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Selection Criterion of Reanalysis Methods for Plane Truss Optimization

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Abstract: Structural reanalysis methods have been proposed to improve the efficiency of structural analysis. However, the methods are typically only applicable to their specific type of structural modification. Since the optimization process often involves multiple types of modifications, it is necessary to establish a criterion for selecting the most suitable reanalysis method for each type of modification, aiming to accelerate the optimization process. In this study, the effects of different types of structural modifications are first analyzed. A qualitative correspondence is established between different types of structural modifications and the mainstream of the reanalysis methods. Secondly, the most suitable reanalysis method for different types of structural modifications is quantitatively analyzed from the aspects of selecting efficiency indicators and clarifying accuracy requirements. Finally, in conjunction with the Structural Topology and Shape Annealing (STSA) algorithm, a criterion for selecting reanalysis methods, which are applicable to the optimization process of plane trusses, is established. To verify the validity of the selection criterion, two types of numerical examples are conducted. The results show that the proposed criterion can effectively improve the efficiency of structural computations.

Keywords: structural optimization; plane truss; structural reanalysis method; selection criterion; computational efficiency

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1. Introduction

During the optimization process of structural design, the structural analysis and solution need to be repeatedly performed due to continuous adjustments and modifications of the design variables, including the decomposition or inverse operation of the structural stiffness matrix. The traditional approach is to treat the structure after each change as a new structure for the complete analysis, even if a minor change is concerned. This causes a large number of unnecessary and repetitive calculations, resulting in low efficiency of structural computation. To solve this problem, reanalysis methods have gradually emerged. The basic idea of these methods is to use the calculation information before the modification (such as the initial stiffness and response of the structure) and then combine it with the modification information (such as incremental stiffness matrix) to quickly solve the structural response after the change. These methods can effectively avoid complete analysis of the structure, significantly reduce the computing cost, and improve the computing efficiency. Currently, structural reanalysis methods are mainly divided into two categories, namely, exact methods and approximate methods, which are based on the accuracy of the calculation results.

Exact methods can generally obtain precise structural response after the modifications through mathematical deduction. The most classic method in the early period is the Sherman–Morrison–Woodbury (SMW) method [1]. The SMW method can quickly obtain the inverse matrix of the modified stiffness matrix by using the inverse matrix of the initial

stiffness matrix and the incremental stiffness matrix. Then the modified structural response can be readily obtained. In recent years, new methods have been developed. Song et al. [2,3] proposed a matrix partitioning method named the UMTF method based on the idea of "filling elements", which shows good performance in large-scale structural calculations. Wang et al. [4] proposed the Independent Coefficient (IC) method, which simplifies the inputs for the calculation and is suitable for local modification of large-scale structures. On this basis, the more applicable Indirect Factorization Updating (IFU) method [5] has been proposed. In addition, a block-based reanalysis method named the BB method proposed by Gao et al. [6] has been applied to the problem of local modification of large structures. For the processing of sparse matrices, Chan et al. [7] proposed a partial re-decomposition method (PR method) for sparse matrices, whereas Davis et al. [8] developed a method for the rapid decomposition of sparse matrices as a whole.

Compared with the exact methods, approximate methods usually estimate the structural response by reducing the solution size (such as reducing the dimension of the stiffness matrix). They can further be classified into four categories: local approximation, global approximation, iterative approximation, and combined approximation [9].

Local approximation methods estimate the structural response based on the calculation information of a single point. The common approach is to perform various series expansions at a single point in order to approximate the exact solution. Typical representative methods include the moving asymptote method [10] and the second-order approximation method [11]. Since these methods only rely on the effective information at a single point, they are suitable for small-scale modifications. Once the modification scale becomes larger, their calculation accuracy may not be guaranteed.

Global approximation methods, also known as multipoint approximation methods, utilize the calculation information of multiple points after structural changes to construct an approximate function of the structural response with respect to design parameters. Classical global approximation methods mainly include the response surface method [12], the reduced basis method [13,14], and the polynomial fitting method [15]. Due to the integration of the calculation information of multiple points, global approximation methods normally have higher calculation accuracy than local approximation methods. However, the acquisition and fitting calculation of multiple-point information often make their calculation efficiency relatively low, especially in solving complex structural problems, which can even be lower than that of complete analysis.

The idea of iterative approximation methods comes from the iterative method for solving linear equations in mathematics [9]. For example, the Preconditioned Conjugate Gradient (PCG) method [16], which is based on the Conjugate Gradient (CG) method [17], introduces a suitable preconditioning matrix to reduce the condition number of the stiffness matrix, thereby improving the overall calculation efficiency. The PCG method is widely used due to its fast convergence rate and high accuracy. Wu et al. [18] extended the PCG method to solve the topology optimization problems with reduced degrees of freedom (DOFs). Combined with Guyan reduction [19], the PCG method can also handle topology optimization problems with increasing DOFs [20]. The principle of Guyan reduction involves partitioning the stiffness matrix to consolidates the newly added DOFs onto the existing structural DOFs, ensuring that the solution scale remains consistent with the existing structural DOFs.

The Combined Approximations (CA) method is a popular and extensively studied method, which is considered to combine the high efficiency of local approximation methods and the high accuracy of global approximation methods. CA was proposed by Kirsch [21], who applied the reduced basis vector method to represent the approximate solutions of displacements [22]. Furthermore, CA can be derived using binomial series as reduced basis vectors and was successfully applied to topology optimization [23]. To improve the accuracy of CA in solving large-scale modifications, Yang et al. [24] proposed the Iterative Combined Approximations (ICA) method. In addition, scholars such as Chen [25], Wu [26], and Huang [27] also conducted a series of studies on CA.

