

Article

Recursively Constructed Uniform Hypergraphs

Frank Gurski ^{1,*}  and Jochen Rethmann ²  and Egon Wanke ¹¹ Institute of Computer Science, Heinrich Heine University, 40225 Duesseldorf, Germany; egon.wanke@hhu.de² Faculty of Engineering and Computer Science, Niederrhein University of Applied Sciences, 47805 Krefeld, Germany; jochen.rethmann@hs-niederrhein.de

* Correspondence: frank.gurski@hhu.de

Abstract

In this work, we introduce and study a generalization for r -uniform hypergraphs of complement-reducible graphs, the so-called co-graphs. The operations for r -join-hypergraphs are the binary disjoint union of two given r -join-hypergraphs and the r -nary join, which inserts all possible hyperedges of cardinality r , each including exactly one vertex from each of r given r -join-hypergraphs. We characterize the primal graphs of r -join-hypergraphs as special co-graphs and give some properties of r -join-hypergraphs. This allows us to give a method that decides whether an r -uniform input hypergraph H is an r -join-hypergraph and, in the case of an affirmative answer, finds a decomposition tree for H in polynomial time. This task proved to be challenging. We show specific formulas for computing various hypergraph parameters for r -uniform hypergraphs defined by the binary disjoint union of two r -uniform hypergraphs and the r -nary join of r r -uniform hypergraphs. The parameters considered are the size of a largest stable set, the size of a largest co-stable set, the size of a largest independent set, the size of a largest co-independent set, the size of a smallest vertex cover, the size of a smallest 2-transversal, the size of a largest matching, the size of a smallest dominating set, the chromatic number, the strong chromatic number, and the upper chromatic number. This yields $\mathcal{O}(n \cdot r)$ -time algorithms to compute these values on r -join-hypergraphs on n vertices given by a decomposition tree. Of particular interest is the development of an efficient algorithm to compute the size of a largest matching. In addition, we infer relations among the considered parameters restricted to r -join-hypergraphs. Our methods generalize and reprove a number of results that are known for co-graphs.

Keywords: hypergraphs; uniform hypergraphs; co-graphs; r -join-hypergraphs; algorithms; special graph classes; system of distinct representatives

MSC: 05C65; 05C15



Academic Editor: Ronghua Li

Received: 19 June 2026

Revised: 6 July 2026

Accepted: 10 July 2026

Published: 14 July 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

Hypergraphs are essential data structures frequently used in modern research to represent and analyze complex and huge amounts of data. A hypergraph is a natural extension of a graph in which an edge can connect any number of vertices, implying that it is a data structure that offers more modeling possibilities than a conventional graph. The theory of hypergraphs is applied to model situations in applied sciences, such as protein-protein interactions [1], epidemics [2], chemistry [3,4], and hypergraph learning [5,6]. Further applications of hypergraphs can be found in [7] (Chapter 7). Hypergraphs are also

very interesting from a theoretical point of view. There are a number of results on various coloring problems on hypergraphs; see [8] (Chapter 11) and [9].

The class of complement-reducible graphs, referred to as co-graphs for short, has been studied since the 1970s in [10–12] as graphs that can be constructed from the one-vertex graph by complementation and disjoint union. One of the first comprehensive studies on this class can be found in [13]. Co-graphs are exactly those graphs that do not have a 4-vertex path as an induced subgraph [13]. Co-graphs lead to a class of graphs that occurs in a variety of applications; see [13–16]. Many difficult graph problems can be solved in linear time if one restricts the problem to co-graphs [13,17]. Co-graphs have been generalized to directed graphs [18,19], which also find applications in the field of genetics; see [20,21]. Many difficult digraph problems can be solved in linear time if they are restricted to directed co-graphs [22,23].

Our contribution. We extend the operations of co-graphs to r -uniform hypergraphs. The operations of r -join-hypergraphs are the disjoint union of two given r -join-hypergraphs and the r -nary join, which inserts all possible hyperedges of cardinality r , each including exactly one vertex from each of r given r -join-hypergraphs.

As our second contribution, we consider structural properties of r -join-hypergraphs. We give a tight upper bound on the number of hyperedges in special r -join-hypergraphs, which allows us to show that for $r \geq 3$ the class of r -join-hypergraphs is not closed under complementation for r -uniform hypergraphs. Furthermore, we characterize the primal graphs of r -join-hypergraphs as special co-graphs. We show that for $r \geq 3$ every hyperedge of a final join of a connected r -join-hypergraph uniquely determines the vertex sets of the r combined hypergraphs, and these sets are even unique. This uniqueness is the basis for our third contribution, which is a method that decides, for $r \geq 3$, whether an r -uniform input hypergraph H is an r -join-hypergraph and, in the case of an affirmative answer, finds a decomposition tree for H . The main idea is, for every height of the decomposition tree (at most $|V(H)|$), to check, for every $e \in E(H)$, whether e induces a partition of $V(H)$ corresponding to a final join. This leads to a running time of $\mathcal{O}(|V(H)|^2 + |E(H)| \cdot |V(H)|^2 + r \cdot |V(H)| \cdot |E(H)|^2)$.

As our fourth contribution, we show that for every r -join-hypergraph H given by a decomposition tree $T(H)$, the size of a largest stable set, the size of a largest co-stable set, the size of a largest independent set, the size of a largest co-independent set, the size of a smallest vertex cover, the size of a smallest 2-transversal, the size of a largest matching, the size of a smallest dominating set, the chromatic number, the strong chromatic number, and the upper chromatic number can be computed in $\mathcal{O}(|V(H)| \cdot r)$ time. The problem of calculating the size of a largest matching is of particular interest from an algorithmic point of view. Since a matching of largest size for the join of r pairwise vertex-disjoint r -uniform hypergraphs can contain up to $\min_{i=1}^r |V(H_i)|$ hyperedges of the final join, we give lower bounds on the number of pairwise disjoint hyperedges generated in a final join occurring in a matching of largest size. This allows us to compute the size of a largest matching for the join of r pairwise vertex-disjoint r -uniform hypergraphs $H_1, \dots, H_r$ from the sizes of largest matchings for $H_1, \dots, H_r$ in $\mathcal{O}(r)$ time. Since for $r = 2$ the set of r -join-hypergraphs corresponds to the well-known class of co-graphs, our solutions extend and prove a variety of results for computing graph parameters on co-graphs.

2. Preliminaries

2.1. Graphs

We consider finite undirected graphs $G = (V(G), E(G))$, where $V(G)$ is a nonempty finite set of vertices and $E(G) \subseteq \{\{u, v\} \mid u, v \in V(G), u \neq v\}$ is a finite set of edges. (Thus, we do not consider graphs with loops, i.e., edges with a single vertex, or repeated

edges, i.e., two or more edges containing the same set of vertices.) For a vertex $v \in V(G)$, we denote by $N_G(v)$ the set of all vertices which are adjacent to v in G , i.e., $N_G(v) = \{w \in V(G) \mid \{v, w\} \in E(G)\}$. The vertex set $N_G(v)$ is called the set of all *neighbors* of v in G or the *neighborhood* of v in G . Please note that v does not belong to $N_G(v)$. The *degree* of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is the number of neighbors of vertex v in G , i.e., $\deg_G(v) = |N_G(v)|$. For a graph G , its *complement graph* is defined by $\overline{G} = (V(G), \{\{u, v\} \mid u, v \in V(G), u \neq v, \{u, v\} \notin E(G)\})$.

We use the following indexed graphs. As usual, we denote by P_n the path on n vertices, by C_n the cycle on n vertices, by K_n the complete graph on n vertices, by $I_n = \overline{K_n}$ the empty graph (also called an edgeless graph or null graph) on n vertices, and by $K_{n_1, \dots, n_k}$ the complete k -partite graph on $n_1 + n_2 + \dots + n_k$ vertices.

2.2. Co-Graphs

A *co-graph*, or *complement-reducible graph*, is a graph that can be generated from the single-vertex graph by complementation and disjoint union; see [13]. There are several equivalent characterizations for co-graphs. From an algorithmic point of view, we recall that co-graphs can be defined from the single-vertex graph by the following two operations. Let G_1 and G_2 be two vertex-disjoint graphs.

- The *disjoint union* of G_1 and G_2 , denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.
- The *join* of G_1 and G_2 , denoted by $G_1 + G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{\{v_1, v_2\} \mid v_1 \in V(G_1), v_2 \in V(G_2)\}$.

Definition 1 (co-graph; [24]). *Every graph $(\{v\}, \emptyset)$ on a single vertex, denoted by v , is a co-graph. If G_1 and G_2 are vertex-disjoint co-graphs, then $G_1 \cup G_2$ and $G_1 + G_2$ are co-graphs, too. There are no other co-graphs.*

Example 1.

1. Every empty graph I_n with n vertices is a co-graph by the expression $(\dots((v_1 \cup v_2) \cup v_3) \cup \dots) \cup v_n$.
2. Every complete graph K_n with n vertices is a co-graph by the expression $(\dots((v_1 + v_2) + v_3) + \dots) + v_n$.
3. Every complete 2-partite graph $K_{n,m}$ with $n + m$ vertices is a co-graph by the expression $((\dots((v_1 \cup v_2) \cup v_3) \cup \dots) \cup v_n) + ((\dots((w_1 \cup w_2) \cup w_3) \cup \dots) \cup w_m)$.

The following closure properties are well known; see [24].

Theorem 1. *Co-graphs are (i) closed under complements and (ii) closed under induced subgraphs, but (iii) not closed under subgraphs.*

By the given definition, every co-graph can be represented by a tree structure, denoted as a *co-tree*. The leaves of the co-tree represent the vertices of the co-graph and the inner nodes of the co-tree correspond to the operations applied to the subexpressions defined by the subtrees.

Theorem 2 ([25]). *For every graph G , in time $\mathcal{O}(|V(G)| + |E(G)|)$, we can decide whether G is a co-graph and, if so, provide a corresponding co-tree.*

With the help of the co-tree, it has been shown that many difficult problems can be solved in polynomial time if they are restricted to co-graphs. For example, the following

well-known graph parameters (which are defined in [26]) can be calculated in linear time for co-graphs:

- the *independence number* α , i.e., the size of a maximum independent set—a subset of the vertices such that no two of them are connected by an edge;
- the *clique number* ω , i.e., the size of a maximum clique—a subset of the vertices such that every two of them are connected by an edge;
- the *clique covering number* θ , i.e., the minimum number of vertex-disjoint cliques needed to cover the vertex set,
- the *chromatic number* χ , i.e., the minimum number of vertex-disjoint independent sets needed to cover the vertex set, or equivalently, the minimum number of colors needed to color the vertices of a graph so that no two adjacent vertices have the same color;
- the *vertex cover number* τ , i.e., the size of a minimum vertex cover—a subset of the vertices that contains at least one vertex of each edge;
- the *domination number* γ , i.e., the size of a maximum dominating set—a subset of the vertices such that every vertex or one of its neighbors is contained in it; and
- the *matching number* ν , i.e., the size of a maximum matching—a subset of the edges such that no two of them share a common vertex.

Please note that some of the given abbreviations for graph parameters such as $\alpha, \omega, \dots$ will also be used to define special vertex sets in hypergraphs in Section 4. The context always makes it clear which parameter is meant.

The previous theorem can be used to efficiently determine all graph parameters listed above for co-graphs. In Table 1 we summarize these results, which are known from [13,27]. Note that the formulas, which are stated in Table 1, are not limited to co-graphs but also hold for general graphs.

Table 1. Graph parameters for single-vertex graphs, the disjoint union, and join operation for graphs and, thus, for co-graphs. The results can be found in [13,27].

Parameter	v	$G_1 \cup G_2$	$G_1 + G_2$
α	1	$\alpha(G_1) + \alpha(G_2)$	$\max(\alpha(G_1), \alpha(G_2))$
ω	1	$\max(\omega(G_1), \omega(G_2))$	$\omega(G_1) + \omega(G_2)$
θ	1	$\theta(G_1) + \theta(G_2)$	$\max(\theta(G_1), \theta(G_2))$
χ	1	$\max(\chi(G_1), \chi(G_2))$	$\chi(G_1) + \chi(G_2)$
τ	0	$\tau(G_1) + \tau(G_2)$	$\min(V(G_1) + \tau(G_2), V(G_2) + \tau(G_1))$
γ	1	$\gamma(G_1) + \gamma(G_2)$	$\min(\gamma(G_1), \gamma(G_2), 2)$
ν	0	$\nu(G_1) + \nu(G_2)$	$\min(V(G_1) + \nu(G_2), V(G_2) + \nu(G_1), \lfloor \frac{ V(G_1) + V(G_2) }{2} \rfloor)$

2.3. Hypergraphs

We use finite undirected *hypergraphs* $H = (V(H), E(H))$, where $V(H)$ is a nonempty finite set of *vertices* and $E(H)$ is a set of subsets of $V(H)$, the *hyperedges*. Edges with fewer than two elements are usually allowed, but will be disregarded here. (Thus, we do not consider hypergraphs with loops, i.e., hyperedges with a single vertex, or repeated edges, i.e., two or more hyperedges containing the same set of vertices.) This is a generalization of the notion of graphs since the hyperedges can have any size instead of size two. In a hypergraph, two vertices are said to be *adjacent* if there is a hyperedge containing both vertices. The *degree* of a vertex $v \in V(H)$, denoted by $\deg_H(v)$, in a hypergraph H is the number of hyperedges that contain v .

Hypergraph H' is a *subhypergraph* of hypergraph H if $V(H') \subseteq V(H)$ and for every $e' \in E(H')$ there is some $e \in E(H)$ such that $e' = e \cap V(H')$ and $|e'| \geq 2$. For every subset $F \subseteq E(H)$, we call the hypergraph $H_F = (V(H), F)$ the *partial subhypergraph* of H . For

every subset $W \subseteq V(H)$, we call $H_W = (W, E')$ with $E' = \{e \in E(H) \mid e \subseteq W\}$ the *induced subhypergraph* of H . That is, E' consists of all hyperedges of $E(H)$ that lie entirely in W .

In a hypergraph H a *path* P from v_0 to v_q is a vertex-hyperedge alternating sequence $v_0, e_1, v_1, e_2, v_2, \dots, e_q, v_q$ such that (i) $v_0, v_1, \dots, v_q \in V(H)$, (ii) $v_0, v_1, \dots, v_{q-1}$ are distinct, (iii) $v_1, \dots, v_q$ are distinct, (iv) $e_1, \dots, e_q \in E(H)$ are distinct, and (v) $v_{i-1} \in e_i$ and $v_i \in e_i$ for $i = 1, 2, \dots, q$. The integer q is the *length* of the path P . Notice that if there is a path from v_0 to v_q , there is also a path from v_q to v_0 . In this case, we say that P *connects* v_0 and v_q . A hypergraph is *connected* if for every pair of vertices, there is a path which connects these vertices; it is *not connected* or *disconnected* otherwise. A *connected component* of a hypergraph is every maximal set of vertices which are pairwise connected by a (possibly trivial) path. Let I_n be the *empty (edgeless) hypergraph* on n vertices and $E(I_n) = \emptyset$.

In the analysis of hypergraphs, one of the most useful concepts is the *primal graph* (which is also called the 2-section, clique graph, representing graph, or Gaifman graph) of a hypergraph. The *primal graph* of a hypergraph H is the graph $P(H) = (V(H), \{\{v, w\} \mid v \neq w, \exists e \in E(H) : \{v, w\} \subseteq e\})$.

2.4. Uniform Hypergraphs

The *rank* $r(H)$ of hypergraph H is the maximum cardinality of a hyperedge in H : $r(H) = \max_{e \in E(H)} |e|$; the *co-rank* $cr(H)$ of H is the minimum cardinality of a hyperedge in H : $cr(H) = \min_{e \in E(H)} |e|$. If $r(H) = cr(H) = r$, then H is *r-uniform* or *uniform*.

For a set X , we denote the set of all r -element subsets of X by $\binom{X}{r}$. For $2 \leq r \leq n$ we define the *complete r-uniform hypergraph* to be the hypergraph K_n^r for $|V(K_n^r)| = n$ and $E(K_n^r) = \binom{V(K_n^r)}{r}$.

Remark 1. Let H be an r -uniform hypergraph. Then, we have $|E(H)| \leq |E(K_{|V(H)|}^r)| = \binom{|V(H)|}{r}$.

For an r -uniform hypergraph H the complement $\bar{H}$ is the r -uniform hypergraph with vertex set $V(H)$ and edge set $\binom{V(H)}{r} \setminus E(H)$.

3. r-Join-Hypergraphs

In this section, we introduce a generalization of co-graphs to r -uniform hypergraphs, consider their structural properties, and give a first recognition algorithm. Therefore, we use two operations for r -uniform hypergraphs. The disjoint union operation of two r -uniform hypergraphs is straightforward, and the join inserts all possible hyperedges of cardinality r , each including exactly one vertex from each of r given hypergraphs.

- Let H_1 and H_2 be two vertex-disjoint r -uniform hypergraphs. The *disjoint union* of H_1 and H_2 , denoted by $\oplus(H_1, H_2)$, is the hypergraph with vertex set $V(H_1) \cup V(H_2)$ and hyperedge set $E(H_1) \cup E(H_2)$.
- Let $H_1, \dots, H_r$ be r pairwise vertex-disjoint r -uniform hypergraphs. The *join* of $H_1, \dots, H_r$, denoted by $\otimes(H_1, \dots, H_r)$, is defined as the hypergraph with vertex set $V(H_1) \cup \dots \cup V(H_r)$ and hyperedge set $E(H_1) \cup \dots \cup E(H_r) \cup \{\{u_1, \dots, u_r\} \mid u_i \in V(H_i), 1 \leq i \leq r\}$.

