algorithms

Article

Equisum Partitions of Sets of Positive Integers †

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† Dedicated to Richard Guy, for his 103rd Birthday, one of many primes of his life.

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Abstract: Let V be a finite set of positive integers with sum equal to a multiple of the integer b. When does V have a partition into b parts so that all parts have equal sums? We develop algorithmic constructions which yield positive, albeit incomplete, answers for the following classes of set V, where n is a given positive integer: (1) an initial interval $\{a \in \mathbb{Z}^+ : a \le n\}$; (2) an initial interval of primes $\{p \in \mathbb{P} : p \le n\}$, where $\mathbb{P}$ is the set of primes; (3) a divisor set $\{d \in \mathbb{Z}^+ : d \big| n\}$; (4) an aliquot set $\{d \in \mathbb{Z}^+ : d \big| n,\ d < n\}$. Open general questions and conjectures are included for each of these classes.

Keywords: integer set partitions; equisum sets; twin primes; perfect numbers; aliquot divisors

1. Introduction

The questions to be addressed here belong to additive number theory. The scope of this field has grown in recent times. We shall consider the possibility of partitioning certain sets of integers into two or more subsets with equal sums. Although this is a very basic question, it appears not to have been previously discussed. A brief survey will show how it fits within the field.

In their classical introduction to number theory, Hardy and Wright [1] devoted three chapters to additive number theory. The first of these chapters, entitled *Partitions*, begins by describing what was then regarded as the general problem of additive number theory, the study of additive representations of positive integers. To paraphrase: Let V be a given subset of the positive integers $\mathbb{Z}^+$, such as $\mathbb{Z}^+$ itself, or the squares $\left\{a^2 :\ a \in \mathbb{Z}^+\right\}$, or the primes $\mathbb{P}$. Let $r(n)$ be the number of representations of an arbitrary positive integer n as a sum, each term of which is an element in V, subject to a variety of possible restrictions: the number of terms in the sum may be fixed or bounded or unbounded; the sum may be permitted to include equal terms, or this may be ruled out; the order of the terms in the sum may be considered relevant or not. The problem then is to determine $r(n)$ or at least to establish some of its properties.

For example, the study of *unrestricted partitions* of positive integers considers $r(n)$ when the set V of possible summands is $\mathbb{Z}^+$, the number of possible terms is unbounded, equal terms are permitted, and order is irrelevant. (Equivalently, in this case, $r(n)$ counts the number of multisets of positive integers with sum n.) This view of additive number theory is endorsed by Nathanson in a more recent work focussing on representations as sums of kth powers ($k \ge 2$) or the primes [2].

Various studies which do not focus on additive representations of integers, but nevertheless rightfully belong to additive number theory, have appeared in modern times. Many studies consider *set partitions* rather than integer partitions. Here, the general problem considers partitions of a given subset V of $\mathbb{Z}^+$ into a fixed or bounded number of disjoint subsets, and seeks conditions in which at least one of those subsets must inevitably exhibit a certain property, or else conditions ensuring the existence of a partition in which none of the subsets has a certain property.

For instance, van der Waerden's classical theorem [3] effectively states that for given positive integers b and k, if V is the set $[1,\ n] := \{a \in \mathbb{Z}^+ : a \le n\}$ comprising all positive integers up to n, and if n

is sufficiently large, then every partition of V into b subsets inevitably includes a subset containing an arithmetic progression of at least k terms. A 2004 major break-through theorem by Green and Tao [4] implies that this result holds if V is the set $\{p \in \mathbb{P} : p \leq n\}$ comprising all primes up to n.

In his survey of unsolved problems in number theory, Guy [5] devotes a chapter to additive number theory, covering a wide range of problems, many of which do not readily fit the general types described above. They include studies seeking a maximal subset V of $[1,\ n]$ with a specified property, such as all subsets of V having distinct sums, or all 2–subsets having distinct sums, or no subset summing to a positive multiple of a prescribed integer m. Related studies consider subsets of $[1,\ n]$ which are determined by the sums of their subsets of fixed size.

Against this background, let us now consider partitions of a finite set $V \subset \mathbb{Z}^+$ with the property that the participating subsets all have the same sum.

The notation and terminology are as follows. Given a finite set of positive integers $V \subset \mathbb{Z}^+$, of cardinality $|V| := v$, let

$$\mathcal{B} := \{B_i | 1 \leq i \leq b\}$$

be a partition of V into b nonempty subsets, called *blocks*, with $1 \leq b \leq v$. Thus,

$$V = \cup_{1 \leq i \leq b} B_i\ , \qquad B_i \cap B_j = \varnothing \iff i \neq j.$$

For brevity, call $\mathcal{B}$ a *b–partition* of V, and call b the *order* of the partition. The partition $\mathcal{B}$ is *proper* when $2 \leq b < v$, and *trivial* when $b = 1$ or $b = v$.

We consider the case when V has sum $\Sigma V := \Sigma\{x : x \in V\}$ which is a multiple of b, say $\Sigma V := bs,\ s \in \mathbb{Z}^+$. A b–partition $\mathcal{B}$ of V is *equisum*, with *block sum* s, if each of its blocks has sum s:

$$\Sigma B_i := \Sigma\{x : x \in B_i\} = s,\ \ 1 \leq i \leq b.$$

When does V have a proper equisum partition? Clearly, $v \geq 3$ and $2 \leq b \leq \lceil v/2 \rceil$ are necessary conditions, and these conditions are sharp. In the following sections, constructive algorithms will be used to show:

1. If $b, n \in \mathbb{Z}^+$, $2 \leq b \leq \lceil n/2 \rceil$ and $b \leq 12$ or b is any prime-power p^a, the initial interval of integers $[1,\ n]$ has an equisum b–partition if and only if $\Sigma[1,\ n]$ is a multiple of b.
2. No product of two odd prime-powers has an equisum 2–partition of its positive divisors; however, for any prime $p \geq 3$ and $a,\ m \in \mathbb{Z}^+$ with m odd, the set of all positive divisors of $2^a p^m$ has an equisum 2–partition when $2^{a+1} \geq \sigma(p^m)$. Even perfect numbers are the "boundary case" of this result.
3. If the set of aliquot divisors of $n \in \mathbb{Z}^+$ has an equisum 2–partition then n has at least two distinct prime factors. For any prime $p \geq 3$ and $a,\ m \in \mathbb{Z}^+$ with m odd, the set of aliquot divisors of $2^a p^m$ has an equisum 2–partition if $2^{a+1} \geq \sigma(p^m)$. Again, even perfect numbers are the "boundary case" of this result.
4. If $n \in \mathbb{Z}^+$ is odd, its set of aliquot divisors can have an equisum 2–partition only when n is a perfect square. Further, if n has exactly two distinct prime factors $p,\ q$, they must either be twin primes or 3 and 7. The aliquot divisors always have an equisum 2–partition when $\{p,\ q\} = \{3,\ 5\}$; this probably also holds when $\{p,\ q\} = \{3,\ 7\}$. However, there may be only finitely many pairs $\{p,\ q\}$ such that the aliquot divisors of n have such a partition.

Some suggestive results are also obtained for initial intervals of the primes. For $b \in \{2,\ 3,\ 4,\ 5,\ 6\}$, it is shown that the smallest feasible initial interval of primes with sum equal to a multiple of b does have an equisum b–partition. For $b = 2$, every odd-sized initial interval of primes $p \leq 127$ does have an equisum 2–partition, and it is conjectured that this holds for every odd-sized initial interval of primes.