Although the above reanalysis methods have significantly improved structural calculation efficiency, previous studies have mainly focused on innovating or improving single methods. Little research has been done on comparing and exploring the most suitable methods for different structural modification types. The purpose of this paper is three-fold. The first is to investigate the effects of different structural modification types on structures during the structural optimization process. The second is to explore the most suitable method that maintains both accuracy and efficiency under different structural modification types from both qualitative and quantitative perspectives. The third is to develop a selection criterion for reanalysis methods, which is suitable for the structural optimization of plane trusses. The criterion will be combined with the Structural Topology and Shape Annealing (STSA) algorithm [28] to "draw on the strengths of all" and further improve the calculation efficiency.

2. Qualitative Classification of Structural Reanalysis Methods

The qualitative classification of reanalysis methods is primarily achieved by analyzing the effects of different types of structural modifications (size, shape, or topology) on the structure. This analysis is to establish the correspondence between the reanalysis methods and the applicable modification types. In this section, CA, ICA, PCG, Epsilon [29], and their variant methods oriented towards changes in DOFs are selected as research objects. The improved PCG method is selected for the reduction of DOFs in the structure, whereas the combinations of the Guyan method and other methods are used for the increase of DOFs.

The STSA algorithm is used for structural generation and optimization in this study. It integrates shape grammar and structural evaluation into an improved simulated annealing algorithm, forming a discrete structural optimization method. The shape grammar defines the basic rules for structural size, shape, and topology transformations. Each rule acts dynamically on the initial structure based on probabilities, achieving free transformation of the structural configuration. The workflow is shown in Figure 1 [28].

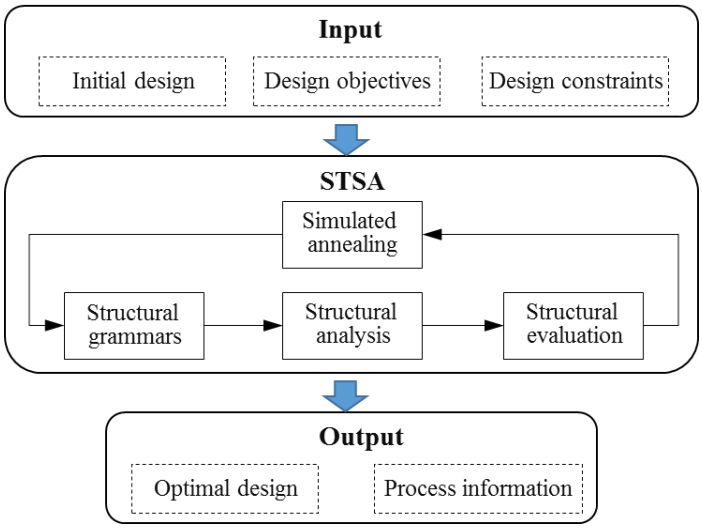


Figure 1. Flow chart of STSA.

According to the principles of the STSA algorithm, structural modifications only involve one of size, shape, or topology transformation at a time. Therefore, determining the effects of a single modification on the structure will be beneficial for the qualitative classification of reanalysis methods. In this paper, the effects of a single modification are divided according to the number of modifications to the elements in the stiffness matrix and according to whether the modifications are local or global in the structure.

The size transformation of a structure involves modifying the cross-section of a single member. The resulting incremental stiffness matrix, ΔK, as shown in Equation (1), contains

non-zero elements only at the corresponding nodes i and j, whereas all the other elements are zero. This indicates that the impact of the cross-section changes in a single member of the structure is relatively limited, especially when the total number of nodes increases. Overall, the effect of size transformation in the structural optimization process is relatively small and can be considered as a local small modification.

$$\Delta K = \frac{E\Delta A}{l_{ij}} \begin{bmatrix} \cdots & i & \cdots & j & \cdots \\ & \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix} & \cdots & \begin{bmatrix} -c^2 & -sc \\ -sc & -s^2 \end{bmatrix} & \vdots \\ & \vdots & \ddots & \vdots & i \\ & \begin{bmatrix} -c^2 & -sc \\ -sc & -s^2 \end{bmatrix} & \cdots & \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix} & \vdots \\ & & & & j \\ & & & & \vdots \end{bmatrix}, \quad (1)$$

where E is the elasticity modulus, L_{ij} is the length of a member, and ΔA represents the changes in the cross-section.

Shape transformation of the structure refers to the modification of a single node's position, assuming that there are m nodes connected to the modified node. The resulting non-zero element count in the incremental stiffness matrix is $12m + 4$. The size of this modification is closely related to the number of connected nodes. In the structural design process, the maximum number of connections for a single node is generally limited due to the consideration of subsequent construction difficulties. In this study, the maximum number of connections for a single node is 5. Overall, the incremental stiffness matrix caused by the changes of node positions exhibits a nonlinear relationship with the design variable (node coordinates). At the same time, its rank is also relatively high. Compared with the size transformation, shape transformation brings relatively greater effects and can be considered as a local moderate change.

Topology transformation includes three cases. The first one is a topology modification where the DOFs remain unchanged, as shown in Figure 2a. The second one is a topology modification that increases the DOFs, as shown in Figure 2b. The third one is a topology modification that reduces the DOFs, as shown in Figure 2c.

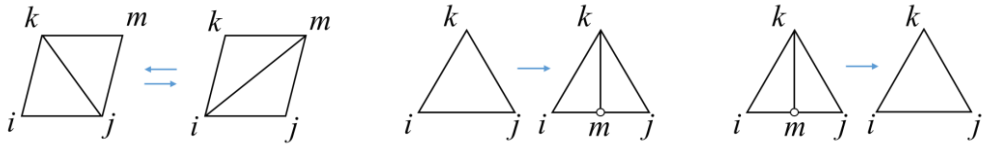


Figure 2. Topology modification types: (**a**) DOFs remain unchanged; (**b**) DOFs increase; (**c**) DOFs decrease.

Regarding the first case, it can be viewed as the superposition of two-size transformations. Firstly, the cross-sectional area of member kj reduces to 0, and secondly, the cross-sectional area of member im changes from 0 to non-zero. Although the number of modified elements in the incremental stiffness matrix is twice that of the size transformation, its overall impact on the structure is still relatively small. Therefore, it can be regarded as a local small change.

