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Geometric Design of a Face Gear Drive with Low Sliding Ratio

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Abstract: A design method of low sliding ratio face gear drive is presented. The rack and gear meshing pair with a low sliding ratio is obtained by constructing the contact path. The face gear is then formed by this gear, and its tooth width characteristics are analyzed. Combined with the explicit solution of the meshing point of the tooth surface, a method to solve the sliding ratio of the proposed face gear pair is given, and the effects of relevant design parameters on the sliding ratio are analyzed. Moreover, the rationality of the tooth profile design is verified by 3D modeling and motion simulation. The proposed face gear pair has a lower sliding ratio, as well as larger tooth width, than those of traditional face gear drive (the pinion is involute spur gear) with the same basic parameters. The results of this study can provide points of reference for reducing friction and improving the mechanical efficiency of face gear drives.

Keywords: low sliding ratio; face gear; tooth width; meshing point; motion simulation

1. Introduction

Face gear drive is a meshing transmission mechanism composed of cylindrical gear and bevel gear. Compared with the traditional spiral bevel gear transmission, it has the advantages of smaller volume, lighter weight, higher load-bearing capacity, better component force, which can meet the development requirements of future aviation equipment [1]. At present, the face gear drive has been successfully applied to the military [2,3] and civil industries [4], and the application scope and occasions will be broader. However, during the meshing process, there is relatively larger sliding friction between tooth surfaces, which will affect the operational life [5,6]. Therefore, it is necessary to develop a face gear drive with a low sliding ratio.

In view of the advantages of face gear transmission, many new face gear drives have been continuously reported. Litvin et al. [7] studied a design and analysis method for face gear drive with an involute helical involute pinion. Combining the characteristics of the worm drive, a larger gear ratio face gear drive was developed, which is composed of conical or cylindrical worms [8]. They also investigated a new geometry different from the existing one by application of asymmetric proles and double-crowned pinion of the drive [9]. This face gear drive can reduce the stresses and transmission errors. Zschippang et al. [10] presented a geometric design method for face gear with helix angle, shaft angle, and axle offset. Lin et al. [11,12] implemented the meshing theory of curve-face gear pair, which can achieve compound motion with much compact structure, better transmission efficiency, and load-bearing capacity than gear-linkage mechanism. For improving the performance and reducing the weight of noncircular gear transmission systems, Liu et al. [13] provided a novel spatial noncircular gear set comprising a cylinder gear and an eccentric face gear. Cui et al. [14] put forward a curved tooth surface gear, which has the
characteristics of relatively simple technology, compact structure, low cost, high coincidence, stable transmission, and high load-bearing capacity than those of the general face gear drive. In addition, Tan [15] introduced a novel face gear drive with a conical involute pinion. Zhang and Wu [16] disclosed the tooth geometry of face gear drive with offset axes.

Thus far, the tooth profile design of the face gear drive with a low sliding ratio is rare, whereas the design of the cylindrical gear drive has been previously considered. Chang and Tsai [17] reported a parametric tooth profile of spur gear drive with a low sliding ratio. Kapelevich [18] advanced an asymmetric gear drive, which has a low sliding ratio. Xu et al. [19] improved the cycloid gear drive, and the sliding ratio is lower than that of involute gear. By controlling the relative curvature of the conjugate tooth profiles, Liu et al. [20] presented a design method for plane meshing tooth profiles, which can also be used as the design of gear drive with a low sliding ratio. Chen et al. [21] presented a pure rolling cylindrical helical gear drive with variable helix angles. Yeh et al. [22] implemented the deviation function (DF) method for the design of the conjugated tooth profiles, which can be used for designing the low sliding ratio gear drive. It is a more effective method to realize the design of a low-sliding plane meshing tooth profile by editing the meshing path, for example, the path combined with a straight line and a curve [23], and a specific arc path [24]. Additionally, Wang et al. [25,26] completed the design of the low sliding ratio tooth profile, which is based on the approximate range of the given sliding ratio.

As the sliding ratio is related to the operational life of the system, a low sliding ratio of the cylindrical gear drive has been developed. However, the sliding ratio has rarely been used as the design purpose to develop a novel face gear drive. In this paper, the sliding ratio is considered as the main design goal for the research of face gear drive. According to Buckingham [27], the face gear can be regarded as a rack with variable pressure angle and variable pitch, which is shown in Figure 1. The shaper of face gear is obtained from the rack, and a specific meshing path is given between the rack and the shaper to achieve the characteristics of a low sliding ratio. For this, the mathematical models of the face gear pair with a low sliding ratio are established. Then, the tooth width restriction conditions are studied, the effects of the given contact path function coefficients on the tooth width characteristics of the face gear are analyzed. The calculation methods of tooth surface meshing point and the sliding ratio are provided, and the effects of design parameters on the sliding ratio are studied. Finally, the rationality of the tooth profile design is verified by accurate modeling and motion simulation.

Figure 1. Approximate tooth profile of face gear.
2. Mathematical Models of the Face Gear Pair with Low Sliding Ratio

2.1. Construction of Shaper

According to the gear meshing theory, the shaper or pinion tooth surface can be achieved based on the meshing of the shaper and rack cutter. Additionally, the conjugate tooth profiles can be obtained from a given path of contact. Consequently, to obtain a low sliding ratio, the contact path $\Sigma$ is represented by a given function, as illustrated in Figure 2. The coordinate systems $S_1(o_1,x_1,y_1)$ and $S_2(o_2,x_2,y_2)$ are rigidly attached to the shaper and rack cutter, respectively. $S(o,x,y)$ is a fixed coordinate system whose origin $o$ coincides with the pitch point $P$.

