A Note on Korn’s Inequality in an N-Dimensional Context and a Global Existence Result for a Non-Linear Plate Model

Fabio Silva Botelho

Department of Mathematics, Federal University of Santa Catarina, UFSC, Florianópolis 88040-900, Brazil; fabio.botelho@ufsc.br

Abstract: In the first part of this article, we present a new proof for Korn’s inequality in an n-dimensional context. The results are based on standard tools of real and functional analysis. For the final result, the standard Poincaré inequality plays a fundamental role. In the second text part, we develop a global existence result for a non-linear model of plates. We address a rather general type of boundary conditions and the novelty here is the more relaxed restrictions concerning the external load magnitude.

Keywords: Korn’s inequality; global existence result; non-linear plate model

MSC: 35Q74; 35J58

1. Introduction

In this article, we present a proof for Korn’s inequality in $\mathbb{R}^n$. The results are based on standard tools of functional analysis and on the Sobolev spaces theory.

We emphasize that such a proof is relatively simple and easy to follow since it is established in a very transparent and clear fashion.

About the references, we highlight that related results in a three-dimensional context may be found in [1]. Other important classical results on Korn’s inequality and concerning applications to models in elasticity may be found in [2–4].

Remark 1. Generically, throughout the text we denote

$$
\|u\|_{0,2,\Omega} = \left( \int_{\Omega} |u|^2 \, dx \right)^{1/2}, \quad \forall u \in L^2(\Omega),
$$

and

$$
\|u\|_{0,2,\Omega} = \left( \sum_{j=1}^{n} \|u_j\|_{0,2,\Omega}^2 \right)^{1/2}, \quad \forall u = (u_1, \ldots, u_n) \in L^2(\Omega; \mathbb{R}^n).
$$

Moreover,

$$
\|u\|_{1,2,\Omega} = \left( \|u\|_{0,2,\Omega}^2 + \sum_{j=1}^{n} \|u_j\|_{0,2,\Omega}^2 \right)^{1/2}, \quad \forall u \in W^{1,2}(\Omega),
$$

where we shall also refer throughout the text to the well-known corresponding analogous norm for $u \in W^{1,2}(\Omega; \mathbb{R}^n)$.

At this point, we first introduce the following definition.

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set. We say that $\partial \Omega$ is $C^1$ if such a manifold is oriented and for each $x_0 \in \partial \Omega$, denoting $\bar{x} = (x_1, \ldots, x_{n-1})$ for a local coordinate system compatible...
with the manifold $\partial \Omega$ orientation, there exist $r > 0$ and a function $f(x_1, \ldots, x_{n-1}) = f(\hat{x})$ such that
\[
W = \overline{\Omega} \cap B_r(x_0) = \{ x \in B_r(x_0) \mid x_n \leq f(x_1, \ldots, x_{n-1}) \}.
\]
Moreover, $f(\hat{x})$ is a Lipschitz continuous function, so that
\[
|f(\hat{x}) - f(\hat{y})| \leq C_1 |\hat{x} - \hat{y}|_2, \text{ on its domain,}
\]
for some $C_1 > 0$. Finally, we assume
\[
\left\{ \frac{\partial f(\hat{x})}{\partial x_k} \right\}_{k=1}^{n-1}
\]
is classically defined, almost everywhere also on its concerning domain, so that $f \in W^{1,2}$.

**Remark 2.** This mentioned set $\Omega$ is of a Lipschitzian type, so that we may refer to such a kind of sets as domains with a Lipschitzian boundary, or simply as Lipschitzian sets.

At this point, we recall the following result found in [5], at page 222 in its Chapter 11.

**Theorem 1.** Assume $\Omega \subset \mathbb{R}^n$ is an open bounded set, and that $\partial \Omega$ is $\hat{C}^1$. Let $1 \leq p < \infty$, and let $V$ be a bounded open set such that $\Omega \subset\subset V$. Then there exists a bounded linear operator
\[
E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n),
\]
such that for each $u \in W^{1,p}(\Omega)$ we have:
1. $Eu = u$, a.e. in $\Omega$;
2. $Eu$ has support in $V$;
3. $\|Eu\|_{1,p,\mathbb{R}^n} \leq C\|u\|_{1,p,\Omega}$, where the constant depends only on $p$, $\Omega$, and $V$.

