Article
Combinatorial Identities with Multiple Harmonic-like Numbers

Kunle Adegoke and Robert Frontczak

1 Department of Physics and Engineering Physics, Obafemi Awolowo University, Ile-Ife 220005, Nigeria; adegoke00@gmail.com
2 Independent Researcher, 72762 Reutlingen, Germany
* Correspondence: robert.frontczak@web.de

Abstract: Multiple harmonic-like numbers are studied using the generating function approach. A closed form is stated for binomial sums involving these numbers and two additional parameters. Several corollaries and examples are presented which are immediate consequences of the main result. Finally, combinatorial identities involving harmonic-like numbers and other prominent sequences like hyperharmonic numbers and odd harmonic numbers are offered.

Keywords: multiple harmonic-like numbers; harmonic number; binomial transform

MSC: 05A19; 11B73; 11B75

1. Preliminaries
Cheon and El-Mikkawy [1,2] defined multiple harmonic-like numbers as follows:

\[ H_n(m) = \sum_{1 \leq k_1 + k_2 + \cdots + k_m \leq n} \frac{1}{k_1 k_2 \cdots k_m}, \]  
(1)

with \( H_n(0) = 1 \) for \( n \geq 0 \) and \( H_0(m) = 0 \) for \( m \geq 1 \). They showed that the generating function of \( H_n(m) \) equals

\[ H(z) = \sum_{n=0}^{\infty} H_n(m) z^n = \frac{(-\ln(1 - z))^m}{1 - z}. \]  
(2)

For \( m = 1 \), these numbers reduce to harmonic numbers \( H_n \) as follows:

\[ H_n(1) = \sum_{1 \leq k_1 \leq n} \frac{1}{k_1} = H_n, \quad H_0 = 0. \]  
(3)

For \( m = 2 \), we see that

\[ H_n(2) = \sum_{1 \leq k_1 + k_2 \leq n} \frac{1}{k_1 k_2} = \sum_{j=1}^{n} \sum_{k_1 + k_2 = j} \frac{1}{k_1 k_2} = \sum_{j=1}^{n} \frac{1}{k_1 (j - k_1)} = \sum_{j=1}^{n} \frac{H_{n-j}}{j} = \sum_{j=1}^{n} \frac{H_{n-j}}{j} = H_n^2 - H_n^{(2)}, \]

where, in the last line, a result of Kargin and Can [3] was used and where \( H_n^{(2)} \) represents the second-order harmonic numbers, i.e.,
The brute-force computation of $H_n(3)$ is tedious. The result is

$$H_n(3) = \sum_{j=1}^{n-1} \frac{H_{n-j-1}}{j}.$$ 

The next lemma is thus helpful.

**Lemma 1.** For all $n \geq 1$ and $m \geq 0$, we have the identity

$$H_n(m+1) = \frac{\sum_{j=1}^{n} H_{n-j}(m)}{j}. \quad (4)$$

**Proof.** Using (2), we have

$$\sum_{n=0}^{\infty} H_n(m+1)z^n = (1-z)^{-1} \left\{ \frac{\ln(1-z)}{1-z} \right\}^m = (1-z) \left( \sum_{n=0}^{\infty} H_n z^n \right) \left( \sum_{n=0}^{\infty} H_n(m) z^n \right)$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} H_j H_{n-j}(m) z^n - \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} H_j H_{n-1-j}(m) z^n.$$

By extracting and comparing the coefficients of $z^n$, we obtain for all $n \geq 1$

$$H_n(m+1) = \sum_{j=0}^{n-1} H_j (H_{n-j}(m) - H_{n-1-j}(m))$$

$$= \sum_{j=1}^{n} H_{n-j}(m) (H_j - H_{j-1})$$

$$= \sum_{j=1}^{n} \frac{H_{n-j}(m)}{j},$$

as claimed. □

Multiple harmonic-like numbers were studied recently by Chen and Guo in [4,5]. For instance, in [4], several summation formulae involving harmonic-like numbers and other combinatorial numbers were derived. In the paper presented in [5], a certain sequence $A_k(n,k)$ was studied, and, as a part of this study, additional interesting combinatorial identities involving harmonic-like numbers were presented.