![Schematic diagram of the rack generating the shaper.](image1.png)  
![Coordinate system of rack generating the shaper.](image2.png)  

**Figure 2.** Generation of the shaper.

The equation of contact path $\Sigma$ in $S(o,x,y)$ can be defined as

$$y = f(x)$$

(1)

It is assumed that the equation of rack profile $\Sigma_r$ in $S_2(o_2,x_2,y_2)$ is expressed as

$$Y = g(X)$$

(2)

Based on the theory of gearing, the profile $\Sigma_r$ can be defined as

$$\begin{cases} x = \int \left( -\frac{y}{x} \right) \, dy + C \\ y = y \end{cases}$$

(3)

where $C$ is an integration constant; let the initial meshing position of conjugated tooth profiles be at the pitch point $P$, then $C = 0$.

For this, Equation (3) can be further expressed as

$$\begin{cases} x = \int \left( -\frac{y}{x} \frac{dy}{dx} \right) \, dx \\ y = y \end{cases}$$

(4)

From Figure 2b, the kinematic relationship between gear and rack can be described: when the rack $\Sigma_r$ translates leftwards to the position II from the start position I by the distance $L$, the shaper $\Sigma_s$ rotates counterclockwise around the point $o_1$ by the angle $\phi_1$ and meshes with $\Sigma_s$ at the point $B(x,y)$ of contact path $\Sigma$. Moreover, the point $B$ corresponds to
point \( B_2(X, Y) \) of tooth profile \( \Sigma \) at the position 1. It is easy to know that \( BB_2 = L = X - x \), and \( \phi_1 \) can be expressed as

\[
\phi_1 = \frac{L}{r_s} = \frac{-x + \int \left( -\frac{y}{x} \cdot \frac{dy}{dx} \right) dx}{r_s}
\]

where \( r_s \) is the radius of the pitch circle of the shaper.

By coordinate transforming and integrating Equation (5), the coordinate of \( B \) can be derived in dashed coordinate system \( S_1(o_1, x_1, y_1) \) as

\[
\begin{align*}
x_1 &= x \cos \phi_1 + y \sin \phi_1 + r_s \sin \phi_1 \\
y_1 &= -x \sin \phi_1 + y \cos \phi_1 + r_s \cos \phi_1
\end{align*}
\]

Thus, the work profiles of shaper can be presented as follows:

\[
\begin{align*}
x^{(w)}_1 &= x \cos \phi_1 + (y + r_s) \sin \phi_1 \\
y^{(w)}_1 &= -x \sin \phi_1 + (y + r_s) \cos \phi_1 \\
\phi_1 &= \frac{-x + \int \left( -\frac{y}{x} \cdot \frac{dy}{dx} \right) dx}{r_s} \\
x \in [x_a, x_b]
\end{align*}
\]

where \( x_a \) and \( x_b \) are the addendum and dedendum parameters of the shaper, which can be solved by the following Equations (8) and (9), respectively.

\[
x^2 + y^2 = (r_s + h_s)^2
\]

\[
a_s x + a_x x^3 = -h_f
\]

where \( h_s \) and \( h_f \) are the addendum height and the dedendum height of shaper, respectively.

The fillet profiles of the gear can be obtained by the envelope movement of the generating rack’s tip. Therefore, the fillet profile of the shaper is expressed as

\[
\begin{align*}
x^{(f)}_1 &= x \cos \phi_1 + (r_s - h_f) \sin \phi_1 \\
y^{(f)}_1 &= -x \sin \phi_1 + (r_s - h_f) \cos \phi_1 \\
\phi_1 &= \frac{-x + \int \left( -\frac{y}{x} \cdot \frac{dy}{dx} \right) dx}{r_s} \\
x \in [0, x_b]
\end{align*}
\]

In order to obtain the rack and gear pair with a low sliding ratio, the relationship between the sliding ratio and meshing curve can be established from the solution method of sliding ratio.

\[
\delta_1 = \frac{y^{(1)}_1 - y^{(2)}_1}{y^{(1)}_1} = 1 - \frac{1}{\frac{y^{(1)}_1}{y^{(2)}_1}} = 1 - \frac{1}{\frac{H + r_s}{h_s}} = \frac{H}{H + r_s}
\]
\[
\delta_2 = \frac{v^{(2)}_t}{v^{(2)}_r} - \frac{v^{(1)}_t}{v^{(2)}_r} = 1 - \frac{v^{(1)}_t}{v^{(2)}_r} = 1 - \frac{H + r_s}{r_s} = -\frac{H}{r_s}
\]  

(12)

where \(v^{(1)}_t\) and \(v^{(2)}_t\) are the tangential velocities of each tooth profile at the meshing point; \(H\) is the ordinate value of the intersection of the common tangent at the meshing point and the axis \(y\).

