Model Predictive Traffic Control by Bi-Level Optimization

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Abstract: A bi-level model for traffic signal optimization is developed. The model predictive framework is applied for traffic control in an urban traffic network. The potential of the bi-level formalization is used to increase the space of control influences with simultaneous evaluation of the green light and cycle durations. Thus, the increased control space allows more traffic parameters to be considered, such as vehicles queues and traffic flows. A particular modification of the bi-level control is applied for the synchronization of the traffic lights in the network. The model predictive approach is used for the real-time management of the traffic in the network. The control implementations are constrained by the shortest evaluated cycle. Thus, a synchronization of the traffic lights is applied for the minimization of the queues and maximization of the outgoing flows from the network. The bi-level model has been numerically tested on a set of intensive crossroads in Sofia. The numerical simulations prove the superiority of the developed bi-level control in comparison with the classical optimization of queue lengths.

Keywords: bi-level optimization; traffic control; signalized intersections; traffic flows; queue length

1. Introduction

The topic of traffic behavior improvement in urban areas remains to attract researchers’ interests. The problem stays relevant due to the continuous increase of vehicle flows and the enormous breakdowns from traffic jams, increased fuel consumption, negatives from emission pollutions and their consequences to climate changes. In brief, traffic control targets the optimization of the transport flows, resulting in a decrease of vehicle congestions and waiting times, and maximization of the flows crossing the transport network, although this is subject to the construction limitations of the roads.

Different approaches have been investigated for traffic control for a long time [1–3]. The control influences, which can be used for the traffic control, are the duration of the green lights per phase, the cycle duration, and the offset between a set of neighbor intersections. Mainly the durations of the traffic lights are applied as control variables in the traffic optimization problems. This causes the control space to be narrow, which restricts the management of the transport system to respect various constraints and/or objectives. If the control space is extended with these three possible influences, the traffic behavior can be improved to respect more requirements for the traffic management. The simultaneous usage of these different control influences in a common optimization problem makes it non-analytically defined and complex for a solution. Due to this complexity, the optimization problems of traffic management apply as control variables only the green lights. Our research targets the modeling of optimal control, which implements both the green lights and cycle durations on the crossroad junctions as control variables and problem solutions simultaneously. The extension of the control set allows more goals for traffic control to be considered for the minimization of queue lengths and maximization of traffic flows. The integration of the two control variables is performed with the usage of a hierarchical definition of a traffic optimization problem. We apply a bi-level formalization, which allows two goal functions to be considered in hierarchical order and to achieve better traffic
parameters in comparison with the classical optimization with one goal function. This research hypothesis was tested in a city network in Sofia. The results, presented below, illustrate the positive potential, which the bi-level optimization formalism demonstrates.

2. Literature Review

The traffic control approaches, and traffic signals control have been classified into three types of strategies [2]:

- Control of isolated intersection,
- Traffic light control with fixed time settings,
- Traffic-responsive coordinated and adaptive signal control.

All these strategies implement the traffic lights durations as control influences. The cycles are not considered as tools for adaptive real-time control of the traffic behavior. For the evolution of the traffic signal control, one can refer to [4]. The simplest form of control is applied to isolated intersections under slow fluctuations of the traffic intensity [5]. Such control targets optimal settings in order to decrease the waiting time in front of the crossroad [6]. This isolated crossroad control with complication was applied as fixed time coordination control in a network of junctions. Traffic plans are evaluated off-line for each junction and the light durations are based on historical data about the traffic demands and statistical evaluations of the origin–destination matrices.

The traffic responsible control strategy is implemented in various control systems, which are continuously cited in the references: OPAC (Optimal Policies for Adaptive Control) [5]; SCOOT (Split Cycle Offset Optimization Technique) [7]; SCATS (Sydney Coordinated Adaptive Traffic System) [5]; RHODES (Real time Hierarchical Optimized Distributed Effective System) [8]; PRODYN (the abbreviation comes from the terms Dynamical Programming) [9]; UTOPIA (Urban Traffic Optimization by Integrated Automation) [10]; TUC (Traffic-responsive Urban Control) [11].

The formal relations, which are used for the definition of the optimization problems, are based on the principles and models, which are taken from fluid dynamics, vehicle-following models, and couple lattice models [12]. Attempts to use different models and methods for modeling the traffic behavior, such as the bio-inspired models and hybrid artificial neural network optimization model, are under consideration, respectively, in [13] and [14]. A challenge for the traffic optimization is the lack of analytical relations between parameters such as delays and off-sets [15], as well as the actual throughput as a function of the green time of movement, cycle length, and time, which is obtained with the simulation environment VISSIM [16]. The advances in technology allow to forecast arrival and discharge rates of traffic flows in real time [17].

Our approach follows the analytic descriptions of the traffic behavior and the definition of the traffic optimization problem. An extended overview of the different forms of traffic optimization problems is given in [18]. The provided analysis classifies the content of the traffic control problems about their objectives and constraints. The main control variables and problem arguments are restricted up to the green lights’ durations. Our specific place for problem characteristics is the simultaneous application of two objectives for the minimization of cycle length and maximization of traffic flows, which are not presented in this overview. Additionally, we extend the control space with both cycle lengths and green durations as arguments in an integrated hierarchically defined optimization problem.