Please note that the disjoint union for r -uniform hypergraphs is identical to the disjoint union for graphs given in Section 2.2. We use different notations because, especially when we work with r -uniform hypergraphs and their primal graphs, it is useful to know whether we are combining hypergraphs or graphs.

Remark 2. Every join operation $\otimes(H_1, \dots, H_r)$ inserts $|\{\{u_1, \dots, u_r\} \mid u_i \in V(H_i), 1 \leq i \leq r\}| = \prod_{i=1}^r |V(H_i)|$ hyperedges of cardinality r .

For $r \geq 2$, the class of r -join-hypergraphs is recursively defined as follows.

Definition 2 (r -join-hypergraphs). Every hypergraph $(\{v\}, \emptyset)$ on a single vertex, denoted by v , is an r -join-hypergraph. If H_1 and H_2 are vertex-disjoint r -join-hypergraphs, then $\oplus(H_1, H_2)$ is an r -join-hypergraph, too. If $H_1, \dots, H_r$ are r pairwise vertex-disjoint r -join-hypergraphs, then $\otimes(H_1, \dots, H_r)$ is an r -join-hypergraph, too. There are no other r -join-hypergraphs.

Remark 3. The set of 2-join-hypergraphs is exactly the set of co-graphs.

Next we give all connected 3-join-hypergraphs on at most five vertices.

Example 2 (3-join-hypergraphs on three, four, and five vertices). Table 2 shows expressions for all connected 3-join-hypergraphs (up to isomorphism) on three, four, and five vertices.

Table 2. Expressions for all connected 3-join-hypergraphs (up to isomorphism) on three, four, and five vertices.

Expression	Figure
$X_1 = \otimes(a, b, c)$	Figure 1a
$X_2 = \otimes(\oplus(a, b), c, d)$	Figure 1b
$X_3 = \otimes(\oplus(a, b), \oplus(c, d), e)$	Figure 1c
$X_4 = \otimes(\oplus(a, b, c), d, e)$	Figure 1d
$X_5 = \otimes(\otimes(a, b, c), d, e)$	Figure 1e

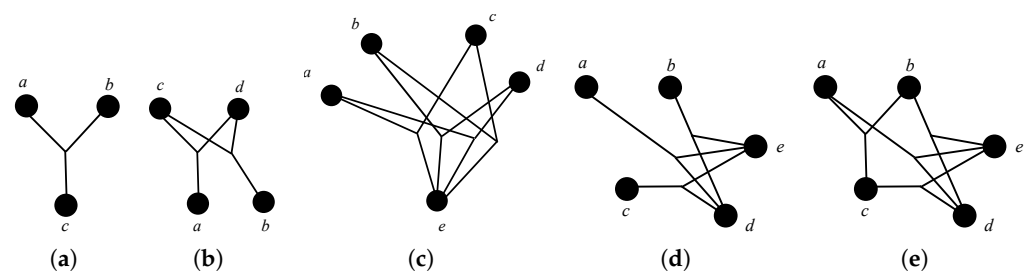


Figure 1. Five 3-join-hypergraphs considered in Example 2. (a) The 3-join-hypergraph defined by expression $X_1 = \otimes(a, b, c)$, (b) the 3-join-hypergraph defined by expression $X_2 = \otimes(\oplus(a, b), c, d)$, (c) the 3-join-hypergraph defined by expression $X_3 = \otimes(\oplus(a, b), \oplus(c, d), e)$, (d) the 3-join-hypergraph defined by expression $X_4 = \otimes(\oplus(a, b, c), d, e)$, and (e) the 3-join-hypergraph defined by expression $X_5 = \otimes(\otimes(a, b, c), d, e)$. We represent the vertices of hypergraphs graphically by black dots and the hyperedges as stars that have the vertices as their leaves.

Lemma 1. The complete r -uniform hypergraph K_n^r is an r -join-hypergraph if and only if $n = r$, i.e., if $|E(K_n^r)| = 1$.

Proof. For $n = r$, the K_n^r is obviously an r -join-hypergraph (For $n = 3$ see expression X_1 in Table 2, which can be generalized for every n). If for $n > r$, the complete r -uniform hypergraph K_n^r is an r -join-hypergraph, the final join operation combines r hypergraphs $H_1, \dots, H_r$ such that at least one H_i contains at least two vertices v' and v'' . All hyperedges of K_n^r , which contain v' and v'' and at least one vertex of some H_j , $j \neq i$, can only be inserted in the final join. Since the join operation of r -join-hypergraphs only inserts hyperedges of cardinality r each, including exactly one vertex from each of the r given hypergraphs, it follows that K_n^r is not an r -join-hypergraph. $\square$

Every r -join-hypergraph can be represented by a binary decomposition tree (also known as the expression tree) that reflects its construction starting with its vertices using disjoint union and join operations as follows:

Definition 3 (decomposition tree of r -join-hypergraphs).

- The decomposition tree $T(H)$ for an r -join-hypergraph H containing only one vertex v consists of a single node w (the root of T) labeled by v .
- The decomposition tree $T(H)$ for an r -join-hypergraph $H = \oplus(H_1, H_2)$ consists of a copy T_1 of the decomposition tree for H_1 , a copy T_2 of the decomposition tree for H_2 , an additional node w (the root of T) labeled by $\oplus$ and two additional edges, each between the roots of T_1 and T_2 and node w . The root of T_1 is the left child of w and the root of T_2 is the right child of w .
- The decomposition tree $T(H)$ for an r -join-hypergraph $H = \otimes(H_1, \dots, H_r)$ consists of a copy T_i of the decomposition tree for H_i for $1 \leq i \leq r$, an additional node w (the root of T) labeled by $\otimes$ and r additional edges, each between the roots of T_i for $1 \leq i \leq r$ and node w . The root of T_i for $1 \leq i \leq r$ is the i -th child of w .

In order to give a method for finding a decomposition tree, we consider properties of a final operation in connected r -join-hypergraphs. For connected r -join-hypergraphs H , we know that the final operation in every expression for H is the join of r vertex-disjoint r -join-hypergraphs. We start with $r = 2$, i.e., co-graphs.

Example 3. The clique K_4 with $V(K_4) = \{a, b, c, d\}$ is a co-graph and can be defined by the expressions

$$X_1 = ((a + b) + c) + d \quad \text{and} \quad X_2 = (a + b) + (c + d).$$

In expression X_1 , the final join is applied to the partition $\{\{a, b, c\}, \{d\}\}$, and in X_2 , the final join is applied to the partition $\{\{a, b\}, \{c, d\}\}$. Thus, the partition is not unique.

Next, we show that for $r \geq 3$ every hyperedge of a final join of a connected r -join-hypergraph uniquely determines the vertex sets of the r combined hypergraphs, and these sets are even unique.

Lemma 2. Let $r \geq 3$, H be a connected r -join-hypergraph defined by some expression X , and e be a hyperedge generated in the final join operation of X . Then the r vertex sets of the r combined hypergraphs in the final join of X can uniquely be determined from e .

Proof. Let $r \geq 3$, H be a connected r -join-hypergraph defined by some expression X , and $e = \{v_1, \dots, v_r\} \in E(H)$ be a hyperedge generated in the final join operation of X . Then starting with $V_i = \{v_i\}$, $1 \leq i \leq r$, the vertex sets $V_1, \dots, V_r$ of the r combined hypergraphs in the final join can be defined as follows. For every $v \in V(H) \setminus \{v_1, \dots, v_r\}$ there is exactly one hyperedge (generated by the final join) in $E(H)$ including v and $r - 1$ vertices $\{v_1, \dots, v_r\} \setminus \{v_j\}$ for some $j \in \{1, \dots, r\}$. The index j of the missing vertex of e leads to the set V_j for vertex v . $\square$

Lemma 3. Let $r \geq 3$, and let H be a connected r -join-hypergraph. Then H is the join of r hypergraphs $H_1, \dots, H_r$, and the vertex sets of the r combined hypergraphs are unique.

Proof by contradiction. Let $r \geq 3$, and H be a connected r -join-hypergraph. We assume that H can be defined by $\otimes(H_1, \dots, H_r)$ and by $\otimes(H'_1, \dots, H'_r)$ such that the two partitions $\{V(H_1), \dots, V(H_r)\}$ and $\{V(H'_1), \dots, V(H'_r)\}$ of $V(H)$ are different. For $e \in \{\{v_1, \dots, v_r\} \mid v_i \in V(H_i), 1 \leq i \leq r\}$ these two different partitions and Lemma 2 imply that $e \notin \{\{v_1, \dots, v_r\} \mid v_i \in V(H'_i), 1 \leq i \leq r\}$. Thus, $e \in E(H'_j)$ for some $1 \leq j \leq r$.

- If e is the only hyperedge generated in the join $\otimes(H_1, \dots, H_r)$, then for every $1 \leq i \leq r$, we have $|V(H_i)| = 1$ and in the family $\{V(H'_1), \dots, V(H'_r)\}$ all sets are empty except for set $V(H'_j)$, which is not possible within a partition.
- If there are at least two hyperedges generated in the join $\otimes(H_1, \dots, H_r)$, then there is some $1 \leq i \leq r$ such that $|V(H_i)| \geq 2$. For $V(H_i) \cap e = \{v\}$ and $v' \in V(H_i) \setminus \{v\}$, we obtain by $e' = e \setminus \{v\} \cup \{v'\}$ a hyperedge such that $e' \in \{\{v_1, \dots, v_r\} \mid v_i \in V(H_i), 1 \leq i \leq r\}$, $e' \neq e$, and $|e \cap e'| \geq 2$. Since $e \in E(H'_j)$ for some $1 \leq j \leq r$, it follows that $e' \notin \{\{v_1, \dots, v_r\} \mid v_i \in V(H'_i), 1 \leq i \leq r\}$ and, thus, $e' \in E(H'_j)$. The same argument for all further vertices of $V(H_i)$ implies $V(H_i) \subseteq V(H'_j)$. The same argument can be applied to all other vertex sets $V(H_{i'})$ with $|V(H_{i'})| \geq 2$, which implies that in the family $\{V(H'_1), \dots, V(H'_r)\}$ all sets are empty except for set $V(H'_j)$, which is not possible within a partition.

Thus, the assumption is not true, and the uniqueness of the vertex sets of the r combined hypergraphs follows. $\square$

Based on the uniqueness of a final join in connected r -join-hypergraphs for $r \geq 3$, we next show how to decide whether a given r -uniform hypergraph H is an r -join-hypergraph and, in the case of a positive answer, to find a decomposition tree $T(H)$. The main idea is, for every height of the decomposition tree (at most $|V(H)|$), to check, for every $e \in E(H)$ whether e induces a partition of $V(H)$ corresponding to a final join.

Theorem 3. *For every r -uniform hypergraph H and every $r \geq 3$, in time $\mathcal{O}(|V(H)|^2 + |E(H)| \cdot |V(H)|^2 + r \cdot |V(H)| \cdot |E(H)|^2)$, we can decide whether H is an r -join-hypergraph and, if so, provide a corresponding decomposition tree.*

Proof. In order to decide whether a given r -uniform hypergraph H is an r -join-hypergraph, we apply Algorithm 1. The correctness is based on the uniqueness of a final join in connected r -join-hypergraphs shown in Lemma 3. We test for every $e \in E(H)$ whether $e = \{v_1, \dots, v_r\}$ leads to a partition of $V(H)$ into r vertex sets of a final join of H . We initialize the partition by $V_i = \{v_i\}$ for $1 \leq i \leq r$. In order to insert the vertices $v \in V(H) \setminus \{v_1, \dots, v_r\}$ into $V_1, \dots, V_r$, we verify (as mentioned in the proof of Lemma 2) whether there is a hyperedge $e' \in E(H)$ such that $|e' \cap e| = r - 1$. We use labeling functions $f : V(H) \rightarrow \{0, 1\}$ and $g : E(H) \rightarrow \mathbb{N}$.

The running time can be bounded as follows. The connectivity of H and the partition of $V(H)$ into connected components can be determined by an auxiliary graph G with $V(G) = V(H)$ and $E(G) = \{\{v_1, v_2\}, \dots, \{v_{r-1}, v_r\} \mid \{v_1, \dots, v_r\} \in E(H)\}$, i.e., every hyperedge of H is replaced by a path in G . The connected components in G correspond to the connected components in H and can be computed by a depth-first search on G in linear time $\mathcal{O}(|V(G)| + |E(G)|) \subseteq \mathcal{O}(|V(H)| + r|E(H)|)$.

In the processing of connected subhypergraphs in Lines (11)–(23), the test whether one hyperedge $e \in E(H)$ leads to a partition of $V(H)$ into r vertex sets of a final join of H requires for the initialization of $V_1, \dots, V_r$ time $\mathcal{O}(|V(H)|)$, for function f time $\mathcal{O}(|V(H)|)$, for function g time $\mathcal{O}(r \cdot |E(H)|)$, and for assigning the vertices in $V(H) \setminus e$ into the sets $V_1, \dots, V_r$ time $\mathcal{O}(|E(H)|)$. Thus, for one hyperedge e , we need time in $\mathcal{O}(|V(H)| + r \cdot |E(H)|)$. Checking for every $e \in E(H)$ whether it leads to a partition of $V(H)$ into r vertex sets of a final join of H can be done in $\mathcal{O}(|E(H)|(|V(H)| + r \cdot |E(H)|)) \subseteq \mathcal{O}(|E(H)| \cdot |V(H)| + r \cdot |E(H)|^2)$.

Since a decomposition tree has a maximum height of $|V(H)|$, the running time can be bounded by

$$\begin{aligned} & \mathcal{O}(|V(H)| \cdot (|V(H)| + r|E(H)| + |E(H)| \cdot |V(H)| + r \cdot |E(H)|^2)) \\ \subseteq & \mathcal{O}(|V(H)|^2 + |E(H)| \cdot |V(H)|^2 + r \cdot |V(H)| \cdot |E(H)|^2) \end{aligned}$$

and the result follows. $\square$

Algorithm 1 Recognition

Input An r -uniform hypergraph H

Output A decomposition tree $T(H)$ or the answer “ H is not an r -join-hypergraph”

```

1: if  $V(H) = \{v\}$  then
2:   create  $T(H)$  consisting of a single node  $w$  (the root of  $T$ ) labeled by  $v$ 
3:   return  $w$ 
4: else
5:   if  $H$  has  $k \geq 2$  connected components  $H_1, \dots, H_k$  then
6:     for  $i = 1, \dots, k$  do create  $T(H_i)$  for component  $H_i$ 
7:     create a node  $w$  labeled by  $\oplus$  as the predecessor of the roots of  $T(H_1), \dots, T(H_k)$ 
8:     return  $w$ 
9:   else  $\triangleright H$  is connected
10:    for all  $e = \{v_1, \dots, v_r\} \in E(H)$  do  $\triangleright$  test whether  $e$  leads to a partition
11:      for  $i = 1, \dots, r$  do  $V_i = \{v_i\}$   $\triangleright$  of  $V(H)$  into  $r$  vertex sets of a final join
12:      for all  $v \in \{v_1, \dots, v_r\}$  do  $f(v) = 1$ 
13:      for all  $v \in V(H) \setminus \{v_1, \dots, v_r\}$  do  $f(v) = 0$ 
14:      for all  $e' \in E(H)$  do  $g(e') = \sum_{v \in e'} f(v)$ 
15:      for all  $e' \in E(H)$  such that  $g(e') = r - 1$  do
16:        let  $v_i$  be the vertex of  $e'$ , which is not in  $e$ 
17:        let  $v_j$  be the vertex of  $e$ , which is not in  $e'$ 
18:         $V_j = V_j \cup \{v_i\}$ 
19:      if  $V(H) = V_1 \cup \dots \cup V_r$  then
20:        for  $i = 1, \dots, r$  do let  $H_i$  be the subhypergraph induced by  $V_i$  in  $H$ 
21:        for  $i = 1, \dots, r$  do create  $T(H_i)$  for component  $H_i$ 
22:        create a node  $w$  labeled by  $\otimes$  as the predecessor of the roots of
23:         $T(H_1), \dots, T(H_r)$ 
24:        return  $w$ 
25:      return “ $H$  is not an  $r$ -join-hypergraph”  $\triangleright$  no hyperedge of a final join found
26:      stop

```

In Remark 1, we consider the number of hyperedges in r -uniform hypergraphs. Next, we give an improved bound for special r -join-hypergraphs, which we then use to show that r -join-hypergraphs are not closed under complementation for r -uniform hypergraphs.

Lemma 4. Let H be an r -join-hypergraph such that $|V(H)| = r^k$ for some integers $r \geq 2$ and $k \geq 2$, then $|E(H)| \leq \frac{|V(H)|^r - |V(H)|}{r^r - r} = \frac{r^{kr} - r^k}{r^r - r}$.

