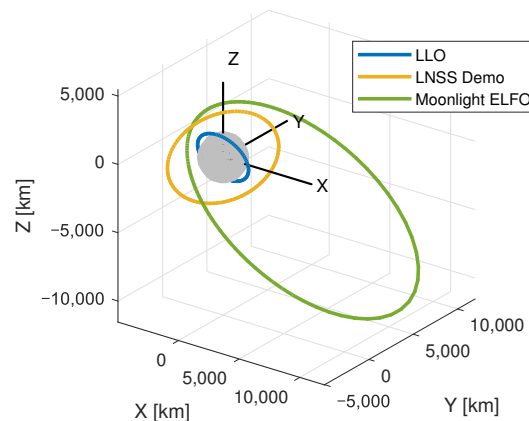


Figure 1. Representative lunar navigation satellite orbits used in the evaluation, drawn to scale in the Moon-centered inertial frame: the circular LLO at an altitude of 300 km, the circular LNSS Demo orbit, and the eccentric Moonlight ELFO. The user receiver is located at the lunar south pole.

3. Relative Positioning Observation Model

This section defines the pseudorange observation model and notation used in the analysis. Let the reference station and user receiver be denoted by s and u , respectively. The position vector of receiver $j \in \{s, u\}$ is denoted by $\mathbf{r}_j$, and the position vector of satellite k at the signal transmission time is denoted by $\mathbf{r}^k$. The geometric range between receiver j and satellite k is

$$\rho_j^k = \|\mathbf{r}^k - \mathbf{r}_j\|. \quad (1)$$

The corresponding LOS unit vector from receiver j to satellite k is defined as

$$\mathbf{e}_j^k = \frac{\mathbf{r}^k - \mathbf{r}_j}{\rho_j^k}. \quad (2)$$

The baseline vector from the reference station to the user receiver is defined as

$$\mathbf{b} = \mathbf{r}_u - \mathbf{r}_s. \quad (3)$$

In terrestrial GNSS relative positioning, $\mathbf{e}_s^k$ and $\mathbf{e}_u^k$ are typically treated identically for short baselines. In the present analysis, however, this approximation is not imposed unless explicitly stated, because the satellite range in lunar navigation may not be sufficiently large compared with the receiver baseline.

The use case considered in this study is a local lunar-surface relative-positioning scenario. A lander or another fixed surface asset is assumed to serve as the reference station, while a rover or other moving target is treated as the user receiver. The analysis does not assume a dense network of reference stations on the lunar surface.

The baseline length considered in this paper was selected to represent a practical scale of lunar-surface mobility. During the extravehicular activities of Apollo 17, the farthest point reached by the Lunar Roving Vehicle (LRV) was reported to be approximately 7.4 km from the Lunar Module (LM) [20]. For future lunar exploration, the Lunar Terrain Vehicle (LTV) in the Artemis program is required to support a driving range of up to 20 km without battery recharging [21]. Assuming a round-trip surface traversing a lander or reference station, this driving range corresponds to a maximum one-way distance of approximately

10 km. Therefore, a maximum receiver baseline length of 10 km is adopted in this paper as a representative scale for the local lunar-surface relative positioning.

3.1. Pseudorange Measurement

The pseudorange measured by receiver j from satellite k is modeled as

$$P_j^k = \rho_j^k + c(\delta t_j - \delta t^k), \quad (4)$$

where c is the speed of light, δt_j is the receiver clock error, and δt^k is the satellite clock error.

In actual pseudorange measurements, additional error sources such as receiver noise, multipath, and residual propagation effects may also be present. These terms are omitted here because the objective is to isolate the geometric residual errors associated with non-identical LOS vectors, broadcast ephemeris errors, and receiver-clock-induced satellite position evaluation errors.

When the receiver computes the satellite position from a broadcast ephemeris, the result generally differs from the true position. Let the broadcast-ephemeris-induced satellite position error be denoted by $\delta \mathbf{r}^k$. To first order, the corresponding pseudorange error at receiver j is obtained by projecting this satellite position error onto the LOS vector:

$$\delta \rho_j^k \simeq (\mathbf{e}_j^k)^T \delta \mathbf{r}^k. \quad (5)$$

This relation is used in later sections to derive the residual effect of broadcast ephemeris errors in the differenced measurements.

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ can be calculated using the inner product, or dot product, which may be expressed in several equivalent forms, such as $\mathbf{a} \cdot \mathbf{b}$, $\langle \mathbf{a}, \mathbf{b} \rangle$, or $\mathbf{a}^T \mathbf{b}$. In this paper, the transpose notation $\mathbf{a}^T \mathbf{b}$ is used, as shown in Equation (5).

3.2. Single-Differenced Measurement Between Receivers

The geometric configuration of the differenced measurements is illustrated in Figure 2, in which the reference station s and the user receiver u , separated by the baseline vector $\mathbf{b}$, simultaneously observe satellites k and l .

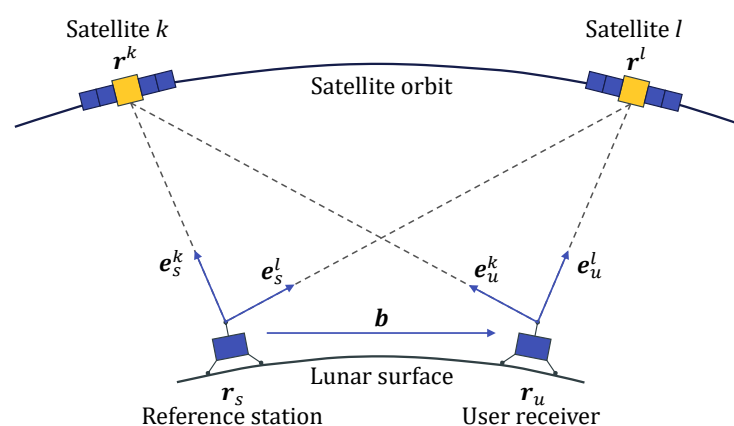


Figure 2. Geometric relation among the receivers, satellites, and differenced range measurements.

The single-differenced pseudorange for satellite k between the user receiver and reference station is defined as

$$\Delta P_{us}^k = P_u^k - P_s^k. \quad (6)$$

Substituting Equation (4) into Equation (6) yields

$$\Delta P_{us}^k = (\rho_u^k - \rho_s^k) + c(\delta t_u - \delta t_s). \quad (7)$$

The satellite clock error is eliminated because the same satellite is observed by both receivers. The receiver clock difference remains in the single-differenced measurement.

