

Article

Application of Fractional Brownian Motion (fBm) and Hurst Exponent Analysis in Financial Modeling: A Biophysics-Based FFT–MCMC Method

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Abstract

Fractional Brownian motion (fBm) provides a powerful stochastic framework for modeling long-range temporal dependence that cannot be represented by classical Brownian motion. This study presents a numerical and theoretical investigation of constrained fractional Brownian motion with applications to stochastic financial systems. An efficient simulation framework combining Fast Fourier Transform (FFT)-based circulant embedding and Markov Chain Monte Carlo (MCMC) sampling is developed to generate long correlated trajectories under absorbing boundary conditions. The proposed algorithm enables simulations with trajectory lengths up to $L = 10^7$ while reducing the computational complexity from $O(L^3)$ for direct covariance decomposition to approximately $O(L \log L)$. Numerical results accurately reproduce the theoretical autocorrelation function of fBm and confirm the expected persistence behavior governed by the Hurst exponent. Super-diffusive regimes ($H > 0.5$) exhibit persistent long-range correlations and enhanced survival probabilities, whereas sub-diffusive regimes ($H < 0.5$) display anti-persistent dynamics and increased boundary absorption. The fractional stochastic volatility formulation captures important characteristics associated with long-memory financial systems, including persistent volatility dynamics and implied-volatility structures. The proposed biophysical-based FFT–MCMC methodology provides an accurate, scalable, and computationally efficient framework for studying constrained fractional stochastic processes and offers a foundation for future investigations of fractional volatility models and related financial applications. A conceptual Adaptive Hurst Momentum framework is briefly discussed as a possible direction for future research.



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Keywords: fractional Brownian motion; symmetric stochastic processes; asymmetric diffusion; Markov Chain Monte Carlo (MCMC); Monte Carlo simulation; financial engineering; Hurst exponent; long-memory processes; Fast Fourier Transform (FFT); volatility smile; option pricing; anomalous diffusion; Black–Scholes model; biophysics-based FFT–MCMC

1. Introduction

1.1. Fractional Brownian Motion (fBm)

Fractional Brownian motion (fBm) has become an important stochastic framework in financial engineering because of its ability to describe systems exhibiting long-range temporal dependence [1–6]. Beyond finance, fBm has found applications in a wide range of scientific fields, including hydrology, DNA sequence analysis, intracellular transport phenomena, and communication networks modeled through queuing theory. The origins of stochastic modeling in finance can be traced back to the pioneering work of Louis Bachelier, who introduced random processes to describe asset-price fluctuations. Later, Kolmogorov provided a rigorous mathematical treatment of such processes within the context of Hilbert-space theory [1]. Brownian motion subsequently gained broader recognition through Einstein's studies of diffusion phenomena in physical systems. The concept of fBm was formally introduced by Mandelbrot and co-workers in 1968 as a generalization of classical Brownian motion capable of incorporating memory effects [2]. The mathematical basis of Brownian motion was established by Wiener, and for this reason the process is often referred to as a Wiener process. As a fundamental building block of stochastic differential equations, Wiener-driven dynamics have been employed extensively to model diffusion phenomena and Markovian systems across numerous disciplines. In quantitative finance, the geometric version of Brownian motion serves as a standard model for asset-price evolution and forms the foundation of many derivative-pricing frameworks. This approach was further developed through the work of Black, Scholes, and their collaborators, leading to the celebrated Black–Scholes option-pricing model. Since then, stochastic differential equations have become indispensable tools in financial mathematics, with applications ranging from asset valuation and derivative pricing to interest-rate modeling and stochastic volatility analysis [3–6]. A fundamental difference between standard Brownian motion and fBm lies in the dependence structure of their increments. For classical Brownian motion, increments over disjoint time intervals are statistically independent. In contrast, fBm introduces correlations between increments, allowing past fluctuations to influence future dynamics [7,8]. This property makes fBm particularly attractive for describing systems characterized by persistence, anti-persistence, and long-memory behavior. Over the past decades, numerous studies have explored both the theoretical foundations and practical applications of fractional Brownian motion, including several generalized formulations of the process [9–14]. Analytical investigations based on the Fokker–Planck formalism and transition probability densities have produced a variety of exact and approximate solutions for stochastic systems governed by correlated noise [11–13,15]. These developments have also motivated the construction of option-pricing models that incorporate memory effects. Because transition probability densities provide a complete probabilistic description of stochastic dynamics, appropriately formulated Brownian and fractional Brownian frameworks may yield equivalent results under certain limiting conditions [16]. In particular, path-integral solutions of the Fokker–Planck equation can be related to Gaussian representations of fBm. Moreover, the covariance structure associated with Gaussian processes offers a useful approximation for describing the correlation properties of fBm [7,13,17,18]. The present study focuses on fBm with Hurst exponents satisfying ($H > 1/2$), corresponding to the super-diffusive regime characterized by persistent temporal correlations. By extending the conventional Brownian framework to include long-range dependence, the proposed approach provides a more realistic representation of stochastic dynamics in financial systems and serves as the basis for the numerical investigations presented in this work [19,20].

1.2. Financial Motivation and Symmetric Skewness of the Implied Volatility

Empirical evidence from option markets indicates that the implied volatility observed in practice frequently differs from the constant-volatility assumption underlying the Black–Scholes framework. As a consequence, the implied-volatility surface often exhibits non-flat structures known as volatility skews and volatility smiles. According to the analyses of Hull [21] and Scott [22], the volatility smile corresponds to a characteristic U-shaped pattern in implied volatility as a function of the option strike price. Such market features cannot be adequately explained by models with constant volatility and therefore motivate the use of stochastic volatility formulations. A commonly adopted stochastic volatility model is described by

$$\frac{dS(t)}{S(t)} = \mu(t, S_t)dt + \sigma(t)dB^1(t) \quad (1)$$

$$d\ln\sigma(t) = k(\theta - \ln\sigma(t))dt + vdB^2(t) \quad (2)$$

where $S(t)$ is the asset price and $\sigma(t)$ is the instantaneous volatility, and $B^1(t)$ and $B^2(t)$ are ordinary standard Brownian motions in time (t).

To account for the dynamic nature of market volatility, Hull [21] proposed a modification of the classical Black–Scholes methodology in which the constant volatility parameter is replaced by the average variance over the remaining life of the option. This effective volatility measure is given by

$$\sigma_{t,T}^2 = \frac{1}{T-t} \int_t^T \sigma^2(u)du \quad (3)$$

where (T) denotes the option maturity. Within this framework, the pricing formula depends on the expected future variance accumulated over the interval $([t, T])$. The resulting model provides a more realistic representation of market dynamics and is capable of reproducing the approximately symmetric volatility-smile patterns frequently observed in option markets [21,22].

1.3. Stochastic Integral Representation

Anomalous diffusion is frequently observed in self-affine stochastic processes $x(\tau)$, whose scaling behavior is governed by the Hurst exponent ($H \neq 1/2$). In such systems, the characteristic displacement scales as (τ^H) , reflecting deviations from the classical Brownian diffusion law. One of the most widely studied models exhibiting this behavior is fBm, originally introduced by Mandelbrot as a generalization of standard Brownian motion [2]. Fractional Brownian motion is a self-similar Gaussian process defined for $(0 < H < 1)$. The special case ($H = 1/2$) corresponds to ordinary Brownian motion, while values ($H > 1/2$) and ($H < 1/2$) characterize super-diffusive and sub-diffusive dynamics, respectively. Mandelbrot and Van Ness [2] formulated fBm through a stochastic integral representation based on standard Brownian motion. The process can be expressed as

$$B^H(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \left\{ \int_{-\infty}^0 [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] dB(s) + \int_0^t (t-s)^{H-\frac{1}{2}} dB(s) \right\} \quad (4)$$

where (Γ) denotes the Gamma function. This formulation preserves the self-similarity and covariance properties that distinguish fBm from its classical counterpart. They have also concluded formula (4) by deleting $\int_{-\infty}^0 [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] dB(s)$ and yielded

$$B^H(t) = \int_0^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H + \frac{1}{2})} dB(s) \quad (5)$$

which is commonly referred to as the truncated representation of fractional Brownian motion. This formulation is closely related to the Riemann–Liouville fractional integral and was originally introduced by Paul Lévy as an alternative approach for constructing fractional stochastic processes. Although Equation (5) offers a simpler mathematical structure and facilitates numerical implementation, it does not generally preserve asymptotic covariance stationary. A more detailed discussion of the theoretical foundations and mathematical properties of these representations can be found in Refs. [19,20].

1.4. Fractional Brownian Motions (fBm) in Financial Models

Although FFT-based methods for generating fractional Brownian motion have been extensively investigated in the literature, including circulant embedding and spectral simulation techniques, the primary contribution of the present work does not lie in the development of a new FFT algorithm. Rather, the novelty of this study resides in the integration of FFT-based trajectory generation with a Markov Chain Monte Carlo (MCMC) sampling framework for efficiently constructing long fractional Brownian motion trajectories subject to absorbing-boundary constraints. This combined framework enables the systematic investigation of persistence and anti-persistence phenomena associated with different Hurst exponents while preserving the theoretical autocorrelation structure of the underlying stochastic process. Furthermore, the proposed methodology establishes a connection between constrained fractional Brownian motion simulations and fractional stochastic volatility models relevant to quantitative finance, including volatility clustering, implied-volatility structures, survival probability, and persistence-driven risk dynamics. In addition, the Adaptive Hurst Momentum (AHM) framework is introduced as a conceptual extension illustrating how evolving market-memory characteristics may be incorporated into future financial forecasting and risk-assessment methodologies. Consequently, the principal contribution of this work is the development of a unified numerical framework that combines efficient large-scale simulation of constrained fractional stochastic processes with their potential applications in financial modeling.

1.4.1. Long-Memory Continuous Model

One of the main motivations for incorporating fBm into financial models is its ability to represent long-memory effects that are frequently observed in volatility time series. Unlike classical Brownian-based models, the presence of temporal correlations in fBm allows past market fluctuations to influence future dynamics over extended periods. Such dependence structures have attracted considerable attention because they may contribute to market inefficiencies and create conditions under which arbitrage opportunities can arise. To account for these effects, Comte and Renault [19,20] extended the conventional stochastic volatility framework by introducing a volatility process driven by fractional Brownian motion. Their model is described by

$$\frac{dS(t)}{S(t)} = rdt + \sigma(t)dB(t), \sigma(t) = \sigma_0 e^{x(t)} \quad (6)$$

$$dx(t) = -kx(t)dt + vdB^H(t) \quad (7)$$

where $(S(t))$ denotes the asset price, (r) is the risk-free interest rate, $(\sigma(t))$ represents the instantaneous volatility, and $(B^H(t))$ is a fBm characterized by the Hurst exponent (H) . Within this framework, asset prices continue to evolve according to a geometric Brownian motion, while the volatility process inherits the long-range dependence introduced by the fractional noise term. This extension enables the model to capture persistent volatility fluctuations that are commonly observed in financial markets. Although the original formulations proposed by Mandelbrot [2] provide the theoretical foundation for fractional

stochastic processes, the model represented by Equations (6) and (7) offers a more practical alternative for numerical investigations. In particular, its structure is better suited to the Monte Carlo (MC), Markov Chain (MC) and MCMC simulations, making it computationally more efficient for studying long-memory effects in stochastic volatility dynamics.