Proof. In order to give an upper bound on the number of hyperedges in an r -join-hypergraph H , we assume that every inner node in a decomposition tree T of H (Definition 3) represents a join operation. Thus, we can assume that T is a complete r -nary tree of height $k = \log_r(|V(H)|)$. The root r of T partitions $V(T)$ into $k + 1$ levels L_ℓ , $0 \leq \ell \leq k$, such that L_ℓ consists of the nodes of distance ℓ from r , i.e., L_0 consists of the root of T and L_k consists of the leaves of T (and, thus, the vertices of $V(H)$). The number of hyperedges in H can be obtained from the number of hyperedges defined by the inner

nodes of T . Therefore, we consider for every level $\ell = 0, \dots, k-1$ the number of nodes in level ℓ and the number of hyperedges generated by the join operations in level ℓ . The number of nodes in level ℓ is r^ℓ . The number of hyperedges generated by the join operations in level ℓ is the number of leaves in subtrees with a root at level $\ell+1$ to the power of r , which leads to $(r^{k-\ell-1})^r$ hyperedges. By Remark 2 we get

$$|E(H)| \leq \sum_{\ell=0}^{k-1} r^\ell \cdot (r^{k-\ell-1})^r = \sum_{\ell=0}^{k-1} r^{\ell+(k-\ell-1)r} = r^{rk-r} \cdot \sum_{\ell=0}^{k-1} r^{\ell(1-r)}.$$

Recognizing that the sum is a geometric series, we obtain

$$|E(H)| \leq r^{rk-r} \frac{1 - r^{(1-r)k}}{1 - r^{1-r}} = \frac{(r^k)^r}{r^r} \cdot \frac{1 - (r^k)^{1-r}}{1 - r^{1-r}} = \frac{(r^k)^r - r^k}{r^r - r}.$$

Since $r^k = |V(H)|$, we obtain

$$|E(H)| \leq \frac{|V(H)|^r - |V(H)|}{r^r - r}.$$

This shows the statements of the lemma. $\square$

Please note that the proof of Lemma 4 shows that the given bound can be achieved.

Corollary 1. Let $r \geq 3$, and let H be an r -uniform hypergraph and $|V(H)| = r^k$ for some integer $k \geq 2$, then H or its complement (w.r.t. r -uniform hypergraphs) $\bar{H}$ is not an r -join-hypergraph.

Proof. Let H be an r -uniform hypergraph on r^k vertices for some integer $k \geq 2$. By definition of the complement $\bar{H}$ of the r -uniform hypergraph H and by Remark 1, it holds that

$$|E(H)| + |E(\bar{H})| = \binom{r^k}{r}. \quad (1)$$

We assume that H and $\bar{H}$ are r -join-hypergraphs. By Lemma 4, it holds that

$$|E(H)| \leq \frac{r^{kr} - r^k}{r^r - r} \quad (2)$$

and

$$|E(\bar{H})| \leq \frac{r^{kr} - r^k}{r^r - r}. \quad (3)$$

If we add Equations (2) and (3), we obtain by Equation (1)

$$\binom{r^k}{r} \leq 2 \frac{r^{kr} - r^k}{r^r - r}. \quad (4)$$

Since, by Lemma A4 in Appendix A, for every $r \geq 3$ and $k \geq 2$, we have $\binom{r^k}{r} > 2 \frac{r^{kr} - r^k}{r^r - r}$ we know that H or $\bar{H}$ is not an r -join-hypergraph. $\square$

There are also r -join-hypergraphs whose complements are r -join-hypergraphs. By expression X_2 in Example 2, we know that the hypergraph H shown in Figure 1b is a 3-join-hypergraph. Since H is self-complementary, its complement is also a 3-join-hypergraph. Furthermore, every complete r -uniform hypergraph K_r^r is an r -join-hypergraph (Lemma 1), and its complement is an empty (edgeless) hypergraph I_r and, thus, also an r -join-hypergraph.

Closure properties for several graph transformations on co-graphs are given in Theorem 1. Next, we consider such properties on r -join-hypergraphs for $r \geq 3$.

Theorem 4. Let $r \geq 3$. The class of all r -join-hypergraphs is (i) not closed under complementation for r -uniform hypergraphs, (ii) closed under induced subhypergraphs, (iii) not closed under subhypergraphs, and (iv) not closed under partial subhypergraphs.

Proof.

- (i) Holds by Corollary 1.
- (ii) Let H be an r -join-hypergraph and $W \subseteq V(H)$. An expression for the induced subhypergraph of H defined by the vertices of W can be obtained from an expression for H by deleting operations corresponding to vertices from $V(H) \setminus W$ and some obvious cleaning transformations. If a join operation loses all vertices of at least one of the involved hypergraphs, it is replaced by the disjoint union of the remaining involved hypergraphs.
- (iii) We show the statement by a counterexample for every $r \geq 3$. The join of r single vertices defines an r -join-hypergraph H_r on exactly r vertices and one hyperedge of cardinality r . Every subset $V' \subseteq V(H_r)$ with $r - 1$ vertices defines a subhypergraph H'_r of H_r . Since H'_r has one hyperedge of cardinality $r - 1$, it is not an r -join-hypergraph.
- (iv) We show the statement by a counterexample for every $r \geq 3$. The join of H_r (defined in (iii)) and $r - 1$ single vertices defines an r -join-hypergraph J_r on exactly $2r - 1$ vertices and (by Remark 2) $r + 1$ hyperedges of cardinality r . Then $J'_r = (V(J_r), E(H_r) \cup \{e\})$, for some hyperedge e defined in the final join of J_r , is a connected partial subhypergraph of J_r on exactly two hyperedges intersecting in one vertex. If J'_r is an r -join-hypergraph, the final join combines r vertex sets $V_1, \dots, V_r$. Since J'_r has $2r - 1$ vertices, at least one of the sets $V_1, \dots, V_r$ has more than one vertex. If there is exactly one set with more than one vertex, then by Remark 2 the final join defines $r > 2$ hyperedges, and if at least two sets have more than one vertex, then by Remark 2 the final join defines at least $4 > 2$ hyperedges. Thus, J'_r is not an r -join-hypergraph.

This shows the statements of the theorem. $\square$

Please note that for a join operation $\otimes(H_1, \dots, H_r)$ removing one vertex $v \in V(H_j)$ implies the removal of all $e \in E(\otimes(H_1, \dots, H_r))$ where $v \in e$, which are $\prod_{i=1, i \neq j}^r |V(H_i)|$ hyperedges of cardinality r . Furthermore, possibly some hyperedges $e \subseteq V(H_j)$ are removed. Thus, an induced subhypergraph of a connected r -join-hypergraph reduces its number of hyperedges remarkably.

Next, we consider the primal graphs $P(H)$ of r -join-hypergraphs H .

Remark 4. If $H = \otimes(H_1, \dots, H_r)$, then the primal graph $P(H)$ has a complete r -partite graph $K_{|V(H_1)|, \dots, |V(H_r)|}$ as subgraph and $P(H)$ is equal to $K_{|V(H_1)|, \dots, |V(H_r)|}$ if and only if all H_i are edgeless.

In order to characterize r -join-hypergraphs by their primal graphs for every $r \geq 2$, we introduce a subclass of co-graphs, which we refer to as r -co-graphs. The only difference lies in the restriction on the join operation, where r graphs must now be combined.

Definition 4 (r -co-graph). The graph $G = (\{v\}, \emptyset)$ is an r -co-graph. If G_1 and G_2 are vertex-disjoint r -co-graphs, then $G_1 \cup G_2$ is an r -co-graph, too. If $G_1, \dots, G_r$ are pairwise vertex-disjoint r -co-graphs, then $+(G_1, \dots, G_r) = (\dots (G_1 + G_2) + \dots) + G_r$ is an r -co-graph, too. There are no other r -co-graphs.

Example 4.

1. Every empty graph I_n with n vertices is an r -co-graph by the expression $X_n = (\dots ((v_1 \cup v_2) \cup v_3) \cup \dots) \cup v_n$.

2. Every complete graph K_r with r vertices is an r -co-graph by the expression $Y_r = +(v_1, \dots, v_r)$.
3. Every complete r -partite graph $K_{n_1, \dots, n_r}$ with $n_1 + n_2 + \dots + n_r$ vertices is an r -co-graph by the expression $+(X_{n_1}, X_{n_2}, \dots, X_{n_r})$.

Remark 5. Every r -co-graph is a co-graph. For $r = 2$ the classes are equal. For every $r \geq 3$, the class of r -co-graphs is a proper subclass of co-graphs. For $r \geq 3$ the complete graph K_ℓ for $2 \leq \ell \leq r - 1$ is a co-graph but no r -co-graph.

In contrast to co-graphs, for $r \geq 3$, the set of r -co-graphs is not closed under induced subgraphs.

Example 5. Let $r \geq 3$. Since K_1 is an r -co-graph, the join of r copies of K_1 , which yields K_r , is also an r -co-graph. Furthermore, the join of $r - 1$ copies of K_1 and one K_r shows that K_{2r-1} is an r -co-graph. As a counterexample, we consider K_{2r-2} , which is a connected induced subgraph of K_{2r-1} . If K_{2r-2} is an r -co-graph, then the final join combines r vertex sets $V_1, \dots, V_r$. Since K_{2r-2} has $2r - 2$ vertices, each of the sets $V_1, \dots, V_r$ has fewer than r vertices. Thus, all edges of K_{2r-2} have to be defined in the final join. Since at least one of the sets $V_1, \dots, V_r$ has at least two vertices, the final join cannot define all edges of K_{2r-2} . Thus, K_{2r-2} is not an r -co-graph and r -co-graphs are not closed under induced subgraphs.

Lemma 5. Let H be an r -uniform hypergraph for $r \geq 2$. Then H is an r -join-hypergraph if and only if the primal graph $P(H)$ is an r -co-graph.

Proof. Let H be an r -join-hypergraph; we show by induction on the recursive structure of H that $P(H)$ is an r -co-graph.

- If H is a single-vertex r -join-hypergraph, then $P(H)$ is a single-vertex graph and, thus, an r -co-graph.
- If $H = \oplus(H_1, H_2)$, then by Definition 2 the two hypergraphs H_1 and H_2 are r -join-hypergraphs and by assumption $P(H_1)$ and $P(H_2)$ are r -co-graphs and $P(H)$ is an r -co-graph by the expression $P(H_1) \cup P(H_2)$.
- If $H = \otimes(H_1, \dots, H_r)$, then by Definition 2 the r hypergraphs $H_1, \dots, H_r$ are r -join-hypergraphs and by assumption $P(H_1), \dots, P(H_r)$ are r -co-graphs and $P(H)$ is an r -co-graph by the expression $+(P(H_1), P(H_2), \dots, P(H_r))$.

Let H be an r -uniform hypergraph and let $P(H)$ be an r -co-graph. We show by induction on the recursive structure of $P(H)$ that H is an r -join-hypergraph.

- If $P(H)$ is a single-vertex r -co-graph, then H is a single-vertex hypergraph and, thus, an r -join-hypergraph.
- If $P(H)$ is the disjoint union of two graphs, then by Definition 4, these two graphs are r -co-graphs, and by assumption, these two graphs are primal graphs of two r -join-hypergraphs H_1 and H_2 , and H is an r -join-hypergraph by the expression $\oplus(H_1, H_2)$.
- If $P(H)$ is the join of r graphs, then by Definition 4, these r graphs are r -co-graphs, and by assumption, these r graphs are primal graphs of r r -join-hypergraphs $H_1, \dots, H_r$, and H is an r -join-hypergraph by the expression $\otimes(H_1, \dots, H_r)$.

This shows the statements of the lemma. $\square$

4. Algorithms on r -Join-Hypergraphs

In the subsequent subsections (Section 4.1–4.11) we show specific formulas for computing several graph parameters for r -uniform hypergraphs defined by the binary disjoint union of two r -uniform hypergraphs and the r -nary join of r many r -uniform hypergraphs.

In the final subsection (Section 4.12), we apply our formulas to compare the considered parameters and conclude that they can be computed in polynomial time when restricted to r -uniform hypergraphs defined by disjoint union and join operation and, thus, on r -join-hypergraphs.

4.1. Max Stable Set

A set $V \subseteq V(H)$ that does not contain all vertices of a hyperedge of $E(H)$, i.e., $e \cap V \neq e$ for all $e \in E(H)$, is called a *stable set*. The *stability number* of a hypergraph H , denoted by $\alpha(H)$, is defined as the maximum cardinality of a stable set in H .

Remark 6. For every r -uniform hypergraph H with $|E(H)| \geq 1$, we have $r - 1 \leq \alpha(H) \leq |V(H)| - 1$.

Lemma 6. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs.

1. $\alpha(\{\{v\}, \emptyset\}) = 1$
2. $\alpha(\oplus(H_1, H_2)) = \alpha(H_1) + \alpha(H_2)$
3. $\alpha(\otimes(H_1, \dots, H_r)) = \sum_{i=1}^r \alpha(H_i) - \min_{i=1}^r \alpha(H_i)$

Proof.

1. $\alpha(\{\{v\}, \emptyset\}) = 1$ holds by definition.
2. The result follows since S is a stable set in $\oplus(H_1, H_2)$ if and only if $S \cap V(H_1)$ is a stable set in H_1 and $S \cap V(H_2)$ is a stable set in H_2 .
3. To show the upper bound $\alpha(\otimes(H_1, \dots, H_r)) \leq \sum_{i=1}^r \alpha(H_i) - \min_{i=1}^r \alpha(H_i)$, we assume that there is a stable set $S \subseteq V(\otimes(H_1, \dots, H_r))$ of size greater than $\sum_{i=1}^r \alpha(H_i) - \min(\alpha(H_1), \dots, \alpha(H_r))$. If S contains more than $\alpha(H_i)$ vertices from some $V(H_i)$, then S is not a stable set. Thus, $|S| > \sum_{i=1}^r \alpha(H_i) - \min(\alpha(H_1), \dots, \alpha(H_r))$ implies that S contains at least one vertex from each of the r combined hypergraphs $H_1, \dots, H_r$. Thus, the considered hyperedge insertion operation would insert at least one hyperedge $e \subseteq S$, which is a contradiction to the stability of S and shows the upper bound. In order to show the lower bound $\alpha(\otimes(H_1, \dots, H_r)) \geq \sum_{i=1}^r \alpha(H_i) - \min_{i=1}^r \alpha(H_i)$, let S_i , $1 \leq i \leq r$, be a stable set for H_i . By definition of the join operation, the disjoint union of $r - 1$ of the sets $S_1, \dots, S_r$ is a stable set in the hypergraph $\otimes(H_1, \dots, H_r)$. By choosing some $r - 1$ largest among these sets, the lower bound follows.

This shows the statements of the lemma. $\square$

For $r = 2$, a stable set in an r -uniform hypergraph is an independent set in a graph. The results of Lemma 6 generalize the results of [13,27] (see Table 1) for computing the size of a largest independent set in a co-graph.

4.2. Max Co-Stable Set

For r -uniform hypergraphs, a *co-stable set* C of hypergraph H is a set of vertices such that every subset of C of order r forms a hyperedge of H . Every set of up to $r - 1$ vertices in an r -uniform hypergraph is a (*trivial*) *co-stable set* in H . (The given definition of a co-stable set in r -uniform hypergraphs extends the definition of a clique in a graph. While a single vertex in a graph is a clique, in r -uniform hypergraphs every set of up to $r - 1$ vertices is a co-stable set, since every possible hyperedge is present.) The *co-stability number* of an r -uniform hypergraph H is defined as the maximum cardinality of a co-stable set in H . We denote this value by $\omega(H)$.

Remark 7. For every r -uniform hypergraph H with $|E(H)| \geq 1$, we have $r \leq \omega(H) \leq |V(H)|$.

Remark 8. For every r -uniform hypergraph H , it holds that $\omega(H) = \alpha(\overline{H})$.

Since for $r \geq 3$ the class of all r -join-hypergraphs is not closed under complementation (Theorem 4) the results of Lemma 6 are not useful to compute the co-stability number of r -join-hypergraphs. For $r = 2$, a co-stable set is a clique in a graph. For $r \geq 3$, the operations of r -join-hypergraphs do not create arbitrarily large co-stable sets by the following lemma.

Lemma 7. Let $H_1, \dots, H_r$ be $r \geq 3$ pairwise vertex-disjoint r -uniform hypergraphs.

1. $\omega(\{\{v\}, \emptyset\}) = 1$
2. $\omega(\oplus(H_1, H_2)) = \max(\omega(H_1), \omega(H_2), \min(|V(H_1)| + |V(H_2)|, r - 1))$
3. $\omega(\otimes(H_1, \dots, H_r)) = \max(\max_{i=1}^r \omega(H_i), r)$

Proof.

1. $\omega(\{\{v\}, \emptyset\}) = 1$ holds by definition.
2. The result follows since V is a co-stable set in $\oplus(H_1, H_2)$ if and only if V is a co-stable set in H_1 or V is a co-stable set in H_2 or V is a trivial co-stable set.
3. The lower bound $\omega(\otimes(H_1, \dots, H_r)) \geq \max(\max_{i=1}^r \omega(H_i), r)$ follows since every co-stable set in the combined hypergraphs $H_1, \dots, H_r$ and every hyperedge of the inserted join is a co-stable set in $\otimes(H_1, \dots, H_r)$.

To show upper bound $\omega(\otimes(H_1, \dots, H_r)) \leq \max(\max_{i=1}^r \omega(H_i), r)$, we observe that the join operation only inserts hyperedges which contain exactly one vertex of every hypergraph H_i for $1 \leq i \leq r$. Thus, the only co-stable sets in $\otimes(H_1, \dots, H_r)$, which have vertices from more than one of the combined hypergraphs, are the hyperedges of the inserted join.

This shows the statements of the lemma. $\square$

Corollary 2. Let H be an r -join-hypergraph, and let $r \geq 3$. Then, we have $\min(|V(H)|, r - 1) \leq \omega(H) \leq r$.

This upper bound on the size of co-stable sets might be a drawback in our definition of r -join-hypergraphs. The co-independent sets behave more naturally when defining large cliques in r -join-hypergraphs; see Lemma 11.