For the definition of our hierarchical optimization problem in this research, we apply relations based on the models of fluid dynamics. These models are applicable mainly to freeway traffic control with ramp metering [19] and applications of the store-and-forward approach [20,21]. For relevant applications of ramp metering, one can refer to [22]. The store-and-forward model is intensively exploited and complicated for usage in centralized and or decentralized control schemes [23]. Due to its simplicity, store-and-forward modeling is applied in traffic control algorithms [24,25].

Store-and-forward modeling is applied for obtaining different control gains in traffic optimization. In [26], green wave optimization was the main target of the control. In [27],
traffic signal coordination in two-way arterial directions is formalized and solved. The control approaches for traffic management become more complex. A representative for such complications is the model predictive control, which simultaneously applies adaptation of the traffic parameters to each control step [11,28]. Distributed control approaches are implemented in [24,25]. The intelligent transportation approaches started to apply machine learning methods such as reinforcement learning [29]. The stochastic character of the traffic demands is explicitly considered and formalized in the traffic control problem [30].

In general, for the traffic control, there are not many influences: the green lights (or the relative split towards the traffic lights cycle), the cycle duration, which contains all phases of the lights, and the offset as time differences between successive intersections [31]. In the cases of traffic signal control, mostly the green light duration is optimized [19,29,30]. The durations of the traffic cycles are mainly evaluated on statistical considerations, analyzing available historical data of the traffic intensities [32,33].

This research targets the development of such a control strategy, which simultaneously evaluates and implements both types of control influences: the green lights and cycle durations. The extended set of control influences gives the opportunity to optimize more parameters for the traffic behavior. Hence, traffic control is formalized as a bi-level optimization problem. This formalism has the potential to control more than one optimization goal and extend the set of traffic constraints.

For the case of consistent presentation of the bi-level problem definition, here, the roots of store-and-forward modeling are derived and presented. This is needed to prove the ability to incorporate the two control influences: the green lights and cycles in a common optimization problem.

3. Methodology
3.1. Theoretical Background of Store-and-Forward Modeling

The cell transmission model is the backbone, over which the freeway traffic control is founded. It originates from the works of [34] and was sequentially improved and complicated in [35,36]. In [37], traffic modeling was performed in the case of dynamical variation optimization. The cell transmission model decomposes the route on sequential cells, $i - 1, i, i + 1$, which have appropriate parameters, given in Figure 1. In this model, the outgoing flow of the $i$th cell is equal to the incoming flow of the $(i + 1)$th cell or $q_{i+1}^{out}(t) = q_{i}^{in}(t)$.

\[ x = i - 1 \quad x = i \quad x = i + 1 \]

**Figure 1.** Cell element and its characteristics.

The sequence of cells substitutes the distance $x$ with the number $i$ of the cell. The parameters which characterize the cell are:
- $q_i(t)$—input flow of the vehicles in time $t$, [veh/time];
- $n_i(t)$—the number of vehicles in the cell $i$ at time $t$, [veh];
- $\rho_i(t)$—the density of the traffic flow in time $t$, [veh/distance].

The cell transmission model is based on the relation for flow continuity in liquids:

\[ \frac{\partial q}{\partial x} + \frac{\partial \rho}{\partial t} = 0. \]
This general relation is substituted by the increase of each partial derivative or:

\[ \frac{\partial \rho}{\partial t} \approx \frac{\Delta \rho}{\Delta t} = \frac{\rho_i(t+1) - \rho_i(t)}{T}, \]  

where \( \rho \) has dimension \( \lbrack \text{veh/km} \rbrack \):

\[ \frac{\partial q}{\partial x} \approx \frac{\Delta q}{\Delta x} = \frac{q(t+1) - q_i(t)}{\Delta x} = \frac{q(t+1) - q_i(t)}{l_i}, \]

where \( q_i, \Delta q \) have dimension \( \lbrack \text{veh/h} \rbrack \).

If \( \Delta t \) is the control step in discrete time, then \( T \) is the duration of the control cycle. The cell length \( l_i \) corresponds to the number of cells in the transmission model. In general, \( l_i \) [km] is the length of the cell.

The difference between the flow densities \([\rho_i(t+1) - \rho_i(t)]\) is numerically expressed by the difference of the number of vehicles \( n_i(t) \), which enter and leave the cell with length \( l_i \) or:

\[ \Delta \rho = \frac{\rho_i(t+1) - \rho_i(t)}{T} = \frac{n_{i+1}(t) - n_i(t)}{l_i}, \]  

Hence, using Equations (2) and (3), it follows:

\[ \frac{\rho_i(t+1) - \rho_i(t)}{T} = \frac{q_i(t+1) - q_i(t)}{l_i} \]

or:

\[ \rho_i(t+1) = \rho_i(t) + \frac{T}{l_i} [q_i(t) - q_{i+1}(t)] \quad [\text{veh/km}] \]  

(4)

Because the density is equal to the number of vehicles for a distance \( l \) (\( \rho = n/l \)), Equation (4) takes the form of:

\[ \frac{n_i(t+1) - n_i(t)}{T_l_i} = \frac{q_i(t+1) - q_i(t)}{l_i} \]

or:

\[ n_i(t+1) = n_i(t) + T_l [q_i(t) - q_{i+1}(t)] \quad \text{[veh]} \]  

(5)

Equation (4) is the main one, which is applied in the LWR (Lighthill–Whitham–Richards) [38,39] modeling for the ramp metering control. Equation (5) is used for the application of store-and-forward modeling.

