

Article

A Multi-Attribute Decision-Making Method Based on Weighted Hesitant Fuzzy Linguistic Frank Aggregation Operators with an Application to Regional Twinning Support Evaluation

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Abstract

Hesitant fuzzy linguistic sets (HFLSs) combine natural language expressions with membership information in a hesitant manner, thereby breaking the barriers between fuzzy evaluation and natural language-based decision-making. Hence, they have become a crucial tool for constructing linguistic decision-making frameworks. However, HFLSs treat all linguistic expressions in the evaluation system as equally important, which reduces the accuracy of linguistic decision models. In practical situations, decision makers usually assign different weights to different linguistic descriptions. Therefore, in this paper, we propose a generalization of HFLSs called the weighted hesitant fuzzy linguistic term set (WHFLTS) to represent the uncertain subjective preference information of experts, and design a multiple-attribute decision-making procedure. Firstly, the definition of WHFLTS is given. Then, under the Frank operational law, the algebraic and geometric aggregation operators for weighted hesitant fuzzy linguistic information are discussed: namely, the WHFLFWA operator and the WHFLFWG operator. On this basis, a linguistic decision-making strategy is proposed. The example presented in this paper shows that WHFLTS can model uncertain problems in a more adequate way, and the Frank aggregation operators defined in this paper can reflect the decision makers' attitudes, making them more broadly applicable in decision-making problems.

Keywords: quantitative evaluation information; Frank t-norm; Frank t-conorm; weighted hesitant fuzzy linguistic term set (WHFLTS); information aggregating operator

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1. Introduction

Multi-attribute decision-making (MADM) refers to the process in which decision makers select the most satisfactory alternative from a set of schemes with diverse attribute values. Limited and ambiguous cognition of decision makers together with the diversity and complexity of attributes often make it difficult to obtain optimal decisions. In recent years, rough set theory has been extended toward more flexible approximation structures to enhance uncertainty modeling. For example, the initial-neighborhood rough set model proposed in [1] improves approximation accuracy by introducing novel neighborhood

systems, while the tritopological approximation framework developed in [2] further strengthens information reduction and knowledge representation through multiple topological structures. These developments indicate that refining approximation mechanisms is an effective way to handle incomplete information in decision environments. Nevertheless, in many practical decision-making problems, uncertainty also stems from the hesitation of decision makers when assigning evaluation values. Hesitant fuzzy sets (HFSs) characterize this behavior by allowing multiple possible membership degrees within a single evaluation, thereby providing a more expressive tool for modeling hesitation in scheme assessment [3]. They enhance flexibility in handling uncertain information and better capture the nature of uncertainty, evolving into an important research branch of fuzzy set theory and its applications. Due to their flexibility and practicality in decision modeling, and in order to further expand their application scope, researchers have proposed several extensions of HFSs [4–9]. However, most of these extended forms are primarily applicable to numerical decision-making problems. In many multi-attribute decision-making (MADM) problems, evaluators are required to provide qualitative assessment information. This arises for several reasons. One is that, due to data involving enterprise (or state) secrets or digital privacy, precise quantitative information cannot be obtained. For example, expressions such as “very many”, “many”, and “few” are commonly used to describe the number of ballistic missiles in a country. Another reason is that the evaluation object itself is difficult to quantify. For instance, terms such as “very excellent”, “ordinary”, and “excellent” are often used to assess the innovation capability of an enterprise. To overcome the limitation that linguistic term evaluation values can only be used individually, Rodríguez et al. introduced the construction method of HFSs into linguistic decision models and proposed the hesitant fuzzy linguistic term set (HFLTS) [9].

Significant progress in theory and applications has been achieved since the introduction of HFLTS. Liao et al. further enriched the theory of HFLTS by defining distance and similarity measures, which facilitate the application of classical MADM methods within the HFLTS context [10]. Based on the VIKOR method, Faizi et al. proposed a multi-attribute hesitant intuitionistic fuzzy linguistic decision model [11]. Cheng et al. combined the fuzzy allocation method with the maximum entropy ordered weighted averaging (OWA) method for HFLTSs to construct MADM models [12]. In recent years, several aggregation operators, including the Heronian mean [13], Hamy mean [14], power geometric mean [15], prioritized Choquet aggregation operator [16], and Maclaurin symmetric mean operator [17], have been extended to aggregate hesitant fuzzy linguistic information.

However, the HFLTS considers that the weights of the linguistic terms provided for the evaluation objects are the same. This may lead to a loss of effective information in the decision-making process. For example, ten experts with the same level of expertise are invited to evaluate the financial risk of an enterprise by using a discrete linguistic term set $S = \{s_0 : \text{extremely dangerous}, s_1 : \text{dangerous}, s_2 : \text{medium}, s_3 : \text{safe}, s_4 : \text{extremely safe}\}$. The evaluation information submitted by the expert panel is not entirely the same: three experts provide s_2 , one expert provides s_3 , and six experts provide s_4 , meaning these ten experts are unable to reach a unified evaluation opinion. If HFLTS is used, then the financial risk measuring value of the company is $\{s_2, s_3, s_4\}$. This means that the choice of s_4 given by the majority of experts and the evaluation information s_3 given by the minority have the same importance, which is obviously contrary to common sense. In addition, due to differences in the evaluators' knowledge levels and familiarity with the evaluated objects, the evaluation values they provide may not be consistent. At this point, assigning different weights to the evaluation information based on the evaluators' knowledge levels and familiarity with the evaluated objects would be more reasonable.

For example, two experts, x_1 and x_2 , are invited to evaluate the financial risk of an enterprise. They have the same professional level, but x_1 is more experienced in financial risk evaluation and more familiar with this enterprise than x_2 . x_1 provides s_3 , and x_2 provides s_4 , and it is difficult for them to persuade each other to change their opinions. If HFLTS is used, then the financial risk measuring value of the company is $\{s_3, s_4\}$, which ignores the problem of information credibility.

To address this, this paper designs a multi-valued evaluation tool called the weighted hesitant fuzzy linguistic set (WHFLS), which allows for differentiated assignment of weights to linguistic terms during decision modeling. The WHFLS utilizes a linguistic term vector to represent the credibility of the preference information, so as to avoid subjective randomness and enhance information reliability in decision-making. In the first financial risk evaluation example, the experts' preference information can be represented by a WHFLS $\{(s_2, 0.3), (s_3, 0.1), (s_4, 0.6)\}$; in the second example, the preference information can be represented by a WHFLS $\{(s_3, w_1), (s_4, w_2)\}$, where $w_i (i=1, 2)$ are the weights of information credibility.

In this paper, we will define a new multiple-valued fuzzy linguistic set called the weighted hesitant fuzzy linguistic term set (WHFLTS), and then introduce some Frank aggregation operators to solve the MADM problem. The evaluation problem is a vast and complex system that has long been of concern to scholars [18–23]. To this end, the framework of this paper is structured as follows: Section 2 reviews some necessary concepts which form the theoretical foundation of this paper. In Sections 3 and 4, some aggregation operators and a decision-making scheme for the WHFLTS are designed, respectively. In Section 5, a risk assessment example is used to illustrate the validity of the designed scheme. Finally, some conclusions are presented in the last section.

2. Preliminaries

2.1. Notation List

To facilitate reading and understanding of the proposed approach, the primary symbols and their corresponding definitions used in this paper are summarized in Table 1.

Table 1. Summary of notations.

Symbol	Description
$X = \{x_1, x_2, \dots, x_n\}$	Set of alternatives
$G = \{g_1, g_2, \dots, g_l\}$	Set of criteria
$\omega = (\omega_1, \omega_2, \dots, \omega_l)^T$	Weight vector of criteria
$S = \{s_\alpha \mid \alpha = 0, 1, 2, \dots, t\}$	Predefined linguistic term set
$H_{S,W}$	A weighted hesitant fuzzy linguistic term set
$h_{S,W}$	A weighted hesitant fuzzy linguistic element
$\eta(h_{S,W})$	Priority degree function of $h_{S,W}$
$\nu(h_{S,W})$	Fluctuation function of $h_{S,W}$
$\oplus_F$	Frank sum
$\otimes_F$	Frank product
$\cdot_F$	Frank scalar multiplication
$\wedge_F$	Frank power operation
Ave	Arithmetic average
WHFLTS	Weighted hesitant fuzzy linguistic term set
WHFLE	Weighted hesitant fuzzy linguistic element
WHFLFWA	Weighted hesitant fuzzy linguistic Frank weighted averaging

2.2. Hesitant Fuzzy Linguistic Term Sets

Let $s_\alpha < s_\beta$, $s_\alpha < s_\beta$, $s_\alpha < s_\beta$, $s_\alpha < s_\beta$ be a linguistic term set, where $s_\alpha < s_\beta$ is a positive odd integer and s_α represents a possible value for a linguistic variable, satisfying the following requirements:

- (1) $s_\alpha < s_\beta$ if $\alpha < \beta$;
- (2) Negation operator: $N(s_i) = s_{t-i}$.

In order to simplify the calculations of the decision-making model, Xu further proposed to treat discrete linguistic variables as continuous ones, and then defined a continuous set $\bar{S} = \{s_\alpha | s_\alpha \in [0, p]\}$, $p(p > t)$ [10].

Let $s_\alpha, s_\beta \in \bar{S}$ and $\lambda, \lambda_1, \lambda_2 > 0$. Xu defined the following operation principles [10]:

- (1) $s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha = s_{\alpha+\beta}$; (2) $\lambda s_\alpha = s_{\lambda\alpha}$; (3) $(\lambda_1 + \lambda_2)s_\alpha = \lambda_1 s_\alpha \oplus \lambda_2 s_\alpha$;
- (4) $s_\alpha \otimes s_\beta = s_\beta \otimes s_\alpha = s_{\alpha\beta}$; (5) $(s_\alpha)^\lambda = s_{(\alpha)^\lambda}$.

