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# Complex Neutrosophic Hypergraphs: New Social Network Models

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**Abstract:** A complex neutrosophic set is a useful model to handle indeterminate situations with a periodic nature. This is characterized by truth, indeterminacy, and falsity degrees which are the combination of real-valued amplitude terms and complex-valued phase terms. Hypergraphs are objects that enable us to dig out invisible connections between the underlying structures of complex systems such as those leading to sustainable development. In this paper, we apply the most fruitful concept of complex neutrosophic sets to theory of hypergraphs. We define complex neutrosophic hypergraphs and discuss their certain properties including lower truncation, upper truncation, and transition levels. Furthermore, we define $T$-related complex neutrosophic hypergraphs and properties of minimal transversals of complex neutrosophic hypergraphs. Finally, we represent the modeling of certain social networks with intersecting communities through the score functions and choice values of complex neutrosophic hypergraphs. We also give a brief comparison of our proposed model with other existing models.

**Keywords:** complex neutrosophic hypergraphs; $T$-related complex neutrosophic hypergraphs; algorithms; comparative analysis

## 1. Introduction

Fuzzy sets (FSs) were originally defined by Zadeh [1] as a novel approach to represent uncertainty arising in various fields that was questioned by many researchers at that time. A FS is characterized by a truth membership function $\mu$ which ranges over $[0,1]$. To generalize the notion of FSs, intuitionistic fuzzy sets (IFSs) were proposed by Atanassov [2] because it is not always true that the falsity degree of an element in a FS is $1-\mu(x)$ as there may be some hesitation part. Therefore, the truth (t) and falsity (f) membership functions are used independently to characterize an IFS such that the sum of truth and falsity degrees should not be greater than one. Fuzzy sets give the degree of membership of an element in a given set (the non-membership of degree equals one minus the degree of membership), while IFSs give both a degree of membership and a degree of non-membership, which are more-or-less independent from each other. Liu et al. [3] introduced different types of centroid transformations of IF values. Furthermore, Feng et al. [4] defined various new operations for generalized IF soft sets. As an extension of IFSs, Smarandache [5] introduced the concept of neutrosophy to study the nature, origin, and neutralities, and the neutrosophic set (NS). A NS is characterized by truth (t), indeterminacy (i), and falsity (f) membership functions. A NS is used as a powerful mathematical tool to deal the inconsistent data that exists in our daily life. For the practical use of NSs in science and engineering, Smarandache [5] and Wang et al. [6] introduced single-valued neutrosophic sets (SVNSs). A SVNS propose an additional choice to handle indeterminate information. Ye [7] proposed a decision-making method by using the weighted correlation coefficient or the weighted cosine similarity measure of SVNSs to rank the alternatives and proposed an illustrative example to demonstrate the application of the proposed decision-making method. The same author defined SVN minimum spanning tree and its clustering method [8]. Ye [9] also proposed a multicriteria decision-making method using aggregation operators for simplified NSs.

The existing models such as FSs, IFSs, SVNSs cannot handle imprecise, inconsistent, and incomplete information of periodic nature. These theories are applicable to different areas of science, but there is one major deficiency in these sets, i.e., a lack of capability to model two-dimensional phenomena. To overcome this difficulty, the concept of complex fuzzy sets (CFSs) was introduced by Ramot et al. [10]. A CFS is characterized by a membership function $\mu(x)$ whose range is not limited to [0, 1] but extends to the unit circle in the complex plane. Hence, $\mu(x)$ is a complex-valued function that assigns a grade of membership of the form $v(x)e^{\iota\alpha(x)}$, $\iota = \sqrt{-1}$ to any element $x$ in the universe of discourse. Thus, the membership function $\mu(x)$ of CFS consists of two terms, i.e., amplitude term $v(x)$ which lies in the unit interval [0, 1] and phase term (periodic term) $\alpha(x)$ which lies in the interval $[0, 2\pi]$. This phase term distinguishes a CFS model from all other models available in the literature. Opposing to a fuzzy characteristic function, the range of CFS's membership degrees is not restricted to $[0, 1]$, but extends to the complex plane with unit circle. Ramot et al. [11] discussed the union, intersection, and compliment of CFSs with the help of illustrative examples. A systematic review of CFSs was proposed by Yazdanbakhsh and Dick [12]. To generalize the concept of CFSs, complex intuitionistic fuzzy sets (CIFSs) were introduced by Alkouri and Salleh [13] by adding non-membership degree $\nu(x) = s(x)e^{\iota\beta(x)}$ to the CFSs subjected to the constraint $r + s \leq 1$. The CIFSs are used to handle the information of uncertainty and periodicity simultaneously. The complex-valued truth and falsity membership degrees can be used to represent uncertainty in many physical quantities such as impedance in electrical engineering, wave function, and decision-making problems. The CFS has only one extra phase term, while CIFS has two additional phase terms which are used in several concepts such as distance measure, projections, and cylindric extensions. To handle imprecise information with a periodic nature, complex neutrosophic sets (CNSs) were proposed by Ali and Smarandache [14]. As we see that uncertainty, inconsistency, and falsity in data are periodic in nature, to handle these types of problems, the CNS plays an important role. A CNS is characterized by a complex-valued truth $t(x)$, complex-valued indeterminate $i(x)$, and complex-valued falsity $f(x)$ membership functions, whose range is extended from [0, 1] to the unit disk in the complex plane. They proposed set theoretic operations such as complement, union, intersection, complex neutrosophic product, Cartesian product, distance measure, and $\delta$-equalities of CNSs and presented an application of CNSs in signal processing.

The vagueness in the representation of various objects and the uncertain interactions between them originated the necessity of fuzzy graphs (FGs) that were first defined by Rosenfeld [15]. He studied several basic graph-theoretic concepts (e.g., bridges and trees), and established some of their properties. Some remarks on FGs were given by Bhattacharya [16] and he proved that results from (crisp) graph theory do not always hold for FGs. To handle the vague and uncertain relations with periodic nature, FGs were extended to complex fuzzy graphs (CFGs) by Thirunavukarasu et al. [17]. They studied the lower and upper bounds of energy of CFGs and illustrated these concepts through numeric examples. Since FGs and CFGs just provide the truth degrees and uncertainties occurring repeatedly, respectively, of pairwise relations. To consider the truth as well as falsity degrees between pairwise relationships simultaneously, intuitionistic fuzzy graphs (IFGs) were defined by Parvathi and Karunambigai [18]. To handle periodic nature of falsity degrees in IFGs, Yaqoob et al. [19] defined complex intuitionistic fuzzy graphs (CIFGs). They studied the homomorphisms of CIFGs and provided an application of CIFGs in cellular network provider companies for the testing of their proposed approach. To extend the concept of IFGs, Broumi et al. [20] defined single-valued neutrosophic graphs (SVNGs) and investigated some of their properties such as strong SVNGs, constant SVNGs, and complete SVNGs. Certain operations on SVNGs were studied by Akram and Shahzadi [21]. Single-valued neutrosophic planar graphs were defined by Akram [22]. Applications of neutrosophic soft graphs were studied by Akram and Shahzadi [23]. To generalize the concept of neutrosophic graphs and CIFGs, complex neutrosophic graphs (CNGs) were defined by Yaqoob and Akram [24]. They discussed some basic operations on CNGs and described these operations with the help of concrete examples. They also presented energy of CNGs.

A hypergraph, as an extension of crisp graph, is considered to be the most developing and powerful tool to model different practical problems in various fields, including biological sciences, computer sciences, and social networks [25]. To deal uncertainty in crisp hypergraphs, fuzzy hypergraphs (FHGs), as an extension of FGs, were defined by Kaufmann [26]. Lee-Kwang and Lee [27] discussed the fuzzy partition using FHGs. A valuable contribution on FGs and FHGs has been proposed by Mordeson and Nair [28]. Fuzzy transversals of FHGs were studied by Goetschel et al. [29]. To discuss the falsity degrees of hypernetworks, intuitionistic fuzzy hypergraphs (IFHGs) were defined by Parvathi et al. [30]. Akram and Dudek [31] proposed some applications of IFHGs. A method for finding the shortest hyperpath in an IFHG (weighted) was proposed by Parvathi et al. [32]. They converted an IFN into intuitionistic fuzzy scores and find the IF shortest hyperpath in the network using the scores and accuracy values. Akram and Shahzadi [33] introduced SVN hypergraphs. Akram and Luqman [34] defined intuitionistic single-valued neutrosophic hypergraphs. The same authors [35] introduced bipolar neutrosophic hypergraphs and discussed the applications of these hypergraphs in marketing and biology. Transversals and minimal transversals of $m$-polar FHGs were studied by Akram and Sarwar [36]. For further studies on FHGs and related extensions, readers are referred to [37–40].

The motivation behind this research work is the existence of indeterminate information of periodic nature in hypernetwork models. A complex neutrosophic hypergraph model plays an important role in handling complicated behavior of indeterminacy and inconsistency with periodic nature. The proposed model generalizes the complex fuzzy model as well as complex intuitionistic fuzzy model. To prove the applicability of our proposed model, we consider two voting procedures. Suppose that 0.6 voters say "yes", 0.2 say "no", and 0.2 are "undecided" in the first voting procedure and 0.3 voters say "yes", 0.3 say "no", and 0.4 are "undecided" in the second voting procedure. We assume that these two procedures held at different days. It is clear that a CFS cannot handle this situation as it only depicts the truth membership 0.6 of voters but fails to represent the falsity and indeterminate degrees. Similarly, a CIFS represents the truth 0.6 and falsity 0.2 degrees of voters but it does not illustrate the 0.2 undecided voters. Now, if we set the amplitude terms as the membership degrees of first voting procedure and phase terms as the membership degrees of second voting procedure, then we can illustrate this information using a complex neutrosophic model as, $\{0.6e^{\iota(0.3)2\pi}, 0.2e^{\iota(0.3)2\pi}, 0.2e^{\iota(0.4)2\pi}\}$. The aim of the proposed work is to apply the most generalized concept of complex neutrosophic sets to hypergraphs to deal periodic nature of inconsistent information existing in hypernetworks. The proposed research generalizes the concepts of CNGs, CFHGs, CIFHGs, and overcomes the drawbacks occurring in previous research. The proposed model is more generalized framework as it does not only deal the reductant nature of imprecise information but also includes the benefits of hypergraphs. Thus, the main objective of this research work is to combine the fruitful effects of CNSs and hypergraph theory.

The contents of this paper are as follows: In Section 2, we define complex neutrosophic hypergraphs, level hypergraphs, lower truncation, upper truncation, and transition levels of these hypergraphs. In Section 3, we define $T$-related complex neutrosophic hypergraphs and discuss certain properties of minimal transversals of complex neutrosophic hypergraphs. We justify the proposed concepts through some concrete examples. Section 4 illustrates the modeling of some social networks with overlapping communities by means of complex neutrosophic hypergraphs. In Section 5, we present a brief comparison of our proposed model with other existing models. In Section 6, we discuss the results of our proposed research. Section 7 deals with conclusions and future directions.

## 2. Complex Neutrosophic Hypergraphs

**Definition 1.** *[5] Let $\mathcal{J}$ be a non-empty set. A neutrosophic set (NS) on $\mathcal{J}$ is defined as,*

$$N = \{(x, t_N(x), i_N(x), f_N(x)) | x \in \mathcal{J}\},$$

*where $t_N, i_N, f_N : \mathcal{J} \to ]0^-, 1^+[$ denote the truth, indeterminacy, and falsity degrees of N such that $0^- \leq t_N(x) + i_N(x) + f_N(x) \leq 3^+$.*

**Definition 2.** *[6] A single-valued neutrosophic set (SVNS) on $\mathcal{J}$ is defined as,*

$$S = \{(x, t_S(x), i_S(x), f_S(x)) | x \in \mathcal{J}\},$$

*where $t_S, i_S, f_S : \mathcal{J} \to [0, 1]$ denote the truth, indeterminacy, and falsity degrees of S such that $0 \leq t_S(x) + i_S(x) + f_S(x) \leq 3$.*

*If $\mathcal{J}$ is continues, then*

$$S = \int_x \frac{(t_S(x), i_S(x), f_S(x))}{x}, \ \forall\, x \in \mathcal{J}.$$

*If $\mathcal{J}$ is discrete, then*

$$S = \sum_x \frac{(t_S(x), i_S(x), f_S(x))}{x}, \ \forall\, x \in \mathcal{J}.$$

**Definition 3.** *[13] A complex intuitionistic fuzzy set (CIFS) I on the universal set $\mathcal{J}$ is defined as,*

$$I = \{(u, t_I(u)e^{\iota\phi_I(u)}, f_I(u)e^{\iota\psi_I(u)}) | u \in \mathcal{J}\},$$

*where $\iota = \sqrt{-1}$, $t_I(u), f_I(u) \in [0, 1]$ are known as amplitude terms, $\phi_I(u), \psi_I(u) \in [0, 2\pi]$ are called phase terms, and for every $u \in \mathcal{J}$, $0 \leq t_I(u) + f_I(u) \leq 1$.*

Complex neutrosophic sets are defined using SVNSs.

**Definition 4.** *[14] A complex neutrosophic set (CNS) $\mathcal{N}$ on the universal set $\mathcal{J}$ is defined as,*

$$\mathcal{N} = \{(u, t_{\mathcal{N}}(u)e^{\iota\phi_{\mathcal{N}}(u)}, i_{\mathcal{N}}(u)e^{\iota\varphi_{\mathcal{N}}(u)}, f_{\mathcal{N}}(u)e^{\iota\psi_{\mathcal{N}}(u)}) | u \in \mathcal{J}\},$$

*where $\iota = \sqrt{-1}$, $t_{\mathcal{N}}(u), i_{\mathcal{N}}(u), f_{\mathcal{N}}(u) \in [0, 1]$ are known as amplitude terms, $\phi_{\mathcal{N}}(u), \varphi_{\mathcal{N}}(u), \psi_{\mathcal{N}}(u) \in [0, 2\pi]$ are called phase terms, and for every $u \in \mathcal{J}$, $0 \leq t_{\mathcal{N}}(u) + i_{\mathcal{N}}(u) + f_{\mathcal{N}}(u) \leq 3$.*

**Definition 5.** *[24] A complex neutrosophic relation (CNR) is a CNS on $\mathcal{J} \times \mathcal{J}$ given as,*

$$R = \{(rs, t_R(rs)e^{\iota\phi_R(rs)}, i_R(rs)e^{\iota\varphi_R(rs)}, f_R(rs)e^{\iota\psi_R(rs)}) | rs \in \mathcal{J} \times \mathcal{J}\},$$

*where $\iota = \sqrt{-1}$, $t_R : \mathcal{J} \times \mathcal{J} \to [0, 1]$, $i_R : \mathcal{J} \times \mathcal{J} \to [0, 1]$, $f_R : \mathcal{J} \times \mathcal{J} \to [0, 1]$ characterize the truth, indeterminacy, and falsity degrees of R, and $\phi_R(rs), \varphi_R(rs), \psi_R(rs) \in [0, 2\pi]$ such that for all $rs \in \mathcal{J} \times \mathcal{J}$, $0 \leq t_R(rs) + i_R(rs) + f_R(rs) \leq 3$.*

**Definition 6.** [24] *A complex neutrosophic graph (CNG) on $\mathcal{J}$ is an ordered pair $G = (A, B)$, where A is a CNS on $\mathcal{J}$ and B is CNR on $\mathcal{J}$ such that*

$$\begin{aligned} &t_B(ab) \leq \min\{t_A(a), t_A(b)\},\\ &i_B(ab) \leq \min\{i_A(a), i_A(b)\},\\ &f_B(ab) \leq \max\{f_A(a), f_A(b)\}, \text{ (for amplitude terms)}\\ &\phi_B(ab) \leq \min\{\phi_A(a), \phi_A(b)\},\\ &\varphi_B(ab) \leq \min\{\varphi_A(a), \varphi_A(b)\},\\ &\psi_B(ab) \leq \max\{\psi_A(a), \psi_A(b)\}, \text{ (for phase terms)} \end{aligned}$$

$0 \leq t_B(ab) + i_B(ab) + f_B(ab) \leq 3$, *for all* $a, b \in \mathcal{J}$.