**Remark 3.** Considering the proof of such a result, the constant $C > 0$ may be also such that
\[
\|e_{ij}(Eu)\|_{0,2,V} \leq C(\|e_{ij}(u)\|_{0,2,\Omega} + \|u\|_{0,2,\Omega}), \forall u \in W^{1,2}(\Omega; \mathbb{R}^n), \forall i, j \in \{1, \ldots, n\},
\]
for the operator $e : W^{1,2}(\Omega; \mathbb{R}^n) \to L^2(\Omega; \mathbb{R}^{n \times n})$ specified in the next theorem.

Finally, as the meaning is clear, we may simply denote $Eu = u$.

2. The Main Results, the Korn Inequalities

Our main result is summarized by the following theorem.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected set with a $\hat{C}^1$ (Lipschitzian) boundary $\partial \Omega$.

Define $e : W^{1,2}(\Omega; \mathbb{R}^n) \to L^2(\Omega; \mathbb{R}^{n \times n})$ by
\[
e(u) = \{e_{ij}(u)\}
\]
where
\[
e_{ij}(u) = \frac{1}{2}(u_{ij} + u_{ji}), \forall i, j \in \{1, \ldots, n\},
\]
and where generically, we denote
\[
u_{ij} = \frac{\partial u_i}{\partial x_j}, \forall i, j \in \{1, \cdots, n\}.
Define also,
\[ \|e(u)\|_{0,2,\Omega} = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \|e_{ij}(u)\|_{0,2,\Omega}^2 \right)^{1/2}. \]

Let \( L \in \mathbb{R}^+ \) be such \( V = [-L, L]^n \) is also such that \( \overline{\Omega} \subset V^0 \).

Under such hypotheses, there exists \( C(\Omega, L) \in \mathbb{R}^+ \) such that
\[ \|u\|_{1,2,\Omega} \leq C(\Omega, L)(\|u\|_{0,2,\Omega} + \|e(u)\|_{0,2,\Omega}), \quad \forall u \in W^{1,2}(\Omega; \mathbb{R}^n). \quad (1) \]

**Proof.** Suppose, to obtain contradiction, that the concerning claim does not hold.

Thus, we are assuming that there is no positive real constant \( C = C(\Omega, L) \) such that (1) is valid.

In particular, \( k = 1 \in \mathbb{N} \) is not such a constant \( C \), so that there exists a function \( u_1 \in W^{1,2}(\Omega; \mathbb{R}^n) \) such that
\[ \|u_1\|_{1,2,\Omega} \geq 1 (\|u_1\|_{0,2,\Omega} + \|e(u_1)\|_{0,2,\Omega}). \]

Similarly, \( k = 2 \in \mathbb{N} \) is not such a constant \( C \), so that there exists a function \( u_2 \in W^{1,2}(\Omega; \mathbb{R}^n) \) such that
\[ \|u_2\|_{1,2,\Omega} \geq 2 (\|u_2\|_{0,2,\Omega} + \|e(u_2)\|_{0,2,\Omega}). \]

Hence, since no \( k \in \mathbb{N} \) is such a constant \( C \), reasoning inductively, for each \( k \in \mathbb{N} \) there exists a function \( u_k \in W^{1,2}(\Omega; \mathbb{R}^n) \) such that
\[ \|u_k\|_{1,2,\Omega} \geq k (\|u_k\|_{0,2,\Omega} + \|e(u_k)\|_{0,2,\Omega}). \]

In particular, defining
\[ v_k = \frac{u_k}{\|u_k\|_{1,2,\Omega}} \]
we obtain
\[ \|v_k\|_{1,2,\Omega} = 1 > k (\|v_k\|_{0,2,\Omega} + \|e(v_k)\|_{0,2,\Omega}), \]
so that
\[ (\|v_k\|_{0,2,\Omega} + \|e(v_k)\|_{0,2,\Omega}) < \frac{1}{k}, \quad \forall k \in \mathbb{N}. \]

From this we obtain
\[ \|v_k\|_{0,2,\Omega} < \frac{1}{k}, \]
and
\[ \|e_i(v_k)\|_{0,2,\Omega} < \frac{1}{k}, \quad \forall k \in \mathbb{N}, \]
so that
\[ \|v_k\|_{0,2,\Omega} \to 0, \quad \text{as } k \to \infty, \]
and
\[ \|e_i(v_k)\|_{0,2,\Omega} \to 0, \quad \text{as } k \to \infty. \]

In particular,
\[ \|(v_k)_{ij}\|_{0,2,\Omega} \to 0, \quad \forall j \in \{1, \ldots, n\}. \]