To address the limitation that the single linguistic term expression mechanism in conventional linguistic decision-making fails to adapt to complex environments, Rodriguez et al. extended fuzzy set theory and proposed hesitant fuzzy linguistic term sets (HFLTSS). This concept abandons the accuracy assumption imposed on decision makers by traditional methods, accommodates their hesitation among multiple linguistic concepts, and enables preference modeling based on continuous linguistic term sets. It serves as a more flexible and practical mathematical tool for expressing cognitive uncertainty in complex decision-making scenarios.

Definition 1. [8] Let X be fixed and $S = \{s_\alpha | \alpha = 0, 1, 2, \dots, t\}$ be a linguistic term set. Then,

$$H_S = \{ \langle x_i, h_S(x_i) \rangle | x_i \in X, i = 1, 2, \dots, n \}$$

is called a hesitant fuzzy linguistic term set on X , where S is a subset of S and is commonly referred to as the hesitant fuzzy linguistic element (HFLE).

2.3. Weighted Hesitant Fuzzy Linguistic Term Set

Definition 2. Let X be a universe of discourse and $S = \{s_\alpha | \alpha = 0, 1, 2, \dots, t\}$ be a linguistic term set. The weighted hesitant fuzzy linguistic term set (WHFLTS) $H_{S,w}$ on X is defined by

$$H_{S,w} = \{ \langle x_i, h_{S,w}(x_i) = (h_S(x_i), w(x_i)) \rangle | x_i \in X \} = \{ \langle x_i, \bigcup_{s_\beta \in h_S(x_i)} \{(s_\beta, w(s_\beta))\} \rangle | x_i \in X \}$$

where $h_S(x_i)$ is an ordered linguistic term set of some values in S , and $w(s_\beta)$ is the weight of s_β such that $0 \leq w(s_\beta) \leq 1$, $\sum_{s_\beta \in h_S(x_i)} w(s_\beta) = 1$. For convenience, $h_{S,w}(x_i) = (h_S(x_i), w(x_i))$ is called a weighted hesitant fuzzy linguistic element (WHFLE). A WHFLE is denoted by $h_{S,w} = (h_S, w)$.

Remark 1. (1) If we take $w(s_\beta) = \frac{1}{l(h_S(x_i))}$, $\forall s_\beta \in h_S(x_i)$, where $l(h_S(x_i))$ denotes the number of elements in $h_S(x_i)$, then the WHFLTS reduces to a typical HFLTS; (2) furthermore, if $l(h_S(x_i)) = 1$, then the WHFLTS reduces to a typical LTS.

Making use of the WHFLTS implies the introduction of the importance of membership degrees into the set description. Such a generalization of HFLTS enriches the methodology and methods of representing fuzzy information, which may lead to modeling uncertainty problems in a more adequate way. It is easy to see from Table 1 that only the

WHFLTS defined in this paper can reflect all the information factors. In practical applications, the number of linguistic terms in WHFLTSs is often not the same. For the convenience of information processing, this article assumes that all WHFLTSs have the same number of elements. If the number of elements in $h_{s,w}(x_1)$ is smaller than that in $h_{s,w}(x_2)$, then we can extend $h_{s,w}(x_1)$ by repeatedly adding $(0.5h_s^+(x_1) \oplus 0.5h_s^-(x_1), 0)$ until it has the same length as $h_{s,w}(x_2)$. Here, $h_s^+(x_1)$ and $h_s^-(x_1)$ represent the maximum and minimum linguistic terms in $h_s(x)$, respectively.

Example 1. With the further perfection and development of the market economy, modern enterprises are faced with a fiercely competitive environment, and operational risk management, as a key factor of core competitiveness, has been highly valued. An enterprise hired two separate decision-maker panels to assess the operational situation. The linguistic evaluation scale is defined as $S = \{s_0: \text{very dangerous}, s_1: \text{dangerous}, s_2: \text{slightly dangerous}, s_3: \text{normal}, s_4: \text{slightly safe}, s_5: \text{safe}, s_6: \text{very safe}\}$. There are $L^{(1)}$ experts in the first team; $L_1^{(1)}$ experts believe “very dangerous” (s_0), $L_2^{(1)}$ experts believe “dangerous” (s_1), and $L_3^{(1)}$ experts believe “slightly dangerous” (s_2), where $\sum_{i=1}^3 L_i^{(1)} = L^{(1)}$. The first team provides the evaluation as $h_{s,w}^1 = \{(s_0, \frac{L_1^{(1)}}{L^{(1)}}), (s_1, \frac{L_2^{(1)}}{L^{(1)}}), (s_2, \frac{L_3^{(1)}}{L^{(1)}})\}$. The second team has $L^{(2)}$ experts; $L_1^{(2)}$ experts believe “dangerous” (s_1), and $L_2^{(2)}$ experts believe “slightly dangerous” (s_2), where $\sum_{i=1}^2 L_i^{(2)} = L^{(2)}$. The second team provides the evaluation as $h_{s,w}^2 = \{(s_1, \frac{L_1^{(2)}}{L^{(2)}}), (s_2, \frac{L_2^{(2)}}{L^{(2)}})\}$. To make $h_{s,w}^1$ and $h_{s,w}^2$ have the same length, $h_{s,w}^2$ is extended to $\bar{h}_{s,w}^2 = \{(s_1, \frac{L_1^{(2)}}{L^{(2)}}), (s_{1.5}, 0), (s_2, \frac{L_2^{(2)}}{L^{(2)}})\}$.

To better utilize the WHFLTS in fuzzy information processing, the binary relation of WHFLTs should be defined.

Definition 3. Let $S = \{s_\alpha \mid \alpha = 0, 1, 2, \dots, t\}$ be a linguistic term set, and let $h_{s,w} = \bigcup_{s_\beta \in h_s(x_i)} \{(s_\beta, w(s_\beta))\}$, $h_{s,w}^1 = \bigcup_{s_\beta^1 \in h_s^1(x_i)} \{(s_\beta^1, w(s_\beta^1))\}$ and $h_{s,w}^2 = \bigcup_{s_\beta^2 \in h_s^2(x_i)} \{(s_\beta^2, w(s_\beta^2))\}$ are three WHFLEs. $\eta(h_{s,w}) = \sum_{s_\beta \in h_s(x_i)} \beta w(s_\beta)$ is called the priority degree function of $h_{s,w}$, and

$\nu(h_{s,w}) = \sqrt{\sum_{s_\beta \neq s_\rho} \left\{ (w(s_\beta) - w(s_\rho))^2 + \left(\frac{\beta - \rho}{t+1} \right)^2 \right\}}$ is called the fluctuation function of $h_{s,w}$. Then, we define the law to compare two WHFLEs ($h_{s,w}^1$ and $h_{s,w}^2$) as below:

- If $\eta(h_{s,w}^1) > \eta(h_{s,w}^2)$, then $h_{s,w}^1 > h_{s,w}^2$;
- If $\eta(h_{s,w}^1) = \eta(h_{s,w}^2)$, then;
 - (i) If $\nu(h_{s,w}^1) = \nu(h_{s,w}^2)$, then $h_{s,w}^1 = h_{s,w}^2$;
 - (ii) If $\nu(h_{s,w}^1) > \nu(h_{s,w}^2)$, then $h_{s,w}^1 < h_{s,w}^2$.

Example 2. Let $h_{s,w}^1 = \{(s_3, 0.5), (s_4, 0.25), (s_5, 0.25)\}$, $h_{s,w}^2 = \{(s_2, 0.35), (s_4, 0.45), (s_6, 0.2)\}$, $h_{s,w}^3 = \{(s_3, 0.5), (s_4, 0.3), (s_5, 0.2)\}$ be three WHFLEs on linguistic term set $S = \{s_0, s_1, s_2, \dots, s_6\}$. With Definition 3, we can get

$$\eta(h_{s,w}^1) = 3.75, \quad \eta(h_{s,w}^2) = 3.7, \quad \eta(h_{s,w}^3) = 3.7$$

and

$$v(h_{s,w}^1) = 0.4974, \quad v(h_{s,w}^2) = 0.7647, \quad v(h_{s,w}^3) = 0.5123$$

Therefore, $h_{s,w}^1 > h_{s,w}^3 > h_{s,w}^2$.

The choice of priority degree function and fluctuation function as ranking indicators is primarily because they reflect the first-order and second-order statistical characteristics of hesitant fuzzy linguistic term sets, respectively. The priority degree function corresponds to the expected value of the linguistic evaluation distribution, serving to characterize the overall evaluation level, while the fluctuation function corresponds to the dispersion of the distribution, reflecting the uncertainty of the evaluation results. When multiple alternatives have identical scores, the fluctuation function is introduced to further distinguish their stability and consistency. Consequently, a ranking mechanism based on the priority degree and fluctuation functions can simultaneously account for both evaluation levels and uncertainty information, ensuring computational simplicity while retaining a significant amount of decision-making information. It should be noted that different ranking mechanisms may yield varying results. This paper adopts the priority degree and fluctuation functions ranking mechanism primarily due to its clear theoretical foundation, computational simplicity, which is also a commonly used approach for ranking linguistic term sets in the current literature.

2.4. Comparison of Language Representation Models

The weighted hesitant fuzzy linguistic model introduces an explicit weighting mechanism into the hesitant linguistic framework. Each linguistic assessment is associated with a corresponding importance weight, enabling simultaneous representation of hesitation and preference heterogeneity. This dual-layer structure significantly improves the realism of decision modeling, as it allows dominant or more reliable judgments to exert greater influence during aggregation. As a result, the weighted framework effectively reduces the over-smoothing effect commonly observed in equal-treatment aggregation strategies and enhances both the robustness and interpretability of final decision outcomes. The comparative characteristics of representative linguistic decision models are summarized in Table 2. It can be observed that while traditional fuzzy linguistic and hesitant fuzzy linguistic models provide foundational representational capabilities, they either suffer from limited uncertainty expression or lack mechanisms for preference differentiation. In contrast, the weighted hesitant fuzzy linguistic model achieves a more balanced trade-off among expressiveness, interpretability, and computational feasibility, making it particularly suitable for complex group decision-making scenarios involving heterogeneous expert reliability.

Table 2. Comparative analysis of linguistic decision models.