The geometric part of the single difference can be expressed as

$$\Delta \rho_{us}^k = \rho_u^k - \rho_s^k. \quad (8)$$

This term is often approximated by using a well-known formula for a short baseline relative to the satellite range [14]:

$$\Delta \rho_{us}^k \simeq -(\mathbf{e}_s^k)^T \mathbf{b}. \quad (9)$$

This approximation assumes that the LOS vector does not change significantly between the reference station and user receiver. In this paper, the validity of this approximation is explicitly examined for the lunar-surface relative positioning.

If the satellite position contains an error $\delta \mathbf{r}^k$, the corresponding error in the single-differenced geometric range is, to first order,

$$\delta(\Delta \rho_{us}^k) \simeq (\mathbf{e}_u^k - \mathbf{e}_s^k)^T \delta \mathbf{r}^k. \quad (10)$$

Equation (10) shows that the satellite position error is completely canceled in the single difference only when the two LOS vectors are identical. Therefore, the LOS-vector difference $\mathbf{e}_u^k - \mathbf{e}_s^k$ is the key quantity governing the residual satellite-related error in relative positioning.

3.3. Double-Differenced Measurement Between Satellites

As illustrated in Figure 2, a double-differenced pseudorange is formed by differencing the single-differenced measurements of two satellites. The double-differenced pseudorange between satellites k and l is defined as

$$\nabla \Delta P_{us}^{kl} = \Delta P_{us}^k - \Delta P_{us}^l. \quad (11)$$

Using Equation (7), the double-differenced observation model becomes

$$\nabla \Delta P_{us}^{kl} = \nabla \Delta \rho_{us}^{kl} = \Delta \rho_{us}^k - \Delta \rho_{us}^l. \quad (12)$$

The receiver clock difference is eliminated by double differencing. Therefore, the double-differenced measurement provides a clock-free observation of the relative receiver geometry.

Using the first-order approximation in Equation (9), the double-differenced geometric range is commonly expressed as

$$\nabla \Delta \rho_{us}^{kl} \simeq -(\mathbf{e}_s^k - \mathbf{e}_s^l)^T \mathbf{b}. \quad (13)$$

This expression is the basis of the conventional linearized relative positioning model. However, its accuracy depends on the assumption that the LOS vectors from the reference and user receivers to each satellite are sufficiently close.

When the satellite position errors are considered, the residual error in the double-differenced geometric range is expressed as

$$\delta(\nabla \Delta \rho_{us}^{kl}) \simeq (\mathbf{e}_u^k - \mathbf{e}_s^k)^T \delta \mathbf{r}^k - (\mathbf{e}_u^l - \mathbf{e}_s^l)^T \delta \mathbf{r}^l. \quad (14)$$

This equation indicates that the satellite-position-error residual in a double-differenced measurement equals the difference between the corresponding single-differenced residuals of the two satellites. Therefore, although the double-differenced error depends on the selected satellite pair, its magnitude can be conservatively bounded by the sum of the magnitudes of the two single-differenced errors. This implies that when the single-differenced error model is characterized, an upper bound for the corresponding double-differenced error can also be estimated. Therefore, the following analysis focuses on the single-differenced error models and their numerical evaluation.

4. Effect of the Identical-LOS-Vector Approximation

When the satellite range is not sufficiently large compared with the receiver baseline, the difference between the LOS vectors at the reference and user receivers can produce a non-negligible residual in the differenced geometric range. In this section, the residual is quantified.

Using Equations (1) and (3), the user-to-satellite geometric range can be expressed as

$$\rho_u^k = \|\mathbf{r}^k - \mathbf{r}_u\| = \|\mathbf{r}^k - \mathbf{r}_s - \mathbf{b}\|. \quad (15)$$

Expanding Equation (15) for the reference station yields

$$\Delta\rho_{us}^k = \rho_u^k - \rho_s^k \simeq -(\mathbf{e}_s^k)^T \mathbf{b} + \frac{1}{2\rho_s^k} \mathbf{b}^T \left(\mathbf{I} - \mathbf{e}_s^k (\mathbf{e}_s^k)^T \right) \mathbf{b}. \quad (16)$$

Appendix A provides a detailed derivation of this expression. This is used here to evaluate the error caused by the identical-LOS-vector approximation.

The first term in Equation (16) is the conventional first-order expression expressed in Equation (9). The second term is the leading-order correction resulting from the non-identical LOS vectors between the reference station and the user receiver. If only the first-order term is used, the residual geometric error in the single-differenced range can be approximated as follows:

$$\delta(\Delta\rho_{us}^k) \simeq \frac{1}{2\rho_s^k} \mathbf{b}^T \left(\mathbf{I} - \mathbf{e}_s^k (\mathbf{e}_s^k)^T \right) \mathbf{b}. \quad (17)$$

This term becomes the largest when the baseline is perpendicular to the LOS direction, and its magnitude is bounded by

$$|\delta(\Delta\rho_{us}^k)| \leq \frac{\|\mathbf{b}\|^2}{2\rho_s^k}. \quad (18)$$

This bound provides a simple criterion for determining whether an identical-LOS-vector approximation is acceptable: When the satellite range is short relative to the squared baseline, the approximation should be checked explicitly.

Note that the residual in Equation (17) is not a measurement noise term. It is a deterministic modeling error caused by truncating the geometric range expansion at first order. This section isolates this geometric effect from other error sources. The residual effects of broadcast ephemeris errors and receiver-clock-induced satellite position errors are analyzed separately in the following sections.

4.1. Single-Differenced Range Error Due to the Identical-LOS-Vector Approximation

In this analysis, the upper bound of Equation (18) is evaluated as a continuous function of the baseline length from 0 to 10 km. This curve represents a conservative envelope of the identical-LOS-vector approximation error. In addition, the actual maximum error predicted by the original time-series error model given by Equation (17) is evaluated for baseline

lengths of 1, 5, and 10 km. The baseline was placed in the local horizontal plane at the lunar south pole and aligned with the $+x$ -axis of the Moon-fixed frame, corresponding to the 0° longitude meridian direction. The same baseline orientation is used in the subsequent error analyses.

For each representative orbit, the exact single-differenced geometric range is computed for each visible time epoch. This comparison distinguishes the conservative geometric upper bound from the maximum error obtained directly from the time-series model. The time-series maximum is expected to be smaller than or equal to the upper-bound curve because the actual satellite geometry does not necessarily realize the worst-case condition.