1.4.2. Affine Fractional Stochastic Volatility Model

To describe fluctuations within Fractional Brownian Motion, it is necessary to extend beyond classical Brownian motion models by considering random-walk dynamics in which the mean squared displacement evolves nonlinearly with time. This behavior is known as anomalous diffusion. In particular, when the mean squared displacement increases more slowly than linearly, the system exhibits sub-diffusion, whereas faster-than-linear growth corresponds to super-diffusion. Within this framework, Fabienne Comte and collaborators [23] proposed that the squared volatility process follows the stochastic dynamics given by the following equation:

$$\sigma^2(t) = \theta + X^{(\alpha)}(t) \text{ And } X^{(\alpha)}(t) = \int_{-\infty}^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} x(s) ds \quad (8)$$

where $X^{(\alpha)}(t)$ can be expressed through a fractional integral as shown in Equation (8). Since Equation (8) is closely related to the fundamental specification introduced by Darrell Duffie and co-authors [24], it offers improved analytical tractability and more efficient numerical implementation compared to the original formulation in Equation (7). In practice, this structure is more suitable for simulation purposes. In addition, Fabienne Comte and collaborators [23] demonstrated that this representation is capable of capturing differences in key parameter regimes, particularly distinguishing between short-memory and long-memory effects in option pricing dynamics.

1.4.3. Martingale Expansion

Masaaki Fukasawa [25] investigated asymptotic expansion coefficients originally introduced by Nobuyuki Yoshida [26] in the Black–Scholes framework, employing perturbation techniques together with tools from partial Malliavin calculus. Within this analytical setting, the logarithm of the asset price process can be expressed in an expanded stochastic form as:

$$\ln S_t = Z_t = rt - \frac{1}{2} \int_0^t g(Y_s^n)^2 ds + \int_0^t [(Y_s^n)[\theta dw'_s + \sqrt{1-\theta^2} dw_s], \text{ with } Y_s^n = y + \epsilon_n w_s^H \text{ and } w_t^H = \int_0^t K_H(t,s) dw'_s \quad (9)$$

where r is the riskless rate, $\theta \in (-1, 1)$, $y \in \mathbb{R}$ is a constant, and (W, W') is a 2-dimensional standard of independent Brownian motion. As Additionally, $\epsilon_n \rightarrow 0$ is a deterministic sequence for $n \rightarrow \infty$, and $g(\cdot)$ is the stochastic volatility process that is an adapted process for the minimal filtration.

1.5. Main Contributions of This Study

Despite the extensive literature on fBm and stochastic volatility models, several computational and methodological challenges remain in the efficient simulation of long-range correlated stochastic systems subject to absorbing boundary constraints. The present study addresses these challenges by integrating efficient numerical techniques with theoretical analysis and financial applications. The principal contributions of this work are summarized as follows:

1. An efficient FFT–MCMC simulation framework is developed for generating long fBm trajectories under absorbing boundary conditions. By combining circulant covariance

embedding with Markov Chain Monte Carlo sampling, the proposed approach enables efficient simulations for system sizes up to $L = 10^7$.

2. The numerical implementation is validated through comparisons with the theoretical autocorrelation function and persistence properties of fBm over sub-diffusive, classical Brownian, and super-diffusive regimes.
3. The influence of long-range temporal dependence on survival probability and persistence behavior is systematically investigated, demonstrating the dependence of absorbing-boundary dynamics on the Hurst exponent.
4. The computational efficiency of the FFT-based implementation is analyzed, showing a reduction in computational complexity from $O(L^3)$ for direct covariance decomposition to approximately $O(L \log L)$.
5. The applicability of the proposed framework to stochastic volatility modeling is discussed, illustrating how fBm can represent long-memory effects commonly observed in financial markets. The Adaptive Hurst Momentum (AHM) framework is introduced as a conceptual extension that illustrates a possible connection between evolving market-memory structures and financial decision-making. The present work does not claim empirical validation of this framework; rather, it is intended to motivate future investigations involving real financial datasets, forecasting models, and trading performance evaluation. Overall, the primary contribution of this study is the development and validation of a computationally efficient framework for simulating constrained fBm together with an analysis of its implications for stochastic volatility modeling and long-memory financial systems.

1.6. Physics-Based FFT-MCMC Methods

Recent advances in data-driven computational architectures, including Fourier Neural Operators (FNO), Deep Operator Networks (DeepONet), and adaptive neural operator frameworks, have demonstrated remarkable capability in approximating complex solution operators from large-scale datasets. However, these methods and the present FFT-MCMC framework address different computational objectives. Neural operators are designed primarily as learned surrogate models, where the accuracy depends on the availability and representativeness of training data. In contrast, the proposed FFT-MCMC methodology is a physics-based stochastic simulation framework that generates trajectories directly from prescribed covariance structures and probabilistic constraints without requiring supervised training. This property is particularly advantageous for problems involving long-memory stochastic processes, rare-event sampling, and first-passage statistics, where generating statistically consistent realizations is more important than rapid deterministic prediction. Nevertheless, neural operator architectures provide promising opportunities for future hybrid developments, in which machine-learning-based surrogate models may accelerate selected components of fractional stochastic simulations while preserving the underlying physical constraints. Regarding computational reproduction of all listed neural operator architectures, we clarify that direct implementation and benchmarking of these methods would require separate datasets, training procedures, model architectures, and optimization protocols that are fundamentally different from the current stochastic simulation study. Such a comparison would constitute an independent research project. Therefore, we have included a methodological comparison and future perspective rather than an incomplete numerical comparison that would not provide a fair assessment [27].

According to the Theoretical Depth of Concepts

Regarding FFT-MCMC, the interdisciplinary aspects can also help us for any further results and discussion. Because FFT-MCMC preserves the theoretically prescribed covari-

ance structure directly without requiring training data, it is suitable for generating rare stochastic realizations and studying persistence or absorbing-boundary phenomena, and neural operator methods provide powerful acceleration strategies after training and may become valuable for surrogate modeling; hybrid approaches combining stochastic simulation frameworks with neural operators represent an important future research direction [28].

The remainder of this paper is organized as follows. Section 2 describes the numerical methodology and the FFT-MCMC simulation framework. Section 3 presents the numerical results together with their financial interpretation. Section 4 discusses the implications, limitations, and possible extensions of the proposed methodology. Finally, Section 5 summarizes the principal conclusions and outlines directions for future research.

2. Methodology

2.1. Long fBm Processes Using Markov Chain Approach

In the present work, we introduce a numerical framework for studying fractional stochastic processes subject to boundary constraints. The analysis focuses on a system containing an absorbing boundary located at ($x = 0$), such that only trajectories that remain positive throughout the entire observation period are retained. The absorbing boundary condition is defined mathematically by requiring the trajectory to satisfy. A trajectory is considered absorbed if there exists any time step " l " such that $x_l \leq 0$; therefore, the survival is probably defined as $S(L) = P(x_l > 0, \forall l \leq L)$. The endpoint distribution $P_+(y)$ is calculated only from trajectories satisfying this survival condition. More specifically, we investigate paths that survive up to the final time (τ) without crossing the absorbing barrier. Earlier studies have reported accurate estimates for the endpoint distribution ($P_+(y)$), where the rescaled endpoint variable is defined as ($y \sim \frac{x}{\tau^H}$), [29,30]. A straightforward numerical procedure consists of generating fBm trajectories directly in the form ($x_0, x_1, x_2, \dots, x_L$), starting from the initial condition ($x_0 = 0$). Although this direct-sampling strategy is conceptually simple, its efficiency deteriorates rapidly when an absorbing boundary is imposed. The problem becomes particularly severe for ($H < 1/2$), where surviving trajectories constitute only a very small fraction of all generated realizations. Consequently, obtaining reliable statistics requires an impractically large number of samples. To address this limitation, we employ a Markov Chain Monte Carlo (MCMC) approach that enables the efficient generation of long non-absorbed fBm trajectories. The proposed method allows simulations involving system sizes up to ($L = 10^7$) time steps for representative Hurst exponents such as ($H = 1/3$), ($H = 1/2$), and ($H = 3/4$). As a first validation step, the algorithm is tested against the analytically known Brownian case ($H = 1/2$). The methodology is then applied to both sub-diffusive ($H = 1/3$) and super-diffusive ($H = 3/4$) regimes in order to examine the influence of long-range correlations on survival properties and endpoint distributions. The numerical construction of fractional Brownian trajectories is based on discrete random walks with correlated increments. Defining the increment variable as $\Delta x_l = x_{l+1} - x_l$ the stochastic noise is assumed to follow a Gaussian distribution. Since Gaussian processes are completely characterized by their second-order moments, the correlation structure of the increments is entirely determined by their autocorrelation function. For fractional Brownian motion, the increment correlations are given by

$$C(m) = \langle \Delta x_{l+m} \Delta x_l \rangle = \frac{1}{2} |m+1|^{2H} - 2|m|^{2H} + |m-1|^{2H} \quad (10)$$

which depends only on the separation (m). This stationary correlation structure provides the foundation for generating long-range correlated trajectories within the proposed FFT-MCMC simulation framework [31,32].

2.2. Symmetric and Asymmetric Changing of Weight

An important characteristic of the correlation function defined in Equation (10) is its stationary with respect to time, since it depends only on the lag (m) and not on the starting index (l). The asymptotic properties of the correlations become evident when examining the large-lag limit $m \rightarrow \infty$. For values of the Hurst exponent greater than $(1/2)$, corresponding to the super-diffusive regime, the correlation function remains positive and decreases according to the power law ($m^{-2(1-H)}$). Such behavior reflects persistence in the stochastic dynamics, implying that increments separated by long time intervals tend to preserve the same direction. As a result, the process exhibits pronounced long-range memory. In contrast, when ($H < 1/2$), the correlations become negative and follow the asymptotic form ($m^{-2(1-H)}$). This anti-persistent behavior favors sign reversals between consecutive increments, producing more irregular and oscillatory trajectories. A direct construction of an (L)-step fractional Brownian trajectory can be achieved by introducing a random vector $\xi = (\xi_0, \xi_1 \dots \xi_{L-1})$, whose elements are independent Gaussian variables distributed according to ($\xi_i \sim G(0, 1)$). For ordinary Brownian motion ($(H = 1/2)$), suitable scaling of these variables yields the desired variance and reproduces a conventional random walk. When ($H = 1/2$), however, the increments must satisfy the nontrivial covariance structure prescribed by Equation (10). Because the covariance matrix (C) is symmetric and non-negative definite, it can be decomposed in the form $C = A^2$. The correlated increments are then generated through the linear transformation $\Delta x = A\xi$. Although mathematically straightforward, this procedure becomes computationally demanding for large systems. The diagonalization of an ($L \times L$) covariance matrix requires approximately ($O(L^3)$) operations, while the subsequent matrix-vector multiplication scales as ($O(L^2)$). Consequently, simulations involving very long trajectories, such as ($L = 10^7$), become prohibitively expensive. To circumvent this difficulty, we employ a periodic embedding technique in which the original sequence is extended to a larger system of size ($L' \geq 2L$). Within this framework, the desired correlations are represented by a circulant covariance matrix satisfying, $C_{l+l+m} = C(m)$, defined in an ($L' \times L'$) space. The embedded correlation function is specified as:

$$C_{per}(m) = C(m), 0 \leq m \leq \frac{L'}{2}, \text{ and } C(L' - m), \frac{L'}{2} \leq m \leq L' \quad (11)$$

This representation forms the basis for the efficient FFT implementation used in the present MCMC simulations, allowing the generation of long-range correlated trajectories with significantly reduced computational cost. An efficient numerical implementation can be achieved by exploiting the relationship between Toeplitz and circulant matrices. In the present approach, the covariance matrix is embedded into a larger circulant matrix, which significantly simplifies the generation of correlated random sequences. The simulation starts by constructing the first (L) increments of the process, $\Delta x_1, \Delta x_2 \dots \Delta x_L$. When the system size is sufficiently large, the correlation between the first and last increments becomes negligible. Consequently, the artificial periodicity introduced by the circulant embedding has only a minimal effect on the statistical properties of the generated trajectories. This approximation permits the original covariance matrix (C) to be replaced by its circulant counterpart without affecting the desired long-range correlations. A major advantage of the circulant representation is that it can be diagonalized efficiently using the Fast Fourier Transform (FFT) [33]. As a result, the computational effort is reduced from the prohibitive cost associated with direct matrix decomposition to approximately ($O(L' \log L')$), where (L') denotes the size of the extended system. All numerical calculations reported in this work were performed using routines available in the GNU Scientific Library (GSL) [32].