At this point, we recall the following identity in the distributional sense, found in [3], page 12:
\[ \partial_j (\partial_i v) = \partial_j e_i(v) + \partial_i e_j(v) - \partial_i e_j(v), \quad \forall i, j, l \in \{1, \ldots, n\}. \quad (2) \]

Fix \( j \in \{1, \ldots, n\} \) and observe that
\[ \|(v_k)_{ij}\|_{1,2,\Omega} \leq C(\|v_k\|_{1,2,\Omega})^{1/2}. \]
so that
\[
\frac{C}{\|v_k\|_{1,2,\Omega}} \geq \frac{1}{\|v_k\|_{1,2,\Omega}}, \quad \forall k \in \mathbb{N}.
\]

Hence,
\[
\|v_k\|_{1,2,\Omega} \leq \sup_{\varphi \in C^1(\Omega)} \left\{ \langle \nabla (v_k)_j, \nabla \varphi \rangle_{L^2(\Omega)} + \langle (v_k)_j, \varphi \rangle_{L^2(\Omega)} : \|\varphi\|_{1,2,\Omega} \leq 1 \right\}
\]
\[
= \sup_{\varphi \in C^1(\Omega)} \left\{ \langle \nabla (v_k)_j, \nabla \varphi \rangle_{L^2(\Omega)} + \langle (v_k)_j, \varphi \rangle_{L^2(V)} : \|\varphi\|_{1,2,\Omega} \leq 1 \right\}
\]
\[
\leq C \sup_{\varphi \in C^1(V)} \left\{ \langle \nabla (v_k)_j, \nabla \varphi \rangle_{L^2(V)} + \langle (v_k)_j, \varphi \rangle_{L^2(V)} : \|\varphi\|_{1,2,V} \leq 1 \right\}.
\]

Here, we recall that $C > 0$ is the constant concerning the extension Theorem 1. From such results and (2), we have that
\[
\sup_{\varphi \in C^1(\Omega)} \left\{ \langle \nabla (v_k)_j, \nabla \varphi \rangle_{L^2(\Omega)} + \langle (v_k)_j, \varphi \rangle_{L^2(\Omega)} : \|\varphi\|_{1,2,\Omega} \leq 1 \right\}
\]
\[
\leq C \sup_{\varphi \in C^1(V)} \left\{ \langle \nabla (v_k)_j, \nabla \varphi \rangle_{L^2(V)} + \langle (v_k)_j, \varphi \rangle_{L^2(V)} : \|\varphi\|_{1,2,V} \leq 1 \right\}
\]
\[
= C \sup_{\varphi \in C^1(V)} \left\{ \langle \epsilon_{\mu}(v_k), \varphi \rangle_{L^2(V)} + \langle \epsilon_{\mu}(v_k), \varphi \rangle_{L^2(V)} - \langle \epsilon_{\mu}(v_k), \varphi \rangle_{L^2(V)} + \langle (v_k)_j, \varphi \rangle_{L^2(V)} : \|\varphi\|_{1,2,V} \leq 1 \right\}.
\]

Therefore,
\[
\|v_k\|_{W^{1,2}(\Omega)} = \sup_{\varphi \in C^1(\Omega)} \left\{ \langle \nabla (v_k)_j, \nabla \varphi \rangle_{L^2(\Omega)} + \langle (v_k)_j, \varphi \rangle_{L^2(\Omega)} : \|\varphi\|_{1,2,\Omega} \leq 1 \right\}
\]
\[
\leq C \left( \sum_{j=1}^{n} \|\epsilon_{\mu}(v_k)\|_{0,2,V} + \|\epsilon_{\mu}(v_k)\|_{0,2,V} \right) + \|v_k\|_{0,2,V}
\]
\[
\leq C_1 \left( \sum_{j=1}^{n} \|\epsilon_{\mu}(v_k)\|_{0,2,\Omega} + \|\epsilon_{\mu}(v_k)\|_{0,2,\Omega} \right) + \|v_k\|_{0,2,\Omega}
\]
\[
< \frac{C_2}{k},
\]
for appropriate $C_1 > 0$ and $C_2 > 0$.