As the upper bound of Equation (18) is inversely proportional to the satellite range ρ_s^k , the minimum visible range determines the conservative envelope for each orbit. Table 2 summarizes the minimum visible satellite ranges for each representative orbit observed from the user receiver at the lunar south pole. The visibility condition is defined with respect to the geometric horizon of the lunar reference sphere. Local terrain masking and additional receiver elevation masks are not considered in this analysis.

Table 2. Minimum visible satellite range for each representative orbit. As the single-differenced measurement residuals are inversely proportional to the satellite range, the minimum range gives the worst-case residual for each orbit.

Orbit	$\rho_{\min}$ [km]
LLO	719.0
LNSS Demo	2229.8
Moonlight ELFO	4411.9

Figure 3 compares the conservative upper-bound curves with the maximum errors obtained from the time-series error model. The upper bound closely approximates the time-series maximum and is not excessively conservative for the representative cases considered here. The approximation error increases quadratically with the baseline length and becomes significant for the lunar-surface relative positioning. For the 10 km baseline, the largest error occurs in the LLO, where the satellite range is the shortest, reaching approximately 70 m. Even for the Moonlight ELFO, where the satellite altitude above the lunar south pole is relatively high, the error is approximately 11 m. These errors are much larger than the level that can be ignored in relative positioning.

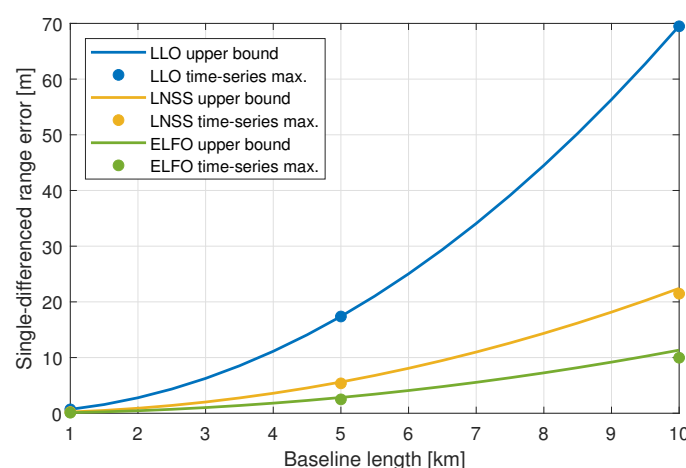


Figure 3. Comparison of the time-series maximum error and conservative upper bound for the identical-LOS-vector approximation.

This result demonstrates that the identical-LOS-vector approximation, which is well-justified in terrestrial GNSS, is not generally applicable to lunar-surface relative positioning with kilometer-level baselines. Therefore, the single-differenced geometric range must not be reduced to the projection of the baseline onto an identical LOS vector.

4.2. Least-Squares Formulation for Single-Differenced Measurements

For baseline estimation, the single-differenced observation equation should not be reduced to projecting the baseline onto an identical LOS vector. Instead, the original nonlinear range-difference expression should be retained and solved iteratively. The general least-squares formulation is summarized in the following.

For the k -th satellite at position $\mathbf{r}^k$, the single-differenced pseudorange measurement between the user receiver and reference station is expressed as

$$\Delta P_{us}^k = \|\mathbf{r}^k - \mathbf{r}_u\| - \|\mathbf{r}^k - \mathbf{r}_s\| + c \Delta \delta t_{us} + \Delta \varepsilon_{us}^k, \quad (19)$$

where $\Delta \delta t_{us}$ is the receiver clock offset difference between the user receiver and reference station, and $\Delta \varepsilon_{us}^k$ includes residual measurement errors. The reference station position $\mathbf{r}_s$ is assumed to be known, and the unknown parameters are the user receiver position and receiver clock difference.

The unknown parameter vector is defined as

$$\mathbf{x} = \begin{bmatrix} \mathbf{r}_u^T & c \Delta \delta t_{us} \end{bmatrix}^T. \quad (20)$$

Given an approximate estimate $\hat{\mathbf{x}}$, the nonlinear single-differenced observation equation is linearized as

$$\Delta \mathbf{P} - \Delta \mathbf{P}(\hat{\mathbf{x}}) = \mathbf{H} \delta \mathbf{x} + \boldsymbol{\varepsilon}, \quad (21)$$

where $\delta \mathbf{x}$ is the correction to the current estimate and $\boldsymbol{\varepsilon}$ is the post-fit residual vector.

For m visible satellites, the single-differenced measurements are collected into the observation vector

$$\Delta \mathbf{P} = \begin{bmatrix} \Delta P_{us}^1 & \Delta P_{us}^2 & \dots & \Delta P_{us}^m \end{bmatrix}^T. \quad (22)$$

The computed measurements based on the current estimate $\hat{\mathbf{x}}$ are expressed analogously as $\Delta \mathbf{P}(\hat{\mathbf{x}})$. The design (or Jacobian) matrix $\mathbf{H}$ is formed by stacking the partial derivatives of the single-differenced measurements with respect to the unknown parameter vector. Therefore,

$$\mathbf{H} = \begin{bmatrix} h^1 \\ h^2 \\ \vdots \\ h^m \end{bmatrix}, \quad (23)$$

For the exact nonlinear single-differenced model, the partial derivative with respect to the user receiver position is obtained by directly differentiating the first range term in Equation (19):

$$\frac{\partial \Delta P_{us}^k}{\partial \mathbf{r}_u} = - \frac{(\mathbf{r}^k - \mathbf{r}_u)^T}{\|\mathbf{r}^k - \mathbf{r}_u\|} = - (\mathbf{e}_u^k)^T. \quad (24)$$

The partial derivative with respect to the receiver clock-difference term $c \Delta \delta t_{us}$ is unity. Therefore, the k -th row of the design matrix for the exact model is

$$h^k = \begin{bmatrix} -(\mathbf{e}_u^k)^T & 1 \end{bmatrix}. \quad (25)$$

As e_u^k depends on the estimated user receiver position, the design matrix is recomputed at each iteration.