As for the second case, it can be converted to a superposition of multiple size transformations. Firstly, the cross-sectional area of member ij is reduced to 0, and then members mi, nj, and mk are added. The third case is the reverse process of the second one. Since the second and third cases involve changes in the size of the stiffness matrix, and the number of modified elements compared to the size transformation is relatively high, these two types of modification can be considered as local moderate changes.

Based on the analysis presented above, the corresponding relationship between various modification types and each method can be preliminarily established, as seen in Table 1. This table provides a summary of the applicable reanalysis methods for each modification type, laying the groundwork for subsequent quantitative analysis.

Table 1. Qualitative selection of reanalysis methods.

Modification Type	The Number of DOFs is Unchanged			The Number of DOFs Is Increased	The Number of DOFs Is Decreased
	Size Modification	Shape Modification	Topology Modification	Topology Modification	Topology Modification
Applicable methods	CA Epsilon PCG	CA ICA Epsilon PCG	CA Epsilon PCG	Guyan + CA Guyan + Epsilon Guyan + PCG	MPCG

DOFs: degrees of freedom.

3. Quantitative Analysis of Reanalysis Methods

In this section, a quantitative analysis is conducted on various reanalysis methods by selecting computational efficiency indicators and specifying accuracy requirements for calculations. This numerical analysis serves to measure the various methods from a numerical perspective and lays a foundation for constructing the ultimate selection criterion.

3.1. Efficiency Study of the Methods

Various factors affect the efficiency of algorithms, such as hardware conditions, computation scale, and complexity. Therefore, it is difficult to find a universal indicator that accurately reflects computational efficiency. Golub et al. [30] proposed using the number of floating point operations (flops) to measure the computational complexity of algorithms, but results relying solely on this indicator have certain limitations. Thus, this study proposes using the storage space required for input, output, and intermediate variables of an algorithm (referred to as the MS value) as an indicator of the algorithm's reading and writing time, in addition to using the flops. This approach enables a more comprehensive assessment of computational efficiency and assists in the measurement of the computational efficiency of various reanalysis methods.

A flop is defined as a single floating point operation that includes one of the four basic arithmetic operations: addition, subtraction, multiplication, or division. For example, if a dot product is performed between two vectors with n elements each, then the number of flops is $2n$ because it involves n multiplications and n additions. Furthermore, real structural stiffness matrices are generally sparse. In order to save storage space, this study stores the stiffness matrix using a two-dimensional banded format and assumes that a single element occupies one unit of storage space. For instance, if the initial stiffness matrix K_0 of size $n \times n$ has a half-bandwidth of b_0, its MS value is $(b_0 + 1)n$. The MS value of other variables is calculated based on their actual size.

Using the CA method as an example, the calculation procedure for flops and MS values is illustrated. Firstly, the initial equilibrium equation of a structure with n DOFs is defined:

$$K_0 r_0 = f_0, \tag{2}$$

where K_0 is the initial stiffness matrix of size $n \times n$, f_0 is the initial load vector, and r_0 is the initial response of the structure.

Due to the symmetry and positive definiteness of K_0, it can be decomposed via Cholesky factorization:

$$K_0 = U_0^T U_0, \tag{3}$$

where U_0 is the upper triangular matrix.

Assume that a structural modification causes an incremental stiffness matrix ΔK, and the modified stiffness matrix is K (with a half-bandwidth of b). The loading vector is denoted by $\boldsymbol{f}$ and the structural response by $\boldsymbol{r}$. In this study, the changes in loading are not considered, i.e., $f = f_0$. The new equilibrium equation is given as follows:

$$Kr = (K_0 + \Delta K)r = f. \tag{4}$$

The CA method mainly reduces the computational complexity by constructing a reduced basis vector. The main steps of the CA method [22] in the structural analysis are given as follows:

1. Approximate the modified structural response $\boldsymbol{r}$ using s linearly independent basis vectors. Define basis vectors r_i and construct a basis vector matrix r_B:

$$r_1 = K_0^{-1} f, \tag{5}$$

$$r_i = B r_{i-1}\ ,\ i = 2, 3, \cdots s, \tag{6}$$

$$r_B = [r_1, r_2, \cdots, r_s], \tag{7}$$

where B is $-K_0^{-1}\Delta K$.

2. Calculate the reduced matrix K_B and load vector f_B:

$$K_B = r_B^{\mathrm{T}} K r_B = r_B^{\mathrm{T}}(K_0 + \Delta K) r_B, \tag{8}$$

$$f_B = r_B^T f \tag{9}$$

3. Calculate the vector of coefficients $\boldsymbol{y}$:

$$y^{\mathrm{T}} = [y_1, y_2, \ldots y_s], \tag{10}$$

$$K_B y = f_B. \tag{11}$$

4. Evaluate the modified displacements $\boldsymbol{r}$:

$$r = y_1 r_1 + y_2 r_2 + \ldots + y_s r_s = r_B y. \tag{12}$$

According to the above-mentioned analysis, the flops can be calculated step by step. By combining Equation (6) in step 1 with Equation (3), the following equation can be obtained:

$$K_0 r_i = U_0^T U_0 r_i = -\Delta K r_{i-1}. \tag{13}$$

Every basis vector can be calculated by forward and backward substitution based on Equation (13). The computational complexity on the right-hand side of the equation is $2n(2b + 1)$. Moreover, the computational complexity for both forward and backward substitution is $2nb$. Therefore, the flops for constructing basis vectors are approximately $4nb + [2n(2b + 1) + 4nb](s - 1)$.

In step 2, Equation (8) involves the multiplication of three matrices. The computational complexity of $r_B^T \cdot K$ is $2ns(2b + 1)$, while the computational complexity of multiplying it by r_B is $2ns^2$. Equation (9) involves the multiplication of a matrix and a vector, and its computational complexity is $2ns$. Therefore, the flops for this step are $4nsb + 4ns + 2ns^2$.