In this paper, we continue the work on harmonic-like numbers by applying the generating function approach. Our first main result is a closed form for binomial sums involving these numbers and two additional parameters $a, b \in \mathbb{C}$. Several corollaries and examples are presented, which are immediate consequences of the main result. Finally, combinatorial identities involving harmonic-like numbers and other prominent sequences like hyperharmonic numbers and odd harmonic numbers are offered.

Odd harmonic numbers $O_n$ are defined as follows:

$$O_n = \sum_{k=1}^{n} \frac{1}{2k-1}, \quad O_0 = 0.$$
Obvious relations between harmonic numbers \( H_n \) and odd harmonic numbers \( O_n \) are given by calculating
\[
H_{2n} = \frac{1}{2} H_n + O_n \quad \text{and} \quad H_{2n-1} = \frac{1}{2} H_{n-1} + O_n.
\]
Additional relations are contained in the next lemma.

**Lemma 2.** If \( n \) is an integer, then
\[
\begin{align*}
H_n - \frac{1}{2} &= 2O_n - 2 \ln 2 \quad (5) \\
H_n - \frac{1}{2} - H_{n-1/2} &= 2(O_n - 1) \quad (6) \\
H_n + 1/2 - H_{1/2} &= 2O_{n+1} - 1 \quad (8) \\
H_{n+1/2} - H_{1/2} &= 2(O_{n+1} - 1) \quad (9) \\
H_{n+1/2} - H_{n-1/2} &= \frac{2}{2n+1}, \quad (10) \\
H_n - H_{-3/2} &= 2(O_n - 1) \quad (11) \\
H_n - H_{-3/2} &= 2(O_{n+1} - 1). \quad (12)
\end{align*}
\]

**Proof.** Let \( \psi(z) = \Gamma'(z)/\Gamma(z) \) be the psi or digamma function, where \( \Gamma(z) \) denotes the Gamma function (see [6]). Then, we have the well-known relation
\[
\psi(n) = H_n - 1/2 - \gamma, \quad (13)
\]
with \( \gamma \) being the Euler–Mascheroni constant. The relation in (13) can be seen as a definition of the harmonic numbers for all complex \( n \) (excluding zero and the negative integers). Having this in mind, we can use the known result for the digamma function at half-integer arguments ([6], Equation (51)), namely,
\[
\psi(n + 1/2) = -\gamma - 2 \ln 2 + 2 \sum_{k=1}^{n} \frac{1}{2k - 1}, \quad (14)
\]
to prove all results stated in the above lemma. For instance, identity (5) follows immediately from combining (13) with (14). Identity (6) is also immediate as \( H_{-1/2} = -2 \ln 2 \). The other identities are proven in a very similar manner. \( \square \)

We conclude this section with a definition of the Stirling numbers of the first kind, \( s(n, k) \), that will also be needed in the sequel. These numbers are defined by the following generating function:
\[
\Sigma_k(z) = \sum_{n=k}^{\infty} \frac{k!}{n!} s(n, k) z^n = \ln^k(1 + z). \quad (15)
\]
Some particular values are
\[
\begin{align*}
s(n, k) &= 0 \quad \text{for } n < k, \\
s(n, 0) &= \begin{cases} 1, & n = 0; \\ 0, & n \geq 1; \end{cases} \\
s(n, 1) &= (-1)^{n-1}(n - 1)! \\
s(n, 2) &= (-1)^{n}(n - 1)!H_{n-1}.
\end{align*}
\]

2. Binomial Sums Involving \( H_n(m) \)

For \( a, b \in \mathbb{C} \), let \( S_n(a, b, m) \) be defined by
\[ S_n(a, b, m) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} H_k(m). \]  

(16)

Then, we have the following result.

**Theorem 1.** For all \( n \geq 0 \), we have

\[ S_n(a, b, m) = \sum_{j=0}^{m} \binom{m}{j} \sum_{k=0}^{n} H_k(j)(a + b)^j \frac{(m - j)!}{(n - k)!} (-1)^{n-k} b^{n-k} s(n - k, m - j). \]  

(17)

where \( s(n, k) \) are the Stirling numbers of the first kind.