Thus, the least-squares correction is computed as

$$\delta \hat{x} = \left(H^T H \right)^{-1} H^T (\Delta P - \Delta P(\hat{x})), \quad (26)$$

Subsequently, the estimate is updated as

$$\hat{x} \leftarrow \hat{x} + \delta \hat{x}. \quad (27)$$

In contrast, the identical-LOS-vector approximation replaces the user-receiver LOS vector with the reference-station LOS vector. The single-differenced measurement is then approximated as

$$\Delta P_{us}^k \simeq - \left(e_s^k \right)^T \mathbf{b} + c \Delta \delta t_{us} + \Delta \epsilon_{us}^k. \quad (28)$$

If the baseline vector and receiver clock-difference term are estimated directly, the unknown vector can be expressed as

$$\mathbf{x} = \begin{bmatrix} \mathbf{b}^T & c \Delta \delta t_{us} \end{bmatrix}^T. \quad (29)$$

The corresponding design matrix row is

$$\mathbf{h}^k = \begin{bmatrix} - \left(e_s^k \right)^T & 1 \end{bmatrix}. \quad (30)$$

Unlike Equation (25), this matrix is constructed from the reference-station LOS vectors and is fixed for a given satellite geometry. The estimated user receiver position is obtained from the estimated baseline as

$$\hat{\mathbf{r}}_u = \mathbf{r}_s + \hat{\mathbf{b}}. \quad (31)$$

The difference between Equations (25) and (30) is the central point of this analysis. The exact model uses the local derivative of the original nonlinear single-differenced range with respect to the user receiver position, whereas the identical-LOS-vector approximation replaces this derivative with the LOS vector at the reference station.

4.3. Simplified Two-Dimensional Simulation

A simplified two-dimensional simulation was conducted to illustrate the effects of these two formulations. Although the least-squares formulation described above is general and applies to three-dimensional positioning with an arbitrary number of satellites, a two-dimensional geometry was used to visualize the deterministic bias caused by the identical-LOS-vector approximation.

In this simulation, the receiver clock-difference term was omitted because the objective was to isolate the geometric effects of the LOS approximation. The reference station was fixed at the origin,

$$\mathbf{r}_s = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \quad (32)$$

and the true user receiver position was placed on the x -axis as

$$\mathbf{r}_u = \begin{bmatrix} b & 0 \end{bmatrix}^T, \quad (33)$$

where b is the baseline length, and three baseline lengths were evaluated: 1, 5, and 10 km. Therefore, in this simplified geometry, the x -axis corresponds to the baseline direction, whereas the y -axis represents the vertical direction. Three satellites were placed at LOS angles of 30° , 90° , and 150° , as viewed from the reference station. For each lunar navigation

scenario, the satellite range from the reference station was set equal to the representative minimum range listed in Table 2. Thus, the position of the k -th satellite was given by

$$\mathbf{r}^k = \rho_{\min} \begin{bmatrix} \cos \theta^k \\ \sin \theta^k \end{bmatrix}, \quad \theta^k \in \{30^\circ, 90^\circ, 150^\circ\}. \quad (34)$$

The noiseless single-differenced pseudorange observation was generated using the exact geometric model,

$$\Delta P_{us}^k = \|\mathbf{r}^k - \mathbf{r}_u\| - \|\mathbf{r}^k - \mathbf{r}_s\|. \quad (35)$$

This simulation was not intended to represent a complete lunar positioning scenario but to demonstrate that modeling errors can be mapped to a deterministic position bias even under ideal, noiseless conditions.

The same observations were then processed using two estimators: the iterative least-squares estimator based on the exact nonlinear model and the linear least-squares estimator based on the identical-LOS-vector approximation. Because the observations contained no noise or other error sources, any position error in the latter solution represented the deterministic bias caused only by the identical-LOS-vector approximation.

Figure 4 shows the position errors obtained from the simplified two-dimensional least-squares simulation. Because the simulation was noise-free and the observations were generated from the exact geometric model, the iterative solution based on the exact single-differenced model converged to the true user position down to the floating-point round-off level. For all three scenarios and baseline lengths, the residual position error of the exact-model solution remained on the order of 10^{-10} m, which reflects the numerical precision of the solver rather than any physical positioning error. Consequently, the corresponding curves overlap near zero error and are almost indistinguishable in the figure.

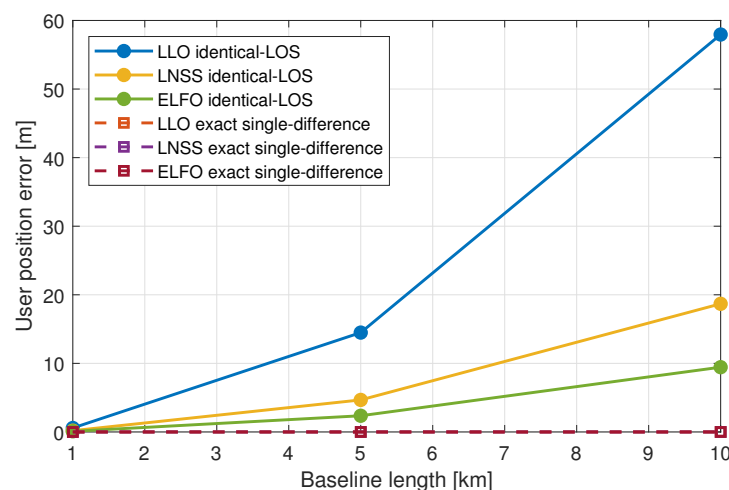


Figure 4. Position error in the two-dimensional noiseless least-squares simulation. The exact single-differenced model converges to the true user position at the numerical-precision floor ($\sim 10^{-10}$ m), so the curves overlap near zero and coincide with the horizontal axis, whereas the identical-LOS-vector approximation produces a deterministic bias that grows with the baseline length.

In contrast, the solution based on the identical-LOS-vector approximation exhibited a deterministic position error. This error increased with baseline length and became more significant for lower-altitude satellites, particularly in the LLO. The position error in the LLO ranged from approximately 0.6 m at a 1 km baseline to 58 m at a 10 km baseline. The result demonstrated that for lunar-surface relative positioning with kilometer-level baselines, the original nonlinear single-differenced observation equation should be used

directly, and the design matrix should be constructed by differentiating the exact range difference rather than by projecting onto an identical LOS vector.

In the following analyses, the single-differenced observation error is used as the primary metric rather than the final user-position error. The user-position error depends on the satellite geometry, number of visible satellites, and estimated state parameters. In contrast, the single-differenced observation error directly represents the residual measurement-domain error introduced by each error source. Therefore, the subsequent analyses focus on the single-differenced observation error to evaluate the fundamental contribution of each error source independently of any particular positioning setup.

5. Residual Effect of Broadcast Ephemeris Errors

In conventional terrestrial GNSS relative positioning, satellite orbit errors are treated as nearly common to the reference and user receivers, and they are substantially suppressed by single differencing. For lunar-surface relative positioning, the satellite range can be much shorter, and the LOS vectors at the two receivers differ more significantly. Consequently, a broadcast ephemeris error can remain in the differenced measurements through its differential projection onto the two LOS vectors.