2. Initial Intervals of $\mathbb{Z}^+$

The most natural class of set V to consider for equisum partition is an initial interval of integers

$$[1,\ n] := \left\{a \in \mathbb{Z}^+ : a \le n\right\}.$$

If $\mathcal{B} := \{B_i | 1 \le i \le b\}$ is a b–partition of $V := [1,\ n]$, its range of block sizes $|B_i| := k_i,\ 1 \le i \le b$ is of interest. The *defect* of $\mathcal{B}$ is the smallest integer δ such that

$$\left|k_i - k_j\right| \le \delta \iff i \ne j.$$

If $\mathcal{B}$ has defect $\delta = 0$ it is a *uniform* partition of V, with *block size* $k := n/b$. It is natural to seek equisum partitions with minimum defect.

Two examples serve to give insight into equisum partitions of $[1,\ n]$.

Example 1. *If $V = [1,\ 14]$ then $\Sigma V = 105$, so any proper equisum partition of V must have order $b \in \{3,\ 5,\ 7\}$. Since $b = 7$ is the only possible order that is a factor of 14, it is the only case where defect $\delta = 0$ is possible. An equisum b–partition does exist in each case*:

$$b = 7,\ s = 15 :\ \{1,\ 14\}, \{2,\ 13\}, \{3,\ 12\}, \{4,\ 11\}, \{5,\ 10\}, \{6,\ 9\}, \{7,\ 8\};$$
$$b = 5,\ s = 21 :\ \{7,\ 14\}, \{1,\ 8,\ 12\}, \{2,\ 9,\ 10\}, \{3,\ 5,\ 13\}, \{4,\ 6,\ 11\};$$
$$b = 3,\ s = 35 :\ \{4,\ 8,\ 10,\ 13\}, \{1,\ 5,\ 6,\ 9,\ 14\}, \{2,\ 3,\ 7,\ 11,\ 12\}.$$

The first partition is uniform, the other two have defect $\delta = 1$.

Example 2. *If $V = [1,\ 15]$ then $\Sigma V = 120$, so any proper equisum partition of V has order $b \in \{2,\ 3,\ 4,\ 5,\ 6,\ 8\}$. An equisum b–partition exists in each case. For instance,*

$$b = 8,\ s = 15 :\ \{15\}, \{1,\ 14\}, \{2,\ 13\}, \{3,\ 12\}, \{4,\ 11\}, \{5,\ 10\}, \{6,\ 9\}, \{7,\ 8\}.$$

By taking unions of consecutive pairs of blocks, then repeating, equisum partitions of orders 4 and 2 are immediately produced. All three have defect $\delta = 1$. This is best possible, since each has even order but $n = 15$ is odd.

Next, for the two possible orders which are divisors of 15 there are uniform equisum partitions:

$$b = 5,\ s = 24 :\ \{1,\ 8,\ 15\}, \{2,\ 9,\ 13\}, \{3,\ 10,\ 11\}, \{4,\ 6,\ 14\},\ \{5,\ 7,\ 12\};$$
$$b = 3,\ s = 40 :\ \{1,\ 5,\ 9,\ 11,\ 14\}, \{2,\ 6,\ 7,\ 10,\ 15\},\ \{3,\ 4,\ 8,\ 12,\ 13\}.$$

Finally, for order 6 we have an equisum partition with defect $\delta = 1$:

$$b = 6,\ s = 20 :\ \{5,\ 15\}, \{6,\ 14\}, \{8,\ 12\}, \{1,\ 9,\ 10\},\ \{2,\ 7,\ 11\},\ \{3,\ 4,\ 13\}.$$

Notice that pairs of blocks of different sizes in this equisum 6–partition could be combined to give an alternative uniform equisum 3–partition.

We now introduce a construction which serves to generalise these examples. For any $n \in \mathbb{Z}^+$ and $X \subseteq \mathbb{Z}^+$, define $X + n := \{x + n | x \in X\}$.

Direct Sum Construction: *Let $\mathcal{B} := \{B_i | 1 \le i \le b\}$ and $\mathcal{B}' := \left\{B_i' \middle| 1 \le i \le b\right\}$ be b–partitions of the sets $V := [1,\ n]$ and $V' := [1,\ n']$, respectively. The direct sum*

$$\mathcal{B}'' := \mathcal{B} \oplus \mathcal{B}' := \left\{B_i \cup \left(B_i' + n\right) \middle| 1 \le i \le b\right\}$$

is a b–partition of the set $V'' := [1,\ n + n']$.

The direct sum construction has several useful special cases. If $\mathcal{B}$ and $\mathcal{B}'$ are uniform, then $\mathcal{B}''$ is uniform. If $\mathcal{B}$ and $\mathcal{B}'$ are equisum, and $\mathcal{B}'$ is uniform, then $\mathcal{B}''$ is equisum.

Let us call $\mathcal{B} := \{B_i | 1 \le i \le b\}$ a *consecsum* b–partition of $V := [1,\ n]$ if its block sums $\Sigma B_i := s_i,\ 1 \le i \le b$ satisfy $s_{i+1} = s_i + 1$ for $1 \le i < b$. We now note two more special cases of the direct sum construction. If $\mathcal{B}$ is consecsum and $\mathcal{B}'$ is uniform equisum, then $\mathcal{B}''$ is consecsum. Finally, if $\mathcal{B}$ and $\mathcal{B}'$ are both consecsum, and $\mathcal{B}'$ is uniform, then $\mathcal{B}^R \oplus \mathcal{B}'$ is equisum when $\mathcal{B}^R$, the *reverse* of $\mathcal{B}$, is defined by $\mathcal{B}^R := \left\{B_i^R := B_{b-i+1} \middle| 1 \le i \le b\right\}$.

Theorem 1. *Let $b \in \mathbb{Z}^+,\ b \ge 2$. If $[1,\ n]$ has an equisum b–partition with defect δ, then so does $[1,\ n']$, for any $n' \in \mathbb{Z}^+,\ n' > n$, such that $n' \equiv n \pmod{2b}$.*

Proof. Let $\mathcal{E} := \{E_i | 1 \le i \le b\}$ be the trivial uniform b–partition of $[1,\ b]$ with $E_i := \{i\}$ for $1 \le i \le b$. Then $\mathcal{E}$ is consecsum, so $\mathcal{A} := \mathcal{E} \oplus \mathcal{E}^R$ is a uniform equisum b–partition of $[1,\ 2b]$. Put $\mathcal{A}^{(1)} := \mathcal{A}$ and $\mathcal{A}^{(m+1)} := \mathcal{A} \oplus \mathcal{A}^{(m)}$ for all $m \in \mathbb{Z}^+$. Then $\mathcal{A}^{(m)}$ is a uniform equisum b–partition of $[1,\ 2mb]$. Let $\mathcal{B} := \{B_i | 1 \le i \le b\}$ be an equisum b–partition of $[1,\ n]$ with defect δ. Then, $\mathcal{B} \oplus \mathcal{A}^{(m)}$ is an equisum b–partition of $[1, n' :=\ n + 2mb]$ with defect δ. □

When $[1,\ n]$ has a b–partition $\mathcal{B}$ of the specified type, the proof of Theorem 1 is, in effect, an unconditional algorithm for constructing from $\mathcal{B}$ such a b–partition for each suitable $[1,\ n']$. The back and forth (boustrophedon) construction producing $\mathcal{A}^{(m)}$ can be described as *knitting*.