**Proof.** Let \( S(z) \) denote the generating function of \( S_n(a, b, m) \). Then (see [7,8]),

\[ S(z) = \sum_{n=0}^{\infty} S_n(a, b, m) z^n = \frac{1}{1 - bz} H \left( \frac{az}{1 - bz} \right). \]

Substituting \( 1 - \frac{az}{1 - bz} = \frac{1 - (a + b)z}{1 - bz} \), we see that

\[ S(z) = \frac{1}{1 - (a + b)z} \left( - \ln(1 - (a + b)z) + \ln(1 - bz) \right)^m \]

\[ = \sum_{j=0}^{m} \binom{m}{j} \frac{(-\ln(1 - (a + b)z))^j}{1 - (a + b)z} \ln^{m-j}(1 - bz) \]

according to the binomial theorem, where \(|1 - (a + b)z| < 1, |1 - bz| < 1\). The expression for \( S_n(a, b, m) \) is now obtained from the generating power series (2) and (15) in conjunction with the Cauchy product for power series. This completes the proof of Theorem 1. \( \square \)

**Remark 1.** When \( m = 1 \), we obtain

\[ S_n(a, b, 1) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} H_k \]

\[ = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} (a + b)^k b^{n-k} s(n - k, 1) + \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} H_k(a + b) b^{n-k} s(n - k, 0) \]  

(18)

\[ = H_n(a + b)^n - \sum_{k=0}^{n-1} (a + b)^k b^{n-k} \frac{1}{n-k}, \]

which reproduces Boyadzhiev's main result (Proposition 6) from [9].

**Corollary 1.** For \( n, m \geq 0 \), we have

\[ \sum_{k=0}^{n} \binom{n}{k} (-1)^k H_k(m) = (-1)^n \frac{n!}{m!} s(n, m) \]  

(19)

and

\[ \sum_{k=m}^{n} \binom{n}{k} \frac{s(k, m)}{k!} = \frac{1}{m!} H_n(m). \]  

(20)

**Proof.** The first result follows immediately by setting \( a = -1 \) and \( b = 1 \) in (17) and simplifying the equation accordingly. The second identity is the inverse binomial transform of the first (for information on the binomial transform see [7]). \( \square \)

**Remark 2.** The identity (19) was first discovered by Chen and Guo (see [5], Corollary 8). The second identity also appears in their paper (albeit with a typo; see the proof of Proposition 13).
Corollary 2. For \( n, m \geq 0 \), we have
\[
\sum_{k=0}^{n} \binom{n}{k} H_k(m) = \sum_{j=0}^{m} \binom{m}{j} \sum_{k=0}^{n} H_k(j)(-1)^{n-k} 2^k \frac{(m-j)!}{(n-k)!} s(n-k, m-j).
\] (21)

In particular, we recover the classical identity [9]:
\[
\sum_{k=0}^{n} \binom{n}{k} H_k = 2^n \left( H_n - \sum_{k=1}^{n} \frac{1}{2^k} \right).
\] (22)

Proof. Set \( a = b \) in (17) and simplify.

Corollary 3. For \( n \geq 0 \), we have
\[
S_n(a, b, 2) = H_n(2)(a+b)^n + 2 \sum_{k=1}^{n} (a+b)^{n-k} b^k H_{k-1} - H_{n-k}.
\] (23)

Proof. Set \( m = 2 \) in (17) and simplify.

Some consequences of Corollary 3 will now be stated as examples.

Example 1. We have
\[
\sum_{k=0}^{n} \binom{n}{k} H_k(2) = 2^n \left( H_n(2) + 2 \sum_{k=1}^{n} \frac{H_{k-1} - H_{n-k}}{2^k} \right).
\] (24)

Example 2. From
\[
\sum_{k=0}^{n} \binom{n}{k} (−1)^k H_k(2) = \frac{2}{n} H_{n−1}.
\] (25)
we deduce that
\[
\sum_{k=0}^{n} \binom{n}{k} (−1)^k H_k^2 = -\frac{H_n}{n}.
\] (26)

To construct the equation above, we used the fact that (see, for instance, [10])
\[
\sum_{k=0}^{n} \binom{n}{k} (−1)^k H_k = \frac{H_n}{n} - \frac{2}{n^2}.
\]
The inverse binomial relation also yields
\[
\sum_{k=1}^{n} \binom{n}{k} (−1)^{k+1} \frac{H_k}{k} = H_n^{(2)}.
\] (27)

Example 3. We have
\[
\sum_{k=0}^{n} \binom{n}{k} 2^k (−1)^{n-k} H_k(2) = H_n(2) + 2 \sum_{k=1}^{n} (−1)^k \frac{H_{k-1} - H_{n-k}}{k}.
\] (28)