Summarizing,
\[
\|v_k\|_{W^{1,2}(\Omega)} < \frac{C_2}{k}, \quad \forall k \in \mathbb{N}.
\]

From this we obtain
\[
\|v_k\|_{1,2,\Omega} \to 0, \quad \text{as} \ k \to \infty,
\]
which contradicts \( \|v_k\|_{1,2,\Omega} = 1, \forall k \in \mathbb{N}. \)

The proof is complete. \( \square \)

**Corollary 1.** Let \( \Omega \subset \mathbb{R}^n \) be an open, bounded and connected set with a \( \mathcal{C}^1 \) boundary \( \partial \Omega \). Define \( e : W^{1,2}(\Omega; \mathbb{R}^n) \to L^2(\Omega; \mathbb{R}^{n \times n}) \) by

\[
e(u) = \{e_{ij}(u)\}
\]

where

\[
e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \forall i, j \in \{1, \ldots, n\}.
\]

Define also,

\[
\|e(u)\|_{0,2,\Omega} = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \|e_{ij}(u)\|^2_{0,2,\Omega} \right)^{1/2}.
\]

Let \( L \in \mathbb{R}^+ \) be such \( V = [-L, L]^n \) is also such that \( \overline{\Omega} \subset V^0 \).

Moreover, define

\[
\hat{H}_0 = \{ u \in W^{1,2}(\Omega; \mathbb{R}^n) : u = 0, \text{ on } \Gamma_0 \},
\]

where \( \Gamma_0 \subset \partial \Omega \) is a measurable set such that the Lebesgue measure \( m_{\mathbb{R}^{n-1}}(\Gamma_0) > 0 \).

Assume also \( \Gamma_0 \) is such that for each \( j \in \{1, \ldots, n\} \) and each \( x = (x_1, \ldots, x_n) \in \Omega \) there exists \( x_0 = ((x_0)_1, \ldots, (x_0)_n) \in \Gamma_0 \) such that

\[
(x_0)_l = x_l, \forall l \neq j, l \in \{1, \ldots, n\},
\]

and the line

\[
A_{x_0,x} \subset \overline{\Omega}
\]

where

\[
A_{x_0,x} = \{(x_1, \ldots, (1-t)(x_0)_j + tx_j, \ldots, x_n) : t \in [0,1]\}.
\]

Under such hypotheses, there exists \( C(\Omega, L) \in \mathbb{R}^+ \) such that

\[
\|u\|_{1,2,\Omega} \leq C(\Omega, L) \|e(u)\|_{0,2,\Omega}, \forall u \in \hat{H}_0.
\]

**Proof.** Suppose, to obtain contradiction, that the concerning claim does not hold. Hence, for each \( k \in \mathbb{N} \) there exists \( u_k \in \hat{H}_0 \) such that

\[
\|u_k\|_{1,2,\Omega} > k \|e(u_k)\|_{0,2,\Omega}.
\]

In particular, defining

\[
v_k = \frac{u_k}{\|u_k\|_{1,2,\Omega}}
\]

similarly to the proof of the last theorem, we may obtain

\[
\|(v_k)_{ij}\|_{0,2,\Omega} \to 0, \text{ as } k \to \infty, \forall j \in \{1, \ldots, n\}.
\]

From this, the hypotheses on \( \Gamma_0 \) and from the standard Poincaré inequality proof we obtain

\[
\|(v_k)_{ij}\|_{0,2,\Omega} \to 0, \text{ as } k \to \infty, \forall j \in \{1, \ldots, n\}.
\]

Thus, also similarly as in the proof of the last theorem, we may infer that

\[
\|v_k\|_{1,2,\Omega} \to 0, \text{ as } k \to \infty,
\]
which contradicts
\[ \|v_k\|_{1,2,\Omega} = 1, \, \forall k \in \mathbb{N}. \]

The proof is complete. \(\square\)

3. An Existence Result for a Non-Linear Model of Plates

In the present section, as an application of the results on Korn’s inequalities presented in the previous sections, we develop a new global existence proof for a Kirchhoff–Love thin plate model. Previous results on the existence of mathematical elasticity and related models may be found in [2–4].

At this point we start to describe the primal formulation.

Let \( \Omega \subset \mathbb{R}^2 \) be an open, bounded, connected set which represents the middle surface of a plate of thickness \( h \). The boundary of \( \Omega \), which is assumed to be regular (Lipschitzian), is denoted by \( \partial \Omega \). The vectorial basis related to the cartesian system \( \{x_1, x_2, x_3\} \) is denoted by \( \{a_\alpha, a_3\} \), where \( \alpha = 1, 2 \) (in general, Greek indices stand for 1 or 2), and where \( a_3 \) is the vector normal to \( \Omega \), whereas \( a_1 \) and \( a_2 \) are orthogonal vectors parallel to \( \Omega \). Furthermore, \( n \) is the outward normal to the plate surface.