**Example 1.** *Consider a CNG $G = (A, B)$ on $\mathcal{J} = \{c_1, c_2, c_3\}$, where $A = \{(c_1, 0.7e^{\iota(0.9)\pi}, 0.6e^{\iota(0.8)\pi}, 0.9e^{\iota(0.7)\pi}), (c_2, 0.5e^{\iota(0.5)\pi}, 0.7e^{\iota(0.9)\pi}, 0.9e^{\iota(0.7)\pi}), (c_3, 0.8e^{\iota(0.8)\pi}, 0.6e^{\iota(0.9)\pi}, 0.5e^{\iota(0.7)\pi})\}$ and $B = \{(c_1c_2, 0.5e^{\iota(0.5)\pi}, 0.6e^{\iota(0.8)\pi}, 0.6e^{\iota(0.6)\pi}), (c_2c_3, 0.5e^{\iota(0.5)\pi}, 0.6e^{\iota(0.8)\pi}, 0.4e^{\iota(0.6)\pi}), (c_1c_3, 0.7e^{\iota(0.8)\pi}, 0.5e^{\iota(0.8)\pi}, 0.4e^{\iota(0.6)\pi})\}$ are CNS and CNR on $\mathcal{J}$, respectively. The corresponding graph is shown in Figure 1.*

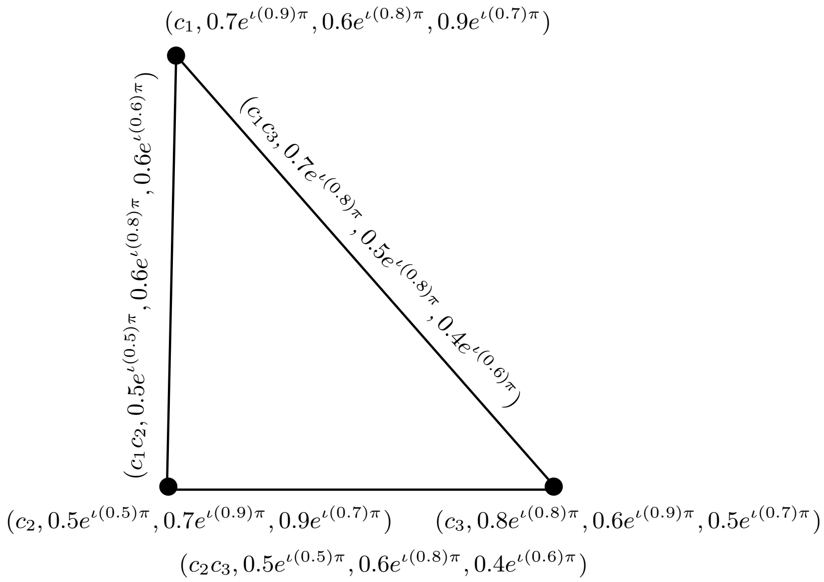


**Figure 1.** Complex neutrosophic graph.

**Definition 7.** [14] *Let $N_1 = \{(u, t_{N_1}(u)e^{\iota\phi_{N_1}(u)}, i_{N_1}(u)e^{\iota\varphi_{N_1}(u)}, f_{N_1}(u)e^{\iota\psi_{N_1}(u)}) | u \in \mathcal{J}\}$ and $N_2 = \{(u, t_{N_2}(u)e^{\iota\phi_{N_2}(u)}, i_{N_2}(u)e^{\iota\varphi_{N_2}(u)}, f_{N_2}(u)e^{\iota\psi_{N_2}(u)}) | u \in \mathcal{J}\}$ be two CNSs in $\mathcal{J}$, then*

- **(i)** $N_1 \subseteq N_2 \Leftrightarrow t_{N_1}(u) \leq t_{N_2}(u)$, $i_{N_1}(u) \leq i_{N_2}(u)$, $f_{N_1}(u) \geq f_{N_2}(u)$, *and* $\phi_{N_1}(u) \leq \phi_{N_2}(u)$, $\varphi_{N_1}(u) \leq \varphi_{N_2}(u)$, $\psi_{N_1}(u) \geq \psi_{N_2}(u)$ *for amplitudes and phase terms, respectively, for all* $u \in \mathcal{J}$.
- **(ii)** $N_1 = N_2 \Leftrightarrow t_{N_1}(u) = t_{N_2}(u)$, $i_{N_1}(u) = i_{N_2}(u)$, $f_{N_1}(u) = f_{N_2}(u)$, *and* $\phi_{N_1}(u) = \phi_{N_2}(u)$, $\varphi_{N_1}(u) = \varphi_{N_2}(u)$, $\psi_{N_1}(u) = \psi_{N_2}(u)$ *for amplitudes and phase terms, respectively, for all* $u \in \mathcal{J}$.
- **(iii)** $N_1 \cup N_2 = \{(u, \max\{t_{N_1}(u), t_{N_2}(u)\}e^{\iota \max\{\phi_{N_1}(u), \phi_{N_2}(u)\}}, \min\{i_{N_1}(u), i_{N_2}(u)\}e^{\iota \min\{\varphi_{N_1}(u), \varphi_{N_2}(u)\}}, \min\{f_{N_1}(u), f_{N_2}(u)\}e^{\iota \min\{\psi_{N_1}(u), \psi_{N_2}(u)\}}) | u \in N_1 \cup N_2\}$.
- **(iv)** $N_1 \cap N_2 = \{(u, \min\{t_{N_1}(u), t_{N_2}(u)\}e^{\iota \min\{\phi_{N_1}(u), \phi_{N_2}(u)\}}, \max\{i_{N_1}(u), i_{N_2}(u)\}e^{\iota \max\{\varphi_{N_1}(u), \varphi_{N_2}(u)\}}, \max\{f_{N_1}(u), f_{N_2}(u)\}e^{\iota \max\{\psi_{N_1}(u), \psi_{N_2}(u)\}}) | u \in N_1 \cap N_2\}$.

**Definition 8.** *The support of a CNS* $N = \{(u, t_N(u)e^{\iota\phi_N(u)}, i_N(u)e^{\iota\varphi_N(u)} f_N(u)e^{\iota\psi_S(u)}) | u \in \mathcal{J}\}$ *is defined as*

$$supp(N) = \{u | t_N(u) \neq 0, i_N(u) \neq 0, f_N(u) \neq 1, 0 < \phi_N(u), \varphi_N(u), \psi_N(u) < 2\pi\}.$$

*The height of a CNS* $N = \{(u, t_N(u)e^{\iota\phi_N(u)}, i_N(u)e^{\iota\varphi_N(u)} f_N(u)e^{\iota\psi_S(u)}) | u \in \mathcal{J}\}$ *is defined as*

$$h(N) = \{\max_{u\in\mathcal{J}} t_N(u)e^{\iota \max_{u\in\mathcal{J}} \phi_N(u)}, \max_{u\in\mathcal{J}} i_N(u)e^{\iota \max_{u\in\mathcal{J}} \varphi_N(u)}, \min_{u\in\mathcal{J}} f_N(u)e^{\iota \min_{u\in\mathcal{J}} \psi_N(u)}\}.$$

**Definition 9.** *A complex neutrosophic hypergraph (CNHG) on* $\mathcal{J}$ *is defined as an ordered pair* $\mathcal{H} = (\mathcal{N}, \lambda)$, *where* $\mathcal{N} = \{N_1, N_2, \cdots, N_k\}$ *is a finite family of CNSs on* $\mathcal{J}$ *and* $\lambda$ *is a CNR on CNSs* $N_j$*'s such that*

**(i)**

$$\begin{aligned}
t_\lambda(\{r_1, r_2, \cdots, r_l\}) &\leq \min\{t_{N_j}(r_1), t_{N_j}(r_2), \cdots, t_{N_j}(r_l)\}, \\
i_\lambda(\{r_1, r_2, \cdots, r_l\}) &\leq \min\{i_{N_j}(r_1), i_{N_j}(r_2), \cdots, i_{N_j}(r_l)\}, \\
f_\lambda(\{r_1, r_2, \cdots, r_l\}) &\leq \max\{f_{N_j}(r_1), f_{N_j}(r_2), \cdots, f_{N_j}(r_l)\}, \text{ (for amplitude terms)} \\
\phi_\lambda(\{r_1, r_2, \cdots, r_l\}) &\leq \min\{\phi_{N_j}(r_1), \phi_{N_j}(r_2), \cdots, \phi_{N_j}(r_l)\}, \\
\varphi_\lambda(\{r_1, r_2, \cdots, r_l\}) &\leq \min\{\varphi_{N_j}(r_1), \varphi_{N_j}(r_2), \cdots, \varphi_{N_j}(r_l)\}, \\
\psi_\lambda(\{r_1, r_2, \cdots, r_l\}) &\leq \max\{\psi_{N_j}(r_1), \psi_{N_j}(r_2), \cdots, \psi_{N_j}(r_l)\}, \text{ (for phase terms)}
\end{aligned}$$

$0 \leq t_\lambda + i_\lambda + f_\lambda \leq 3$, *for all* $r_1, r_2, \cdots, r_l \in \mathcal{J}$.

**(ii)** $\bigcup_j supp(N_j) = \mathcal{J}$, *for all* $N_j \in \mathcal{N}$.

*Please note that* $E_k = \{r_1, r_2, \cdots, r_l\}$ *is the crisp hyperedge of* $\mathcal{H} = (\mathcal{N}, \lambda)$.

**Definition 10.** *Let* $\mathcal{H} = (\mathcal{N}, \lambda)$ *be a CNHG. The height of* $\mathcal{H}$, *denoted by* $h(\mathcal{H})$, *is defined as* $h(\mathcal{H}) = (\max \lambda_l e^{\iota \max \phi}, \max \lambda_m e^{\iota \max \varphi}, \min \lambda_n e^{\iota \min \psi})$, *where* $\lambda_l = \max t_{\xi_j}(v_k)$, $\phi = \max \phi_{\xi_j}(v_k)$, $\lambda_m = \max i_{\xi_j}(v_k)$, $\varphi = \max \varphi_{\xi_j}(v_k)$, $\lambda_n = \min f_{\xi_j}(v_k)$, $\psi = \min \psi_{\xi_j}(v_k)$. *Here,* $t_{\xi_j}(v_k)$, $i_{\xi_j}(v_k)$, $f_{\xi_j}(v_k)$ *denote the truth, indeterminacy, and falsity degrees of vertex* $v_k$ *to hyperedge* $\xi_j$, *respectively.*

**Definition 11.** *Let* $\mathcal{H} = (\mathcal{N}, \lambda)$ *be a CNHG. Suppose that* $\alpha, \beta, \gamma \in [0,1]$ *and* $\Theta, \Phi, \Psi \in [0, 2\pi]$ *such that* $0 \leq \alpha + \beta + \gamma \leq 3$. *The* $(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})$*-level hypergraph of* $\mathcal{H}$ *is defined as an ordered pair* $\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} = (\mathcal{N}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}, \lambda^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})})$, *where*

**(i)** $\lambda^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} = \{\lambda_j^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} : \lambda_j \in \lambda\}$ *and* $\lambda_j^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} = \{u \in \mathcal{J} : t_{\lambda_j}(u) \geq \alpha, \phi_{\lambda_j}(u) \geq \Theta,\ i_{\lambda_j}(u) \geq \beta, \varphi_{\lambda_j}(u) \geq \Phi, \text{ and } f_{\lambda_j}(u) \leq \gamma, \psi_{\lambda_j}(u) \leq \Psi\}$,

**(ii)** $\mathcal{N}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} = \bigcup_{\lambda_j \in \lambda} \lambda_j^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}$.

*Please note that* $(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})$*-level hypergraph of* $\mathcal{H}$ *is a crisp hypergraph.*

**Definition 12.** *Let* $\mathcal{H} = (\mathcal{N}, \lambda)$ *be a CNHG and for* $0 < \alpha \leq t(h(\mathcal{H}))$, $0 < \beta \leq i(h(\mathcal{H}))$, $\gamma \geq f(h(\mathcal{H})) > 0$, $0 < \Theta \leq \phi(h(\mathcal{H}))$, $0 < \Phi \leq \varphi(h(\mathcal{H}))$, *and* $\Psi \geq \psi(h(\mathcal{H})) > 0$, *let* $\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} = (\mathcal{N}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}, \lambda^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})})$ *be the level hypergraph of* $\mathcal{H}$. *The sequence of complex numbers* $\{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1}), (\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2}), \cdots, (\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})\}$ *such that* $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n = t(h(\mathcal{H}))$, $0 < \beta_1 < \beta_2 < \cdots < \beta_n = i(h(\mathcal{H}))$, $\gamma_1 > \gamma_2 > \cdots > \gamma_n = f(h(\mathcal{H})) > 0$, $0 < \Theta_1 < \Theta_2 < \cdots < \Theta_n = \phi(h(\mathcal{H}))$, $0 < \Phi_1 < \Phi_2 < \cdots < \Phi_n = \varphi(h(\mathcal{H}))$, *and* $\Psi_1 > \Psi_2 > \cdots > \Psi_n = \psi(h(\mathcal{H})) > 0$ *satisfying the conditions,*

**(i)** *if* $\alpha_{k+1} < \alpha' \leq \alpha_k$, $\beta_{k+1} < \beta' \leq \beta_k$, $\gamma_{k+1} > \gamma' \geq \gamma_k$, $\Theta_{k+1} < \phi \leq \Theta_k$, $\Phi_{k+1} < \varphi \leq \Phi_k$, $\Psi_{k+1} > \psi \geq \Psi_k$, *then* $\lambda^{(\alpha' e^{\iota\phi}, \beta' e^{\iota\varphi}, \gamma' e^{\iota\psi})} = \lambda^{(\alpha_k e^{\iota\Theta_k}, \beta_k e^{\iota\Phi_k}, \gamma_k e^{\iota\Psi_k})}$, *and*

**(ii)** $\lambda^{(\alpha_k e^{\iota\Theta_k},\beta_k e^{\iota\Phi_k},\gamma_k e^{\iota\Psi_k})} \subset \lambda^{(\alpha_{k+1} e^{\iota\Theta_{k+1}},\beta_{k+1} e^{\iota\Phi_{k+1}},\gamma_{k+1} e^{\iota\Psi_{k+1}})}$,

*is called the fundamental sequence of* $\mathcal{H} = (\mathcal{N}, \lambda)$, *denoted by* $\mathcal{F}_s(\mathcal{H})$. *The set of* $(\alpha_j e^{\iota\Theta_j}, \beta_j e^{\iota\Phi_j}, \gamma_j e^{\iota\Psi_j})$ *-level hypergraphs* $\{\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1},\beta_1 e^{\iota\Phi_1},\gamma_1 e^{\iota\Psi_1})}, \mathcal{H}^{(\alpha_2 e^{\iota\Theta_2},\beta_2 e^{\iota\Phi_2},\gamma_2 e^{\iota\Psi_2})}, \cdots, \mathcal{H}^{(\alpha_n e^{\iota\Theta_n},\beta_n e^{\iota\Phi_n},\gamma_n e^{\iota\Psi_n})}\}$ *is called the set of core hypergraphs or the core set of* $\mathcal{H}$, *denoted by* $c(\mathcal{H})$.

**Example 2.** *Consider a CNHG* $\mathcal{H} = (\mathcal{N}, \lambda)$ *on* $\mathcal{J} = \{r_1, r_2, r_3, r_4, r_5, r_6\}$. *The CNR* $\lambda$ *is given as,* $\lambda(\{r_1, r_2, r_3\}) = (0.6e^{\iota(0.6)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.3e^{\iota(0.3)2\pi})$, $\lambda(\{r_1, r_4\}) = (0.8e^{\iota(0.8)2\pi}, 0.5e^{\iota(0.5)2\pi}, 0.4e^{\iota(0.4)2\pi})$, $\lambda(\{r_3, r_4, r_5\}) = (0.3e^{\iota(0.3)2\pi}, 0.2e^{\iota(0.2)2\pi}, 0.1e^{\iota(0.1)2\pi})$, *and* $\lambda(\{r_1, r_5, r_6\}) = (0.3e^{\iota(0.3)2\pi}, 0.2e^{\iota(0.2)2\pi}, 0.1e^{\iota(0.1)2\pi})$. *The corresponding CNHG is shown in Figure 2.*

*Let*

$$
\begin{aligned}
(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1}) &= (0.9e^{\iota(0.9)2\pi}, 0.7e^{\iota(0.7)2\pi}, 0.6e^{\iota(0.6)2\pi}),\\
(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2}) &= (0.8e^{\iota(0.8)2\pi}, 0.5e^{\iota(0.5)2\pi}, 0.4e^{\iota(0.4)2\pi}),\\
(\alpha_3 e^{\iota\Theta_3}, \beta_3 e^{\iota\Phi_3}, \gamma_3 e^{\iota\Psi_3}) &= (0.6e^{\iota(0.6)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.3e^{\iota(0.3)2\pi}),\\
(\alpha_4 e^{\iota\Theta_4}, \beta_4 e^{\iota\Phi_4}, \gamma_4 e^{\iota\Psi_4}) &= (0.3e^{\iota(0.3)2\pi}, 0.2e^{\iota(0.2)2\pi}, 0.1e^{\iota(0.1)2\pi}).
\end{aligned}
$$

*Please note that the sequence* $\{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1}), (\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2}), (\alpha_3 e^{\iota\Theta_3}, \beta_3 e^{\iota\Phi_3}, \gamma_3 e^{\iota\Psi_3}), (\alpha_4 e^{\iota\Theta_4}, \beta_4 e^{\iota\Phi_4}, \gamma_4 e^{\iota\Psi_4})\}$ *satisfies all the conditions of Definition 12. Thus, it is a fundamental sequence of* $\mathcal{H}$. *The corresponding* $(\alpha_j e^{\iota\Theta_j}, \beta_j e^{\iota\Phi_j}, \gamma_j e^{\iota\Psi_j})$*-level hypergraphs are shown in Figures 3–5.*