Our research uses the last relation. Due to its simplicity, it is applied in analytically defined traffic control problems. Unfortunately, the size of the network makes the overall optimization problem quite difficult due to the mutual interconnections between the transport crossroads. The meaning of the term \( T_l q_i(t) \) concerns the inflow in a junction, and the corresponding \( T_l q_{i+1}(t) \) refers to the outgoing flow. These terms can be expressed with the green phases and cycle durations in a control problem. Our approach is to decompose these control relations in a bi-level formal description. This can give an opportunity to define an optimization problem, which optimizes simultaneously both variables as green lights and cycle durations according to different goal functions in a hierarchical order.

### 3.2. Bi-Level Formalization in Traffic Control Problems

The bi-level formalization is a comparatively new way of extension and definition of optimization problems. The classical optimization problem contains an analytically defined goal function. The target of the optimization is to find an extreme of the goal function towards a set of variables. However, the latter has to belong to a feasible region, defined by a set of analytically defined constraints. The bi-level formalization makes an extension to the definition of the feasible area of the initial optimization problem. It
includes an additional optimization problem, which is a part to the feasible domain of the initial problem. Thus, two optimization problems are interconnected in hierarchical order: upper- and lower-level optimization problems. The solutions of the upper-level problem change parameters in the goal function and/or constraints of the lower-level problem and vice-versa. The benefit of this bi-level problem definition comes from the extended set of optimal variables, which are given from the upper- and lower-level problems. For technical reasons, the extended set of controls allows achieving more goals in the control process and respectively satisfying an extended set of constraints. This is beneficial for the behavior of the control object. Particularly, for the domain of transportation, this can improve the traffic dynamics by increasing the intensity of the traffic flows, decreasing the waiting time of vehicles, to influence driver behavior.

The bi-level formalization is currently intensively applied for the solution of different tasks, related to the effective management of transportation resources and for the identification of transport parameters.

In [40], the bi-level optimization framework is applied for the estimation of the origin-destination matrix, which is needed for the definition of the inflows in urban networks. In [41], bi-level formalization was used for coordinating the vehicle motion at roundabouts. In [42], the signals for traffic control were evaluated in a bi-level problem, according to the requests of crossing pedestrians. The bi-level optimization program has been applied to cope with the dynamically changes in time of the urban traffic flows [43]. An attempt for traffic signal modeling and optimization using bi-level programming was made in [44]. Because the defined problem is quite complex, heuristics were developed for finding a suboptimal solution.

Bi-level optimization was used in minimizing the pricing and carbon emissions in green transportation [45]. The policy for transport exploitation has been evaluated under multi-objective bi-level optimization [46]. The traffic signal optimization with bi-level formalization was used in [47].

The signal control for the case of transport of hazardous goods is also formalized by definition in a bi-level framework [48].

The bi-level optimization graphically can be presented as a two-level hierarchical system, of which the levels solve appropriate optimization problems. Both hierarchical levels influence the parameters of the other problem; see Figure 2.

![Figure 2](image-url)

**Figure 2.** A bi-level hierarchical optimization problem.

Using the notations for the upper-level problem for the arguments \( \mathbf{y} \), the goal function \( f_u(\mathbf{x}, \mathbf{y}) \), and the domain of the constraints \( S_u(\mathbf{x}, \mathbf{y}) \), both sets of relations \( f_u(\mathbf{x}, \mathbf{y}) \) and \( S_u(\mathbf{x}, \mathbf{y}) \) depend on a set of parameters \( \mathbf{x} \). The lasts are given as problem solutions of the lower-level optimization problem:

\[
\min_{\mathbf{x}} f_u(\mathbf{x}, \mathbf{y}) \\
\mathbf{x} \in S_u(\mathbf{x}, \mathbf{y})
\]

which in its turn is also parameterized by the set of solutions \( \mathbf{y} \) of the upper-level problem. Hence, both upper- and lower-level problems are interconnected by their solutions, and
they change each other’s parameters in the goal functions and constraints. The overall formal description of the bi-level problem can be given in the form:

\[
\min_y f_y(x, y)
\]

\[y \in S_y(x, y),\]

where:

\[
x \equiv \arg\min_x \{ \min f_x(x, y) \} \quad \left( x \in S_x(x, y) \right),
\]

where part of the constraints for \(y\) is given as a set of solutions of \(x\), from the lower-level problem. The advantages of this bi-level formalization come from the extended set of optimal solutions \((x, y)\), which allows the optimization to be performed by two goal functions in hierarchical order: \(f_y(x, y)\) and \(f_x(x, y)\), which simultaneously satisfy an enlarged set of constraints \(S_y(x, y) \cap S_x(x, y)\). Thus, the quality of the control can have better optimal properties.