4.3. Max Independent Set

A set $V \subseteq V(H)$ in which no two vertices are allowed to be adjacent, i.e., are not contained in some common hyperedge, is called an *independent set* or *strongly stable set*. The *independence number* of a hypergraph H , denoted by $\beta(H)$, is defined as the maximum cardinality of an independent set in H . (Using $\beta(H)$, we follow the notations of [28]. In [29], the maximum cardinality of a strongly stable set in H is denoted by $\bar{\alpha}(H)$. In [7], the maximum cardinality of a strongly stable set in H is denoted by $\alpha'(H)$.)

The following relation has been shown in [28].

Remark 9. For every hypergraph H , it holds that $\alpha(H) \geq \beta(H)$. If H is a graph, then equality holds. Let H be the 3-join-hypergraph shown in Figure 1a. Then, we have $\beta(H) = 1$ and $\alpha(H) = 2$. Thus, even for 3-join-hypergraphs, the equality does not always hold.

Remark 10. For every r -uniform hypergraph H with $|E(H)| \geq 1$, we have $1 \leq \beta(H) \leq |V(H)| - r + 1$.

There is a close relation between the independence number of a hypergraph and the independence number of its primal graph.

Lemma 8 ([30]). For every hypergraph H , it holds that $\alpha(P(H)) = \beta(H)$.

Since the primal graph of an r -join-hypergraph is a special co-graph (Lemma 5) we can compute the independence number of an r -join-hypergraph by computing the independence number of a co-graph; see [13,27] or Table 1.

Lemma 9. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs.

1. $\beta(\{\{v\}, \emptyset\}) = 1$
2. $\beta(\oplus(H_1, H_2)) = \beta(H_1) + \beta(H_2)$
3. $\beta(\otimes(H_1, \dots, H_r)) = \max_{i=1}^r \beta(H_i)$

Proof.

1. $\beta(\{\{v\}, \emptyset\}) = 1$ holds by definition.
2. By Lemmas 8 and 5, it holds that

$$\begin{aligned} \beta(\oplus(H_1, H_2)) &= \alpha(P(\oplus(H_1, H_2))) \\ &= \alpha(P(H_1) \cup P(H_2)) \\ &= \alpha(P(H_1)) + \alpha(P(H_2)) \\ &= \beta(H_1) + \beta(H_2). \end{aligned}$$

3. By Lemmas 8 and 5, it holds that

$$\begin{aligned} \beta(\otimes(H_1, \dots, H_r)) &= \alpha(P(\otimes(H_1, \dots, H_r))) \\ &= \alpha(P(H_1) + \dots + P(H_r)) \\ &= \max_{i=1}^r \alpha(P(H_i)) \\ &= \max_{i=1}^r \beta(H_i). \end{aligned}$$

This shows the statements of the lemma. $\square$

4.4. Max Co-Independent Set

A set $V \subseteq V(H)$ in which every two vertices are adjacent is called a *co-independent set*. The *co-independence number* of a hypergraph H , denoted by $\delta(H)$, is defined as the maximum cardinality of a co-independent set in H .

Remark 11. For every r -uniform hypergraph H with $|E(H)| \geq 1$ we know $r \leq \delta(H) \leq |V(H)|$.

There is a close relation between the co-independence number of a hypergraph and the clique number of its primal graph.

Lemma 10 ([30]). For every hypergraph H , it holds that $\omega(P(H)) = \delta(H)$.

Since the primal graph of an r -join-hypergraph is a special co-graph (Lemma 5), we can compute the co-independence number of an r -join-hypergraph by computing the clique number of a co-graph; see [13,27] or Table 1.

Lemma 11. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs.

1. $\delta(\{\{v\}, \emptyset\}) = 1$
2. $\delta(\oplus(H_1, H_2)) = \max(\delta(H_1), \delta(H_2))$
3. $\delta(\otimes(H_1, \dots, H_r)) = \sum_{i=1}^r \delta(H_i)$

Proof.

1. $\delta(\{\{v\}, \emptyset\}) = 1$ holds by definition.
2. The statement can be proven as Lemma 9(2) using Lemmas 10 and 5.
3. The statement can be proven as Lemma 9(3) using Lemmas 10 and 5.

This shows the statements of the lemma. $\square$

For $r = 2$, a co-independent set is a clique in a graph. The results of Lemma 11 generalize the results of [13,27] (see Table 1) for computing the size of a largest clique in a co-graph.

4.5. Min Vertex Cover

A *vertex cover* in a hypergraph is a set of vertices such that every hyperedge of the hypergraph contains at least one vertex of that set. The *vertex-cover number* (also known as the *transversal number*) of a hypergraph H , denoted by $\tau(H)$, is the smallest size of a vertex cover in H .

Gallai's Theorem [31] on the close relation between the vertex cover number and the independence number of graphs also holds for hypergraphs, if we consider the stability number.

Lemma 12 ([30]). *For every hypergraph H , it holds that $\alpha(H) + \tau(H) = |V(H)|$.*

Remark 12. *For every r -uniform hypergraph H with $|E(H)| \geq 1$, we have $1 \leq \tau(H) \leq |V(H)| - r + 1$.*

By Lemma 12, our results given in Lemma 6 can be used to compute the vertex-cover number of an r -join-hypergraph using its stability number. Next we give concrete formulas for the calculation of the vertex-cover number of an r -join-hypergraph, which will be modified for computing a 2-transversal in Section 4.6.

Lemma 13. *Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs.*

1. $\tau(\{\{v\}, \emptyset\}) = 0$
2. $\tau(\oplus(H_1, H_2)) = \tau(H_1) + \tau(H_2)$
3. $\tau(\otimes(H_1, \dots, H_r)) = \sum_{i=1}^r \tau(H_i) + \min_{i=1}^r (|V(H_i)| - \tau(H_i))$

Proof.

1. $\tau(\{\{v\}, \emptyset\}) = 0$ holds by definition.
2. By Lemmas 12 and 6(2), it holds that

$$\begin{aligned} \tau(\oplus(H_1, H_2)) &= |V(\oplus(H_1, H_2))| - \alpha(\oplus(H_1, H_2)) \\ &= |V(H_1)| + |V(H_2)| - \alpha(H_1) - \alpha(H_2) \\ &= \tau(H_1) + \tau(H_2). \end{aligned}$$

3. By Lemmas 12 and 6(3), it holds that

$$\begin{aligned} \tau(\otimes(H_1, \dots, H_r)) &= \sum_{i=1}^r |V(H_i)| - \alpha(\otimes(H_1, \dots, H_r)) \\ &= \sum_{i=1}^r |V(H_i)| - \sum_{i=1}^r \alpha(H_i) + \min_{i=1}^r \alpha(H_i) \\ &= \sum_{i=1}^r (|V(H_i)| - \alpha(H_i)) + \min_{i=1}^r \alpha(H_i) \\ &= \sum_{i=1}^r \tau(H_i) + \min_{i=1}^r (|V(H_i)| - \tau(H_i)). \end{aligned}$$

This shows the statements of the lemma. $\square$

4.6. Min 2-Transversal

A 2-transversal in a hypergraph is a set of vertices such that every hyperedge of the hypergraph contains at least two vertices of that set. The 2-transversal number of a hypergraph H , denoted by $\tau_2(H)$, is the smallest size of a 2-transversal in H .

Remark 13. For every hypergraph H , we have $0 \leq \tau(H) \leq \tau_2(H)$.

For $r = 2$, a minimum 2-transversal of a connected graph is the vertex set. Next we assume that $r \geq 3$.

Remark 14. For every r -uniform hypergraph H with $|E(H)| \geq 1$ for $r \geq 3$, we have $2 \leq \tau_2(H) \leq |V(H)| - r + 2$.

In our following results we denote by $\min(a_1, \dots, a_r)$ the smallest value among a multiset of integers $\langle a_1, \dots, a_r \rangle$ and by $\underline{\min}(a_1, \dots, a_r)$ the smallest value among the multiset of integers $\langle a_1, \dots, a_r \rangle \setminus \langle \min(a_1, \dots, a_r) \rangle$. That is, if $\min(a_1, \dots, a_r)$ occurs twice in $\langle a_1, \dots, a_r \rangle$ both values are equal.

Lemma 14. Let $H_1, \dots, H_r$ be $r \geq 3$ pairwise vertex-disjoint r -uniform hypergraphs.

1. $\tau_2(\{\{v\}, \emptyset\}) = 0$
2. $\tau_2(\oplus(H_1, H_2)) = \tau_2(H_1) + \tau_2(H_2)$
3. $\tau_2(\otimes(H_1, \dots, H_r)) = \sum_{i=1}^r \tau_2(H_i) + \min_{i=1}^r (|V(H_i)| - \tau_2(H_i)) + \underline{\min}_{i=1}^r (|V(H_i)| - \tau_2(H_i))$

Proof.

1. $\tau_2(\{\{v\}, \emptyset\}) = 0$ holds by definition.
2. The result follows since V is a 2-transversal in $\oplus(H_1, H_2)$ if and only if $V \cap V(H_1)$ is a 2-transversal in H_1 and $V \cap V(H_2)$ is a 2-transversal in H_2 .
3. Let $k = \sum_{i=1}^r \tau_2(H_i) + \min_{i=1}^r (|V(H_i)| - \tau_2(H_i)) + \underline{\min}_{i=1}^r (|V(H_i)| - \tau_2(H_i))$ be the claimed 2-transversal number.

To show the upper bound $\tau_2(\otimes(H_1, \dots, H_r)) \leq k$, we note that for $1 \leq i \leq r$, there is a 2-transversal V_i of size $\tau_2(H_i)$ in H_i . As a 2-transversal for $\otimes(H_1, \dots, H_r)$, we choose

$$V = \bigcup_{i=1}^r V_i \setminus (V_{j'} \cup V_{j''}) \cup (V(H_{j'}) \cup V(H_{j''})),$$

where $j' = \arg \min (|V(H_i)| - \tau_2(H_i))$ and $j'' = \arg \underline{\min} (|V(H_i)| - \tau_2(H_i))$. That means that we choose the disjoint union of all 2-transversals of the r combined hypergraphs, with two of these sets, $V_{j'}$ and $V_{j''}$, replaced by the vertex sets of the corresponding hypergraphs $H_{j'}$ and $H_{j''}$. The replacement is chosen such that $|V(H_i)| - \tau_2(H_i)$ is minimal. The vertex set V obviously has at least two vertices of all hyperedges in $\bigcup_{i=1}^r E(H_i)$. Furthermore, the replacement ensures that V also has at least two vertices of all hyperedges inserted by the join operation.

In order to show the lower bound $\tau_2(\otimes(H_1, \dots, H_r)) \geq k$ we recall that for $1 \leq i \leq r$ there is a 2-transversal V_i of size $\tau_2(H_i)$ in H_i . The vertices in every 2-transversal for $\otimes(H_1, \dots, H_r)$ have to cover the hyperedges in $\bigcup_{i=1}^r E(H_i)$ of the combined hypergraphs and the hyperedges of the considered join operation $J = \{\{v_1, \dots, v_r\} \mid v_i \in V(H_i), 1 \leq i \leq r\}$. Since all H_i are vertex-disjoint induced subhypergraphs of $\otimes(H_1, \dots, H_r)$, every 2-transversal for $\otimes(H_1, \dots, H_r)$ has to contain the vertices of some 2-transversal V_i of H_i for $1 \leq i \leq r$, which leads to at least $\sum_{i=1}^r \tau_2(H_i)$ vertices.

In order to meet the conditions of a 2-transversal for the hyperedges of J , we observe that, by Remark 14, for none of the r combined hypergraphs H_i with at least one hyperedge is a minimum 2-transversal V_i the full vertex set $V(H_i)$. This obviously also holds for H_i , which are empty (edgeless). Thus, $\min_{i=1}^r (|V(H_i)| - \tau_2(H_i)) > 0$ and $\underline{\min}_{i=1}^r (|V(H_i)| - \tau_2(H_i)) > 0$. Thus, for all of the r combined hypergraphs, there is some $v_i \in V(H_i)$, which is not included in a minimum 2-transversal for H_i . The hyperedge $\{v_1, \dots, v_r\}$ between these r vertices generated by the considered join operation contains no vertex of the union of optimal vertex covers of H_i for $1 \leq i \leq r$. This implies that we have to insert $\min_{i=1}^r (|V(H_i)| - \tau_2(H_i)) + \underline{\min}_{i=1}^r (|V(H_i)| - \tau_2(H_i))$ vertices into the union of optimal 2-transversals of H_i for $1 \leq i \leq r$ to obtain a minimum 2-transversal of size k for $\otimes(H_1, \dots, H_r)$.

This shows the statements of the lemma. $\square$

Remark 15. A k -transversal in a hypergraph is a set of vertices such that every hyperedge of the hypergraph contains at least k vertices of that set. The k -transversal number of a hypergraph H , denoted by $\tau_k(H)$, is the smallest size of a k -transversal in H . Our solution can be generalized to find k -transversals in r -join-hypergraphs for every $k \geq 2$ and $r \geq k$. Obviously, for $r = k$ a k -transversal of a connected r -uniform hypergraph is the vertex set. The solution in Lemma 14(3) can be generalized to $\tau_k(H_1 \otimes H_2) = \sum_{i=1}^r \tau_k(H_i) + \sum_{j=1}^k \min_{i=1}^r (|V(H_i)| - \tau_k(H_i))$. Here we denote by $\min(\ell)_{i=1}^r a_i$ the ℓ -th minimum (ℓ -th smallest element) with respect to multiple occurrences among $a_1, \dots, a_r$.

4.7. Maximum Matching

A *matching* in a hypergraph is a set of hyperedges, in which every two hyperedges are vertex-disjoint. It is an extension of the notion of matching in a graph. The *matching number* of a hypergraph H , denoted by $\nu(H)$, is the largest size of a matching in H .

While for graphs the matching number can be found in polynomial time, the problem of finding the matching number in hypergraphs is NP-hard, even for 3-uniform hypergraphs [32]. Hypergraph matching on r -uniform hypergraphs is a well-studied problem. It is known to be impossible to approximate within a factor of $\Omega(\frac{1}{r})$ in polynomial time, unless $\text{NP} \subseteq \text{BPP}$ [33]. Furthermore, the factor between the optimal solution and the optimal value of the natural LP relaxation is at least $\frac{1}{r-1+\frac{1}{r}}$ [34].

Remark 16. For every r -uniform hypergraph H with $|E(H)| \geq 1$ and $r \geq 2$ we know $1 \leq \nu(H) \leq \lfloor \frac{|V(H)|}{r} \rfloor$. For every hypergraph H with $|E(H)| = 0$, we have $\nu(H) = 0$.

First, we discuss the size of a matching of the largest size of a join of r vertex-disjoint r -uniform hypergraphs $H_1, \dots, H_r$. For $r = 2$ and two vertex-disjoint graphs H_1 and H_2 in [27] (Section 14.4), the following formula is given:

$$\nu(H_1 + H_2) = \min(|V(H_1)| + \nu(H_2), |V(H_2)| + \nu(H_1), \lfloor \frac{|V(H_1)| + |V(H_2)|}{2} \rfloor). \quad (5)$$

We will reprove Equation (5) below, which is based on the fact that for connected graphs it is best possible to choose a maximum number, i.e., $\min(|V(H_1)|, |V(H_2)|)$, of pairwise disjoint edges of a final join into a matching M of the largest size. For $r \geq 3$, choosing $\min_{i=1}^r |V(H_i)|$ pairwise disjoint hyperedges of a final join into a matching of largest size is not always the best possible by the following example. We will now use the abbreviations $n_i = |V(H_i)|$ and $n_{\min} = \min_{i=1}^r n_i$.

Example 6. Let $r \geq 3$, and let H_1 be a hypergraph on $r - 2$ isolated vertices. Furthermore, for $2 \leq i \leq r$, each hypergraph H_i has r vertices that define a hyperedge. Then we can obtain a matching for $H = \otimes(H_1, \dots, H_r)$ by choosing $m_j \leq n_{\min} = r - 2$ hyperedges of the final join. Choosing a hyperedge of the final join in a matching makes it impossible to choose the hyperedges of $H_2, \dots, H_r$ into the matching. But choosing none of the hyperedges of the final join allows us to choose the hyperedges of $H_2, \dots, H_r$ into the matching, which even leads to a matching of size $r - 1$ for $H = \otimes(H_1, \dots, H_r)$.

Next we give an upper bound and several lower bounds on the number of pairwise disjoint hyperedges generated in a final join occurring in a matching of the largest size for $H = \otimes(H_1, \dots, H_r)$. The following upper bound is obvious.

Lemma 15. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs and $H = \otimes(H_1, \dots, H_r)$. Then every matching of largest size for H contains at most $n_{\min}$ pairwise disjoint hyperedges of a final join in H .

For $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs $H_1, \dots, H_r$ and $H = \otimes(H_1, \dots, H_r)$ we will frequently consider the function

$$f(k) = k + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor), \quad (6)$$

which will be shown to be the size of a largest possible matching in H including $0 \leq k \leq n_{\min}$ pairwise disjoint hyperedges of a final join of H .

Lemma 16. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs, $H = \otimes(H_1, \dots, H_r)$, and $0 \leq k \leq n_{\min}$. Then $f(k)$ defined in Equation (6) is equal to the maximum number of possible hyperedges in a matching for H when we use k pairwise disjoint hyperedges of a final join of H .