Using the baseline vector defined by Equation (3), the LOS-vector difference in Equation (10) can be approximated as

$$\mathbf{e}_u^k - \mathbf{e}_s^k \simeq -\frac{1}{\rho_s^k} \left(\mathbf{I} - \mathbf{e}_s^k (\mathbf{e}_s^k)^T \right) \mathbf{b}. \quad (36)$$

Appendix B provides a detailed derivation of this expression.

Substituting Equation (36) into Equation (10) yields

$$\delta(\Delta \rho_{us}^k) \simeq -\frac{1}{\rho_s^k} \mathbf{b}^T \left(\mathbf{I} - \mathbf{e}_s^k (\mathbf{e}_s^k)^T \right) \delta \mathbf{r}^k. \quad (37)$$

Here, $\mathbf{I} - \mathbf{e}\mathbf{e}^T$ is the orthogonal projection matrix on the plane perpendicular to the unit vector $\mathbf{e}$. This expression indicates that the component of the satellite position error parallel to the LOS is largely removed from the single-differenced residual, whereas the component perpendicular to the LOS remains through its projection in the baseline direction. The upper bound follows directly from Equation (37):

$$\left| \delta(\Delta \rho_{us}^k) \right| \leq \frac{\|\mathbf{b}\|}{\rho_s^k} \|\delta \mathbf{r}^k\|. \quad (38)$$

The residual scales as $\|\mathbf{b}\|/\rho_s^k$; therefore, it is often negligible for terrestrial GNSS but non-trivial for the shorter ranges encountered in lunar navigation. This result motivates a quantitative assessment of the broadcast ephemeris requirements for lunar-surface relative positioning.

Single-Differenced Range Error Due to Broadcast Ephemeris Errors

In the time-series evaluation, the baseline length is fixed at 10 km, and the magnitude of the satellite position error is varied as 1, 10, and 100 m. The satellite position error is assumed to be in the tangential direction to the RTN frame. This direction is selected because the tangential errors along the satellite velocity direction are typically a dominant component of orbit prediction errors. At each visible time epoch, the single-differenced range error is computed using Equation (37).

The time-series maxima are compared with the upper-bound lines obtained from Equation (38). Because the upper bound assumes the most unfavorable alignment between

the satellite position error and the LOS-vector difference, the time-series maxima for the specified tangential error direction are expected to be smaller than or equal to the upper-bound curves. This comparison clarifies both the expected residual error for a representative ephemeris-error direction and the conservative worst-case level for each representative lunar navigation orbit.

Figure 5 shows that the residual single-differenced range error caused by broadcast ephemeris errors increases linearly with the satellite position error magnitude. For a 100 m satellite position error, the time-series maximum error is approximately 1.4 m for the LLO, where the satellite range is the shortest, and approximately 0.2 m for the Moonlight ELFO, where the satellite range is the largest among the representative orbits. Although these errors are not negligible for precise relative positioning, they are much smaller than the original satellite position error because the common component of the broadcast ephemeris error is largely canceled by single differencing.

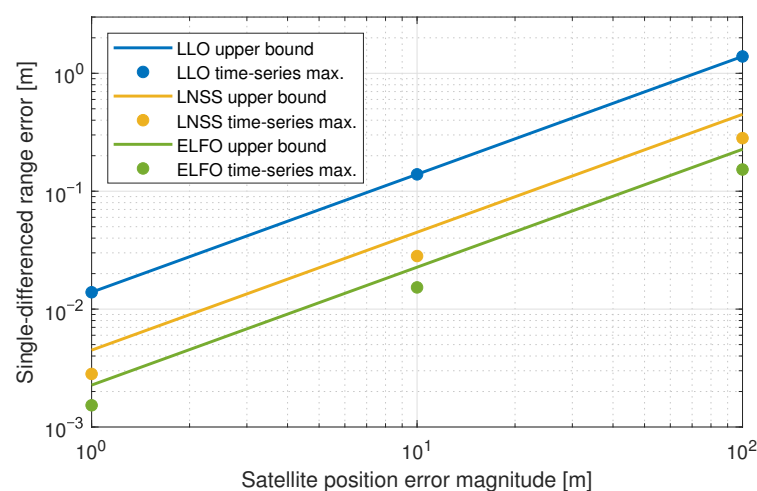


Figure 5. Comparison of the time-series maximum error and the conservative upper bound for the single-differenced range error caused by broadcast ephemeris errors.

The sensitivity of the single-differenced residual also depends on the direction of the satellite position error. As indicated by Equation (37), the residual is governed by the component of the satellite position error projected onto the plane perpendicular to the LOS vector and subsequently onto the baseline direction. Therefore, a satellite position error nearly parallel to the LOS, such as the radial component in ordinary standalone positioning, is largely removed as a common-mode error in single differencing. In contrast, tangential or normal components, which are more perpendicular to the LOS, remain more directly in the single-differenced residual. This explains why the time-series maxima for the tangential error direction agree well with the conservative upper bound.

This result is favorable for early lunar navigation systems, where achieving GNSS-like orbit-determination accuracy may be challenging. The reduced sensitivity of single-differenced measurements to satellite position errors implies that relative navigation can tolerate larger broadcast ephemeris errors than standalone positioning. The close agreement between the tangential-error time-series maxima and conservative upper bound confirms that the bound provides a practical tool for assessing the broadcast ephemeris accuracy requirements without being overly conservative.

6. Receiver Clock Error as Satellite Position Evaluation Time Error

In the double-differenced measurement model (Equation (12)), the direct receiver-clock term is removed by differencing across satellites. However, the receiver clock error can still affect the relative positioning through a different mechanism: If the receiver time used

to determine the signal transmission time is biased, the satellite position is evaluated at an incorrect epoch. This produces an apparent satellite position error proportional to the satellite velocity and time error. Unlike the direct clock term, this satellite-position error is projected onto the LOS vectors and is not necessarily canceled by differencing.

In terrestrial GNSS, this effect is usually negligible because the satellite range is large relative to typical baselines and because a sufficient number of visible satellites enables the receiver clock to be estimated and corrected with high accuracy. However, in early lunar navigation systems, limited satellite visibility may render standalone positioning unavailable. In addition, if the user receiver and reference station initialize their clocks independently, the satellite position evaluation time errors of the two receivers should not be treated as a common quantity.