Model	Representation Form	Hesitation Modeling	Preference Heterogeneity	Uncertainty Type	Information Richness	Interpretability
Fuzzy linguistic set	Single transformed value	No	No	Fuzziness	Low	Moderate
Interval-valued fuzzy linguistic set	Linguistic interval	Partial	No	Bounded uncertainty	Moderate	Moderate
Hesitant fuzzy linguistic term set	Linguistic set	Yes	No (equal)	Hesitation	High	High
Interval-valued hesitant fuzzy linguistic term set	Set of intervals	Yes	No	Hybrid uncertainty	High	Moderate–High
Weighted hesitant fuzzy linguistic term set	Weighted linguistic set	Yes	Yes (explicit)	Hesitation + preference heterogeneity	Very high	Very high

Overall, linguistic decision models exhibit a progressive enhancement in uncertainty representation capability and aggregation sophistication. While early fuzzy linguistic models provide a basic quantification framework, and hesitant and extended linguistic models improve expressiveness, they still lack an explicit mechanism to distinguish the relative importance of multiple linguistic evaluations. The weighted hesitant fuzzy linguistic model addresses this limitation by introducing a unified weighting structure, enabling simultaneous modeling of hesitation and preference heterogeneity. This leads to significantly improved aggregation rationality, decision stability, and interpretability, making it a more suitable and general framework for complex linguistic multi-criteria group decision-making problems.

2.5. Frank Operations of WHFLEs

Designed to overcome the limitations of algebraic T-norms and S-norms, t-norms and t-conorms—serving as an extension of Archimedean operators with built-in parameter adjustment mechanisms—have become one of the most representative parametric operators in the field of fuzzy information aggregation [21]. The introduction of Frank operators endows multi-attribute decision-making models with flexibility and generalization capabilities that go beyond traditional algebraic operators. Through targeted parameter configuration, these operators can precisely regulate the measure and process of information fusion, resulting in a higher degree of sufficiency and compatibility in the integration of decision information. This characteristic not only broadens their application boundaries in multi-attribute decision-making problems but also provides powerful mathematical tools and diversified research paradigms for the deepening and innovation of multi-criteria decision theory. The Frank operations consist of two parts: the Frank sum (denoted by $\oplus_F$) and the Frank product (denoted by $\otimes_F$), which belong to the t-conorm and t-norm, respectively, as follows:

$$x \oplus_F y = 1 - \log_{\lambda} \left(1 + \frac{(\lambda^{1-x} - 1)(\lambda^{1-y} - 1)}{\lambda - 1} \right), \quad \forall (x, y) \in [0, 1] \times [0, 1] \quad (1)$$

$$x \otimes_F y = \log_{\lambda} \left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1} \right), \quad \forall (x, y) \in [0, 1] \times [0, 1] \quad (2)$$

The Frank operations for WHFLEs are defined as follows.

Definition 4. Let $h_{S,W} = \bigcup_{s_{\beta} \in h_S^1(x_i)} \{(s_{\beta}, w(s_{\beta}))\}$, $h_{S,W}^1 = \bigcup_{s_{\beta}^1 \in h_S^1(x_i)} \{(s_{\beta}^1, w(s_{\beta}^1))\}$, $h_{S,W}^2 = \bigcup_{s_{\beta}^2 \in h_S^2(x_i)} \{(s_{\beta}^2, w(s_{\beta}^2))\}$ be three WHFLEs on linguistic term set $S = \{s_{\alpha} | \alpha = 0, 1, \dots, t\}$, and

$k > 0$, $\lambda > 1$, $l = t + 1$, then,

$$(1) \quad h_{S,W}^1 \oplus_F h_{S,W}^2 = \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \left\{ \left(s_{\frac{l(1-\log_{\lambda}(1+\frac{(\lambda^{1-\beta_1/l}-1)(\lambda^{1-\beta_2/l}-1)}{\lambda-1}))}{\lambda-1}}, \overline{\text{Ave}}(w_{s_{\beta_1}}, w_{s_{\beta_2}}) \right) \right\};$$

$$(2) \quad h_{S,W}^1 \otimes_F h_{S,W}^2 = \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \left\{ \left(s_{\frac{l \log_{\lambda}(1+\frac{(\lambda^{\beta_1/l}-1)(\lambda^{\beta_2/l}-1)}{\lambda-1}))}{\lambda-1}}, \overline{\text{Ave}}(w_{s_{\beta_1}}, w_{s_{\beta_2}}) \right) \right\};$$

$$(3) \quad k \cdot_F h_{S,W} = \bigcup_{s_{\beta} \in h_{S,W}} \left\{ \left(s_{\frac{l(1-\log_{\lambda}(1+\frac{(\lambda^{1-\beta/l}-1)^k}{(\lambda-1)^{k-1}}))}{(\lambda-1)^{k-1}}}, w_{s_{\beta}} \right) \right\};$$

$$(4) \quad h_{S,W}^{\wedge_F k} = \bigcup_{s_{\beta} \in h_{S,W}} \left\{ \left(s_{\frac{l \log_{\lambda}(1+\frac{(\lambda^{\beta/l}-1)^k}{(\lambda-1)^{k-1}})}{(\lambda-1)^{k-1}}}, w_{s_{\beta}} \right) \right\}.$$

Remark 2. $\overline{\text{Ave}}(w_{s_{\alpha_1}}, w_{s_{\alpha_2}})$ is the arithmetic average of $w_{s_{\alpha_1}}$ and $w_{s_{\alpha_2}}$. Suppose $h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n$ are WHFLEs on S , and $\odot$ is a Frank binary operation on WHFLEs. The weight of $h_{S,W}^1 \odot h_{S,W}^2 \odot \dots \odot h_{S,W}^n$ is calculated in two steps: (1) we first calculate $w_{s_{\alpha_1}} + w_{s_{\alpha_2}} + \dots + w_{s_{\alpha_n}}$; (2)

after the membership degrees calculation of $h_{s,w}^1 \odot h_{s,w}^2 \odot \cdots \odot h_{s,w}^n$ are completed, then we get their weights $\frac{w_{s_{\alpha_1}} + w_{s_{\alpha_2}} + \cdots + w_{s_{\alpha_n}}}{n}$.

Theorem 1. Let $h_{s,w} = \bigcup_{s_{\beta} \in h_S^1(x_i)} \{(s_{\beta}, w(s_{\beta}))\}$, $h_{s,w}^1 = \bigcup_{s_{\beta} \in h_S^1(x_i)} \{(s_{\beta}^1, w(s_{\beta}^1))\}$, $h_{s,w}^2 = \bigcup_{s_{\beta} \in h_S^2(x_i)} \{(s_{\beta}^2, w(s_{\beta}^2))\}$ be three WHFLEs on linguistic term set $S = \{s_{\alpha} | \alpha = 0, 1, \dots, t\}$, and $k, k_1, k_2 > 0$, then we have:

- (1) $h_{s,w}^1 \oplus_F h_{s,w}^2 = h_{s,w}^2 \oplus_F h_{s,w}^1$;
- (2) $k \cdot_F (h_{s,w}^1 \oplus_F h_{s,w}^2) = k \cdot_F h_{s,w}^1 \oplus_F k \cdot_F h_{s,w}^2$;
- (3) $(k_1 \cdot_F h_{s,w}) \oplus_F (k_2 \cdot_F h_{s,w}) = (k_1 + k_2) \cdot_F h_{s,w}$;
- (4) $(k_1 k_2) \cdot_F h_{s,w} = k_1 \cdot_F (k_2 \cdot_F h_{s,w})$.

Proof. (1) Obviously, this conclusion has been established.

(2) According to $h_{s,w}^1 \oplus_F h_{s,w}^2 = \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \{(s_{\beta_1}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2})\}$, we have

$$\begin{aligned} k \cdot_F (h_{s,w}^1 \oplus_F h_{s,w}^2) &= k \cdot_F \left(\bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \{(s_{\beta_1}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2})\} \right) \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \{(s_{\beta_1}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2})\} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \{(s_{\beta_1}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2})\} \end{aligned}$$

In addition, based on

$$k \cdot_F h_{s,w}^1 = \bigcup_{s_{\beta_1} \in h_{S,W}^1} \{(s_{\beta_1}, w_{s_{\beta_1}})\} \text{ and } k \cdot_F h_{s,w}^2 = \bigcup_{s_{\beta_2} \in h_{S,W}^2} \{(s_{\beta_2}, w_{s_{\beta_2}})\},$$

we have

$$\begin{aligned} k \cdot_F h_{s,w}^1 \oplus_F k \cdot_F h_{s,w}^2 &= \bigcup_{s_{\beta_1} \in h_{S,W}^1} \{(s_{\beta_1}, w_{s_{\beta_1}})\} \oplus_F \bigcup_{s_{\beta_2} \in h_{S,W}^2} \{(s_{\beta_2}, w_{s_{\beta_2}})\} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \{(s_{\beta_1}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2})\} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \{(s_{\beta_1}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2})\} = k \cdot_F (h_{s,w}^1 \oplus_F h_{s,w}^2). \end{aligned}$$