From Equations (11) and (12), it can be seen that when the absolute values of \(H\) at the entry and exit points of engagement are as small as possible, the absolute value of the sliding ratio will also be as small as possible. Thus, the equation for the engagement path can be described by a cubic function as in Equation (13).

\[
y = a_1 x + a_2 x^3, \quad (a_1 < 0, a_2 < 0)
\]  

(13)

where the coefficients \(a_1\) and \(a_2\) should meet the following conditions: (1) the shaper tooth tip thickness should be greater than 0.3 times the modulus and (2) the contact ratio must be greater than 1.

To illustrate the low sliding ratio that can result from the provided meshing trajectory profile, a gear with module \(m = 4\) mm, gear tooth number \(N_s = 32\), coefficients \(a_1 = -\tan 20^\circ\) and \(a_2 = -0.001\) is used as an example. The tooth profile and sliding ratio of the gear pair are shown in Figure 3.

![Figure 3](image_url)

(a) Tooth profile of gear. (b) Sliding ratio of gear pair.

Figure 3. Tooth profile and sliding ratio of gear pair.

It can be seen that the proposed rack-and-gear transmission has a lower sliding ratio, and the maximum sliding ratio is about 47% lower than that of the involute gear with the same basic parameters.

2.2. Construction of Face Gear Tooth Surface

The tooth surface of the face gear can be generated by the shaper. For this, first, four coordinate systems are set up, as depicted in Figure 4. \(S_0(0_o, x_o, y_o, z_o)\) and \(S_0(0_o, x_o, y_o, z_o)\) are fixed coordinate systems of shaper and face gear, respectively. Likewise, \(S_0(0_s, x_s, y_s, z_s)\) and \(S_0(0_s, x_s, y_s, z_s)\) are rigidly attached to the shaper and the face gear. Then, the origins of the four coordinate systems are at the same position, and the shaper and face gear perform rotation about the axis \(z_o\) and \(z_s\) respectively.
Figure 4. Generation of the face gear.

The kinematic relationship between shaper and face gear can be described as follows: when the shaper rotates from coordinate system \(S_0\) to \(S\) by angle \(\phi_s\), the face gear rotates from \(S_0\) to \(S_f\) by angle \(\phi_f\). Then, the basic transformation matrices between the coordinate systems are as follows:

\[
\begin{align*}
M_{s0,s} &= \begin{pmatrix}
\cos \phi_s & -\sin \phi_s & 0 & 0 \\
\sin \phi_s & \cos \phi_s & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \\
M_{f0,s} &= \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{align*}
\]
\[
M_{f,f_0} = \begin{pmatrix}
\cos \phi_f & \sin \phi_f & 0 & 0 \\
-\sin \phi_f & \cos \phi_f & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (16)

\[
M_{f,s} = M_{f,f_0} \cdot M_{f_0,s} = \begin{pmatrix}
-\sin \phi_s \cos \phi_f & -\cos \phi_s \cos \phi_f & \sin \phi_f & 0 \\
\sin \phi_s \sin \phi_f & \cos \phi_s \sin \phi_f & \cos \phi_f & 0 \\
-\cos \phi_s & \sin \phi_s & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (17)

where \(M_{0,s}\) represents the transformation from coordinate system \(S_s\) to coordinate system \(S_0\); the others can then be compared.

In view of the production method, the two surfaces of the shaper are often set symmetrically with respect to the plane \(x_0z_0\). Then, the equation of shaper in \(S_s(0, x_s, y_s)\) is expressed as

\[
r_i(x, h) = \begin{pmatrix}
-x \sin \phi_0 + (a_r x + a_s x^3 + r_s) \cos \phi_0 \\
\pm [x \cos \phi_0 + (a_r x + a_s x^3 + r_s) \sin \phi_0] \\
h \\
1
\end{pmatrix}
\] (18)

where ‘\(\pm\)’ represents the left and the right side of the alveolar surface, respectively. \(\phi_0\) is defined as

\[
\phi_0 = \frac{-(a_i^2 + 1)x + \frac{4}{3}a_i a_s x^3 + \frac{3}{5}a_i^2 x^5}{r_s} + \frac{\pi}{2N_i}
\] (19)

According to the differential geometry theory, the unit normal vector \(n_s\) of shaper tooth profile can be defined as

\[
n_s = \frac{\frac{\partial r_s}{\partial x} \times \frac{\partial r_s}{\partial h}}{|\frac{\partial r_s}{\partial x} \times \frac{\partial r_s}{\partial h}|}
\] (20)

Then, the meshing equation between the shaper and the face gear is expressed as follows:

\[
f(x, h, \phi_s) = n_s \cdot v_{s}^{(f)} = 0
\] (21)

where \(v^{(f)}\) is the relative velocity vector of the shaper and the face gear in \(S_s(0, x_s, y_s)\).

\[
v_{s}^{(f)} = v_{s}^{(s)} - v_{s}^{(f)}
\] (22)

where \(v^{(s)}\) and \(v^{(f)}\) are the velocity vector of the shaper and the face gear.

