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On the Četaev Condition for Nonholonomic Systems

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Abstract: In the context of holonomic systems, the identification of virtual displacements is clear and consolidated. This provides the possibility, once the class of displacements have been coupled with Newton's equations, for us to write the correct equations of motion. This method combines the d'Alembert principle with Lagrange formalism. As far as nonholonomic systems are concerned, the conjecture that dates back to Četaev actually defines a class of virtual displacements through which the d'Alembert–Lagrange method can be applied again. A great deal of literature is dedicated to the Četaev rule from both the theoretical and experimental points of view. The absence of a rigorous (mathematical) validation of the rule inferable from the constraint equations has been declared to have expired in a recent publication; one of our objectives is to produce a critical comment on this stated result. Finally, we explore the role of the Četaev condition within the significant class of nonholonomic homogeneous constraints.

Keywords: nonholonomic mechanical systems; d'Alembert–Lagrange principle; Četaev condition; displacements for linear and nonlinear nonholonomic constraints

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1. Introduction

Our study concerns discrete systems constrained by nonholonomic constraints. Among the various approaches used to deal with the problem (virtual displacements, variational principles in differential or integral form, algebraic and geometric methods), our attention is directed to the d'Alembert–Lagrange principle (d'A–L P.). The classic formulation of the principle will be presented in the next Section; here, we address the issue from a substantial and qualitative point of view.

As is known (see, for instance, [1] or [2]), the formulation of d'A–L P. starts from Newton's equations of motion and selects a particular category of displacements $\delta\mathbf{r}$ (we will define this notation later on), which are consistent with the dynamics of the system (virtual displacements). The situation is clear and universally accepted when we are dealing with systems constrained by geometric conditions (holonomic constraints): the displacements are distinctly identifiable and spontaneously linked to simple concepts in geometry (tangent space of a manifold).

The approach through d'A–L P. is not so common in the literature when we move on to the more complex category of kinematic constraints, i.e., constraints which involve the velocities of the system and which cannot be reduced to geometric constraints via integration. Starting from the fundamental text [3], the problem of how to define displacements for nonholonomic systems is rarely addressed, and alternative approaches (methods based on differential geometry or on a variational principle) might prevail in the study of nonholonomic systems. We refer to [4] for a useful review and an updated state of the art with respect to virtual displacements in nonlinear nonholonomic constraints.

In examining the role and potentiality of the d'A–L P. when dealing with nonholonomic systems, we find a clear line of demarcation dividing the linear case and the nonlinear one; if, in the first case, the application of the d'A–L P. method is a natural and simple extension of the holonomic model, in the latter case, the task of assigning displacements to the system is more difficult and unclear. From a mathematical point of view, the question is simple to introduce; the concern is to deduce appropriate integer conditions in terms of $\delta\mathbf{r}$ starting from a set of constraints on the state $(\mathbf{r}, \dot{\mathbf{r}})$ of the system, which cannot be integrated. The main difficulty lies in the fact that the displacements that work for a straightforward derivation of the equations of motion are not necessarily possible (i.e., compatible with the constraints) displacements.

Nevertheless, it must be said that the topic of nonlinear kinematic constraints is quite problematic, and it is not confined to the perspective of the d'A–L P. we are focusing on; actually, starting from the very existence of physical models of this type, the issues under debate concern the concrete feasibility of physical models [5], the distinct role from control forces [6], the axiomatic and theoretical features that provide a valid generalization of the linear case [7], and the mathematical aspect of expressing a certain condition in multiple equivalent ways and in different sets of coordinates [8], to mention just a few. In addition to the references mentioned above, we indicate the texts [9,10] and articles [11–15], which examine the topic of nonlinear constraints from a historical and axiomatic point of view.

Returning to the themes of this work, our analysis aims to examine a condition that establishes the class of virtual displacements $\delta\mathbf{r}$ in a very simple way, which is known as the Četaev rule ([16], and we also refer to [17,18]). Naturally, as we will recall later, the rule properly extends the existing conditions for the holonomic case and the linear kinematic case. An interesting aspect is that historically, the hypothesis was born with the aim of making the d'A–L P. and the Gauss principle equivalent in order to achieve agreement through the Hertz–Helder principle.

Going through the specific literature, we understand that the Četaev rule is in the state of a postulate; that is, there does not seem to be a rigorous theoretical justification starting from the laws of mechanics [4]. At the same time, the debate is also open in the context of experimental tests, and there are conflicting opinions even when the hypothesis is tested directly with experimental procedures [19,20]. For this reason, we were impressed by the content of [21], where a theoretical argument is intended to extend the validity of the d'A–L P. to treat general nonholonmic systems; essentially, the Četaev rule is claimed to be derived (through mathematical steps) directly from the constraint equations, and in this way, the displacements defined by the rule itself are fully justified. The procedure is also extended to higher-order constraints, which involve the accelerations of the system.

The dissertation in [21] is undoubtedly outstanding and exhaustive since it explores the most remarkable aspects of the theory on nonholonomic constraints (including various types of principles, the debated transpositional rule, ans the pseudovelocity formalism). This is why we find it appropriate, as the initial purpose of our work, to dwell on the mathematical tools by which the rule seems to find its demonstration.

The second aim of this paper is to investigate the effects of the Četaev rule on relevant physical aspects, particularly the overlap of virtual displacements and instantaneous velocities, the vanishing of virtual work carried out by the constraint forces, and the definition of a Hamilton function restricted to the independent velocities.

For the sake of clarity, we present the organization of the work and indicate the main steps as follows:

- In Section 2, we present the formal method and ordinary method of virtual displacements, starting from Newton's equations of motion. We explicitly write the equations of motion for systems constrained with nonlinear kinematic constraints whenever the Četaev condition is assumed to hold.

- The first paragraph of Section 3 is devoted to commenting on the theoretical justification of the Četaev rule asserted in [21]. In the next two paragraphs, we analyze the role of this rule with respect to virtual displacements such as possible instantaneous velocities, virtual work, and the energy of the system. A special condition with respect to the explicit constraint equations (condition (40)) turns out to outline the right class of nonlinear constraints for which the extension of the method virtual displacements is appropriate.
- In Section 4, we see that the just-mentioned condition identifies the constraints formulated by homogeneous functions (of any degree) with respect to the generalized velocities. Finally, the results are commented on in Section 5.

2. The Mathematical Model

2.1. The d'Alembert–Lagrange Principle

We consider a system of N material points governed by Newton's equations

$$m_s \ddot{\mathbf{r}}_s = \mathbf{F}_s + \boldsymbol{R}_s, \qquad s = 1, \ldots, N, \tag{1}$$

where $\mathbf{r}_s$ is the radius vector in $\mathbb{R}^3$ locating the s-th point, m_s is the mass, $\mathbf{F}_s$ is the corresponding active force, and $\mathbf{R}_s$ is the unknown force (exerted on the s-th point) due to restrictions imposed to the system.