(3) According to

$$k_1 \cdot_F h_{s,w} = \bigcup_{s_{\beta} \in h_{S,W}} \{(s_{\beta}, w_{s_{\beta}})\} \text{ and } k_2 \cdot_F h_{s,w} = \bigcup_{s_{\beta} \in h_{S,W}} \{(s_{\beta}, w_{s_{\beta}})\},$$

we have

$$\begin{aligned}
 k_1 \cdot_F h_{S,W} \oplus_F k_2 \cdot_F h_{S,W} &= \bigcup_{s_\beta \in h_{S,W}} \left\{ \left(s_{\frac{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_1}{(\lambda-1)^{k_1-1}}))}{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_2}{(\lambda-1)^{k_2-1}}))}}, w_{s_\beta} \right) \right\} \oplus_F \bigcup_{s_\beta \in h_{S,W}} \left\{ \left(s_{\frac{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_2}{(\lambda-1)^{k_2-1}}))}{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_1}{(\lambda-1)^{k_1-1}}))}}, w_{s_\beta} \right) \right\} \\
 &= \bigcup_{s_\beta \in h_{S,W}} \left\{ \left(s_{\frac{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_1}{(\lambda-1)^{k_1-1}}))}{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_2}{(\lambda-1)^{k_2-1}}))}}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2} \right) \right\} \\
 &= \bigcup_{s_\beta \in h_{S,W}} \left\{ \left(s_{\frac{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_1+k_2}{(\lambda-1)^{k_1+k_2-1}}))}{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_1+k_2}{(\lambda-1)^{k_1+k_2-1}}))}}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2} \right) \right\} = (k_1 + k_2) \cdot_F h_{S,W} \\
 (4) \quad (k_1 k_2) \cdot_F h_{S,W} &= k_1 \cdot_F \bigcup_{s_\beta \in h_{S,W}} \left\{ \left(s_{\frac{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_2}{(\lambda-1)^{k_2-1}}))}{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_2}{(\lambda-1)^{k_2-1}}))}}, w_{s_\beta} \right) \right\} \\
 &= \bigcup_{s_\beta \in h_{S,W}} \left\{ \left(s_{\frac{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_2}{(\lambda-1)^{k_2-1}}))}{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_2}{(\lambda-1)^{k_2-1}}))}}, w_{s_\beta} \right) \right\} = \bigcup_{s_\beta \in h_{S,W}} \left\{ \left(s_{\frac{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_2}{(\lambda-1)^{k_2-1}}))}{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_2}{(\lambda-1)^{k_2-1}}))}}, w_{s_\beta} \right) \right\} \\
 &= \bigcup_{s_\beta \in h_{S,W}} \left\{ \left(s_{\frac{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_2}{(\lambda-1)^{k_2-1}}))}{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta/l}-1)k_2}{(\lambda-1)^{k_2-1}}))}}, w_{s_\beta} \right) \right\} = k_1 \cdot_F (k_2 \cdot_F h_{S,W}).
 \end{aligned}$$

Thus, Theorem 1 holds. $\square$

3. Frank Aggregation Operators for WHFLE

Aggregation operators, as convenient and highly effective information aggregation tools, are widely used in the fields of medical diagnosis, decision-making, image recognition, intelligent algorithms, and so on. To aggregate the multidimensional weighted hesitant fuzzy linguistic preference values into a comprehensive value, we develop several aggregation operators in this section.

3.1. WHFLFWA Operator

Definition 5. Let $h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n$ be WHFLEs on $S = \{s_\alpha | \alpha = 0, 1, \dots, t\}$ and $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$ be the weight vector of $h_{S,W}^{(k)}$ ($k = 1, 2, \dots, n$), satisfying $\omega_i \in [0, 1], \sum_{i=1}^n \omega_i = 1$. A weighted hesitant fuzzy linguistic Frank weighted averaging (WHFLFWA) operator is a mapping WHFLFWA: $H_{S,W}^n \rightarrow H_{S,W}$, such that

$$\text{WHFLFWA} (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n) = \omega_1 \cdot_F h_{S,W}^1 \oplus_F \omega_2 \cdot_F h_{S,W}^2 \oplus_F \dots \oplus_F \omega_n \cdot_F h_{S,W}^n$$

Specifically, if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then

$$\text{WHFLFA} (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n) = \frac{1}{n} \cdot_F (h_{S,W}^1 \oplus_F h_{S,W}^2 \oplus_F \dots \oplus_F h_{S,W}^n)$$

is called the weighted hesitant fuzzy linguistic Frank averaging (WHFLFA) operator.

Theorem 2. Let $h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n$ be WHFLEs on $S = \{s_\alpha | \alpha = 0, 1, \dots, t\}$ and $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$ be the weight vector of $h_{S,W}^{(k)}$ ($k = 1, 2, \dots, n$), satisfying $\omega_i \in [0, 1], \sum_{i=1}^n \omega_i = 1$, then the aggregated value by the WHFLFWA operator is also a WHFLE and

$$\text{WHFLFWA} (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n) = \bigoplus_{j=1}^n (\omega_j \cdot_F h_{S,W}^j)$$

$$= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2, \dots, s_{\beta_n} \in h_{S,W}^n} \left\{ (s_{l(1-\log_\lambda(1+\prod_{j=1}^n (\lambda^{1-\beta_j/l} - 1)^{\omega_j}))}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_n}}}{n}) \right\}, l = t+1 \quad (3)$$

Proof. (1) For $n = 2$, we have

$$\omega_1 \cdot_F h_{S,W}^1 = \bigcup_{s_{\beta_1} \in h_{S,W}^1} \left\{ (s_{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta_1/l} - 1)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}))}, w_{s_{\beta_1}}) \right\}$$

$$\text{and } \omega_2 \cdot_F h_{S,W}^2 = \bigcup_{s_{\beta_2} \in h_{S,W}^2} \left\{ (s_{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta_2/l} - 1)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}))}, w_{s_{\beta_2}}) \right\},$$

Therefore,

$$\begin{aligned} \text{WHFLFWA } (h_{S,W}^1, h_{S,W}^2) &= \omega_1 \cdot_F h_{S,W}^1 \oplus_F \omega_2 \cdot_F h_{S,W}^2 \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1} \left\{ (s_{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta_1/l} - 1)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}))}, w_{s_{\beta_1}}) \right\} \oplus_F \bigcup_{s_{\beta_2} \in h_{S,W}^2} \left\{ (s_{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta_2/l} - 1)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}))}, w_{s_{\beta_2}}) \right\} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \left\{ (s_{l(1-\log_\lambda(1+\frac{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta_1/l} - 1)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}))}{(\lambda-1)^{\omega_1-1}} - 1)(\lambda^{\frac{1-\log_\lambda(1+\frac{(\lambda^{1-\beta_2/l} - 1)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}))}{l} - 1)}{\lambda-1})}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2}) \right\} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \left\{ (s_{l(1-\log_\lambda(1+\prod_{j=1}^2 (\lambda^{1-\beta_j/l} - 1)^{\omega_j}))}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2}) \right\} \end{aligned}$$

i.e., Equation (3) holds for $n = 2$.

(2) If Equation (3) holds for $n = k$ and $k \geq 2$, then we have

$$\begin{aligned} \text{WHFLFWA } (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^k) &= \omega_1 \cdot_F h_{S,W}^1 \oplus_F \omega_2 \cdot_F h_{S,W}^2 \oplus_F \dots \oplus_F \omega_{k+1} \cdot_F h_{S,W}^{k+1} \\ &= (\omega_1 \cdot_F h_{S,W}^1 \oplus_F \omega_2 \cdot_F h_{S,W}^2 \oplus_F \dots \oplus_F \omega_k \cdot_F h_{S,W}^k) \oplus_F \omega_{k+1} \cdot_F h_{S,W}^{k+1} \\ &= \text{WHFLFWA } (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^k) \oplus_F \omega_{k+1} \cdot_F h_{S,W}^{k+1} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2, \dots, s_{\beta_k} \in h_{S,W}^k} \left\{ (s_{l(1-\log_\lambda(1+\frac{\prod_{j=1}^k (\lambda^{1-\beta_j/l} - 1)^{\omega_j}}{\sum_{j=1}^k \omega_j - 1} - 1)}), w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_k}}) \right\} \\ &\quad \oplus_F \bigcup_{s_{\beta_{k+1}} \in h_{S,W}^{k+1}} \left\{ (s_{l(1-\log_\lambda(1+\frac{(\lambda^{1-\beta_{k+1}/l} - 1)^{\omega_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}))}, w_{s_{\beta_{k+1}}}) \right\} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2, \dots, s_{\beta_{k+1}} \in h_{S,W}^{k+1}} \left\{ (s_{l(1-\log_\lambda(1+\prod_{j=1}^{k+1} (\lambda^{1-\beta_j/l} - 1)^{\omega_j}))}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_{k+1}}}}{k+1}) \right\}. \end{aligned}$$

i.e., $n = k+1$, Equation (3) holds.

Therefore, Equation (3) holds for all n ($n \geq 2$). $\square$

It is worth noting that the WHFLFWA operator has the following properties.

Proof.

$$\lim_{\lambda \rightarrow +\infty} \text{HFLFWA} (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n)$$

$$= \bigcup_{s_{\beta_1} \in h_{S,W}^1, \dots, s_{\beta_n} \in h_{S,W}^n} \left\{ \left(s, \frac{\lim_{\lambda \rightarrow +\infty} l(1 - \log_{\lambda} (1 + \prod_{j=1}^n (\lambda^{1-\beta_j/l} - 1)^{\omega_j}))}{n}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_n}}}{n} \right) \right\}$$

According to L'Hôpital's rule, we have

$$\lim_{\lambda \rightarrow +\infty} l(1 - \log_{\lambda} (1 + \prod_{j=1}^n (\lambda^{1-\beta_j/l} - 1)^{\omega_j})) = l - l \lim_{\lambda \rightarrow +\infty} \frac{\ln(1 + \prod_{j=1}^n (\lambda^{1-\beta_j/l} - 1)^{\omega_j}))}{\ln \lambda}$$

$$= l - l \lim_{\lambda \rightarrow +\infty} \frac{\frac{\prod_{j=1}^n (\lambda^{1-\beta_j/l} - 1)^{\omega_j}}{1 + \prod_{j=1}^n (\lambda^{1-\beta_j/l} - 1)^{\omega_j}} \cdot \sum_{j=1}^n \omega_j (1 - \beta_j / l) \cdot \frac{\lambda^{-\beta_j/l}}{\lambda^{1-\beta_j/l} - 1}}{\frac{1}{\lambda}}$$

$$= l - l \sum_{j=1}^n \omega_j (1 - \beta_j / l) = \sum_{j=1}^n \omega_j \beta_j$$

Therefore,

$$\lim_{\lambda \rightarrow +\infty} \text{HFLFWA} (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n)$$

$$= \bigcup_{s_{\beta_1} \in h_{S,W}^1, \dots, s_{\beta_n} \in h_{S,W}^n} \left\{ \left(s, \frac{\lim_{\lambda \rightarrow +\infty} l(1 - \log_{\lambda} (1 + \prod_{j=1}^n (\lambda^{1-\beta_j/l} - 1)^{\omega_j}))}{n}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_n}}}{n} \right) \right\}$$

$$= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2, \dots, s_{\beta_n} \in h_{S,W}^n} \left\{ \left(s, \frac{\sum_{j=1}^n \omega_j \beta_j}{n}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_n}}}{n} \right) \right\}$$

$$= \text{WHFLWA} (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n). \square$$

3.2. WHFLFWG Operator

To provide more flexibility for aggregating weighted hesitant fuzzy linguistic information, a weighted hesitant fuzzy linguistic Frank-weighted geometric (WHFLFWG) operator is introduced, which is the dual form of the WHFLFWA operator.