\[
v_{s}^{(s)} = \omega_s \times r_s = \omega_s k_s \times r_s
\] (23)

\[
v_{s}^{(f)} = \omega_f \times r_s = \omega_f k_f \times r_s
\] (24)
Considering the machining coordinate transforming relation, $k_f$ can be expressed at coordinate system $S_o(o, x_o, y_o)$ as

$$k_f = -\cos \phi \mathbf{i} + \sin \phi \mathbf{j}$$  \hspace{1cm} (25)$$

Thus, $v^{(\theta)}$ is further expressed as

$$v_s^{(\theta)} = \omega_s \begin{pmatrix} -i f z_s \sin \phi_s - y_s \\ -i f z_s \cos \phi_s + x_s \\ i f (x_s \sin \phi_s + y_s \cos \phi_s) \end{pmatrix}$$  \hspace{1cm} (26)$$

where $i_s$ is the gear ratio and satisfies the following equation:

$$i_s = \frac{\phi_f}{\phi_s} = \frac{\omega_f}{\omega_s} = \frac{N_f}{N_s}$$  \hspace{1cm} (27)$$

Through the coordinate transformation and meshing principle, the work surface equation the face gear coordinate system is defined as

$$r_f^{(w)}(x, h, \phi_s, \phi_f) = M_{fs}(\phi_s, \phi_f) \cdot r_f(x, h)$$

$$= \begin{pmatrix} -x_s \sin \phi_s \cos \phi_f - y_s \cos \phi_s \cos \phi_f + z_s \sin \phi_s \\ x_s \sin \phi_s \sin \phi_f + y_s \cos \phi_s \sin \phi_f + z_s \cos \phi_f \\ -x_s \cos \phi_s + y_s \sin \phi_s \end{pmatrix}$$  \hspace{1cm} (28)$$

The fillet surface of face gear can be formed by the edge of the addendum of shaper tooth profile. Therefore, the fillet surface equation is expressed as follows:

$$r_f^{(f)}(x^{(f)}, h, \phi_s, \phi_f) = M_{fs}(\phi_s, \phi_f) \cdot r_f(x^{(f)}, h)$$

$$= \begin{pmatrix} -x_s \sin \phi_s \cos \phi_f - y_s \cos \phi_s \cos \phi_f + z_s \sin \phi_s \\ x_s \sin \phi_s \sin \phi_f + y_s \cos \phi_s \sin \phi_f + z_s \cos \phi_f \\ -x_s \cos \phi_s + y_s \sin \phi_s \end{pmatrix}$$  \hspace{1cm} (29)$$

where $x^{\theta}$ is the parameter of the shaper addendum circle, which is solved by the following equation:

$$x_s^2 + y_s^2 = (r_s + h_y)^2$$  \hspace{1cm} (30)$$

3. Tooth Width Characteristics

3.1. Undercutting

Undercutting is usually used as a criterion to restrict the face gear tooth inwards. The minimum inner radius can be determined by calculating the position of singular points on the generated face gear flank surfaces, which is no undercutting. According to a previous publication, the tooth surface will undercut, when the velocity of the two tooth surfaces meets the following relationship:

$$v_s^{(f)} + v_s^{(\theta)} = v_f = 0$$  \hspace{1cm} (31)$$

The differentiated equation of meshing is given by
Both Equations (31) and (32) yield a system of four linear equations of two unknowns \( (dx/dt, dh/dt) \), where \( df/dt \) is defined as chosen. This system of linear equations has a solution for the two unknowns if the following matrix \( K \) has the rank \( r = 2 \):

\[
K = \begin{pmatrix}
\partial f/\partial x & \partial f/\partial h & -\partial f/\partial \phi_t \\
\partial f/\partial x & \partial f/\partial h & -\partial f/\partial \phi_t \\
\partial f/\partial x & \partial f/\partial h & -\partial f/\partial \phi_t \\
\partial f/\partial x & \partial f/\partial h & -\partial f/\partial \phi_t \\
\end{pmatrix}
\]

\[
\Delta_1 = \begin{pmatrix}
\partial x/\partial x & \partial x/\partial h & -\partial x/\partial \phi_t \\
\partial x/\partial x & \partial x/\partial h & -\partial x/\partial \phi_t \\
\partial x/\partial x & \partial x/\partial h & -\partial x/\partial \phi_t \\
\partial x/\partial x & \partial x/\partial h & -\partial x/\partial \phi_t \\
\end{pmatrix} = 0
\]

\[
\Delta_2 = \begin{pmatrix}
\partial y/\partial x & \partial y/\partial h & -\partial y/\partial \phi_t \\
\partial y/\partial x & \partial y/\partial h & -\partial y/\partial \phi_t \\
\partial y/\partial x & \partial y/\partial h & -\partial y/\partial \phi_t \\
\partial y/\partial x & \partial y/\partial h & -\partial y/\partial \phi_t \\
\end{pmatrix} = 0
\]

\[
\Delta_3 = \begin{pmatrix}
\partial z/\partial x & \partial z/\partial h & -\partial z/\partial \phi_t \\
\partial z/\partial x & \partial z/\partial h & -\partial z/\partial \phi_t \\
\partial z/\partial x & \partial z/\partial h & -\partial z/\partial \phi_t \\
\partial z/\partial x & \partial z/\partial h & -\partial z/\partial \phi_t \\
\end{pmatrix} = 0
\]

\[
\Delta_4 = \begin{pmatrix}
\partial x/\partial x & \partial x/\partial h & -\partial x/\partial \phi_t \\
\partial x/\partial x & \partial x/\partial h & -\partial x/\partial \phi_t \\
\partial x/\partial x & \partial x/\partial h & -\partial x/\partial \phi_t \\
\partial x/\partial x & \partial x/\partial h & -\partial x/\partial \phi_t \\
\end{pmatrix} = 0
\]