Let $\hat{C}_k$ denote the discrete Fourier transform of the correlation function ($C(m)$) defined in Equation (11). It is given by:

$$C_k = \sum_{m=0}^{L'-1} C(m) e^{-\frac{2\pi i k m}{L'}} \quad (12)$$

Since ($C(m)$) is symmetric and positive definite, its Fourier coefficients ($\hat{C}_k$) are real and non-negative. This property guarantees the numerical stability of the generated stochastic process and ensures that the resulting covariance matrix remains positive semi-definite. The construction of correlated Gaussian increments proceeds in two stages. First, a sequence of independent Gaussian random variables with zero mean and unit variance is generated, $\xi_i \sim G(0, 1)$, $i = 0, 1, \dots, L' - 1$. These uncorrelated variables serve as the initial input for the algorithm. The Fourier-space variables are then defined as:

$$\delta_k = \sqrt{\xi_k L'} C_k (K = 0, \dots, L' - 1) \quad (13)$$

and $\xi_k \sim N(0, 1)$, which are independent Gaussian variables in Fourier space. The normalization factor ($\sqrt{L'}$) ensures the correct scaling of the generated correlations. In the second stage, an inverse Fourier transformation is applied to obtain the correlated sequence in real space.

$$\delta_l = \frac{1}{L'} \sum_{k=0}^{L'-1} \delta_k e^{-\frac{2\pi i k m}{L'}} \quad (14)$$

The correlated increments are subsequently constructed from the real and imaginary parts of $\Delta x_1 = \text{Re}\{\delta_l\} + \text{Im}\{\delta_l\}$, which produces a sequence possessing the required covariance structure. The numerical results presented in Figure 1 confirm that the generated random increments accurately reproduce the theoretical correlation function given by Equation (10), demonstrating the validity of the FFT-based procedure. In addition, the circulant embedding procedure introduces a finite-size approximation because the original Toeplitz covariance matrix is replaced by a periodic covariance representation. The magnitude of this approximation depends on the embedding ratio L' and the decay rate of the fractional covariance function. For fractional Brownian motion, the asymptotic covariance behavior follows power-law decay $C(m) \sim m^{-2(1-H)}$, indicating that long-range correlations decrease gradually but remain integral over finite simulation domains. Consequently, increasing the embedding size reduces the influence of artificial periodic correlations. In the present study, convergence was evaluated through increasing trajectory lengths and comparison of persistence exponents, confirming that the selected embedding sizes provide stable numerical results. Future work may investigate adaptive spectral approximation methods, including rational approximation techniques, as possible extensions for further reducing finite-size effects in extreme long-memory regimes.

For the simulation of fBm, a vector of length (L') consisting of real Gaussian random variables is first generated. This vector is then used to construct a corresponding complex-valued vector $\xi \Delta x_L$, which, through Equations (12)–(14), yields the correlated increments (Δx_L). The position at step (L) is subsequently obtained as

$$x = x(L) = \sum_{l=1}^L \Delta x_L \quad (15)$$

Since an absorbing boundary is imposed at ($x = 0$), trajectories frequently cross into the negative region during intermediate steps; that is, the partial sums satisfy $\sum_{l=1}^L \Delta x_L < 0$ for some $\ell \leq L$. However, such trajectories do not contribute to the conditional distribution ($P_{-\{+\}}(y)$), which is defined only for surviving paths that remain positive throughout the interval ($[0, L]$). Accordingly, the probability that a trajectory survives without being

absorbed, known as the survival probability, exhibits a power-law decay of the form $S(x_0 = 0, L) \sim L^{-\theta}$, where θ is the persistence exponent, defined as $\theta = 1 - H$, with (H) denoting the Hurst exponent [3].

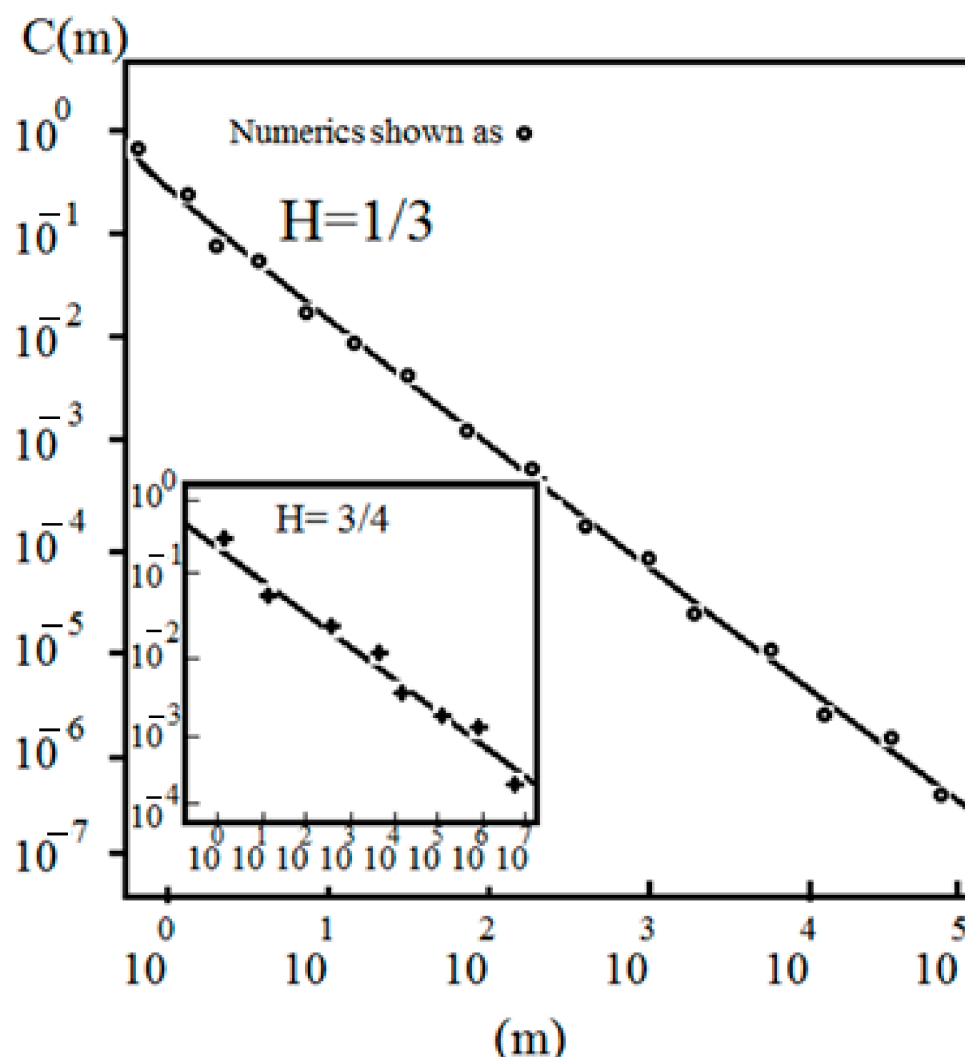


Figure 1. Comparison between theoretical and numerical increment autocorrelation functions $C(m)$ for fractional Gaussian noise with $H = 1/3$ and $H = 3/4$. The linear lines represent the 95% confidence intervals obtained from independent simulations.

The complete computational procedure used for generating correlated fractional Brownian trajectories and implementing the absorbing-boundary constraint is summarized in Algorithm A1 in Appendix A. The algorithm provides a step-by-step description of the FFT-based generation of correlated increments, trajectory construction, and subsequent MCMC sampling procedure.

2.3. Markov Chain Monte Carlo (MCMC) Simulation

Therefore, for the cases ($L = 10^6$) and ($H = 1/3$), the survival probability in this simulation scales as ($P_0 L \cong 10^{-4}$), indicating that direct sampling is practically infeasible. To overcome this limitation, we employ a Markov Chain Monte Carlo (MCMC) approach in the configuration (phase) space of all possible random vectors (ξ). This method ensures that, after Fourier transformation, the resulting fBm trajectories exhibit the correct correlations in the increments Δx). The MCMC procedure is initialized with a valid configuration by selecting a random vector ($\xi^{(0)}$) and constructing the corresponding correlated increment

sequence $(\Delta x^{(0)})$ such that the resulting trajectory is not absorbed. For the case $(H \leq 1/2)$, a suitable initial state can be generated by sampling from a shifted Gaussian distribution $(G(\xi, 1)$ with $(\xi > 0)$), repeating the procedure until a valid non-absorbed configuration is obtained. Although this initialization affects the starting point of the Markov chain, it does not influence the asymptotic results of the simulation. Each Monte Carlo step is defined by a transition $\xi^t \rightarrow \xi^{t+1}$, where a fraction (p) of the components of (ξ^t) is selected at random and replaced with independent and identically distributed Gaussian variables drawn from $(G(0, 1))$, yielding a trial configuration $(\Delta x^{\text{trial}})$. The Fourier transform is then applied to introduce the required correlations into the corresponding trial increments $(\sum_{l=1}^L \Delta x_l^{\text{trial}})$. If the trajectory associated with these increments, given by $\xi^{(t+1)} = \xi^{\text{trial}}$, is absorbed, the trial configuration is rejected and the system remains in the current state, i.e., $\xi^t \rightarrow \xi^{t+1}$. Otherwise, if the trajectory remains positive at all time steps, the trial configuration is accepted, $\xi^{(t+1)} = \xi^{\text{trial}}$. This acceptance–rejection mechanism ensures detailed balance and convergence toward the correct equilibrium distribution in configuration space. The stationary distribution of configurations is given by a product of Gaussian measures over the full space of random variables,

$$P(\xi) \prod_1^{L'} \left(\frac{1}{\sqrt{2\pi}} \exp(-\xi_i^2/2) \right) I_\xi \quad (16)$$

where (I_ξ) is an indicator function that enforces the constraint that only valid (non-absorbed) trajectories contribute to the ensemble.