Example 4. We have
\[
\sum_{k=0}^{n} \binom{n}{k} 2^k H_k(2) = 3^n \left( H_n(2) + 2 \sum_{k=1}^{n} \frac{H_{k-1} - H_{n-k}}{2^k} \right).
\] (29)

Example 5. Let \( F_n \) and \( L_n \) be the Fibonacci and Lucas numbers, respectively (see Koshy [11] or Vajda [12]). Then, we have
Theorem 2. If \( m \) and \( n \) are non-negative integers, then

\[
\sum_{k=0}^{n} \binom{n}{k} F_k H_k(2) = H_n(2) F_{2n} + 2 \sum_{k=1}^{n} F_{2(n-k)} \frac{H_{k-1} - H_{n-k}}{k},
\]

(30)

\[
\sum_{k=0}^{n} \binom{n}{k} L_k H_k(2) = H_n(2) L_{2n} + 2 \sum_{k=1}^{n} L_{2(n-k)} \frac{H_{k-1} - H_{n-k}}{k},
\]

(31)

and

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} F_k H_k(2) = H_n(2) F_{n} + 2 \sum_{k=1}^{n} F_{n-k} \frac{H_{k-1} - H_{n-k}}{k},
\]

(32)

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^{k} L_k H_k(2) = H_n(2) L_{n} + 2 \sum_{k=1}^{n} L_{n-k} \frac{H_{k-1} - H_{n-k}}{k}.
\]

(33)

Corollary 4. For \( n \geq 0 \), we have

\[
S_n(a, b, 3) = H_n(3)(a + b)^n - 3 \sum_{k=1}^{n} (a + b)^{n-k} b^k H_{k-1}^2 - H_{k-1}^{(2)} - 2H_{k-1} H_{n-k} + H_{n-k}^2 - H_{n-k}^{(2)}.
\]

(34)

Proof. Set \( m = 3 \) in (17) and simplify using

\[
s(n, 3) = \frac{1}{2} (-1)^{n-1} (n - 1)! \left( H_{n-1}^2 - H_{n-1}^{(2)} \right).
\]

\[\square\]

3. Combinatorial Identities from Partial Summation

For sequences \((a_k)_{k \geq 0}\) and \((b_k)_{k \geq 0}\), let \( \Delta a_k = a_k - a_{k-1} \). Then, we have the following well-known partial summation formula:

\[
\sum_{k=1}^{n} b_k \Delta a_{k+1} = b_n a_{n+1} - b_0 a_1 - \sum_{k=1}^{n} a_k \Delta b_k.
\]

In particular, with \( b_k = H_k \), we obtain

\[
\sum_{k=1}^{n} H_k (a_{k+1} - a_k) = H_n a_{n+1} - \sum_{k=1}^{n} \frac{a_k}{k}.
\]

(35)

Remark 3. Lemma 1 is also a partial summation identity. Indeed, with \( a_k = H_{n-k}(m) \) \( (a_n = H_0(m) = 0) \), (35) reads

\[
\sum_{k=1}^{n-1} H_k (H_{n-(k+1)}(m) - H_{n-k}(m)) = - \sum_{k=1}^{n-1} \frac{H_{n-k}(m)}{k}
\]

or

\[
\sum_{k=1}^{n-1} \frac{H_{n-k}(m)}{k} = \sum_{k=1}^{n-1} H_k (H_{n-k}(m) - H_{n-1-k}(m)),
\]

from which Lemma 1 can be derived easily.

Theorem 2. If \( m \) and \( n \) are non-negative integers, then

\[
\sum_{k=1}^{n} H_k \sum_{j=m}^{k} \binom{k-1}{j-1} s(j, m) \frac{1}{j!} = \frac{1}{m!} H_n(m) H_n - \frac{1}{m!} \sum_{k=1}^{n} \frac{H_{k-1}(m)}{k}.
\]

(36)

Proof. Use (35) with \( a_k = H_{k-1}(m) \) while noting from (20) that
$$H_k(m) - H_{k-1}(m) = m! \sum_{j=m}^{k} \binom{k-1}{j-1} \frac{s(j,m)}{j!}.$$  \tag{37}