A virtual displacement $\delta \mathbf{r}_s$ is a variation of the configuration of the system at a given moment of time which is compatible with the constraints imposed on the system [3]. The standard formulation of the equations of motion using the d'A–L P. passes through the definition of ideal constraint (we use the summation symbol Σ for greater clarity given that the range of indices is changeable):

$$\sum_{s=1}^{N} \boldsymbol{R}_s \cdot \delta \mathbf{r}_s = 0, \tag{2}$$

so that it can be written as

$$\sum_{s=1}^{N} (m_s \ddot{\mathbf{r}}_s - \mathbf{F}_s) \cdot \delta \mathbf{r}_s = 0, \tag{3}$$

from which the correct equations of motion should be deduced. If generalized independent coordinates $q_1, \ldots, q_n$, $n = 3N$ are used, then $\delta \mathbf{r}_s = \sum\limits_{j=1}^{n} \frac{\partial \mathbf{r}_s}{\partial q_j} \delta q_j$ for each $s = 1, \ldots, N$, and (3) is

$$\sum_{j=1}^{n} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - F^{(j)} \right) \delta q_j = 0, \tag{4}$$

where the kinetic energy $T = \frac{1}{2} \sum\limits_{s=1}^{N} m_s \dot{\mathbf{r}}_s^2$ is a function of the generalized coordinates $\mathbf{q} = (q_1, \ldots, q_n)$ and the generalized velocities $\dot{\mathbf{q}} = (\dot{q}_1, \ldots, \dot{q}_n)$ by means of

$$\mathbf{r}_s = \mathbf{r}_s(\mathbf{q}, t), \qquad \dot{\mathbf{r}}_s(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_s}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_s}{\partial t}, \qquad s = 1, \ldots, N, \tag{5}$$

and $F^{(j)} = \sum_{s=1}^{N} \mathbf{F}_s \cdot \frac{\partial \mathbf{r}_s}{\partial q_j}$ is the generalized force along the j-th direction, $j = 1, \ldots, n$. The generalized constraint forces are $R^{(j)} = \sum_{s=1}^{N} \mathbf{R_s} \cdot \frac{\partial \mathbf{r}_s}{\partial q_j}$, and in the Lagrangian coordinates, the condition (2) of ideal constraint is

$$\sum_{j=1}^{n} R^{(j)} \delta q_j = 0. \tag{6}$$

Still within the standard premises of the problem, we recall that if part of the force comes from a generalized potential in such a way that

$$F_P^{(j)} = \frac{\partial U}{\partial q_j} - \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j}, \qquad j = 1, \ldots, n;$$

then, the introduction of the Lagrangian function $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T + U$ makes us write (4) as

$$\sum_{j=1}^{n} \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} - F_{NP}^{(j)} \right) \delta q_j = 0, \tag{7}$$

where the term $F_{NP}^{(j)}$ takes into account the remaining active forces not deriving from a potential.

Remark 1. *In the transfer from the Cartesian coordinates $\mathbf{r}_s$ to the generalized ones, no further geometric constraint occurs; i.e., $N = 3n$; if this happens, the selection of a smaller number of generalized coordinates $n < 3N$ does not involve substantial changes other than the reformulation of the constraint forces $\mathbf{R}_s$.*

2.2. The Constraint Conditions

We assume that the restrictions to the dynamics of the system can be formulated by the constraints equations

$$\psi_\nu(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t) = 0, \qquad \nu = 1, \ldots, k < 3N; \tag{8}$$

hence, we are considering kinematic constraints (linear or nonlinear) involving the positions and the velocities os the points. We will use the following notation: for a real-valued function $f : \mathbb{R}^\mu \to \mathbb{R}$, if $f = f(\boldsymbol{\zeta})$, $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_\mu)$, we denote by $\frac{\partial f}{\partial \boldsymbol{\zeta}}$ the $\mathbb{R}^\mu$-vector $\left(\frac{\partial f}{\partial \zeta_1}, \ldots, \frac{\partial f}{\partial \zeta_\mu}\right)$. Moreover, for a vector function $\mathbf{f} = (f_1, \ldots, f_\sigma) : \mathbb{R}^\mu \to \mathbb{R}^\sigma$, where $f_i = f_i(\boldsymbol{\zeta}) : \mathbb{R}^\mu \to \mathbb{R}$ for each $i = 1, \ldots, \sigma$, we denote by $\frac{\partial \mathbf{f}}{\partial \boldsymbol{\zeta}}$ the $\sigma \times \mu$ Jacobian matrix with entries $\left(\frac{\partial \mathbf{f}}{\partial \boldsymbol{\zeta}}\right)_{i,j} = \frac{\partial f_i}{\partial \zeta_j}$, $i = 1, \ldots, \sigma$, $j = 1, \ldots, \mu$.

The constraints are independent according to the following assumption:

$$\text{the } k \text{ vectors } \frac{\partial \psi_\nu}{\partial \dot{\mathbf{r}}} \in \mathbb{R}^{3N}, \ \nu = 1, \ldots, k \quad \text{are independent,} \tag{9}$$

where $\dot{\boldsymbol{r}} = (\dot{\boldsymbol{r}}_1, \ldots, \dot{\boldsymbol{r}}_N) \in \mathbb{R}^{3N}$. Equivalently, the constraints are independent if the rank of the $(k \times 3N)$ Jacobian matrix $\frac{\partial \boldsymbol{\psi}}{\partial \dot{\mathbf{r}}}$, $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_k)$, attains its maximum value k.

By means of (5), the constraint Equation (8) can be written in terms of the generalized coordinates as follows:

$$\phi_\nu(\mathbf{q}, \dot{\mathbf{q}}, t) = 0, \qquad \nu = 1, \ldots, k. \tag{10}$$

Owing to the chain relation $\dfrac{\partial \boldsymbol{\phi}}{\partial \dot{\mathbf{q}}} = \dfrac{\partial \boldsymbol{\psi}}{\partial \dot{\mathbf{r}}}\dfrac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{q}}} = \dfrac{\partial \boldsymbol{\psi}}{\partial \dot{\mathbf{r}}}\dfrac{\partial \mathbf{r}}{\partial \mathbf{q}}$, where $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_k)$, we see that the constraints (10) are independent with respect to the generalized velocities $\dot{\mathbf{q}}$ by virtue of (9) and the non-singularity of $\partial \mathbf{r}/\partial \mathbf{q}$ since the generalized coordinates $\mathbf{q}$ are independent.