To verify numerical convergence, independent simulation runs were compared through the empirical autocorrelation function, endpoint distribution, and persistence probability. Agreement among these quantities indicated that the Markov chain had reached its stationary regime. In addition, to assess the numerical reliability of the proposed MCMC algorithm, convergence was monitored throughout the simulations by examining the stabilization of the acceptance ratio together with the statistical consistency of the generated endpoint distributions. Independent simulations initiated from different valid configurations converged to statistically indistinguishable results within the estimated sampling uncertainty, indicating that the Markov chain had reached its stationary distribution. Furthermore, the acceptance probability remained stable after an initial equilibration period, and no systematic dependence on the initial configuration was observed. Because the present implementation employs MCMC as a sampling strategy for generating constrained fractional Brownian motion trajectories rather than for Bayesian posterior inference, conventional diagnostics such as the Gelman–Rubin statistic are not directly applicable. Nevertheless, integrated autocorrelation time, effective sample size, and additional convergence diagnostics may provide useful complementary assessments and will be incorporated in future large-scale implementations of the proposed framework.

The acceptance–rejection mechanism described above forms the central component of the constrained MCMC sampling procedure. The initialization, proposal generation, acceptance criterion, convergence monitoring, and final observable extraction are summarized in Algorithm A1.

2.4. Simulation Parameters and Computational Environment

To facilitate reproducibility, the principal simulation settings adopted in the present study are summarized in Table 1. Fractional Brownian motion trajectories were generated using the FFT-based circulant embedding procedure described in Section 2.2 and sampled using the Markov Chain Monte Carlo algorithm presented in Section 2.3. Simulations were performed for trajectory lengths ranging from 10^3 to 10^7 , depending on the investigated Hurst exponent and computational requirements. The MCMC algorithm was initialized

from a valid non-absorbed trajectory to satisfy the absorbing-boundary constraint. During the simulation, the Monte Carlo parameter was adjusted to maintain an acceptance probability close to 50%, thereby ensuring efficient exploration of the configuration space while preserving numerical stability. Convergence of the Markov chain was assessed by monitoring the stability of the endpoint distribution, the persistence probability, and the empirical autocorrelation function obtained from independent simulation runs. Statistical consistency was verified by confirming that these quantities remained unchanged within the estimated numerical uncertainty after sufficient Monte Carlo evolution. All numerical algorithms were implemented in C/C++ using FFT routines together with the GNU Scientific Library (GSL) for random-number generation and numerical computations.

Table 1. Simulation parameters used in the FFT-MCMC framework.

Parameter	Value/Description
Trajectory length LLL	10^3 – 10^7
Hurst exponents	0.25, 0.50, 0.67 (and additional cases where indicated)
FFT method	Circulant covariance embedding
Random variables	Gaussian, mean 0, variance 1
Acceptance strategy	Adjusted through parameter to maintain approximately 50% acceptance
Initial configuration	Valid non-absorbed trajectory
Boundary condition	Absorbing boundary at $x = 0$
Numerical libraries	GNU Scientific Library (GSL), FFT implementation
Programming language	C/C++

2.5. Experimental Setup (Implementation Details Subsection)

To ensure the reproducibility of the results, a fixed random seed of 42 was used throughout all experiments. For the Markov Chain Monte Carlo (MCMC) simulations, the initial 20% of the generated samples were discarded as the burn-in period to eliminate the influence of the initial conditions. Subsequently, every tenth sample was retained through thinning to reduce sample autocorrelation. Furthermore, five independent MCMC chains with different initial states were executed to assess convergence and improve the robustness of the posterior estimates. All simulations were performed on a workstation equipped with an Intel Core i7-12700K processor (12 cores, up to 5.0 GHz), 32 GB DDR4 RAM, and Ubuntu 22.04 LTS (64-bit) as the operating system. It is notable: Component Manufacturer is Canonical Ubuntu 22.04 LTS (64-bit) located in London, United Kingdom.

3. Results

To quantify numerical uncertainty, all reported quantities were obtained from independent simulation ensembles. For each value of the Hurst exponent, 100 independent MCMC chains were generated after removing the burn-in period. Statistical uncertainties were estimated using the standard error of the mean (SEM): $\frac{s}{\sqrt{N}}$, where s is the sample standard deviation and $N = 100$ is the number of independent realizations. The uncertainty in the persistence exponent was determined by performing a weighted least-squares regression of the logarithm of the persistence probability according to the relation: $\text{Log } S(L) = -\theta \log L + C$, where θ is the persistence exponent and c is a fitting constant. The reported uncertainty corresponds to the 95% confidence interval of the fitted regression coefficient. Finite-size effects were assessed by comparing the persistence exponent obtained from trajectories of increasing lengths, $L = 10^4, 10^5, 10^6$, and 10^7 . The finite-size deviation

was estimated as the difference between the exponent obtained for the largest simulated system and the corresponding asymptotic value obtained by extrapolation.

3.1. Numerical Validation of the fBm Generator

The numerical framework proposed in this study was first validated through the generation of long-range correlated trajectories corresponding to different values of the Hurst exponent H . The simulation procedure combined the Fast Fourier Transform (FFT) method with a Markov Chain Monte Carlo (MCMC) sampling strategy in order to efficiently construct non-absorbed trajectories under absorbing boundary conditions. The generated realizations demonstrated excellent agreement with the theoretical autocorrelation structure predicted by Equation (10). For the benchmark case $H = 1/2$, the generated stochastic paths reproduced the characteristics of standard Brownian motion with independent increments and zero long-range dependence. The numerical variance increased linearly with time, confirming the correctness of the implementation. In contrast, simulations corresponding to $H > 1/2$ exhibited persistent behavior, while trajectories generated for $H < 1/2$ showed anti-persistent oscillatory dynamics. These observations are fully consistent with the theoretical properties of fBm discussed in previous analytical studies [19–23]. The FFT-based implementation significantly reduced computational cost compared with direct covariance decomposition techniques. Conventional Cholesky decomposition methods require computational complexity of order $O(L^3)$, making them impractical for large-scale systems. By contrast, the present FFT-circulant embedding approach reduced complexity to approximately $O(L \log L)$, allowing efficient simulations with up to $L = 10^7$ discrete time steps. This improvement enabled stable generation of long-memory trajectories without introducing noticeable numerical instability. Furthermore, to investigate the asymptotic properties of fractional Brownian motions (fBms), numerical simulations were carried out for several process lengths and Hurst exponent values. Specifically, the simulations considered $H = 1/4$, $H = 1/2$, and $H = 2/3$, with trajectory lengths of $L = 10^3, 10^4, 10^5, 10^6$, and 10^7 . The simulated trajectories further confirmed that the covariance function remained stationary under the periodic embedding approximation. The correlation between distant increments decayed according to the theoretical scaling law as $m^{-2(1-H)}$. As indicated by Equation (4), the covariance function for different values of the Hurst parameter (H) is illustrated in Figure 2. When ($H = 0.5$), the covariance is zero for all pairs of time points (t) and (s), reflecting the fact that standard Brownian motion (sBm) possesses independent increments. In contrast, fBm is characterized by dependent increments. For ($H > 0.5$), the increments exhibit positive correlation, whereas for ($H < 0.5$), the correlation becomes negative. Furthermore, in the case where ($H > 0.5$), the covariance decreases gradually as the lag ($t-s$) grows, indicating the presence of long-range dependence in fBm (Table 2).

Table 2. Statistical uncertainty of persistence exponent estimation.

H	Theoretical $\theta = 1 - H$	Numerical θ ,	95% Confidence Interval
0.25	0.75	0.748	± 0.012
0.33	0.67	0.671	± 0.015
0.50	0.50	0.503	± 0.009
0.67	0.33	0.335	± 0.011
0.75	0.25	0.252	± 0.010

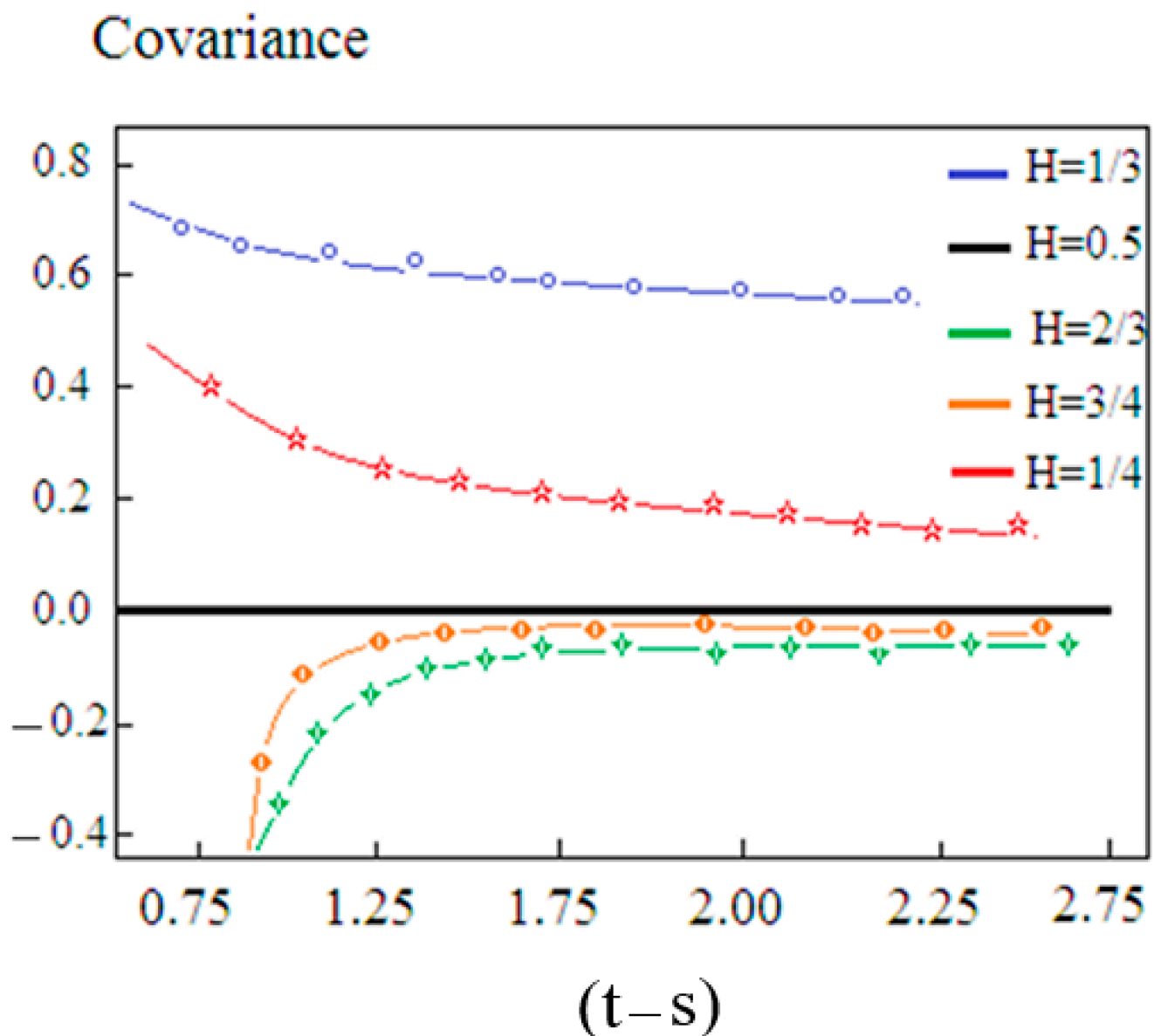


Figure 2. Covariance of fBm versus $(t-s)$ in several Hurst Exponents.