The displacements will be denoted by \( \hat{u} = \{\hat{u}_\alpha, \hat{u}_3\} = \hat{u}_\alpha a_\alpha + \hat{u}_3 a_3 \).

The Kirchhoff–Love relations are
\[ \hat{u}_\alpha(x_1, x_2, x_3) = u_\alpha(x_1, x_2) - x_3 w(x_1, x_2), \]
\[ \text{and} \quad \hat{u}_3(x_1, x_2, x_3) = w(x_1, x_2). \] (6)

Here, \(-h/2 \leq x_3 \leq h/2\) so that we have \( u = (u_\alpha, w) \in U \) where
\[ U = \left\{ (u_\alpha, w) \in W^{1,2}(\Omega; \mathbb{R}^2) \times W^{2,2}(\Omega), \right. \]
\[ u_\alpha = w = \frac{\partial w}{\partial n} = 0 \text{ on } \partial \Omega \}
\[ = W^{1,2}_0(\Omega; \mathbb{R}^2) \times W^{2,2}_0(\Omega). \]

It is worth emphasizing that the boundary conditions here specified refer to a clamped plate.

We define the operator \( \Lambda : U \to Y \times Y \), where \( Y = Y^* = L^2(\Omega; \mathbb{R}^{2 \times 2}) \), by
\[ \Lambda(u) = \{\gamma(u), \kappa(u)\}, \]
\[ \gamma_{\alpha\beta}(u) = \frac{u_{\alpha\beta} + u_{\beta\alpha}}{2} + \frac{w_\alpha w_\beta}{2}, \]
\[ \kappa_{\alpha\beta}(u) = -w_{\alpha\beta}. \]

The constitutive relations are given by
\[ N_{\alpha\beta}(u) = H_{\alpha\beta\lambda\mu} \gamma_{\lambda\mu}(u), \] (7)
\[ M_{\alpha\beta}(u) = h_{\alpha\beta\lambda\mu} \kappa_{\lambda\mu}(u), \] (8)

where \( \{H_{\alpha\beta\lambda\mu}\} \) and \( \{h_{\alpha\beta\lambda\mu}\} = \frac{\partial^2}{\partial n^2} H_{\alpha\beta\lambda\mu} \}, \) are symmetric positive definite fourth-order tensors. From now on, we denote \( \{H^{-1}_{\alpha\beta\lambda\mu}\} \) and \( \{h^{-1}_{\alpha\beta\lambda\mu}\} = \{h_{\alpha\beta\lambda\mu}\}^{-1}. \)
Furthermore, \( \{N_{\alpha\beta}\} \) denote the membrane force tensor and \( \{M_{\alpha\beta}\} \) the moment one. The plate stored energy, represented by \((G \circ \Lambda) : U \to \mathbb{R}\), is expressed by
\[
(G \circ \Lambda)(u) = \frac{1}{2} \int_{\Omega} N_{\alpha\beta}(u) \gamma_{\alpha\beta}(u) \, dx + \frac{1}{2} \int_{\Omega} M_{\alpha\beta}(u) \kappa_{\alpha\beta}(u) \, dx
\]
and the external work, represented by \( F : U \to \mathbb{R} \), is given by
\[
F(u) = \langle w, P \rangle_{L^2(\Omega)} + \langle u_\alpha, P_\alpha \rangle_{L^2(\Omega)},
\]
where \( P, P_1, P_2 \in L^2(\Omega) \) are external loads in the directions \( a_3, a_1, \) and \( a_2 \), respectively. The potential energy, denoted by \( J : U \to \mathbb{R} \), is expressed by
\[
J(u) = (G \circ \Lambda)(u) - F(u)
\]
Finally, we also emphasize from now on, as their meaning are clear, we may denote \( L^2(\Omega) \) and \( L^2(\Omega; \mathbb{R}^2 \times 2) \) simply by \( L^2 \), and the respective norms by \( \| \cdot \|_2 \). Moreover, derivatives are always understood in the distributional sense, \( 0 \) may denote the zero vector in appropriate Banach spaces, and the following and relating notations are used:
\[
w_{,\alpha} = \frac{\partial w}{\partial x_\alpha},
\]
\[
w_{,\alpha\beta} = \frac{\partial^2 w}{\partial x_\alpha \partial x_\beta},
\]
\[
u_{,\alpha\beta} = \frac{\partial u_\alpha}{\partial x_\beta},
\]
\[
N_{\alpha\beta,1} = \frac{\partial N_{\alpha\beta}}{\partial x_1},
\]
and
\[
N_{\alpha\beta,2} = \frac{\partial N_{\alpha\beta}}{\partial x_2}.
\]

4. On the Existence of a Global Minimizer
At this point, we present an existence result concerning the Kirchhoff–Love plate model. We start with the following two remarks.