Setting $m = 1$ in (36) gives the known result

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2} \left( H_n^2 + H_n^{(2)} \right),$$

while $m = 2$ gives

$$2 \sum_{k=1}^{n} H_k \sum_{j=1}^{k} \frac{(-1)^j (k-1)!}{j!} H_{j-1} = H_n^3 - H_n^{(2)} H_n - \sum_{k=1}^{n} \frac{H_{k-1} - H_k^{(2)}}{k}.$$

**Theorem 3.** If $m$ and $n$ are positive integers, then

$$\sum_{k=1}^{n} H_k \sum_{j=m}^{n-k} \frac{n-k}{j!} s(j,m) = \frac{1}{m!} H_{n+1}(m+1).$$  \tag{38}

In particular,

$$\sum_{k=1}^{n} \frac{H_k}{n-k+1} = H_{n+1}^2 - H_{n+1}^{(2)}$$  \tag{39}

and

$$\sum_{k=1}^{n} H_k \sum_{j=2}^{n-k+1} \frac{(-1)^j}{j} H_{j-1} = \frac{1}{2} \sum_{k=1}^{n+1} \left( \sum_{j=1}^{n+1-k} H_{n-k-j+1} \right).$$  \tag{40}

**Proof.** Set $a_k = H_{n-k+1}(m)$ in (35) and use (4) and (37). \hfill \Box

**Lemma 3.** If $(a_n)_{n \geq 0}$ is a sequence, then

$$\sum_{k=1}^{n} \frac{a_k - a_{k-1}}{k} = \sum_{k=1}^{n} \frac{a_k}{k(k+1)} - a_0 + \frac{a_n}{n+1}.$$  \tag{41}

**Proof.** This is a consequence of

$$\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}.$$

Many results can be derived from Lemma 3. For instance, setting $a_k = H_k + p$ (with $p$ being a real number) gives

$$\sum_{k=1}^{n} \frac{H_k + p}{k(k+1)} = \begin{cases} H_n^{(2)} - \frac{H_n}{n+1}, & p = 0; \\ H_n + H_p - H_{n+p} + \frac{H_{n+p}}{n+1} & \text{otherwise.} \end{cases}$$  \tag{42}

For the equation above, we used the following result:

$$\sum_{k=1}^{n} \frac{1}{k(k+p)} = \begin{cases} \frac{H_n^{(2)}}{p}, & p = 0; \\ \frac{H_n + H_p - H_{n+p}}{p} & \text{otherwise.} \end{cases}$$  \tag{43}
Theorem 4. If \( m \) is a non-negative integer and \( n \) is a positive integer, then
\[
\sum_{k=1}^{n} \frac{H_{n-k}(m)}{k(k+1)} = H_n(m) + H_n(m+1) - H_{n+1}(m+1).
\] (44)

In particular,
\[
\sum_{k=1}^{n} \frac{H_{n-k}}{k(k+1)} = H_n + H_n^2 - H_{n+1}^2 - H_{n+1}^2
\] (45)
and
\[
\sum_{k=1}^{n} \frac{H_{n-k}^2 - H_{n-k}^{(2)}}{k(k+1)} = H_n^2 - H_n^{(2)} + \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n-k} \frac{H_{n-k-j}}{j} - \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n-k} \frac{H_{n+1-k-j}}{j}.
\] (46)

Proof. Use \( a_k = H_{n-k}(m) \) in Lemma 3 and invoke Lemma 1. \( \square \)

Lemma 4. Let \( (a_n)_{n \geq 0} \) be a sequence. If \( r \) is a complex number and \( n \) is a non-negative number, then
\[
\sum_{k=0}^{n} (-1)^k \binom{r-1}{k} (a_{k+1} - a_k) = (-1)^n \binom{r-1}{n} a_{n+1} - \sum_{k=0}^{n} (-1)^k \binom{r}{k} a_k.
\] (47)

Proof. This is a variation on an identity of Kollár ([13], Lemma 1). \( \square \)

Theorem 5. If \( r \) is a complex number and \( m \) and \( n \) are positive integers, then
\[
\sum_{k=0}^{n} (-1)^k \binom{r-1}{k} \sum_{j=m}^{k+1} \binom{k}{j-1} \frac{S(j,m)}{j!} = (-1)^n \binom{r-1}{n} \frac{1}{m!} H_{n+1}(m) - \frac{1}{m!} \sum_{k=0}^{n} (-1)^k \binom{r}{k} H_k(m).
\] (48)