With the convention that $E_1 - 1 := \varnothing$, we *define* $\mathcal{E}' := \mathcal{E} - 1$ to be a consecsum b–partition of $[1, b-1]$ with defect $\delta = 1$, admitting one empty block. The knitting step $\mathcal{A}' := \mathcal{E}' \oplus \mathcal{E}^R$ produces an equisum b–partition of $[1,\ 2b-1]$ with defect $\delta = 1$, so with ordinary knitting $\mathcal{A}^{(m)}$ we have

Theorem 2. *For any $b \in \mathbb{Z}^+,\ b \ge 2$, and $\delta \in \{0,\ 1\}$, the set $[1, n]$ has an equisum b–partition, with defect δ, for all $n \in \mathbb{Z}^+,\ n + \delta \equiv 0 \pmod{2b}$.*

A parallel construction, proceeding from consecsum to equisum, now establishes.

Lemma 1. *Let $b \in \mathbb{Z}^+,\ b \ge 2$. If $[1,\ n]$ has a consecsum b–partition with defect δ, then $[1,\ n']$ has an equisum b–partition with defect δ for any $n' \in \mathbb{Z}^+,\ n' > n$, such that $n' \equiv n + b \pmod{2b}$.*

Proof. As in the previous proof, $\mathcal{E}$ is a uniform consecsum b–partition of $[1,\ b]$, and $\mathcal{A} := \mathcal{E} \oplus \mathcal{E}^R$ is a uniform equisum b–partition of $[1,\ 2b]$. For all $m \in \mathbb{Z}^+$ let $C^{(1)} := \mathcal{E}$, $C^{(m+1)} := \mathcal{E} \oplus \mathcal{A}^{(m)}$. Then, $C^{(m)}$ is a uniform consecsum b–partition of $[1,\ (2m-1)b]$. Suppose that $\mathcal{B} := \{B_i | 1 \le i \le b\}$ is a consecsum b–partition of $[1,\ n]$ with defect δ. Then, $\mathcal{B}^R \oplus C^{(m)}$ is an equisum b–partition of $[1,\ n + (2m-1)b]$ with defect δ. □

When $[1,\ n]$ has a consecsum b–partition $\mathcal{B}$ with defect δ, the proof of Lemma 1 is, in effect, an unconditional algorithm for constructing from $\mathcal{B}$ an equisum b–partition with defect δ for each suitable $[1,\ n']$. Knitting is the key.

Let $C'^{(1)} := \mathcal{E}'$, $C'^{(m+1)} := \mathcal{E}' \oplus \mathcal{A}^{(m)}$, for all $m \in \mathbb{Z}^+$. As $\mathcal{E}'$ is a consecsum partition of $[1,\ b-1]$ with defect $\delta = 1$, so $C'^{(m)}$ is a consecsum b–partition of $[1, (2m-1)b - 1]$ with defect $\delta = 1$. Hence

Theorem 3. *For any $b \in \mathbb{Z}^+,\ b \ge 2$, and $\delta \in \{0,\ 1\}$, the set $[1,\ n]$ has a consecsum b–partition, with defect δ, for all $n \in \mathbb{Z}^+,\ n + \delta \equiv b\ (mod\ 2b)$.*

The next construction proceeds from equisum to consecsum.

Lemma 2. *For any odd $b \in \mathbb{Z}^+,\ b \ge 3$, if $[1,\ n]$ has an equisum b–partition with defect δ, then $[1,\ n + b]$ and $[1,\ n + 2b]$ have consecsum b–partitions with defect δ.*

Proof. For any $k \in \mathbb{Z}^+,\ k \geq 2$, let $b := 2k-1$. Let $\mathcal{D} := \{D_i | 1 \leq i \leq 2k-1 = b\}$ be the b–partition of $[1,\ 2b]$ with blocks

$$D_i := \{i,\ 3k+i-2\},\ \ 1 \leq i \leq k;$$
$$D_{k+i} := \{k+i,\ 2k+i-1\},\ 1 \leq i \leq k-1.$$

Clearly $\Sigma D_i - 3k + 2$ runs through even members of $[1,\ 2k]$ as i runs through $[1,\ k]$, and $\Sigma D_{k+i} - 3k$ runs through odd members of $[1,\ 2k]$ as i runs through $[1,\ k-1]$. For each $j \in \mathbb{Z}^+,\ 3k \leq j \leq 5k-2$, there is an $i \in \mathbb{Z}^+, 1 \leq i \leq 2k-1 = b$ such that D_i has block sum $\Sigma D_i = j$. Thus there is a permutation $\pi : [1,\ b] \to [1,\ b]$ such that

$$\Sigma D_{\pi(i)} = 3k + i - 1,\ \ 1 \leq i \leq b,$$

so $\mathcal{D}^\pi := \left\{D_{\pi(i)} \middle| 1 \leq i \leq b\right\}$ is a uniform consecsum b–partition of $[1,\ 2b]$. Let $\mathcal{B} := \{B_i | 1 \leq i \leq b\}$ be an equisum b–partition of $[1,\ n]$ with defect δ. Then, $\mathcal{B} \oplus \mathcal{E}$ and $\mathcal{B} \oplus \mathcal{D}^\pi$ are consecsum b–partitions of $[1,\ n+b]$ and $[1,\ n+2b]$, respectively, both with defect δ. □

The construction in the proof of Lemma 2, together with a modified version in which $\mathcal{D}$ is replaced by $\mathcal{D} - 1$ throughout, yields

Lemma 3. *For any odd $b \in \mathbb{Z}^+,\ b \geq 3,$ and $\delta \in \{0,\ 1\}$, the set $[1,\ n]$ has a consecsum b–partition, with defect δ, for all $\in \mathbb{Z}^+, n + \delta \equiv 0 \pmod b$.*

Forming direct sums with $\mathcal{E}^R$ now yields

Theorem 4. *For any odd $b \in \mathbb{Z}^+,\ b \geq 3,$ and $\delta \in \{0,\ 1\}$, the set $[1,\ n]$ has an equisum b–partition, with defect δ, for all $n \in \mathbb{Z}^+,\ n > b$, such that $n + \delta \equiv 0 \pmod b$.*

The interval $[1,\ n]$ can have an equisum b–partition only if at most one block is a singleton, so $b \leq \lceil n/2 \rceil$. Together, Theorems 2 and 4 establish

Theorem 5. *For any $b, n \in \mathbb{Z}^+,\ 2 \leq b \leq \lceil n/2 \rceil$, and $\delta \in \{0,\ 1\}$, the set $[1,\ n]$ has an equisum partition of order b, with defect δ, if (1) $n + \delta \equiv 0 \pmod{2b}$, or else if (2) $n + \delta \equiv 0 \pmod b$, and b is odd.*

Let $\mathbb{P}$ be the set of primes. As n and $n+1$ are coprime, we have

Corollary 1. *For any $a, n \in \mathbb{Z}^+,\ p \in \mathbb{P}$, with $n > p^a$, the set $[1,\ n]$ has an equisum partition of order p^a, with defect $\delta \in \{0,\ 1\}$, if and only if $2p^a \big| n(n+1)$.*

For any proper divisor b of n or $n+1$, Theorem 5 settles when $[1,\ n]$ has an equisum partition of order b. This leaves open the question of whether $[1,\ n]$ has an equisum partition of order $b := ac$ when $a \geq 2, c \geq 2$ are coprime positive integers such that $a|n,\ c|n+1$ and $2ac\big|n(n+1)$.