Proof. We denote by $m_i(k)$ the maximum number of hyperedges from $E(H_i)$ that we can include in a matching for H if we use $0 \leq k \leq n_{\min}$ pairwise disjoint hyperedges of a final join of H . We next show that $m_i(k) = \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor)$. The upper bound $m_i(k) \leq \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor)$ holds, since $m_i(k)$ obviously is at most $v(H_i)$ and by choosing k hyperedges of the final join $m_i(k)$ is at most $\lfloor \frac{n_i - k}{r} \rfloor$. The lower bound $m_i(k) \geq \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor)$ holds, since we can define a matching of size $\min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor)$ for H_i by choosing the k hyperedges of the final join such that they have common vertices with the fewest number of hyperedges in $E(H_i)$. This choice is possible since the join operation adds all possible hyperedges of cardinality r each, including exactly one vertex from each of the r hypergraphs $H_1, \dots, H_r$.

Thus, the largest size of a matching for H when we use k pairwise disjoint hyperedges of a final join of H is $k + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor) = f(k)$. $\square$

By Lemmas 15 and 16, we obtain the following result.

Corollary 3. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs and $H = \otimes(H_1, \dots, H_r)$. Then, it holds that $v(H) = \max_{0 \leq k \leq n_{\min}} f(k)$.

Next, we consider lower bounds on the number of pairwise disjoint hyperedges generated in a final join occurring in a matching of largest size for H .

Lemma 17. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs and $H = \otimes(H_1, \dots, H_r)$. Then every matching of largest size for H has at least $\max(n_{\min} - r + 1, 0)$ pairwise disjoint hyperedges of a final join in H .

Proof. If $n_{\min} \leq r - 1$, then the statement is that every matching of largest size for H has at least $\max(n_{\min} - r + 1, 0) = 0$ pairwise disjoint hyperedges of a final join in H , which is obviously true.

For $r \leq k \leq n_{\min}$ and function f defined in Equation (6), we can estimate the following:

$$\begin{aligned} f(k) &= k + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor) \\ &= k - r + \sum_{i=1}^r (1 + \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor)) \\ &\geq k - r + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor + 1) \\ &= k - r + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - k + r}{r} \rfloor) \\ &= (k - r) + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - (k - r)}{r} \rfloor) \\ &= f(k - r) \end{aligned}$$

Thus, for $k = n_{\min}$, we know that $f(n_{\min}) \geq f(n_{\min} - r)$. Further, by iterating the argument, we obtain that for every $0 \leq k' < n_{\min} - r$ there is some $n_{\min} - r \leq k \leq n_{\min}$ such that $f(k') \leq f(k)$. Thus, the maximum value of $f(k)$ for $0 \leq k \leq n_{\min}$ can be found by considering $f(k)$ for $n_{\min} - r + 1 \leq k \leq n_{\min}$. This leads to the lower bound of $\max(n_{\min} - r + 1, 0)$ for the number of pairwise disjoint hyperedges of a final join in every matching of largest size for H . $\square$

Next, we consider improvements of the given lower bound.

Lemma 18. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs and $H = \otimes(H_1, \dots, H_r)$. Then every matching of largest size for H contains at least $\max(n_{\min} - r + 2, 0)$ pairwise disjoint hyperedges of a final join in H .

Proof. If $n_{\min} \leq r - 2$, then the statement is that every matching of largest size for H has at least $\max(n_{\min} - r + 2, 0) = 0$ pairwise disjoint hyperedges of a final join in H , which is obviously true.

For $n_{\min} \geq k \geq r - 1$ and function f defined in Equation (6) we consider

$$\begin{aligned} f(k) - f(k - r + 1) &= k + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor) \\ &\quad - (k - r + 1) - \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - (k - r + 1)}{r} \rfloor) \\ &= k - (k - r + 1) \\ &\quad + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor) - \min(v(H_i), \lfloor \frac{n_i - (k - r + 1)}{r} \rfloor) \\ &= r - 1 \\ &\quad + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor) - \min(v(H_i), \lfloor \frac{n_i - k - 1}{r} \rfloor + 1) \end{aligned}$$

By the abbreviations $x_i = n_i - k$ and $t_i = \min(v(H_i), \lfloor \frac{x_i}{r} \rfloor) - \min(v(H_i), \lfloor \frac{x_i - 1}{r} \rfloor + 1)$, we obtain

$$\begin{aligned} f(k) - f(k - r + 1) &= r - 1 + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{x_i}{r} \rfloor) - \min(v(H_i), \lfloor \frac{x_i - 1}{r} \rfloor + 1) \\ &= r - 1 + \sum_{i=1}^r t_i \end{aligned}$$

- If x_i is a multiple of r , then $\lfloor \frac{x_i}{r} \rfloor = q$ and $\lfloor \frac{x_i - 1}{r} \rfloor = q - 1$ for some $q \in \mathbb{N}$. Then, we have $t_i = \min(v(H_i), q) - \min(v(H_i), (q - 1) + 1) = \min(v(H_i), q) - \min(v(H_i), q) = 0$.
- If x_i is no multiple of r , then $\lfloor \frac{x_i}{r} \rfloor = \lfloor \frac{x_i - 1}{r} \rfloor = q$ for some $q \in \mathbb{N}$. Then, we have $t_i = \min(v(H_i), q) - \min(v(H_i), q + 1)$. Since $v(H_i)$ is an integer for $v(H_i) \leq q$ we have $t_i = v(H_i) - v(H_i) = 0$ and for $v(H_i) \geq q + 1$ we have $t_i = q - (q + 1) = -1$.

Thus, for $1 \leq i \leq r$, we have $t_i \geq -1$. For the special case, $k = n_{\min}$ and i^* the index of a hypergraph with $n_{\min}$ vertices, we obtain $x_{i^*} = 0$ and

$$\begin{aligned} t_{i^*} &= \min(v(H_{i^*}), \lfloor \frac{x_{i^*}}{r} \rfloor) - \min(v(H_{i^*}), \lfloor \frac{x_{i^*}-1}{r} \rfloor + 1) \\ &= \min(v(H_{i^*}), \lfloor \frac{0}{r} \rfloor) - \min(v(H_{i^*}), \lfloor \frac{0-1}{r} \rfloor + 1) \\ &= \min(v(H_{i^*}), 0) - \min(v(H_{i^*}), -1 + 1) \\ &= 0. \end{aligned}$$

Thus, $t_{i^*} = 0$ and for $i \in \{1, \dots, r\} \setminus \{i^*\}$ we only know $t_i \geq -1$, which leads to

$$f(n_{\min}) - f(n_{\min} - r + 1) = r - 1 + \sum_{i=1}^r t_i \geq r - 1 + 0 + (r - 1)(-1) = 0.$$

This implies the lower bound of $\max(n_{\min} - r + 2, 0)$ for the number of pairwise disjoint hyperedges of a final join in every matching of largest size for H . $\square$

The bound of Lemma 18 can obviously be further improved if there is more than one hypergraph H_j for which $n_j = n_{\min}$.

Lemma 19. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs.

1. $v(\{v\}, \emptyset) = 0$
2. $v(\oplus(H_1, H_2)) = v(H_1) + v(H_2)$
3. $v(\otimes(H_1, \dots, H_r)) = \max_{n_{\min}-r+2 \leq k \leq n_{\min}} k + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor)$

Proof.

1. $v(\{v\}, \emptyset) = 0$ holds by definition.
2. The result follows since M is a matching in $\oplus(H_1, H_2)$ if and only if $M \cap E(H_1)$ is a matching in H_1 and $M \cap E(H_2)$ is a matching in H_2 .
3. The result follows by Lemmas 15, 16, and 18.

This shows the statements of the lemma. $\square$

By Lemmas 15 and 18, for $r = 2$, it is best to choose a maximum number of pairwise disjoint edges of a final join in a matching of the largest size, i.e., $k = \min(|V(H_1)|, |V(H_2)|)$ edges; thus, we have

$$\begin{aligned} v(\otimes(H_1, H_2)) &= \min(|V(H_1)|, |V(H_2)|) \\ &\quad + \sum_{i=1}^2 \min(v(H_i), \lfloor \frac{|V(H_i)| - \min(|V(H_1)|, |V(H_2)|)}{2} \rfloor) \\ &= \min(|V(H_1)| + v(H_2), |V(H_2)| + v(H_1), \lfloor \frac{|V(H_1)| + |V(H_2)|}{2} \rfloor). \end{aligned}$$

This shows that for $r = 2$ our result corresponds to the known result for co-graphs of [27] (Section 14.4) in Equation (5). Thus, our result generalizes that result and also the algorithm shown in [17], which computes a maximum matching on co-graphs in time $\mathcal{O}(n)$.

By Lemma 19(3), we can compute the size of a largest matching of a join of r pairwise vertex-disjoint r -uniform hypergraphs from the matching numbers of the r combined hypergraphs in $\mathcal{O}(r^2)$ time. Next, we give an improved method, which is based on the calculation in the following example.

Example 7. We consider the join of $r = 10$ hypergraphs $H_1, \dots, H_{10}$ with the following number of vertices and matching numbers.

i	1	2	3	4	5	6	7	8	9	10
n_i	119	121	124	113	149	129	126	131	120	147
$v(H_i)$	5	6	1	5	9	5	4	8	5	5

Every matching for $H = \otimes(H_1, \dots, H_{10})$ contains at most $k = n_{\min} = 113$ pairwise disjoint hyperedges of the final join in H . Furthermore, for $1 \leq i \leq 10$ from $E(H_i)$, we can choose at most $m_i(k) = \min(v(H_i), \lfloor \frac{n_i - k}{r} \rfloor)$ hyperedges into a matching for H . We consider H_7 .

- Choosing $n_{\min} = 113$ pairwise disjoint hyperedges of the final join allows us to choose $m_7(113) = \min(4, \lfloor \frac{126 - 113}{10} \rfloor) = \min(4, 1) = 1$ hyperedge from $E(H_7)$ into a matching for H .
- Choosing only $n_{\min} - 7 = 113 - 7 = 106$ pairwise disjoint hyperedges of the final join allows us to choose even $m_7(106) = \min(4, \lfloor \frac{126 - 106}{10} \rfloor) = \min(4, 2) = 2$ hyperedges from $E(H_7)$ into a matching for H .

Thus, by removing 7 hyperedges from the maximum number of 113 pairwise disjoint hyperedges of a final join in H , we can use one additional hyperedge from $E(H_7)$ in a matching for H . The computation of this value $x(i)$ for every H_i , $1 \leq i \leq r$, is fundamental to our subsequent solution.

Theorem 5. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs and $H = \otimes(H_1, \dots, H_r)$. Then we can compute the size of a largest matching for H from the matching numbers of $H_1, \dots, H_r$ in $\mathcal{O}(r)$ time.

Proof. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs and $H = \otimes(H_1, \dots, H_r)$. In order to compute $v(H)$, we compute a number j^* of hyperedges that have to be removed from a set of $n_{\min}$ pairwise disjoint hyperedges of a final join of H such that the size of a possible matching can be maximized, i.e., $v(H) = f(n_{\min} - j^*)$. By Lemma 18 we can assume that $0 \leq j^* \leq r - 2$.

1. Every matching for H has at most $n_{\min}$ pairwise disjoint hyperedges of a final join in H , and there is always a matching M' for H (not necessarily of maximum size) that contains this maximum number of hyperedges of a final join in H . For every hypergraph H_i , $1 \leq i \leq r$, we compute the value $x(i)$ as the number of join hyperedges we have to remove from M' in order to use one additional hyperedge from $E(H_i)$ in a matching for H . This is obviously only reasonable if $v(H_i) \cdot r > n_i - n_{\min}$. Thus, we define

$$x(i) = \begin{cases} r - ((n_i - n_{\min}) \bmod r) & \text{if } v(H_i) \cdot r > n_i - n_{\min} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

2. For every $1 \leq j \leq r - 2$, we compute the value $y(j) = |\{i \mid 1 \leq i \leq r, x(i) = j\}|$, i.e., the number of hypergraphs H_i for which the number of join hyperedges that must be removed to use one additional hyperedge from $E(H_i)$ is equal to j .
3. For every $1 \leq j \leq r - 2$, we compute the value $z(j)$ by $z(1) = y(1)$ and for $2 \leq j \leq r - 2$ by $z(j) = y(j) + z(j - 1)$, i.e., the sum of all additional hyperedges from some $E(H_i)$, $1 \leq i \leq r$, when we remove j join hyperedges.
4. For every $1 \leq j \leq r - 2$ we compute the difference $d(j) = z(j) - j$, i.e., the advantage of inserting $z(j)$ hyperedges of $E(H_1), \dots, E(H_r)$ and removing j hyperedges of the join.
5. If $\max_{j=1}^{r-2} d(j) > 0$, we obtain $j^* = \arg \max d(j)$ and otherwise $j^* = 0$.

Since every step runs in time $\mathcal{O}(r)$, we can compute the size of a largest matching for a join of r pairwise vertex-disjoint r -uniform hypergraphs in $\mathcal{O}(r)$ time. $\square$

Example 8. We consider the $r = 10$ hypergraphs $H_1, \dots, H_{10}$ with the following number of vertices and matching numbers from Example 7. The values $x(i)$ can be computed following the proof of Theorem 5.

i	1	2	3	4	5	6	7	8	9	10
n_i	119	121	124	113	149	129	126	131	120	147
$v(H_i)$	5	6	1	5	9	5	4	8	5	5
$x(i)$	4	2	—	10	4	4	7	2	3	6

The values $y(i)$, $z(j)$, and $d(j)$ for $1 \leq j \leq 8$ are given in the proof of Theorem 5 as follows:

j	1	2	3	4	5	6	7	8
$y(j)$	0	2	1	3	0	1	1	0
$z(j)$	0	2	3	6	6	7	8	8
$d(j)$	−1	0	0	2	1	1	1	0

Thus, the removal of $j^* = \arg \max d(j) = 4$ join hyperedges from $n_{\min} = 113$ hyperedges allows us to use $z(4) = 6$ further hyperedges from $E(H_1), \dots, E(H_r)$ in a matching for H . This leads to

$$\begin{aligned}
 v(H) &= f(n_{\min} - \arg \max d(j)) = f(113 - 4) \\
 &= f(109) \\
 &= 109 + \sum_{i=1}^r \min(v(H_i), \lfloor \frac{n_i - 109}{r} \rfloor) \\
 &= 109 + 16 \\
 &= 125.
 \end{aligned}$$

By contrast, choosing $n_{\min} = 113$ hyperedges from the final join only leads to $f(113) = 113 + 10 = 123$. (For the values $\min(v(H_i), \lfloor \frac{n_i - 109}{r} \rfloor)$ and $\min(v(H_i), \lfloor \frac{n_i - 113}{r} \rfloor)$ see Appendix B.)

The relation of the vertex-cover number and the matching number is as follows.

Lemma 20 ([9]). For every hypergraph H , it holds that $v(H) \leq \tau(H)$.

Lemma 21 ([9]). For every r -uniform hypergraph H , it holds that $v(H) \leq \tau(H) \leq r \cdot v(H)$.

The following examples show that for r -join-hypergraphs the matching number can be arbitrarily smaller than the vertex-cover number.

Example 9.

- For $r = 2$, we consider the 2-join-hypergraphs (co-graphs) $H_1 = \otimes(K_2, K_2)$, i.e., a K_4 , and for $i \geq 2$, we define $H_i = \oplus(H_{i-1}, H_1)$, i.e., the disjoint union of i copies of graph K_4 . Then, we have

$$v(H_i) = 2i < 3i = \tau(H_i).$$

- For $r \geq 3$, we consider the r -join-hypergraphs $J_1 = \otimes(K_r^r, \dots, K_r^r)$, the join of r complete r -uniform hypergraphs on r vertices, and for $i \geq 2$, we define $J_i = \oplus(J_{i-1}, J_1)$, i.e., the disjoint union of i copies of hypergraph J_1 . Then, we have

$$v(J_i) = r \cdot i < (2r - 1) \cdot i = \tau(J_i).$$

4.8. Min Dominating Set

A set $D \subseteq V(H)$ is a dominating set of hypergraph H if for every $v \in V(H) \setminus D$ there exists $u \in D$ such that u and v are adjacent in H ; that is, if there exists $e \in E(H)$ such that

$u, v \in e$. The domination number of a hypergraph H , denoted by $\gamma(H)$, is defined as the minimum cardinality of a dominating set in H .

Remark 17. For every r -uniform hypergraph H with $|E(H)| \geq 1$, we have $1 \leq \gamma(H) \leq |V(H)| - r + 1$.

Lemma 22. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs.

1. $\gamma(\{v\}, \emptyset) = 1$
2. $\gamma(\oplus(H_1, H_2)) = \gamma(H_1) + \gamma(H_2)$
3. $\gamma(\otimes(H_1, \dots, H_r)) = \min(\min_{i=1}^r \gamma(H_i), 2)$

Proof.

1. $\gamma(\{v\}, \emptyset) = 1$ holds by definition.
2. The result follows since V is a dominating set in $\oplus(H_1, H_2)$ if and only if $V \cap V(H_1)$ is a dominating set in H_1 and $V \cap V(H_2)$ is a dominating set in H_2 .
3. To show the upper bound $\gamma(\otimes(H_1, \dots, H_r)) \leq \min(\min_{i=1}^r \gamma(H_i), 2)$ let D_i be a dominating set of size $\gamma(H_i)$ for H_i . By definition of the join operation, each of the sets D_i is a dominating set for $\otimes(H_1, \dots, H_r)$. Choosing some D_i of smallest size shows an upper bound for $\gamma(\otimes(H_1, \dots, H_r))$. Furthermore, any set $D = \{x, y\}$ with $x \in V(H_i)$ and $y \in V(H_j)$ for $1 \leq i \neq j \leq r$ is a dominating set for $\otimes(H_1, \dots, H_r)$. Thus, 2 is also an upper bound for $\gamma(\otimes(H_1, \dots, H_r))$.