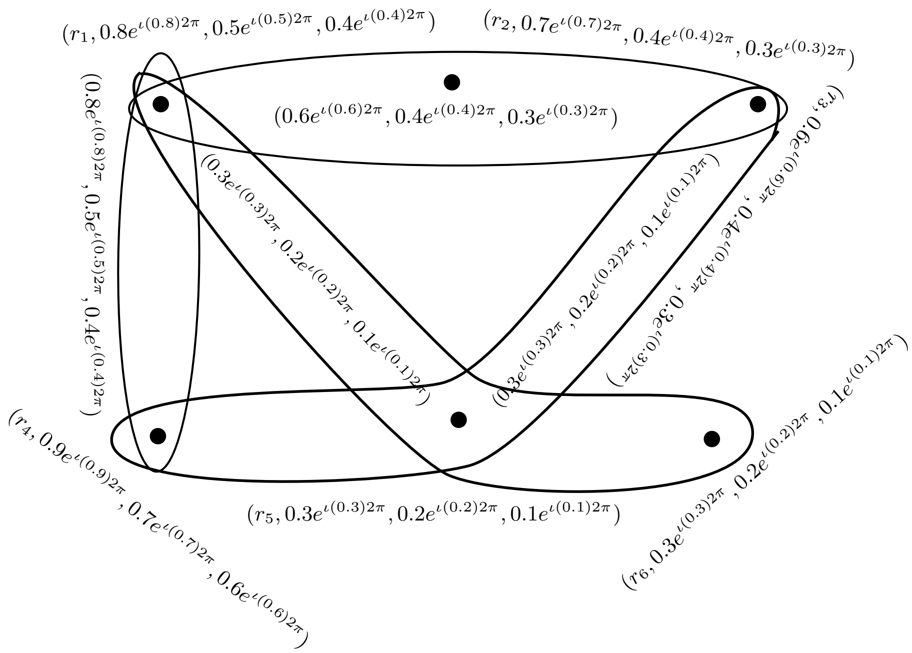


**Figure 2.** Complex neutrosophic hypergraph $\mathcal{H}$.

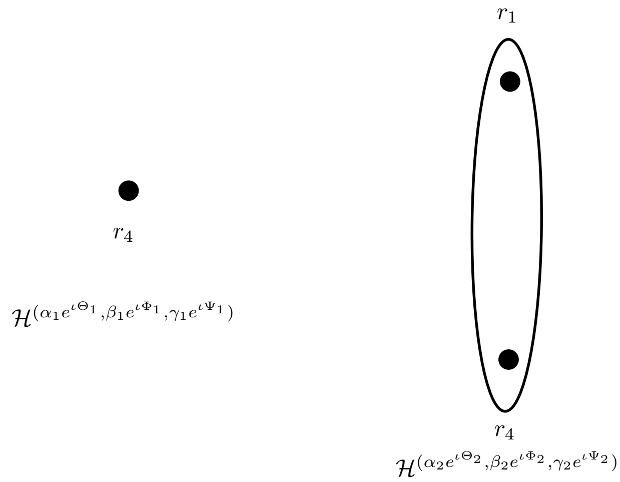


**Figure 3.** $\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})}$, $\mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})}$-level hypergraphs.

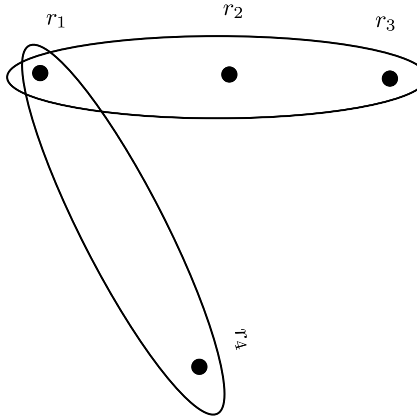


**Figure 4.** $\mathcal{H}^{(\alpha_3 e^{\iota\Theta_3}, \beta_3 e^{\iota\Phi_3}, \gamma_3 e^{\iota\Psi_3})}$-level hypergraph.

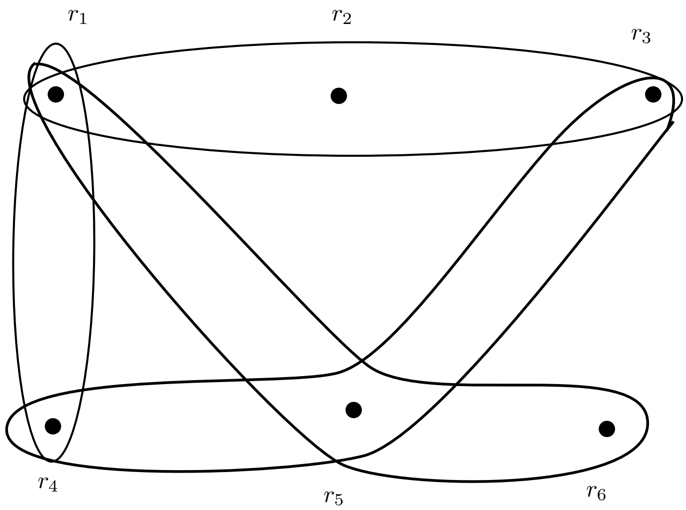


**Figure 5.** $\mathcal{H}^{(\alpha_4 e^{\iota\Theta_4}, \beta_4 e^{\iota\Phi_4}, \gamma_4 e^{\iota\Psi_4})}$-level hypergraph.

**Definition 13.** *A CNHG* $\mathcal{H} = (\mathcal{N}, \lambda)$ *is ordered if* $c(\mathcal{H})$ *is ordered, i.e., if* $c(\mathcal{H}) = \{\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})}, \mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})}, \cdots, \mathcal{H}^{(\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})}\}$*, then* $\{\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})} \subset \mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})} \subset \cdots \subset \mathcal{H}^{(\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})}\}$*.*

*A CNHG* $\mathcal{H} = (\mathcal{N}, \lambda)$ *is simply ordered if* $c(\mathcal{H})$ *is simply ordered, i.e., if* $e \in E_{j+1} \setminus E_j$*, then* $e \nsubseteq \mathcal{J}_j$*.*

**Example 3.** *Consider a CNHG $\mathcal{H} = (\mathcal{N}, \lambda)$ as shown in Figure 2. The set of core hypergraphs is given as,*

$$c(\mathcal{H}) = \{\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})}, \mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})}, \mathcal{H}^{(\alpha_3 e^{\iota\Theta_3}, \beta_3 e^{\iota\Phi_3}, \gamma_3 e^{\iota\Psi_3})}, \mathcal{H}^{(\alpha_4 e^{\iota\Theta_4}, \beta_4 e^{\iota\Phi_4}, \gamma_n e^{\iota\Psi_4})}\},$$

*where*

$$\begin{aligned}
\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})} &= (\mathcal{J}_1, E_1),\ \mathcal{J}_1 = \{r_4\},\ E_1 = \{\}, \\
\mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})} &= (\mathcal{J}_2, E_2),\ \mathcal{J}_2 = \{r_1, r_4\},\ E_2 = \{\{r_1, r_4\}\}, \\
\mathcal{H}^{(\alpha_3 e^{\iota\Theta_3}, \beta_3 e^{\iota\Phi_3}, \gamma_3 e^{\iota\Psi_3})} &= (\mathcal{J}_3, E_3),\ \mathcal{J}_3 = \{r_1, r_2, r_3, r_4\},\ E_3 = \{\{r_1, r_4\}, \{r_1, r_2, r_3\}\}, \\
\mathcal{H}^{(\alpha_4 e^{\iota\Theta_4}, \beta_4 e^{\iota\Phi_4}, \gamma_4 e^{\iota\Psi_4})} &= (\mathcal{J}_4, E_4),\ \mathcal{J}_4 = \{r_1, r_2, r_3, r_4, r_5, r_6\},\ E_4 = \{\{r_1, r_4\}, \{r_1, r_2, r_3\}, \{r_1, r_5, r_6\} \\
&\qquad , \{r_3, r_4, r_5\}\}.
\end{aligned}$$

*Please note that*

$$\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})} \subseteq \mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})} \subseteq \mathcal{H}^{(\alpha_3 e^{\iota\Theta_3}, \beta_3 e^{\iota\Phi_3}, \gamma_3 e^{\iota\Psi_3})} \subseteq \mathcal{H}^{(\alpha_4 e^{\iota\Theta_4}, \beta_4 e^{\iota\Phi_4}, \gamma_4 e^{\iota\Psi_4})}.$$

*Hence, $\mathcal{H} = (\mathcal{N}, \lambda)$ is an ordered CNHG. Also, $\mathcal{H} = (\mathcal{N}, \lambda)$ is simply ordered.*

**Definition 14.** *A CNHG $\mathcal{H} = (\mathcal{N}, \lambda)$ with $\mathcal{F}_s(\mathcal{H}) = \{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1}), (\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2}), \cdots, (\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})\}$ is called sectionally elementary if for every $\lambda_j \in \lambda$ and for $k \in \{1, 2, \cdots, n\}$, $\lambda_j^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} = \lambda_j^{(\alpha_k e^{\iota\Theta_k}, \beta_k e^{\iota\Phi_k}, \gamma_k e^{\iota\Psi_k})}$, for all $\alpha \in (\alpha_{k+1}, \alpha_k]$, $\beta \in (\beta_{k+1}, \beta_k]$, $\gamma \in (\gamma_{k+1}, \gamma_k]$, $\Theta \in (\Theta_{k+1}, \Theta_k]$, $\Phi \in (\Phi_{k+1}, \Phi_k]$, and $\Psi \in (\Psi_{k+1}, \Psi_k]$.*

**Definition 15.** *Let $N$ be a CNS on $\mathcal{J}$. The lower truncation of $N$ at level $(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})$, $0 < \alpha, \beta, \gamma \leq 1$, $0 < \Theta, \Phi, \Psi \leq 2\pi$, is the CNSS $N_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}$ defined by,*

$$t_{N_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x) e^{i\phi_{N_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x)} = \begin{cases} t_N(x) e^{i\phi_N(x)}, & \text{if} \quad x \in N^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}, \\ 0, & \text{otherwise.} \end{cases}$$

$$i_{N_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x) e^{i\varphi_{N_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x)} = \begin{cases} i_N(x) e^{i\varphi_N(x)}, & \text{if} \quad x \in N^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_{N_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x) e^{i\psi_{N_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x)} = \begin{cases} f_N(x) e^{i\psi_N(x)}, & \text{if} \quad x \in N^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}, \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 16.** *Let $N$ be a CNS on $\mathcal{J}$. The upper truncation of $N$ at level $(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})$, $0 < \alpha, \beta, \gamma \leq 1$, $0 < \Theta, \Phi, \Psi \leq 2\pi$, is the CNSS $N^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}$ defined by,*

$$t_{N^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x) e^{i\phi_{N^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x)} = \begin{cases} \alpha e^{\iota\Theta}, & \text{if} \quad x \in N^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}, \\ t_N(x) e^{i\phi_N(x)}, & \text{otherwise.} \end{cases}$$

$$i_{N^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x) e^{i\varphi_{N^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x)} = \begin{cases} \beta e^{\iota\Phi}, & \text{if} \quad x \in N^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}, \\ i_N(x) e^{i\varphi_N(x)}, & \text{otherwise.} \end{cases}$$

$$f_{N^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x) e^{i\psi_{N^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}}(x)} = \begin{cases} \gamma e^{\iota\Psi}, & \text{if} \quad x \in N^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}, \\ f_N(x) e^{i\psi_N(x)}, & \text{otherwise.} \end{cases}$$

**Definition 17.** *Let* $\mathcal{H} = (\mathcal{N}, \lambda)$ *be a CNHG. The lower truncation* $\mathcal{H}_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}$ *of* $\mathcal{H}$ *at level* $(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})$ *is defined as,* $\mathcal{H}_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]} = (\mathcal{N}_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}, \lambda_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}))$*, where* $\mathcal{N}_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]} = \{N_{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]} | N \in \mathcal{N}\}$.

*The upper truncation* $\mathcal{H}^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}$ *of* $\mathcal{H}$ *at level* $(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})$ *is defined as,* $\mathcal{H}^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]} = (\mathcal{N}^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}, \lambda^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]}))$*, where* $\mathcal{N}^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]} = \{N^{[(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})]} | N \in \mathcal{N}\}$.

**Definition 18.** *Let* $N$ *be a CNS on* $\mathcal{J}$*. Then, each* $(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})$*, such that* $\alpha \in (0, t(h(N)))$*,* $\beta \in (0, i(h(N)))$*,* $\gamma \in (0, f(h(N)))$*,* $\Theta \in (0, \phi(h(N)))$*,* $\Psi \in (0, \varphi(h(N)))$*, and* $\Psi \in (0, \psi(h(N)))$*, for which* $N^{(\alpha e^{\iota\theta}, \beta e^{\iota\phi}, \gamma e^{\iota\psi})} \subset N^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}$*, is called a transition level of* $N$.

**Example 4.** *Consider a CNHG* $\mathcal{H} = (\mathcal{N}, \lambda)$ *as shown in Figure 2. The* $(0.6e^{\iota(0.6)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.3e^{\iota(0.3)2\pi})$*-level hypergraph of* $\mathcal{H}$ *is shown in Figure 4. Then, the lower truncation* $\mathcal{H}_{[(0.6e^{\iota(0.6)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.3e^{\iota(0.3)2\pi})]} = (\mathcal{N}_{[(0.6e^{\iota(0.6)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.3e^{\iota(0.3)2\pi})]}, \lambda_{[(0.6e^{\iota(0.6)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.3e^{\iota(0.3)2\pi})]})$ *of* $\mathcal{H}$ *is a CNHG on* $\mathcal{J}_1 = \{r_1, r_2, r_3, r_4\}$ *as given in Figure 6.*

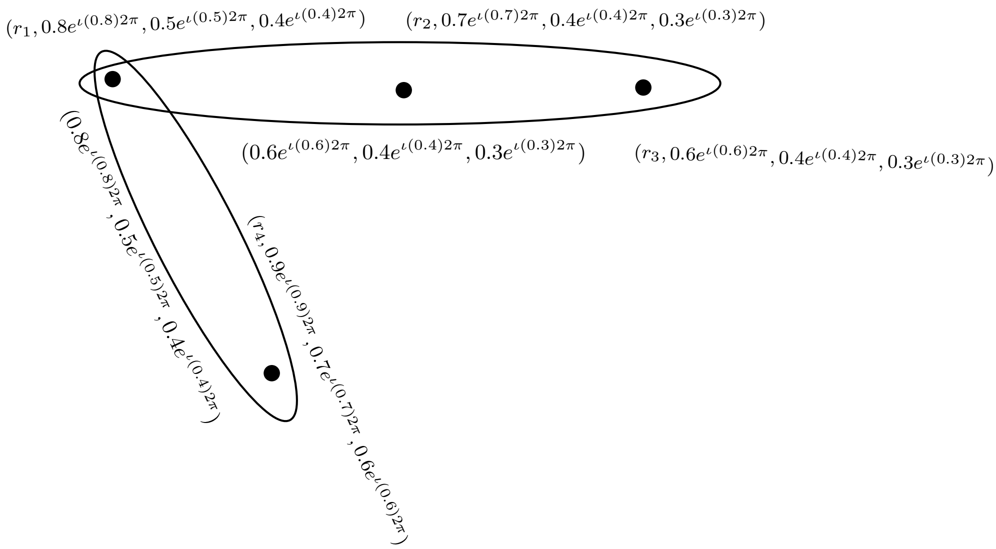


**Figure 6.** Lower truncation of $\mathcal{H}$.

*Not that* $\mathcal{J}_1 = \bigcup_{N \in \mathcal{N}} N_{[(0.6e^{\iota(0.6)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.3e^{\iota(0.3)2\pi})]}$. *The upper truncation* $\mathcal{H}^{[(0.6e^{\iota(0.6)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.3e^{\iota(0.3)2\pi})]} = (\mathcal{N}^{[(0.6e^{\iota(0.6)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.3e^{\iota(0.3)2\pi})]}, \lambda^{[(0.6e^{\iota(0.6)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.3e^{\iota(0.3)2\pi})]})$ *of* $\mathcal{H}$ *is a CNHG on* $\mathcal{J} = \{r_1, r_2, r_3, r_4, r_5, r_6\}$ *as given in Figure 7.*

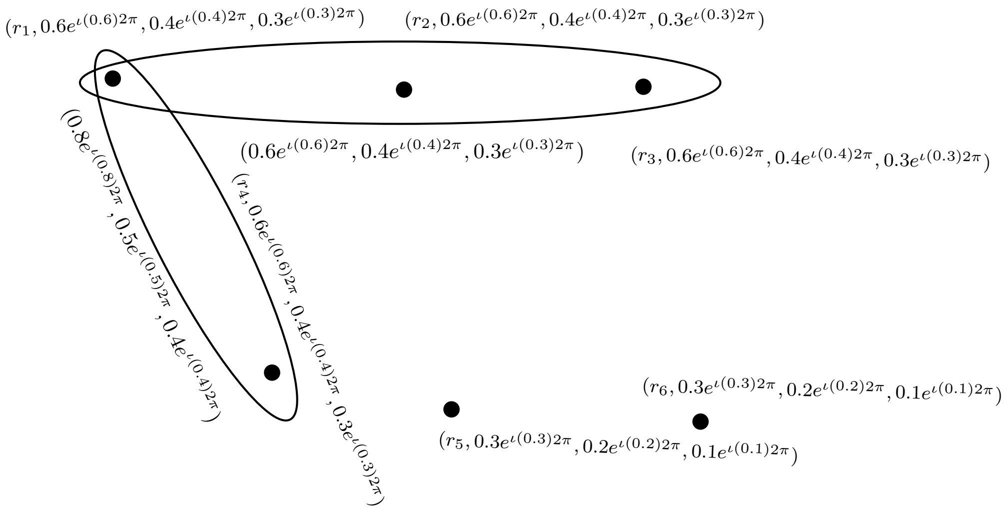


**Figure 7.** Upper truncation of $\mathcal{H}$.

**Definition 19.** *Let $\mathcal{H} = (\mathcal{N}, \lambda)$ be a CNHG. A complex neutrosophic transversal (CNT) $\tau$ is a CNS of $\mathcal{J}$ satisfying the condition $\xi^{h(\xi)} \cap \tau^{h(\xi)} \neq \emptyset$, for all $\xi \in \lambda$, where $h(\xi)$ is the height of $\xi$.*

*A minimal complex neutrosophic transversal $\tau_1$ is the CNT of $\mathcal{H}$ with the property that if $\tau \subset \tau_1$, then $\tau$ is not a CNT of $\mathcal{H}$.*

Let us denote the family of minimal CNTs of $\mathcal{H}$ by $T_r(\mathcal{H})$.