Example 3. *For $V = [1,\ 14]$ the following is a consecsum partition of order 10, with defect $\delta = 1$:*

$$\{1,\ 5\}, \{7\}, \{2,\ 6\}, \{9\}, \{10\}, \{3,\ 8\}, \{12\}, \{13\}, \{14\}, \{4,\ 11\}.$$

Its direct sum with the reversed trivial consecsum 10–partition of $[1,\ 10]$ yields an equisum 10–partition of $[1,\ 24]$ with defect $\delta = 1$:

$$b = 10,\ s = 30:\ \{1,\ 5,\ 24\}, \{7,\ 23\}, \{2,\ 6,\ 22\}, \{9,\ 21\}, \{10,\ 20\},$$
$$\{3,\ 8,\ 19\}, \{12,\ 18\}, \{13,\ 17\}, \{14,\ 16\}, \{4,\ 11,\ 15\}.$$

Here $n = 24,\ a = 2,\ c = 5$.

To generalise this example, it is notationally convenient to denote the set of *triangular numbers* by $T := \{t_k := (k-1)k/2 : k \in \mathbb{Z}^+\}$. Note that $t_k + k = t_{k+1}$.

Lemma 4. *For $k,\ m \in \mathbb{Z}^+,\ k \geq 3$, let $b := t_k$. The set $[1, (k-1)(mk+1)]$ has an equisum b–partition with defect $\delta = 1$.*

Proof. For any $k \in \mathbb{Z}^+,\ k \geq 3$, let $\mathcal{F} := \{F_i | 1 \leq i \leq b := t_k\}$ be the b–partition of $I_{k+1} := [1,\ t_{k+1} - 1] = [1,\ k - 1 + b]$ with blocks

$$F_i := \{k + i\},\ 1 < i < t_k,\ i \notin T;$$
$$F_{t_j} := \left\{j,\ k + t_j\right\},\ \ 1 \leq j \leq k - 1.$$

Since $\Sigma F_{t_j} = k + t_{j+1}$ for $1 \leq j \leq k-1$, it follows that $\{\Sigma F_i | i \in I_k\} = [k+1,\ t_{k+1}]$, so $\mathcal{F}$ is consecsum with defect $\delta = 1$. Let $\mathcal{E} := \{E_i := \{i\} | i \in I_k\}$, so $\mathcal{F}^* := \mathcal{F} \oplus \mathcal{E}^R$ is a defect $\delta = 1$ equisum b–partition of $[1,\ k-1+2b] = \left[1,\ k^2 - 1\right]$. Let $\mathcal{A} := \mathcal{E} \oplus \mathcal{E}^R$ and $\mathcal{A}^{(1)} := \mathcal{A}$, $\mathcal{A}^{(m+1)} := \mathcal{A}^{(m)} \oplus \mathcal{A}$ for all $m \in \mathbb{Z}^+$. Then $\mathcal{A}^{(m)}$ is a uniform b–partition of $[1,\ 2mb] = [1,\ mk(k-1)]$. Hence $[1,\ (k-1)(mk+1)]$ has the defect $\delta = 1$ equisum b–partition $\mathcal{F}^* \oplus \mathcal{A}^{(m-1)}$, for any $m \geq 2$. □

Lemma 4 describes a family of sets $[1, n := (k-1)(mk+1)]$ with an equisum partition of order b such that $2b = (k-1)k$ and $k - 1 | n,\ k | n+1$. In particular, $k = 4$ leads to an order $b = 6$ equisum partition of $[1,\ 3 + 12m]$ with defect $\delta = 1$, starting with the order 6 partition of $[1,\ 15]$ given in Example 2. Similarly, $k = 5$ leads to an order $b = 10$ equisum partition of $[1,\ 4 + 20m]$ with defect $\delta = 1$, starting with the order 10 partition of $[1,\ 24]$ in Example 3.

Example 4. *Here is an order 6 consecsum partition for $V = [1,\ 14]$, with defect $\delta = 1$:*

$$n = 14,\ b = 6:\ \{2,\ 4,\ 9\}, \{3,\ 13\}, \{5,\ 12\}, \{1,\ 7,\ 10\}, \{8,\ 11\}, \{6,\ 14\}$$

and an order 10 consecsum partition for $V = [1,\ 25]$, with defect $\delta = 1$:

$$n = 25,\ b = 10:\ \{4,\ 11,\ 13\}, \{1,\ 8,\ 20\}, \{2,\ 9,\ 19\}, \{6,\ 7,\ 18\}, \{15,\ 17\},$$
$$\{3,\ 5,\ 25\},\ \{10,\ 24\}, \{12,\ 23\}, \{14,\ 22\}, \{16,\ 21\}.$$

Each of these partitions yields a defect $\delta = 1$ equisum b–partition when we form its direct sum with the reversed trivial consecsum b–partition of $[1, b]$. With Lemma 1 it now follows for all $m \in \mathbb{Z}^+$ that $[1,\ 8 + 12m]$ has an equisum 6–partition, and $[1,\ 15 + 20m]$ has an equisum 10–partition, with defect $\delta = 1$, in all instances.

As 12 is not a triangular number, we cannot use Lemma 4 for the case $b = 12$, but the method used in Example 4 can be applied.

Example 5. *Here are defect $\delta = 1$ order 12 consecsum partitions for the intervals $V = [1,\ 20]$ and $V = [1,\ 27]$:*

$$n = 20,\ b = 12:\ \{12\}, \{13\}, \{14\}, \{5,\ 10\}, \{1,\ 15\}, \{8,\ 9\},$$
$$\{7,\ 11\},\ \{19\},\ \{2,\ 18\},\ \{4,\ 17\},\ \{6,\ 22\},\ \{3,\ 20\}.$$
$$n = 27,\ b = 12:\ \{4,\ 22\}, \{11,\ 16\}, \{3,\ 25\}, \{10,\ 19\}, \{6,\ 24\}, \{1,\ 12,\ 18\},$$
$$\{9,\ 23\},\ \{7,\ 26\},\ \{14,\ 20\},\ \{5,\ 13,\ 17\},\ \{15,\ 21\},\ \{2,\ 8,\ 27\}.$$

Equisum 12–partitions with defect $\delta = 1$ result by forming direct sums with the reversed trivial consecsum 12–partition of $[1,\ 12]$. For all $m \in \mathbb{Z}^+$, Lemma 1 now shows that $[1,\ 8 + 24m]$ and $[1,\ 15 + 24m]$ have equisum 12–partitions with defect $\delta = 1$.

For $b \in \{6,\ 10,\ 12\}$, these results demonstrate the existence of order b equisum partitions of $[1,\ n]$ for all four residue classes of $n \pmod{2b}$ such that $2b \big| n(n+1)$.