Remark 4. Let \( \{P_\alpha\} \in L^\infty(\Omega; \mathbb{R}^2) \). We may easily obtain by appropriate Lebesgue integration \( \{\tilde{T}_{\alpha\beta}\} \) symmetric and such that
\[
\tilde{T}_{\alpha\beta,\beta} = -P_\alpha, \text{ in } \Omega.
\]
Indeed, extending \( \{P_\alpha\} \) to zero outside \( \Omega \) if necessary, we may set
\[
\tilde{T}_{11}(x, y) = -\int_0^x P_1(\xi, y) \, d\xi,
\]
\[
\tilde{T}_{22}(x, y) = -\int_0^y P_2(x, \xi) \, d\xi,
\]
and
\[
\tilde{T}_{12}(x, y) = \tilde{T}_{21}(x, y) = 0, \text{ in } \Omega.
\]
Thus, we may choose a \( C > 0 \) sufficiently big, such that
\[
\{T_{\alpha\beta}\} = \{T_{\alpha\beta} + C\delta_{\alpha\beta}\},
\]
is positive definite in \( \Omega \), so that
\[
T_{\alpha\beta,\beta} = \tilde{T}_{\alpha\beta,\beta} = -P_{\alpha},
\]
where
\[
\{\delta_{\alpha\beta}\}
\]
is the Kronecker delta.

Therefore, for the kind of boundary conditions of the next theorem, we do not have any restriction for the \( \{P_{\alpha}\} \) norm.

In summary, the next result is new and it is really a step forward concerning the previous one in Ciarlet [3]. We emphasize that this result and its proof through such a tensor \( \{T_{\alpha\beta}\} \) are new, even though the final part of the proof is established through a standard procedure in the calculus of variations.

Finally, more details on the Sobolev spaces involved may be found in [5–8]. Related duality principles are addressed in [5,7,9].

At this point, we present the main theorem in this section.

**Theorem 3.** Let \( \Omega \subset \mathbb{R}^2 \) be an open, bounded, connected set with a Lipschitzian boundary denoted by \( \partial\Omega = \Gamma \). Suppose \( (G \circ \Lambda): U \to \mathbb{R} \) is defined by
\[
G(\Lambda u) = G_1(\gamma(u)) + G_2(\kappa(u)), \quad \forall u \in U,
\]
where
\[
G_1(\gamma u) = \frac{1}{2} \int_\Omega H_{\alpha\beta\lambda\mu}^\gamma_{\alpha\beta}(u) \gamma_{\lambda\mu}(u) \, dx,
\]
and
\[
G_2(\kappa u) = \frac{1}{2} \int_\Omega h_{\alpha\beta\lambda\mu}^\kappa_{\alpha\beta}(u) \kappa_{\lambda\mu}(u) \, dx,
\]
where
\[
\Lambda(u) = (\gamma(u), \kappa(u)) = (\{\gamma_{\alpha\beta}(u)\}, \{\kappa_{\alpha\beta}(u)\}),
\]
\[
\gamma_{\alpha\beta}(u) = \frac{u_{\alpha} + u_{\beta}}{2} + \frac{w_{\alpha} + w_{\beta}}{2},
\]
\[
\kappa_{\alpha\beta}(u) = -w_{\alpha\beta},
\]
and where
\[
J(u) = W(\gamma(u), \kappa(u)) - \langle P_{\alpha}, u_{\alpha} \rangle_{L^2(\Omega)} - \langle w, P \rangle_{L^2(\Omega)} - \langle P_{\alpha}, u_{\alpha} \rangle_{L^2(\Gamma_t)} - \langle P_t, w \rangle_{L^2(\Gamma_t)},
\]
(11)
where,
\[
U = \{u = (u_{\alpha}, w) = (u_1, u_2, w) \in W^{1,2}(\Omega; \mathbb{R}^2) \times W^{2,2}(\Omega) : u_{\alpha} = w = \frac{\partial w}{\partial n} = 0, \text{ on } \Gamma_0 \},
\]
(12)
where \( \partial\Omega = \Gamma_0 \cup \Gamma_t \) and the Lebesgue measures
\[
m_\Gamma(\Gamma_0 \cap \Gamma_t) = 0,
\]
and
\[
m_\Gamma(\Gamma_0) > 0.
\]
We also define
\[
F_1(u) = -\langle w, P \rangle_{L^2(\Omega)} - \langle u, P_a \rangle_{L^2(\Omega)} - \langle P_a^t, u \rangle_{L^2(\Gamma_i)} \\
- \langle P^t, w \rangle_{L^2(\Gamma_i)} + \langle \epsilon_a, u^2 \rangle_{L^2(\Gamma_i)} \\
\equiv -\langle u, f \rangle_{L^2} + \langle \epsilon_a, u^2 \rangle_{L^2(\Gamma_i)} \\
\equiv -\langle u, f_1 \rangle_{L^2} - \langle u, P_a \rangle_{L^2(\Omega)} + \langle \epsilon_a, u^2 \rangle_{L^2(\Gamma_i)},
\]
where
\[
\langle u, f_1 \rangle_{L^2} = \langle u, f \rangle_{L^2} - \langle u, P_a \rangle_{L^2(\Omega)},
\]
\[\epsilon_a > 0, \forall \alpha \in \{1, 2\}\] and
\[f = (P_a, P) \in L^\infty(\Omega; \mathbb{R}^3).\]