The equality of the fourth sub-determinant \( \Delta_4 = 0 \) (based on the first three rows of matrix \( K \)) yields Equation (32) of meshing and is, therefore, not considered. Once Equation (38) is satisfied, undercutting will occur. If only one of the three Equations (34)–(37) is satisfied, singularities will appear.

\[
g(x, h, \phi_t) = \Delta_1^2 + \Delta_2^2 + \Delta_3^2 = 0
\]
Both Equations (21) and (38) are used to solve the position of undercutting on the tooth surface. It should be mentioned that the search method is applied to solve the parameter $x$ from the tip to the root of the shaper. The first value obtained is the tooth profile parameter of the minimum radius that does not undercut. Substitute the parameter $x$ into the working tooth surface Equation (28) to obtain the position coordinates of the key points. The minimum inner radius $R_i$ to avoid undercutting can be obtained as

$$R_i = \sqrt{x_i^2 + y_i^2}$$  \hspace{1cm} (39)

3.2. Pointing

Compared with cylindrical gears, the top width of face gears is not constant. In the enveloping process, the shaper tooth tip will cause the tooth surfaces on both sides of the tooth to intersect at a point, so the outer diameter tooth of the face gear is restricted. To accurately determine the position of the cusp, it is generally necessary to consider the equations of the tooth surfaces on both sides. However, due to the symmetrical distribution of the tooth surfaces on both sides, Equation (40) is adopted to analyze the tooth surface on one side.

$$\begin{align*}
x_f(x^{(f)}, h, \phi_f) &= 0 \\
z_f(x^{(f)}, h, \phi_f) &= h_a - r_s \\
f(x^{(f)}, h, \phi_f) &= 0
\end{align*}$$  \hspace{1cm} (40)

After solving the cusp position parameters, they are substituted into the face gear working tooth surface equation to obtain the position coordinates of the cusp. The outer radius $R_o$ of the tip of the face gear can be expressed as

$$R_o = x_s \sin \phi_s \sin \phi_f + y_s \cos \phi_s \sin \phi_f + z_s \cos \phi_f$$  \hspace{1cm} (41)

Considering the undercutting and the pointing, the tooth width $W_t$ is defined as

$$W_t = R_o - R_i$$  \hspace{1cm} (42)

3.3. Analysis of Tooth Width

The characteristics of the tooth width are widely concerned while designing the tooth profile of the face gear, which is related to the carrying capacity of the gear. The basic design parameters of the proposed face gear drive are listed in Table 1, and the coefficients of the contact path function as shown in Table 2.

| Table 1: The basic design parameters of the face gear drive. |
|---|---|
| **Design Parameters** | **Value** |
| Module $m$ (mm) | 4 |
| Tooth number of pinion $N_p$ | 30 |
| Tooth number of shaper $N_s$ | 32 |
| Tooth number of face gear $N_f$ | 90 |
| Addendum coefficient $h^*$ | 1.0 |
| Head clearance coefficient $c^*$ | 0.25 |
Table 2. Coefficients of the contact path function.

<table>
<thead>
<tr>
<th>Design Parameters</th>
<th>Coefficient $a_1$</th>
<th>Coefficient $a_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D-1</td>
<td>$-\tan 18^\circ$</td>
<td>-0.001</td>
</tr>
<tr>
<td>D-2</td>
<td>$-\tan 19^\circ$</td>
<td>-0.001</td>
</tr>
<tr>
<td>D-3</td>
<td>$-\tan 20^\circ$</td>
<td>-0.001</td>
</tr>
<tr>
<td>D-4</td>
<td>$-\tan 21^\circ$</td>
<td>-0.001</td>
</tr>
<tr>
<td>D-5</td>
<td>$-\tan 22^\circ$</td>
<td>-0.001</td>
</tr>
<tr>
<td>D-6</td>
<td>$-\tan 20^\circ$</td>
<td>0</td>
</tr>
<tr>
<td>D-7</td>
<td>$-\tan 20^\circ$</td>
<td>-0.0005</td>
</tr>
<tr>
<td>D-8</td>
<td>$-\tan 20^\circ$</td>
<td>-0.0015</td>
</tr>
<tr>
<td>D-9</td>
<td>$-\tan 20^\circ$</td>
<td>-0.002</td>
</tr>
</tbody>
</table>

From the above tables, the parameter groups D-1 to D-5 were used to analyze the effects of coefficient $a_1$ on the tooth width. Similarly, through the parameter groups D-6 to D-9 and D-3, the effects of the coefficient $a_2$ on these geometric features were studied, which are shown in Figure 5. It can be found that the inner radius and the outer radius increased, but the tooth width of the face gear decreased, as the coefficients $a_1$ and $a_2$ increased.

![Figure 5. Effects of $a_1$ and $a_2$ on tooth width characteristics.](image-url)
Moreover, from Figure 5b, it can be seen that the tooth width of the proposed gear drive D-3 is 8.2%, greater than that of the general gear drive D-6, due to the smaller inner radius of the proposed gear drive, as illustrated in Table 3.

Table 3. Comparison of two face gears tooth width.