**Definition 20.** *A CNT $\tau$ with the property that $\tau^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} \in t_r(\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})})$, for all $\alpha, \beta, \gamma \in [0, 1]$, and $\Theta, \Phi, \Psi \in [0, 2\pi]$ is called the locally minimal CNT of $\mathcal{H}$. The collection of all locally minimal CNTs of $\mathcal{H}$ is represented by $T_r^*(\mathcal{H})$.*

*Please note that $T_r^*(\mathcal{H}) \subseteq T_r(\mathcal{H})$, but the converse is not generally true.*

**Definition 21.** *Let $N$ be a CNS on $\mathcal{J}$. Then, the basic sequence of $N$ determined by $N$, denoted by $B_s(N)$, is defined as $\{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})^N, (\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})^N, \cdots, (\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})^N\}$, where*

- **(i)** $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, $\beta_1 > \beta_2 > \cdots > \beta_n$, $\gamma_1 < \gamma_2 < \cdots < \gamma_n$, $\Theta_1 > \Theta_2 > \cdots > \Theta_n$, $\Phi_1 > \Phi_2 > \cdots > \Phi_n$, $\Psi_1 < \Psi_2 < \cdots < \Psi_n$,
- **(ii)** $(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1}) = h(N)$,
- **(iii)** $\{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})^N, \cdots, (\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})^N\}$ *are the transition levels of $N$.*

**Definition 22.** *Let $B_s(N) = \{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})^N, (\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})^N, \cdots, (\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})^N\}$ be the basic sequence of $N$. Then, the set of basic cuts $B_c(N)$ is defined as, $B_c(N) = \{N^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} | (\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi}) \in B_s(N)\}$.*

**Lemma 1.** *Let $\mathcal{H} = (\mathcal{N}, \lambda)$ be a CNHG with $\mathcal{F}_s(\mathcal{H}) = \{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1}), (\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2}), \cdots, (\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})\}$. Then,*

- **(i)** *If $(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})$ is a transition level of $\tau \in T_r(\mathcal{H})$, then there exists an $\epsilon > 0$ such that for all $\alpha_1 \in (\alpha, \alpha + \epsilon]$, $\beta_1 \in (\beta, \beta + \epsilon]$, $\gamma_1 \in (\gamma, \gamma + \epsilon]$, $\Theta_1 \in (\Theta, \Theta + \epsilon]$, $\Phi_1 \in (\Phi, \Phi + \epsilon]$, $\Psi_1 \in (\Psi, \Psi + \epsilon]$, $\tau^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}$ is a minimal $\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}$ transversal extension of $\tau^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})}$, i.e., if $\tau^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})} \subseteq C \subset \tau^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}$, then $C$ is not a transversal of $\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}$.*
- **(ii)** *$T_r(\mathcal{H})$, i.e., the collection of minimal transversals of $\mathcal{H}$ is sectionally elementary.*
- **(iii)** *$\mathcal{F}_s(T_r(\mathcal{H}))$ is properly contained in $\mathcal{F}_s(\mathcal{H})$.*
- **(iv)** *$\tau^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} \in T_r(\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})})$, for all $\tau \in T_r(\mathcal{H})$ and for every $\alpha_2 < \alpha \leq \alpha_1$, $\beta_2 < \beta \leq \beta_1$, $\gamma_2 > \gamma \geq \gamma_1$, $\Theta_2 < \Theta \leq \Theta_1$, $\Phi_2 < \Phi \leq \Phi_1$, $\Psi_2 > \Psi \geq \Psi_1$.*

**Definition 23.** *Let $\mathcal{H} = (\mathcal{N}, \lambda)$ be a CNHG. The complex neutrosophic line graph of $\mathcal{H}$ is defined as an ordered pair $l(\mathcal{H}) = (\mathcal{N}_l, \lambda_l)$, where $\mathcal{N}_l = \lambda$ and there exists an edge between two vertices in $l(\mathcal{H})$ if $|supp(\lambda_j) \cap supp(\lambda_k)| \geq 1$, for all $\lambda_j, \lambda_k \in \lambda$. The membership degrees of $l(\mathcal{H})$ are given as,*

**(i)** $\mathcal{N}_l(E_k) = \lambda(E_k)$,

**(ii)** $\lambda_l(E_jE_k) = (\min\{t_\lambda(E_j), t_\lambda(E_k)\}e^{\iota \min\{\phi_\lambda(E_j), \phi_\lambda(E_k)\}}, \min\{i_\lambda(E_j), i_\lambda(E_k)\}e^{\iota \min\{\varphi_\lambda(E_j), \varphi_\lambda(E_k)\}},$ $\max\{f_\lambda(E_j), f_\lambda(E_k)\}e^{\iota \max\{\psi_\lambda(E_j), \psi_\lambda(E_k)\}})$.

## 3. *T*-Related Complex Neutrosophic Hypergraphs

**Definition 24.** *A CNHG $\mathcal{H} = (\mathcal{N}, \lambda)$ is N-tempered CNHG of $H = (\mathcal{J}, E)$ if there exists $H = (\mathcal{J}, E)$, a crisp hypergraph, and a CNS N such that $\lambda = \{\delta_e | e \in E\}$, where*

$$t_\delta(u)e^{\iota\phi_\delta(u)} = \begin{cases} \min\{t_N(x)e^{\iota \min\{\phi_N(x)\}} | x \in e\}, & \text{if} \quad u \in e, \\ 0, & \text{otherwise.} \end{cases}$$

$$i_\delta(u)e^{\iota\varphi_\delta(u)} = \begin{cases} \min\{i_N(x)e^{\iota \min\{\varphi_N(x)\}} | x \in e\}, & \text{if} \quad u \in e, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_\delta(u)e^{\iota\psi_\delta(u)} = \begin{cases} \max\{f_N(x)e^{\iota \max\{\psi_N(x)\}} | x \in e\}, & \text{if} \quad u \in e, \\ 0, & \text{otherwise} \end{cases}$$

*An N-tempered CNHG $\mathcal{H} = (\mathcal{N}, \lambda)$ determined by H and CNS N is denoted by $N \otimes H$.*

**Definition 25.** *A pair $(G, J)$ of crisp hypergraphs is T-related if whenever g is a minimal transversal of G, k is any transversal of J, and $g \subseteq k$, then there exists a minimal transversal t of J such that $g \subseteq t \subseteq k$.*

**Definition 26.** *Let $\mathcal{H} = (\mathcal{N}, \lambda)$ be a CNHG with $\mathcal{F}_s(\mathcal{H}) = \{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1}), (\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2}), \cdots, (\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})\}$. Then, $\mathcal{H}$ is T-related if from the core set*

$$c(\mathcal{H}) = \{\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})}, \mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})}, \cdots, \mathcal{H}^{(\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})}\}$$

*of $\mathcal{H}$, every successive ordered pair $(\mathcal{H}^{(\alpha_j e^{\iota\Theta_j}, \beta_j e^{\iota\Phi_j}, \gamma_j e^{\iota\Psi_j})}, \mathcal{H}^{(\alpha_{j-1} e^{\iota\Theta_{j-1}}, \beta_{j-1} e^{\iota\Phi_{j-1}}, \gamma_{j-1} e^{\iota\Psi_{j-1}})})$ is T-related.*

*If $\mathcal{F}_s(\mathcal{H})$ contains only one element, $\mathcal{H}$ is considered to be trivially T-related.*

**Theorem 1.** *Let $\mathcal{H} = (\mathcal{N}, \lambda)$ be a T-related CNHG, then $T_r(\mathcal{H}) = T_r^*(\mathcal{H})$.*

**Proof.** Let $\mathcal{H} = (\mathcal{N}, \lambda)$ be a $T$-related CNHG with $\mathcal{F}_s(\mathcal{H}) = \{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1}), (\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2}), \cdots, (\alpha_1 e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})\}$. Then, there arises two cases:

**Case (i)** First we consider that $\mathcal{F}_s(\mathcal{H}) = \{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})\}$. Then, Lemma 1 implies that for each $\xi \in T_r(\mathcal{H})$, $\xi^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} \in T_r(\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})})$, for all $0 < \alpha \leq t(h(\mathcal{H}))$, $0 < \beta \leq i(h(\mathcal{H}))$, $\gamma \geq f(h(\mathcal{H})) > 0$, $0 < \Theta \leq \phi(h(\mathcal{H}))$, $0 < \Phi \leq \varphi(h(\mathcal{H}))$, and $\Psi \geq \psi(h(\mathcal{H})) > 0$. Thus, $T_r(\mathcal{H}) = T_r^*(\mathcal{H})$.

**Case (ii)** We now suppose that $|\mathcal{F}_s(\mathcal{H})| \geq 2$. Since, $T_r^*(\mathcal{H}) \subseteq T_r(\mathcal{H})$, we just have to prove that $T_r(\mathcal{H}) \subseteq T_r^*(\mathcal{H})$. Let $\xi \in T_r(\mathcal{H})$, and $\xi^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})} \subset \xi^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})}$. AS $\xi^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})} \in T_r(\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})})$, $\xi^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})} \in T_r(\mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})})$, and the ordered pair $(\mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})}, \mathcal{H}^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})})$ is $T$-related. If $\xi^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})} \notin T_r(\mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})})$, then there exists a minimal transversal $\tau$ of $\mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})}$ such that $\xi^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})} \subseteq \tau_2 \subset \xi^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})}$. Hence, we obtain a CNT $\delta$ of $\mathcal{H}$ such that $\delta \subset \xi$. Let $\xi^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})} = \tau_1$ and

$\delta = \xi^{(\alpha_3 e^{\iota\Theta_3}, \beta_3 e^{\iota\Phi_3}, \gamma_3 e^{\iota\Psi_3})} \cup \rho_2 \cap \rho_1$, where $\rho_k$ is an elementary CNS with support $\tau_k$ and height $(\alpha_k e^{\iota\Theta_k}, \beta_k e^{\iota\Phi_k}, \gamma_k e^{\iota\Psi_k})$, $k = 1, 2$. This contradiction shows that $\xi^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})} \in T_r(\mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})})$. Then, Lemma 1 implies that $\xi^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} \in T_r(\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})})$, for $\alpha \in (\alpha_3, \alpha_1]$, $\beta \in (\beta_3, \beta_1]$, $\gamma \in (\gamma_3, \gamma_1]$, $\Theta \in (\Theta_3, \Theta_1]$, $\Phi \in (\Phi_3, \Phi_1]$, $\Psi \in (\Psi_3, \Psi_1]$. Continuing the same recursive procedure, we show that $\xi^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} \in T_r(\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})})$, for each $\alpha \in (0, \alpha_1]$, $\beta \in (0, \beta_1]$, $\gamma \in (0, \gamma_1]$, $\Theta \in (0, \Theta_1]$, $\Phi \in (0, \Phi_1]$, $\Psi \in (0, \Psi_1]$.

□

**Example 5.** *Let $\mathcal{H} = (\mathcal{N}, \lambda)$ be a CNHG represented by the incidence matrix as given in Table 1.*

*Please note that*

$$\begin{aligned}
\lambda^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} &= \{\{j_1, j_2\}, \{j_1, j_3\}, \{j_2, j_3\}\}, \\
\lambda^{(0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi})} &= \{\{j_1, j_2, j_4\}, \{j_1, j_3, j_4\}, \{j_2, j_3, j_5\}\}, \\
\lambda^{(0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi})} &= \{\{j_1, j_2, j_4, j_5\}, \{j_1, j_3, j_4, j_5\}, \{j_2, j_3, j_4, j_5\}\}.
\end{aligned}$$

*Clearly, $\mathcal{F}_s(\mathcal{H}) = \{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), (0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi}), (0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi})\}$. Also, $T_r(\mathcal{H}) = \{\tau_1, \tau_2, \tau_3\} = T_r^*(\mathcal{H})$, where*

$$\begin{aligned}
\tau_1 &= \{(j_1, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), (j_2, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})\}, \\
\tau_2 &= \{(j_1, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), (j_3, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})\}, \\
\tau_3 &= \{(j_2, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), (j_3, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})\}.
\end{aligned}$$

*Since, $\{j_4, j_5\} \in T_r(\mathcal{H}^{(0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi})})$ and $\{j_4\} \in T_r(\mathcal{H}^{(0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi})})$, i.e., no minimal transversal of $\mathcal{H}^{(0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi})}$ contains $\{j_4, j_5\}$. Thus, $(\mathcal{H}^{(0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi})}, \mathcal{H}^{(0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi})})$ is not T-related, therefore $\mathcal{H}$ is not T-related.*

**Table 1.** Incidence matrix of CNHG $\mathcal{H} = (\mathcal{N}, \lambda)$.

| $I$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|---|---|---|---|
| $j_1$ | $(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})$ | $(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})$ | $(0, 0, 1)$ |
| $j_2$ | $(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})$ | $(0, 0, 1)$ | $(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})$ |
| $j_3$ | $(0, 0, 1)$ | $(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})$ | $(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})$ |
| $j_4$ | $(0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi})$ | $(0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi})$ | $(0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi})$ |
| $j_5$ | $(0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi})$ | $(0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.3)2\pi})$ | $(0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi}, 0.6e^{\iota(0.6)2\pi})$ |

**Theorem 2.** *Let $\mathcal{H} = (\mathcal{N}, \lambda)$ be an ordered CNHG, then $T_r(\mathcal{H}) = T_r^*(\mathcal{H}) \Leftrightarrow \mathcal{H}$ is T-related.*

**Proof.** In view of Theorem 1, this is enough to prove that $T_r(\mathcal{H}) = T_r^*(\mathcal{H})$ implies $\mathcal{H}$ is $T$-related. Suppose that $\mathcal{F}_s(\mathcal{H}) = \{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1}), (\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2}), \cdots, (\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})\}$ and $\mathcal{H}$ is not $T$-related. Here, we obtain $\xi \in T_r(\mathcal{H})$ such that $\xi \notin T_r^*(\mathcal{H})$. Assume that the ordered pair $(\mathcal{H}^{(\alpha_j e^{\iota\Theta_j}, \beta_j e^{\iota\Phi_j}, \gamma_j e^{\iota\Psi_j})}, \mathcal{H}^{(\alpha_{j+1} e^{\iota\Theta_{j+1}}, \beta_{j+1} e^{\iota\Phi_{j+1}}, \gamma_{j+1} e^{\iota\Psi_{j+1}})})$ is not $T$-related and $c(\mathcal{H}) = \{\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})}, \mathcal{H}^{(\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2})}, \cdots, \mathcal{H}^{(\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})}\}$. Then, there exists a CNT $\tau_k$ such that $\tau_k \in T_r(\mathcal{H}^{(\alpha_k e^{\iota\Theta_k}, \beta_k e^{\iota\Phi_k}, \gamma_k e^{\iota\Psi_k})})$ and $\tau_k \subset \tau_{k+1}$, where

$$\tau_{k+1} \in T_r(\mathcal{H}^{(\alpha_{k+1} e^{\iota\Theta_{k+1}}, \beta_{k+1} e^{\iota\Phi_{k+1}}, \gamma_{k+1} e^{\iota\Psi_{k+1}})})$$