Theorem 6. *For $b \in \{6,\ 10,\ 12\}$ and $n \in \mathbb{Z}^+,\ n \geq 2b-1$, the set $[1,\ n]$ has an equisum partition of order b, with defect $\delta \in \{0,\ 1\}$, if and only if $2b \big| n(n+1)$.*

It appears likely that for all $b,\ n \in \mathbb{Z}^+$, if $2 \leq b \leq \lceil n/2 \rceil$ and $2b \big| n(n+1)$, then $[1,\ n]$ has an equisum partition of order b, with defect $\delta \in \{0,\ 1\}$. However, a proof with this level of generality seems to be elusive.

3. Initial Intervals of $\mathbb{P}$

An apparently unlikely class of set V to consider for equisum partition is an initial interval of the primes $\mathbb{P}$,

$$[\mathbb{P}:\ q] := \{p \in \mathbb{P} : p \leq q\}.$$

Because $\mathbb{P}$ appears to have an irregular fine structure, regularity in partitions of its initial intervals is unexpected. However, $\mathbb{P}$ has a regular large scale structure, such as the asymptotic equidistribution of primes into the reside classes coprime with k for any $k \in \mathbb{Z}^+,\ k \geq 3$. Hence, for any $b \in \mathbb{Z}^+,\ b \geq 2$, we might hope that there are infinitely many primes q such that $[\mathbb{P}:\ q]$ has an equisum b–partition with a relatively small defect. Closer study gives support to this hope.

If $[\mathbb{P}:\ q]$ has an equisum b–partition with $b \in \mathbb{Z}^+,\ b \geq 2$, then $\Sigma[\mathbb{P}:\ q] = bs$ for some $s \in \mathbb{Z}^+,\ s \geq q$. Perhaps, for each $b \geq 2$, there might be a prime q_0 such that $[\mathbb{P}:\ q]$ has an equisum b–partition if and only if $b \big| \Sigma[\mathbb{P}:\ q]$ and $q \geq q_0$. The next example is consistent with this possibility.

Example 6. *For $2 \leq b \leq 5$, the earliest instances of $q \in \mathbb{P}$ such that $b \big| \Sigma[\mathbb{P}:\ q]$ are the following, noted for each b as pairs $(q,\ s)$ with $s := \Sigma[\mathbb{P}:\ q]/b$, for $s \geq q$.*

$$b = 2:\ (5,\ 5),\ (11,\ 14),\ (17,\ 29),\ (23,\ 50),\ (31,\ 80),\ (41,\ 119),\ (47,\ 164),\ldots$$
$$b = 3:\ (29,\ 43),\ (53,\ 127),\ (61,\ 167),\ (71,\ 213),\ (89,\ 321),\ldots$$
$$b = 4:\ (23,\ 25),\ (31,\ 40),\ (47,\ 82),(59,\ 110),(67,\ 142),(73,\ 178),(97,\ 265),\ldots$$
$$b = 5:\ (31,\ 32),\ (59,\ 88),\ (97,\ 212),\ \ldots$$

For $b = 6$ the instances are much less frequent: $(269,\ 1145)$ is the first.

Evidently, $b = 4:\ (23,\ 25)$ and $b = 5:\ (31,\ 32)$ do not have equisum partitions of order b because s is not large enough. For each order $b \leq 6$, the following are the earliest possible equisum b–partitions, with the corresponding possible q_0. Considerations of parity show that the block containing 2 must be the only block of its size, so each partition has a positive defect:

$$b = 2,\ q_0 = 5:\ \{2,\ 3\},\ \{5\};\ \{2,\ 5,\ 7\},\{3,\ 11\};\ \{2,\ 3,\ 7,\ 17\},\{5,\ 11,\ 13\};\ldots$$
$$b = 3,\ q_0 = 29:\ \{2,\ 5,\ 17,\ 19\},\ \{7,\ 13,\ 23\},\ \{3,\ 11,\ 29\};\ldots$$
$$b = 4,\ q_0 = 31:\ \{2,\ 7,\ 31\},\ \{3,\ 5,\ 13,\ 19\},\ \{11,\ 29\},\{17,\ 23\};\ldots$$
$$b = 5,\ q_0 = 59:\ \{2,\ 7,\ 19,\ 23,\ 37\},\ \{3,\ 11,\ 31,\ 43\},\{5,\ 13,\ 17,\ 53\},\{29,\ 59\},\ \{41,\ 47\};\ldots$$

For $b = 6:\ (269,\ 1145)$, an equisum partition of order 6 for $q_0 = 269$ is

$$\{2,\ 103,\ 251,\ 257,\ 263,\ 269\},\{5,\ 97,\ 101,\ 229,\ 233,\ 239,\ 241\},$$
$$\{17,\ 71,\ 197,\ 199,\ 211,\ 223,\ 227\},\ \{61,\ 167,\ 173,\ 179,\ 181,\ 191,\ 193\},$$
$$\{11,\ 107,\ 131,\ 137,\ 139,\ 149,\ 151,\ 157,\ 163\},\ B,$$

where the final block B is the subset of 21 remaining primes from $[\mathbb{P}:\ 127]$. This solution was produced using a greedy algorithm approach. It is easy to see that order 6 equisum partitions with smaller defect do exist: for example, the subset $\{7,\ 13,\ 23,\ 29,\ 31\} \subset B$ could be exchanged for Guy's prime $\{103\}$ from the first block to reduce the defect of the above partition.

Let us examine the case $b = 2$ more closely. Let $\mathbb{P} := \{p_n : n \in \mathbb{Z}^+\}$ with $p_n < p_{n+1}$ for all $n \in \mathbb{Z}^+$. Since 2 is the only even prime, $2\big|\Sigma[\mathbb{P} :\ p_n]$ if and only if n is odd. Suppose, for some particular k, that $[\mathbb{P} :\ p_{2k+1}]$ has an equisum 2–partition

$$\mathcal{B}_k := \Big\{B_{k,\ i} : i \in \{1,\ 2\}\Big\}.$$

Say that $\mathcal{B}_k$ has the *extension property* if there are subsets $A \subset B_{k,1},\ A' \subset B_{k,2}$ such that $\Sigma A' - \Sigma A = \varepsilon_k$, where $p_{2k+3} - p_{2k+2} := 2\varepsilon_k$. (This is somewhat related to the classical Goldbach conjecture.) Then,

$$B_{k+1,1} := \Big(B_{k,1} \backslash A\Big) \cup A' \cup \{p_{2k+2}\},\ B_{k+1,2} := \Big(B_{k,2} \backslash A'\Big) \cup A \cup \{p_{2k+3}\}$$

is an equisum 2–partition $\mathcal{B}_{k+1}$ of $[\mathbb{P} :\ p_{2k+3}]$. (To suit the notation, at times it might be necessary to interchange the two blocks of $\mathcal{B}_k$.) If $\varepsilon_k = p \in \mathbb{P}$, the construction always works with $A' = \{p\},\ A = \varnothing$. If $\varepsilon_k = 1$, and 2, 3 are in separate blocks of $\mathcal{B}_k$, then $A = \{2\},\ A' = \{3\}$ works and 2, 3 are in separate blocks of $\mathcal{B}_{k+1}$. Beginning with $B_{1,\ 1} = \{2,\ 3\},\ B_{1,2} = \{5\}$, the sequence