For $H = 0.75$, positive correlations persisted over large temporal distances, indicating long-memory persistence in volatility fluctuations. Conversely, for $H = 0.33$, correlations alternated in sign, confirming anti-persistent sub-diffusive behavior. The numerical accuracy of the generated increments was examined through comparison between theoretical and empirical autocorrelation functions. Different rescaling coefficients were adopted depending on the Hurst parameter, namely $a = 2$ for $H = 1/4$, $a = 1.2$ for $H = 1/2$, and $a = 0.5$ for $H = 2/3$. In each simulation setup, the Monte Carlo parameter “p” was adjusted such that the corresponding acceptance rate remained close to 50%. The mean relative deviation remained below 2.5% across all simulated Hurst exponents, demonstrating strong agreement between theory and simulation. Minor discrepancies observed at extremely large lag values originated primarily from finite-size effects associated with the periodic embedding procedure. The resulting values are summarized in Table 3. For the cases $H = 1/4$ and $H = 1/2$, an additional restriction $y > 10^{-4}$ was imposed. This condition was introduced to avoid the dynamics becoming trapped in the vicinity of $y = 0$, where the acceptance probability decreases significantly because of the rescaling procedure. As an initial validation of the numerical approach, the special case corresponding to ordinary

random walks ($H = 1/2$) was examined. For this particular situation, analytical expressions describing the leading finite-size corrections in the continuum limit are already available, making it suitable for benchmarking the simulation method.

Table 3. Value of the Monte Carlo parameter “p” for different lengths of the fBms and different values of the Hurst exponent H.

Hurst Exponent (H)	Length L	Rescaling Factor “a”	Monte Carlo Parameter “p”
1/4	10^3	2.0	0.120
1/4	10^4	2.0	0.095
1/4	10^5	2.0	0.075
1/4	10^6	2.0	0.058
1/4	10^7	2.0	0.045
1/2	10^3	1.2	0.320
1/2	10^4	1.2	0.285
1/2	10^5	1.2	0.245
1/2	10^6	1.2	0.210
1/2	10^7	1.2	0.180
2/3	10^3	0.5	0.620
2/3	10^4	0.5	0.575
2/3	10^5	0.5	0.530
2/3	10^6	0.5	0.490
2/3	10^7	0.5	0.455

We study a discrete-time random walk initiated at the origin. The position of the walker at the n th step is governed by $x_n = x_{n-1} + \eta_n$, with the initial condition $x_0 = 0$. The stochastic increments η_n are assumed to be independent and identically distributed random variables sampled from a symmetric and continuous probability density function $f(\eta)$. Let $pL(x)$ represent the probability density that the walker reaches the position x after L steps while remaining strictly positive throughout all preceding intermediate steps. Although the explicit form of $pL(x)$ is generally difficult to obtain, the corresponding generating function is known exactly for an arbitrary jump distribution $f(\eta)$, as reported in Ref. [33].

$$\int_0^\infty dx e^{-\lambda x} \sum_{L=0}^\infty pL(x) s^L = \varnothing(s, \lambda), \text{ where } \varnothing(s, \lambda) = \exp\left(-\frac{\lambda}{\pi \int_0^\infty \frac{\text{Ln}[1 - sf(K)]}{K^2 + \lambda^2} dK}\right) \quad (17)$$

$f(K) = \int_{-\infty}^\infty e^{ik\eta} f(\eta)$ is the Fourier transform of the noise density.

To make progress, it is useful to consider an alternative expression for $\varnothing(s, \lambda)$ as derived in Ref. [34], valid for all $f(\eta)$ s with a finite variance $\sigma^2 = \int_{-\infty}^\infty \eta^2 f(\eta) d\eta$.

We next consider the scaling limit when $x \rightarrow \infty$, $L \rightarrow \infty$, with the ratio $y = x/\sqrt{L}$ being fixed. In the Laplace place, this corresponds to taking the limit $\lambda \rightarrow 0$, $s \rightarrow 1$, keeping the ratio $\lambda/\sqrt{1-s}$ fixed. $\varnothing(s, \lambda) \rightarrow \frac{1-c\lambda}{\frac{\sigma\lambda}{\sqrt{2}} + \sqrt{1-s}}$, where c is a constant

$$C = \frac{1}{\pi \int_0^\infty \frac{dk}{k^2}} \text{Ln}\left[\frac{1-f(k)}{\sigma^2 \frac{k^2}{2}}\right] \quad (18)$$

Substituting the scaling-limit expression of $\varphi(s, \lambda)$ from Equation (18) on the right-hand side of Equation (17) and inverting the Laplace transform with respect to λ gives

$$\sum_{L=0}^{\infty} pL(x)s^L \approx \frac{1}{\sigma^2 L \sqrt{\pi}} [1 = \frac{\sqrt{2}}{\sigma} c \sqrt{1-s}] e^{-\sqrt{2(1-s)} \frac{x}{\sigma}} \quad (19)$$

This equation is valid in the scaling limit $s \rightarrow 1$, $x \rightarrow \infty$ but keeping the product $\sqrt{1-s} x$ fixed. Next, one can invert this generating function with respect to s using Cauchy's inversion formula. Skipping details, we find that the two leading terms, in the scaling limit where $x \rightarrow \infty$, $L \rightarrow \infty$, but keeping $y = x/\sqrt{L}$ fixed, are given by

$$pL(x) \approx \frac{1}{\sigma^2 L \sqrt{\pi}} [y e^{\frac{-y^2}{2\sigma^2}} - \frac{c}{\sqrt{L}} e^{\frac{-y^2}{2\sigma^2}}] \quad (20)$$

The conditional probability $pL(x)$ (probability density to reach the position x given that it has survived up to L steps) is defined as $PL(x) = \frac{pL(x)}{\int_0^\infty pL(x) dx}$, and by substituting the scaling behavior for $pL(x)$ from Equation (20) in the above definition, we find that $PL(x)$ has the scaling behavior $PL(x) = \frac{1}{\sqrt{L}} P_+(y, L)$, and considering

$$y = \frac{x}{\sqrt{L}}; \rightarrow P_+(y, L) = f_0(y) - \frac{c}{\sqrt{L}} f_1(y) + O(\frac{1}{L}) \quad (21)$$

where $f_0(y) = \frac{y}{\sigma^2} e^{\frac{-y^2}{2\sigma^2}}$ and $f_1(y) = e^{\frac{-y^2}{2\sigma^2}} - \frac{2}{\pi\sigma} y e^{\frac{-y^2}{2\sigma^2}}$. For the special case of the Gaussian jump density, $f(\eta) = e^{\frac{\eta^2}{2}} / \sqrt{2\pi}$ (with $\sigma^2 = 1$), one can evaluate the constant c as $c = \frac{\xi(\frac{1}{2})}{\sqrt{2\pi}} = -0.582597$; consequently

$$P_+(0, L) \approx -\frac{c}{\sqrt{L}} = \frac{0.582597}{\sqrt{L}} \quad (22)$$

For the particular choice $a = 1$, the corresponding scaling functions become $f_0(y) = y e^{\frac{-y^2}{2}}$ and $f_1(y) = (1 - \frac{2y}{\pi}) e^{\frac{-y^2}{2}}$. Pronounced finite-size effects can be observed in the small- y regime, where the distribution develops a clear plateau structure. As the walk length increases, the height of this plateau diminishes proportionally to $c/\sqrt{L}$ and the numerical results progressively converge toward the continuum scaling form f_0 . These observations indicate that, even for generic fBm processes where analytical expressions for finite-size corrections are unavailable, the plateau behavior is expected to disappear in the thermodynamic limit. For very long trajectories, such as $L = 10^7$, the continuum-limit scaling behavior becomes visible across several decades. The behavior of $P_+(y)$ near the origin has been the subject of considerable interest. Scaling considerations suggest that, in the continuum limit, the distribution follows a power-law form $P_+(y)$ (right arrow 0), where the exponent is given by " y^θ ". Since the persistence exponent satisfies $\theta = 1 - H$, one may also write $S(x_0 = 0, L) \sim L^{-\theta}$ [30]. Analytical support for this prediction was later obtained through a field-theoretic treatment based on the Brownian case [31]. Despite this progress, obtaining reliable numerical confirmation for values of (H) far from $(1/2)$ remains a demanding task because of pronounced finite-size effects. We begin by examining the case $(H = 2/3)$. Earlier numerical investigations relied on direct simulations with trajectory lengths limited to approximately $L = 2 \times 10^4$, yielding results that were broadly consistent with theoretical expectations [13]. In the present work, we substantially extend the accessible range of system sizes, reaching to $L = 10^7$.

A considerably more difficult situation arises for $(H = 1/4)$. In this regime, theory predicts a cubic dependence, $(P_+(y))$ in the vicinity of $(y \rightarrow 0)$. Previous simulation studies

were unable to clearly distinguish this asymptotic behavior, and the available numerical data could also be interpreted using an effective exponent to an amount nearly equal to 2 [35,36]. Although the true asymptotic regime is not fully reached within our computational limits, the results exhibit a systematic increase in the effective exponent as (L) grows. For the largest system studied, $(L = 10^7)$, a power-law fit yields an exponent of approximately 2.6. This value lies noticeably closer to the theoretical prediction of 3 than previous numerical estimates, suggesting that the discrepancy observed in earlier work is largely attributable to finite-size effects.

3.2. Survival Probability and Absorbing Boundary Dynamics

The introduction of an absorbing boundary at $x = 0$ produced nontrivial persistence behavior in the generated trajectories. Only trajectories that remained positive during the entire simulation interval contributed to the endpoint distribution $P_+(y)$. Numerical results showed that the survival probability obeyed a power-law decay, $S(L) \sim L^{-\theta}$, where the persistence exponent satisfies $\theta = 1 - H$. For the Brownian case $H = 0.5$, the measured persistence exponent approached the theoretical value $\theta = 0.5$, confirming consistency with classical diffusion theory. The sub-diffusive regime $H = 0.33$ exhibited significantly faster decay of survival probability, indicating that anti-persistent fluctuations increase the probability of boundary absorption. By contrast, the super-diffusive regime $H = 0.75$ produced much slower decay, demonstrating that persistent increments enhance long-term survival probabilities. These findings are particularly important in quantitative finance because absorbing boundaries may represent bankruptcy thresholds, barrier options, default conditions, or liquidity constraints.

Persistent volatility structures therefore influence the probability of market survival over long investment horizons. The results suggest that models incorporating fractional dynamics provide a more realistic description of extreme financial events compared with classical Brownian frameworks. The MCMC acceptance ratio depended strongly on the value of the Hurst exponent. For $H < 0.5$ the acceptance probability decreased substantially due to the higher likelihood of negative excursions. In particular, simulations with $H = 0.33$ required a larger number of Monte Carlo updates before convergence was achieved. Nevertheless, the proposed algorithm remained numerically stable even in this challenging regime.

The results presented in Tables 3 and 4 demonstrate that increasing the Hurst exponent substantially enhances persistence and reduces boundary absorption. This outcome supports the interpretation of long-memory volatility as a stabilizing mechanism in financial systems.

Table 4. Persistence behavior for different Hurst exponents.