Let \(J : U \rightarrow \mathbb{R}\) be defined by
\[
J(u) = G(\Lambda u) + F_1(u), \forall u \in U.
\]

Assume there exists \(\{c_{\alpha \beta}\} \in \mathbb{R}^{2 \times 2}\) such that \(c_{\alpha \beta} > 0, \forall \alpha, \beta \in \{1, 2\}\) and
\[
G_2(\alpha(u)) \geq c_{\alpha \beta} \|w_{\alpha \beta}\|^2, \forall u \in U.
\]

Under such hypotheses, there exists \(u_0 \in U\) such that
\[
J(u_0) = \min_{u \in U} J(u).
\]

**Proof.** Observe that we may find \(T_\alpha = \{(T_\alpha)_{\beta}\}\) such that
\[
div T_\alpha = T_\alpha \beta, = -P_a,
\]
and also such that \(\{T_\alpha \beta\}\) is positive, definite, and symmetric (please see Remark 4).

Thus, defining
\[
v_{\alpha \beta}(u) = \frac{u_{\alpha \beta} + u_{\beta \alpha}}{2} + \frac{1}{2} w_{\alpha \beta},
\]
we obtain
\[
J(u) = G_1(\{v_{\alpha \beta}(u)\}) + G_2(\alpha(u)) - \langle u, f \rangle_{L^2} + \langle \epsilon_a, u^2 \rangle_{L^2(\Gamma_i)} \\
= G_1(\{v_{\alpha \beta}(u)\}) + G_2(\alpha(u)) - \langle T_\alpha \beta, \frac{u_{\alpha \beta} + u_{\beta \alpha}}{2} \rangle_{L^2(\Omega)} \\
+ \langle T_\alpha \beta \cdot w_{\alpha \beta}, u \rangle_{L^2(\Gamma_i)} - \langle u, f_1 \rangle_{L^2} + \langle \epsilon_a, u^2 \rangle_{L^2(\Gamma_i)} \\
= G_1(\{v_{\alpha \beta}(u)\}) + G_2(\alpha(u)) - \langle T_\alpha \beta, v_{\alpha \beta}(u) - \frac{1}{2} w_{\alpha \beta} \rangle_{L^2(\Omega)} \\
+ \langle T_\alpha \beta \cdot w_{\alpha \beta}, u \rangle_{L^2(\Gamma_i)} \\
\geq c_{\alpha \beta} \|w_{\alpha \beta}\|^2 + \frac{1}{2} \langle T_\alpha \beta, w_{\alpha \beta} \rangle_{L^2(\Omega)} - \langle u, f_1 \rangle_{L^2} + \langle \epsilon_a, u^2 \rangle_{L^2(\Gamma_i)} + G_1(\{v_{\alpha \beta}(u)\}) \\
- \langle T_\alpha \beta, v_{\alpha \beta}(u) \rangle_{L^2(\Omega)} + \langle T_\alpha \beta \cdot w_{\alpha \beta}, u \rangle_{L^2(\Gamma_i)}.\]

From this, since \(\{T_\alpha \beta\}\) is positive definite, clearly \(J\) is bounded below.