Definition 6. Let $h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n$ be WHFLEs on $S = \{s_{\alpha} | \alpha = 0, 1, \dots, t\}$ and $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$ be the weight vector of $h_{S,W}^{(k)}$ ($k = 1, 2, \dots, n$), satisfying $\omega_i \in [0, 1], \sum_{i=1}^n \omega_i = 1$.

A weighted hesitant fuzzy linguistic Frank weighted geometric (WHFLFWG) operator is a mapping WHFLFWG: $H_{S,W}^n \rightarrow H_{S,W}$, such that

$$\text{WHFLFWG} (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n) = h_{S,W}^{1 \wedge_F \omega_1} \otimes_F h_{S,W}^{2 \wedge_F \omega_2} \otimes_F \dots \otimes_F h_{S,W}^{n \wedge_F \omega_n}$$

Specifically, if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then

$$\text{WHFLFWG} (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n) = (h_{S,W}^1 \otimes_F h_{S,W}^2 \otimes_F \dots \otimes_F h_{S,W}^n)^{\wedge_F (1/n)}$$

is called the weighted hesitant fuzzy linguistic Frank geometric (WHFLFG) operator.

Theorem 4. Let $h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n$ be WHFLEs on $S = \{s_\alpha | \alpha = 0, 1, \dots, t\}$ and $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$ be the weight vector of $h_{S,W}^{(k)}$ ($k = 1, 2, \dots, n$), satisfying $\omega_i \in [0, 1], \sum_{i=1}^n \omega_i = 1$, then the aggregated value by the WHFLFWG operator is also a WHFLE and

$$\begin{aligned} \text{WHFLFWG}(h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n) &= \bigotimes_{j=1}^n (h_{S,W}^{j \wedge_F \omega_j}) \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2, \dots, s_{\beta_n} \in h_{S,W}^n} \left\{ \left(s_{\log_\lambda \left(1 + \prod_{j=1}^n (\lambda^{\beta_j/l} - 1)^{\omega_j} \right)}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_n}}}{n} \right) \right\}, l = t+1 \end{aligned} \quad (4)$$

Proof. (1) For $n = 2$, we have

$$h_{S,W}^{1 \wedge_F \omega_1} = \bigcup_{s_{\beta_1} \in h_{S,W}^1} \left\{ \left(s_{\log_\lambda \left(1 + \frac{(\lambda^{\beta_1/l} - 1)^{\omega_1}}{(\lambda - 1)^{\omega_1 - 1}} \right)}, w_{s_{\beta_1}} \right) \right\},$$

$$h_{S,W}^{2 \wedge_F \omega_2} = \bigcup_{s_{\beta_2} \in h_{S,W}^2} \left\{ \left(s_{\log_\lambda \left(1 + \frac{(\lambda^{\beta_2/l} - 1)^{\omega_2}}{(\lambda - 1)^{\omega_2 - 1}} \right)}, w_{s_{\beta_2}} \right) \right\}$$

Therefore,

$$\begin{aligned} \text{WHFLFWG}(h_{S,W}^1, h_{S,W}^2) &= h_{S,W}^{1 \wedge_F \omega_1} \otimes_F h_{S,W}^{2 \wedge_F \omega_2} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1} \left\{ \left(s_{\log_\lambda \left(1 + \frac{(\lambda^{\beta_1/l} - 1)^{\omega_1}}{(\lambda - 1)^{\omega_1 - 1}} \right)}, w_{s_{\beta_1}} \right) \right\} \otimes_F \bigcup_{s_{\beta_2} \in h_{S,W}^2} \left\{ \left(s_{\log_\lambda \left(1 + \frac{(\lambda^{\beta_2/l} - 1)^{\omega_2}}{(\lambda - 1)^{\omega_2 - 1}} \right)}, w_{s_{\beta_2}} \right) \right\} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \left\{ \left(s_{\log_\lambda \left(1 + \frac{\log_\lambda \left(1 + \frac{(\lambda^{\beta_1/l} - 1)^{\omega_1}}{(\lambda - 1)^{\omega_1 - 1}} \right)}{\frac{(\lambda - 1)^{\omega_1 - 1}}{(\lambda - 1)^{\omega_1 - 1}}} - 1 \right) \frac{\log_\lambda \left(1 + \frac{(\lambda^{\beta_2/l} - 1)^{\omega_2}}{(\lambda - 1)^{\omega_2 - 1}} \right)}{\frac{(\lambda - 1)^{\omega_2 - 1}}{(\lambda - 1)^{\omega_2 - 1}}} - 1 \right)}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2} \right\} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \left\{ \left(s_{\log_\lambda \left(1 + (\lambda^{\beta_1/l} - 1)^{\omega_1} (\lambda^{\beta_2/l} - 1)^{\omega_2} \right)}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2} \right) \right\} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2} \left\{ \left(s_{\log_\lambda \left(1 + \prod_{j=1}^2 (\lambda^{\beta_j/l} - 1)^{\omega_j} \right)}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}}}{2} \right) \right\} \end{aligned}$$

i.e., Equation (4) holds for $n = 2$.

(2) If Equation (4) holds for $n = k$ ($k \geq 2$), then we have

$$\begin{aligned} \text{WHFLFWG}(h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n) &= h_{S,W}^{1 \wedge_F \omega_1} \otimes_F h_{S,W}^{2 \wedge_F \omega_2} \otimes_F \dots \otimes_F h_{S,W}^{k+1 \wedge_F \omega_{k+1}} \\ &= (h_{S,W}^{1 \wedge_F \omega_1} \otimes_F h_{S,W}^{2 \wedge_F \omega_2} \otimes_F \dots \otimes_F h_{S,W}^{k \wedge_F \omega_k}) \otimes_F h_{S,W}^{k+1 \wedge_F \omega_{k+1}} \\ &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2, \dots, s_{\beta_k} \in h_{S,W}^k} \left\{ \left(s_{\log_\lambda \left(1 + \prod_{j=1}^k (\lambda^{\beta_j/l} - 1)^{\omega_j} \right)}, w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_k}} \right) \right\} \\ &\quad \otimes_F \bigcup_{s_{\beta_{k+1}} \in h_{S,W}^{k+1}} \left\{ \left(s_{\log_\lambda \left(1 + \frac{(\lambda^{\beta_{k+1}/l} - 1)^{\omega_{k+1}}}{(\lambda - 1)^{\omega_{k+1} - 1}} \right)}, w_{s_{\beta_{k+1}}} \right) \right\} \end{aligned}$$

$$\begin{aligned}
 &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2, \dots, s_{\beta_{k+1}} \in h_{S,W}^{k+1}} \left\{ (s, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_{k+1}}}}{k+1}) \right\} \\
 &\quad \frac{\prod_{j=1}^k (\lambda^{\beta_j/l} - 1)^{\omega_j}}{l \log_\lambda (1 + \frac{\sum_{j=1}^k \omega_j - 1}{(\lambda-1)^{l-1}})} \cdot \frac{l \log_\lambda (1 + \frac{(\lambda^{\alpha_{k+1}/l} - 1)^{\alpha_{k+1}}}{(\lambda-1)^{l-1}})}{l \log_\lambda (1 + \frac{(\lambda^{\alpha_{k+1}/l} - 1)^{\alpha_{k+1}}}{(\lambda-1)^{l-1}})} \\
 &= \bigcup_{s_{\beta_1} \in h_{S,W}^1, s_{\beta_2} \in h_{S,W}^2, \dots, s_{\beta_{k+1}} \in h_{S,W}^{k+1}} \left\{ (s, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_{k+1}}}}{k+1}) \right\} \\
 &\quad \frac{1}{l \log_\lambda (1 + \prod_{j=1}^{k+1} (\lambda^{\beta_j/l} - 1)^{\omega_j})}
 \end{aligned}$$

Therefore, Equation (4) holds for all n ($n \geq 2$). $\square$

Property 3 can be proven along the same lines as the proof of Property 2; therefore, we omit the details here.