For the particular case of traffic signal optimization, the variables \(y\) of the upper problem can correspond to the duration of the cycle, and the variables \(x\) of the lower level can be the green signals. Respectively, the upper-level problem can target the maximization of the traffic flows in the network, while the lower-level problem can minimize the queue lengths and, consequently, the waiting times. The analytical formalization of the traffic optimization problem with simultaneous optimization of green and cycle times is formally presented according to the network topology, corresponding to an area in Sofia.

4. Results

4.1. Traffic Network Topology

The network topology corresponds to a busy part of the traffic in Sofia, where business, administrative, and trading centers are situated. As there are several residential areas, a lot of people cross the streets of the network. We consider a traffic network consisting of eight crossroad sections with saturations \(s_k, k = 1, \ldots, 16\), which give the maximal throughput per direction, [veh/per unit time]; see Figure 3. We note the traffic flows as \(x_i, I = 1, \ldots, 30\). Each traffic flow can move straight ahead or turn to the right or to the left. The traffic light cycles and the green light durations are, respectively, \(y_j, j = 1, \ldots, 8\), and \(u_k, k = 1, \ldots, 16\). The traffic light cycles and the green light durations have fixed values now. This causes long vehicle queue lengths in front of junctions, leading to side effects such as air and noise pollution, slow driving, and delays, leading to economic and social losses.

Our goal is to decrease traffic jams by changing the duration of the traffic light cycles and green lights in accordance with an appropriate optimization problem. As the control variables are two types and the traffic flows pass through the linked crossroads of the network, the classical optimization is not appropriate. The last is suitable for an isolated crossroad and this is the known control practice. Here, integration and coordination of the unknown variables are needed. Our suggestion is to apply bi-level optimization as a suitable methodology for this control strategy. We have to determine the lower-level and upper-level optimization problem of the bi-level optimization according to Figure 2. Bi-level optimization is based on the store-and-forward model and traffic dynamics following the continuity of flows, given above.
We have in mind that the vehicles move straight ahead and turn to the left and right in the optimization problem with simultaneous optimization of green and cycle times. The analytical formalization of the traffic efficiency, administrative, and trading centers are situated. As there are several residential areas, the green signals. Respectively, the upper-level problem can target the maximization of eight crossroad sections with saturations and, consequently, the waiting times. The analytical formalization of the traffic networks. The advantages of this bi-level formalization come from the extended set of optimal solutions (lem). The traffic flows are evaluated for cycle duration, their values correspond to the number of flows of each direction to be bigger than the incoming to the junction traffic flow. Because the traffic flows are evaluated for cycle duration, their values correspond to the number of vehicles for direction \( i \), which formalizes the proximity of the model to reality:

\[
\begin{align*}
\mathbf{x}_1 &= \mathbf{x}_{10} + \mathbf{x}_{1in} - \mathbf{x}_{1out},
\end{align*}
\]

where the notation \( \mathbf{x}_{10} \) means vehicles for direction \( i = 1, 30 \) and “0” is the notation of the initial number of vehicles.

The incoming flow \( \mathbf{x}_{1in} \) depends on the duration of the traffic light cycle \( y_1 \), the road saturation \( s_1 \), and a coefficient \( a_3 \), which formalizes the proximity of the model to reality:

\[
\mathbf{x}_{1in} = a_3 s_1 y_1.
\]

The outgoing flow depends on the green light duration \( u_1 \) and the road saturation \( s_1 \). We have in mind that the vehicles move straight ahead and turn to the left and right in the vertical directions. The part of the turning cars is denoted by coefficient \( a_2 \), and because

Figure 3. Network topology.

4.2. Definition of the Lower-Level Problem

The lower-level optimization problem targets optimization of the duration of the green light of the traffic lights \( u_k \), \( k = 1, \ldots, 16 \), and superiority of the outgoing over the incoming flow of each junction of the network. The traffic light cycles are denoted by \( y_j \), \( j = 1, \ldots, 8 \). The amber light is 1/10th part of the traffic light cycle. For the first crossroad section, the duration of the green lights \( u_1 \) and \( u_2 \) are, respectively, for the horizontal and vertical direction. The sum \( u_1 \) and \( u_2 \) of the first traffic light represents 9/10th part of the cycle duration \( y_1 \). This condition can be formalized by the following equation:

\[
\begin{align*}
\mathbf{u}_1 + \mathbf{u}_2 &= 0.9\mathbf{y}_1.
\end{align*}
\]

The optimization goal is the maximization of the green light durations, satisfying a set of constraints. A part of the constraints represents a modification of the store-and-forward model, applied for each queue of the junction. In this model, the intention is the outgoing flow of each direction to be bigger than the incoming to the junction traffic flow. Because the traffic flows are evaluated for cycle duration, their values correspond to the number of vehicles which can pass or make queues in the network links.