The crucial point can be summarized by the following question: as long as the d'Alembert–Lagrange approach (3) is exerted in a general context of nonholonomic (possibly nonlinear) constraints, which is the proper set of virtual displacements to define so that (2) or, equivalently, (6) is valid? Once this has been established, the right equations of motion can be explicitly deduced from (7).

It is natural to keep in mind the holonomic case (whose theory is long-established) as a starting point for the correct generalization: we recall that [1] in the case of geometric constraints of the following type:

$$\psi_\nu(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) = 0 \qquad \Longrightarrow \quad f_\nu(\mathbf{q}, t) = 0 \tag{11}$$

(the Lagrangian formulation by f_ν is due to (5)). The consolidated theory, in this regard, provides the following category of virtual displacements:

$$\sum_{j=1}^{n} \frac{\partial f_\nu}{\partial q_j} \delta q_j = 0. \tag{12}$$

It is worth noting that setting the constraint in the form of (10) via differentiation:

$$\phi_\nu(\mathbf{q}, \dot{\mathbf{q}}, t) = \dot{f}_\nu(\mathbf{q}, t) = \sum_{j=1}^{n} \frac{\partial f_\nu}{\partial q_j} \dot{q}_j + \frac{\partial f_\nu}{\partial t} = 0, \tag{13}$$

(12) coincides with

$$\delta^{(c)} \phi_\nu = \sum_{j=1}^{n} \frac{\partial \phi_\nu}{\partial \dot{q}_j} \delta q_j = 0. \tag{14}$$

The form (14) for identifying virtual displacements is known in the literature as the Četaev condition, and the superscript (c) recalls it. More generally, passing from the holonomic case to that of linear kinematic constraints:

$$\phi_\nu(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{j=1}^{n} A_{\nu,j}(\mathbf{q}, \mathbf{t}) \dot{q}_j + b_\nu(\mathbf{q}, t), \tag{15}$$

the condition (14) formally reproduces the standard way of treating linear kinematic constraints based on the definition of virtual displacements:

$$\sum_{j=1}^{n} A_{\nu,j} \delta q_j = 0 \tag{16}$$

(indeed $\partial \phi_\nu / \partial \dot{q}_j = A_{\nu,j}$), which is introduced in most of the treatises on kinematic constraints (it suffices to mention [3]). Hence, the Četaev condition properly includes holonomic and linear kinematics constrained systems as special cases. Obviously, (16) is (12) when (15) is (13). A debated topic ([20,22,23] et al.) concerns precisely the fact of accepting condition (14) even in the general case of nonlinear kinematic constraints (10).

Remark 2. *Obviously, (14) is not the only way of defining the virtual displacements in terms of the functions (10); a second way which has received attention in the literature requires the introduction of the virtual variations $\delta\dot{q}_j$, and as a whole, the virtual displacements must verify the condition*

$$\delta^{(v)}\phi_\nu = \sum_{j=1}^{n} \left(\frac{\partial \phi_\nu}{\partial q_j} \delta q_j + \frac{\partial \phi_\nu}{\partial \dot{q}_j} \delta \dot{q}_j \right) = 0. \tag{17}$$

The superscript (v) refers to "vakonomic" approach, which is founded on the theory presented in [24–26]. Even though we will not deal with (17), we briefly compare (14) and (17). First of all, the holonomic case (11) falls within (17) only if one sets $\phi_\nu = f_\nu$ in (17) so that (12) is obtained since $\partial f_\nu / \partial \dot{q}_j = 0$. On the other hand, for holonomic constraints, the direct definition $\phi_\nu = f_\nu$ in (14) (rather than the derivative (13)) would not make sense. In other words, the two formulas match in the holonomic case $f = f(\mathbf{q}, t)$ as long as

$$\delta^{(v)} f_\nu = \delta^{(c)} \dot{f}_\nu.$$

A second difference emerges in the case of linear kinematic constraints (15): on the one hand, $\delta^{(c)}\phi_\nu = 0$ is (16); on the other hand, in the linear case (15) condition, (17) obtains

$$\delta^{(v)}\phi_\nu = \sum_{i,j=1}^{n} \left(\frac{\partial A_{\nu,i}}{\partial q_j} \dot{q}_i + \frac{\partial b_\nu}{\partial q_j} \right) \delta q_j + \sum_{j=1}^{n} A_{\nu,j} \delta \dot{q}_j = 0, \tag{18}$$

which is apparently very different.

2.3. The Equations of Motion

The coupling of the d'A–L. P. principle (7) and the Četaev condition leads straightforwardly to the equations of motion. A simple argument lies on the geometric meaning of (14): the whole set of virtual variations $\delta\mathbf{q}$ builds the vector space orthogonal to the space generated by $\frac{\partial \phi_\nu}{\partial \dot{\mathbf{q}}}$, $\nu = 1, \ldots, k$. Since the latter vectors are independent (as we remarked just after (10)), their linear combinations cover the totality of the vectors orthogonal to all the $\delta\mathbf{q}$; among them, we find the vector in round brackets in (4) so that it is necessarily

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} - F_{NP}^{(j)} = \sum_{\nu=1}^{k} \lambda_\nu \frac{\partial \phi_\nu}{\partial \dot{q}_j}, \qquad j = 1, \ldots, n, \tag{19}$$

with λ_ν unknown multiplying factors (Lagrange multipliers). The just-written equations have to be coupled with the constraint Equation (10) to form a a system of $n + k$ equations in the $n + k$ unknowns $\mathbf{q}$, λ_ν, $\nu = 1, \ldots, k$. The constraint forces $(R^{(1)}, \ldots, R^{(n)})$ are smoooth according to (6).

The possibility of writing the equations of motion without the multipliers (which are uncomfortable unknown quantities) passes through the explicit version of (14): in a more abstract and general way, we indicate (14) as

$$\sum_{j=1}^{n} \beta_{\nu,j} \delta q_j = 0, \qquad \nu = 1, \ldots, k, \tag{20}$$

where $\beta_{\nu,j}$ are the elements of a $k \times n$ matrix with full rank k. Without losing generality, we can determine the displacements as a function of the first independent displacements $\delta q_1, \ldots, \delta q_m$, $m = n - k$:

$$\delta q_{m+\nu} = \sum_{r=1}^{m} \gamma_{\nu,r} \delta q_r, \qquad \nu = 1, \ldots, k, \tag{21}$$

where $m = n - k$ and $\delta q_1, \ldots, \delta q_m$ are arbitrary. By replacing (21) in (7) for each $\nu = 1, \ldots, k$ and taking into account that $\delta q_1, \ldots, \delta q_m$ are independent parameters, the equations of motion can be written as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}}{\partial q_r} + \sum_{\nu=1}^{k} \gamma_{\nu,r}\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_{m+\nu}} - \frac{\partial \mathcal{L}}{\partial q_{m+\nu}}\right) = F_{NP}^{(r)} + \sum_{\nu=1}^{k} \gamma_{\nu,r} F_{NP}^{(m+\nu)} \qquad (22)$$
$$r = 1, \ldots, m,$$

to be combined, this time, with (10). The $m + k = n$ equations contain n unknown functions $q_1, \ldots, q_n$; in addition to the lower number of equations, the obvious advantage of the second system (19) with respect to (19) is the absence of the multipliers λ_ν.