Property 2. (1) (Commutativity) Let $\bar{h}_{S,W}^1, \bar{h}_{S,W}^2, \dots, \bar{h}_{S,W}^n$ be any permutation of $h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n$, then

$$\text{WHFLFWG}(\bar{h}_{S,W}^1, \bar{h}_{S,W}^2, \dots, \bar{h}_{S,W}^n) = \text{WHFLFWG}(h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n)$$

(2) (Idempotency) Let $h_{S,W}^1 = h_{S,W}^2 = \dots = h_{S,W}^n = h_{S,W} = \bigcup_{s_\beta \in h_S(x_i)} \{(s_\beta, w(s_\beta))\}$, then

$$\text{WHFLFWG}(h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n) = h_{S,W}$$

Theorem 5. Let $h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n$ be WHFLEs on $S = \{s_\alpha | \alpha = 0, 1, \dots, t\}$ and $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$ be the weight vector of $h_{S,W}^{(k)}$ ($k = 1, 2, \dots, n$), satisfying $\omega_i \in [0, 1], \sum_{i=1}^n \omega_i = 1$, then as $\lambda \rightarrow 1$, the WHFLFWG operator reduces to the weighted hesitant fuzzy linguistic weighted geometric (WHFLWG) operator

$$\lim_{\lambda \rightarrow 1} \text{WHFLFWG}(h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n) = \text{WHFLWG}(h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n)$$

Proof. Since $\lambda \rightarrow 1$, we obtain $\prod_{j=1}^n (\lambda^{\beta_j/l} - 1)^{\omega_j} \rightarrow 0$

and so

$$\ln(1 + \prod_{j=1}^n (\lambda^{\beta_j/l} - 1)^{\omega_j}) \sim \prod_{j=1}^n (\lambda^{\beta_j/l} - 1)^{\omega_j}, \quad \lambda \rightarrow 1$$

By Taylor's expansion,

$$\lambda^{\beta_j/l} = 1 + \frac{\beta_j}{l} \ln \lambda + \frac{(\beta_j)^2}{2l^2} (\ln \lambda)^2 + \dots \quad (5)$$

It follows from (5) that

$$\begin{aligned}
 &\lambda^{\beta_j/l} - 1 \rightarrow \frac{\beta_j}{l} \ln \lambda \\
 &\Rightarrow (\lambda^{\beta_j/l} - 1)^{\omega_j} \rightarrow \left(\frac{\beta_j}{l} \ln \lambda\right)^{\omega_j} = (\beta_j)^{\omega_j} \left(\frac{\ln \lambda}{l}\right)^{\omega_j} \\
 &\Rightarrow \prod_{j=1}^n (\lambda^{\beta_j/l} - 1)^{\omega_j} \rightarrow \frac{\ln \lambda}{l} \cdot \prod_{j=1}^n (\beta_j)^{\omega_j}
 \end{aligned}$$

According to L'Hôpital's rule, we have

$$\lim_{\lambda \rightarrow 1} l \log_{\lambda} (1 + \prod_{j=1}^n (\lambda^{\beta_j/l} - 1)^{\omega_j}) = \lim_{\lambda \rightarrow 1} l \frac{\frac{\ln \lambda}{l} \cdot \prod_{j=1}^n (\beta_j)^{\omega_j}}{\ln \lambda} = \prod_{j=1}^n (\beta_j)^{\omega_j}$$

Therefore,

$$\begin{aligned} & \lim_{\lambda \rightarrow 1} \text{WHFLFWG} (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n) \\ &= \bigcup_{s_{\beta_1} \in h_1, s_{\beta_2} \in h_2, \dots, s_{\beta_n} \in h_n} \left\{ s_{\lim_{\lambda \rightarrow 1} l \log_{\lambda} (1 + \prod_{j=1}^n (\lambda^{\beta_j/l} - 1)^{\omega_j})}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_{k+1}}}}{n} \right\} \\ &= \bigcup_{s_{\beta_1} \in h_1, s_{\beta_2} \in h_2, \dots, s_{\beta_n} \in h_n} \left\{ s_{\prod_{j=1}^n \beta_j^{\omega_j}}, \frac{w_{s_{\beta_1}} + w_{s_{\beta_2}} + \dots + w_{s_{\beta_{k+1}}}}{n} \right\} \\ &= \text{WHFLG} (h_{S,W}^1, h_{S,W}^2, \dots, h_{S,W}^n) \end{aligned}$$

and the proof is complete. $\square$

4. Method of Aggregating Weighted Hesitant Fuzzy Linguistic Multi-Criteria Information

Considering a weighted hesitant fuzzy linguistic multiple attribute decision-making (MADM) problem, suppose $X = \{x_1, x_2, \dots, x_n\}$ is a set of n alternatives. Let $G = \{g_1, g_2, \dots, g_l\}$ be a set of attributes with the weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_l)^T$, where $\omega_i \in [0, 1]$, $\sum_{i=1}^n \omega_i = 1$. Let $H = (h_{S,W}^{i,j})_{n \times l}$ be a weighted hesitant fuzzy linguistic decision matrix, where $h_{S,W}^{i,j} = \bigcup_{s_{\beta} \in h_S^{i,j}} \{(s_{\beta}, w(s_{\beta}))\}$ is a weighted hesitant fuzzy linguistic term for the alternative $x_i \in X$ with respect to the attribute $g_j \in G$.

Step 1: Transform $H = (h_{S,W}^{i,j})_{n \times l}$ into a normalized matrix $\bar{H} = (\bar{h}_{S,W}^{i,j})_{n \times l}$, where $\bar{h}_{S,W}^{i,j} = h_{S,W}^{i,j}$, for benefit attribute g_j ; $\bar{h}_{S,W}^{i,j} = N(h_{S,W}^{i,j}) = \bigcup_{s_{\beta} \in h_S^{i,j}} \{(s_{t-\beta}, w(s_{\beta}))\}$, where t is the subscript of the biggest linguistic term in $S = \{s_{\alpha} \mid \alpha = 0, 1, 2, \dots, t\}$, for cost attribute g_j , $i = 1, 2, \dots, n$, $j = 1, 2, \dots, l$.

Step 2: Utilize the WHFLFWA (or WHFLFWG) to aggregate the i th line of $\bar{H} = (\bar{h}_{S,W}^{i,j})_{n \times l}$:

$$h_{S,W}^i = \text{WHFLFWA} (\bar{h}_{S,W}^{i,1}, \bar{h}_{S,W}^{i,2}, \dots, \bar{h}_{S,W}^{i,l}), \quad i = 1, 2, \dots, n$$

or

$$h_{S,W}^i = \text{WHFLFWG} (\bar{h}_{S,W}^{i,1}, \bar{h}_{S,W}^{i,2}, \dots, \bar{h}_{S,W}^{i,l}), \quad i = 1, 2, \dots, n$$

and obtain the comprehensive evaluation value $h_{S,W}^i$ for the alternative $h_{S,W}^i$.

Step 3: Utilize the binary relation of WHFLEs described in Definition 3 to rank $h_{S,W}^i$ ($i = 1, 2, \dots, n$) in descending order.

Step 4: Rank and optimize the alternatives $i = 1, 2, (i = 1, 2, \dots, n)$.

5. An Illustrative Example

In what follows, we use an example of regional twinning–support performance evaluation, adapted from [19], to illustrate the proposed method.

Example 3. (Adapted from [19]) *Regional twinning support has played a positive role in promoting regional economic development, balancing the distribution of educational resources, and enhancing cooperation and exchange between regions.*

Promoting regional economic development: By transferring resources and expertise from developed areas to less developed regions, this initiative assists these regions in learning from successful experiences, developing their own advantageous industries, and enhancing their economic development.

Promoting a balanced distribution of educational resources: Conveying high-quality educational resources to underdeveloped areas can successfully enhance educational standards in impoverished regions. For example, measures such as supporting high-quality educational groups to operate schools across regions, collaboratively organizing comprehensive training for all teachers, and arranging for college students to conduct internships and provide teaching support can be implemented to enhance regional educational capabilities.

Strengthening talent training and labor coordination: Through talent training, labor coordination, and market expansion, this support improves the skills and knowledge of local talent, driving the development of urban–rural planning and design, cultural tourism product planning, ecological agriculture, and hydroelectric resource development.

Improving infrastructure and public services: Pairing assistance also involves the concentrated implementation of major infrastructure projects to enhance the comprehensive carrying capacity of towns, focusing specifically on improvements in public health, the living environment, and public services.

To further improve the effectiveness and quality of regional twinning support, regular evaluations of the performance of regional twinning support should be conducted, providing important support and guidance for the effective conduct of pairing assistance work. Suppose that a group of experts intend to evaluate the impact of twinning support measures implemented in five designated areas $\{x_1, x_2, \dots, x_5\}$. There are four attributes (with the weight vector $\omega = (0.4, 0.2, 0.2, 0.2)^T$) that need to be considered: c_1 -promoting regional economic development; c_2 -promoting the balanced distribution of educational resources; c_3 -strengthening talent training and labor coordination; and c_4 -improving infrastructure and public services. We use a finite and completely ordered linguistic term set $S = \{s_0: \text{terrible}, s_1: \text{very bad}, s_2: \text{bad}, s_3: \text{medium}, s_4: \text{well}, s_5: \text{very well}, s_6: \text{perfect}\}$. $H = (h_{S,W}^{i,j})_{5 \times 4}$. The weighted hesitant fuzzy linguistic decision matrix $H = (h_{S,W}^{i,j})_{5 \times 4}$ is constructed as shown in Table 3.

Table 3. Decision matrix.

	C_1	C_2	C_3	C_4
x_1	$\{(s_3, 0.2), (s_4, 0.8)\}$	$\{(s_3, 0.3), (s_4, 0.7)\}$	$\{(s_4, 0.2), (s_5, 0.2), (s_6, 0.6)\}$	$\{(s_4, 0.4), (s_5, 0.6)\}$
x_2	$\{(s_3, 0.35), (s_4, 0.3), (s_5, 0.35)\}$	$\{(s_4, 0.4), (s_5, 0.6)\}$	$\{(s_3, 0.4), (s_4, 0.6)\}$	$\{(s_4, 0.2), (s_5, 0.3), (s_6, 0.5)\}$
x_3	$\{(s_3, 0.7), (s_4, 0.3)\}$	$\{(s_4, 0.6), (s_5, 0.3), (s_6, 0.1)\}$	$\{(s_4, 0.6), (s_5, 0.4)\}$	$\{(s_5, 1)\}$
x_4	$\{(s_3, 0.15), (s_4, 0.25), (s_5, 0.6)\}$	$\{(s_2, 0.1), (s_3, 0.2), (s_4, 0.7)\}$	$\{(s_4, 0.4), (s_5, 0.35), (s_6, 0.25)\}$	$\{(s_4, 0.5), (s_5, 0.5)\}$
x_5	$\{(s_4, 0.45), (s_5, 0.55)\}$	$\{(s_3, 0.35), (s_5, 0.2), (s_6, 0.45)\}$	$\{(s_3, 0.4), (s_4, 0.2), (s_5, 0.4)\}$	$\{(s_3, 0.5), (s_4, 0.5)\}$

5.1. Rank the Alternatives by the WHFLFWG Operator

Step 1: The attributes involved in this example are all benefit types, so $H = (h_{S,W}^{i,j})_{n \times l} = \bar{H}$.