In that manner, we formalize the requirement for decreasing the queue lengths in front of the junction. Let us describe the constraints for the first crossroad. According to the store-and-forward model and Equation (5), the queue length \( x_1 \) is determined by the initial number of vehicles \( x_{10} \) plus the number of incoming vehicles \( x_{1in} \) decreased by the number of the outgoing vehicles \( x_{1out} \). Equation (5) can be presented in the following form:

\[
\begin{align*}
x_1 &= x_{10} + x_{1in} - x_{1out},
\end{align*}
\]
there are two turning directions, the turning vehicles are $2a_2$. The outgoing traffic flow from the first crossroad in a horizontal direction from west to east is:

$$x_{1\text{out}} = (1 + 2a_2)s_1u_1.$$ 

The flows from the vertical direction are not considered, because they do not belong to the internal traffic in the network in Figure 3.

Equation (6) becomes:

$$x_1 = x_{1\text{in}} + a_3s_1y_1 - (1 + 2a_2)s_1u_1. \quad (7)$$

Our intention is for the outgoing flow to be bigger than the incoming flow in order to decrease the queue length $x_1$. This can be formalized by the inequality:

$$a_3s_1y_1 - (1 + 2a_2)s_1u_1 \leq -x_{1\text{in}}.$$ 

The outgoing flow is:

$$x_{3\text{in}} = s_3u_3 + 2a_2s_4u_4.$$ 

The outgoing flow is:

$$x_{3\text{out}} = (1 + 2a_2)s_1u_1.$$ 

The constraint formalizing the condition the outgoing flow of $x_3$ to be bigger than the incoming flow is:

$$s_3u_3 + 2a_2s_4u_4 - (1 + 2a_2)s_1u_1 \leq -x_{3\text{in}}.$$ 

By the same way, the rest traffic flows of the network are formalized. They represent a part of the constraints of the lower-level optimization problem.

The lower-level optimization problem is in the form:

$$\max_u u^T u$$

subject to:

$$a_3s_1y_1 - (1 + 2a_2)s_1u_1 \leq -x_{1\text{in}}$$

$$a_3s_2y_1 - (1 + 2a_2)s_2u_2 \leq -x_{2\text{in}}$$

$$s_3u_3 + 2a_2s_4u_4 - (1 + 2a_2)s_1u_1 \leq -x_{3\text{in}}$$

$$a_3s_2y_1 - (1 + 2a_2)s_2u_2 \leq -x_{4\text{in}}$$

$$s_1u_1 + 2a_2s_2u_2 - (1 + 2a_2)s_3u_3 \leq -x_{3\text{in}}$$

$$s_{12}u_{12} + 2a_2s_{13}u_{13} - (1 + 2a_2)s_4u_4 \leq -x_{6\text{in}}$$

$$s_5u_5 + 2a_2s_6u_6 - (1 + 2a_2)s_5u_5 \leq -x_{7\text{in}}$$

$$a_3s_4y_2 - (1 + 2a_2)s_4u_4 \leq -x_{8\text{in}}$$

$$s_3u_3 + 2a_2s_4u_4 - (1 + 2a_2)s_5u_5 \leq -x_{9\text{in}}$$

$$s_{14}u_{14} + 2a_2s_{13}u_{13} - (1 + 2a_2)s_6u_6 \leq -x_{10\text{in}}$$

$$s_7u_7 + 2a_2s_8u_8 - (1 + 2a_2)s_7u_7 \leq -x_{11\text{in}}$$

$$a_3s_6y_3 - (1 + 2a_2)s_6u_6 \leq -x_{12\text{in}}$$

$$s_5u_5 + 2a_2s_6u_6 - (1 + a_2)s_7u_7 \leq -x_{13\text{in}}$$
4.3. Definition of the Upper-Level Problem

The upper-level optimization problem targets optimization of the traffic light cycles, satisfying a set of constraints:

\[ \begin{align*}
&s_{16}u_{16} + 2a_2s_{15}u_{15} - 2a_2s_8u_8 \leq -x_{14} \\
& s_{9}u_{9} + a_2s_{10}u_{10} - (1 + a_2)s_{7}u_{7} \leq -x_{15} \\
& s_{7}u_{7} + a_2s_{8}u_{8} - (1 + a_2)s_{9}u_{9} \leq -x_{16} \\
& a_3s_9y_5 - (1 + a_2)s_9u_9 \leq -x_{17} \\
& a_3s_{10}y_5 - 2a_2s_{10}u_{10} \leq -x_{18} \\
& a_3s_{11}y_6 - (1 + a_2)s_{11}u_{11} \leq -x_{19} \\
& a_3s_{12}y_6 - (1 + 2a_2)s_{11}u_{11} \leq -x_{20} \\
& s_{13}u_{13} + 2a_2s_{14}u_{14} - (1 + 2a_2)s_{11}u_{11} \leq -x_{21} \\
& s_{4}u_{4} + 2a_2s_{3}u_{3} - (1 + 2a_2)s_{12}u_{12} \leq -x_{22} \\
& s_{11}u_{11} + 2a_2s_{12}u_{12} - (1 + 2a_2)s_{13}u_{13} \leq -x_{23} \\
& a_3s_{14}y_7 - (1 + 2a_2)s_{14}u_{14} \leq -x_{24} \\
& s_{15}u_{15} + 2a_2s_{16}u_{16} - (1 + 2a_2)s_{13}u_{13} \leq -x_{25} \\
& s_{6}u_{6} + 2a_2s_{5}u_{5} - (1 + 2a_2)s_{14}u_{14} \leq -x_{26} \\
& s_{13}u_{13} + 2a_2s_{14}u_{14} - (1 + 2a_2)s_{15}u_{15} \leq -x_{27} \\
& a_3s_{16}y_8 - (1 + 2a_2)s_{16}u_{16} \leq -x_{28} \\
& s_{15}y_8 - (1 + 2a_2)s_{15}u_{15} \leq -x_{29} \\
& 2a_2s_{7}u_{7} - (1 + 2a_2)s_{16}u_{16} \leq -x_{30} \\
& u_1 + u_2 = 0.9y_1 \\
& u_3 + u_4 = 0.9y_2 \\
& u_5 + u_6 = 0.9y_3 \\
& u_7 + u_8 = 0.9y_4 \\
& u_9 + u_{10} = 0.9y_5 \\
& u_{11} + u_{12} = 0.9y_6 \\
& u_{13} + u_{14} = 0.9y_7 \\
& u_{15} + u_{16} = 0.9y_8 \\
& 0 \leq u \leq 80
\end{align*} \]

Equation (8) has two unknown variables: the green light durations \( u_k, k = 1, \ldots, 16 \), and the cycle durations \( y_j, j = 1, \ldots, 8 \). The green light durations \( u_k \), are limited between 0 and 40 s from practical considerations for the considered network. The bi-level optimization solves Equation (8) towards \( u_k, k = 1, \ldots, 16 \), where \( y_j, j = 1, \ldots, 8 \), are solutions of the upper-level optimization problem, and during the solution of Equation (8), \( y_j, j = 1, \ldots, 8 \), are regarded as known parameters. The solutions of Equation (8) \( u_k, k =1, \ldots, 16 \), are transferred to the upper-level problem whereas for the upper-level problem they are regarded as known parameters.
This means that we can neglect the distance between the traffic lights and observe the 

\[ x \]

These values are applied for estimation of each queue length 

\[ u_i \]

problem), and green light duration 

\[ y \]

variants of the estimated optimization results:

\[ y \]

the following equations:

\[ x \]

ing to the store-and-forward model. The first queue length

\[ x \]

drives from one junction to the next is approximately the duration of a traffic light cycle. 

\[ x \]

short, and according to the included limits of changes of the variables in the problems’ 

\[ x \]

are used for updating both bi-level problems for the next iteration/control cycle. In this updating, the 

\[ x \]

Here, we describe this idea only for the first and second crossroads, but this is done for all 

crossroad couples of the network.

The network traffic control has the peculiarity of synchronization of the traffic lights 

\[ x \]

in order to provide a green wave of moving vehicles. In our network, the distances are 

\[ x \]

the neighbor junctions are short enough to consider that the estimated cycles and green 

\[ y \]

Here is the check of the durations of the estimated values of the cycles and green light 

\[ s \]

The cycle durations \( y \), are limited between 40 and 200 s from local normative requirements.

5. Solution of the Bi-Level Optimization Problem

The bi-level optimization problem is solved in an iterative manner. Each calculation 

\[ x \]

\[ x \]

\[ x \]

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Case I: \( y_1 = y_2 \). In this case, the values of the queue lengths are calculated according 

to the store-and-forward model. The first queue length \( x_1 \) is determined by Equation (7). The queue lengths of the first and second junctions are evaluated according to the following equations:

\[ x_1 = x_{10} + a_3 s_1 y_1 - (1 + 2a_2) s_1 u_1 \]

\[ x_2 = x_{20} + a_3 s_2 y_1 - (1 + 2a_2) s_2 u_2 \]

\[ x_3 = x_{30} + s_3 u_3 + 2a_2 s_4 u_4 - (1 + 2a_2) s_1 u_1 \]

\[ x_4 = x_{40} + a_3 s_2 y_1 - (1 + 2a_2) s_2 u_2 \]

\[ x_5 = x_{50} + s_1 u_1 + 2a_2 s_2 u_2 - (1 + 2a_2) s_3 u_3 \]

\[ x_6 = x_{60} + s_2 u_1 + 2a_2 s_1 u_1 - (1 + 2a_2) s_4 u_4 \]
Figure 5. The ratio between the cycle and green light durations when $y_1 > y_2$ (a) $u_1 < y_2$ and (b) $u_1 > y_2$.