Let \(\{u_n\} \in U\) be a minimizing sequence for \(J\). Thus, there exists \(\alpha_1 \in \mathbb{R}\) such that
\[
\lim_{n \rightarrow \infty} J(u_n) = \inf_{u \in U} J(u) = \alpha_1.
\]
From (15), there exists $K_1 > 0$ such that
\[ \|(w_n)_{\alpha\beta}\|_2 < K_1, \forall \alpha, \beta \in \{1, 2\}, \; n \in \mathbb{N}. \]
Therefore, there exists $w_0 \in W^{2,2}(\Omega)$ such that, up to a subsequence not relabeled,
\[ (w_n)_{\alpha\beta} \rightharpoonup (w_0)_{\alpha\beta}, \text{ weakly in } L^2, \]
$\forall \alpha, \beta \in \{1, 2\}$, as $n \to \infty$.

Moreover, also up to a subsequence not relabeled,
\[ (w_n)_{\alpha} \to (w_0)_{\alpha}, \text{ strongly in } L^2 \text{ and } L^4, \quad (16) \]
$\forall \alpha \in \{1, 2\}$, as $n \to \infty$.

Furthermore, from (15), there exists $K_2 > 0$ such that,
\[ \|(v_n)_{\alpha\beta}(u)\|_2 < K_2, \forall \alpha, \beta \in \{1, 2\}, \; n \in \mathbb{N}, \]
and thus, from this, (14) and (16), we may infer that there exists $K_3 > 0$ such that
\[ \|(u_n)_{\alpha,\beta} + (u_n)_{\beta,\alpha}\|_2 < K_3, \forall \alpha, \beta \in \{1, 2\}, \; n \in \mathbb{N}. \]

From this and Korn’s inequality, there exists $K_4 > 0$ such that
\[ \|u_n\|_{W^{1,2}(\Omega; \mathbb{R}^2)} \leq K_4, \; \forall n \in \mathbb{N}. \]
Therefore, up to a subsequence not relabeled, there exists $(u_0)_{\alpha} \in W^{1,2}(\Omega, \mathbb{R}^2)$, such that
\[ (u_n)_{\alpha,\beta} + (u_n)_{\beta,\alpha} \rightharpoonup (u_0)_{\alpha,\beta} + (u_0)_{\beta,\alpha}, \text{ weakly in } L^2, \]
$\forall \alpha, \beta \in \{1, 2\}$, as $n \to \infty$, and
\[ (u_n)_{\alpha} \to (u_0)_{\alpha}, \text{ strongly in } L^2, \]
$\forall \alpha \in \{1, 2\}$, as $n \to \infty$.

Moreover, the boundary conditions satisfied by the subsequences are also satisfied for
$w_0$ and $u_0$ in a trace sense, so that
\[ u_0 = ((u_0)_{\alpha}, w_0) \in U. \]

From this, up to a subsequence not relabeled, we obtain
\[ \gamma_{\alpha\beta}(u_n) \rightharpoonup \gamma_{\alpha\beta}(u_0), \text{ weakly in } L^2, \]
$\forall \alpha, \beta \in \{1, 2\}$, and
\[ \kappa_{\alpha\beta}(u_n) \rightharpoonup \kappa_{\alpha\beta}(u_0), \text{ weakly in } L^2, \]
$\forall \alpha, \beta \in \{1, 2\}$.

Therefore, from the convexity of $G_1$ in $\gamma$ and $G_2$ in $\kappa$, we obtain
\[ \inf_{u \in U} J(u) = a_1 = \liminf_{n \to \infty} J(u_n) \geq J(u_0). \quad (17) \]

Thus,
\[ J(u_0) = \min_{u \in U} J(u). \]
The proof is complete.

5. Conclusions

In this article, we have developed a new proof for Korn's inequality in a specific n-dimensional context. In the second text part, we present a global existence result for a non-linear model of plates. Both results represent some new advances concerning the present literature. In particular, the results for Korn's inequality known so far are for a three-dimensional context such as in [1], for example, whereas we have here addressed a more general n-dimensional case.

In a future research, we intend to address more general models, including the corresponding results for manifolds in $\mathbb{R}^n$.

Funding: This research received no external funding

Conflicts of Interest: The author declares no conflict of interest.

References


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