satisfying the condition that $N$ is not a minimal transversal of $\mathcal{H}^{(\alpha_{k+1} e^{\iota\Theta_{k+1}}, \beta_{k+1} e^{\iota\Phi_{k+1}}, \gamma_{k+1} e^{\iota\Psi_{k+1}})}$, for every $N$, $\tau_k \subseteq N \subseteq \tau_{k+1}$. Since, $\mathcal{H} = (\mathcal{N}, \lambda)$ is an ordered CNHG, then $\mathcal{H}^{(\alpha_k e^{\iota\Theta_k}, \beta_k e^{\iota\Phi_k}, \gamma_k e^{\iota\Psi_k})} \subseteq \mathcal{H}^{(\alpha_{k+1} e^{\iota\Theta_{k+1}}, \beta_{k+1} e^{\iota\Phi_{k+1}}, \gamma_{k+1} e^{\iota\Psi_{k+1}})}$, therefore $\tau_k$ is not a transversal of $\mathcal{H}^{(\alpha_{k+1} e^{\iota\Theta_{k+1}}, \beta_{k+1} e^{\iota\Phi_{k+1}}, \gamma_{k+1} e^{\iota\Psi_{k+1}})}$, for otherwise $\tau_k \in T_r(\mathcal{H}^{(\alpha_{k+1}e^{\iota\Theta_{k+1}},\beta_{k+1}e^{\iota\Phi_{k+1}},\gamma_{k+1}e^{\iota\Psi_{k+1}})})$, which is a contradiction to our assumption. Let $\delta$ be an arbitrary CNT of $\mathcal{H}^{(\alpha_{k+1}e^{\iota\Theta_{k+1}},\beta_{k+1}e^{\iota\Phi_{k+1}},\gamma_{k+1}e^{\iota\Psi_{k+1}})}$ such that $\tau_k \subseteq \delta \subseteq \tau_{k+1}$. Now, if $\tau_k \subseteq Q \subset \delta$, then $Q$ is not a crisp transversal of $\mathcal{H}^{(\alpha_{k+1}e^{\iota\Theta_{k+1}},\beta_{k+1}e^{\iota\Phi_{k+1}},\gamma_{k+1}e^{\iota\Psi_{k+1}})}$. As we know that $\delta \notin T_r(\mathcal{H}^{(\alpha_{k+1}e^{\iota\Theta_{k+1}},\beta_{k+1}e^{\iota\Phi_{k+1}},\gamma_{k+1}e^{\iota\Psi_{k+1}})})$ and $\tau_k \subset \delta$. Thus, we can obtain a minimal CNT $\xi$ of $\mathcal{H}$ such that $\xi \notin T_r^*(\mathcal{H})$. First, we compute a minimal CNT $\xi_1$ of $\mathcal{H}^{(\alpha_k e^{\iota\Theta_k},\beta_k e^{\iota\Phi_k},\gamma_k e^{\iota\Psi_k})}$, where $\tau_k$ is the top level cut of $\xi_1$ at level $(\alpha_k e^{\iota\Theta_k}, \beta_k e^{\iota\Phi_k}, \gamma_k e^{\iota\Psi_k})$ and satisfies $\xi_1^{(\alpha_{k+1}e^{\iota\Theta_{k+1}},\beta_{k+1}e^{\iota\Phi_{k+1}},\gamma_{k+1}e^{\iota\Psi_{k+1}})} \subseteq \tau_{k+1}$. Then, Lemma 1 implies that the $(\alpha_{k+1}e^{\iota\Theta_{k+1}}, \beta_{k+1}e^{\iota\Phi_{k+1}}, \gamma_{k+1}e^{\iota\Psi_{k+1}})$-cut, $\xi_1^{(\alpha_{k+1}e^{\iota\Theta_{k+1}},\beta_{k+1}e^{\iota\Phi_{k+1}},\gamma_{k+1}e^{\iota\Psi_{k+1}})}$ of $\xi_1$ should equal to some $\delta$ that satisfies $\tau_k \subseteq \delta \subseteq \tau_{k+1}$ and $\tau_k \subseteq Q \subset \delta$, then $Q$ is not a crisp transversal of $\mathcal{H}^{(\alpha_{k+1}e^{\iota\Theta_{k+1}},\beta_{k+1}e^{\iota\Phi_{k+1}},\gamma_{k+1}e^{\iota\Psi_{k+1}})}$. Thus, we obtain $\xi_1 \in T_r(\mathcal{H}^{(\alpha_k e^{\iota\Theta_k},\beta_k e^{\iota\Phi_k},\gamma_k e^{\iota\Psi_k})}) \setminus T_r^*(\mathcal{H}^{(\alpha_k e^{\iota\Theta_k},\beta_k e^{\iota\Phi_k},\gamma_k e^{\iota\Psi_k})})$.

We now assume that $(\alpha_k e^{\iota\Theta_k}, \beta_k e^{\iota\Phi_k}, \gamma_k e^{\iota\Psi_k}) \subset (\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1})$. Since, $\mathcal{H}$ is ordered, then there exists an ordered sequence $t_k \supseteq t_{k-1} \supset \cdots \supseteq t_1$ of crisp minimal transversals of $\mathcal{H}^{(\alpha_k e^{\iota\Theta_k},\beta_k e^{\iota\Phi_k},\gamma_k e^{\iota\Psi_k})}$, $\mathcal{H}^{(\alpha_{k-1}e^{\iota\Theta_{k-1}},\beta_{k-1}e^{\iota\Phi_{k-1}},\gamma_{k-1}e^{\iota\Psi_{k-1}})}$, $\ldots$, $\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1},\beta_1 e^{\iota\Phi_1},\gamma_1 e^{\iota\Psi_1})}$, respectively. Let $\rho_l$ be an elementary CNSS with support $t_l$ and height $\xi_l$. Then, $\xi = \rho_1 \cup \cdots \cup \rho_{l-1} \cup \delta$ such that $\xi \in T_r(\mathcal{H})$ and $\xi \notin T_r^*(\mathcal{H})$. □

**Corollary 1.** *Let $\mathcal{H} = (\mathcal{N}, \lambda)$ be an ordered CNHG with $\mathcal{F}_s(\mathcal{H}) = \{(\alpha_1 e^{\iota\Theta_1}, \beta_1 e^{\iota\Phi_1}, \gamma_1 e^{\iota\Psi_1}), (\alpha_2 e^{\iota\Theta_2}, \beta_2 e^{\iota\Phi_2}, \gamma_2 e^{\iota\Psi_2}), \cdots, (\alpha_n e^{\iota\Theta_n}, \beta_n e^{\iota\Phi_n}, \gamma_n e^{\iota\Psi_n})\}$ and $c(\mathcal{H}) = \{\mathcal{H}^{(\alpha_1 e^{\iota\Theta_1},\beta_1 e^{\iota\Phi_1},\gamma_1 e^{\iota\Psi_1})}, \mathcal{H}^{(\alpha_2 e^{\iota\Theta_2},\beta_2 e^{\iota\Phi_2},\gamma_2 e^{\iota\Psi_2})}, \cdots, \mathcal{H}^{(\alpha_n e^{\iota\Theta_n},\beta_n e^{\iota\Phi_n},\gamma_n e^{\iota\Psi_n})}\}$.*

*If an ordered pair $(\mathcal{H}^{(\alpha_j e^{\iota\Theta_j},\beta_j e^{\iota\Phi_j},\gamma_j e^{\iota\Psi_j})}, \mathcal{H}^{(\alpha_{j+1}e^{\iota\Theta_{j+1}},\beta_{j+1}e^{\iota\Phi_{j+1}},\gamma_{j+1}e^{\iota\Psi_{j+1}})})$ is not T-related, then*

**(i)** $(\alpha_{j+1}e^{\iota\Theta_{j+1}}, \beta_{j+1}e^{\iota\Phi_{j+1}}, \gamma_{j+1}e^{\iota\Psi_{j+1}}) \in \mathcal{F}_s(T_r(\mathcal{H}))$.

**(ii)** $(\alpha_{j+1}e^{\iota\Theta_{j+1}}, \beta_{j+1}e^{\iota\Phi_{j+1}}, \gamma_{j+1}e^{\iota\Psi_{j+1}})$ *is a transition level for* $\xi \in T_r(\mathcal{H}) \setminus T_r^*(\mathcal{H})$.

**Example 6.** *Let $N = \{(u, t_N(u)e^{\iota\phi_N(u)}, i_N(u)e^{\iota\varphi_N(u)}, f_N(u)e^{\iota\psi_N(u)}) | u \in \mathcal{J}\}$ be a CNS on $\mathcal{J} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ such that $N(a_7) = (0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})$ and $N(a) = (0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})$, for all $a \in \mathcal{J} \setminus \{a_7\}$. Let $H = (\mathcal{J}, E)$ be a crisp hypergraph on $\mathcal{J}$, where $E_1 = \{a_1, a_2, a_4\}$, $E_2 = \{a_1, a_3, a_4\}$, $E_3 = \{a_4, a_5, a_6\}$, $E_4 = \{a_1, a_5\}$, and $E_5 = \{a_5, a_7\}$. Then, N-tempered CNHG $\mathcal{H} = (\mathcal{N}, \lambda)$ is given by the incidence matrix as shown in Table 2.*

*Here, $\mathbf{0} = (0, 0, 1)$, $\mathbf{0.9e^{\iota(0.9)2\pi}}$=$(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})$, and $\mathbf{0.4e^{\iota(0.4)2\pi}}$= $(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})$. Please note that $\mathcal{F}_s(\mathcal{H}) = \{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), (0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})\}$ and $c(\mathcal{H}) = \{\mathcal{H}^{(0.9e^{\iota(0.9)2\pi},0.9e^{\iota(0.9)2\pi},0.9e^{\iota(0.9)2\pi})}, \mathcal{H}^{(0.4e^{\iota(0.4)2\pi},0.4e^{\iota(0.4)2\pi},0.4e^{\iota(0.4)2\pi})}\}$, where*

$$\begin{aligned}
\mathcal{H}^{(0.9e^{\iota(0.9)2\pi},0.9e^{\iota(0.9)2\pi},0.9e^{\iota(0.9)2\pi})} &= (\mathcal{J}_1\}, \mathcal{E}_1),\ \mathcal{J}_1 = \{a_1, a_2, a_3, a_4, a_5, a_6\},\\
\mathcal{E}_1 &= \{\{a_1, a_2, a_4\}, \{a_1, a_3, a_4\}, \{a_4, a_5, a_6\}, \{a_1, a_5\}\},\\
\mathcal{H}^{(0.4e^{\iota(0.4)2\pi},0.4e^{\iota(0.4)2\pi},0.4e^{\iota(0.4)2\pi})} &= (\mathcal{J}_2, \mathcal{E}_2),\ \mathcal{J}_2 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\},\\
\mathcal{E}_2 &= \{\{a_1, a_2, a_4\}, \{a_1, a_3, a_4\}, \{a_4, a_5, a_6\}, \{a_1, a_5\}\{a_5, a_7\}\}.
\end{aligned}$$

*Please note that*

$$\{a_1, a_4\} \in T_r(\mathcal{H}^{(0.9e^{\iota(0.9)2\pi},0.9e^{\iota(0.9)2\pi},0.9e^{\iota(0.9)2\pi})}), \{a_1, a_4\} \notin T_r(\mathcal{H}^{(0.4e^{\iota(0.4)2\pi},0.4e^{\iota(0.4)2\pi},0.4e^{\iota(0.4)2\pi})}),$$

*i.e., $\{a_1, a_4, a_5\}$ is a transversal of $\mathcal{H}^{(0.4e^{\iota(0.4)2\pi},0.4e^{\iota(0.4)2\pi},0.4e^{\iota(0.4)2\pi})}$ but not a minimal transversal. Therefore, the ordered pair $(\mathcal{H}^{(0.9e^{\iota(0.9)2\pi},0.9e^{\iota(0.9)2\pi},0.9e^{\iota(0.9)2\pi})}, \mathcal{H}^{(0.4e^{\iota(0.4)2\pi},0.4e^{\iota(0.4)2\pi},0.4e^{\iota(0.4)2\pi})})$ as well as $\mathcal{H}$ is not T-related.*

**Table 2.** Incidence matrix of $N$-tempered CNHG $\mathcal{H}$.

| $\mathcal{H}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ |
|---|---|---|---|---|---|
| $a_1$ | $0.9e^{\iota(0.9)2\pi}$ | $0.9e^{\iota(0.9)2\pi}$ | 0 | $0.9e^{\iota(0.9)2\pi}$ | 0 |
| $a_2$ | $0.9e^{\iota(0.9)2\pi}$ | 0 | 0 | 0 | 0 |
| $a_3$ | 0 | $0.9e^{\iota(0.9)2\pi}$ | 0 | 0 | 0 |
| $a_4$ | $0.9e^{\iota(0.9)2\pi}$ | $0.9e^{\iota(0.9)2\pi}$ | $0.9e^{\iota(0.9)2\pi}$ | 0 | 0 |
| $a_5$ | 0 | 0 | $0.9e^{\iota(0.9)2\pi}$ | $0.9e^{\iota(0.9)2\pi}$ | $0.4e^{\iota(0.4)2\pi}$ |
| $a_6$ | 0 | 0 | $0.9e^{\iota(0.9)2\pi}$ | 0 | 0 |
| $a_7$ | 0 | 0 | 0 | 0 | $0.4e^{\iota(0.4)2\pi}$ |

**Remark 1.**

- *Example 6 shows that there exists some ordered CNHGs that are not T-related.*
- *Every simply ordered CNHG* $\mathcal{H} = (\mathcal{N}, \lambda)$ *satisfies* $(T_r^*(\mathcal{H})^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} = T_r(\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})})$, *for all* $\alpha \in (0, t(h(\mathcal{H}))]$, $\beta \in (0, i(h(\mathcal{H}))]$, $\gamma \in (0, f(h(\mathcal{H}))]$, $\Theta \in (0, \phi(h(\mathcal{H}))]$, $\Phi \in (0, \varphi(h(\mathcal{H}))]$, $\Psi \in (0, \psi(h(\mathcal{H}))]$.

**Lemma 2.** *Let* $H = (\mathcal{J}, E)$ *be a crisp hypergraph and* $j$ *be an arbitrary vertex of* $H$. *Then* $j \in \mathcal{E} \in T_r(H) \Leftrightarrow j \in E_k \in E$ *such that for any hyperedge* $E_l \neq E_k$ *of* $H$, $E_l \nsubseteq E_k$.

**Proposition 1.** *Let* $H_1 = (\mathcal{J}_1, E_1)$ *be a crisp partial hypergraph of* $H = (\mathcal{J}, E)$ *that is obtained by removing those hyperedges of* $H = (\mathcal{J}, E)$ *that contain any other edges properly. Then,*

**(i)** $T_r(H_1) = T_r(H)$,
**(ii)** $\cup T_r(H) = \mathcal{J}_1$.

**Definition 27.** *Let* $\mathcal{H} = (\mathcal{N}, \lambda)$ *be a CNHG. The join of* $\mathcal{H}$, *denoted by* $J(\mathcal{H})$, *is defined as,* $J(\mathcal{H}) = \bigcup\limits_{\rho \in \lambda} \rho$, *where* $\lambda$ *is the CN hyperedge set of* $\mathcal{H}$.

*For every* $\alpha \in (0, t(h(\mathcal{H}))]$, $\beta \in (0, i(h(\mathcal{H}))]$, $\gamma \in (0, f(h(\mathcal{H}))]$, $\Theta \in (0, \phi(h(\mathcal{H}))]$, $\Phi \in (0, \varphi(h(\mathcal{H}))]$, $\Psi \in (0, \psi(h(\mathcal{H}))]$, *the* $(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})$*-level cut of* $J(\mathcal{H})$, *i.e.,* $(J(\mathcal{H}))^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})}$ *is the set of vertices of* $(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})$*-level hypergraph of* $\mathcal{H}$, *i.e.,* $(J(\mathcal{H}))^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} = \mathcal{J}(\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})})$.