In this case, the queue lengths are evaluated according to the relations:

$$x_1 = x_{10} + a_3s_1y_2 - (1 + 2a_2)s_1\min(u_1, y_2)$$
$$x_2 = x_{20} + a_3s_2y_2 - (1 + 2a_2)s_2\max(y_2 - (u_1 + 0.1y_1), 0)$$
$$x_3 = x_{30} + s_3u_3 + 2a_2s_4u_4 - (1 + 2a_2)s_3\min(u_1, y_2)$$
$$x_4 = x_{40} + a_3s_2y_2 - (1 + 2a_2)s_2\max(y_2 - (u_1 + 0.1y_1), 0)$$
$$x_5 = x_{50} + s_1\min(u_1, y_2) + 2a_2s_2\max(y_2 - (u_1 + 0.1y_1), 0) - (1 + 2a_2)s_3u_3$$
$$x_6 = x_{60} + s_1u_1 + 2a_2s_{12}u_{12} - (1 + 2a_2)s_4u_4$$
$$x_7 = x_{70} + s_3u_3 + 2a_2s_5u_5 - (1 + 2a_2)s_3u_3$$
$$x_8 = x_{80} + a_3s_4y_2 - (1 + 2a_2)s_4u_4$$

The estimation of the current calculated iteration/control values above and for the whole network becomes initial values $x_{io}$, $i = 1, \ldots, 30$, for the next iteration.

Case III: $y_1 < y_2$. Because $y_1 < y_2$, the green light duration of the first cycle $u_1$ and $u_2$ will be fully implemented. Here, the ratios $y_1$ and $u_3$ have to be compared. When $y_1 > u_3$, the implementation of $u_4$ is only a part of it: $u_4 = y_1 - (u_3 + 0.1y_2)$; see Figure 5a. When $y_1 < u_3$, then $u_4$ will not start; see Figure 5b.

Figure 5. Ratio between cycle and green light durations when $y_1 < y_2$, (a) $u_3 < y_1$, and (b) $u_3 > y_1$. 

$$x_7 = x_{70} + s_5u_5 + 2a_2s_5u_6 - (1 + 2a_2)s_3u_3$$
$$x_8 = x_{80} + a_3s_4y_2 - (1 + 2a_2)s_4u_4$$
In this case, the queue lengths are evaluated according to the relations:

\[
\begin{align*}
x_1 &= x_{10} + a_3 s_1 y_1 - (1 + 2a_2) s_1 u_1 \\
x_2 &= x_{20} + a_3 s_2 y_1 - (1 + 2a_2) s_2 u_2 \\
x_3 &= x_{30} + s_3 \min(u_3, y_1) + 2a_2 s_4 \max((y_1 - u_3 - 0.1 y_2), 0) - (1 + 2a_2) s_1 u_1 \\
x_4 &= x_{40} + a_3 s_2 y_1 - (1 + 2a_2) s_2 u_2 \\
x_5 &= x_{50} + s_1 u_1 + 2a_2 s_2 u_2 - (1 + 2a_2) s_3 \min(u_3, y_1) \\
x_6 &= x_{60} + s_1 u_{12} + 2a_2 s_{11} u_{11} - (1 + 2a_2) s_4 \max((y_1 - u_3 - 0.1 y_2), 0) \\
x_7 &= x_{70} + a_3 s_3 y_1 - (1 + 2a_2) s_3 \min(u_3, y_1) \\
x_8 &= x_{80} + a_3 s_4 y_2 - (1 + 2a_2) s_4 \max((y_1 - u_3 - 0.1 y_2), 0).
\end{align*}
\]

These three cases, which are embedded into the computational algorithm, formalizes the requirement of synchronization of the traffic control influences toward a couple of crossroads of the traffic network. At each iteration/control cycle, the queue lengths are calculated in order to “observe” the traffic dynamics.

The bi-level optimization results are compared with nonlinear optimization problem (classical optimization), where the control variable is only one—the green light duration. The comparison is made on the basis of waiting vehicles in front of the junctions of the traffic network.

The nonlinear problem is similar to Equation (8), and solved by the lower-level optimization problem. However, the cycle durations \( y_j, j = 1, \ldots, 8 \) are not given like solutions to the upper-level optimization problem. They are regarded as fixed preliminary known values.

6. Experiments and Results

The bi-level optimization, Equations (8) and (9), is solved in a MATLAB environment, using the YALMIP extension [51]. The solutions of the upper-level optimization problem \( y_j, j = 1, \ldots, 8 \), of the first 15 control cycles (calculation iterations) are given in Table 1. The duration of the cycles [sec] for each intersection is different, which corresponds with the definition of the bi-level optimization problem. The synchronization procedure implements the minimal cycle length. This is illustrated in Figure 6, where the values of the higher evaluated cycle durations for \( y_2, y_3 \), and \( y_4 \) are decreased up to the minimal evaluated cycle and applied to the control process. This procedure for problem definition, evaluation, and synchronization is repeated 15 times; see Figure 6.

<table>
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<tr>
<th>( y_1 )</th>
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</tbody>
</table>
6. Experiments and Results

The synchronization procedure implemented cycle and applied to the control process. This procedure for problem definition, evaluation, and implementation is illustrated in Figure 6, where the values of the calculated in order to “observe” the traffic dynamics. From the bi-level problem are higher. Because the corresponding cycle duration of the cycles [sec] for each intersection is different, which corresponds with the requirement of synchronization of the traffic control influences toward a couple of increased from synchronization, the resulting green light duration of the green light duration of u4, and the implemented values are illustrated in Figures 8–10. This is a consequence of the changes in the corresponding cycle durations of y3, y4, and y7 from the synchronization procedure.