6.1. Satellite Position Error Caused by Receiver Clock Error

Let Δt_j be the satellite position evaluation time error associated with receiver j , where $j \in \{s, u\}$. For a sufficiently small time error, the resulting satellite position error can be linearized as

$$\delta \mathbf{r}_j^k \simeq \mathbf{v}^k \Delta t_j, \quad (39)$$

where $\mathbf{v}^k$ is the satellite velocity at the transmission time. Equation (39) shows that the receiver time error is converted into a satellite position error in the direction of the satellite velocity. The corresponding range error at receiver j is

$$\delta \rho_j^k \simeq \left(\mathbf{e}_j^k \right)^T \mathbf{v}^k \Delta t_j. \quad (40)$$

Therefore, the resulting single-differenced range error is expressed as

$$\delta \left(\Delta \rho_{us}^k \right) \simeq \left(\mathbf{e}_u^k \right)^T \mathbf{v}^k \Delta t_u - \left(\mathbf{e}_s^k \right)^T \mathbf{v}^k \Delta t_s. \quad (41)$$

Introducing the differential time error between the two receivers:

$$\Delta t_{us} = \Delta t_u - \Delta t_s, \quad (42)$$

Equation (41) can be decomposed as

$$\delta \left(\Delta \rho_{us}^k \right) \simeq \left(\mathbf{e}_u^k - \mathbf{e}_s^k \right)^T \mathbf{v}^k \Delta t_s + \left(\mathbf{e}_u^k \right)^T \mathbf{v}^k \Delta t_{us}. \quad (43)$$

The first term is the LOS-difference component. Using Equation (37), it can be expressed as

$$\delta \left(\Delta \rho_{us}^k \right)_{1st} \simeq -\frac{\Delta t_s}{\rho_s^k} \mathbf{b}^T \left(\mathbf{I} - \mathbf{e}_s^k \left(\mathbf{e}_s^k \right)^T \right) \mathbf{v}^k. \quad (44)$$

The corresponding upper bound is

$$\left| \delta \left(\Delta \rho_{us}^k \right)_{1st} \right| \leq \frac{\|\mathbf{b}\|}{\rho_s^k} \|\mathbf{v}^k\| |\Delta t_s|. \quad (45)$$

It clarifies that the receiver clock error is suppressed by $\|\mathbf{b}\|/\rho_s^k$ and can be small in the first term.

The second term is the range-rate component expressed as

$$\delta \left(\Delta \rho_{us}^k \right)_{2nd} \simeq \dot{\rho}_u^k \Delta t_{us}, \quad (46)$$

where $\dot{\rho}_u^k$ is the range rate given by

$$\dot{\rho}_u^k = \left(\mathbf{e}_u^k \right)^T \mathbf{v}^k. \quad (47)$$

The corresponding upper bound is

$$\left| \delta \left(\Delta \rho_{us}^k \right)_{2\text{nd}} \right| \leq \left| \dot{\rho}_u^k \right| \left| \Delta t_{us} \right|. \quad (48)$$

This expression shows that the second term can be much larger than the first term, because it is governed by the satellite range rate without any suppression.

6.2. Single-Differenced Range Error Due to Receiver Clock Initialization

In this evaluation, the receiver clock initialization error is modeled as an error in the assumed signal propagation delay. The receiver is assumed to initialize its clock by adding an average propagation delay to the signal transmission time of the received navigation message. As the actual propagation delay is unknown before positioning, the average propagation delay for each representative orbit can serve as the nominal delay.

Let ρ_j be the visible satellite range from receiver j and $\bar{\rho}_j$ be its average value over the visible period. The assumed average propagation delay is then given by

$$\bar{\tau}_j = \frac{\bar{\rho}_j}{c}. \quad (49)$$

The resulting receiver clock initialization error is expressed as

$$\Delta t_j = \frac{\rho_j - \bar{\rho}_j}{c}. \quad (50)$$

This time error is used to compute the time series of the single-differenced range error shown in Equation (43) for each representative lunar navigation satellite orbit, as listed in Table 1.

In the numerical evaluation, the user receiver is located at the lunar south pole, and the baseline length is fixed at 10 km. For each visible satellite arc, the satellite velocity $\mathbf{v}^k$ and LOS vectors $\mathbf{e}_j^k$ are computed at each time epoch. Subsequently, the maximum absolute value of the resulting time series is used as the exact time-series result of the error model.

The maximum errors obtained from the first term of Equation (43) and corresponding upper bounds are listed in Table 3. In the table, “time-series max.” denotes the maximum absolute value of the single-differenced range error computed directly from Equation (44), whereas “upper bound” denotes the conservative estimate obtained from Equation (45).

Table 3. Maximum single-differenced range error caused by the LOS-difference component of receiver clock initialization error for a 10 km baseline at the lunar south pole.

Orbit	$\bar{\rho}$ [km]	$ \Delta t_s _{\text{max}}$ [ms]	$\ \mathbf{v}^k\ _{\text{max}}$ [km/s]	Time-Series Max. [m]	Upper Bound [m]
LLO	844.2	0.73	1.55	0.009	0.016
LNSS Demo	2693.6	2.55	1.13	0.005	0.013
Moonlight ELFO	12,692.7	27.6	1.25	0.044	0.078

Note that the upper bound in Equation (45) is conservative because the minimum satellite range, maximum satellite velocity, and maximum clock initialization error are generally not simultaneously attained. The time-series maxima in Table 3 confirm that the first term contribution is small under the evaluated conditions.

However, Table 3 does not represent the full effect when the two receiver clocks are initialized independently. In that case, the second term of Equation (43) must also be evaluated. As the receiver clocks are initialized independently, their satellite position evaluation time errors are not common. With the differential time error of Equation (42), its magnitude is bounded as

$$|\Delta t_{us}| \leq |\Delta t_u| + |\Delta t_s|. \quad (51)$$

Because both receiver clocks are initialized using the same procedure, the maximum values of $|\Delta t_u|$ and $|\Delta t_s|$ are assumed to be equal. Therefore, the conservative maximum differential time error is approximated as

$$|\Delta t_{us}| \simeq 2|\Delta t_s|. \quad (52)$$

The resulting upper bounds in Equation (48) are summarized in Table 4.

Table 4. Maximum single-differenced range error caused by the range-rate component for a 10 km baseline at the lunar south pole. The two receiver clocks are initialized independently.