Step 3 solves the coefficient vector $\boldsymbol{y}$, and Equation (11) uses LU decomposition with flops of approximately $2s^3/3 + 2s^2$.

In step 4, the approximation response only involves the multiplication of a matrix and a vector, with flops of $2ns$.

Combining the above-mentioned analysis, the flops of the CA method as a whole are $2s^3/3 + 2s^2 + 12nsb + 8ns + 2ns^2 - 4nb - 2n$.

The inputs, outputs, and intermediate variables involved in the above process include U_0, ΔK, K, f, r_B, K_B, f_B, etc. Thus, the MS value is approximately $s^2 + (n + 2)s + (3b + 5)n$.

After analyzing the flops and MS values of various reanalysis methods, the relevant results are summarized in Table 2. Please refer to [31] for detailed information.

Table 2. Summary of index results.

Method	Flops	MS
CA	$2s^3/3 + 2s^2 + 12nsb + 8ns + 2ns^2 - 4nb - 2n$	$s^2 + (n + 2)s + (3b + 5)n$
ICA	$2nb + n + [2s^3/3 + 2s^2 + 12nsb + 8ns + 2ns^2 - 4nb - 2n]k$	$(3b + 5)n + (s^2 + ns + 2s)k$
PCG	$[8nb + 14n + 3]k + 2n$	$2nb + 9n$
Epsilon	$8nsb + 11ns/2 + 5ns^2/2 - 4nb - 2n$	$[s^2/2 + 5s/(2 + 2b + 3)]n$
MPCG	$(4mb + 4nb + 14m + 2n)k$	$(m + n)b + n + 9m$
Guyan + CA	$n^2 + 4nt^2 + 2n^2t + 7t^2/3 + 4nt - n + t + 2s^3/3 + 2s^2 + 12nsb + 8ns + 2ns^2 - 4nb$	$t^2 + s^2 + 3bn + 2nm + 2n + 2m + s$
Guyan + Epsilon	$n^2 + 4nt^2 + 2n^2t + 7t^2/3 + 4nt - n + t + 11ns/2 + 5ns^2/2 + 8nsb - 4nb$	$ns^2/2 + 5ns/2 + 2nb + 2nm + 3n + m$
Guyan + PCG	$4tb + 2t + b^2 + b + 2n + (2 + 8nb + 14n + 4tb + 2t)k$	$t^2 + 2nb + 2nm + 8n + 2m$
Cholesky	$nb^2 + 7nb + 2n$	$2nb + 4n$

k: number of iterations; n: number of DOFs of K_0; m: number of DOFs of K; $t = m - n$.

3.2. Computational Accuracy Requirements

The accuracy of the structural reanalysis method is typically measured by the error between the computed response $\widetilde{r}$ and the actual response r, as shown in Equation (14).

$$\varepsilon = \frac{\|\widetilde{r} - r\|}{\|r\|}. \tag{14}$$

However, the actual response is often unknown in practical computations. Therefore, the following equation is generally used to estimate the computational accuracy in place of the actual response. The calculation formula for the error in this study is also based on Equation (15).

$$\varepsilon = \frac{\|K\widetilde{r} - f\|}{\|f\|}. \tag{15}$$

Both excessively large and excessively small values of error will impact the performance of the algorithm. An error that is too small can compromise the effectiveness of reanalysis methods by sacrificing computational efficiency to ensure accuracy, sometimes even resulting in lower efficiency than complete analysis. On the other hand, an excessively large error will affect the optimization process and results of the algorithm. In order to determine a reasonable value of error, a series of numerical tests was conducted. As a result, a value of 0.01 was chosen, indicating that the overall error ε of this study should not exceed 0.01. For structural calculations during the optimization process, once the error exceeds 0.01, the reanalysis method should be discontinued, and the accurate solution method should be applied to eliminate the error in the structural response calculation.

4. Construction of Selection Criterion

4.1. Method Comparison Study

Based on the information from Table 2, the flop ratio (Fr) and Ms ratio (Mr) were defined to compare the computational efficiency of different methods, as shown in Equations (16) and (17). The numerators of the two indicators represent the flops and MS values of the first reanalysis method, while the denominators represent these of the second method.

$$Fr = \frac{flop_1}{flop_2}, \tag{16}$$

$$Mr = \frac{MS_1}{MS_2}. \tag{17}$$

If the *Fr* value is greater than 1, it indicates that the flops of the first method are higher than that of the second, and it can be considered that the second method has relatively higher computational efficiency. Similarly, if the *Mr* value is greater than 1, then the second method has a smaller storage space and shorter read and write time, which contributes to the improvement of the method's computational efficiency to a certain extent. The impact of the *Mr* value is limited and is only used as an auxiliary judgment factor.

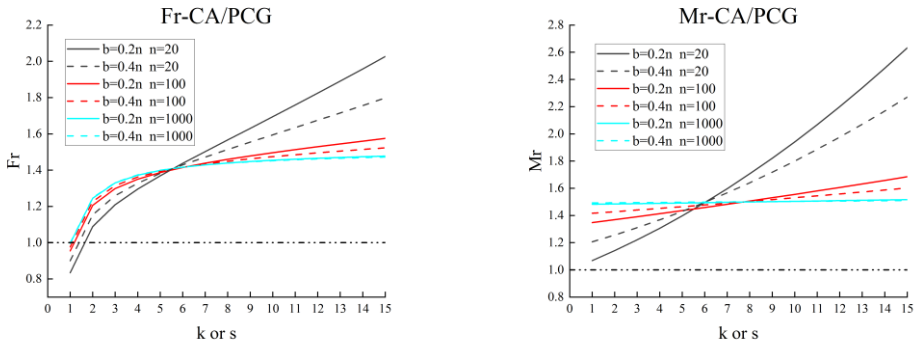
4.1.1. Unchanged Number of DOFs

According to Table 1, there are four candidate methods including CA, ICA, Epsilon, and PCG methods when the structural DOFs remain unchanged. Previous studies have suggested that the CA method and the PCG method have certain equivalences, i.e., when the number of basis vectors (s) in the CA method is equal to the iteration steps (k) in the PCG method, the two methods can achieve the same computational accuracy [16]. Therefore, the CA method and the PCG method can be directly compared at the same level of accuracy, and the calculations of the *Fr* and *Mr* for the two methods are given as follows:

$$Fr_{CA/PCG} = \frac{2s^3/3 + 2s^2 + 12nsb + 8ns + 2ns^2 - 4nb - 2n}{[8nb + 14n + 3]k + 2n}, \tag{18}$$

$$Mr_{CA/PCG} = \frac{s^2 + (n+2)s + (3b+5)n}{2nb + 9n}. \tag{19}$$

In this section, three cases were selected for the structural DOFs (n), namely, 20, 100, and 1000, and the half-bandwidth b was divided into two cases: $0.2n$ and $0.4n$. The calculation results are shown in Figure 3.



(**a**) (**b**)

Figure 3. Indicator curves of CA/PCG: (**a**) *Fr*; (**b**) *Mr*.

It can be observed in Figure 3a that the *Fr* values of CA/PCG gradually increase with the increase of k or s and gradually converge to around 1.5 with the increase of structural DOFs. Only when the structural DOFs are small, and k or s is equal to 1, are the *Fr* values of CA/PCG less than 1. However, the accuracy of the two methods is generally low at this time, so the situation where k or s is 1 can be ignored. Based on this, it can be seen from a flops perspective that the computational efficiency of the CA method is lower than that of the PCG method. From the *Mr* curve in Figure 3b, it can be seen that the values of *Mr* for CA/PCG are greater than 1 under different DOFs and k or s, indicating that the storage space of the CA method is relatively high, which further affects its efficiency. Therefore, the PCG method is more suitable than the CA method for the case of unchanged DOFs.

For the Epsilon, PCG, and ICA methods, since the three methods do not have theoretical equivalence, the efficiency indicator of complete analysis (Cholesky factorization) is

used as the numerator for calculation. In order to present the comparison results clearly, a log scale is used for the vertical axis. The relevant results are shown below.

From the figures above, it can be observed that the ln(*Fr*) values of the ICA method are the smallest among the three methods. Especially in cases of lower structural DOFs, even with smaller iteration steps, the corresponding ln(*Fr*) values for the ICA method are smaller than 0. This implies that for structures with low DOFs, the computational efficiency of the ICA method is generally lower than that of complete analysis. Compared to the other two methods, the performance of ICA is poor. On the other hand, as shown in Figure 4f, its *Mr* values are below 0.7 both for different DOFs and iteration steps, indicating its low performance compared to the other two methods. Therefore, the ICA method is not used for structures with unchanged DOFs.