**Lemma 3.** *Let* $\mathcal{H} = (\mathcal{N}, \lambda)$ *be a CNHG and* $\xi \in T_r(\mathcal{H})$. *If* $j \in supp(\xi)$, *then there exists a CN hyperedge* $\rho$ *of* $\mathcal{H}$ *such that*

**(i)** $\rho(j) = h(\rho) = \xi(j) > 0$,
**(ii)** $\xi^{h(\rho)} \cap \rho^{h(\rho)} = \{j\}$.

**Proof.** Let $j_0 \in supp(\xi)$ such that $\xi \in T_r(\mathcal{H})$ and $\xi(j_0) = (\alpha_0 e^{\iota\phi_0}, \beta_0 e^{\iota\varphi_0}, \gamma_0 e^{\iota\psi_0})$. Since every $\xi_1$ that is a transversal of $\mathcal{H}$ contains a transversal $\xi$ such that $\xi \subseteq j(\mathcal{H})$. This implies that $j_0 \in \mathcal{N}^{(\alpha_0 e^{\iota\phi_0}, \beta_0 e^{\iota\varphi_0}, \gamma_0 e^{\iota\psi_0})} = \mathcal{J}(\mathcal{H}^{(\alpha_0 e^{\iota\phi_0}, \beta_0 e^{\iota\varphi_0}, \gamma_0 e^{\iota\psi_0})})$. Therefore, there exists at least one hyperedge $\rho$ of $\mathcal{H}$ such that $\rho(j_0) \geq (\alpha_0 e^{\iota\phi_0}, \beta_0 e^{\iota\varphi_0}, \gamma_0 e^{\iota\psi_0})$. Let $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ be the set of hyperedges of $\mathcal{H}$ and $\rho(j_0) \geq (\alpha_0 e^{\iota\phi_0}, \beta_0 e^{\iota\varphi_0}, \gamma_0 e^{\iota\psi_0})$. We now prove that there exists at least one $\lambda_k \in \lambda$ such that $h(\lambda_j) = (\alpha_0 e^{\iota\phi_0}, \beta_0 e^{\iota\varphi_0}, \gamma_0 e^{\iota\psi_0})$. For otherwise, we have $h(\lambda_k) = (\alpha_k e^{\iota\phi_k}, \beta_k e^{\iota\varphi_k}, \gamma_k e^{\iota\psi_k}) \geq (\alpha_0 e^{\iota\phi_0}, \beta_0 e^{\iota\varphi_0}, \gamma_0 e^{\iota\psi_0})$, for all $\lambda_k \in \lambda$, $k = 1, 2, \cdots, m$. This implies that for every $\lambda_k \in \lambda$, there exists an element $u_k \in supp(\xi)$ such that $u_k \in (\lambda_k)^{(\alpha_k e^{\iota\phi_k}, \beta_k e^{\iota\varphi_k}, \gamma_k e^{\iota\psi_k})} \cap \xi^{(\alpha_k e^{\iota\phi_k}, \beta_k e^{\iota\varphi_k}, \gamma_k e^{\iota\psi_k})}$, for $(\alpha_k e^{\iota\phi_k}, \beta_k e^{\iota\varphi_k}, \gamma_k e^{\iota\psi_k}) \geq (\alpha_0 e^{\iota\phi_0}, \beta_0 e^{\iota\varphi_0}, \gamma_0 e^{\iota\psi_0})$. Since, $\xi(j_0) = (\alpha_0 e^{\iota\phi_0}, \beta_0 e^{\iota\varphi_0}, \gamma_0 e^{\iota\psi_0})$, then $h(\lambda_k) = (\alpha_k e^{\iota\phi_k}, \beta_k e^{\iota\varphi_k}, \gamma_k e^{\iota\psi_k}) \geq (\alpha_0 e^{\iota\phi_0}, \beta_0 e^{\iota\varphi_0}, \gamma_0 e^{\iota\psi_0})$ and $u_k \in (\lambda_k)^{(\alpha_k e^{\iota\phi_k}, \beta_k e^{\iota\varphi_k}, \gamma_k e^{\iota\psi_k})} \cap \xi^{(\alpha_k e^{\iota\phi_k}, \beta_k e^{\iota\varphi_k}, \gamma_k e^{\iota\psi_k})}$ imply that $u_k \neq j_0$, $k = 1, 2, \cdots, m$. If these hold, it could be shown that $\xi \notin T_r(\mathcal{H})$ by computing a CNT $\delta$ of $\mathcal{H}$ that satisfies $\delta \subset \xi$. This argument follows form the fact that $\mathcal{J}$ and $\lambda$ are finite, there exist intervals $(\alpha_0 - \epsilon, \alpha_0]$, $(\beta_0 - \epsilon, \beta_0]$, $(\gamma_0 - \epsilon, \gamma_0]$, $(\phi_0 - 2\pi\epsilon, \phi_0]$, $(\varphi_0 - 2\pi\epsilon, \varphi_0]$, and $(\psi_0 - 2\pi\epsilon, \psi_0]$

such that $\mathcal{H}^{(\alpha e^{\iota\phi},\beta e^{\iota\varphi},\gamma e^{\iota\psi})} = \mathcal{H}^{(\alpha_0 e^{\iota\phi_0},\beta_0 e^{\iota\varphi_0},\gamma_0 e^{\iota\psi_0})}$ on $(\alpha_0-\epsilon,\alpha_0]$, $(\beta_0-\epsilon,\beta_0]$, $(\gamma_0-\epsilon,\gamma_0]$, $(\phi_0-2\pi\epsilon,\phi_0]$, $(\varphi_0-2\pi\epsilon,\varphi_0]$, and $(\psi_0-2\pi\epsilon,\psi_0]$.

Define $\delta(u)$ as,

$$t_\delta(u) = \begin{cases} t_\xi(u), & \text{if} \quad u \neq j_0, \\ \alpha_0-\epsilon, & \text{if} \quad u = j_0. \end{cases}, \; i_\delta(u) = \begin{cases} i_\xi(u), & \text{if} \quad u \neq j_0, \\ \beta_0-\epsilon, & \text{if} \quad u = j_0. \end{cases},$$

$$f_\delta(u) = \begin{cases} f_\xi(u), & \text{if} \quad u \neq j_0, \\ \gamma_0-\epsilon, & \text{if} \quad u = j_0. \end{cases}, \; \phi_\delta(u) = \begin{cases} \phi_\xi(u), & \text{if} \quad u \neq j_0, \\ \phi_0-2\pi\epsilon, & \text{if} \quad u = j_0. \end{cases},$$

$$\varphi_\delta(u) = \begin{cases} \varphi_\xi(u), & \text{if} \quad u \neq j_0, \\ \varphi_0-2\pi\epsilon, & \text{if} \quad u = j_0. \end{cases}, \; \psi_\delta(u) = \begin{cases} \psi_\xi(u), & \text{if} \quad u \neq j_0, \\ \psi_0-2\pi\epsilon, & \text{if} \quad u = j_0. \end{cases}.$$

Clearly $\delta \subset \xi$ and $\delta$ is a transversal of $\mathcal{H}$. Also, $\xi^{(\alpha_0 e^{\iota\phi_0},\beta_0 e^{\iota\varphi_0},\gamma_0 e^{\iota\psi_0})} \setminus \{j_0\}$ contains $\{u_k | k = 1,2,\cdots,m\}$. Therefore, $\xi^{(\alpha_0 e^{\iota\phi_0},\beta_0 e^{\iota\varphi_0},\gamma_0 e^{\iota\psi_0})} \setminus \{j_0\}$ is a transversal of $\mathcal{H}^{(\alpha_0 e^{\iota\phi_0},\beta_0 e^{\iota\varphi_0},\gamma_0 e^{\iota\psi_0})}$. The same argument holds for every $\mathcal{H}^{(\alpha e^{\iota\phi},\beta e^{\iota\varphi},\gamma e^{\iota\psi})}$, where $\alpha \in (\alpha_0-\epsilon,\alpha_0]$, $\beta \in (\beta_0-\epsilon,\beta_0]$, $\gamma \in (\gamma_0-\epsilon,\gamma_0]$, $\phi \in (\phi_0-2\pi\epsilon,\phi_0]$, $\varphi \in (\varphi_0-2\pi\epsilon,\varphi_0]$, $\psi \in (\psi_0-2\pi\epsilon,\psi_0]$. Since, $\delta^{(\alpha e^{\iota\phi},\beta e^{\iota\varphi},\gamma e^{\iota\psi})} = \xi^{(\alpha e^{\iota\phi},\beta e^{\iota\varphi},\gamma e^{\iota\psi})}$, for all $\alpha \in (0, t(h(\mathcal{H}))] \setminus (\alpha_0-\epsilon,\alpha_0]$, $\beta \in (0, i(h(\mathcal{H}))] \setminus (\beta_0-\epsilon,\beta_0]$, $\gamma \in (0, f(h(\mathcal{H}))] \setminus (\gamma_0-\epsilon,\gamma_0]$, $\phi \in (0,\phi(h(\mathcal{H}))] \setminus (\phi_0-2\pi\epsilon,\phi_0]$, $\varphi \in (0,\varphi(h(\mathcal{H}))] \setminus (\varphi_0-2\pi\epsilon,\varphi_0]$, $\psi \in (0,\psi(h(\mathcal{H}))] \setminus (\psi_0-2\pi\epsilon,\psi_0]$. This establishes the existence of $\rho \in \mathcal{H}$ for which $\rho(j_0) = h(\rho) = \xi(j_0) > 0$.

We now suppose that every hyperedge from the set $\lambda = \{\lambda_1,\lambda_2,\cdots,\lambda_m\}$ with height $\xi(j_0)$ contain two or more than two elements of $\xi^{(\alpha_0 e^{\iota\phi_0},\beta_0 e^{\iota\varphi_0},\gamma_0 e^{\iota\psi_0})} \setminus \{j_0\}$. BY repeating the above procedure, we can establish that $\xi \notin T_r(\mathcal{H})$, which is a contradiction. □

**Example 7.** *Consider a CNHG $\mathcal{H} = (\mathcal{N},\lambda)$ on $\mathcal{J} = \{u_1,u_2,u_3,u_4\}$ as represented by incidence matrix given in Table 3.*

*Here,* $\mathbf{0.7e^{\iota(0.7)2\pi}} = (0.7e^{\iota(0.7)2\pi}, 0.7e^{\iota(0.7)2\pi}, 0.7e^{\iota(0.7)2\pi})$, $\mathbf{0.9e^{\iota(0.9)2\pi}} = (0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})$, $\mathbf{0.4e^{\iota(0.4)2\pi}} = (0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})$. *Then, we see that $\lambda_1$, $\lambda_3$, and $\lambda_5$ have no transitions levels and $(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})$ is the transition level of $\lambda_2$ and $\lambda_4$. The basic sequences are given as,*

$$\begin{aligned}
B_s(\lambda_1) &= \{(0.7e^{\iota(0.7)2\pi}, 0.7e^{\iota(0.7)2\pi}, 0.7e^{\iota(0.7)2\pi})\}, \\
B_s(\lambda_2) &= \{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), (0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})\}, \\
B_s(\lambda_3) &= \{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})\}, \\
B_s(\lambda_4) &= \{(0.7e^{\iota(0.7)2\pi}, 0.7e^{\iota(0.7)2\pi}, 0.7e^{\iota(0.7)2\pi}), (0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})\}, \\
B_s(\lambda_5) &= \{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})\}.
\end{aligned}$$

*Thus,*

$$\begin{aligned}
B_c(\lambda_1) &= \{\lambda_1^{(0.7e^{\iota(0.7)2\pi}, 0.7e^{\iota(0.7)2\pi}, 0.7e^{\iota(0.7)2\pi})}\}, \\
B_c(\lambda_2) &= \{\lambda_2^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})}, \lambda_2^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})}\}, \\
B_c(\lambda_3) &= \{\lambda_3^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})}\}, \\
B_c(\lambda_4) &= \{\lambda_4^{(0.7e^{\iota(0.7)2\pi}, 0.7e^{\iota(0.7)2\pi}, 0.7e^{\iota(0.7)2\pi})}, \lambda_4^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})}\}, \\
B_c(\lambda_5) &= \{\lambda_5^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})}\}.
\end{aligned}$$

*Also, we have* $\mathcal{F}_s(\mathcal{H}) = \{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), (0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})\}$ *and* $c(\mathcal{H}) = \{\mathcal{H}^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})}, \mathcal{H}^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})}\}$*, where*

$$\begin{aligned}
\lambda^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} &= \{\{u_1, u_2, u_3\}, \{u_1, u_2\}, \{u_2, u_3\}\}, \\
\lambda^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})} &= \{\{u_1, u_2, u_3, u_4\}, \{u_1, u_2\}, \{u_1, u_2, u_4\}, \{u_2, u_3\}, \{u_2, u_3, u_4\}\}.
\end{aligned}$$

*We now determine* $T_r(\mathcal{H})$ *and* $T_r^*(\mathcal{H})$*. If* $\tau \in T_r(\mathcal{H})$*, then* $\tau^{h(\lambda_1)} \cap \{u_1, u_2\} \neq \emptyset$, $\tau^{h(\lambda_2)} \cap \{u_1, u_2\} \neq \emptyset$, $\tau^{h(\lambda_3)} \cap \{u_2, u_3\} \neq \emptyset$, $\tau^{h(\lambda_4)} \cap \{u_2, u_3\} \neq \emptyset$*, and* $\tau^{h(\lambda_5)} \cap \{u_1, u_3, u_4\} \neq \emptyset$*. Please note that* $T_r(\mathcal{H}) = \{\tau_1, \tau_2, \tau_3, \tau_4\}$*, where*

$$\begin{aligned}
\tau_1 &= \{(u_1, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), (u_3, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})\}, \\
\tau_2 &= \{(u_2, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), (u_3, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}\}, \\
\tau_3 &= \{(u_2, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), (u_4, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}\}, \\
\tau_4 &= \{(u_2, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), (u_1, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}\}.
\end{aligned}$$

*Now* $T_r(\mathcal{H}^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})}) = \{\{u_2\}, \{u_1, u_3\}\}$ *and* $T_r(\mathcal{H}^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})}) = \{\{u_1, u_3\}, \{u_2, u_3\}, \{u_2, u_4\}, \{u_1, u_2\}, \{u_1, u_3, u_4\}\}$ *and* $\tau_k^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})} \in T_r(\mathcal{H}^{(\alpha e^{\iota\Theta}, \beta e^{\iota\Phi}, \gamma e^{\iota\Psi})})$*, for all* $\alpha \in (0, t(h(\mathcal{H}))]$, $\beta \in (0, i(h(\mathcal{H}))]$, $\gamma \in (0, f(h(\mathcal{H}))]$, $\Theta \in (0, \phi(h(\mathcal{H}))]$, $\Phi \in (0, \varphi(h(\mathcal{H}))]$, $\Psi \in (0, \psi(h(\mathcal{H}))]$*. Hence,* $T_r^*(\mathcal{H}) = \{\tau_1\}$*.*

*We now illustrate Lemma 3 through the above example.*

$$\begin{aligned}
\lambda_2(u_1) = h(\lambda_2) = \tau_1(u_1) &= (0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), \\
\lambda_3(u_3) = h(\lambda_3) = \tau_1(u_3) &= (0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), \\
\lambda_2(u_2) = h(\lambda_2) = \tau_2(u_2) &= (0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), \\
\lambda_5(u_3) = h(\lambda_5) = \tau_2(u_3) &= (0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}), \\
\lambda_3(u_2) = h(\lambda_3) = \tau_3(u_2) &= (0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}), \\
\lambda_5(u_4) = h(\lambda_5) = \tau_3(u_4) &= (0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}), \\
\lambda_5(u_1) = h(\lambda_5) = \tau_4(u_2) &= (0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}), \\
\lambda_3(u_2) = h(\lambda_3) = \tau_4(u_2) &= (0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}).
\end{aligned}$$

*Also note that*

$$\begin{aligned}
\tau_1^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} \cap \lambda_2^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} &= \{u_1\}, \\
\tau_1^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} \cap \lambda_3^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} &= \{u_3\}, \\
\tau_2^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} \cap \lambda_2^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} &= \{u_2\}, \\
\tau_2^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})} \cap \lambda_5^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})} &= \{u_3\}, \\
\tau_3^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} \cap \lambda_3^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} &= \{u_2\}, \\
\tau_3^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})} \cap \lambda_5^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})} &= \{u_4\}, \\
\tau_4^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})} \cap \lambda_5^{(0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi}, 0.4e^{\iota(0.4)2\pi})} &= \{u_1\}, \\
\tau_4^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} \cap \lambda_3^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} &= \{u_2\}.
\end{aligned}$$

*Hence,* $(T_r(\mathcal{H}))^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})} = \{\tau_1^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})}, \tau_2^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})}, \tau_3^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})}, \tau_4^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})}\} = \{\{u_1, u_3\}, \{u_2\}, \{u_2\}, \{u_2\}\} = \{\{u_1, u_3\}, \{u_2\}\} = T_r(\mathcal{H}^{(0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi}, 0.9e^{\iota(0.9)2\pi})})$.

**Table 3.** Incidence matrix of $\mathcal{H}$.

| $I_{\mathcal{H}}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ |
|---|---|---|---|---|---|
| $u_1$ | $\mathbf{0.7}e^{\iota(\mathbf{0.7})2\pi}$ | $\mathbf{0.9}e^{\iota(\mathbf{0.9})2\pi}$ | $(0,0,1)$ | $(0,0,1)$ | $\mathbf{0.4}e^{\iota(\mathbf{0.4})2\pi}$ |
| $u_2$ | $\mathbf{0.7}e^{\iota(\mathbf{0.7})2\pi}$ | $\mathbf{0.9}e^{\iota(\mathbf{0.9})2\pi}$ | $\mathbf{0.9}e^{\iota(\mathbf{0.9})2\pi}$ | $\mathbf{0.7}e^{\iota(\mathbf{0.7})2\pi}$ | $(0,0,1)$ |
| $u_3$ | $(0,0,1)$ | $(0,0,1)$ | $\mathbf{0.9}e^{\iota(\mathbf{0.9})2\pi}$ | $\mathbf{0.7}e^{\iota(\mathbf{0.7})2\pi}$ | $\mathbf{0.4}e^{\iota(\mathbf{0.4})2\pi}$ |
| $u_4$ | $(0,0,1)$ | $\mathbf{0.4}e^{\iota(\mathbf{0.4})2\pi}$ | $(0,0,1)$ | $\mathbf{0.4}e^{\iota(\mathbf{0.4})2\pi}$ | $\mathbf{0.4}e^{\iota(\mathbf{0.4})2\pi}$ |

**Theorem 3.** *Let $\mathcal{H} = (\mathcal{N}, \lambda)$ be a CNHG and $j \in \mathcal{J}$. If $\xi \in T_r(\mathcal{H})$ with $j \in supp(\xi)$, then there exists an hyperedge $\rho \in \lambda$ such that*

**(i)** $\rho(j) = h(\rho)$,

**(ii)** *For $\lambda_1 \in \lambda$ such that $h(\lambda_1) \geq h(\rho)$, $\lambda_1^{h(\lambda_1)} \nsubseteq \rho^{h(\rho)}$,*

**(iii)** *$\mathcal{E}_k \nsubseteq \rho^{h(\rho)}$, where $\mathcal{E}_k$ is an arbitrary hyperedge of $\mathcal{H}^{h(\rho)}$,*

**(iv)** $\xi(j) = \rho(j)$.

**Corollary 2.** *Let $\mathcal{H} = (\mathcal{N}, \lambda)$ be a CNHG. If $\lambda_1 \in \lambda$ satisfies $h(\lambda_1) \geq h(\rho)$, $\lambda_1^{h(\lambda_1)} \nsubseteq \rho^{h(\rho)}$, then $h(\lambda_1) \in \mathcal{F}_s(\mathcal{H})$.*

## 4. Applications

In this section, we propose the modeling of overlapping communities that exist in different social networks through CNHGs. These communities intersect each other when one person belongs to multiple communities at the same time. The vertices of the CNHGs are used to represent different communities and the hyperlinks of individuals who participate in more than one community are illustrated using hyperedges of CNHGs. Here, we define a score function for ranking CNSs by considering the truth, indeterminacy, and falsity degrees.