$$\begin{gathered}\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 2,\ \varepsilon_4 = 1, \varepsilon_5 = \varepsilon_6 = 2, \varepsilon_7 = \varepsilon_8 = 3, \\ \varepsilon_9 = 1, \varepsilon_{10} = 2, \varepsilon_{11} = 4, \varepsilon_{12} = \varepsilon_{13} = 1,\ \varepsilon_{14} = 7, \ldots\end{gathered}$$

leads to equisum 2–partitions with $B_{2,\ 1} = \{3,\ 11\},\ B_{2,2} = \{2,\ 5,\ 7\}$, and subsequently:

$$\begin{gathered}3 \in B_{4,\ 1},\ 2 \in B_{4,2};\ 2 \in B_{5,\ 1}, 3 \in B_{5,2};\ 2 \in B_{7,\ 1}, 3 \in B_{7,2}; 2 \in B_{9,\ 1}, 3 \in B_{9,2}; \\ 3 \in B_{10,\ 1}, 2 \in B_{10,2}; 2, 3 \in B_{11,\ 1},\ 7 \in B_{11,\ 2}; 2, 7 \in B_{12,\ 1}, 3 \in B_{12,2}; \\ 2, 7 \in B_{14,\ 1}, 3 \in B_{14,2}; 2 \in B_{15,\ 1}, 3, 7 \in B_{15,2}; \ldots\end{gathered}$$

Thus, $[\mathbb{P} :\ p_{2k+1}]$ has an equisum 2–partition for $1 \le k \le 15$, since the first 14 cases have the extension property, and apart from adjoining the two new primes, at each step, except $k = 11$ and $k = 14$, it suffices to move 2, or move 3, or interchange 2 and 3; when $k = 11$ we interchange 3 and 7, and when $k = 14$ we move 7. This covers all primes to $p_{31} = 127$. Even the famously large gap $\varepsilon_{14} = 7$ is accommodated. The construction is heuristic rather than algorithmic, as the extension property has not been proved to continue to hold, though it is highly plausible that it will do so.

Conjecture: $[\mathbb{P} :\ p_{2k+1}]$ *has an equisum* 2*–partition for all* $k \in \mathbb{Z}^+$.

Similar, but more complicated, heuristic constructions can be given for higher order partitions, but we leave the details for the reader.

4. Divisor Sets

For any $n \in \mathbb{Z}^+$ the divisor set of n is $D(n) := \{d \in \mathbb{Z}^+ : d\big|n\}$, and

$$\Sigma D(n) := \sigma(n) = \prod\left\{\sigma(p^a) = \frac{p^{a+1} - 1}{p - 1} :\ \ p^a \in P(n)\right\},$$

where $P(n) := \{p^a : p \in \mathbb{P},\ a \ge 1,\ p^a \parallel n\}$ is the set of maximal prime-power divisors of n. We call $\omega(n) := \big|P(n)\big|$ the *rank* of n : it is simply the number of distinct prime factors of n.

For any $b \in \mathbb{Z}^+,\ b \ge 2$, a necessary condition for $D(n)$ to have an equisum b–partition is $b\big|\sigma(n)$ and $s := \sigma(n)/b \ge n$, since $n \in D(n)$. Thus, we need $\sigma(n) \ge bn$, so the "boundary case" $\sigma(n) = 2n$ requires n to be *perfect* [6]; in all other cases, $\sigma(n) > 2n$, so n is *abundant*. No prime-power is perfect or abundant, since $\sigma(p^a) < 2p^a$ always holds: for $D(n)$ to have an equisum b–partition with $b \ge 2$, the rank of n must be at least 2. Indeed, order $b \ge 3$ requires a rank of at least 3, order $b \ge 5$ requires rank at least 6, and so on. If n is odd and has rank 2 then $\sigma(n) < 15n/8$, so an equisum 2–partition of $D(n)$ for n of rank 2 is only possible if $2|n$. If $n = 2^a p$ for some $a \in \mathbb{Z}^+,\ p \in \mathbb{P},\ p \geq 3$, then

$$\sigma(n) = \left(2^{a+1} - 1\right)(p+1) \geq 2n \Longleftrightarrow 2^{a+1} \geq p+1$$

so $D(2^a p)$ can have an equisum 2–partition only if $2^{a+1} \geq p + 1$.

Lemma 5. *Let $a \in \mathbb{Z}^+,\ p \in \mathbb{P},\ 2^{a+1} - 1 \geq p \geq 3$. For every $m \in \mathbb{Z}^+,\ m \geq a$, the divisor set $D(2^m p)$ has an equisum 2–partition.*

Proof. Let $k := (p+1)/2$. Choose any $m \in \mathbb{Z}^+, m \geq a$, and let

$$B_1 := D\left(2^{m-1}p\right),\ B_2 := \{2^m,\ 2^m p\}.$$

Then $\mathcal{B}(m) := \{B_1, B_2\}$ is a 2–partition of $D(2^m p)$, with block sums

$$s_1 := \Sigma B_1 = (2^m - 1)(p+1),\ \ s_2 := \Sigma B_2 = 2^m(p+1) = s_1 + 2k.$$

To begin, suppose $k = 2^a$ and $m = a$. (This is the case in which $2^a p$ is perfect.) Then the block sums of $\mathcal{B}(a)$ are

$$s_1 = 2k(k-1),\ \ s_2 = 2k^2$$

so the transfer

$$B_1' := B_1 \cup \{2^a\} = B_1 \cup \{k\},\ B_2' := B_2 \backslash \{2^a\} = \{2^a p\} = \{k(2k-1)\}$$

produces a new 2–partition $\mathcal{B}'(a) := \left\{B_1', B_2'\right\}$ of $D(2^a p)$ which is equisum, since its block sums are

$$s_1' := \Sigma B_1' = s_1 + k = k(2k-1),\ s_2' := \Sigma B_2' = s_2 - k = k(2k-1).$$

Note that $B_1' = D(2^a p) \backslash \{2^a p\}$, $B_2' = \{2^a p\}$ and $s_1' = s_2' = 2^a p$. Now suppose $k = 2^a$ and $m > a$, or $k < 2^a$ and $m \geq a$. (In this case $2^m p$ is abundant.) Recall that the block sums of $\mathcal{B}(m) := \{B_1, B_2\}$ satisfy $s_2 = s_1 + 2k$. Let $r := 2^m - k$. Then $0 < r < 2^m$ because either $k = 2^a < 2^m$ or $k < 2^a \leq 2^m$. Since B_1 contains $\left\{2^i : 0 \leq i \leq m-1\right\}$, there is some set $R \subseteq B_1$ such that $\Sigma R = r$ matches the binary representation of r. Now the exchange

$$B_1' := (B_1 \backslash R) \cup \{2^m\},\ \ B_2' := (B_2 \backslash \{2^m\}) \cup R$$

defines a 2–partition $\mathcal{B}'(m) := \left\{B_1', B_2'\right\}$ of $D(2^m p)$ which is equisum, since

$$s_1' := \Sigma B_1' = s_1 - r + 2^m = s_1 + k,\ \ s_2' := \Sigma B_2' = s_2 - 2^m + r = s_2 - k = s_1 + k.$$

This construction explicitly satisfies the claim. □

When $a, m \in \mathbb{Z}^+$, the divisor set $D(2^a p^m)$ contains exactly $m+1$ odd factors of $2^a p^m$, so the block sums of any 2–partition of $D(2^a p^m)$ have the same parity if and only if m is odd. Hence, an equisum 2–partition of $D(2^a p^m)$ is only possible when m is odd.