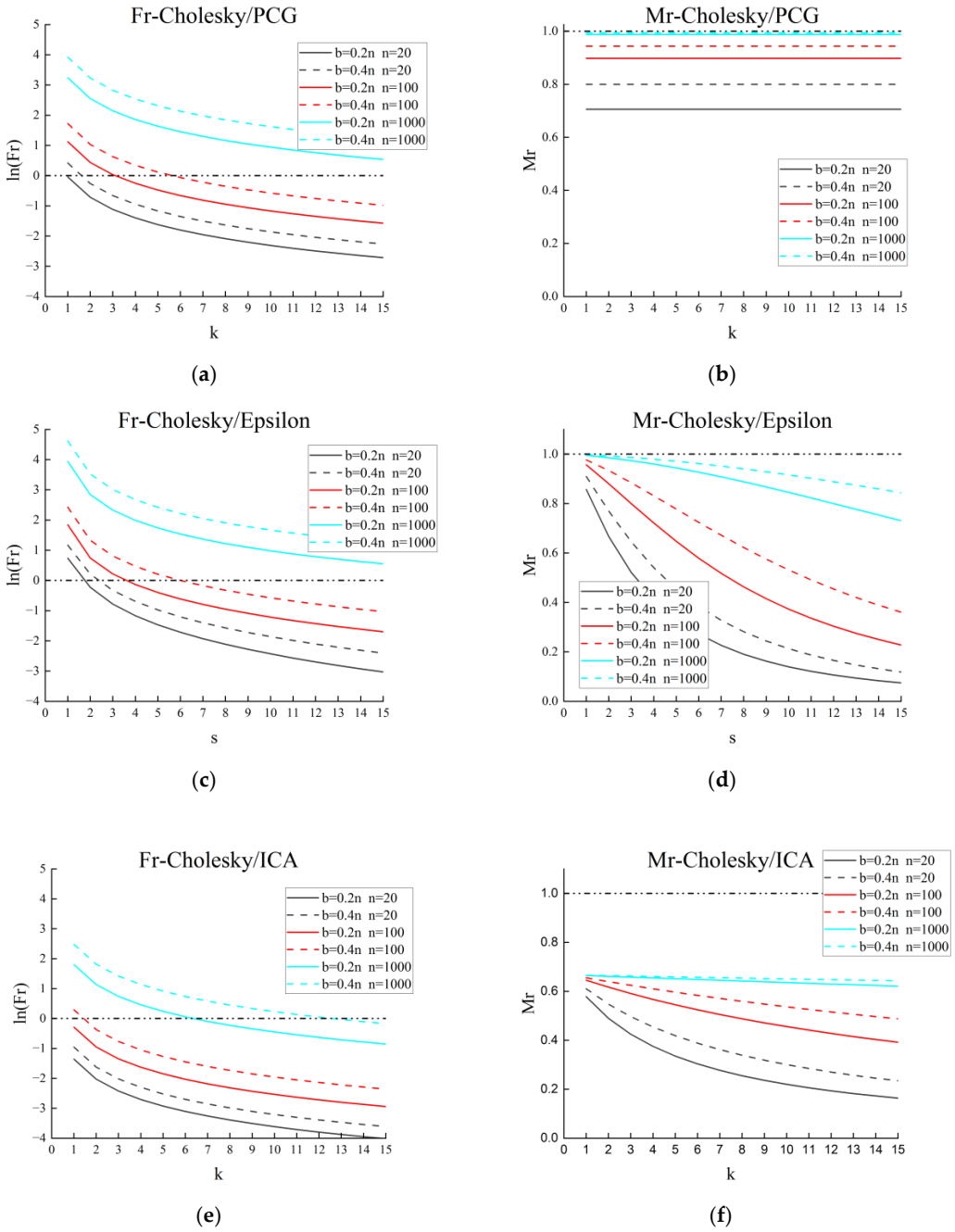


Figure 4. Indicator curves of three combinations: (**a**) *Fr*–Cholesky/PCG; (**b**) *Mr*–Cholesky/PCG; (**c**) *Fr*–Cholesky/Epsilon; (**d**) *Mr*–Cholesky/Epsilon; (**e**) *Fr*–Cholesky/ICA; (**f**) *Mr*–Cholesky/ICA.

As for structures with higher DOFs, as shown in Figure 4a,c, both the Epsilon method and the PCG method have high computational efficiency, especially when there are small changes in high DOF structures (which require fewer basis vectors or iteration steps). Furthermore, when the number of basis vectors in the Epsilon method is less than or equal to 3, its ln(*Fr*) value is significantly higher than that of the PCG method. This indicates that its computational efficiency is higher than that of PCG. However, as the number of basis vectors or iteration steps exceeds three, the difference between the two methods gradually narrows, and the PCG method proves to be more efficient.

Since the number of iteration steps of the PCG method is generally smaller than the number of basis vectors in the Epsilon method when achieving similar accuracy (Epsilon usually takes an odd number of basis vectors), it is stipulated that when using the Epsilon method, only three basis vectors should be used. Otherwise, the PCG method should be employed. In order to verify this conclusion, three examples with different DOFs (n = 20, 96, and 1368) are designed for validation (Figure 5).

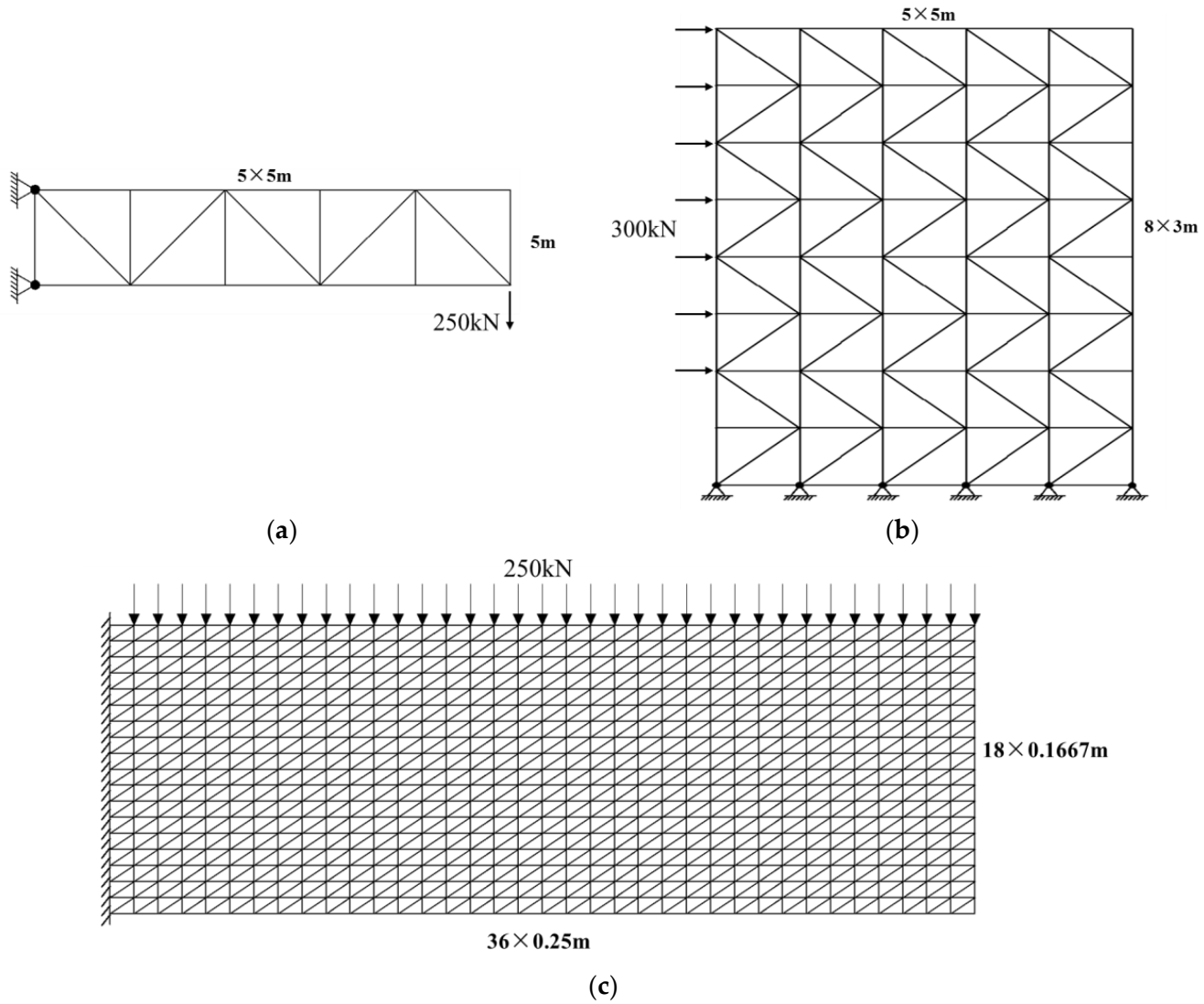


Figure 5. Initial layouts of the three structural cases based on number of DOFs: (**a**) n = 20; (**b**) n = 96; (**c**) n = 1368.