**Definition 28.** *Let $N = (te^{\iota\phi}, ie^{\iota\varphi}, fe^{\iota\psi})$ be a CNN, the score function $S$ of $N$ is defined as,*

$$S(N) = \frac{1+t-2i-f}{2} + \frac{2\pi+\phi-2\varphi-\psi}{4\pi},$$

*where $S(N) \in [-2, 2]$.*

### *4.1. Modeling of Intersecting Research Communities*

Research scholars have different fields of interest and these multiple research interests make researchers parts of different research communities at the same time. For example, Mathematics, Physics, and Computer Science may be the fields of interest for one researcher at the same time. That is how overlapping communities occur in research fields. We use a CNHG to model intersecting communities that emerge in different research fields. The vertices of a CNHG represent the different research fields and these fields are connected through an hyperedge that represents a research scholar who works in the corresponding fields. The corresponding model of intersecting research communities is shown in Figure 8.

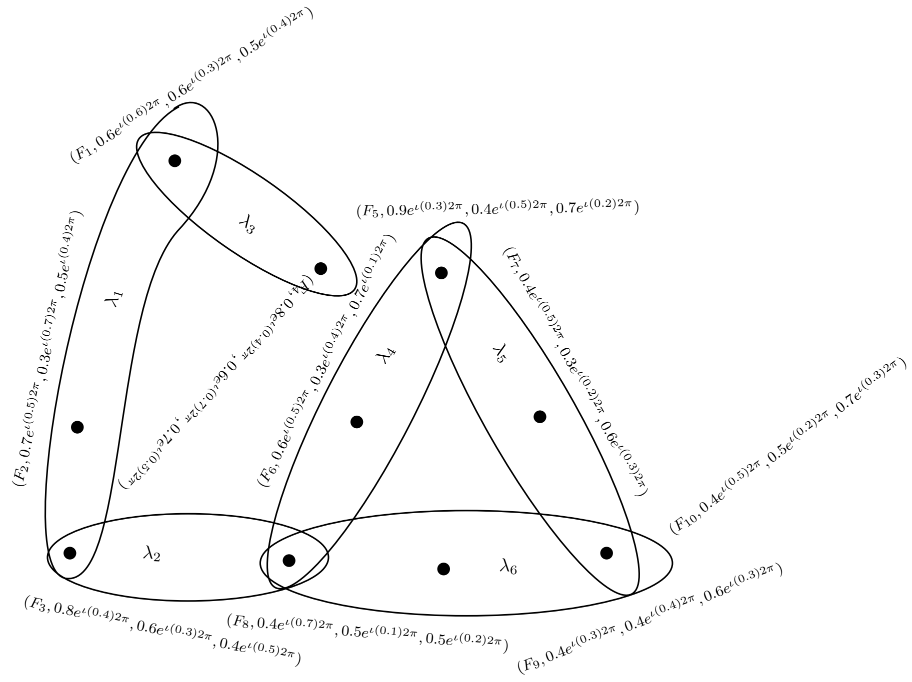


**Figure 8.** Intersecting research communities.

Here, the truth, indeterminacy, and falsity degrees of each vertex represent the accepted, submitted, and rejected articles of that community in a specific period of time that is represented by the phase terms. This inconsistent information with periodic nature is given in Table 4.

**Table 4.** Periodic behavior of research communities.

| Research Fields | Accepted Articles | Submitted Articles | Rejected Articles |
|---|---|---|---|
| $F_1$ | $0.6e^{\iota(0.6)2\pi}$ | $0.6e^{\iota(0.3)2\pi}$ | $0.5e^{\iota(0.4)2\pi}$ |
| $F_2$ | $0.7e^{\iota(0.5)2\pi}$ | $0.3e^{\iota(0.7)2\pi}$ | $0.5e^{\iota(0.4)2\pi}$ |
| $F_3$ | $0.8e^{\iota(0.4)2\pi}$ | $0.6e^{\iota(0.3)2\pi}$ | $0.4e^{\iota(0.5)2\pi}$ |
| $F_4$ | $0.8e^{\iota(0.4)2\pi}$ | $0.6e^{\iota(0.7)2\pi}$ | $0.7e^{\iota(0.5)2\pi}$ |
| $F_5$ | $0.9e^{\iota(0.3)2\pi}$ | $0.4e^{\iota(0.5)2\pi}$ | $0.7e^{\iota(0.2)2\pi}$ |
| $F_6$ | $0.6e^{\iota(0.5)2\pi}$ | $0.3e^{\iota(0.4)2\pi}$ | $0.7e^{\iota(0.1)2\pi}$ |
| $F_7$ | $0.4e^{\iota(0.5)2\pi}$ | $0.3e^{\iota(0.2)2\pi}$ | $0.6e^{\iota(0.3)2\pi}$ |
| $F_8$ | $0.4e^{\iota(0.7)2\pi}$ | $0.5e^{\iota(0.1)2\pi}$ | $0.5e^{\iota(0.2)2\pi}$ |
| $F_9$ | $0.4e^{\iota(0.3)2\pi}$ | $0.4e^{\iota(0.4)2\pi}$ | $0.6e^{\iota(0.3)2\pi}$ |
| $F_{10}$ | $0.4e^{\iota(0.5)2\pi}$ | $0.5e^{\iota(0.2)2\pi}$ | $0.7e^{\iota(0.3)2\pi}$ |

Please note that number of accepted, submitted, and rejected articles of community $F_1$ are 0.6, 0.6, and 0.5, and the corresponding behaviors repeat after $(0.6)2\pi$, $(0.3)2\pi$, and $(0.4)2\pi$ periods of time, respectively, and so on. The research scholar $\lambda_1$ belongs to communities $F_1$, $F_2$, and $F_3$ as he shares these three fields of interest. Similarly, $\lambda_2$ belongs to $F_3$ and $F_8$ and the communities overlap with each other. The indeterminate information about a researcher is calculated using CNRs given as,

$$\begin{aligned}
\lambda_1(\{F_1, F_2, F_3\}) &= (0.6e^{\iota(0.2)2\pi}, 0.3e^{\iota(0.3)2\pi}, 0.4e^{\iota(0.2)2\pi}),\\
\lambda_2(\{F_3, F_8\}) &= (0.4e^{\iota(0.3)2\pi}, 0.5e^{\iota(0.1)2\pi}, 0.4e^{\iota(0.2)2\pi}),\\
\lambda_3(\{F_1, F_4\}) &= (0.6e^{\iota(0.3)2\pi}, 0.4e^{\iota(0.2)2\pi}, 0.7e^{\iota(0.4)2\pi}),\\
\lambda_4(\{F_5, F_8, F_6\}) &= (0.4e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.1)2\pi}, 0.7e^{\iota(0.2)2\pi}),\\
\lambda_5(\{F_5, F_7, F_{10}\}) &= (0.4e^{\iota(0.3)2\pi}, 0.3e^{\iota(0.2)2\pi}, 0.7e^{\iota(0.3)2\pi}),\\
\lambda_6(\{F_8, F_9, F_{10}\}) &= (0.4e^{\iota(0.3)2\pi}, 0.4e^{\iota(0.1)2\pi}, 0.7e^{\iota(0.3)2\pi}).
\end{aligned}$$

It shows the researcher represented by $\lambda_1$ has 0.6 accepted, 0.3 submitted, and 0.4 rejected articles within some specific periods of time. The line graph of intersecting communities as given in Figure 8 is shown in Figure 9. Here, the nodes represent the individuals and the communities are described by the links of same color.

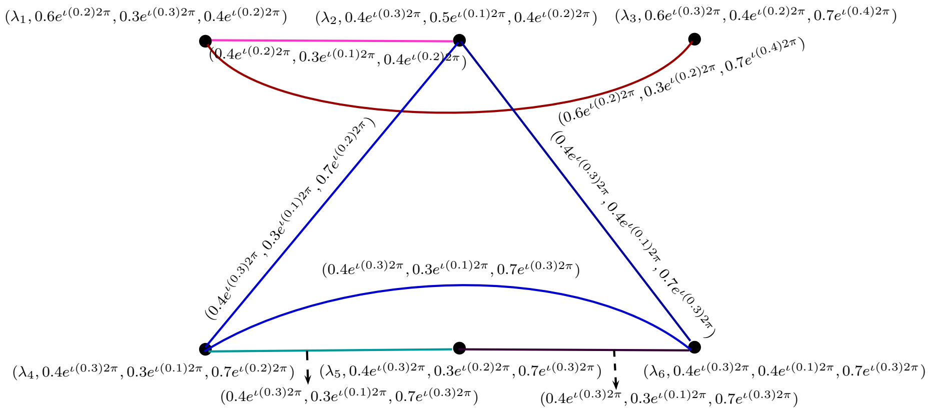


**Figure 9.** Line graph of intersecting research communities.

This line graph represents the relationships between researchers. The researchers that belong to the community $F_3$ are connected through pink edge, members of $F_1$ are linked by red edge, members of $F_{10}$ are connected by purple links, cyan and blue edges are used to represent the relation between the members of $F_5$ and $F_8$, respectively. The absence of $F_2$, $F_4$, $F_6$, $F_7$, and $F_9$ in the above graph shows that these communities share no common researchers as their members. The membership degrees of each edge of this line graph represent the collective work of corresponding researchers. The score functions and choice values of a CNG are given as,

$$\begin{aligned}
S_{jk} &= \frac{1}{2}[1 + t_{jk} - 2i_{jk} - f_{jk}] + \frac{1}{4\pi}[2\pi + \phi_{jk} - 2\varphi_{jk} - \psi_{jk}],\\
C_j &= \sum_k S_{jk} + \frac{1}{2}[1 + t_j - 2i_j - f_j] + \frac{1}{4\pi}[2\pi + \phi_j - 2\varphi_j - \psi_j],
\end{aligned}$$

respectively. The score functions and choice values of researchers represented by the line graph given in Figure 9 are calculated in Table 5.

**Table 5.** Score and choice values of complex neutrosophic line graph.

| $S_{jk}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ | $\lambda_6$ | $C_j$ |
|---|---|---|---|---|---|---|---|
| $\lambda_1$ | 0 | 0.600 | 0.350 | 0 | 0 | 0 | 0.450 |
| $\lambda_2$ | 0.600 | 0 | 0 | 0.500 | 0 | 0.350 | 0.900 |
| $\lambda_3$ | 0.350 | 0 | 0 | 0 | 0 | 0 | −0.350 |
| $\lambda_4$ | 0 | 0.500 | 0 | 0 | 0.450 | 0.450 | 0.900 |
| $\lambda_5$ | 0 | 0 | 0 | 0.450 | 0 | 0.450 | 0.200 |
| $\lambda_6$ | 0 | 0.350 | 0 | 0 | 0.450 | 0 | −0.050 |

The choice values of Table 5 show that $\lambda_2$ and $\lambda_4$ are the most active and efficient participants of these research communities. Also, the score values show that $\lambda_1$ and $\lambda_2$ are the members with the strongest interactions between them and can share the most fruitful ideas relevant to their corresponding research fields being the participants of intersecting research communities. The procedure adopted in our application is described in Algorithm 1.

**Algorithm 1:** Selection of a systematic member from intersecting research communities

1. Input the set of vertices (research communities) $F_1, F_2, \cdots, F_j$.
2. Input the CNS $N$ of vertices such that $N(F_k) = (t_k e^{\iota \phi_k}, i_k e^{\iota \varphi_k}, f_k e^{\iota \psi_k}), 1 \leq k \leq j$, $0 \leq t_k + i_k + f_k \leq 3$.
3. Input the number of hyperedges (researchers) $r$ of a CNHG $\mathcal{H} = (\mathcal{N}, \lambda)$.
4. Input the membership degrees of the hyperedges $E_1, E_2, \cdots, E_r$.
5. Construct a complex neutrosophic line graph $l(\mathcal{H}) = (\mathcal{N}_l, \lambda_l)$ whose vertices are the $r$ hyperedges $E_1, E_2, \cdots, E_n$ such that $\mathcal{N}_l(E_n) = \lambda(E_n)$.
6. If $|supp(\lambda_j) \cap supp(\lambda_k)| \geq 1$, then draw an edge between $E_j$ and $E_k$ and $\lambda_l(E_j E_k) = (\min\{t_\lambda(E_j), t_\lambda(E_k)\} e^{\iota \min\{\phi_\lambda(E_j), \phi_\lambda(E_k)\}}, \min\{i_\lambda(E_j), i_\lambda(E_k)\} e^{\iota \min\{\varphi_\lambda(E_j), \varphi_\lambda(E_k)\}}, \max\{f_\lambda(E_j), f_\lambda(E_k)\} e^{\iota \max\{\psi_\lambda(E_j), \psi_\lambda(E_k)\}})$.
7. Input the adjacency matrix $I = [(t_{mn}, i_{mn}, f_{mn})]_{r \times r}$ of vertices of complex neutrosophic line graph $l(\mathcal{H})$.
8. **do** $m$ from $1 \rightarrow r$
9. $C_m = 0$
10. **do** $n$ from $1 \rightarrow r$
11. $S_{mn} = \frac{1}{2}[1 + t_{mn} - 2i_{mn} - f_{mn}] + \frac{1}{4\pi}[2\pi + \phi_{mn} - 2\varphi_{mn} - \psi_{mn}]$
12. $C_m = C_m + S_{mn}$
13. **end do**
14. $C_m = C_m + \frac{1}{2}[1 + t_m - 2i_m - f_m] + \frac{1}{4\pi}[2\pi + \phi_m - 2\varphi_m - \psi_m]$
15. **end do**
16. The vertex with highest choice value in $l(\mathcal{H})$ is the most effective researcher among all the participants.

### *4.2. Influence of Modern Teaching Strategies on Educational Institutes*

Teaching strategies are defined as the methods, techniques, and procedures that an educational institute use to improve its performance. An educational institute can be judged according to its inputs and outputs that are highly affected through the teaching techniques adopted by that institute. Traditional teaching methods mainly depends on textbooks and emphasizes on basic skills while the modern techniques are based on technical approach and emphasizes on creative ideas. Thus, modern teaching is very important and most effective in this technological era. Presently, educational institutes are modified through modern teaching strategies to enhance their outputs and these modern techniques play a vital role for teachers to explain the concepts in more effective and radiant manner.

Here, we consider a CNHG model $\mathcal{H} = (\mathcal{N}, \lambda)$ to study the influence of modern teaching methods on a specific group of institutes in a time frame of 12 months. The vertices of a CNHG represent the different teaching strategies and these techniques are grouped through an hyperedge if they are applied in the same institute. Since more than one institute can adopt a same strategy so the intersecting communities occur in this case. Each strategy is different form the other in terms of its positive, neutral, and negative impacts on students. The truth, indeterminacy, and falsity degrees of each strategy represent the positive, neutral, and negative effects of the corresponding technique on some institute during the time period of 12 months. The indeterminate information about modern teaching strategies with periodic nature is given in Table 6.

**Table 6.** Impacts of modern teaching strategies.

| Teaching Strategy | Positive Effects | Neutral Behavior | Negative Effects |
|---|---|---|---|
| Brain storming | $0.8e^{\iota(10/12)2\pi}$ | $0.7e^{\iota(7/12)2\pi}$ | $0.1e^{\iota(1/12)2\pi}$ |
| Micro technique | $0.6e^{\iota(4/12)2\pi}$ | $0.6e^{\iota(3/12)2\pi}$ | $0.1e^{\iota(1/12)2\pi}$ |
| Mind map | $0.6e^{\iota(6/12)2\pi}$ | $0.3e^{\iota(5/12)2\pi}$ | $0.7e^{\iota(7/12)2\pi}$ |
| Cooperative learning | $0.8e^{\iota(10/12)2\pi}$ | $0.7e^{\iota(7/12)2\pi}$ | $0.1e^{\iota(1/12)2\pi}$ |
| Dramatization | $0.5e^{\iota(3/12)2\pi}$ | $0.3e^{\iota(3/12)2\pi}$ | $0.2e^{\iota(2/12)2\pi}$ |
| Educational software | $0.8e^{\iota(10/12)2\pi}$ | $0.3e^{\iota(3/12)2\pi}$ | $0.2e^{\iota(1/12)2\pi}$ |

Please note that the membership values $(0.8e^{\iota(10/12)2\pi}, 0.7e^{\iota(7/12)2\pi}, 0.1e^{\iota(1/12)2\pi})$ of brainstorming show that this teaching technique has positive influence with degree 0.8 and this effect spreads over ten months, the indeterminacy value represents the neutral effect or indeterminate behavior with degree 0.7 with time interval of seven months, and the falsity degree 0.1 illustrates some negative effects of this strategy that spreads over one month. Similarly, the effects of all other strategies can be seen form Table 6 along with their time periods. An hyperedge of a CNHG represent some institute in which the corresponding techniques are applied. The model of CNHG grouping these strategies is shown in Figure 10.