When (20) assumes the form (14); that is, $\beta_{\nu,j} = \frac{\partial \phi_\nu}{\partial \dot{q}_j}$, then the functions appearing in (21) are

$$\gamma_{\nu,r} = \frac{\partial \alpha_\nu}{\partial \dot{q}_r}, \qquad \nu = 1, \ldots, k, \quad r = 1, \ldots, m \qquad (23)$$

with

$$\alpha_\nu = \dot{q}_{m+\nu}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_m, t), \qquad \nu = 1, \ldots, k \qquad (24)$$

deduced from (10) by virtue of the non-singularity condition (save for enumerating the variables in a different way),

$$det\left(\frac{\partial \boldsymbol{\phi}}{\partial \dot{\mathbf{q}}^{(k)}}\right) \neq 0 \qquad \boldsymbol{\phi} = (\phi_1, \ldots, \phi_k), \quad \dot{\mathbf{q}}^{(k)} = (\dot{q}_1, \ldots, \dot{q}_k) \qquad (25)$$

(the adopted notation for the Jacobian matrix is explained just after Equation (8)). In order to check (23), it suffices to calculate the derivatives of $\phi_\nu = 0$ with respect to the independent velocities $\dot{q}_r$, $r = 1, \ldots, m$:

$$\frac{\partial \phi_\nu}{\partial \dot{q}_r} + \sum_{s=1}^{k} \frac{\partial \phi_\nu}{\partial \dot{q}_{m+s}}\frac{\partial \alpha_s}{\partial \dot{q}_r} = 0 \;\Rightarrow\; \sum_{r=1}^{m}\left(\frac{\partial \phi_\nu}{\partial \dot{q}_r}\delta q_r + \sum_{s=1}^{k} \frac{\partial \phi_\nu}{\partial \dot{q}_{m+s}}\frac{\partial \alpha_s}{\partial \dot{q}_r}\delta q_r\right) = 0$$

and to compare with (14), so that we can obtain

$$\delta q_{m+\nu} = \sum_{r=1}^{m} \frac{\partial \alpha_\nu}{\partial \dot{q}_r}\delta q_r, \qquad \nu = 1, \ldots, k \qquad (26)$$

where the virtual variations $\delta q_1, \ldots, \delta q_r$ are independent.

The relations (24) assign to $\dot{q}_1, \ldots, \dot{q}_m$ the role of independent velocities, while the remaining ones $\dot{q}_{m+1}, \ldots, \dot{q}_n$ are dependent velocities. By virtue of (23), the equations of motion (22) take the following form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}}{\partial q_r} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r}\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_{m+\nu}} - \frac{\partial \mathcal{L}}{\partial q_{m+\nu}}\right) = F_{NP}^{(r)} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} F_{NP}^{(m+\nu)} \qquad (27)$$
$$r = 1, \ldots, m.$$

3. The Četaev Rule

3.1. A Reported Proof of the Četaev Condition

As we read in [22], the equations of motion (19) can be obtained from Hertz's principle of least curvature [27] or from a variational principle [28]. On the other hand, if we adopt the technique of virtual displacements for the d'A–L P. (4), the conjecture (14) has an axiomatic role and seems to have no real theoretical justification. For this reason, a claimed demonstration [21] based on the constraint equations caught our attention.

The proof in question (Section III, B of [21]) can be reported as follows. By differentiating (8) with respect to t and taking into account the partition among independent generalized velocities $\dot{q}_i$, $i = 1, \dots, m$ and dependent ones $\dot{q}_{m+s}$, $s = 1, \dots, k$, we can write

$$\sum_{j=1}^{n} \frac{\partial \phi_\nu}{\partial q_j} \dot{q}_j + \sum_{i=1}^{m} \frac{\partial \phi_\nu}{\partial \dot{q}_i} \ddot{q}_i + \sum_{s=1}^{k} \frac{\partial \phi_\nu}{\partial \dot{q}_{m+s}} \ddot{q}_{m+s} + \frac{\partial \phi_\nu}{\partial t} = 0, \qquad \nu = 1, \dots, k,$$

whence we deduce, by deriving with respect to $\ddot{q}_r$, $r = 1, \dots, m$:

$$\frac{\partial \phi_\nu}{\partial \dot{q}_r} + \sum_{s=1}^{k} \frac{\partial \phi_\nu}{\partial \dot{q}_{m+s}} \frac{\partial \ddot{q}_{m+s}}{\partial \ddot{q}_r} = 0, \qquad r = 1, \dots, m. \tag{28}$$

Now, the point we criticize in [21] states (adapting the symbols to our notations): "Although the coordinate function $q_{m+s} = q_{m+s}(q_1, \dots, q_m, t)$, $s = 1, \dots, k$ is unknown for nonintegrable constraint (10), the dependent displacements

$$\delta q_{m+s} = \sum_{i=1}^{m} \frac{\partial q_{m+s}}{\partial q_i} \delta q_i = \sum_{i=1}^{m} \frac{\partial \dot{q}_{m+s}}{\partial \dot{q}_i} \delta q_i = \sum_{i=1}^{m} \frac{\partial \ddot{q}_{m+s}}{\partial \ddot{q}_i} \delta q_i \tag{29}$$

can be obtained in terms of the independent variations δq_r, $r - 1, \dots, m$". The doubtful step fails, in our opinion, to correctly interpret what is not known via what cannot be obtained. In fact, the (known or unknown) dependence is attributed only to geometric constraints; it cannot be mathematically inferred from a non-integrable differential expression (10).