The above examples all consist of circular solid members with a cross-sectional area of 250 cm^2 and an elastic modulus of 210 $\times$ 10^9 Pa. The following types of modifications are applied to all three examples:

(1) Size modification: one randomly selected member's area is changed to twice that of its original cross-sectional area. (2) Shape modification: one randomly selected node undergoes vertical displacement. (3) Topology modification: one randomly selected member undergoes a connection node transformation (see Figure 2a).

Complete analysis, the Epsilon method, and the PCG method were used to perform calculations on the three examples under the three modifications. In order to ensure the reliability of the results, the calculations for Examples 1, 2, and 3 were iterated 10,000 times, 5000 times, and 5 times, respectively, with a required accuracy of 1% error limit. These examples were implemented using the MATLAB programming language and executed on a computer with IntelI CoreI i7-6700k CPU @ 4.00GHz. The results are given in Table 3.

Table 3. Results of structural examples with unchanged DOFs.

Example No.	Modification Type	Indicator	Cholesky	Epsilon	PCG
1	(1)	s or k	-	3	2
		Computation time	1.5196 s	0.9577 s	1.0967 s
		Proportion	100%	63.0%	72.2%
	(2)	s or k	-	13	8
		Computation time	1.5786 s	5.6932 s	4.6567 s
		Proportion	100%	360.6%	295.0%
	(3)	s or k	-	5	3
		Computation time	1.4979 s	2.0093 s	1.7246 s
		Proportion	100%	134.1%	115.1%
2	(1)	s or k	-	3	2
		Computation time	39.3519 s	6.7370 s	7.6099 s
		Proportion	100%	17.1%	19.3%
	(2)	s or k	-	13	8
		Computation time	39.5669 s	42.5712 s	30.0116 s
		Proportion	100%	107.6%	75.9%
	(3)	s or k	-	13	8
		Computation time	40.1709 s	13.5988 s	11.3561 s
		Proportion	100%	33.9%	28.3%
3	(1)	s or k	-	3	2
		Computation time	153.3645 s	2.0374 s	2.3564 s
		Proportion	100%	1.3%	1.5%
	(2)	s or k	-	9	6
		Computation time	150.3360 s	6.8872 s	6.6025 s
		Proportion	100%	4.6%	4.4%
	(3)	s or k	-	5	3
		Computation time	149.7456 s	3.6079 s	3.4680 s
		Proportion	100%	2.4%	2.3%

It can be seen that under different DOFs, the Epsilon method can always complete the calculation and meet accuracy requirements with only three basis vectors when dealing with changes in structural size (modification type (1)), while the corresponding PCG method requires two iteration steps. In terms of computation time, the Epsilon method outperforms the PCG method. However, for the other two modification types, the Epsilon method requires more than three basis vectors to meet the accuracy requirements, and the computation time for the Epsilon method is longer than that for the PCG method. Therefore, the PCG method is more effective for shape and topology modifications (with unchanged DOFs). It is worth noting that when the DOFs of the structure are low, the iteration steps or the number of basis vectors required by both methods for shape modification (modification type (2)) are relatively large, resulting in significantly lower computational efficiency than complete analysis. Therefore, the complete analysis should be considered for structural calculations with low DOFs in shape modifications.

Based on this analysis, the selection criterion for the reanalysis method under the condition of unchanged DOFs can be summarized as follows: The Epsilon method can be used for size modifications, while the PCG method can be used for shape and topology modifications. However, when the DOFs of the structure are below 50, the complete analysis should be used for shape modifications.

4.1.2. Increased Number of DOFs

Under the condition of increased DOFs, there are three candidate methods, namely, the Guyan + CA method, the Guyan + Epsilon method, and the Guyan + PCG method. The Guyan reduction [19] is an independent method that reduces the size of the stiffness matrix. The stiffness matrix processed by this method can be solved using those reanalysis methods under unchanged DOFs. Following the comparison between the CA method and the PCG method in the previous section, the Guyan + CA method can be eliminated first. This section focuses on comparing the Guyan + Epsilon method and Guyan + PCG method, and it compares the ln(*Fr*) and *Mr* values obtained using the complete analysis and the two methods. The results are shown below.

As can be seen from Figure 6, when the number of iteration steps is small, the ln(*Fr*) value of the Guyan + PCG method is significantly larger than that of the corresponding Guyan + Epsilon method, and this phenomenon occurs for various DOFs. Considering that the iteration steps of the PCG method are generally smaller than the number of basis vectors in the Epsilon method under similar accuracy, the difference between ln(*Fr*) values will further widen in this case. Therefore, it can be considered that under the condition of increased DOFs, the Guyan + PCG method has better computational efficiency. As for the *Mr* values, both methods are smaller than those obtained by complete analysis. However, as the number of iteration steps or basis vectors increases, the storage space of the Guyan + Epsilon method gradually becomes larger.

The same examples as in Section 4.1.1 were used for validation, but with only one modification type, i.e., adding a node to a non-boundary member in each of the three examples. Other conditions remained the same. The results are shown in Table 4.