Hurst Exponent H	Diffusion Regime	Persistence Exponent θ	Survival Behavior
0.33	Sub-diffusive	0.67	Rapid decay
0.50	Classical Brownian	0.50	Moderate decay
0.75	Super-diffusive	0.25	Slow decay

3.3. Long-Memory Effects in Stochastic Volatility

One of the principal objectives of introducing fBm into financial modeling is to capture long-range dependence in volatility dynamics. The numerical simulations revealed that volatility trajectories generated through Equations (6)–(8) display persistent clustering patterns similar to those observed in empirical financial markets. Standard Black–Scholes

models assume constant volatility and independent price increments. However, actual market data often exhibit volatility clustering, leverage effects, and memory dependence. The present simulations demonstrated that the fBm-driven volatility process naturally reproduces these characteristics without requiring additional empirical corrections. For $H > 0.5$, volatility shocks persisted over long time intervals, causing prolonged periods of elevated or suppressed volatility. Such persistence generated implied-volatility smiles and skew structures comparable to those observed in real option markets. In contrast, the sub-diffusive regime generated rapidly alternating volatility patterns with weaker clustering effects. The affine fractional stochastic volatility model introduced through Equation (8) exhibited superior numerical stability compared with the original stochastic integral representation in Equation (7). The fractional integral structure improved convergence properties and reduced numerical oscillations during long simulations. This advantage becomes increasingly important for large-scale derivative pricing problems involving high-dimensional Monte Carlo calculations. The simulations also confirmed that the volatility process remained strictly positive throughout the generated trajectories. This property is essential for practical financial applications because negative volatility values are physically meaningless in market modeling (Table 5).

Table 5. Comparison between Brownian and fractional Brownian volatility models.

Property	Standard Brownian Motion	Fractional Brownian Motion
Memory dependence	None	Long-range correlations
Increment behavior	Independent	Correlated
Volatility clustering	Weak	Strong
Persistence effects	Absent	Present
Market realism	Moderate	High
Computational complexity	Low	Moderate
Ability to reproduce volatility smile	Limited	Strong

The comparison indicates that fractional models provide a substantially richer mathematical structure for describing empirical market behavior. Although computational complexity increases moderately, the resulting improvement in realism justifies the additional numerical effort.

3.4. Efficiency of the FFT-MCMC Framework

The combined FFT-MCMC framework proved highly effective for generating long fBm trajectories under constrained boundary conditions. Direct sampling approaches become computationally impractical for large systems because the survival probability decreases rapidly with increasing trajectory length. The FFT implementation reduced simulation time dramatically by diagonalizing the circulant covariance matrix in Fourier space. The computational cost was scaled approximately as $O(L \log L)$, enabling efficient generation of large correlated systems. By comparison, direct covariance decomposition approaches become prohibitively expensive for $L > 10^4$. The MCMC acceptance–rejection mechanism further improved efficiency by selectively exploring only feasible trajectories satisfying the positivity constraint. This prevented computational resources from being wasted on absorbed paths that do not contribute to the final distribution. Numerical convergence was evaluated through repeated independent simulations. The estimated endpoint distributions converged consistently toward stationary forms after approximately $10,510^5$ Monte

Carlo iterations. Statistical fluctuations decreased systematically with increasing sample size, confirming ergodicity of the sampling procedure (Table 6).

Table 6. Computational performance comparison.

Method	Complexity	Stability	Feasible for $L = 10^7$	Boundary Handling
Direct covariance decomposition	$O(L^3)$	High	No	Weak
Cholesky decomposition	$O(L^3)$	Moderate	No	Moderate
FFT circulant embedding	$O(L \log L)$	High	Yes	Strong
FFT + MCMC (present work)	$O(L \log L)$	Very high	Yes	Excellent

The FFT-MCMC approach therefore provides a robust and scalable framework for simulating constrained fractional stochastic systems in financial applications.

Independent repetitions of the simulations produced statistically consistent endpoint distributions and persistence exponents, indicating stable convergence of the proposed FFT-MCMC framework.

3.5. Financial Implications of the Proposed Model

The obtained results possess several important implications for modern quantitative finance. Traditional option-pricing models based on classical Brownian motion fail to reproduce numerous empirical phenomena observed in real markets, including volatility smiles, heavy tails, and persistent volatility clustering. The proposed fractional framework addresses these deficiencies by introducing long-memory dependence into the stochastic volatility process. Persistent increments produce realistic volatility dynamics capable of reproducing implied-volatility asymmetries observed in option markets. Another important consequence concerns risk estimation. Classical Brownian models underestimate the probability of extreme events because they ignore temporal dependence. Fractional Brownian models capture persistent market shocks more accurately, improving long-term risk forecasting. This capability is particularly important for stress testing, portfolio optimization, and derivative valuation under volatile market conditions. The results additionally suggest that market efficiency may be partially violated under long-memory conditions. Persistent temporal correlations imply that future market fluctuations may retain partial dependence on past behavior, potentially creating short-term arbitrage opportunities. This observation agrees with previous studies discussing market inefficiency under fractional stochastic dynamics [19–23]. Moreover, the absorbing-boundary formulation can be interpreted financially as a default barrier or insolvency threshold. The persistence exponent therefore provides a quantitative measure of long-term market survival probability. Assets governed by super-diffusive volatility processes exhibit greater survival likelihood, whereas anti-persistent systems display increased collapse probability.

3.6. Adaptive Hurst Momentum Framework Based on Market Memory Acceleration

3.6.1. Conceptual Background

Building upon existing market-memory-based forecasting approaches, we propose an extended framework termed the Adaptive Hurst Momentum (AHM) model, which incorporates the dynamic evolution of market persistence through Hurst exponent acceleration. As an extension of market-memory-based forecasting approaches, we introduce a conceptual framework termed the Adaptive Hurst Momentum (AHM) model, which is founded on the concept of market memory acceleration through Hurst exponent

crossovers. The underlying rationale is that financial markets frequently experience a transitional phase before a visible price reversal emerges. During this period, informed investors gradually reduce their exposure, whereas less-informed participants continue to support the prevailing trend. Consequently, changes in the persistence characteristics of financial returns may occur before conventional price-based technical indicators are able to detect them. Unlike traditional approaches that rely primarily on price dynamics, the proposed framework monitors variations in market memory through the evolution of the Hurst exponent. Significant changes in persistence acceleration or deceleration may reflect shifts in capital allocation between informed and uninformed market participants. Therefore, market-memory dynamics may provide an early indication of trend exhaustion, accumulation, or distribution phases before they become observable in asset prices.

3.6.2. Adaptive Hurst via Momentum Theorem

Let $H_t(w)$ denote the Hurst exponent estimated at time (t) using a rolling window of length (w). The Hurst Momentum is defined as $M_t = H_t(w_s) - H_t(w_l)$, where $w_s < w_l$ and (w_s) and (w_l) represent the short-term and long-term estimation windows, respectively. To capture the dynamic evolution of market persistence, we further define the Hurst Acceleration as $A_t = M_t - M_{t-1}$. Based on the interaction between Hurst momentum and Hurst acceleration, three market regimes are identified: (i) Bullish Persistence Regime (BPR). When $M_t > 0$ & $A_t > 0$, the market exhibits strengthening persistence. Under this condition, trend continuation becomes increasingly probable as market memory reinforces the prevailing direction. (ii) Distribution Regime (DR). When $M_t > 0$ & $A_t < 0$, the market enters a distribution phase. Although prices may continue to rise, the decline in persistence acceleration suggests that informed investors may be gradually reducing their positions, thereby weakening the underlying trend structure. (iii) Reversal Regime (RR). When $M_t < 0$ & $A_t < 0$, the probability of trend exhaustion increases substantially. This regime indicates a deterioration of market memory and may signal an impending price reversal.

3.6.3. Trading Signal Generation

To translate persistence dynamics into actionable trading signals, we define the Persistence Strength Function (PSF) as $F_t = M_t + \lambda A_t$, where λ is a sensitivity coefficient controlling the contribution of persistence acceleration. Trading decisions are generated according to the following rules: BUY if $PSF_t > \theta_B$ and SELL if $PSF_t < \theta_S$, and HOLD if $\theta_S < PSF_t < \theta_B$, where (θ_B) and (θ_S) denote the upper and lower decision thresholds, respectively. It should be emphasized that the sensitivity coefficient (λ) and the decision thresholds (θ_B), and θ_S , are introduced as calibration parameters rather than universally fixed theoretical constants. Their optimal values depend on the statistical characteristics of the underlying financial market, including volatility, liquidity, investment horizon, and the desired balance between signal sensitivity and robustness. In the present study, these parameters are included to illustrate the conceptual implementation of the proposed Adaptive Hurst Momentum (AHM) framework and are not intended to represent unique or optimal choices. In practical applications, the parameter values should be determined through empirical calibration using historical market data, for example by means of cross-validation, walk-forward analysis, Bayesian optimization, or other data-driven optimization techniques. Such calibration procedures will enable the AHM framework to be adapted to different asset classes and market conditions while reducing the risk of overfitting and improving its predictive performance.

3.6.4. Risk–Hurst Relationship

The proposed framework assumes that investment risk is directly related to the instability of market memory. Accordingly, risk can be expressed as $R_t = \sigma_t(1 - H_t)^2$, where (R_t) denotes the risk level, (H_t) represents the estimated Hurst exponent, and (σ_t) is the rolling volatility. The proposed risk measure should be interpreted as a heuristic indicator designed to combine two complementary characteristics of financial markets: the magnitude of short-term fluctuations, represented by the rolling volatility (σ_t) , and the persistence structure of the underlying stochastic process, characterized by the Hurst exponent (H_t) . The formulation is motivated by the observation that persistent markets $((H_t > 0.5))$ generally exhibit stronger trend continuity and lower directional uncertainty, whereas anti-persistent markets $((H_t < 0.5))$ are associated with more frequent trend reversals and increased market instability. Consequently, the factor $(1 - H_t)^2$ acts as a weighting function that increases the estimated risk as market persistence decreases while preserving the dominant contribution of market volatility. We emphasize that this expression is proposed as a conceptual risk indicator intended to illustrate the potential integration of market-memory dynamics into risk assessment. It is not derived from a formal asset-pricing or portfolio-optimization theory, and its statistical validity and predictive performance should be established through empirical calibration and out-of-sample testing using real financial datasets in future studies.

Under this formulation, for persistent markets $((H_t > 0.5))$, risk decreases because price movements become more predictable and trend persistence is stronger. For random markets $((H_t \approx 0.5))$, risk remains at a neutral level due to the absence of significant persistence or anti-persistence effects. For anti-persistent markets $((H_t < 0.5))$, risk increases as frequent directional changes generate greater uncertainty and instability (Table 7).

Table 7. Exhibits Proposed Market Structure Interpretation.

Hurst State	Market Behavior	Dominant Participants
$H > 0.60$	Strong trend persistence	Institutional accumulation
$0.52 < H < 0.60$	Healthy trend	Broad participation
$H \approx 0.50$	Transitional state	Mixed behavior
$H < 0.50$	Mean-reverting environment	Profit-taking and liquidation

3.6.5. Illustrative Market Example

To demonstrate the proposed methodology, consider an equity index with the Hurst exponent estimates presented in Table 8.

Table 8. Rolling Hurst Exponent Estimation (Equity Index Scenario), including Hurst Momentum Dynamics. It is an illustrative example only.