In particular,
\[
\sum_{k=0}^{n} (-1)^k \binom{r-1}{k} = (-1)^n \binom{r-1}{n} H_{n+1} - \sum_{k=0}^{n} (-1)^k \binom{r}{k} H_k
\] (49)
and
\[
\sum_{k=0}^{n} (-1)^k \binom{r-1}{k} \sum_{j=2}^{k+1} (-1)^j \binom{k}{j-1} \frac{H_{j-1}}{j} = (-1)^n \binom{r-1}{n} \frac{1}{2} (H_{n+1}^2 - H_n^{(2)}) - \frac{1}{2} \sum_{k=0}^{n} (-1)^k \binom{r}{k} (H_k^2 - H_k^{(2)}).
\] (50)

Proof. Use \( a_k = H_k(m) \) in Lemma 4. \( \square \)

4. More Identities Involving \( H_n(m) \) and Other Sequences

Hyperharmonic numbers \( H_{n,p} \) (or \( h_i^{(p)} \), \( p \geq 1 \), are another generalization of harmonic numbers [14–16]. They are defined by
\[
H_{n,p} = \sum_{i=1}^{n} H_{i,p-1} \quad \text{with} \quad H_{n,0} = \frac{1}{n}, \quad H_{0,p} = 0, \quad H_{n,1} = H_n.
\] (51)

They can be written in the compact form as
Theorem 6. For all \( n, p, m \geq 0 \), we have
\[
\sum_{k=0}^{n} \binom{k+p}{k} H_{n-k}(m)(H_{k+p} - H_p) = \sum_{k=0}^{n} \binom{k+p}{k} H_{n-k}(m+1). 
\] (53)

In particular,
\[
\sum_{k=0}^{n} \binom{k+p}{k} (H_{k+p} - H_p) = \sum_{k=0}^{n} \binom{k+p}{k} H_{n-k} 
\] (54)
and
\[
\sum_{k=0}^{n} \binom{k+p}{k} H_{n-k}(H_{k+p} - H_p) = \sum_{k=0}^{n} \binom{k+p}{k} (H^2_{n-k} - H^2_{n-k}). 
\] (55)

Proof. Let \( B(z) \) be the generating function of \( H_{n,p+1} \). We can then calculate
\[
H(z) \cdot B(z) = \left( \sum_{n=0}^{\infty} H_n(m)z^n \right) \left( \sum_{n=0}^{\infty} H_{n,p+1}z^n \right)
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} H_k(m)H_{n-k,p+1}z^n
\]
\[
= (\ln(1-z))^m - \ln(1-z) \frac{1}{1-z} (1-z)^{p+1}
\]
\[
= \frac{1}{(1-z)^{p+1}} (-\ln(1-z))^{m+1}
\]
\[
= \left( \sum_{n=0}^{\infty} \binom{n+p}{n} z^n \right) \left( \sum_{n=0}^{\infty} H_n(m+1)z^n \right)
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{k+p}{k} H_{n-k}(m+1)z^n
\]
and the statement follows. \( \square \)

The next lemma provides a relation between hyperharmonic numbers and odd harmonic numbers.

Lemma 5. If \( p \) and \( r \) are non-negative integers, then
\[
H_{r,p+1/2} = \frac{1}{2^r-1} \binom{2p}{p}^{-1} \binom{2(r+p)}{r+p} \binom{r+p}{r} (O_{r+p} - O_p). 
\] (56)

Proof. Identity (7) gives
\[
H_{r+p-1/2} - H_{r-1/2} = 2(O_{r+p} - O_p). 
\] (57)

Using this and
\[
\binom{r+p - 1/2}{r} = \frac{1}{2^r} \binom{2p}{p}^{-1} \binom{2(r+p)}{r+p} \binom{r+p}{r} 
\] (58)
in (52) gives (56). □

**Theorem 7.** If \( n \) and \( p \) are non-negative integers, then

\[
\sum_{k=1}^{n} \frac{1}{2^k} \binom{2(k+p)}{k+p} \binom{k+p}{k} \left( O_{k+p} - O_p \right) = \frac{1}{2^{2n+1}} \frac{p+1}{2p+1} \binom{2(n+p+1)}{n+p+1} \left( O_{n+p+1} - O_{p+1} \right). \tag{59}
\]

In particular,

\[
\sum_{k=1}^{n} \frac{O_k}{2^k} \binom{2k}{k} = \frac{n+1}{2^{2n+1}} \binom{2(n+1)}{n+1} (O_{n+1} - 1). \tag{60}
\]