In order to show the lower bound $\gamma(\otimes(H_1, \dots, H_r)) \geq \min(\min_{i=1}^r \gamma(H_i), 2)$, we assume that there is a dominating set of size $k < \min(\min_{i=1}^r \gamma(H_i), 2)$ for $\otimes(H_1, \dots, H_r)$. We distinguish between the following two cases.

- If $\min_{i=1}^r \gamma(H_i) < 2$, then $\min(\min_{i=1}^r \gamma(H_i), 2) = \min_{i=1}^r \gamma(H_i) = 1$ and a dominating set of size $k < 1$ for $\otimes(H_1, \dots, H_r)$ is not possible.
- If $\min_{i=1}^r \gamma(H_i) \geq 2$, then for all $1 \leq i \leq r$, we have $|V(H_i)| \geq 2$. Furthermore, since $\min(\min_{i=1}^r \gamma(H_i), 2) = 2$, a dominating set of size $k < 2$ for $\otimes(H_1, \dots, H_r)$ has exactly one vertex $v \in V(H_j)$ for some $1 \leq j \leq r$. Since $\min_{i=1}^r \gamma(H_i) \geq 2$ vertex v is not adjacent to all further vertices in $V(H_j)$. Thus, a dominating set of size $k < 2$ for $\otimes(H_1, \dots, H_r)$ is not possible.

This leads to a contradiction in both cases and the lower bound follows.

This shows the statements of the lemma. $\square$

Corollary 4. For every $r \geq 2$ and every connected r -join-hypergraph H , we have $1 \leq \gamma(H) \leq 2$.

4.9. Min Coloring

A proper k -coloring or non-monochromatic k -coloring of a hypergraph H is a function $c : V(H) \rightarrow \{1, \dots, k\}$ for which every hyperedge $e \in E(H)$ has at least two vertices of different colors. The (non-monochromatic) chromatic number of H , denoted by $\chi(H)$, is the minimum value of k such that H has a proper k -coloring. A hypergraph H is k -colorable if $\chi(H) \leq k$. The vertex sets $V_i = \{v \in V \mid c(v) = i\}$, with $1 \leq i \leq k$, divide $V(H)$ into a partition of so-called color classes. For every proper k -coloring c , the color classes are stable sets.

For every $k \geq 2$ it is NP-complete to decide whether a hypergraph is k -colorable. It is even NP-complete for $k = 2$ on 3-uniform hypergraphs [35]. Well-known facts about the chromatic number of hypergraphs can be found in the survey [8] (Chapter 11).

Remark 18. For graphs, i.e., 2-uniform hypergraphs, non-monochromatic colorings correspond to colorings such that the definitions of hypergraph coloring and graph coloring coincide.

For some hypergraph H , some function $c : V(H) \rightarrow \{1, \dots, k\}$, and $V \subseteq V(H)$, we denote by $c(V) = \{c(v) \mid v \in V\}$ the set of used colors for vertex set V .

Lemma 23. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs.

1. $\chi(\{v\}, \emptyset) = 1$
2. $\chi(\oplus(H_1, H_2)) = \max(\chi(H_1), \chi(H_2))$
3. $\chi(\otimes(H_1, \dots, H_r)) = \max(\max_{i=1}^r \chi(H_i), \lceil \frac{\sum_{i=1}^r \chi(H_i)}{r-1} \rceil)$

Proof.

1. $\chi(\{v\}, \emptyset) = 1$ holds by definition.
2. The upper bound $\chi(\oplus(H_1, H_2)) \leq \max(\chi(H_1), \chi(H_2))$ holds, since there obviously is a coloring for $\oplus(H_1, H_2)$ using $\max(\chi(H_1), \chi(H_2))$ colors.
The lower bound $\chi(\oplus(H_1, H_2)) \geq \max(\chi(H_1), \chi(H_2))$ holds, since otherwise H_1 or H_2 has a proper coloring using fewer than $\chi(H_1)$ or $\chi(H_2)$ colors, respectively.
3. Let $k = \max(\max_{i=1}^r \chi(H_i), \lceil \frac{\sum_{i=1}^r \chi(H_i)}{r-1} \rceil)$ be the claimed number of colors.

In order to show the lower bound $\chi(\otimes(H_1, \dots, H_r)) \geq k$ we observe that $H_1, \dots, H_r$ are induced subhypergraphs of $\otimes(H_1, \dots, H_r)$; thus, their chromatic numbers lead to lower bounds of $\chi(\otimes(H_1, \dots, H_r))$, and it holds that $\chi(\otimes(H_1, \dots, H_r)) \geq \max_{i=1}^r \chi(H_i)$. Further, let $c : V(\otimes(H_1, \dots, H_r)) \rightarrow \{1, \dots, k\}$ be a proper k -coloring for $\otimes(H_1, \dots, H_r)$. By definition of the join operation, every color in $c(V(\otimes(H_1, \dots, H_r)))$ occurs in at most $r - 1$ of the r sets $c(V(H_1)), \dots, c(V(H_r))$, which implies

$$\sum_{i=1}^r \chi(H_i) \leq \sum_{i=1}^r |c(V(H_i))| \leq k \cdot (r - 1).$$

Thus, for $k = \chi(\otimes(H_1, \dots, H_r))$, we obtain $\chi(\otimes(H_1, \dots, H_r)) \geq \lceil \frac{\sum_{i=1}^r \chi(H_i)}{r-1} \rceil$ and the lower bound is shown.

To show the upper bound $\chi(\otimes(H_1, \dots, H_r)) \leq k$, we define a proper coloring for $\otimes(H_1, \dots, H_r)$ using k colors. We define $s_i = k - \chi(H_i) \geq 0$ as the number of unnecessary colors in H_i for $1 \leq i \leq r$. A suitable assignment of the colors $\{1, \dots, k\}$ to the sets $V(H_i)$ where they are not used, can be defined as a matching in the complete bipartite graph G with $V(G) = V_C \cup V_U$ with $V_C = \{1, \dots, k\}$ representing the existing colors and $V_U = \{(i, j) \mid 1 \leq i \leq r, 1 \leq j \leq s_i\}$ representing possible missing colors in $H_i, 1 \leq i \leq r$. Then we have

$$|V_U| = \sum_{i=1}^r s_i = \sum_{i=1}^r k - \chi(H_i) = r \cdot k - \sum_{i=1}^r \chi(H_i) \geq r \cdot k - (r - 1) \cdot k = k.$$

This implies for every (non-empty) $V' \subseteq V_C$ the inequality $|N(V')| = |V_U| \geq k \geq |V'|$. Hall's marriage theorem [36] states that there is a V_C -perfect matching $M \subseteq E(G)$, i.e., a set of pairwise vertex-disjoint edges, which covers every vertex in V_C . For $1 \leq i \leq r$, we use M to define the set of colors for H_i by $S_i = \{1, \dots, k\} \setminus \{c \mid \{c, (i, j)\} \in M, 1 \leq j \leq s_i\}$.

In order to define a proper coloring for $\otimes(H_1, \dots, H_r)$ using k colors for every $1 \leq i \leq r$, we choose a proper coloring for H_i using the colors in S_i . This is possible since $|S_i| \geq k - s_i = \chi(H_i)$. Further, this is even a proper coloring for $\otimes(H_1, \dots, H_r)$ since every hyperedge inserted by the join operation has at least two vertices of different colors since matching $M \subseteq E(G)$ ensures that for every color $c \in \{1, \dots, k\}$ there is some i , such that $c \notin S_i$.

This shows the statements of the lemma. $\square$

For $r = 2$, a proper k -coloring of an r -uniform hypergraph corresponds to a proper coloring in a graph. The results of Lemma 23 generalize the results of [13,27] (see Table 1) for computing the chromatic number in a co-graph.

Berge [29] has shown the following relations.

Lemma 24 ([29]). *For every hypergraph H , it holds that $\chi(H) \cdot \alpha(H) \geq |V(H)|$.*

Lemma 25 ([29]). *For every hypergraph H , it holds that $\chi(H) + \alpha(H) \leq |V(H)| + 1$.*

4.10. Min Strong Coloring

A *strong k -coloring* of a hypergraph H is a function $c : V(H) \rightarrow \{1, \dots, k\}$ such that if $u, v \in e$ for some $e \in E(H)$ and $u \neq v$, then we have that $c(u) \neq c(v)$. Thus, now we assign distinct colors to vertices that are contained in a common hyperedge. The *strong chromatic number* of H , denoted by $s\chi(H)$, is the minimum value of k such that H has a strong k -coloring. A hypergraph H is *strongly k -colorable* if $s\chi(H) \leq k$. The strong chromatic number was independently introduced by Fellows [37] and Alon [38].

Remark 19. *Since every strong coloring is also a proper coloring, we have $\chi(H) \leq s\chi(H)$ and for graphs H , i.e., 2-uniform hypergraphs, we have $\chi(H) = s\chi(H)$.*

There is a close relation between the strong chromatic number of a hypergraph and the chromatic number of its primal graph [29].

Lemma 26 ([29]). *For every hypergraph H , it holds that $\chi(P(H)) = s\chi(H)$.*

Since the primal graph of an r -join-hypergraph is a special co-graph (Lemma 5), we can compute the strong chromatic number of an r -join-hypergraph by computing the chromatic number of a co-graph; see [13,27] or Table 1.

Lemma 27. *Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs.*

1. $s\chi(\{v\}, \emptyset) = 1$
2. $s\chi(\oplus(H_1, H_2)) = \max(s\chi(H_1), s\chi(H_2))$
3. $s\chi(\otimes(H_1, \dots, H_r)) = \sum_{i=1}^r s\chi(H_i)$

Proof.

1. $s\chi(\{v\}, \emptyset) = 1$ holds by definition.
2. The statement can be proven as Lemma 9(2) using Lemmas 26 and 5.
3. The statement can be proven as Lemma 9(3) using Lemmas 26 and 5.

This shows the statements of the lemma. $\square$

For $r = 2$, a strong k -coloring of a hypergraph corresponds to a proper coloring of a graph. The results of Lemma 27 generalize the results of [13,27] on computing the chromatic number of a co-graph.

4.11. Max $\mathcal{C}$ -Coloring

A $\mathcal{C}$ -coloring of a hypergraph applies an opposite coloring constraint, prescribing the presence of two vertices with a *common* color inside each hyperedge. A *proper k - $\mathcal{C}$ -coloring* of a hypergraph H is a function $c : V(H) \rightarrow \{1, \dots, k\}$ for which every hyperedge $e \in E(H)$ has at least two vertices of the same color, which means that we have to avoid *polychromatic* hyperedges. (We use the notation *polychromatic hyperedge* in the sense of [8], Chapter 11). The same notation is also used to ensure that all k colors of a function $c : V(H) \rightarrow \{1, \dots, k\}$

are used for every hyperedge; see, e.g., [39]. The *upper chromatic number* of H , denoted by $\bar{\chi}(H)$, is the maximum value of k such that H has a proper k - $\mathcal{C}$ -coloring. A hypergraph H is k - $\mathcal{C}$ -colorable if $\bar{\chi}(H) \geq k$.

Remark 20. If we restrict to r -uniform hypergraphs and $r = 2$ (graphs), the problem is simple. Every connected graph G has $\bar{\chi}(G) = 1$. Furthermore, any graph G on k connected components has $\bar{\chi}(G) = k$.

For $r \geq 3$, the r -uniform hypergraphs H with $\bar{\chi}(H) = 1$ are no more than single-vertex hypergraphs.

Remark 21. For $k \geq 2$, a proper k - $\mathcal{C}$ -coloring can always be modified to a proper $(k - 1)$ - $\mathcal{C}$ -coloring, e.g., by merging two color classes into one.

Determination of the upper chromatic number is an NP-hard problem, even for the class of 3-uniform hypertrees [8] (Chapter 11). Well-known facts about the upper chromatic number of hypergraphs can be found in the survey [8] (Chapter 11) and [9].

Remark 22. By [9] every r -uniform hypergraph H with at least one hyperedge admits a proper $(r - 1)$ - $\mathcal{C}$ -coloring, because the pigeonhole principle implies that each hyperedge contains a repeated color in every coloring $c : V(H) \rightarrow \{1, \dots, r - 1\}$ of the vertices. Therefore, for r -uniform hypergraphs H with at least one hyperedge we have $\bar{\chi}(H) \geq r - 1$ and for r -uniform hypergraphs H with no hyperedges we have $\bar{\chi}(H) = |V(H)|$.

First we discuss the upper chromatic number of a join of r vertex-disjoint hypergraphs $H_1, \dots, H_r$. The existence of a polychromatic hyperedge when assigning colors to the vertices of $H = \otimes(H_1, \dots, H_r)$ is very closely related to a well-known concept in combinatorics. Given are r sets $S_1, \dots, S_r$. If there exist pairwise distinct $a_1, \dots, a_r$ such that for $1 \leq i \leq r$ it holds that $a_i \in S_i$, then $\{a_1, \dots, a_r\}$ is a *system of distinct representatives* (SDR) for $S_1, \dots, S_r$. For some hypergraph H , some function $c : V(H) \rightarrow \{1, \dots, k\}$, and $V \subseteq V(H)$, we denote by $c(V) = \{c(v) \mid v \in V\}$ the set of colors used on the vertex set V . This leads to the following characterization of proper k - $\mathcal{C}$ -colorings for $H = \otimes(H_1, \dots, H_r)$.

Lemma 28. Let $H = \otimes(H_1, \dots, H_r)$ for $r \geq 2$ vertex-disjoint hypergraphs $H_1, \dots, H_r$ and $c : V(H) \rightarrow \{1, \dots, k\}$ be a function. Then the following conditions are equivalent. (i) Function c is a proper k - $\mathcal{C}$ -coloring of H ; (ii) Function c leads to no polychromatic hyperedge in H ; and (iii) $c(V(H_1)), \dots, c(V(H_r))$ has no system of distinct representatives.

Lemma 29. Let $H = \otimes(H_1, \dots, H_r)$ for $r \geq 2$ vertex-disjoint hypergraphs $H_1, \dots, H_r$. Then, we have $\bar{\chi}(H) = \sum_{i=1}^r \bar{\chi}(H_i) - \min_{i=1}^r \bar{\chi}(H_i) - \underline{\min}_{i=1}^r \bar{\chi}(H_i) + 1$.

Proof. Let $k = \sum_{i=1}^r \bar{\chi}(H_i) - \min_{i=1}^r \bar{\chi}(H_i) - \underline{\min}_{i=1}^r \bar{\chi}(H_i) + 1$ be the claimed number of colors for $H = \otimes(H_1, \dots, H_r)$. We assume that the r combined hypergraphs H_i are indexed in descending order with respect to their upper chromatic number, i.e., we assume that $\bar{\chi}(H_1) \geq \bar{\chi}(H_2) \geq \dots \geq \bar{\chi}(H_{r-1}) \geq \bar{\chi}(H_r)$. Thus, we have $k = 1 + \sum_{i=1}^{r-2} \bar{\chi}(H_i)$.

To show the lower bound $\bar{\chi}(H) \geq k$, we give a proper k - $\mathcal{C}$ -coloring for H . For $1 \leq i \leq r$ let c_i be a proper k_i - $\mathcal{C}$ -coloring for H_i such that $k_i = \bar{\chi}(H_i)$. Then we define a k - $\mathcal{C}$ -coloring $c : V(\otimes(H_1, \dots, H_r)) \rightarrow \{1, \dots, k\}$ for H by

$$c(v) = \begin{cases} c_1(v) & \text{if } v \in V(H_1) \\ \bar{\chi}(H_1) + c_2(v) & \text{if } v \in V(H_2) \\ \bar{\chi}(H_1) + \bar{\chi}(H_2) + c_3(v) & \text{if } v \in V(H_3) \\ \vdots & \\ \sum_{i=1}^{r-3} \bar{\chi}(H_i) + c_{r-2}(v) & \text{if } v \in V(H_{r-2}) \\ k & \text{if } v \in V(H_{r-1}) \\ k & \text{if } v \in V(H_r). \end{cases}$$

For the correctness of a proper k - $\mathcal{C}$ -coloring, we observe that function c either keeps the colors (H_1), increases the colors by the same certain value ($H_2, \dots, H_{r-2}$), or changes all colors to k (H_{r-1} and H_r). Thus, for $1 \leq i \leq r$, every hyperedge $e \in E(H_i)$ still has at least two vertices of the same color. Furthermore, the hyperedges generated by the final join operation $J = \{\{v_1, \dots, v_r\} \mid v_i \in V(H_i), 1 \leq i \leq r\}$ combine exactly one vertex from each of the r combined hypergraphs $H_1, \dots, H_r$. Since all vertices of H_{r-1} and H_r are labeled with k , every new hyperedge also has at least two vertices of the same color.

To show the upper bound $\bar{\chi}(H) \leq k$ let c be a proper $\mathcal{C}$ -coloring for $H = \otimes(H_1, \dots, H_r)$. By Lemma 28 we know that $c(V(H_1)), \dots, c(V(H_r))$ has no system of distinct representatives. Hall's Theorem [36] characterizes the existence of a system of distinct representatives by verifying *Hall's condition*, i.e., whether every union of $r' \leq r$ sets contains at least r' distinct elements. Thus, there is some $I \subseteq \{1, \dots, r\}$ with $|I| \geq 2$ such that $|\bigcup_{i \in I} c(V(H_i))| \leq |I| - 1$. Therefore,

$$|\bigcup_{i=1}^r c(V(H_i))| \leq |\bigcup_{i \in I} c(V(H_i))| + \sum_{i \notin I} |c(V(H_i))| \leq |I| - 1 + \sum_{i \notin I} \bar{\chi}(H_i).$$

Since $\bar{\chi}(H_i) \geq 1$ for $1 \leq i \leq r$, the right-hand side is maximized when $|I| = 2$ and $I = \{r-1, r\}$. Thus, taking $I = \{r-1, r\}$ yields $|\bigcup_{i=1}^r c(V(H_i))| \leq 1 + \sum_{i=1}^{r-2} \bar{\chi}(H_i)$. Hence, every proper $\mathcal{C}$ -coloring of H uses at most $1 + \sum_{i=1}^{r-2} \bar{\chi}(H_i) = k$ colors. $\square$

Lemma 30. Let $H_1, \dots, H_r$ be $r \geq 2$ pairwise vertex-disjoint r -uniform hypergraphs.