Returning to (29), it is clear that the first equality is the characteristic condition for the holonomic case and it corresponds to (12) written by explicit functions. The same equality leads to (14) (which holds in the holonomic case, as we have already noticed in (13)) in the following way:

$$\sum_{j=1}^{n} \frac{\partial \phi_\nu}{\partial \dot{q}_j} \delta q_j = \sum_{i=1}^{m} \frac{\partial \phi_\nu}{\partial \dot{q}_i} \delta q_i + \sum_{s=1}^{k} \frac{\partial \phi_\nu}{\partial \dot{q}_{m+s}} \delta q_{m+s} = \sum_{i=1}^{m} \left(\frac{\partial \phi_\nu}{\partial \dot{q}_i} + \sum_{s=1}^{k} \frac{\partial \phi_\nu}{\partial \dot{q}_{m+s}} \frac{\partial \ddot{q}_{m+s}}{\partial \ddot{q}_i} \right) \delta q_i = 0.$$

The first step simply separates the independent variables from the dependent ones; the second step makes use of (29), last equality, in order to rewrite δq_{m+s}. Finally, the entire expression is null due to (28).

We stress that the key point is actually $\delta q_{m+k} = \sum\limits_{i=1}^{m} \frac{\partial q_{m+k}}{\partial q_i} \delta q_i$, which is characteristic of holonomic systems, and, in principle, it is not necessarily compatible with (14). Using such an identity in a non-proper way is, in our opinion, the point at which the reasoning falls. This property, in fact, is valid for holonomic systems, but for them, we already know that the Četaev rule holds as the standard condition on displacements.

We further remark that if the procedure were rigorous, the second derivatives would not even be needed: the procedure is much simpler if we replace (28) with

$$\sum_{i=1}^{m} \left(\frac{\partial \phi_\nu}{\partial \dot{q}_i} + \sum_{s=1}^{k} \frac{\partial \phi_\nu}{\partial \dot{q}_{m+s}} \frac{\partial \dot{q}_{m+s}}{\partial \dot{q}_i} \right) = 0,$$

and we conclude that

$$\sum_{j=1}^{n} \frac{\partial \phi_\nu}{\partial \dot{q}_j} \delta q_j = \sum_{i=1}^{m} \frac{\partial \phi_\nu}{\partial \dot{q}_i} \delta q_i + \sum_{s=1}^{k} \frac{\partial \phi_\nu}{\partial \dot{q}_{m+s}} \delta q_{m+s} = \sum_{i=1}^{m} \left(\frac{\partial \phi_\nu}{\partial \dot{q}_i} + \sum_{s=1}^{k} \frac{\partial \phi_\nu}{\partial \dot{q}_{m+s}} \frac{\partial \dot{q}_{m+s}}{\partial \dot{q}_i} \right) \delta q_i = 0$$

using, on this occasion, the second equality in (29).

The same argument is performed in [21], Section III, C, for higher order constraints of the type:

$$h_\nu(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t) = 0$$

in order to prove the relation (similar to (14)):

$$\sum_{j=1}^{n} \frac{\partial h_\nu}{\partial \ddot{q}_j} \delta q_j = 0. \tag{30}$$

The derivative of the constraint equation involves the third derivative $\dddot{h}_\nu$, and the property

$$\delta q_{m+s} = \sum_{i=1}^{m} \frac{\partial q_{m+s}}{\partial q_i} \delta q_i = \sum_{i=1}^{m} \frac{\partial \dddot{q}_{m+s}}{\partial \dddot{q}_i} \delta q_i,$$

analogous to (29), is claimed in order to prove the validity of (30). Nevertheless in our opinion, "unknown" is confused with "not existing", once again, at the moment in which the dependent functions $q_{m+s}(q_1, \ldots, q_m, t)$ have, although unknown, been assumed to exist. A comprehensive and remarkable point of reference in regard to higher-order constraints is [29].

Remark 3. *The claimed proof of (14) does not actually require (17) to hold, as one seems to understand from [21]; as a matter of fact, the reasoning starts from the differential expression $\dot{g}_\nu$ (the one preceding (28)), regardless of the condition in terms of displacements (17).*

Still, according to [21], the theoretical justification of the Četaev rule for a general constraint $\phi_\nu(\mathbf{q}, \dot{\mathbf{q}}, t) = 0$ is supported by the fact that the same Equation (19) can be deduced from the Gauss principle; although we will not go into the details of the mathematical reasoning that justifies this derivation from the principle, we may observe that the validity of the identity

$$\frac{\partial \mathbf{r}}{\partial q_j} = \frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}_j} = \frac{\partial \ddot{\mathbf{r}}}{\partial \ddot{q}_j}$$

is invoked; whereas, in our opinion, it can be used only in the holonomic case. In the next paragraph, we will deal with the just-written identity.

3.2. An Argument in Support of the Rule

Regardless of a rigorous theoretical support, hypothesis (14) is widely considered in the literature for nonholonomic mechanical models; moreover, it is particularly suitable for extending the concept of virtual displacements for higher order constraints, as (30) suggests. Here we are going to add a heuristic consideration in favor of the condition (14).

The question can be formulated as follows: coming back to the notation (5), we consider a system $\mathbf{r}_s(\mathbf{q}, t)$ and the virtual displacements

$$\delta \mathbf{r}_s = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_s}{\partial q_j} \delta q_r \tag{31}$$

in terms of the virtual variations δq_r. In the case of a holonomic system, the property $\dfrac{\partial \dot{\mathbf{r}}_s}{\partial \dot{q}_j} = \dfrac{\partial \mathbf{r}_s}{\partial q_j}$ makes (31) equivalent to

$$\delta \mathbf{r}_s = \sum_{j=1}^{n} \frac{\partial \dot{\mathbf{r}}_s}{\partial \dot{q}_j} \delta q_r. \tag{32}$$

In nonholonomic systems, the additional kinematic constraints (10) prevent (31) and (32) from being equivalent since the velocities $\dot{q}_1, \ldots, \dot{q}_n$ are not independent.

The following property establishes that the definition of virtual displacements based on (14) is the only one that extends the equivalence between (31) and (32) to the case of nonholonomic constraints, provided that only the independent velocities $\dot{q}_r$, $r = 1, \ldots, m$ are considered.

Property 1. *Assume that (14) holds for the set of kinematic constraints (10) $\nu = 1, \ldots, k$ imposed on the system $\mathbf{r}_s(\mathbf{q}, t)$, $s = 1, \ldots, N$. Then, $\delta\mathbf{r}_s$, defined in (31), verifies*

$$\delta\mathbf{r}_s = \sum_{r=1}^{m} \frac{\partial \dot{\mathbf{r}}_s}{\partial \dot{q}_r}\delta q_r, \qquad s = 1, \ldots, N \tag{33}$$

where $\dot{q}_1, \ldots, \dot{q}_m$ is the set of independent velocities, as in (24).