Figure 6. Dynamics of the cycles y2, y3, and y4.

The dynamics of the cycles y2, y3, and y4 are presented in Figure 6.

Following the synchronization procedure for the cycle durations, the green lights u2, i = 1, …, 15, also change their values between their optimal solutions and the real implemented ones, corrected according to the synchronization procedure.

Figure 7 illustrates the changes of the green light u4. The evaluated optimal values from the bi-level problem are higher. Because the corresponding cycle y2 has been decreased from synchronization, the resulting green light duration of u4 also decreases. Thus, u4 adapts its value, according to the real implemented cycle y2.

Figure 7. Difference between the evaluated and implemented green light duration u4.

In the same way, the changes between the evaluated optimal solutions u6, u8, and u14 and the implemented ones are illustrated in Figures 8–10. This is a consequence of the changes in the corresponding cycle durations of y3, y4, and y7 from the synchronization procedure.
Following the synchronization procedure for the cycle durations, the green lights duration $u_6$ has been decreased. The evaluated optimal values $u_8$, $u_1$, ..., $u_{15}$ also change their values between their optimal solutions and the implemented ones, corrected according to the synchronization procedure.

Figure 8. Difference between the evaluated and implemented green light duration $u_6$.

Figure 9. Difference between the evaluated and implemented green light duration $u_8$.

Figure 10. Difference between the evaluated and implemented green light duration $u_{14}$.

For all cases in Figures 8–10, the implemented green lights durations are smaller in comparison with their optimal evaluated values.

The dynamics of the queue lengths $x_6$ and $x_{10}$ according to the applied control cycles are presented in Figure 11.
In order to assess the results of the developed control policy, the bi-level implemented solutions are compared with the solutions of the classical nonlinear optimization problem. The last is defined to minimize the queue lengths on an intersection and the arguments are the duration of the green lights for fixed values of the cycle durations. A comparison has been made by integrating the sum of all queue lengths in the urban network for 15 control cycles. This is the value of the total number of waiting cars in the network. This is used as an integrated criterion for the functionality and effectiveness of the developed control approach. A comparison between the total number of waiting cars in the network, obtained after the bi-level control (solid line) and the classical optimization one (dashed lines), is given in Figure 12.

Figure 11. Dynamics of queue lengths $x_6$ and $x_{10}$.

The classical (one-level) optimization is applied for fixed cycle durations with 40, 50, 60, and 70 s.

The queue lengths, resulting from the bi-level optimization (blue solid line), is less than the values of the one-level optimization (dashed lines). This is proof of the positive potential of the suggested bi-level optimization. Figure 12 illustrates the one-level optimization when the cycle duration is close to the implemented bi-level value of $y = 40$ s and the resulting waiting vehicles (red dashed line) are closest to the bi-level case. When the cycle duration increases (50, 60, 70), the sum of the resulting queue lengths also increases, which gives worse transport behavior in the network.

7. Discussion and Conclusions

This research presents a new, bi-level formalization of traffic control. A bi-level optimization problem is defined and solved. The hierarchical approach of the problem
definition allows extending the control arguments of the problem both by the cycle and green lights durations. The results obtained confirm that the extended set of controls allows more objectives to be optimized in hierarchical order: minimization of the cycle durations, maximization of the traffic flows, and, as a consequence, minimization of the total number of vehicles in the network. The comparisons between the bi-level and classical one-level optimization give proof of the superiority of this new formalization of traffic control. The applied synchronization of the cycles allows the network to operate as a unit, where the integral number of vehicles decreases to steady-state about three control cycles; see Figure 12. The number of vehicles is lower in comparison with the one-level optimization approach.

The advantages of these comparisons come from the newly defined hierarchical, bi-level optimization problem.

The bi-level optimization problem integrates and coordinates two goals, more variables, and more constraints, which is a prerequisite for improving and adapting the control process to the dynamic changes in traffic behavior. The bi-level optimization estimates the green light duration of the traffic lights of the network as a solution of the lower-level problem, and at the same time, the traffic light cycle durations as a solutions of the upper-level optimization problem. In order to synchronize the control in the network, an additional technique is embedded in the bi-level optimization algorithm. The model predictive approach is used for the real-time management of the traffic in the network. The implementation of the predictive control is based on the comparison and the choice of the shorter cycle of each couple of neighbor junctions. This allows synchronization of the traffic lights, minimization of the queues in front of the traffic lights, and maximization of the outgoing flows of the network. The developed bi-level model has been numerically simulated on a set of intensive crossroads in Sofia. The bi-level solutions have been compared with the solutions of one-level classical optimization related to the sum of the queue lengths in the network. The results give advantages to the developed bi-level control in comparison to the classical optimization.

The potential and future extension of this modeling method should consider actions regarding the travel delays of vehicles between two neighboring junctions. These delays will result in differences between the incoming and outgoing traffic flows. The explicit inclusion of the time delays in vehicle motion by additional constraints will complicate the formal definition of the bi-level problem.

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