1. $\bar{\chi}(\{v\}, \emptyset) = 1$
2. $\bar{\chi}(\oplus(H_1, H_2)) = \bar{\chi}(H_1) + \bar{\chi}(H_2)$
3. $\bar{\chi}(\otimes(H_1, \dots, H_r)) = \sum_{i=1}^r \bar{\chi}(H_i) - \min_{i=1}^r \bar{\chi}(H_i) - \underline{\min}_{i=1}^r \bar{\chi}(H_i) + 1$

Proof.

1. $\bar{\chi}(\{v\}, \emptyset) = 1$ holds by definition.
2. The lower bound $\bar{\chi}(\oplus(H_1, H_2)) \geq \bar{\chi}(H_1) + \bar{\chi}(H_2)$ holds, since there obviously is a proper $\mathcal{C}$ -coloring for $\oplus(H_1, H_2)$ using $\bar{\chi}(H_1) + \bar{\chi}(H_2)$ colors.
The upper bound $\bar{\chi}(\oplus(H_1, H_2)) \leq \bar{\chi}(H_1) + \bar{\chi}(H_2)$ holds, since otherwise H_1 or H_2 has a proper $\mathcal{C}$ -coloring using more than $\bar{\chi}(H_1)$ or $\bar{\chi}(H_2)$ colors, respectively.
3. Holds by Lemma 29.

This shows the statements of the lemma. $\square$

Voloshin [40] mentioned the following relation between the upper chromatic number and the stability number of hypergraphs.

Lemma 31 ([40]). For every hypergraph H , it holds that $\bar{\chi}(H) \leq \alpha(H)$.

Even for r -join-hypergraphs H , the difference $\alpha(H) - \bar{\chi}(H)$ can be arbitrarily large by the following examples.

Example 10. For $r \geq 3$, we consider the r -join-hypergraphs $H = \otimes(H_{1,i}, H_{2,i}, \dots, H_{r,i})$, where, for $1 \leq j \leq r$, each hypergraph $H_{j,i}$ is the disjoint union of i isolated vertices. Then, for $1 \leq j \leq r$, we have $\alpha(H_{j,i}) = i$ and $\bar{\chi}(H_{j,i}) = i$. Furthermore, by Lemmas 30(3) and 6(3), we have

$$\bar{\chi}(\otimes(H_{1,i}, \dots, H_{r,i})) = (r-2) \cdot i + 1 < (r-1) \cdot i = \alpha(\otimes(H_{1,i}, \dots, H_{r,i})).$$

For $r = 2$, the remark regarding the difference between the stability number and the upper chromatic number also holds true, but the upper chromatic number is constant for connected graphs.

Example 11. For $r = 2$, we consider the r -join-hypergraphs (co-graphs) $H = \otimes(H_{1,i}, H_{2,i})$, where $H_{1,i}$ and $H_{2,i}$ are each the disjoint union of i isolated vertices. Then, we have $\alpha(H_{1,i}) = \alpha(H_{2,i}) = i$ and $\bar{\chi}(H_{1,i}) = \bar{\chi}(H_{2,i}) = i$. Furthermore, since the result of the join is a connected graph, it holds that

$$\bar{\chi}(\otimes(H_{1,i}, H_{2,i})) = 1 < i = \alpha(\otimes(H_{1,i}, H_{2,i})).$$

A 2-transversal can be used to give a proper k -C-coloring of a hypergraph H as follows, as shown in [8] (Chapter 11) and [9].

Lemma 32. For every hypergraph H , it holds that $\bar{\chi}(H) \geq n - \tau_2(H) + 1$.

4.12. Results

Next, we apply our formulas to compute and compare the considered parameters.

Theorem 6. For every $r \geq 2$ and every r -join-hypergraph H on n vertices given by a decomposition tree $T(H)$, the size of a largest stable set, the size of a largest co-stable set, the size of a largest independent set, the size of a largest co-independent set, the size of a smallest vertex cover, the size of a smallest 2-transversal, the size of a largest matching, the size of a smallest dominating set, the chromatic number, the strong chromatic number, and the upper chromatic number can be computed in $\mathcal{O}(n \cdot r)$ time.

Proof. The result holds by the formulas shown in Section 4.1–4.11. For computing the size of a largest matching for $H = \otimes(H_1, \dots, H_r)$ from the sizes of largest matchings for $H_1, \dots, H_r$ in $\mathcal{O}(r)$ time see Theorem 5. $\square$

Besides computing the sizes of the special vertex or hyperedge sets, we can additionally compute a vertex or hyperedge set of this size using the results of Section 4.1–4.8.

Theorem 7. Let H be an r -join-hypergraph for $r \geq 3$, then the following relations hold

$$\gamma(H) - 1 \leq \beta(H) \leq \bar{\chi}(H) \leq \alpha(H) \quad \text{and} \quad \chi(H) \leq s\chi(H) = \delta(H).$$

Proof. Let H be an r -join-hypergraph for $r \geq 3$. The relation $\gamma(H) - 1 \leq \beta(H)$ holds by Lemmas 22 and 9, $\beta(H) \leq \bar{\chi}(H)$ holds by Lemmas 9 and 30, and $\bar{\chi}(H) \leq \alpha(H)$ holds by Lemmas 30 and 6. Furthermore, $\chi(H) \leq s\chi(H)$ holds by Lemmas 23 and 27, and the equality $s\chi(H) = \delta(H)$ holds by Lemmas 27 and 11. $\square$

5. Conclusions and Outlook

We introduced r -join-hypergraphs. The operations used to construct r -join-hypergraphs are the binary disjoint union of two given r -join-hypergraphs and the r -nary join, which inserts all possible hyperedges of cardinality r , each including exactly one vertex from each of r given r -join-hypergraphs. For $r = 2$, the set of r -join-hypergraphs corresponds to the well-known class of co-graphs, thus providing a generalization of co-graphs to r -uniform hypergraphs. Our results include properties of r -join-hypergraphs and a characterization of their primal graphs as special co-graphs. While for $r = 2$, every r -join-hypergraph can be defined using disjoint union and complementation, for $r \geq 3$, this does not hold true.

Using their recursive structure, we give dynamic programming algorithms that demonstrate that, for every r -join-hypergraph given by a decomposition tree, many basic hypergraph problems can be solved in linear time. In Table 3, we summarize the results obtained for r -join-hypergraphs. Our solutions extend and provide new proofs of a number of results that are known for co-graphs (see Table 1).

Table 3. Hypergraph parameters for single-vertex hypergraphs, the disjoint union and join operation for r -uniform hypergraphs and, thus, for r -join-hypergraphs. We use the abbreviations $n_i = |V(H_i)|$ and $n_{\min} = \min_{i=1}^r n_i$.

Parameter	Section	v	$\oplus(H_1, H_2)$	$\otimes(H_1, \dots, H_r)$
α	Section 4.1	1	$\alpha(H_1) + \alpha(H_2)$	$\sum_{i=1}^r \alpha(H_i) - \min_{i=1}^r \alpha(H_i)$
ω	Section 4.2	1	$\max(\omega(H_1), \omega(H_2), \min(n_1 + n_2, r - 1))$	$\max(\max_{i=1}^r \omega(H_i), r)^*$
β	Section 4.3	1	$\beta(H_1) + \beta(H_2)$	$\max_{i=1}^r \beta(H_i)$
δ	Section 4.4	1	$\max(\delta(H_1), \delta(H_2))$	$\sum_{i=1}^r \delta(H_i)$
τ	Section 4.5	0	$\tau(H_1) + \tau(H_2)$	$\sum_{i=1}^r \tau(H_i) + \min_{i=1}^r (n_i - \tau(H_i))$
τ_2	Section 4.6	0	$\tau_2(H_1) + \tau_2(H_2)$	$\sum_{i=1}^r \tau_2(H_i) + \min_{i=1}^r (n_i - \tau_2(H_i)) + \min_{i=1}^r (n_i - \tau_2(H_i))^*$
ν	Section 4.7	0	$\nu(H_1) + \nu(H_2)$	$\max_{n_{\min}-r+2 \leq k \leq n_{\min}} k + \sum_{i=1}^r \min(\nu(H_i), \lfloor \frac{n_i-k}{r} \rfloor)$
γ	Section 4.8	1	$\gamma(H_1) + \gamma(H_2)$	$\min(\min_{i=1}^r \gamma(H_i), 2)$
χ	Section 4.9	1	$\max(\chi(H_1), \chi(H_2))$	$\max(\max_{i=1}^r \chi(H_i), \lceil \frac{\sum_{i=1}^r \chi(H_i)}{r-1} \rceil)$
$s\chi$	Section 4.10	1	$\max(s\chi(H_1), s\chi(H_2))$	$\sum_{i=1}^r s\chi(H_i)$
$\bar{\chi}$	Section 4.11	1	$\bar{\chi}(H_1) + \bar{\chi}(H_2)$	$\sum_{i=1}^r \bar{\chi}(H_i) - \min_{i=1}^r \bar{\chi}(H_i) - \min_{i=1}^r \bar{\chi}(H_i) + 1$

* Applies only for $r \geq 3$.

Besides the parameters considered above, there are a number of so-called *structural parameters*, which measure the structure of a graph when it is decomposed into a tree structure. For graphs, tree-width [41] and clique-width [42] are among the most important structural parameters.

The notions of tree-width and path-width for graphs can also be used for hypergraphs by considering the tree-width or path-width of the primal graph, as already done in [43]. By Lemma 5, and the results of [44] for graphs, we can obtain formulas for computing the tree-width and path-width of the disjoint union of two hypergraphs and the join of r hypergraphs in terms of the widths of the hypergraphs involved. Since the formulas for path-width and tree-width are equal, the equality between path-width and tree-width known for co-graphs from [44] for co-graphs also holds for every $r \geq 2$ for r -join-hypergraphs.

A further approach to defining tree-width for hypergraphs is the notion of hypertree-width, which was defined by Gottlob, Leone, and Scarcello in [45] and was motivated by algorithmic problems in artificial intelligence and database theory. Hypertree-width is based on the same tree decompositions as tree-width, but its width is measured differently.

Hypertree-width is the minimum number of hyperedges needed to cover all bags in a tree decomposition. It remains to investigate hypertree-width for r -join-hypergraphs.

Following the idea of [43] for tree-width, the clique-width of hypergraphs can be defined as the clique-width of the primal graph using the definition in [42]. Since, by Remark 5 and Lemma 5, the primal graphs of r -join-hypergraphs are co-graphs, the clique-width of r -join-hypergraphs is at most 2.

In [46], a generalization of clique-width to r -uniform hypergraphs was suggested. The operations of hyper- r -clique-width are the creation of a new labeled vertex, the vertex-disjoint union of two already constructed hypergraphs, the insertion of all possible hyperedges of cardinality r between vertices of r different labels in a defined hypergraph, and the uniform relabeling of vertices. Under this definition, every r -join-hypergraph has hyper- r -clique-width at most r .

There are a lot of further interesting hypergraph problems that can be considered on r -join-hypergraphs. Some examples are rainbow coloring [47], harmonious coloring [48], non-monochromatic coloring [8] (Chapter 11), spanning hypertree [49], spanning hypergraph [50], matching [51], edge clique covering [52,53], Hamiltonian cycle [54], Hamiltonian path [54], and Hamiltonian chain [55]. The aim of this paper is not to provide a complete list of hypergraph problems that can be solved by dynamic programming on r -join-hypergraphs, but to show that many of these problems can be handled using dynamic programming based on the same scheme.

While co-graphs have been well researched, many interesting questions remain open for r -join-hypergraphs. In Section 3, we showed a method to decide whether an r -uniform input hypergraph H is an r -join-hypergraph and, in the case of an affirmative answer, to find a decomposition tree $T(H)$ in polynomial time. Since our method does not run in linear time, it remains to determine whether such a method can be found. Can we find a characterization of r -join-hypergraphs by forbidden induced subhypergraphs for $r \geq 3$? By the closure property shown in Theorem 4(ii), the set of r -join-hypergraphs is a so-called *hereditary* graph class. A graph class is hereditary if and only if it admits a characterization in terms of minimal forbidden induced subgraphs, i.e., minimal graphs that do not belong to the class; see [56] (Chapter 2).

In [40] the notion of a $\mathcal{C}$ -perfect hypergraph was introduced. A hypergraph H is $\mathcal{C}$ -perfect if $\bar{\chi}(H') = \alpha(H')$ for every induced subhypergraph H' of H . Examples 10 and 11 give, for every $r \geq 2$, a set of r -join-hypergraphs H such that $\bar{\chi}(H) < \alpha(H)$, which implies that for $r \geq 2$ the set of r -join-hypergraphs is not $\mathcal{C}$ -perfect.

An alternative way of defining a binary join for two r -uniform hypergraphs H_1 and H_2 is given in [57]. The *join* of H_1 and H_2 is the hypergraph H with $V(H) = V(H_1) \cup V(H_2)$ and $E(H) = E(H_1) \cup E(H_2) \cup \{e \subseteq V(H) \mid |e| = r, e \cap V(H_1) \neq \emptyset, e \cap V(H_2) \neq \emptyset\}$. By combining this with our binary disjoint union (Section 3), we get an alternative way of defining co-graphs for r -uniform hypergraphs, which is called the set of *r -co-hypergraphs*. An interesting property of r -co-hypergraphs is that they are closed under complementation (w.r.t. r -uniform hypergraphs); see [30].

For co-graphs and also for directed co-graphs, a number of subclasses have been defined and investigated; see [13,58–63]. Such restrictions are also possible for r -join-hypergraphs by limiting the size or structure of one of the combined hypergraphs.

Furthermore, co-graphs and also directed co-graphs have a number of real applications; see [13–16,20,21]. It remains to identify practical applications for r -join-hypergraphs.

Author Contributions: Conceptualization, F.G.; methodology, F.G., J.R. and E.W.; validation, F.G. and J.R.; writing—original draft preparation, F.G.; writing—review and editing, F.G., J.R. and E.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: We acknowledge support from Heinrich Heine University Duesseldorf.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Estimations

Lemma A1. For every $r \geq 3$, it holds that $(r + 1)^r > 2r^r + r^2$.

Proof. Let $r \geq 3$. We can estimate the following:

$$\begin{aligned} (r + 1)^r &= \sum_{i=0}^r \binom{r}{i} r^i && \text{binomial theorem} \\ &= r^0 + \binom{r}{1} r + \binom{r}{2} r^2 + \binom{r}{2} r^2 + \dots + \binom{r}{r-1} r^{r-1} + r^r \\ &= r^0 + r^2 + \binom{r}{2} r^2 + \binom{r}{2} r^2 + \dots + r^r + r^r && \text{since } \binom{r}{1} = \binom{r}{r-1} = r \\ &> 2r^r + r^2 && \text{since } r \geq 3 \end{aligned}$$

This shows the statement of the lemma. $\square$

Lemma A2. For every $r \geq 3$, it holds that $(2r^r + r^2) \cdot (r^r - r) > 2(r^r)^2$.

Proof. Let $r \geq 3$. We can estimate the following:

$$\begin{aligned} (2r^r + r^2) \cdot (r^r - r) &= 2(r^r)^2 + r^{r+2} - 2r^{r+1} - r^3 \\ &= 2(r^r)^2 + r \cdot r^{r+1} - 2r^{r+1} - r^3 \\ &= 2(r^r)^2 + 2 \cdot r^{r+1} + (r - 2) \cdot r^{r+1} - 2r^{r+1} - r^3 \\ &\geq 2(r^r)^2 + 2 \cdot r^{r+1} + 1 \cdot r^4 - 2r^{r+1} - r^3 && \text{since } r \geq 3 \\ &= 2(r^r)^2 + 1 \cdot r^4 - r^3 \\ &> 2(r^r)^2 \end{aligned}$$

This shows the statement of the lemma. $\square$

Lemma A3. For every $r \geq 3$ and $k \geq 2$, it holds that $\binom{r^k}{r} \geq (r^{k-1} + r^{k-2})^r$.