<table>
<thead>
<tr>
<th>Tooth Width Parameters</th>
<th>The Proposed Face Gear(D-3)</th>
<th>The General face Gear(D-6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner radius (mm)</td>
<td>170.490</td>
<td>173.059</td>
</tr>
<tr>
<td>Outer radius (mm)</td>
<td>203.135</td>
<td>203.231</td>
</tr>
<tr>
<td>Tooth width (mm)</td>
<td>32.645</td>
<td>30.172</td>
</tr>
</tbody>
</table>

4. Sliding Ratio Characteristics
4.1. Explicit Solution for Meshing Point

Generally, the face gear drive engages in the form of point contact, which can avoid eccentric load. In this paper, a pinion was chosen with 1–3 teeth less than the shaper as the driving wheel of the face gear pair, which is the same as that of the general face gear drive [28]. At present, the common idea for solving the contact points of the tooth surface is to establish the assembly relationship between the shaper, the pinion, and the face gear. The movement relationship between the pinion and the shaper is regarded as an imaginary internal meshing. Then, in the process of producing, the tooth surfaces of the shaper and face gear are in line contact at any time; the contact lines of shaper and face gear are denoted as $L_{SF}$ and $L_{S}$, as displayed in Figure 6. Similarly, the instantaneous contact lines for the meshing of the shaper and the pinion are denoted as $L_{sp}$ and $L_{sp}$, respectively. Therefore, when the pinion meshes with the face gear, the contact that forms between the tooth surfaces is point contact. Additionally, the meshing points of face gear can be obtained by calculating the intersection point of the contact lines $L_{SF}$ and $L_{sp}$, supplemented by coordinate transformation. This involves the solution of a system of equations, but the solution requires harsh initial values, which brings inconvenience to the solution [29]. Therefore, an explicit solution method is described below.

![Figure 6. Instantaneous meshing lines on the shaper.](image)

As for the above-mentioned meshing process, which is expressed in Equation (21), the parameter $h$ can be obtained as follows:

$$ h = \frac{C}{A \sin(\phi_0 \pm \phi_1) + B \cos(\phi_0 \pm \phi_1)} $$(43)

with

$$ A = 1 - (a_1x + a_2x^3 + r_s)(a_1^2 + 1 + 4a_1a_2x^2 + 3a_2^2x^4) / r_s $$ (44)

$$ B = -(a_1 + 3a_2x^3) - (a_1^2x + x + 4a_1a_2x^3 + 3a_2^2x^5) / r_s $$ (45)

$$ C = [-x - (a_1x + a_2x^3 + r_s)(a_1 + 3a_2x^3)] / i_{fi} $$ (46)
\[
\varphi_0 = \frac{- \left( a_1^2 + 1 \right) x + \frac{4}{3} a_1 a_2 x^3 + \frac{3}{5} a_2^2 x^5}{r_s} + \frac{\pi}{2N_s} 
\]

Here, it contains two unknown quantities \(x\) and \(\varphi_s\).

According to the meshing relationship between the shaper and the pinion, the relationship between the parameter \(x\) and rotation angle \(\varphi_s\) can be determined with Equation (47). Therefore, Equation (43) is further defined as Equation (48), and therefore, the rotation angle \(\varphi_s\) of the pinion can be expressed, which is the key point.

\[
\varphi_s = \frac{(a_1^2 + 1)x + \frac{4}{3} a_1 a_2 x^3 + \frac{3}{5} a_2^2 x^5}{r_s} - \frac{N_s}{2\pi} = -\varphi_0 
\]

Due to \(\phi_s/\phi_p = r_p/r_s\), the rotation angle \(\phi_p\) of pinion can be determined. Therefore, it is necessary to solve the value range of \(x\). Combining Equations (7)–(9), it can be known that the parameters \(x\) of meshing point at the tooth tip (\(x = x_c\)) and root (\(x = x_d\)) of the pinion can be obtained as follows:

\[
x_i^2 + y_i^2 = (r_p + h_o)^2 \quad (49)
\]

\[
a_4 x + a_2 x^3 = -h_a \quad (50)
\]

Where \(x_1\) and \(y_1\) are defined by replacing \(r_i\) with \(r_p\) in Equation (7).

Consequently, according to \(x \in [x_c, x_d]\), the range of angle \(\phi_p\) can be solved by Equation (47) and gear ratio, and \(h\) also can be obtained by Equation (48). On the other hand, based on the range of angle \(\phi_p\), for each given angle \(\phi_p\), the corresponding \(x\) can be solved by Equation (47) and gear ratio, and then \(h\) also can be obtained by Equation (48). Thus far, the explicit solution of the meshing points for the tooth surface is completed.

Using the calculation method of the tooth surface meshing point, the meshing points of the two face gears were carried out, as shown in Figure 7. Moreover, the effectiveness of the proposed method was verified by the intersection of the lines \(L_{SF}\) and \(L_{SP}\) (by coordinate transformation). The coordinate positions of the tooth surface meshing points of the proposed face gear were similar to those of the general face gear (the pinion was involute spur gear).
4.2. Solution of Sliding Ratio

The sliding ratio between the pinion $\Sigma_p$ and face gear $\Sigma_f$ can be interpreted, in a particularly short time, as the ratio of the relative arc length of the two surfaces sliding over the arc length of the tooth surface, as shown in Figure 8. At a certain instant, the two surfaces mesh at point $M$. After a short period of time, $\Sigma_p$ and $\Sigma_f$ move to $\Sigma_p'$ and $\Sigma_f'$, respectively. Moreover, the two surfaces mesh at point $M'$, and $MM'$ is the meshing track. The corresponding points of $M'$ on the $\Sigma_p$ and $\Sigma_f$ are $M_p$ and $M_f$.