Here, each hyperedge represents an institute which groups the strategies adopted by that institute and the membership degrees of each hyperedge represent the combined effects of teaching strategies on corresponding institute. We now want to find a strategy or a collection of those techniques which are easy to apply, less in cost, and have higher positive effects on the performance of educational institutes. To find such methods, we calculate the minimal transversal of CNHG given in Figure 10 using Algorithm 2.

**Algorithm 2:** Find a minimal complex neutrosophic transversal

1. Input the CNSs $\lambda_1, \lambda_2, \cdots, \lambda_r$ of hyperedges.
2. Input the membership degrees of hyperedges.
3. **do** $j$ from $1 \to r$
4. $S_j = \lambda_j^{h(\lambda_j)}$
5. $S = S \cup S_j$
6. **end do**
7. Take $\tau$ as the CNSS with support $S$.

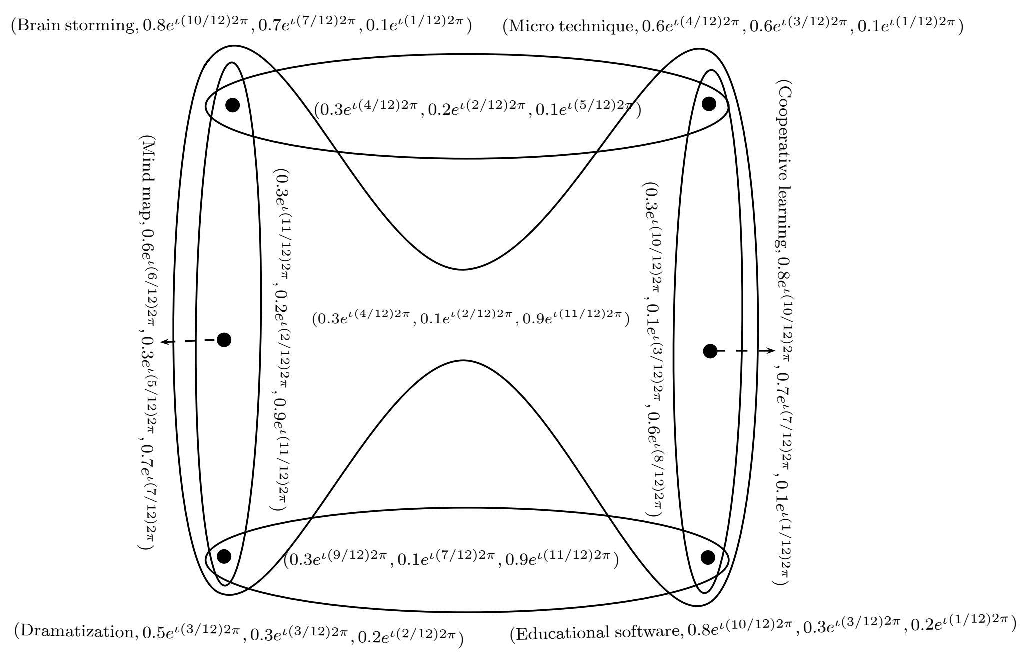


**Figure 10.** Complex neutrosophic hypergraph model of modern teaching strategies.

By following Algorithm 2, we construct a minimal CNT of $\mathcal{H} = (\mathcal{N}, \lambda)$.

We have five hyperedges $E_1, E_2, E_3, E_4, E_5$ of $\mathcal{H}$. The heights of all complex neutrosophic hyperedges are given as,

$$h(\lambda_1) = (0.8e^{\iota(10/12)2\pi}, 0.7e^{\iota(7/12)2\pi}, 0.1e^{\iota(1/12)2\pi}),\ \lambda_1^{h(\lambda_1)} = \{\text{Brain storming}\},$$
$$h(\lambda_2) = (0.7e^{\iota(10/12)2\pi}, 0.6e^{\iota(7/12)2\pi}, 0.1e^{\iota(1/12)2\pi}),\ \lambda_2^{h(\lambda_2)} = \{\text{Brain storming}\},$$
$$h(\lambda_3) = (0.8e^{\iota(10/12)2\pi}, 0.3e^{\iota(3/12)2\pi}, 0.2e^{\iota(1/12)2\pi}),\ \lambda_3^{h(\lambda_3)} = \{\text{Educational software}\},$$
$$h(\lambda_4) = (0.8e^{\iota(10/12)2\pi}, 0.7e^{\iota(7/12)2\pi}, 0.1e^{\iota(1/12)2\pi}),\ \lambda_4^{h(\lambda_4)} = \{\text{Cooperative learning}\},$$
$$h(\lambda_5) = (0.8e^{\iota(10/12)2\pi}, 0.7e^{\iota(7/12)2\pi}, 0.1e^{\iota(1/12)2\pi}),\ \lambda_5^{h(\lambda_5)} = \{\text{Brainstorming, Cooperative learn.}\}.$$

$$\begin{aligned} S &= \lambda_1^{h(\lambda_1)} \cup \lambda_2^{h(\lambda_2)} \cup \lambda_3^{h(\lambda_3)} \cup \lambda_4^{h(\lambda_4)} \cup \lambda_5^{h(\lambda_5)} \\ &= \{\text{Brainstorming, Cooperative learning, Educational software}\}. \end{aligned}$$

The CNS with support $S$ is given as,

$$\{(\text{Brain storming}, 0.8e^{\iota(10/12)2\pi}, 0.7e^{\iota(7/12)2\pi}, 0.1e^{\iota(1/12)2\pi}), (\text{Cooperative learning}, 0.8e^{\iota(10/12)2\pi}, 0.7e^{\iota(7/12)2\pi}, 0.1e^{\iota(1/12)2\pi}), (\text{Educational software}, 0.8e^{\iota(10/12)2\pi}, 0.3e^{\iota(3/12)2\pi}, 0.2e^{\iota(1/12)2\pi})\},$$

which is the minimal CNT of $\mathcal{H} = (\mathcal{N}, \lambda)$ and it shows that brainstorming, cooperative learning, and educational software are the most influential teaching strategies for the given period of time. Thus, for some certain period of time, an influential and effective collection of modern teaching techniques can be determined.

## 5. Comparative Analysis

A CNS is characterized by truth, indeterminacy, and falsity degrees which are the combination of real-valued amplitude terms and complex-valued phase terms. To prove the flexibility and generalization of our proposed model CNHGs, we propose the modeling of social networks through CNGs, CFHGs, and CIFHGs. Consider a part of the social network as described in Section 4.2. Here, we consider only three modern techniques that are brainstorming, cooperative learning, and educational software. A CFHG model of these techniques is given in Figure 11.

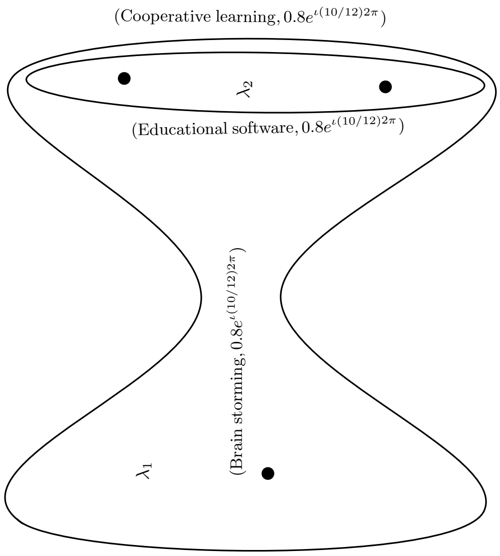


**Figure 11.** Complex fuzzy hypergraph model of teaching techniques.

Please note that a CFHG model of intersecting techniques just illustrates the positive effects of these methods during a specific time interval. We see that a CFHG model fails to describe the negative effects of teaching strategies. To describe the positive as well as negative effects of these strategies, we use a CIFHG model as shown in Figure 12.

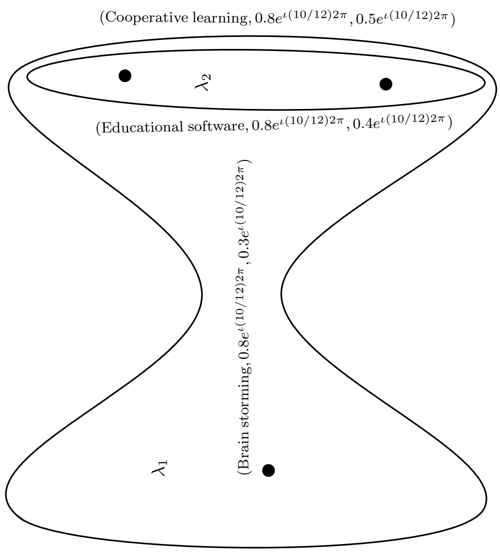


**Figure 12.** Complex intuitionistic fuzzy hypergraph model of teaching techniques.

This shows that a CIFHG model can well describe the positive and negative impacts of modern techniques on educational institutes but it cannot handle the situations when there is no effect during some time interval or there is indeterminate behavior. To handle such type of situations, we use a complex neutrosophic model as shown in Figure 13.

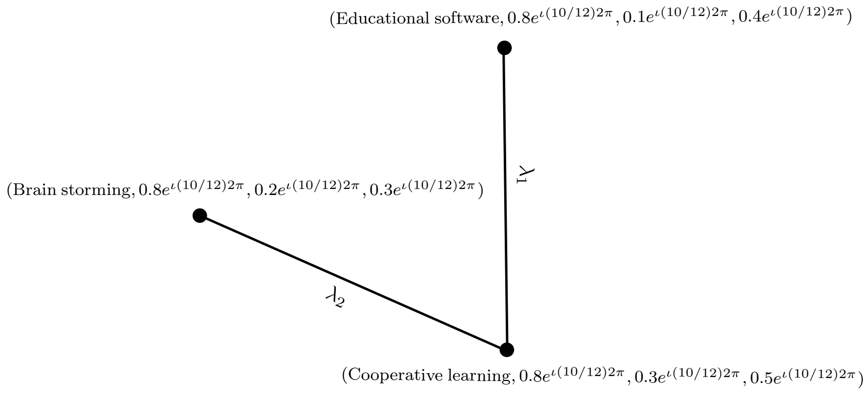


**Figure 13.** Complex neutrosophic graph model of modern techniques.

Please note that a CNG model describe the truth, indeterminacy, and falsity degrees of impacts of teaching methods for some specific interval of time and proves to be a more generalized model as compared to CF and CIF models. Figure 13 shows that $\lambda_1$ institute adopts the modern methods such as educational software and cooperative learning. Now, if an institute wants to use more than two strategies then this model fails to model the required situation. For example, $\lambda_1$ wants to adopt the all three modern teaching techniques. Then, we cannot model this social network using a simple graph. To handle such type of difficulties, i.e., for the modeling of indeterminate information with periodic nature existing in social hypernetworks, we have proposed CNHGs. The applicability and flexibility of our proposed model can be seen from Table 7.

**Table 7.** Comparative analysis.

| Models | Edges | Hyperedge Containing Three Strategies | Positive Effect | Neutral Behavior | Negative Effect |
|---|---|---|---|---|---|
| CFHG model | $\lambda_1$ | {Brain storming,<br>Cooperative learning,<br>Educational software} | $0.8e^{\iota(10/12)2\pi}$<br>$0.8e^{\iota(10/12)2\pi}$<br>$0.8e^{\iota(10/12)2\pi}$ | -<br>-<br>- | -<br>-<br>- |
| CIFHG model | $\lambda_1$ | {Brain storming,<br>Cooperative learning,<br>Educational software} | $0.8e^{\iota(10/12)2\pi}$<br>$0.8e^{\iota(10/12)2\pi}$<br>$0.8e^{\iota(10/12)2\pi}$ | -<br>-<br>- | $0.3e^{\iota(10/12)2\pi}$<br>$0.5e^{\iota(10/12)2\pi}$<br>$0.4e^{\iota(10/12)2\pi}$ |
| CNG model | Cannot combine more than two elements | -<br>-<br>- | $0.8e^{\iota(10/12)2\pi}$<br>$0.8e^{\iota(10/12)2\pi}$<br>$0.8e^{\iota(10/12)2\pi}$ | $0.2e^{\iota(10/12)2\pi}$<br>$0.3e^{\iota(10/12)2\pi}$<br>$0.1e^{\iota(10/12)2\pi}$ | $0.3e^{\iota(10/12)2\pi}$<br>$0.5e^{\iota(10/12)2\pi}$<br>$0.4e^{\iota(10/12)2\pi}$ |
| CNHG model | $\lambda_1$ | {Brain storming,<br>Cooperative learning,<br>Educational software} | $0.8e^{\iota(10/12)2\pi}$<br>$0.8e^{\iota(10/12)2\pi}$<br>$0.8e^{\iota(10/12)2\pi}$ | $0.2e^{\iota(10/12)2\pi}$<br>$0.3e^{\iota(10/12)2\pi}$<br>$0.1e^{\iota(10/12)2\pi}$ | $0.3e^{\iota(10/12)2\pi}$<br>$0.5e^{\iota(10/12)2\pi}$<br>$0.4e^{\iota(10/12)2\pi}$ |

## 6. Discussions

It can be seen clearly from Table 7 that all existing models, including CNGs, CFHGs, and CIFHGs lack some information to handle the periodic and indeterminate data in case of hypernetworks. Table 7 shows that a CFHG model can illustrate the combine effects of three different techniques through a hyperedge. The truth degrees $0.8e^{\iota(10/12)2\pi}$ of these techniques show that these methods provide very good influence which spread over ten months but this model fails to describe the negative effects of some teaching technique happening periodically. A CIFHG model is then used to overcome such type of deficiencies. The falsity degree $0.4e^{\iota(10/12)2\pi}$ of "educational software" shows that this technique has some negative effects that spread over ten months. The failure of CIFHG model appears when neither positive nor negative effects or neutral effects of periodic nature are experienced because some information does not have only truth and falsity degrees but also some indeterminacy degrees which are independent of each other. For example, a $20^o$ temperature in summer means a cool day and in winter means a warm day but neither cool nor warm day in spring. This phenomenon indicates that some real-life situations may have indeterminacy and periodicity along with uncertainty. To handle such type of phenomena, a CNS model is more flexible and applicable. As we have seen from Table 7 that a CNG illustrates the positive and negative as well as indeterminate effects of under consideration teaching strategies applied to different institutes. The membership degrees $(0.8e^{\iota(10/12)2\pi}, 0.2e^{\iota(10/12)2\pi}, 0.3e^{\iota(10/12)2\pi})$ show that some particular technique has 0.8 positive effects, 0.2 neutral effects, and 0.3 negative effects on some institute and all these effects spread over ten months. The main drawback of a CNG model is that a single edge can connect only two vertices, i.e., if we consider the teaching techniques as vertices and these vertices (techniques) are connected through an edge if they are adopted by a same institute. Then, a CNG model cannot illustrate the situation when more than two techniques are applied to a single institute. In modeling of such type of hypernetworks with indeterminacy of periodic nature, we propose a CNHG model. It can be seen clearly from Table 7 that our proposed model is more generalized framework as it does not only deal the reductant nature of imprecise information but also includes the benefits of hypergraphs. Hence, a CNHG model combines the fruitful effects of CNSs and hypergraph theory.

## 7. Conclusions and Future Directions

A CNS extends the concept of SVNS from real unit interval $[0, 1]$ to the complex plane and is used to represent two-dimensional imprecise and indeterminate information. A CNS plays a vital role in modeling the real-life applications where the truth, indeterminacy, and falsity degrees of given data are periodic in nature. Thus, a CNS is more effective and generalized framework to deal the periodic nature of indeterminacy where the CFS and CIFS fail. For example, a wave particle such as an electron can be in two different positions at the same time and the CFS is not able to deal with this phenomenon. A CIFS can only represent the information involving the information of the type: "yes" or "no" occurring periodically. These models fail to deal the information that is neither true nor false or true and false at the same time. A CNS model is more effectively used to deal such type of situations in our daily life. In this paper, we have defined CNHGs which generalize the concepts of CFHGs and CIFHGs. We have studied the level hypergraphs, lower truncation, upper truncation, and transition levels of CNHGS. Furthermore, we have defined $T$-related CNHGs and discussed their certain properties. We have illustrated the proposed ideas through some concrete examples. Moreover, we have presented the modeling of certain social networks with intersecting communities using CNHGs. We have determined a strong participant in overlapping research communities by defining the score and choice values of CNGs. We have also determined the collection of most influential teaching strategies using the minimal transversals of CNHGs. Finally, we have proved the novelty and applicability of this work by giving a brief comparison of our proposed model with other existing models. We have seen that the main drawback of CFHG models is that they cannot deal the falsity and indeterminate information existing in a periodic manner. Similarly, a CIFHG fails to handle the situations when the indeterminate and inconsistent information is happening repeatedly. The proposed analysis proved the dominance of CNHG model to all other existing models by comparing the applicability of CFHGs, CIFHGs, CNGs, and CNHGs using numeric examples as well as some theoretic results.

We aim to broaden our study to (1) Complex bipolar fuzzy hypergraphs, (2) Complex bipolar neutrosophic hypergraphs, (3) Complex fuzzy soft hypergraphs and (4) Complex Pythagorean fuzzy soft hypergraphs.

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