Proof. We have

$$\binom{r^k}{r} = \frac{r^k \cdot (r^k - 1) \cdot (r^k - 2) \cdot (r^k - 3) \cdot \dots \cdot (r^k - r + 1)}{r \cdot (r - 1) \cdot (r - 2) \cdot (r - 3) \cdot \dots \cdot 1} = \prod_{i=0}^{r-1} \frac{r^k - i}{r - i}$$

For $i = 1, \dots, r - 1$ and $k \geq 2$, we have

$$\begin{aligned} \frac{r^k - i}{r - i} &\geq \frac{r^k - i \cdot r^{k-2}}{r - i} && \text{since } k \geq 2 \\ &\geq \frac{r^k - i \cdot r^{k-1} + r^{k-1} - i \cdot r^{k-2}}{r - i} && \text{since } i \geq 1 \\ &= \frac{r^{k-2} \cdot (r^2 - i \cdot r + r - i)}{r - i} \\ &= \frac{r^{k-2} (r + 1) (r - i)}{r - i} \\ &= r^{k-2} (r + 1) \\ &= r^{k-1} + r^{k-2} \end{aligned}$$

If this were also true for $i = 0$, the proof would be done. But for $i = 0$, the inequality $\frac{r^k - i}{r - i} \geq r^{k-1} + r^{k-2}$ is not true. Therefore, we combine the first three factors ($i = 0, 1, 2$), and we show

$$\frac{r^k \cdot (r^k - 1) \cdot (r^k - 2)}{r \cdot (r - 1) \cdot (r - 2)} \geq (r^{k-2} \cdot (r + 1))^3 \quad (\text{A1})$$

by considering the equivalent inequality

$$r^{k-1} \cdot (r^k - 1) \cdot (r^k - 2) \geq r^{3k-6} \cdot (r+1)^3 \cdot (r-1) \cdot (r-2). \quad (\text{A2})$$

- First we show (A2) for $k = 2$.

$$\begin{aligned} r \cdot (r^2 - 1) \cdot (r^2 - 2) &= (r^2 - 1) \cdot (r^3 - 2r) \\ &> (r^2 - 1) \cdot (r^3 - 2r - r - 2) \\ &= (r^2 - 1) \cdot (r^3 - 3r - 2) \\ &= (r^2 - 1) \cdot (r+1)^2(r-2) \\ &= (r+1) \cdot (r-1) \cdot (r+1)^2 \cdot (r-2) \\ &= (r+1)^3 \cdot (r-1) \cdot (r-2) \end{aligned}$$

- Next we show (A2) for $k \geq 3$.

$$\begin{aligned} r^{k-1} \cdot (r^k - 1) \cdot (r^k - 2) &= r^{k-1} \cdot (r^{2k} - 3r^k + 2) \\ &= r^{3k-1} - 3r^{2k-1} + 2r^{k-1} \\ &> r^{3k-1} - 3r^{2k-1} \\ &\geq r^{3k-1} - r \cdot r^{2k-1} && \text{since } r \geq 3 \\ &= r^{3k-1} - r^{2k} \\ &\geq r^{3k-1} - r^{3k-3} && \text{since } k \geq 3 \\ &\geq r^{3k-1} - 2r^{3k-3} \\ &= r^{3k-1} - 4r^{3k-3} + r^{3k-3} + r^{3k-3} \\ &> r^{3k-1} - 4r^{3k-3} + r^{3k-4} + r^{3k-5} \\ &= r^{3k-1} - 4r^{3k-3} + r \cdot r^{3k-5} + r \cdot r^{3k-6} \\ &\geq r^{3k-1} - 4r^{3k-3} + 3r^{3k-5} + 2r^{3k-6} && \text{since } r \geq 3 \\ &> r^{3k-1} - 4r^{3k-3} - 2r^{3k-4} + 3r^{3k-5} + 2r^{3k-6} \\ &= r^{3k-6} \cdot (r^5 - 4r^3 - 2r^2 + 3r + 2) \\ &= r^{3k-6} \cdot (r^2 + 2r + 1) \cdot (r^3 - 2r^2 - r + 2) \\ &= r^{3k-6} \cdot (r^2 + 2r + 1) \cdot (r^2 - 1) \cdot (r - 2) \\ &= r^{3k-6} \cdot (r+1)^2 \cdot (r^2 - 1) \cdot (r - 2) \\ &= r^{3k-6} \cdot (r+1)^2 \cdot (r+1) \cdot (r-1) \cdot (r - 2) \\ &= r^{3k-6} \cdot (r+1)^3 \cdot (r-1) \cdot (r - 2) \end{aligned}$$

This shows the statement of the lemma. $\square$

Lemma A4. For every $r \geq 3$ and $k \geq 2$, it holds that $\binom{r^k}{r} > 2 \frac{r^{kr} - r^k}{r^r - r}$.

Proof. Let $r \geq 3$ and $k \geq 2$. We can estimate the following:

$$\begin{aligned} \binom{r^k}{r} &\geq (r^{k-1} + r^{k-2})^r && \text{Lemma A3} \\ &= (r^{k-2} \cdot (r+1))^r \\ &= r^{r(k-2)} \cdot (r+1)^r \\ &= (r+1)^r \cdot (r^r)^{k-2} \\ &= \frac{(r+1)^r}{(r^r)^2} (r^r)^k \\ &> \frac{2r^r + r^2}{(r^r)^2} (r^r)^k && \text{Lemma A1} \\ &> \frac{2}{r^r - r} (r^r)^k && \text{Lemma A2} \\ &> \frac{2}{r^r - r} ((r^r)^k - r^k) \\ &= 2 \frac{(r^r)^k - r^k}{r^r - r} \\ &= 2 \frac{r^{kr} - r^k}{r^r - r} \end{aligned}$$

This shows the statement of the lemma. $\square$

Appendix B. Values Used in Example 8

In order to compute $f(109)$ and $f(113)$ in Example 8, we use the following values for $\min(v(H_i), \lfloor \frac{n_i-k}{r} \rfloor)$:

i	$k = 109$	$k = 113$
1	$\min(5, 1) = 1$	$\min(5, 0) = 0$
2	$\min(6, 1) = 1$	$\min(6, 0) = 0$
3	$\min(1, 1) = 1$	$\min(1, 1) = 1$
4	$\min(5, 0) = 0$	$\min(5, 0) = 0$
5	$\min(9, 4) = 4$	$\min(9, 3) = 3$
6	$\min(5, 2) = 2$	$\min(5, 1) = 1$
7	$\min(4, 1) = 1$	$\min(4, 1) = 1$
8	$\min(8, 2) = 2$	$\min(8, 1) = 1$
9	$\min(5, 1) = 1$	$\min(5, 0) = 0$
10	$\min(5, 3) = 3$	$\min(5, 3) = 3$
Σ	16	10

References

- Eidi, M.; Farzam, A.; Leal, W.; Samal, A.; Jost, J. Edge-based analysis of networks: Curvatures of graphs and hypergraphs. *Theory Biosci.* **2020**, *139*, 337–348.
- Higham, D.; de Kergorlay, H.L. Epidemics on hypergraphs: Spectral thresholds for extinction. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2021**, *477*, 20210232.
- Jost, J.; Mulas, R. Hypergraph Laplace operators for chemical reaction networks. *Adv. Math.* **2019**, *351*, 870–896.
- Konstantinova, E.; Skorobogatov, V. Application of hypergraph theory in chemistry. *Discret. Math.* **2001**, *235*, 365–383.
- Huang, J.; Zhang, R.; Yu, J. Scalable Hypergraph Learning and Processing. In Proceedings of the 2015 IEEE International Conference on Data Mining, Atlantic City, NJ, USA, 14–17 November 2015; pp. 775–780.
- Zhou, D.; Huang, J.; Schölkopf, B. Learning with Hypergraphs: Clustering, Classification, and Embedding. In *Proceedings of the Advances in Neural Information Processing Systems*; Schölkopf, B., Platt, J., Hoffman, T., Eds.; MIT Press: Cambridge, MA, USA, 2006; Volume 19.
- Bretto, A. *Hypergraph Theory*; Springer: Berlin, Germany, 2013.
- Beineke, L.; Wilson, R.J. (Eds.) *Topics in Chromatic Graph Theory*; Encyclopedia of Mathematics and Its Applications; Cambridge University Press: Cambridge, UK, 2015.
- Bujtas, C.; Tuza, Z. Maximum number of colors: C-coloring and related problems. *J. Geom.* **2011**, *101*, 83–97.
- Jung, H. On a class of posets and the corresponding comparability graphs. *J. Comb. Theory Ser. B* **1978**, *24*, 125–133.
- Lerchs, H. *On Cliques and Kernels*; Technical Report; Department of Computer Science, University of Toronto: Toronto, ON, Canada, 1971.
- Seinsche, D. On a property of the class of n-colorable graphs. *J. Comb. Theory B* **1974**, *16*, 191–193.
- Corneil, D.; Lerchs, H.; Stewart-Burlingham, L. Complement Reducible Graphs. *Discret. Appl. Math.* **1981**, *3*, 163–174.
- Abrahamson, K.; Dadoun, N.; Kirkpatrick, D.; Przytycka, T. A simple parallel tree contraction algorithm. *J. Algorithms* **1989**, *10*, 287–302.
- Adhar, G.; Peng, S. Parallel Algorithms for Path Covering, Hamiltonian Path and Hamiltonian Cycle in Cographs. In *Proceedings of the 1990 International Conference on Parallel Processing, Urbana-Champaign, IL, USA, 13–17 August 1990*; Volume 3: Algorithms and Applications; Yew, P., Ed.; Pennsylvania State University Press: University Park, PA, USA, 1990; pp. 364–365.
- Hellmuth, M.; Hernandez-Rosales, M.; Huber, K.; Moulton, V.; Stadler, P.; Wieseke, N. Orthology Relations, Symbolic Ultrametrics, and Cographs. *J. Math. Biol.* **2013**, *66*, 399–420.
- Yu, M.S.; Yang, C.H. An $O(n)$ time algorithm for maximum matching on cographs. *Inf. Process. Lett.* **1993**, *47*, 89–93.
- Bechet, D.; de Groote, P.; Retoré, C. A complete axiomatisation of the inclusion of series-parallel partial orders. In *Proceedings of the Rewriting Techniques and Applications*; LNCS; Springer: Berlin, Germany, 1997; Volume 1232, pp. 230–240.
- Crespelle, C.; Paul, C. Fully dynamic recognition algorithm and certificate for directed cographs. *Discret. Appl. Math.* **2006**, *154*, 1722–1741.
- Hellmuth, M.; Stadler, P.; Wieseke, N. The mathematics of xenology: Di-cographs, symbolic ultrametrics, 2-structures and tree-representable systems of binary relations. *J. Math. Biol.* **2017**, *75*, 199–237.

21. Nojgaard, N.; El-Mabrouk, N.; Merkle, D.; Wieseke, N.; Hellmuth, M. Partial Homology Relations - Satisfiability in terms of Di-Cographs. In *Proceedings of International Computing and Combinatorics Conference (COCOON)*; Springer: Berlin, Germany, 2018; LNCS; Volume 10976, pp. 403–415.
22. Gurski, F.; Komander, D.; Rehs, C. How to compute Digraph Width Measures on Directed Co-Graphs. *Theor. Comput. Sci.* **2021**, *855*, 161–185.
23. Gurski, F.; Komander, D.; Rehs, C.; Rethmann, J.; Wanke, E. Computing Directed Steiner Path Covers. *J. Comb. Optim.* **2022**, *43*, 402–431.
24. Brandstädt, A.; Le, V.; Spinrad, J. *Graph Classes: A Survey*; SIAM Monographs on Discrete Mathematics and Applications; SIAM: Philadelphia, PA, USA, 1999.
25. Corneil, D.; Perl, Y.; Stewart, L. A linear recognition algorithm for cographs. *SIAM J. Comput.* **1985**, *14*, 926–934.
26. Garey, M.; Johnson, D. *Computers and Intractability: A Guide to the Theory of NP-Completeness*; W.H. Freeman and Company: San Francisco, CA, USA, 1979.
27. Borie, R.; Parker, R.; Tovey, C. Solving Problems on Recursively Constructed Graphs. *ACM Comput. Surv.* **2009**, *41*, 1–51.
28. Acharya, B. Domination in Hypergraphs. *AKCE J. Graphs. Combin.* **2007**, *4*, 117–126.
29. Berge, C. *Hypergraphs*; North-Holland Publishing Company: Amsterdam, The Netherlands; New York, NY, USA; Oxford, UK; Tokyo, Japan, 1989.
30. Gurski, F.; Rethmann, J. Complement reducible uniform hypergraphs. *Axioms* **2026**, *15*, 278.
31. Gallai, T. Über extreme Punkt- und Kantenmengen. *Ann. Univ. Sci. Budapest Eotvos Sect. Math.* **1959**, *2*, 133–138.
32. Karp, R. Reducibility among combinatorial problems. In *Proceedings of the Complexity of Computer Computations*; Miller, R., Thatcher, J., Eds.; Springer: Berlin, Germany, 1972; pp. 85–103.
33. Lee, E.; Svensson, O.; Thiery, T. Asymptotically optimal hardness for k -set packing and k -matroid intersection. In *Proceedings of the Annual ACM Symposium on Theory of Computing (STOC)*; ACM: New York, NY, USA, 2025; pp. 54–61.
34. Chan, Y.; Lau, L. On linear and semidefinite programming relaxations for hypergraph matching. *Math. Program.* **2012**, *135*, 123–148.
35. Lovász, L. Coverings and colourings of hypergraphs. *Congr. Numer.* **1973**, *8*, 3–12.
36. Hall, P. On representation of subsets. *Quart. J. Math.* **1935**, *10*, 26–30.
37. Fellows, M. Transversals of Vertex Partitions in Graphs. *SIAM J. Discret. Math.* **1990**, *3*, 206–215.
38. Alon, N. The strong chromatic number of a graph. *Random Struct. Algorithms* **1992**, *3*, 1–7.
39. Pálvölgyi, D. Note on Polychromatic Coloring of Hereditary Hypergraph Families. *Graphs Comb.* **2024**, *40*, 131.
40. Voloshin, V. On the upper chromatic number of a hypergraph. *Australas. J. Combin.* **1995**, *11*, 25–45.
41. Robertson, N.; Seymour, P. Graph Minors II. Algorithmic aspects of tree width. *J. Algorithms* **1986**, *7*, 309–322.
42. Courcelle, B.; Olariu, S. Upper bounds to the clique width of graphs. *Discret. Appl. Math.* **2000**, *101*, 77–114.
43. Adler, I.; Gottlob, G.; Grohe, M. Hypertree width and related hypergraph invariants. *Eur. J. Comb.* **2007**, *28*, 2167–2181.
44. Bodlaender, H.; Möhring, R. The pathwidth and treewidth of cographs. *SIAM J. Disc. Math.* **1993**, *6*, 181–188.
45. Gottlob, G.; Leone, N.; Scarcello, F. Hypertree Decompositions and Tractable Queries. *J. Comput. Syst. Sci.* **2002**, *64*, 579–627.
46. Fujita, T.; Al-Hawary, T. Short note of SuperHyperClique-width and Local Superhypertree-width. *Neutrosophic Sets Syst.* **2025**, *86*, 379–392.
47. Guruswami, V.; Saket, R. Hardness of Rainbow Coloring Hypergraphs. In *Proceedings of the 37th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2017, Kanpur, India, 11–15 December 2017*; Schloss Dagstuhl—Leibniz-Zentrum für Informatik: Wadern, Germany, 2017; LIPIcs; Volume 93, pp. 33:33–33:15.
48. Czerwinski, S. On harmonious coloring of hypergraphs. *Discret. Math. Theor. Comput. Sci.* **2024**, *26*, 1–5. <https://doi.org/10.46298/DMTCS.11101>.
49. Tomescu, I.; Zimand, M. Minimum spanning hypertrees. *Discret. Appl. Math.* **1994**, *54*, 67–76.
50. Budden, M.; Hiller, J.; Penland, A. Minimally connected r -uniform hypergraphs. *Australas. J. Comb.* **2022**, *82*, 1–20.
51. Tang, Z.; Zou, H.; Diao, Z. On the Matching Number of k -Uniform Connected Hypergraphs with Maximum Degree. In *Proceedings of the Frontiers of Algorithmics-17th International Joint Conference, IJTCS-FAW 2023 Macau, China, 14–18 August 2023, Proceedings*; Li, M., Sun, X., Wu, X., Eds.; Lecture Notes in Computer Science; Springer: Berlin, Germany, 2023; Volume 13933, pp. 71–84.
52. Lehel, J. Covers in hypergraphs. *Combinatorica* **1982**, *2*, 305–309.
53. Rödl, V.; Ruciński, A. Covering the edges of a random hypergraph by cliques. *Discuss. Math. Graph Theory* **2021**, *42*, 275–288.
54. Liu, G.; Liu, X. Hamiltonian Paths and Cycles in Some 4-Uniform Hypergraphs. *Graphs Comb.* **2022**, *38*, 122.
55. Katona, G.Y.; Kierstead, H.A. Hamiltonian chains in hypergraphs. *J. Graph Theory* **1999**, *30*, 205–212.
56. Kitaev, S.; Lozin, V. *Words and Graphs*; Springer: Berlin, Germany, 2015.
57. Sarkar, A.; Banerjee, A. Joins of hypergraphs and their spectra. *Linear Algebra Its Appl.* **2020**, *603*, 101–129.
58. Golumbic, M. Trivially perfect graphs. *Discret. Math.* **1978**, *24*, 105–107.

59. Chvátal, V.; Hammer, P. Aggregation of inequalities in integer programming. *Ann. Discret. Math.* **1977**, *1*, 145–162.
60. Nikolopoulos, S.; Papadopoulos, C. A Simple Linear-Time Recognition Algorithm for Weakly Quasi-Threshold Graphs. *Graphs Comb.* **2011**, *27*, 557–565.
61. Heggernes, P.; Meister, D.; Papadopoulos, C. Graphs of linear clique-width at most 3. *Theor. Comput. Sci.* **2011**, *412*, 5466–5486.
62. Boeckner, D. Oriented Threshold Graphs. *Australas. J. Comb.* **2018**, *71*, 43–53.
63. Gurski, F.; Komander, D.; Rehs, C. On Characterizations for Subclasses of Directed Co-Graphs. *J. Comb. Optim.* **2021**, *41*, 234–266.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.