Suppose m is odd and $2^{a+1} \geq \sigma(p^m)$. Continue with the construction used to prove Lemma 5, now adjusted by taking

$$B_1 := D\left(2^{a-1}p^m\right),\ B_2 := \left\{2^a p^i : 0 \leq i \leq m\right\}.$$

As m is odd, so $\sigma(p^m)$ is even. Let $k := \sigma(p^m)/2$ and $r := 2^a - k$, so $0 \leq r < 2^a$. There is a subset $R \subseteq \left\{2^i : 0 \leq i < a\right\} \subset B_1$ such that $\Sigma R = r$, and the exchange

$$B_1' := (B_1 \backslash R) \cup \{2^a\},\ \ B_2' := (B_2 \backslash \{2^a\}) \cup R$$

produces an equisum 2–partition of $D(2^a p^m)$. If $r = 0$, then $R = \varnothing$. Hence, we have

Theorem 7. *Let $a,\ m \in \mathbb{Z}^+,\ 2 \nmid m,\ p \in \mathbb{P},\ p \geq 3,\ 2^{a+1} \geq \sigma(p^m)$. Then the divisor set $D(2^a p^m)$ has an equisum 2–partition.*

It is worth noting that the equisum 2–partition explicitly constructed to prove Theorem 7 is not necessarily unique. For instance, when $n = 2^5 3^3 = 864$, the constructed equisum 2–partition of $D(n)$ has $\{4,\ 8,\ 96,\ 288,\ 864\}$ as the block containing n, while an alternative has $\{108,\ 288,\ 864\}$ as the block containing n. (This alternative essentially results from the identity $2^3 + 1 = 3^2$.)

We shall forego discussion of equisum b–partitions of divisor sets $D(n)$ for cases when n has rank greater than 2. However, there is a nice question to note. Presumably for each $b \in \mathbb{Z}^+,\ b \geq 2$, there are divisor sets $D(n)$ which have an equisum b–partition; if so, what is the smallest such n?

5. Aliquot Sets

For any $n \in \mathbb{Z}^+$, the *aliquot set* $D'(n) := \{d \in \mathbb{Z}^+ : d \big| n,\ d < n\}$ is the set of divisors less than n, often called its *aliquot parts*. Its sum is $s(n) := \Sigma D'(n) = \sigma(n) - n$. Unlike $\sigma(n)$, the function $s(n)$ is not multiplicative, so it is less straightforward to use the structure of n to predict when $s(n)$ will be a multiple of any given $b \in \mathbb{Z}^+$. However, for the simplest case $b = 2,\ n \geq 3$, evidently $2 \big| s(n)$ if and only if $\sigma(n)$ and n have the same parity: this occurs precisely when n is an odd square, or n is even and is neither a square nor twice a square.

The construction used for Lemma 5 and Theorem 7 easily adapts to the aliquot case, and shows in particular that the aliquot set of an even perfect number has an equisum 2–partition.

Theorem 8. *Let $a,\ m \in \mathbb{Z}^+,\ 2 \nmid m,\ p \in \mathbb{P},\ p \geq 3,\ 2^{a+1} \geq \sigma(p^m)$. Then, the aliquot set $D'(2^a p^m)$ has an equisum 2–partition.*

Proof. Let $\mathcal{B}' := \left\{B_1', B_2'\right\}$ be the equisum 2–partition of the divisor set $D(2^a p^m)$ used to establish Theorem 7. Then $\left\{2^{a-1} p^i : 1 \leq i \leq m\right\} \subset B_1'$ and $2^a p^m \in B_2'$. Let

$$B_1'' := B_1' \backslash \left\{2^{a-1} p^m\right\},\ \ B_2'' := \left(B_2' \backslash \{2^a p^m\}\right) \cup \left\{2^{a-1} p^m\right\}.$$

Evidently, $\mathcal{B}'' := \left\{B_1'', B_2''\right\}$ is an equisum 2–partition of $D'(2^a p^m)$. □

For any $n \in \mathbb{Z}^+$, suppose the divisor set $D(2n)$ has an equisum 2–partition $\mathcal{B} := \{B_1, B_2\}$ such that $n \in B_1, 2n \in B_2$. Then, the construction used for Theorem 8 modifies to yield an equisum 2–partition $\mathcal{B}' := \left\{B_1', B_2'\right\}$ for the aliquot set $D'(2n)$, thus:

$$B_1' := B_1 \backslash \{n\},\ \ B_2' := (B_2 \backslash \{2n\}) \cup \{n\}.$$

Note that if $\mathcal{B}$ is an equisum 2–partition of $D(2n)$, the blocks of $\mathcal{B}$ do not necessarily separate n and $2n$. For instance, $D(600)$ has an equisum 2–partition into $\{30,\ 300,\ 600\}$ and its complement. On the other hand, the equisum 2–partition into $\{30,\ 100,\ 200,\ 600\}$ and its complement does separate 300 and 600, so easily modifies to give an equisum 2–partition of $D'(600)$.

For any $a, b \in \mathbb{Z}^+,\ p \in \mathbb{P}, b \geq 2$, we have $D'(p^a) = \{1\} \cup pD'\left(p^{a-1}\right)$, so any b–partition of $D'(p^a)$ has one block sum in the residue class $1 (\mathrm{mod}\ p)$ and all others in $0 (\mathrm{mod}\ p)$; hence, the block sums cannot be equal. Thus, if $D'(n)$ has an equisum b–partition, n must have a rank of at least 2.

For brevity, we shall confine the remaining discussion to equisum 2–partitions of $D'(n)$ for odd n of rank 2.

Theorem 9. *Let $n \in \mathbb{Z}^+,\ 2 \nmid n$, have just two distinct prime factors $p < q$. If the aliquot set $D'\left(n^2\right)$ has an equisum 2–partition then $p,\ q$ are twin primes or $p = 3,\ q = 7$.*

Proof. Let $n^2 := p^{2a}q^{2c}$ for some $a,\ c \in \mathbb{Z}^+$. Clearly, $D'\left(n^2\right)$ has no equisum 2–partition if its largest aliquot divisor $p^{2a-1}q^{2c}$ is greater than the sum of all other aliquot divisors. The sum of those other divisors is

$$S := \sigma\left(p^{2a}q^{2c-1}\right) + q^{2c}\sigma\left(p^{2a-2}\right) < \frac{p^{2a+1}q^{2c}}{(p-1)(q-1)} + \frac{p^{2a-1}q^{2c}}{p-1}.$$

Hence, $S < p^{2a-1}q^{2c}$ certainly holds if $p^2 \le (p-2)(q-1)$. For $p \in \{3,\ 5\}$, this holds when $q \ge 11$. For $p \ge 7$, it holds when $q \ge p+4$; since q is prime, the only case not then excluded is $q = p+2$. □

Theorem 10. *For every $a,\ c \in \mathbb{Z}^+$, the aliquot set $D'\left(3^{2a}5^{2c}\right)$ has an equisum 2–partition.*