Orbit	$2 \Delta t_s _{\max}$ [ms]	$ \dot{\rho}_u^k _{\max}$ [m/s]	Upper Bound [m]
LLO	1.46	1003.9	1.47
LNSS Demo	5.10	484.6	2.47
Moonlight ELFO	55.2	537.3	29.7

The results in Table 4 exhibit that the second term contribution is much larger than that of the first term in Table 3. For the LLO and LNSS Demo, the single-differenced range error is approximately 1.5–2.5 m. For the Moonlight ELFO, the larger variation in receiver clock initialization yields a maximum of approximately 29.7 m. This residual is essentially a range-rate error caused by using a biased receiver time to evaluate the satellite position.

The LOS-difference component of the receiver clock initialization error is indeed small, but the range-rate component due to independent receiver clock initialization can introduce a meter-level or larger bias in single-differenced measurements. For lunar-surface relative positioning, the receiver time used for satellite ephemeris evaluation should therefore be synchronized between the reference and user receivers, corrected using the navigation-message transmission time, or estimated explicitly as part of the relative navigation solution when sufficient observations are available.

7. Discussion

7.1. Single-Differenced Range Error Sources and Their Upper Bounds

This study established closed-form bounds relating short-range relative positioning errors to the receiver baseline and satellite geometry and demonstrated their utility through time-series evaluation under representative lunar orbital scenarios. The analyses in the preceding sections clarify the effects of the three error sources on the lunar-surface relative positioning through a non-identical LOS geometry. Although the specific mechanisms differ, the residual errors share a common geometric feature: they are governed by the difference between the LOS vectors at the reference station and user receiver.

This shared geometric structure does not, however, imply that the three mechanisms are mutually coupled. In the first-order formulations derived in this paper, their contributions can be treated as independent and additive at the measurement level. The identical-LOS-vector approximation error (Equation (17)) is a deterministic modeling error that exists even with perfect ephemerides and clocks, whereas the other two residuals are driven by independent external error sources: the broadcast ephemeris error and the re-

ceiver clock initialization error. Consequently, the three bounds can be assessed separately and, in the worst case, conservatively combined linearly into an overall single-differenced error budget.

Among the three error sources, the identical-LOS-vector approximation carries the most fundamental implication. The approximation neglects the higher-order term associated with the baseline component perpendicular to the LOS, and the resulting single-differenced range error grows quadratically with the baseline length (Section 4, Table 2, Figure 3). A simplified least-squares simulation shows that this range-level error maps to a deterministic positioning bias, whereas the estimator based on the exact nonlinear single-differenced equation converges to its true position in a noise-free case. Therefore, for kilometer-level lunar baselines, the observation model should retain the original nonlinear single-differenced range, and the design matrix should be constructed from its exact derivative.

Broadcast ephemeris errors produce a different but related residual. The component of the satellite position error parallel to the LOS is largely canceled by single differencing, whereas the perpendicular component remains through the LOS-vector difference. The residual scales as $\|\mathbf{b}\|/\rho_s^k$ (Equation (38)), and the numerical evaluation confirms that the proposed bound is not overly conservative (Section 5, Figure 5). This confirms that single differencing remains effective in lunar navigation, and broadcast ephemeris requirements can be relaxed relative to standalone positioning, provided that the residual term is included in the error budget.

The receiver clock initialization error has two distinct contributions (Section 6). The LOS-difference component is small because it is suppressed by the baseline-to-range ratio. However, if the user receiver and the reference station initialize their clocks independently, the range-rate component can become meter-level or larger. Periodic clock correction, inter-receiver synchronization, or explicit estimation of the differential receiver time error is therefore important when receiver time is used for evaluating broadcast ephemerides.

7.2. Receiver Clock Error Mitigation Approaches

Unlike the other two error sources, the receiver clock initialization error cannot be suppressed by the observation modeling alone. The identical-LOS-vector approximation error is eliminated by retaining the original nonlinear single-differenced range and solving the least-squares problem iteratively (Section 4.2), and the satellite orbit error contained in the broadcast ephemeris is largely canceled by single differencing (Section 5). In contrast, when the number of visible satellites is insufficient for a standalone position solution, the receiver clock error cannot be estimated as part of the navigation solution, and the receiver clock initialization error can therefore become the dominant error source in lunar-surface relative positioning. Such sparse visibility is expected in the early deployment phase: a single satellite in any of the three representative orbits cannot provide the four-satellite visibility required for a standalone solution, and previous studies have therefore examined measurement accumulation over time under one- or two-satellite visibility [16,17]. In this regime, managing the receiver clock initialization error is essential.

According to Table 4, periodic clock correction using the navigation-message transmission time can reduce the single-differenced range error to about 1.5 m for the LLO, which is comparable to the broadcast ephemeris error shown in Figure 5. For the Moonlight ELFO, however, the residual can remain nearly 30 m, whereas the upper bound of the broadcast ephemeris error can be below 0.2 m. To make the receiver clock contribution comparable to this ephemeris bound and achieve a single-differenced range accuracy of 0.2 m, the corresponding differential receiver time error Δt_{us} must be kept to approximately 0.4 ms (Equation (48)). Such accuracy is difficult to attain by periodic

clock correction alone, but inter-receiver synchronization via two-way time transfer over a surface communication link between the reference station and the user can readily achieve sub-millisecond accuracy [22,23]. In the longer term, once enough satellites are deployed to allow a standalone position solution, the differential time error can instead be obtained directly as a by-product of the navigation solution through explicit estimation. In summary, inter-receiver synchronization over a communication link is recommended for the early deployment phase with limited satellite visibility, explicit estimation of the differential time error becomes attractive as constellation coverage matures, and periodic clock correction serves as a complementary measure.

7.3. Implications for a Complete Positioning Scenario

In the single-differenced pseudorange measurement model of Equation (19), the unknowns are the three-dimensional user position, or equivalently the three baseline components when the reference-station position is known, and the differential receiver clock error Δt_{us} . Therefore, at least four satellites in view are required for an instantaneous unconstrained three-dimensional relative navigation solution.

When enough satellites are visible, the situation contrasts with the sparse-visibility case in the preceding subsection. The receiver clock error is estimated as part of the navigation solution and is no longer the dominant error. The satellite position error contained in the broadcast ephemeris then becomes the dominant error source. In standalone positioning, this error maps directly into the position solution, whereas single differencing effectively removes it, leaving only the small residual bounded in Section 5. Although the accuracy of lunar satellite orbit determination is expected to improve as the infrastructure matures, for example, through the deployment of tracking stations on the lunar surface, the broadcast ephemeris error will remain a limiting factor in the early operational phase. Single-differenced range measurements are therefore effective for improving relative positioning accuracy both under the sparse visibility of the early deployment phase and under the full visibility of a mature constellation, which supports their use throughout the development of lunar navigation systems.