**Proof.** Write \( p + 1 + 1/2 \) for \( p \) in (51) to obtain

\[
\sum_{k=1}^{n} O_k \binom{2k}{k} = \sum_{k=0}^{n} H_{n-k}(m+1) \binom{2k}{k} \frac{H_{n-k}(m+1)}{4^k}.
\]

and use (56). □

**Theorem 8.** For all \( n, m \geq 0 \) we have

\[
\sum_{k=0}^{n} \frac{\binom{2k}{k} O_k H_{n-k}(m)}{4^k} = \frac{1}{2} \sum_{k=0}^{n} \frac{\binom{2k}{k} H_{n-k}(m+1)}{4^k}. \tag{61}
\]

In particular,

\[
\sum_{k=0}^{n} \frac{\binom{2k}{k} O_k H_{n-k}}{4^k} = \frac{1}{2} \sum_{k=0}^{n} \frac{\binom{2k}{k} H_{n-k}}{4^k} \tag{62}
\]

and

\[
\sum_{k=0}^{n} \frac{\binom{2k}{k} O_k H_{n-k}}{4^k} = \frac{1}{2} \sum_{k=0}^{n} \frac{\binom{2k}{k}}{4^k} \left( H_{n-k}^2 - H_{n-k}^{(2)} \right). \tag{63}
\]

**Proof.** We use the fact that [17,18]

\[
\sum_{n=0}^{\infty} \frac{\binom{2n}{n} z^n}{n!} = \frac{1}{\sqrt{1 - 4z}}.
\]

It is also known that [17]

\[
O(z) = \sum_{n=0}^{\infty} \binom{2n}{n} O_n z^n = \frac{1}{2} \sqrt{1 - 4z} \left( \frac{-\ln(1 - 4z)}{1 - 4z} \right).
\]

This yields

\[
O(z) \cdot H(4z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2k}{k} O_k 4^{n-k} H_{n-k}(m) z^n = \frac{1}{2} \frac{(-\ln(1 - 4z))^{m+1}}{1 - 4z}
\]

\[
= \frac{1}{2} \left( \sum_{n=0}^{\infty} \binom{2n}{n} z^n \right) \left( \sum_{n=0}^{\infty} H_n(m+1) 4^n z^n \right)
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2k}{k} 4^{n-k} H_{n-k}(m+1) z^n
\]

and the proof is completed. □
Note that from (60) and (62), we also have

$$\sum_{k=0}^{n} \binom{2k}{k} \frac{H_{n-k}}{2^{2k}} = \frac{n+1}{2n} \left(\binom{2(n+1)}{n+1} O_{n+1} - 1\right).$$  \(\text{(64)}\)

The next theorem, based on Theorem 6, generalizes Theorem 8.

**Theorem 9.** If \(n\) and \(p\) are non-negative integers, then

$$\sum_{k=0}^{n} \frac{1}{2^{2k}} \binom{2(k+p)}{k+p} \left(\frac{k+p}{k}\right) H_{n-k}(m) \left(O_{k+p} - O_{p}\right)$$

$$= \frac{1}{2} \sum_{k=0}^{n} \frac{1}{2^{2k}} \binom{2(k+p)}{k+p} \left(\frac{k+p}{k}\right) H_{n-k}(m+1).$$  \(\text{(65)}\)

In particular, setting \(m = 0\) and using (59) gives the following generalization of (64):

$$\sum_{k=0}^{n} \frac{1}{2^{2k}} \binom{2(k+p)}{k+p} \left(\frac{k+p}{k}\right) H_{n-k}$$

$$= \frac{1}{2} \sum_{k=0}^{n} \frac{1}{2^{2k}} \binom{2(k+p)}{k+p} \left(\frac{k+p}{k}\right) H_{n-k}(m+1).$$  \(\text{(66)}\)

**Proof.** Write \(p - 1/2\) for \(p\) in (53) and use (57) and (58). \(\square\)

**Lemma 6.** If \((a_n)_{n \geq 0}\) is a sequence, then

$$\sum_{k=1}^{n} k(a_k - a_{k-1}) = na_n - \sum_{k=1}^{n} a_{k-1}.$$

**Theorem 10.** If \(n \in \mathbb{N}_0\) and \(p \in \mathbb{C} \setminus \mathbb{Z}^+\), then

$$\sum_{k=1}^{n} kH_{k,p} = nH_{n,p+1} - H_{n-1,p+2}.$$  \(\text{(67)}\)