Proof. When $n = 3^2 5^2 = 225$, the aliquot set $D'(n)$ has the equisum 2–partition

$$D'(225) = \{1,\ 3,\ 15,\ 25,\ 45\} \cup \{5,\ 9,\ 75\}.$$

Fix $c \in \mathbb{Z}^+$ and suppose $\mathcal{B} := \{B_1, B_2\}$ is an equisum 2–partition of $D'\left(3^2 5^{2c}\right)$ with $3 := 5^{2c} \in B_1$. Since

$$D'\left(3^2 5^{2c+2}\right) \backslash D'\left(3^2 5^{2c}\right) = \left\{3^2 5^{2c}\right\} \cup 5^{2c+1} D'\left(3^2 5\right),$$

then $D'\left(3^2 5^{2c+2}\right)$ has 2–partition $\mathcal{B}' := \left\{B_1', B_2'\right\}$ such that

$$B_1' := \left(B_1 \backslash \left\{3{\cdot}5^{2c}\right\}\right) \cup \left\{3{\cdot}5^{2c+1}, 3{\cdot}5^{2c+2}\right\},$$
$$B_2' := B_2 \cup \left\{3{\cdot}5^{2c}, 3^2 5^{2c}\right\} \cup \left\{5^{2c+1}, 5^{2c+2}, 3^2 5^{2c+1}\right\}.$$

Moreover, $\mathcal{B}'$ is equisum because

$$\Sigma B_1' - \Sigma B_1 - 3{\cdot}5^{2c} = 87{\cdot}5^{2c},\ \ \Sigma B_2' - \Sigma B_2 = 87{\cdot}5^{2c}.$$

Now fix $a,\ c \in \mathbb{Z}^+$ and suppose $\mathcal{C} := \{C_1, C_2\}$ is an equisum 2–partition of $D'\left(3^{2a}5^{2c}\right)$ with $3^{2a} \in C_1$. Then $D'\left(3^{2a+2}5^{2c}\right)$ has 2–partition $\mathcal{C}' := \left\{C_1', C_2'\right\}$ such that

$$C_1' := \left(C_1 \backslash \left\{3^{2a}\right\}\right) \cup 3^{2a+1} E\left(5^{2c}\right) \cup 3^{2a+2} E\left(5^{2c-2}\right),$$
$$C_2' := C_2 \cup \left\{3^{2a}, 3^{2a}5^{2c}\right\} \cup 3^{2a+1} 5 E\left(5^{2c-2}\right) \cup 3^{2a+2} 5 E\left(5^{2c-2}\right),$$

where $E\left(5^{2c}\right) := \left\{5^{2i} : i \in \mathbb{Z},\ 0 \le i \le c\right\}$. Furthermore, $\mathcal{C}'$ is equisum because

$$\Sigma C_1' - \Sigma C_1 - 3^{2a} = 7K{\cdot}3^{2a},\ \Sigma B_2' - \Sigma B_2 = 7K{\cdot}3^{2a},\ \ K := \left(5^{2c} - 1\right)/2.$$

Induction on a and c completes the argument. □

As before, the proof of Theorem 10, and that which follows for Theorem 11, do essentially provide unconditional algorithms for constructing an equisum partition in any concrete instance.

Theorem 11. *For $a \in \{1,\ 2\}$ and every $c \in \mathbb{Z}^+$, the aliquot set $D'\left(3^{2a}7^{2c}\right)$ has an equisum 2–partition.*

Proof. When $n = 3^2 7^2 = 441$, the aliquot set $D'(n)$ has the equisum 2–partition

$$D'(441) = \{3,\ 147\} \cup \{1,\ 7,\ 9,\ 21,\ 49,\ 63\}.$$

Fix $c \in \mathbb{Z}^+$ and suppose $\mathcal{B} := \{B_1, B_2\}$ is an equisum 2–partition of $D'\left(3^2 7^{2c}\right)$ with $7^{2c} \in B_2$. Then $D'\left(3^2 7^{2c+2}\right)$ has 2–partition $\mathcal{B}' := \left\{B'_1, B'_2\right\}$ such that

$$B'_1 := B_1 \cup \left\{7^{2c},\ 3{\cdot}7^{2c+2}\right\},$$
$$B'_2 := \left(B_2 \backslash \left\{7^{2c}\right\}\right) \cup \left\{3^2 7^{2c},\ 7^{2c+1},\ 3{\cdot}7^{2c+1},\ 3^2 7^{2c+1},\ 7^{2c+2}\right\}.$$

Moreover $\mathcal{B}'$ is equisum because

$$\Sigma B'_1 - \Sigma B_1 = 148{\cdot}7^{2c},\ \ \Sigma B'_2 - \Sigma B_2 - 7^{2c} = 148{\cdot}7^{2c}.$$

When $n = 3^4 7^2 = 3969$, the aliquot set $D'(n)$ has an equisum 2–partition comprising

$$\{1,\ 7,\ 49,\ 63,\ 147,\ 189,\ 441,\ 567\}$$

and its complement. When $n = 3^4 7^4 = 194481$, the aliquot set $D'(n)$ has an equisum 2–partition comprising $\left\{1,\ 3^3 7,\ 3{\cdot}7^4,\ 3^3 7^4\right\}$ and its complement. Fix $c \geq 2$ and assume $\mathcal{C} := \{C_1, C_2\}$ is an equisum 2–partition of $D'\left(3^4 7^{2c}\right)$ with $3^3 7^{2c} \in C_1,\ 7^{2c} \in C_2$. Then $D'\left(3^4 7^{2c+2}\right)$ has the 2–partition $\mathcal{C}' := \left\{C'_1, C'_2\right\}$ such that

$$C'_1 := \left(C_1 \backslash \left\{3^3 7^{2c}\right\}\right) \cup \left\{7^{2c}\right\} \cup \left\{3{\cdot}7^{2c+2}, 3^3 7^{2c+2}\right\},$$

with complement $C'_2 := D'\left(3^4 7^{2c+2}\right) \backslash C'_1$. Furthermore,

$$\Sigma C'_1 - \Sigma C_1 - 3^3 7^{2c} = 1444{\cdot}7^{2c},\ \ \Sigma C'_2 - \Sigma C_2 - 7^{2c} = 1444{\cdot}7^{2c},$$

so $\mathcal{C}'$ is equisum. The claim now follows by induction on c. □

Probably the aliquot set $D'\left(3^{2a} 7^{2c}\right)$ has an equisum 2–partition for every $a,\ c \in \mathbb{Z}^+$, although a general construction seems to be elusive. However, perhaps there is only a finite number of prime pairs p, q such that $D'\left(p^{2a} q^{2c}\right)$ has an equisum 2–partition for any $a,\ c \in \mathbb{Z}^+$.

As with divisor sets, we forego discussion of aliquot sets $D'(n)$ when n has rank greater than 2, except to report that $D'(5040)$ has both an equisum 3–partition and an equisum 4–partition, so we can ask the following general question. Presumably for each $b \in \mathbb{Z}^+,\ b \geq 2$, there are aliquot sets $D'(n)$ which have an equisum b–partition; if so, what is the smallest such n?

6. Concluding Remarks

Except in the case of aliquot divisor sets of odd integers, it has been shown in a variety of contexts that when a finite set of positive integers meets simple necessary conditions, it turns out to have an equisum partition of an appropriate order. The cases studied have been treated constructively. Intriguingly, although the constructions are strong evidence that such partitions will be possible in general, existence proofs of significant generality seem to be elusive.

Footnote: This study was motivated by the author's recent geometric work on divisor sets and aliquot sets, including partitioning an appropriate rectangle into rectangles of areas equal to the relevant divisors [7].

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