Proof. Since, owing to (24),

$$\dot{\mathbf{r}}_s = \sum_{r=1}^{m} \frac{\partial \mathbf{r}_s}{\partial q_r}\dot{q}_r + \sum_{\nu=1}^{k} \frac{\partial \mathbf{r}_s}{\partial q_{m+\nu}}\alpha_\nu + \frac{\partial \mathbf{r}_s}{\partial t}, \qquad s = 1, \ldots, N, \tag{34}$$

we see that

$$\frac{\partial \dot{\mathbf{r}}_s}{\partial \dot{q}_r} = \frac{\partial \mathbf{r}_s}{\partial q_r} + \sum_{\nu=1}^{k} \frac{\partial \mathbf{r}_s}{\partial q_{m+\nu}}\frac{\partial \alpha_\nu}{\partial \dot{q}_r}, \qquad s = 1, \ldots, N, \quad r = 1, \ldots, m;$$

hence,

$$\begin{aligned}\sum_{r=1}^{m} \frac{\partial \dot{\mathbf{r}}_s}{\partial \dot{q}_r}\delta q_r &= \sum_{r=1}^{m}\left(\frac{\partial \mathbf{r}_s}{\partial q_r} + \sum_{\nu=1}^{k}\frac{\partial \alpha_\nu}{\partial \dot{q}_r}\frac{\partial \mathbf{r}_s}{\partial q_{m+\nu}}\right)\delta q_r \\ &= \sum_{r=1}^{m}\frac{\partial \mathbf{r}_s}{\partial q_r}\delta q_r + \sum_{\nu=1}^{k}\frac{\partial \mathbf{r}_s}{\partial q_{m+\nu}}\underbrace{\sum_{r=1}^{m}\frac{\partial \alpha_\nu}{\partial \dot{q}_r}\delta q_r}_{=\delta q_{m+\nu}}\end{aligned} \tag{35}$$

by virtue of (26), and (33) is proven. □

The essential hypothesis is (14), which implies (26); otherwise, the property is not verified.

3.3. Virtual Displacements and Possible Instantaneous Velocities

A possible instantaneous velocity $\hat{\dot{\mathbf{r}}}_s$ is a velocity allowed by the constraints at a given instant of time. Its expression can be inferred from (5) by blocking the time t:

$$\hat{\dot{\mathbf{r}}}_s = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_s}{\partial q_j}\dot{q}_j = \dot{\mathbf{r}}_s - \partial\mathbf{r}_s/\partial t, \qquad s = 1, \ldots, N. \tag{36}$$

In a nonholonomic system constrained by (10), the corresponding explicit functions (24) assign (36) the form

$$\hat{\dot{\mathbf{r}}}_s = \sum_{r=1}^{m} \frac{\partial \mathbf{r}_s}{\partial q_r}\dot{q}_r + \sum_{\nu=1}^{k} \frac{\partial \mathbf{r}_s}{\partial q_{m+\nu}}\dot{q}_{m+\nu} = \sum_{r=1}^{m} \frac{\partial \mathbf{r}_s}{\partial q_r}\dot{q}_r + \sum_{\nu=1}^{k} \frac{\partial \mathbf{r}_s}{\partial q_{m+\nu}}\alpha_\nu. \tag{37}$$

On the other hand, whenever the condition (14) is assumed to hold, the virtual displacements (31) must satisfy (26); hence,

$$\delta \mathbf{r}_s = \sum_{r=1}^{n} \frac{\partial \mathbf{r}_s}{\partial q_r}\delta q_r + \sum_{\nu=1}^{k} \frac{\partial \mathbf{r}_s}{\partial q_{m+\nu}}\delta q_{m+\nu} = \sum_{r=1}^{m}\left(\frac{\partial \mathbf{r}_s}{\partial q_r} + \sum_{\nu=1}^{k} \frac{\partial \mathbf{r}_s}{\partial q_{m+\nu}} \sum_{r=1}^{m} \frac{\partial \alpha_\nu}{\partial \dot{q}_r}\right)\delta q_r \tag{38}$$

Let us define the functions

$$\overline{\alpha}_\nu(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t) := \sum_{r=1}^{m} \frac{\partial \alpha_\nu}{\partial \dot{q}_r}\dot{q}_r \qquad \nu = 1, \dots, k. \tag{39}$$

The comparison of (37) with (38) leads to the following.

Property 2. *Assume that (14) holds for the set (10) of kinematic constraints imposed on the system $\mathbf{r}_s(\mathbf{q}, t)$, $s = 1, \dots, N$. Then, the virtual displacements $\delta \mathbf{r}_s$ and the possible velocities $\widehat{\dot{\mathbf{r}}}_s$, $s = 1, \dots, N$ exhibit the same direction if and only if*

$$\overline{\alpha}_\nu = \alpha_\nu \quad \textit{for any} \;\; \nu = 1, \dots k, \tag{40}$$

on varying, in the same way, the independent generalized velocities $\dot{q}_1, \dots, \dot{q}_m$ and the independent displacements $\delta q_1, \dots, \delta q_m$.

Indeed, in the case (40), the functions α_ν, $\nu = 1, \dots, k$, can be replaced by $\alpha_\nu = \sum\limits_{r=1}^{m} \frac{\partial \alpha_\nu}{\partial \dot{q}_r}\dot{q}_r$, and the two expressions (37) and (38) are formally identical.

In the class of constraints satisfying (40), the Četaev rule (14) enables the overlap of virtual displacements and possible velocities. As a consequence, the condition of ideal constraint $\sum\limits_{s=1}^{N} \mathbf{R}_s \cdot \delta \mathbf{r}_s = \sum\limits_{j=1}^{n} \mathcal{R}^{(j)} \delta q_j = 0$ (see (2) and (6)) actually represents the absence of virtual work $\sum\limits_{s=1}^{N} \mathbf{R}_s \cdot \widehat{\dot{\mathbf{r}}}_s = \sum\limits_{j=1}^{n} \mathcal{R}^{(j)} \dot{q}_j = 0$ whenever (39) is held.

Remark 4. *More generally, the following relation can be deduced from (19) and (27):*

$$\sum_{j=1}^{n} \mathcal{R}^{(j)} \dot{q}_j = \sum_{\nu=1}^{k} (\alpha_\nu - \overline{\alpha}_\nu)\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_{m+\nu}} - \frac{\partial \mathcal{L}}{\partial q_{m+\nu}} - F_{NP}^{(m+\nu)}\right). \tag{41}$$

We see that the virtual work vanishes if (39) holds.