Step 2: Utilize the WHFLFWG operator (suppose $\lambda=2$) to aggregate the i th line of $\bar{H} = (\bar{h}_{S,W}^{i,j})_{n \times l}$, and we can get the comprehensive evaluation value.

$$h_{S,W}^1 = \{(s_{3.3713}, 0.275), (s_{3.9613}, 0.05), (s_{4.5493}, 0.675)\};$$

$$h_{S,W}^2 = \{(s_{3.7313}, 0.3375), (s_{4.1753}, 0.15), (s_{4.9685}, 0.5125)\};$$

$$h_{S,W}^3 = \{(s_{3.7405}, 0.475), (s_{4.2556}, 0.075), (s_{4.7553}, 0.45)\};$$

$$h_{S,W}^4 = \{(s_{3.1177}, 0.2875), (s_{4.0535}, 0.2), (s_{4.9685}, 0.5125)\};$$

$$h_{S,W}^5 = \{(s_{3.3713}, 0.425), (s_{4.2749}, 0.1), (s_{4.9685}, 0.475)\}.$$

Step 3: Calculate the priority degree $\eta(h_{S,W}^i)$ of the comprehensive evaluation values $h_{S,W}^i$ ($i=1, 2, \dots, n$).

$$\eta(h_{S,W}^1) = 4.1959, \quad \eta(h_{S,W}^2) = 4.3105, \quad \eta(h_{S,W}^3) = 4.2358, \quad \eta(h_{S,W}^4) = 4.2534, \quad \eta(h_{S,W}^5) = 4.2203.$$

Step 4: Rank and optimize the alternatives $i=1, 2, \dots, n$.

$$x_2 \succ x_4 \succ x_3 \succ x_5 \succ x_1$$

So, the impact of twinning support measures implemented in area x_2 is the best.

5.2. Parameter Analysis and Computational Complexity

Here, we want to show the impact of selecting different parameters and operators in the decision scheme on the recommendation results.

Table 4 and Figure 1 show that, under different values of the parameter λ , the rankings of the urban assistance effects remain unchanged when the WHFLFWG operator is used, indicating that the operator is highly stable. Table 5 and Figure 2 indicate that the optimal alternative remains unchanged when the WHFLFWA operator is used under different values of the parameter λ . However, the rankings of the assistance effects in other cities undergo slight changes as λ varies.

Table 4. The calculation results of the WHFLFWG operator.

Parameter	The Priority Degree	Ranking
	$\eta(h_{S,W})=(\eta(h_{S,W}^1),\eta(h_{S,W}^2),\eta(h_{S,W}^3),\eta(h_{S,W}^4),\eta(h_{S,W}^5))$	
$\lambda \rightarrow 1$	(4.1850, 4.3031, 4.2234, 4.2425, 4.2131)	$x_2 \succ x_4 \succ x_3 \succ x_5 \succ x_1$
$\lambda = 2$	(4.1959, 4.3105, 4.2358, 4.2534, 4.2203)	$x_2 \succ x_4 \succ x_3 \succ x_5 \succ x_1$
$\lambda = 5$	(4.2080, 4.3185, 4.2494, 4.2654, 4.2282)	$x_2 \succ x_4 \succ x_3 \succ x_5 \succ x_1$

$\lambda = 10$	(4.2154, 4.3233, 4.2577, 4.2728, 4.2330)	$x_2 \succ x_4 \succ x_3 \succ x_5 \succ x_1$
$\lambda = 50$	(4.2273, 4.3311, 4.2715, 4.2852, 4.2408)	$x_2 \succ x_4 \succ x_3 \succ x_5 \succ x_1$
$\lambda = 100$	(4.2306, 4.3333, 4.2754, 4.2889, 4.2430)	$x_2 \succ x_4 \succ x_3 \succ x_5 \succ x_1$
$\lambda \rightarrow +\infty$	(4.2400, 4.3400, 4.2875, 4.3025, 4.2500)	$x_2 \succ x_4 \succ x_3 \succ x_5 \succ x_1$

Table 5. The calculation results of the WHFLFWA operator

Parameter	The Priority Degree	Ranking
	$\eta(h_{s,w})=(\eta(h_{s,w}^1), \eta(h_{s,w}^2), \eta(h_{s,w}^3), \eta(h_{s,w}^4), \eta(h_{s,w}^5))$	
$\lambda \rightarrow 1$	(4.3743, 4.4164, 4.4081, 4.3955, 4.3227)	$x_2 \succ x_3 \succ x_4 \succ x_1 \succ x_5$
$\lambda = 2$	(4.3594, 4.4083, 4.3939, 4.3847, 4.3148)	$x_2 \succ x_3 \succ x_4 \succ x_1 \succ x_5$
$\lambda = 5$	(4.3417, 4.3986, 4.3771, 4.3719, 4.3054)	$x_2 \succ x_3 \succ x_4 \succ x_1 \succ x_5$
$\lambda = 10$	(4.3297, 4.3920, 4.3658, 4.3633, 4.2990)	$x_2 \succ x_3 \succ x_4 \succ x_1 \succ x_5$
$\lambda = 50$	(4.3063, 4.3791, 4.3440, 4.3468, 4.2866)	$x_2 \succ x_4 \succ x_3 \succ x_1 \succ x_5$
$\lambda = 100$	(4.2980, 4.3744, 4.3364, 4.3410, 4.2822)	$x_2 \succ x_4 \succ x_3 \succ x_1 \succ x_5$
$\lambda \rightarrow +\infty$	(4.2507, 4.3405, 4.2880, 4.3030, 4.2505)	$x_2 \succ x_4 \succ x_3 \succ x_1 \succ x_5$

When $\lambda \leq 20.55$, the ranking obtained by the WHFLFWA operator is

$$x_2 \succ x_3 \succ x_4 \succ x_1 \succ x_5.$$

When $20.55 < \lambda$, the ranking becomes

$$x_2 \succ x_4 \succ x_3 \succ x_1 \succ x_5.$$

The main reason for this phenomenon is that the variation in the priority degree, $\Delta\eta(h_{s,w})$, of the WHFLFWA operator is greater than that of the WHFLFWG operator as the parameter increases, as illustrated in Figures 1 and 2. For example, when λ increases from three to five, the changes in the priority degree of the WHFLFWA and WHFLFWG operators are as follows:

$$\begin{aligned}\Delta\eta(h_{s,w})|_{\text{WHFLFWG}} &= (\Delta\eta(h_{s,w}^1), \Delta\eta(h_{s,w}^2), \Delta\eta(h_{s,w}^3), \Delta\eta(h_{s,w}^4), \Delta\eta(h_{s,w}^5)) = (0.0014, 0.0009, 0.0016, 0.0014, 0.0009) \\ \Delta\eta(h_{s,w})|_{\text{WHFLFWA}} &= (\Delta\eta(h_{s,w}^1), \Delta\eta(h_{s,w}^2), \Delta\eta(h_{s,w}^3), \Delta\eta(h_{s,w}^4), \Delta\eta(h_{s,w}^5)) = (0.0019, 0.0011, 0.0019, 0.0014, 0.001)\end{aligned}$$

From Figures 1 and 2, it can be observed that the priority degrees calculated by the WHFLFWG operator are significantly smaller than those calculated by the WHFLFWA operator. Further analysis reveals that, for the same alternative, the priority degrees obtained by the WHFLFWA operator decrease gradually as the parameter λ increases, whereas those obtained by the WHFLFWG operator increase gradually with increasing values of λ . Therefore, the Frank operators for WHFLEs possess the excellent property of varying monotonically with the parameter λ , indicating that the parameter λ can be interpreted as the decision makers' attitude toward future development prospects. If the experts are more optimistic about the future prospects of the decision problem, a larger value of λ should be adopted for the WHFLFWG operator. Conversely, if they are more pessimistic about the future prospects of the decision problem, a smaller value of λ should be adopted for the WHFLFWA operator.

Let the number of alternatives be m , the number of attributes be n , and the average number of linguistic terms contained in each hesitant fuzzy linguistic element be l . For ordinary hesitant fuzzy linguistic methods, when calculating the score function, distance measure, or aggregation operator, it is generally necessary to traverse all linguistic terms within each element. Therefore, the computational complexity for processing a single element is $O(l)$, and the overall computational complexity of the decision matrix is $O(mnl)$. For the weighted hesitant fuzzy linguistic method, although corresponding weights need

to be assigned to each linguistic term and weighted multiplication or normalization operations are required during the computation, these operations are still performed within the same traversal of the linguistic terms and do not introduce an additional level of iteration. Therefore, the computational complexity of processing a single element remains $O(l)$, and the overall computational complexity also remains $O(mnl)$. Compared with the ordinary hesitant fuzzy linguistic method, the weighted hesitant fuzzy linguistic method only introduces a constant number of additional arithmetic operations and does not alter the asymptotic time complexity of the algorithm. Consequently, when the weights are predefined or can be obtained as a preprocessing step, the proposed weighted hesitant fuzzy linguistic method has the same order of computational complexity as the ordinary hesitant fuzzy linguistic method, while providing greater flexibility in representing the relative importance of evaluation information.

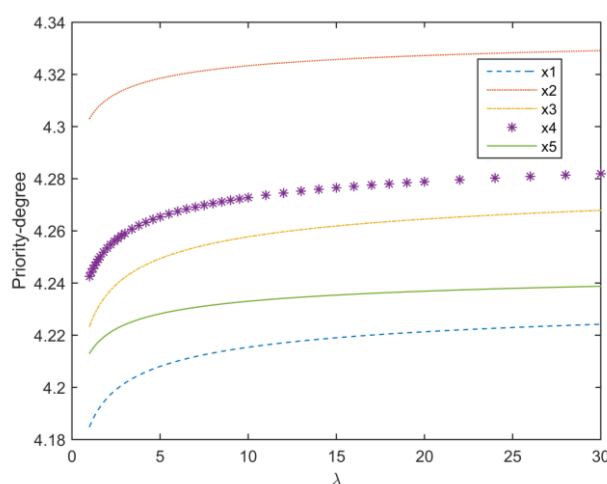


Figure 1. Priority degrees obtained by the WHFLFWG operator.