8. Conclusions

This paper derived analytical upper bounds for the three error sources affecting lunar-surface relative positioning under non-identical LOS geometry: the identical-LOS-vector approximation error, which grows quadratically with the baseline length; the broadcast ephemeris residual, which is suppressed by the baseline-to-range ratio; and the receiver clock initialization error, whose range-rate component can reach the meter level or larger when the receiver clocks are initialized independently. The bounds were verified through time-series evaluations under three representative lunar orbital scenarios.

Altogether, the results indicate that single differencing is effective in reducing common satellite-related errors in lunar navigation, but that an identical-LOS-vector approximation is not a safe assumption for kilometer-level lunar-surface baselines. The key recommendation is to retain the exact receiver-dependent LOS geometry in the observation model and to use the analytical bounds derived in this paper when budgeting the broadcast ephemeris and receiver clock synchronization requirements. In particular, inter-receiver synchronization or explicit estimation of the differential receiver time error should be considered when receiver time errors are not sufficiently suppressed by standalone positioning, while improved orbit determination and monitoring infrastructure are required to reduce the remaining broadcast-ephemeris-related residuals.

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Abbreviations

The following abbreviations are used in this manuscript:

ELFO	Elliptical Lunar Frozen Orbit
ESA	European Space Agency
GNSS	Global Navigation Satellite System
JAXA	Japan Aerospace Exploration Agency
LCNS	Lunar Communications and Navigation Services
LCRNS	Lunar Communications Relay and Navigation Systems
LLO	Low Lunar Orbit
LM	Lunar Module
LNSS	Lunar Navigation Satellite System
LOS	Line-of-Sight
LRV	Lunar Roving Vehicle
LTV	Lunar Terrain Vehicle
NASA	National Aeronautics and Space Administration
NRHO	Near-Rectilinear Halo Orbit
PNT	Positioning, Navigation, and Timing
RTN	Radial–Tangential–Normal

Appendix A

The range vector from the reference station to satellite k is defined as

$$\mathbf{R}_s^k = \mathbf{r}^k - \mathbf{r}_s. \quad (\text{A1})$$

The LOS unit vector is then expressed as

$$\mathbf{e}_s^k = \frac{\mathbf{R}_s^k}{\rho_s^k}. \quad (\text{A2})$$

Using the baseline vector, the user-to-satellite range can be expressed as

$$\rho_u^k = \|\mathbf{R}_s^k - \mathbf{b}\|. \quad (\text{A3})$$

Squaring the range yields

$$(\rho_u^k)^2 = (\mathbf{R}_s^k - \mathbf{b})^T (\mathbf{R}_s^k - \mathbf{b}) = (\rho_s^k)^2 - 2\rho_s^k (\mathbf{e}_s^k)^T \mathbf{b} + \mathbf{b}^T \mathbf{b}. \quad (\text{A4})$$

Therefore,

$$\rho_u^k = \rho_s^k \sqrt{1 - \frac{2(\mathbf{e}_s^k)^T \mathbf{b}}{\rho_s^k} + \frac{\mathbf{b}^T \mathbf{b}}{(\rho_s^k)^2}}. \quad (\text{A5})$$

Let

$$x = -\frac{2(\mathbf{e}_s^k)^T \mathbf{b}}{\rho_s^k} + \frac{\mathbf{b}^T \mathbf{b}}{(\rho_s^k)^2}. \quad (\text{A6})$$

Using the Taylor expansion

$$\sqrt{1+x} \simeq 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3) \quad (\text{A7})$$

and retaining the terms up to the second order in $\|\mathbf{b}\|/\rho_s^k$, the user-to-satellite range can be rewritten as

$$\begin{aligned} \rho_u^k &\simeq \rho_s^k - (\mathbf{e}_s^k)^T \mathbf{b} + \frac{1}{2\rho_s^k} \mathbf{b}^T \mathbf{b} - \frac{1}{2\rho_s^k} \left((\mathbf{e}_s^k)^T \mathbf{b} \right)^2 \\ &= \rho_s^k - (\mathbf{e}_s^k)^T \mathbf{b} + \frac{1}{2\rho_s^k} \mathbf{b}^T \left(\mathbf{I} - \mathbf{e}_s^k (\mathbf{e}_s^k)^T \right) \mathbf{b}. \end{aligned} \quad (\text{A8})$$

Subtracting ρ_s^k from both sides yields Equation (16).

Appendix B

The LOS unit vector from the user receiver to satellite k is defined as

$$\mathbf{e}_u^k = \frac{\mathbf{r}^k - \mathbf{r}_u}{\|\mathbf{r}^k - \mathbf{r}_u\|}. \quad (\text{A9})$$

Substituting $\mathbf{r}_u = \mathbf{r}_s + \mathbf{b}$ yields

$$\mathbf{e}_u^k = \frac{\mathbf{r}^k - \mathbf{r}_s - \mathbf{b}}{\|\mathbf{r}^k - \mathbf{r}_s - \mathbf{b}\|}. \quad (\text{A10})$$

For $\|\mathbf{b}\| \ll \rho_s^k$, this expression can be linearized with respect to $\mathbf{b}$. For a general unit vector $\mathbf{f}(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$, the perturbation is expressed as

$$\delta \mathbf{f} = \frac{1}{\|\mathbf{x}\|} \left(\mathbf{I} - \mathbf{f} \mathbf{f}^T \right) \delta \mathbf{x}. \quad (\text{A11})$$

The first-order approximation of the unit vector function is then expressed as

$$\mathbf{f}(\mathbf{x} + \delta \mathbf{x}) \simeq \mathbf{f}(\mathbf{x}) + \frac{1}{\|\mathbf{x}\|} \left(\mathbf{I} - \mathbf{f} \mathbf{f}^T \right) \delta \mathbf{x}. \quad (\text{A12})$$

Substituting

$$\mathbf{x} = \mathbf{r}^k - \mathbf{r}_s, \quad \delta \mathbf{x} = -\mathbf{b}, \quad \mathbf{f} = \mathbf{e}_s^k \quad (\text{A13})$$

into this expression, the user LOS vector is approximated as

$$\mathbf{e}_u^k \simeq \mathbf{e}_s^k - \frac{1}{\rho_s^k} \left(\mathbf{I} - \mathbf{e}_s^k (\mathbf{e}_s^k)^T \right) \mathbf{b}. \quad (\text{A14})$$

Subtracting $\mathbf{e}_s^k$ from both sides yields Equation (36).

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