![Figure 8. Relative sliding of tooth surfaces.](image)

In a fixed coordinate system, $v^{(p)}$ and $v^{(f)}$ are the speeds of the two gears at the meshing point $M$. Similarly, $v_{t}^{(p)}$ and $v_{t}^{(f)}$ parameters represent the moving speed of point $M$ along the meshing lines of $MM_p$ and $MM_f$, which is also the velocity along the common tangent plane. $v_{n}^{(p)}$ and $v_{n}^{(f)}$ parameters represent the velocity along the common normal, both of which have the same mode length and direction. These vectors have the following relationship:

$$\theta_p = \arccos \frac{v^{(p)} \cdot n_0}{|v^{(p)}||n_0|}$$  \hspace{1cm} (51)

$$\theta_f = \arccos \frac{v^{(f)} \cdot n_0}{|v^{(f)}||n_0|}$$  \hspace{1cm} (52)

$$v^{(p)}_t = v^{(p)} \sin \theta_p$$  \hspace{1cm} (53)

$$v^{(f)}_t = v^{(f)} \sin \theta_f$$  \hspace{1cm} (54)

where $n_0$ is the common normal vector at meshing point; $\theta_p$ ($\theta_f$) is the angle between $v^{(p)}$ ($v^{(f)}$) and $n_0$. 

**Figure 7.** Meshing points of two face gears.
Thus, combined with the definition of sliding ratio, the sliding ratios \( d_p \) and \( d_f \) of \( \Sigma_p \) and \( \Sigma_f \) are expressed as follows:

\[
\delta_p = \lim_{MM_f \to 0} \frac{MM_p - MM_f}{MM_p} = \frac{V_{f}^{(p)} - V_{f}^{(f)}}{V_{f}^{(p)}} = 1 - \frac{V_{f}^{(p)}}{V_{f}^{(f)}}
\]

\[
\delta_f = \lim_{MM_f \to 0} \frac{MM_f - MM_p}{MM_f} = \frac{V_{f}^{(f)} - V_{f}^{(p)}}{V_{f}^{(f)}} = 1 - \frac{V_{f}^{(p)}}{V_{f}^{(f)}}
\]

Here, \( v^{(p)} \) and \( v^{(f)} \) need to be solved. The solution of them is described below.

As evident from Figure 4, in the fixed coordinate system \( S_f(0, y_0, z_0) \), the vector equation of the engagement point \( M \) is \((x, 0.5 mN_p a_1 x + a_2 x^3 - r_p)\), and the angular velocity vectors of the pinion and face gears are \((0, w_p, 0)\) and \((0, 0, w_f)\).

\[
v_p = \omega_p \times r_M = \omega_p [(a_1 x + a_2 x^3 - r_p)i - xk]
\]

\[
v_f = \omega_f \times r_M = -\omega_f [r_f i - xj]
\]

The common normal vector \( n \) at that point is \((x, 0, a_1 x + a_2 x^3)\), and inserting \( n \), Equations (57) and (58) into Equations (51)–(54) yields

\[
v_{f}^{(p)} = \omega_{p} \sqrt{\frac{[x^2 + (a_1 x + a_2 x^3)^2] \cdot [(a_1 x + a_2 x^3 - r_p)^2 + x^2] - r_p^2 x^2}{[x^2 + (a_1 x + a_2 x^3)^2]}}
\]

\[
v_{f}^{(f)} = \omega_{f} \sqrt{\frac{[x^2 + (a_1 x + a_2 x^3)^2] \cdot [r_f^2 + x^2] - r_f^2 x^2}{[x^2 + (a_1 x + a_2 x^3)^2]}}
\]

At this point, substituting \( v^{(p)} \) and \( v^{(f)} \) into Equations (55) and (56), the sliding ratios of pinion and face gear can be further expressed as

\[
\delta_p = 1 - \frac{r_f \sqrt{(x^2 + y^2)(y^2 - 2r_p y + x^2) + r_p^2 y^2}}{r_p \sqrt{(x^2 + y^2)x^2 + r_p^2 y^2}}
\]

\[
\delta_f = 1 - \frac{r_f \sqrt{(x^2 + y^2)x^2 + r_f^2 y^2}}{r_f \sqrt{(x^2 + y^2)(y^2 - 2r_p y + x^2) + r_f^2 y^2}}
\]

### 4.3. Analysis of Sliding Ratio

According to Tables 1 and 2, the parameter groups D-1 to D-5 were used to analyze the effects of coefficient \( a_1 \) on the sliding ratio. Similarly, the parameter groups D-6 to D-9 and D-3 were applied to study the effects of the coefficient \( a_2 \) on the sliding ratio, the results of which are shown in Figure 9. It can be seen that, as the coefficient \( a_1 \) or \( a_2 \) decreased, the sliding ratio of face gear drive decreased.
Moreover, the maximum sliding ratios (absolute value) of this gear drive were lower than those of the general gear drive, as displayed in Table 4, through groups D-3 and D-6.

Table 4. Comparison of sliding ratio for the two gear drives.