A further point which confers interest upon the condition (40) comes from the calculation of the Hamiltonian as the Legendre tranform of $\mathcal{L}$ with respect to all the generalized velocities $\dot{q}_j$, $j = 1, \dots, n$:

$$\mathcal{H}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = \sum_{j=1}^{n} \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L}. \tag{42}$$

Consider now the Hamiltonian restricted to independent generalized velocities:

$$\mathcal{H}^*(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t) = \sum_{r=1}^{m} \dot{q}_r \frac{\partial \mathcal{L}^*}{\partial \dot{q}_r} - \mathcal{L}^*, \tag{43}$$

where

$$\mathcal{L}^*(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t) = \mathcal{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, \alpha_1(\cdot), \dots, \alpha_k(\cdot), t)$$

is the restricted Lagrangian calculated by means of (24) and $(\cdot)$ stands for $(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_m, t)$.

The two functions, (42) and (43), do not coincide, meaning that $\mathcal{H}|_{\dot{q}_{m+\nu}=\alpha_\nu} \neq \mathcal{H}^*$. The exact relation, which can be easily checked, is

$$\mathcal{H}^* = \mathcal{H} + \sum_{\nu=1}^{k} (\overline{\alpha}_\nu - \alpha_\nu) \frac{\partial \mathcal{L}}{\partial \dot{q}_{m+\nu}}. \tag{44}$$

Once again, we see that the hypothesis (40) plays a unifying role in the definition since, in that case, the physical quantities and (42) and (43) do coincide; this allows the energy of the system to be uniquely defined.

4. Homogeneous Constraints

Rather than searching for a mathematical derivation of the elusive Cetaev condition (as long as it exists), we assume the Četaev rule (14) as a postulate and we examine in more depth the class of constraints fulfilling (40): actually, Properties 1 and 2 of the previous Section assign to nonholonomic systems of the class (40) the same features as we meet in holonomic and linear nonholonomic systems. Hence, as long as the Četaev rule holds, condition (40) provides a category of constraints which best "fit in" with the rule, which extend in a natural way the theory of holonomic systems. We will conclude (Proposition 1) that this category of constraints coincides with (10), where the functions ϕ_ν are homogeneous functions (even of different degrees) with respect to the generalized velocities $\dot{\mathbf{q}}$.

We recall that a real-valued function $f(\xi_1, \ldots, \xi_\ell)$ defined on a domain $\mathcal{D} \subseteq \mathbb{R}^\ell$ is a positive homogeneous function of degree σ if

$$f(\lambda\xi_1, \ldots, \lambda\xi_\ell) = \lambda^\sigma f(\xi_1, \ldots, \xi_\ell) \quad \forall\, (\xi_1, \ldots, \xi_\ell) \in \mathcal{D} \text{ and } \forall\, \lambda > 0. \tag{45}$$

Here, we pay attention to constraints which are homogeneous functions with respect to the generalized velocities $\dot{q}_j$, $j = 1, \ldots, n$; that is, with respect to (10):

$$\phi_\nu(\mathbf{q}, \lambda\dot{\mathbf{q}}, t) = \lambda_\nu^\sigma \phi_\nu(\mathbf{q}, \dot{\mathbf{q}}, t) \qquad \forall\, \lambda > 0, \qquad \nu = 1, \ldots, k, \tag{46}$$

where the degree σ_ν can be different for each constraint equation. The vast majority of examples and applications in literature of nonholonomic systems with kinematic restrictions belong to the category of constraints (46).

Example 1. *Linear kinematic constraints (15) with $b_\nu = 0$ are positive homogeneous functions of degree 1 with respect to the variables $\dot{\mathbf{q}}$ (in this case, (45) holds also for $\lambda < 0$) in particular holonomic constraints (13) for f_ν , not depending explicitly on time. As an example, for two points, P_1 and P_2, at a constant distance ℓ,the geometric constraint $(q_1 - q_4)^2 + (q_2 - q_5)^2 + (q_3 - q_6)^2 - \ell^2 = 0$ (where (q_1, q_2, q_3) and (q_4, q_5, q_6) are the Cartesian coordinates of P_1 and P_2, respectively) is converted by derivation to the homogeneous function of degree 1 $(q_1 - q_4)^2(\dot{q}_1 - \dot{q}_4) + (q_2 - q_5)^2(\dot{q}_2 - \dot{q}_5) + (q_3 - q_6)^2(\dot{q}_3 - \dot{q}_6) = 0$.*

Example 2. *Given a system of two material points, P_1 and P_2, the following nonholonomic restrictions, i.e.,*

$$\begin{array}{lll}
(i) & \dot{P}_1 \wedge \dot{P}_2 = \mathbf{0} & \textit{parallel velocities} \\
(ii) & \dot{P}_1 \cdot \dot{P}_2 = 0 & \textit{perpendicular velocities} \\
(iii) & |\dot{P}_1| = |\dot{P}_2| & \textit{same magnitude of the velocities} \\
(iv) & \dot{B} \wedge \overrightarrow{P_1P_2} = 0 & \textit{the velocity of midpoint B is parallel to the joining line} \\
(v) & \dot{B} \cdot \overrightarrow{P_1P_2} = 0 & \textit{the velocity of B is perpendicular to the joining line} \\
(vi) & \dot{P}_1 \cdot \overrightarrow{P_1P_2} = \dot{P}_2 \cdot \overrightarrow{P_1P_2} = 0 & \textit{the velocities are perpendicular to the joining line}
\end{array}$$

correspond to homogeneous constraints; actually, in fixing the Lagrangian parameters $\mathbf{q}$ *as the Cartesian coordinates* $P_1 = (q_1, q_2, q_3)$, $P_2 = (q_4, q_5, q_6)$, *the constraints are*

$$\begin{array}{ll}
(i) & \dot{q}_2\dot{q}_6 - \dot{q}_3\dot{q}_5 = 0,\ \dot{q}_3\dot{q}_4 - \dot{q}_1\dot{q}_6 = 0,\ \dot{q}_1\dot{q}_5 - \dot{q}_2\dot{q}_4 = 0 \\
(ii) & \dot{q}_1\dot{q}_4 + \dot{q}_2\dot{q}_5 + \dot{q}_3\dot{q}_6 = 0 \\
(iii) & \sqrt{\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2} - \sqrt{\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2} = 0 \\
(iv) & \begin{cases} (q_3 - q_6)(\dot{q}_2 + \dot{q}_5) - (q_2 - q_5)(\dot{q}_3 + \dot{q}_6) = 0 \\ (q_1 - q_4)(\dot{q}_3 + \dot{q}_6) - (q_3 - q_6)(\dot{q}_1 + \dot{q}_4) = 0 \\ (q_2 - q_5)(\dot{q}_1 + \dot{q}_4) - (q_1 - q_4)(\dot{q}_2 + \dot{q}_5) = 0 \end{cases} \\
(v) & (q_1 - q_4)(\dot{q}_1 + \dot{q}_4) + (q_2 - q_5)(\dot{q}_2 + \dot{q}_5) + (q_3 - q_6)(\dot{q}_3 + \dot{q}_6) = 0 \\
(vi) & \begin{cases} (q_1 - q_2)\dot{q}_1 + (q_2 - q_5)\dot{q}_2 + (q_3 - q_6)\dot{q}_3 = 0 \\ (q_1 - q_2)\dot{q}_4 + (q_2 - q_5)\dot{q}_5 + (q_3 - q_6)\dot{q}_6 = 0 \end{cases}
\end{array}$$

and they all show positive homogeneous functions of degree 1 *or* 2 *with respect to the variables* $\dot{\mathbf{q}}$.