**Proof.** Identity (52) gives the following recurrence relation:

$$H_{k,p+1} - H_{k-1,p+1} = H_{k,p}.$$  \(\text{(68)}\)

Use \(a_k = H_{k,p+1}\) in Lemma 6, keeping (68) in mind. \(\square\)

**Theorem 11.** If \(n\) and \(p\) are non-negative integers, then

$$\sum_{k=1}^{n} \frac{k}{2^{2k}} \binom{2(k+p)}{k+p} \left(\frac{1}{k}\right) \left(O_{k+p} - O_{p}\right)$$

$$= \frac{n}{2^{2n+1}} \binom{2(p+1)}{p+1}^{-1} \binom{2(p+2)}{p+2}^{-1} \binom{2(n+p+1)}{n+p+1} \left(\frac{n+p+1}{n} \left(O_{n+p+1} - O_{p+1}\right)\right)$$

$$- \frac{1}{2^{2n+2}} \binom{2(p+2)}{p+2}^{-1} \binom{2(p+1)}{p} \binom{2(n+p+1)}{n+p+1} \left(\frac{n+p+1}{n} \left(O_{n+p+1} - O_{p+2}\right)\right).$$  \(\text{(69)}\)

In particular,

$$\sum_{k=1}^{n} \frac{k}{2^{2k}} \binom{2k}{k} O_k = \frac{n(n+1)}{2^{2n+1}} \binom{2(n+1)}{n+1} \left(O_{n+1} - 1\right)$$

$$- \frac{n(n+1)}{2^{2n+3}} \binom{2(n+1)}{n+1} \left(O_{n+1} - \frac{4}{3}\right).$$  \(\text{(70)}\)
Proof. Write $p + 1/2$ for $p$ in (67) and use (56).

**Theorem 12.** If $n$ and $p$ are non-negative integers, then

$$
\sum_{k=1}^{n} \binom{2k}{k}^{-1} \binom{2(k + p)}{k} \binom{k + p}{k} (O_{k+p} - O_k) = \frac{1}{4} \binom{2n}{n}^{-1} \binom{2(n + p + 1)}{n + p + 1} \left( \frac{n + p + 1}{n} \right) (O_{n+p+1} - O_n) - \frac{1}{4} \binom{2(p + 1)}{p + 1} O_{p+1}.
$$

(71)

**Proof.** Rearrange (68), interchange $k$ and $p$, and write $k + 1/2$ for $k$ to obtain

$$
H_{p,k+1/2} = H_{p+1,k+1/2} - H_{p+1,k-1/2},
$$

which telescopes to give

$$
\sum_{k=1}^{n} H_{p,k+1/2} = H_{p+1,n+1/2} - H_{p+1,1/2},
$$

from which (71) follows, in view of (56).

5. Conclusions

In this study, we offer new contributions to the field of combinatorics of multiple harmonic-like numbers. To prove most of the main results, the generating function approach was applied, making the derivations concise and easily accessible. We have also revealed connections between harmonic-like numbers, hyperharmonic numbers, and odd harmonic numbers. In the future, it would be interesting to explore combinatorial identities involving harmonic-like numbers and other prominent sequences of numbers (and polynomials), such as Catalan numbers, Lah numbers, Bernoulli numbers, Euler numbers, or Genocchi numbers [19–22]. Recall that Catalan numbers $C_n$ are given by

$$
C_n = \binom{2n}{n} \frac{1}{n+1}, \quad 0 \leq n,
$$

whereas the (unsigned) Lah numbers $L(n,k)$ can be expressed by the following formula:

$$
L(n,k) = \binom{n-1}{k-1} \frac{n!}{k!}, \quad 1 \leq k \leq n.
$$

Bernoulli numbers $B_n$, Euler numbers $E_n$, and Genocchi numbers $G_n$ are defined by their respective generating functions:

$$
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad \frac{2e^x}{e^{2x} + 1} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}, \quad \frac{2x}{e^x + 1} = \sum_{n=1}^{\infty} G_n \frac{x^n}{n!}.
$$

Studying relations between multiple harmonic-like numbers and these sequences could be an interesting task for future research.

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