<table>
<thead>
<tr>
<th>Sliding Ratio</th>
<th>The Proposed Face Gear Drive (D-3)</th>
<th>The General Face Gear Drive (D-6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root of tooth</td>
<td>-0.6041</td>
<td>-0.3931</td>
</tr>
<tr>
<td>Top of tooth</td>
<td>0.2822</td>
<td>0.3766</td>
</tr>
</tbody>
</table>

Given a face gear drive with $a_1 = -\tan 20^\circ$, $a_2 = -0.001$, $m = [2, 2.5, 3, 3.5, 4]$ mm, and with other parameters being the same as Table 1, the sliding ratio of face gear pair were calculated, as shown in Figure 10. With the pinion parameters kept unchanged, and given the gear ratio as $i_{fr} = [2, 2.5, 3, 3.5, 4]$, the sliding ratio of the gear pair under different transmission ratio conditions are displayed in Figure 11. From Figures 10 and 11, it can be deduced that, as the gear ratio or modulus increased, the sliding ratio of the proposed gear drive decreased.
Figure 11. Effect of gear ratio \(i_{gf}\) on sliding ratio.

5. Model Validation

5.1. Modeling of Face Gear Pair

Generally, it is difficult to obtain the tooth surface model of a face gear pair directly from 3D software due to the complexity of the tooth surface. Commercial software CATIA V5 (produced by Dassault System) and MATLAB R2019a (produced by MathWorks) can be used for the parameterized modeling of complex surfaces, which mainly depends on the numerical calculation of MATLAB software and the accurate modeling of CATIA software for complex surfaces [30].

To realize the parametric modeling of the proposed pinion, the key steps are as follows:

1. Parameter calculation: the ranges of parameters \(x\) corresponding to the working and the fillet tooth profile were calculated according to Equations (8) and (9);

2. Solving tooth profile coordinates: the tooth surface coordinates \(x, y\) of the working tooth profile and the transition tooth profile were calculated according to Equations (7) and (10);

3. Data output: the point data stored in the matrix were outputted to a specific Excel file. Then, they were imported into a 3D modeling software to continue the subsequent tooth surface modeling process. Additionally, the specific steps are shown in Figure 12a.

Similarly, the key steps for accurate modeling of face gear are as follows:

1. Parameter calculation: the parameters corresponding to the minimum inner radius (to avoid undercutting) and the maximum outer radius (to avoid pointing) were calculated according to Equations (36) and (39), and then the range of \(z\) was determined corresponding to the tooth height;

2. Discrete tooth surface coordinates: coordinate \(y\) was discretized along the tooth width direction to obtain a series of \(y\); coordinate \(z\) was discretized along the tooth height direction to obtain a series of \(z\);

3. Solving the tooth surface coordinate \(x\): According to \(y\) and \(z\), coordinates obtained by discretization, the surface equation was substituted to obtain the coordinate \(x\);

4. Data output: the point data stored in the matrix were cyclically outputted to a specific Excel file. Then, they were imported into a 3D modeling software to continue the subsequent tooth surface modeling process. Additionally, the specific steps are shown in Figure 12b.

Finally, the pinion and face gear models were used to build the gear pair assembly model, as shown in Figure 12c.
Figure 12. Modeling of the proposed face gear pair.

5.2. Motion Simulation

Motion simulation can be used to judge the rationality of the meshing pair movement, and also to determine whether there is interference between components [31]. In this study, based on the movement relationship between the two parts, motion simulation was carried out, as displayed in Figure 13. It can be observed that there was no interference between the two gears, as the rotation angle $\phi$ of the pinion changed. When a pair
of gear teeth entered into meshing, the other pair of teeth had not yet exited meshing. Thus, the necessary continuity of action was ensured.

![Figure 13. Motion simulation of the proposed face gear pair.](image)

6. Conclusions

This paper presented a design method for face gear drive associated with a low sliding ratio. Based on the analysis and results, the conclusions are as follows:

1. Based on the provided contact path function \( y = ax + ax^3 \), general mathematical models of a pair of the conjugated spur gear and rack with low sliding ratio were given, including the working and the fillet profiles;

2. Using the established spur gear as the shaper to envelope the face gear, the mathematical models of the working and the fillet surfaces of the face gear were obtained, and the tooth width restriction conditions were established, including the undercutting of the inner and the pointing of the outer. The results show that the tooth width of the proposed face gear was larger than that of general face gear, due to the smaller inner radius. Furthermore, as the coefficients \( a_1 \) or \( a_2 \) increased, the inner radius and the outer radius increased; conversely, the tooth width of the face gear increased.

3. An explicit solution method for the contact point of the tooth surface of the face gear pair was provided, which is more convenient than the general solution method. Additionally, this method is also suitable for solving the meshing points of the general face gear drive, in which case, let \( a_2 = 0 \);

4. The calculation method of the sliding ratio of the spatial meshing of the face gear pair was studied, and the effects of design parameters on the sliding ratio were analyzed. The sliding ratio of the proposed face gear pair was lower than those of the general face gear pair. Moreover, as the coefficient \( a_1 \) or \( a_2 \) decreased, the sliding ratio of the face gear drive decreased.

5. The rationality of the proposed face gear was demonstrated by accurate 3D modeling and motion simulation. However, to determine the optimum design and precision machining, further studies are needed.

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