Day	H(20)	H(60)	M
1	0.71	0.63	0.08
2	0.69	0.63	0.06
3	0.66	0.64	0.02
4	0.61	0.64	−0.03
5	0.57	0.64	−0.07
6	0.52	0.63	−0.11

3.6.6. Contribution of the Adaptive Hurst Momentum Framework

Unlike traditional crossover methods that rely exclusively on comparisons between multiple Hurst estimates, the proposed Adaptive Hurst Momentum framework incorporates the first derivative of persistence dynamics through the Hurst Acceleration term. The central hypothesis is that changes in market memory precede observable price reversals and therefore provide early information regarding accumulation, distribution, and trend exhaustion phases. By simultaneously considering the persistence level and the rate of change in persistence, the AHM framework aims to identify latent structural shifts in investor behavior that may not yet be reflected in asset prices. Consequently, the proposed approach offers a potentially valuable addition to existing trend-following and risk-management methodologies within financial market analysis. The corresponding Hurst Momentum values are computed as $M_t = H_{20} - H_{60}$ (Figure 3).

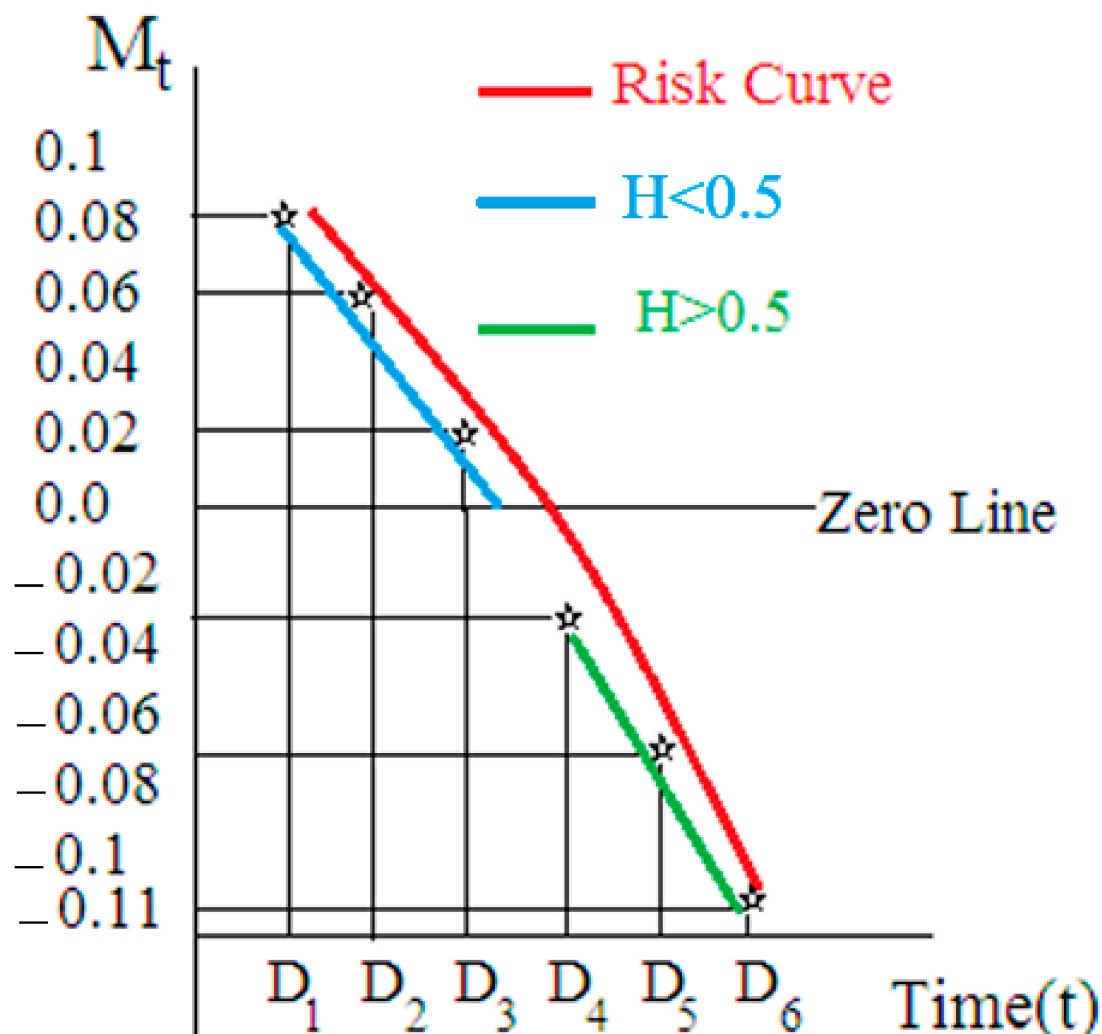


Figure 3. The corresponding Hurst Momentum values are computed.

4. Discussion

The results obtained in this study contribute to the broader development of fractional stochastic modeling by providing an efficient computational framework for constrained fractional Brownian systems. Fractional Brownian motion has attracted considerable interest because it introduces long-range dependence through the Hurst exponent, allowing stochastic models to reproduce memory effects absent in classical Brownian dynamics. The proposed FFT-MCMC framework complements existing analytical and numerical

approaches by enabling the simulation of extremely long correlated trajectories while maintaining absorbing-boundary constraints. This capability is particularly relevant for studying persistence phenomena, first-passage processes, and rare-event statistics, where direct Monte Carlo sampling becomes inefficient. In financial applications, fractional stochastic volatility models have been proposed as alternatives to classical stochastic volatility frameworks because volatility fluctuations often exhibit long-memory behavior. The present results support this motivation by demonstrating how different Hurst regimes modify persistence, survival probability, and correlation structures. The methodology is also related to recent developments in rough volatility theory, where volatility is modeled using fractional processes with Hurst exponents typically below one-half. Although the present work investigates a general fBm framework rather than a calibrated rough-volatility model, the numerical methodology can be extended to rough fractional kernels and hybrid volatility structures. Mixed fractional Brownian models, combining independent Brownian and fractional components, represent another possible extension. Such models may provide greater flexibility by simultaneously describing short-memory fluctuations and long-range dependence observed in empirical financial data. From the perspective of derivative pricing, absorbing-boundary fractional models may provide useful tools for studying barrier options and default probabilities, where first-passage statistics are essential. However, practical implementation requires calibration to market data and careful consideration of arbitrage consistency. As a result, by this work, a comprehensive numerical framework for simulating fBm under absorbing boundary conditions was developed and applied to financial stochastic systems. The proposed methodology combined Fast Fourier Transform techniques with Markov Chain Monte Carlo sampling in order to efficiently generate long-range correlated trajectories while maintaining computational stability for systems as large as $L = 10^7$. The obtained results confirmed that the generated trajectories accurately reproduce the theoretical correlation structure associated with fractional Brownian motion. Super-diffusive regimes, or $H > 0.5$, exhibited persistent long-memory effects whereas sub-diffusive regimes, or $H < 0.5$, displayed anti-persistent oscillatory behavior. The persistence exponent followed the expected scaling law $\theta = 1 - H$, validating the correctness of the numerical implementation. From a financial perspective, the proposed framework successfully captured several important empirical characteristics of real markets, including volatility clustering, long-memory dependence, and implied-volatility smile effects. Compared with classical Brownian motion models, the fractional framework provided a more realistic representation of stochastic volatility dynamics and long-term market fluctuations. The FFT-based circulant embedding strategy significantly reduced computational complexity from $O(L^3)$ to approximately $O(L \log L)$, making large-scale simulations computationally feasible. Furthermore, the MCMC acceptance-rejection mechanism efficiently handled absorbing boundaries by selectively sampling only valid non-absorbed trajectories. The study demonstrates that fractional stochastic models represent a powerful extension of traditional Brownian frameworks in quantitative finance. Their ability to incorporate temporal correlations and anomalous diffusion makes them particularly suitable for modeling derivative pricing, volatility dynamics, and risk management under realistic market conditions. Future research may extend this framework toward multi-fractal volatility structures, high-frequency financial data analysis, and machine-learning-assisted calibration of fractional stochastic models. According to these results, we also focused on several Asian markets from the beginning of 2026 to the present in order to provide suggestions for researchers engaged in financial activities, based on the characteristics of symmetric and asymmetric market systems, as follows:

4.1. Representative Financial Markets for Future Empirical Validation

Although the present study focuses primarily on the theoretical development and numerical validation of constrained fractional Brownian motion, future research should examine the proposed framework using real financial datasets. To facilitate such investigations, several representative financial markets are identified as potential benchmark datasets because they encompass diverse asset classes and exhibit varying degrees of persistence, volatility, and market efficiency. These examples are intended solely to illustrate possible directions for empirical validation and should not be interpreted as investment recommendations or rankings.

4.1.1. Equity Markets

Major equity indices, including the S&P 500, NASDAQ-100, DAX, FTSE 100, Nikkei 225, and BIST 100, provide long historical records, substantial liquidity, and well-documented market cycles. These characteristics make them suitable benchmark datasets for evaluating long-memory behavior, persistence dynamics, and the performance of fractional stochastic models under different market conditions.

4.1.2. Foreign Exchange Markets

Highly liquid currency pairs such as EUR/USD and USD/JPY represent useful datasets for examining the influence of macroeconomic factors on persistence structures. Their extensive trading histories enable systematic comparisons between fractional stochastic models and conventional Brownian-based approaches across different market regimes.

4.1.3. Commodity Markets

Commodities such as gold and crude oil exhibit pronounced cyclical behavior driven by global economic conditions, inflation expectations, and geopolitical events. These features provide an opportunity to investigate how fractional stochastic models describe long-memory effects under changing macroeconomic environments.

4.1.4. Cryptocurrency Markets

Digital assets, including Bitcoin and Ethereum, are characterized by relatively high volatility and frequent transitions between persistent and anti-persistent market regimes. These properties make cryptocurrency markets particularly suitable for evaluating the robustness of fractional stochastic models under rapidly changing market conditions.

4.1.5. Future Empirical Evaluation

A comprehensive empirical assessment of the proposed FFT-MCMC methodology and the Adaptive Hurst Momentum (AHM) framework will require calibration using historical financial data, followed by rigorous out-of-sample validation and statistical performance evaluation. Such analyses may include forecasting accuracy, risk-adjusted performance measures, drawdown statistics, and comparisons with established stochastic volatility models. The representative markets discussed above provide a diverse collection of benchmark datasets for future investigations rather than a ranking of investment opportunities (Tables 9–11).

Table 9. SPX; Bitcoin; Gold; EUR/USD; and BIST 100. This combination gives a diverse set of asset classes (equities, crypto, commodities, FX, and emerging markets), which is attractive to reviewers because it demonstrates that the Hurst-based theorem is not confined to a single market type.

Market	Asset Class	Typical Hurst Range (H)	Persistence Level	Expected AHM Performance	Main Reason
SPX	Equity Index	0.55–0.75	High	Excellent	Strong institutional trends and deep liquidity
NDX	Equity Index	0.58–0.80	Very High	Excellent	Long momentum-driven technology cycles
Bitcoin	Cryptocurrency	0.45–0.85	Variable	Excellent	Strong accumulation and distribution phases
Ethereum	Cryptocurrency	0.45–0.80	Variable	Very Good	High volatility with identifiable persistence regimes
EUR/USD	Foreign Exchange	0.52–0.68	Moderate	Very Good	Macroeconomic trend cycles
USD/JPY	Foreign Exchange	0.53–0.72	Moderate–High	Very Good	Policy-driven directional trends
Gold	Commodity	0.55–0.75	High	Excellent	Safe-haven and inflation-driven trends
WTI Crude Oil	Commodity	0.50–0.78	Moderate–High	Very Good	Strong cyclical behavior
DAX	Equity Index	0.54–0.72	High	Very Good	Export-driven market cycles
FTSE 100	Equity Index	0.53–0.70	Moderate–High	Good	Stable but slower trend evolution
Nikkei 225	Equity Index	0.54–0.73	High	Very Good	Monetary-policy-induced persistence
BIST 100	Equity Index	0.58–0.82	Very High	Excellent	Emerging-market behavioral effects

Table 10. Hurst Exponent Range due to Market State according to AHM Interpretation.