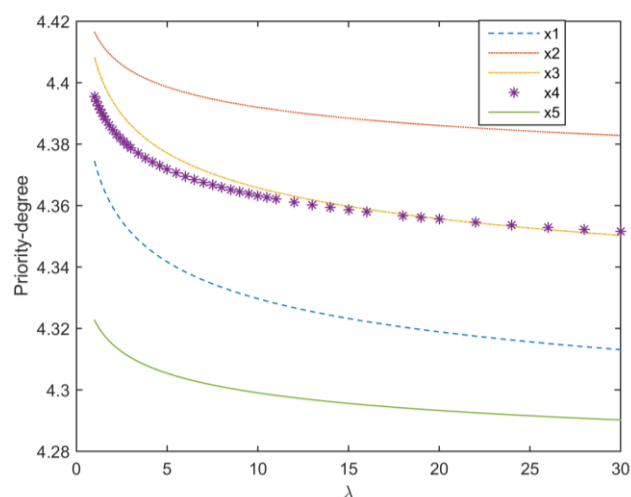


Figure 2. Priority degrees obtained by the WHFLFWA operator.

5.3. Compared with Other Methods

To demonstrate the effectiveness, stability, and superiority of the proposed decision-making method, three classical aggregation methods widely used in multi-attribute decision-making, namely the ordered weighted averaging (OWA) operator, the ordered weighted geometric (OWG) operator, and the technique for order preference by similarity

to ideal solution (TOPSIS), were extended to the weighted hesitant fuzzy linguistic environment for comparative analysis. The computational results obtained by these methods are summarized in Table 6.

Table 6. Comparison with other methods.

	Method 1	Method 2	Method 3
Method	OWA [12]	OWG [24]	TOPSIS [25]
Result	(4.24, 4.34, 4.04, 4.3, 4.25)	(4.18, 4.3, 3.97, 4.24, 4.21)	(0.58, 0.58, 0.53, 0.57, 0.47)
Ranking of alternatives	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3$	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3$	$x_2 \sim x_1 \succ x_4 \succ x_3 \succ x_5$

As shown in Table 1, under the weighted hesitant fuzzy linguistic environment, the OWA and OWG methods identify the same optimal alternative as the proposed method, namely x_2 , demonstrating a high degree of consistency among the three methods in identifying the optimal solution. In contrast, the TOPSIS method identifies both x_2 and x_1 as the optimal alternatives, resulting in a ranking that differs from that of the proposed method. This discrepancy can be attributed to the different decision-making mechanisms employed by the methods. Specifically, TOPSIS ranks alternatives according to their relative distances from the positive and negative ideal solutions, whereas the OWA, OWG, and the proposed method aggregate decision information using different aggregation operators. Such differences in aggregation mechanisms inevitably lead to variations in the final ranking results. Overall, the ranking results obtained by the proposed method exhibit a high level of consistency with those produced by the classical decision-making methods, particularly in correctly identifying the optimal alternative. These results demonstrate the rationality, effectiveness, and stability of the proposed method. Furthermore, compared with the TOPSIS method, the proposed approach provides greater discrimination among alternatives, thereby offering a more reliable decision-making tool for multi-attribute decision-making problems in a weighted hesitant fuzzy linguistic environment.

5.4. Compared with HFLE Method

Let us reconsider the problem which is described in Example 3. If the decision experts do not consider the importance of possible membership degrees of each alternative under every attribute and use the HFLEs to represent the decision-making information, then the hesitant fuzzy linguistic decision matrix $H = (h_S^{i,j})_{n \times l}$ is constructed as shown in Table 7.

Table 7. Decision matrix for HFLEs.

	c_1	c_2	c_3	c_4
x_1	$\{s_3, s_4\}$	$\{s_3, s_4\}$	$\{s_4, s_5, s_6\}$	$\{s_4, s_5\}$
x_2	$\{s_3, s_4, s_5\}$	$\{s_4, s_5\}$	$\{s_3, s_4\}$	$\{s_4, s_5, s_6\}$
x_3	$\{s_3, s_4\}$	$\{s_4, s_5, s_6\}$	$\{s_4, s_5\}$	$\{s_5\}$
x_4	$\{s_3, s_4, s_5\}$	$\{s_2, s_3, s_4\}$	$\{s_4, s_5, s_6\}$	$\{s_4, s_5\}$
x_5	$\{s_4, s_5\}$	$\{s_3, s_5, s_6\}$	$\{s_3, s_4, s_5\}$	$\{s_3, s_4\}$

First, two new operators are proposed. When $w(s_\beta) = \frac{1}{l(h_S(x_i))}$, the proposed WHFLFWG and WHFLFWA operators reduce to the HFLFWG and HFLFWA operators, respectively.

$$\text{HFLFWA } (h_S^1, h_S^2, \dots, h_S^n) = \bigoplus_{j=1}^n (\omega_j \cdot h_S^j)$$

$$= \bigcup_{s_{\beta_1} \in h_S^1, s_{\beta_2} \in h_S^2, \dots, s_{\beta_n} \in h_S^n} \{s_{l(1-\log_\lambda(1+\prod_{j=1}^n (\lambda^{1-\beta_j/l}-1)^{\omega_j}))}\}, l=t+1$$

$$\text{HFLFWG} (h_S^1, h_S^2, \dots, h_S^n) = \bigotimes_{j=1}^n (h_S^{j \wedge_F \omega_j})$$

$$= \bigcup_{s_{\beta_1} \in h_S^1, s_{\beta_2} \in h_S^2, \dots, s_{\beta_n} \in h_S^n} \{s_{l \log_\lambda(1+\prod_{j=1}^n (\lambda^{\beta_j/l}-1)^{\omega_j})}\}, l=t+1$$

Then, the operators used in the decision-making method defined in Section 4 are replaced by the HFLFWA and HFLFWG operators, and the rankings of the alternatives under different values of the parameter are listed in Table 8. It can be seen from Table 6 that the optimal alternative obtained by both the HFLFWA and HFLFWG operators is x_3 for different values of the parameter, which is completely different from the result in Example 3. Although the same decision-making process is adopted, the difference in the results arises because the experts use different approaches to represent the evaluation information. In Example 3, the decision matrix is constructed using WHFLEs by taking the weights of the possible membership degrees into account. This representation preserves more evaluation information provided by the experts, making the decision-making results more reliable and convincing. In practical decision-making problems, it is important to select an information representation tool that can adequately preserve the evaluation information provided by the experts. If the decision-makers cannot reach a consensus on the preference values or are able to assign different weights to fuzzy linguistic terms according to their professional knowledge and experience, WHFLEs are a suitable choice. On the other hand, if the decision-makers reach a consensus on the evaluation information and assume that all fuzzy linguistic terms have equal weights, HFLEs may be a more appropriate choice.

Then, the operators used in the decision-making method defined in Section 4 are replaced by the HFLFWA operator and the HFLFWG operator, and the rankings of alternatives under different parameters are listed in Table 6. It can be seen from Table 6 that the best choice given by the HFLFWA operator and the HFLFWG operator in different parameters is x_3 , which is completely different from Example 3. Both use the same decision-making process, and the reason for the difference in results is that the expert panel adopted different ways to represent the information. Considering the weight of possible membership degrees, Example 3 uses WHFLEs to construct a decision matrix. It is better to retain the expert information in decision-making model, which can make the results more convincing. In the actual decision-making, we need to fully recognize the importance of using a tool that can adequately represent the evaluation information provided by the expert panel. If the decision experts cannot agree on preference values or they can reasonably assign the weight of fuzzy linguistic terms according to their professional knowledge and experience, then the WHFLEs may be a good choice. If the decision experts have one man on information evaluation and believe that the weights of the fuzzy linguistic terms are the same, then the HFLEs may be a good choice.

Table 8. Decision-making result.

Parameter	Operator	
	HFLFWA	HFLFWG
$\lambda \rightarrow 1$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$
$\lambda = 2$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$
$\lambda = 5$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$
$\lambda = 10$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$
$\lambda = 50$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$
$\lambda = 100$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$
$\lambda \rightarrow +\infty$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$	$x_3 \succ x_5 \succ x_2 \succ x_4 \succ x_1$

6. Conclusions

Precisely representing preference information is essential for ensuring the accuracy and reliability of decision-making models. Although various types of fuzzy sets have been developed to represent information in decision-making problems, most linguistic fuzzy set models overlook the potential differences in the weights of linguistic terms. In this paper, the weighted hesitant fuzzy linguistic set (WHFLS) is introduced as a new multi-valued linguistic evaluation tool for representing preference information. It extends the classical hesitant fuzzy linguistic set (HFLS) and linguistic term set (LTS) by incorporating a weight vector into each hesitant fuzzy linguistic element, making it more suitable for modeling preference information in complex fuzzy environments. Furthermore, the Frank operational rules for WHFLSs are investigated. On this basis, two Frank aggregation operators for weighted hesitant fuzzy linguistic information, namely the WHFLFWA operator and the WHFLFWG operator, are developed, and their desirable properties are analyzed. Moreover, a multiple attribute decision-making (MADM) approach is proposed to solve decision-making problems involving weighted hesitant fuzzy linguistic information while effectively characterizing the weights of linguistic terms. Finally, a pairing assistance performance evaluation example is presented to verify the effectiveness of the proposed method and to illustrate how to determine the parameter in the proposed operators according to the decision-makers' attitudes toward the future development prospects of the decision problem.

Although weighted hesitant fuzzy linguistic sets (WHFLS) provide a flexible framework for modeling linguistic evaluations with hesitation and importance weights, several limitations still exist in practical applications. The representation process is relatively complex due to the simultaneous consideration of linguistic terms, hesitant information, and weights, which increases computational burden and reduces interpretability. Moreover, the determination of weights is often subjective, and existing aggregation and comparison methods may become unstable when handling highly inconsistent or large-scale data. Future research is expected to focus on developing distance and similarity measures for WHFLS and exploring their applications in clustering analysis, option pricing, dynamic programming, pattern recognition, and group decision-making problems.

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