Remark 5. *The just-mentioned conditions may appear abstract, complying only with theoretical requirements. We refer to [8] for a useful and accurate description of devices (rods, blades, wheels, sleds, etc.), which put in practice the constraints listed in Example 2 or a combination of them. Another example of a physically realizable nonlinear nonholonomic mechanical system is the nonholonomic pendulum proposed in [5]. On the other hand, examples of nonlinear nonholonomic models constrained by nonhomogeneous velocity conditions and corresponding to physically realizable systems are, to our knowledge, uncommon, if not fully absent, in literature.*

The connection between the condition (40) and the homogeneity of the constraint functions is explained by the following:

Proposition 1. *These three statements are equivalent:*

(1) *The constraint functions* ϕ_ν *of (10),* $\nu = 1, \ldots, k$, *are positive homogeneous functions with respect to the variables* $(\dot{q}_1, \ldots, \dot{q}_n)$, *even of different degrees* σ_ν.
(2) *Any set of explicit functions* α_ν *defined in (24) and deduced from (10) are positive homogeneous functions of degree* 1 *with respect to the variables* $(\dot{q}_1, \ldots, \dot{q}_m)$.
(3) *The condition (40) holds for any set of explicit functions* α_ν, $\nu = 1, \ldots, k$, *calculated from (10).*

Proof. (Essential steps): the implications (2) $\Longleftrightarrow$ (3) correspond to the Euler's homogeneous function Theorem, stating that the definition (45) is equivalent to the condition

$$\sum_{i=1}^{\ell} \xi_i \frac{\partial f}{\partial \xi_i}(\xi_1, \ldots, \xi_\ell) = \sigma f(\xi_1, \ldots, \xi_\ell) \quad \forall\, (\xi_1, \ldots, \xi_\ell) \in \mathcal{D},$$

which coincides with (40) for $\sigma = 1$ and $(\xi_1, \ldots, \xi_\ell) = (\dot{q}_1, \ldots, \dot{q}_m)$. Furthermore, the implications (1) $\Longleftrightarrow$ (2) can be proved by virtue of the fact that each derivative $\frac{\partial f}{\partial xi_i}$,

$i = 1, \ldots, \ell$, of a homogeneous function f of degree σ is a homogeneous function of degree $\sigma - 1$. $\square$

Remark 6. *A significant point in favor of the category (46) of homogeneous constraints is that the formal structure is invariant if a transformation of Lagrangian coordinates* $\overline{\mathbf{q}} = \overline{\mathbf{q}}(\mathbf{q})$ *is applied; indeed, the induced linear transformation of the generalized velocities* $\dot{\overline{\mathbf{q}}} = \left(\frac{\partial \overline{\mathbf{q}}}{\partial \mathbf{q}}\right)\dot{\mathbf{q}}$ *(where the quantity in round brackets indicates the Jacobian matrix) makes the property (46) still valid for the generalized velocities* $\dot{\overline{\mathbf{q}}}$.

5. Conclusions and Next Investigation

The d'Alembert–Lagrange principle combined with the class of displacements (14) offers a simple way of inferring the equations of motion for nonholonomic systems, even with nonlinear kinematic constraints. Actually, the linear conditions (14) on the δq_j allow us an easy transition from the principle (7) to the equations of motion (19) simply by using an argument of linear algebra. A further advantage of (14) combined with the d'A–L P. is that it is not required to express about the possible commutation $\delta \dot{q}_j - \frac{d}{dt}\delta q_j$ (this is actually a debated question), which, instead, is indispensable whenever (17) is assumed, and necessarily combined with the integral formulation of a principle, owing to the presence of $\delta \dot{q}_j$.

A long-standing problem is whether there exists the possibility of deriving the set (14) from the constraint Equation (10). In our opinion, the problem has not been resolved in [21], and the Četaev rule remains, at least for the moment, without a theoretical explanation. However, this circumstance does not detract from a series of advantages that the rule offers: The use of the d'A–L P. allows the equations of motion to be written in a simple and direct way. The generalization to higher orders (see (30)) occurs through the δq_j alone; therefore, nothing changes from a formal point of view; many other advantageous aspects could be mentioned. We also highlighted how the validity of (39) (which, as reported in Proposition 1, is characteristic of homogeneous constraints of any order) places the mechanical system in a natural physical context, where the displacements coincide with the virtual velocities, the virtual work of the constraint forces is zero, and the energy of the system can be univocally defined (see (41) and (44)). The class of displacements (14) joined with the requirement (39) (homogeneous constraints) appears to be the natural extension of the standard holonomic case to the general case of nonholonomic nonlinear constraints.

The conditions listed in Example 2 show various nonholonomic constraints (some of them nonlinear) of systems that are certainly not marginal in the context of kinematic restrictions. This encourages us to think that (39) essentially covers the set of physically feasible nonholonomic constraints. From a mathematical point of view, the topic to be investigated is the possibility of formulating a zero level set through homogeneous functions. Obviously, the topic concerns stationary constraints and time-dependent ones requires a separate study.

A second issue we are investigating concerns the correlation between the two conditions (14) and (17); on the one hand, it is simple to write the mathematical identity that links them (the so-called transpositional relation); on the other, it is not obvious, for example, which category of systems admits both displacements, or what the role of the commutation rule is. In recent decades, this debate has been animated in the literature [4,22,30].

A further interesting point is to compare the procedure offered by the d'Alembert–Lagrange principle with different approaches (as variational methods) which generalize standard methods for holonomic systems to the case of nonholonomic systems, especially with nonlinear constraints. A stimulating starting point is the recent paper [31], where the Hamiltonian and the action functional are extended by introducing an extra variable performing a sort of dissipation. The corresponding Lagrange problem leads to equations comparable with (19). Furthermore, the method used in [31] is suitable to deal with the energy question, which we simply mention via Formulae (27)–(44).

In the present work, we focused on the theoretical questions about the Cetaev condition and the link with homogeneous constraints omitting a numerical approach: moving to these kinds of questions (Lagrangian mechanics versus vakonomic mechanics (transpositional relation)), neither numerical simulations nor experimental validations provide strong support for the theoretical conclusions.

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