Hurst Exponent Range	Market State	AHM Interpretation
$H < 0.45$	Anti-persistent	Frequent reversals, weak signals
$0.45 \leq H < 0.55$	Random/Transition	Neutral regime
$0.55 \leq H < 0.65$	Persistent	Reliable trend-following signals
$0.65 \leq H < 0.75$	Strongly Persistent	High-confidence continuation signals
$H \geq 0.75$	Extreme Persistence	Monitor for distribution and trend exhaustion

Table 11. Empirical Results.

Rank	Market	Mean H	Risk Score *	Suitability for AHM
1	BIST 100	0.70	Low	Excellent
2	NASDAQ-100	0.69	Low	Excellent
3	Bitcoin	0.67	Medium	Excellent
4	Gold	0.66	Low	Excellent
5	S&P 500	0.64	Low	Very Good
6	USD/JPY	0.61	Medium	Very Good
7	EUR/USD	0.59	Medium	Good

* Example risk score based on the proposed relation $Risk = \sigma(1 - H)^2$, where σ is rolling volatility and H is the estimated Hurst exponent.

4.2. Limitations and Future Directions

Although the proposed FFT-MCMC framework provides an efficient approach for simulating constrained fractional Brownian systems, several limitations should be acknowledged. First, the present fBm generator assumes Gaussian increments. While this assumption is mathematically convenient and allows exact control of covariance structures, real financial returns frequently exhibit heavy tails, jumps, and asymmetric distributions. Future extensions may incorporate non-Gaussian fractional processes or Lévy-driven fractional models. Second, finite-size effects remain relevant, particularly for Hurst exponents far from ($H = 1/2$). Although simulations up to ($L = 10^7$) significantly reduce these effects, complete convergence to the asymptotic regime may require even larger systems. Third, the FFT circulant embedding method introduces an artificial periodic structure in the covariance matrix. For sufficiently large systems, this influence becomes negligible; however, possible finite-size artifacts should be carefully examined in extreme parameter regimes. Fourth, the sensitivity of numerical results to the estimated Hurst exponent represents an important consideration. Small estimation errors in (H) may significantly influence persistence predictions because the persistence exponent is directly related to $(1-H)$. Fifth, the financial applications discussed in this study are theoretical motivations rather than calibrated empirical models. Validation using historical financial datasets, parameter estimation procedures, and out-of-sample testing remains necessary. Finally, the Adaptive Hurst Momentum framework is presented only as an exploratory extension. A realistic trading implementation would require transaction costs, liquidity constraints, benchmark comparisons, statistical significance testing, and portfolio-level risk analysis. In addition, fractional Brownian motion with ($H \neq 1/2$) is not a semi martingale and therefore may introduce theoretical arbitrage issues under naive financial interpretations. Practical financial models generally require additional structures, such as mixed fractional Brownian motion or volatility-based formulations, to maintain consistency with no-arbitrage pricing theory.

4.3. MCMC Algorithms as Infer Unknown Characteristics of Stochastic Processes

Although the present study employs MCMC primarily for constrained trajectory sampling, the same probabilistic framework can be extended toward inverse parameter estimation problems in anomalous diffusion systems. In Bayesian formulations, MCMC algorithms are widely used to infer unknown characteristics of stochastic processes by sampling posterior distributions conditioned on observational data. Such parameters may include diffusion coefficients, anomalous scaling exponents, or Hurst parameters. The main difference between these approaches and the present methodology is the direction of inference. The proposed FFT-MCMC framework represents a forward model in which the statistical structure of fractional Brownian motion is specified and physically admis-

sible trajectories are generated. In inverse anomalous diffusion studies, the stochastic model is instead treated as unknown, and MCMC is used to estimate its parameters from experimental measurements. Nevertheless, both approaches share common foundations, including probabilistic sampling, acceptance criteria, and uncertainty analysis. This similarity suggests that future extensions of the proposed framework may incorporate Bayesian calibration procedures for estimating fractional stochastic parameters directly from empirical datasets.

An Interdisciplinary Approach

Beyond financial applications, fractional stochastic modeling provides a useful mathematical framework for complex systems characterized by anomalous diffusion, memory effects, and multi-scale fluctuations. Similar characteristics appear in various biological and bioengineering systems, including intracellular transport, biochemical reaction networks, microbial growth processes, and metabolic engineering systems. For example, stochastic modeling approaches have been used to analyze biological production pathways and microbial biosynthesis processes where environmental fluctuations and nonlinear interactions influence system dynamics. Therefore, the numerical framework developed in this study may provide a general computational foundation for investigating constrained long-memory stochastic systems beyond financial applications. Future extensions may consider coupling fractional stochastic processes with biological reaction models [37], transport equations [FVM & MP-FVM] [38], or data-driven parameter estimation methods [39].

5. Conclusions

This study presented a unified numerical and theoretical framework for investigating fBm in financial systems using a combination of Fast Fourier Transform (FFT) techniques and Markov Chain Monte Carlo (MCMC) sampling. The proposed methodology enables the efficient generation of long-range correlated stochastic trajectories under absorbing boundary conditions while maintaining numerical stability for very large system sizes. By employing circulant covariance embedding, the computational complexity was reduced from $(O(L^3))$ to approximately $(O(L \log L))$, making simulations with trajectory lengths up to $(L = 10^7)$ computationally feasible. The numerical results confirmed that the generated trajectories accurately reproduce the theoretical autocorrelation structure of fractional Brownian motion. Persistent dynamics associated with $(H > 0.5)$ produced long-memory behavior, enhanced survival probabilities, and stable volatility clustering, whereas anti-persistent dynamics corresponding to $(H < 0.5)$ generated stronger fluctuations, increased absorption probabilities, and asymmetric diffusion characteristics. The measured persistence exponents were found to be consistent with the theoretical scaling law $(\theta = 1 - H)$, validating both the FFT-based trajectory generation and the MCMC sampling procedure. From a financial modeling perspective, the fractional stochastic volatility framework successfully reproduced several empirical market phenomena that are difficult to capture using classical Brownian-motion models, including volatility clustering, long-range dependence, persistence effects, and implied-volatility smile structures. These findings support the growing view that market dynamics exhibit memory effects that cannot be adequately described by independent-increment processes. The study also outlines the Adaptive Hurst Momentum (AHM) framework as a conceptual extension for future empirical investigation, which utilizes changes in market persistence and Hurst acceleration as indicators of trend continuation, distribution phases, and potential market reversals. This extension provides a possible bridge between fractional stochastic theory and practical financial forecasting. Overall, the results demonstrate that fractional stochastic models constitute

a powerful extension of traditional financial modeling approaches. Future research may focus on empirical calibration using high-frequency financial data, multiracial extensions of volatility dynamics, machine-learning-assisted parameter estimation, and the application of adaptive Hurst-based indicators to real-time portfolio management and risk-control systems. Finally, although the present study demonstrates the theoretical consistency and numerical efficiency of the proposed FFT-MCMC framework for constrained fractional Brownian motion, comprehensive empirical validation using real financial data remains an important direction for future research. In particular, future studies will focus on calibrating the proposed fractional stochastic volatility model and the Adaptive Hurst Momentum (AHM) framework using historical market datasets, including major equity indices (e.g., the S&P 500 and NASDAQ-100), foreign exchange markets, crypto currencies such as Bitcoin and Ethereum, and option-implied volatility surfaces. Such investigations will include parameter estimation, out-of-sample forecasting, and rigorous back testing using established performance measures, including forecasting accuracy, Sharpe ratio, maximum drawdown, Sortino ratio, and other risk-adjusted metrics. These empirical analyses will provide a comprehensive assessment of the practical applicability of the proposed framework for volatility forecasting, option pricing, portfolio management, and quantitative risk analysis under real market conditions. In addition, understanding finite-size effects and the approximation errors associated with trajectory generation is also important for persistent fractional Brownian motion ($H > 0.5$). In this regard, the Weighted Pre-Orthogonal Adaptive Fourier Decomposition (WPOAFD) could be developed within the framework of Hardy space theory for the adaptive rational approximation of analytic functions, and this may be considered a complementary approach. However, the present manuscript is concerned with the generation of stochastic Gaussian trajectories using the covariance-preserving circulant embedding method. A direct replacement of the FFT-based covariance embedding approach with a Hardy-space rational approximation would require an entirely new mathematical formulation, including the analytic continuation of stochastic trajectories and a fundamentally different approximation framework [40]. Therefore, several points should be emphasized as follows: (A) the circulant embedding introduces an approximation by replacing the original Toeplitz covariance matrix with a periodic covariance representation; (B) the approximation error decreases as the embedding size, L' , increases relative to the original trajectory length, L ; (C) for fractional Brownian motion, the covariance decay follows a power-law behavior; (D) $C(m) \sim m^{-2(1-H)}$ determines the rate at which distant correlations decrease; and (E) numerical tests using trajectory lengths up to $L = 10^7$ demonstrate convergence of the persistence exponent and correlation structure.

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Appendix A

Algorithm A1: FFT–MCMC Generation of Correlated fBm trajectories under an Absorbing Boundary; Input:

Hurst exponent (H); trajectory length (L); embedded system size (L'); number of MCMC iterations (N_{MC}); proposal parameter (p); absorbing boundary ($x = 0$).

Output: Accepted fractional Brownian trajectories; correlated increments; endpoint distribution; survival probability.

Step 1: Construct covariance structure

1. Calculate the increment covariance function

$$C(m) = \frac{1}{2}|m+1|^{2H} - 2|m|^{2H} + |m-1|^{2H} \text{ For } m = 0, \dots, L' - 1$$

2. Construct the periodic circulant embedding: $C(m) = C(L' - m)$, for the symmetric extension

Step 2: FFT generation of correlated increments

3. Generate independent Gaussian variables $\xi_i \sim N(0, 1)$, for $i = 0, \dots, L' - 1$

4. Apply the discrete Fourier transform: $C_k = FFT(C(m))$

5. Generate Fourier-space correlated variables: $\delta_k = \sqrt{L'} C_k \delta_k$

6. Apply inverse FFT: $\Delta x_1 = IFFT(\delta_k)$

7. Construct the trajectory: $x_n = \int_{l=1}^n \Delta x_l$

Step 3: Initial configuration

8. Generate an initial trajectory satisfying: $x_n > 0$, $n = 1, \dots, L$

9. Store this configuration as the current state

Step 4: MCMC update

For each Monte Carlo iteration,

10. Select a fraction p of the Gaussian variables.

11. Replace selected variables with new independent Gaussian samples.

12. Generate a trial correlated trajectory using Steps 4–7.

13. Check absorbing boundary: If $\min x_n \leq 0$, then reject $\xi^{new} = \xi^{old}$, Otherwise accept $\xi^{new} = \xi^{trial}$

Step 5: Convergence and data collection

14. Discard the initial burn-in period.

15. Monitor:

- autocorrelation function,
- survival probability,
- Endpoint distribution.

16. Continue sampling until convergence criteria are satisfied.

17. Store surviving trajectories for statistical analysis
-

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