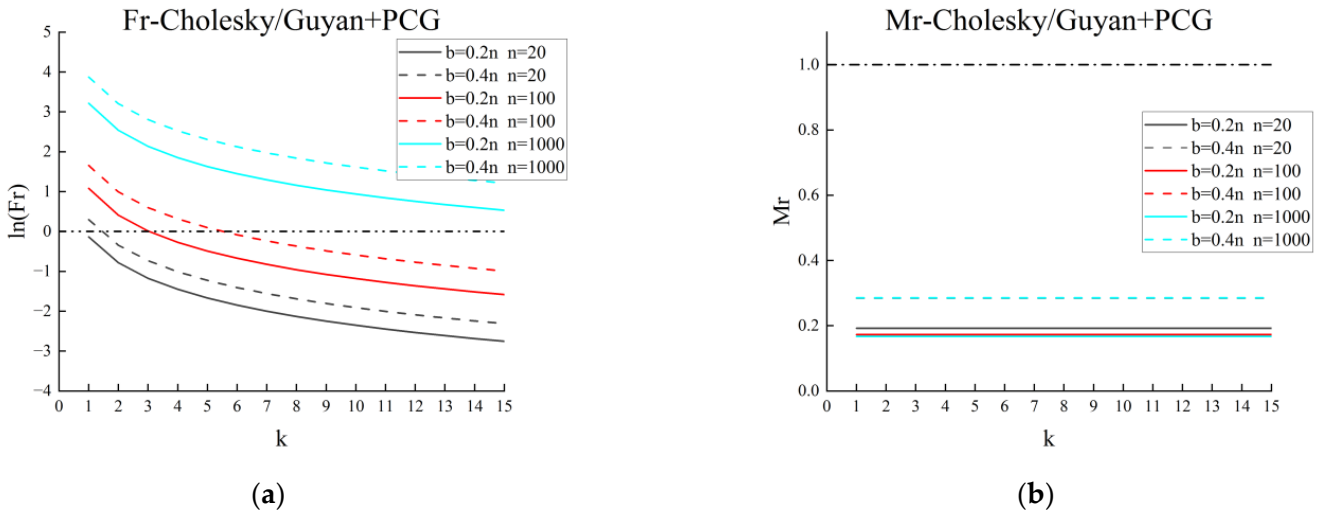
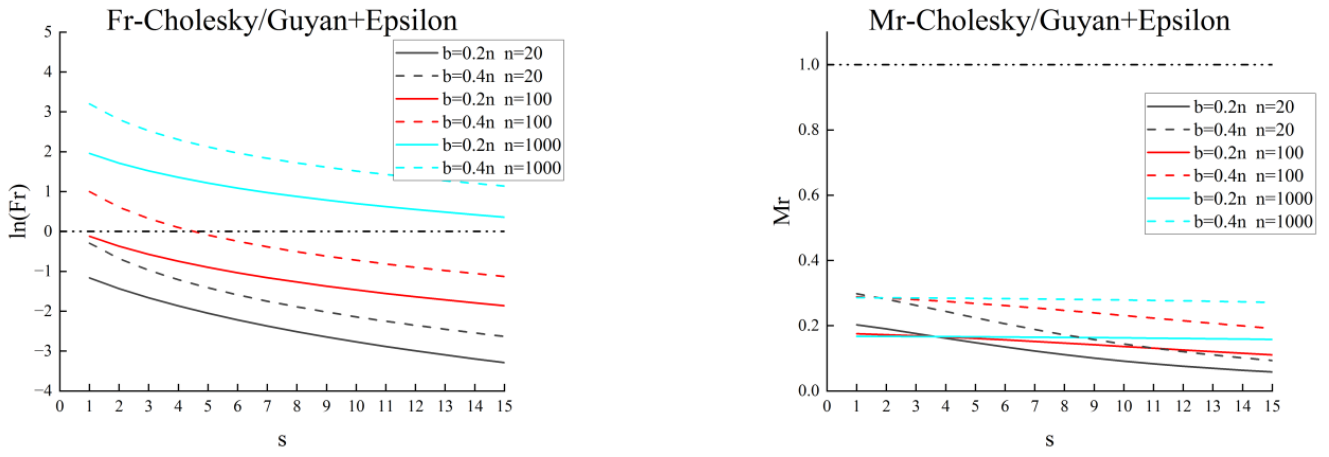


Figure 6. *Cont.*



(**c**) (**d**)

Figure 6. Indicator curves of two combinations: (**a**) *Fr*–Cholesky/Guyan + PCG; (**b**) *Mr*–Cholesky/Guyan + PCG; (**c**) *Fr*–Cholesky/Guyan + Epsilon; (**d**) *Mr*–Cholesky/Guyan + Epsilon.

Table 4. Results of structural cases with increased DOFs.

Case No.	Indicator	Cholesky	Guyan + Epsilon	Guyan + PCG
1	*s* or *k*	-	3	2
	Computation time	1.8388 s	2.1057 s	2.0882 s
	Proportion	100%	114.5%	113.6%
2	*s* or *k*	-	3	2
	Computation time	63.7727 s	18.3677 s	17.6684 s
	Proportion	100%	28.8%	27.7%
3	*s* or *k*	-	3	2
	Computation time	149.7456 s	2.6594 s	2.1351 s
	Proportion	100%	1.8%	1.4%

According to Table 4, it can be seen that under the condition of increased DOFs, the number of basis vectors or iteration steps in both methods remains consistent with that of size modification under unchanged DOFs. Additionally, the computational time of the Guyan + Epsilon method is longer than that of the Guyan + PCG method. For lower DOFs, the computational time of complete analysis is shorter than that of both methods. These results support the conclusions presented earlier. Therefore, the selection criterion for increased DOFs is summarized as follows: when the structure undergoes a topology modification that increases DOFs, the Guyan + PCG method should be used for solving. However, when the DOFs of the structure are below 50, the complete analysis should be adopted.

4.1.3. Decreased Number of DOFs

For the situation where the DOFs decrease, the MPCG method [18] was employed in this study. To evaluate its effectiveness, the examples presented in Section 4.1.1 were used for verification. Specifically, the top-right corner node was removed from examples 1 and 2, and the bottom-right corner node was removed from example 3 while keeping all other conditions constant. The complete analysis and MPCG methods were utilized for the calculations, and the results are presented in Table 5.

Table 5. Results of structural cases with decreased DOFs.

Case No.	Indicator	Cholesky	MPCG
1	k	-	3
	Computation time	1.0027 s	1.2604 s
	Proportion	100%	125.7%
2	k	-	1
	Computation time	42.1804 s	2.7074 s
	Proportion	100%	6.4%
3	k	-	4
	Computation time	153.4685 s	3.2429 s
	Proportion	100%	2.1%

According to Table 5, the computational efficiency of the MPCG method increases with the increase of DOFs. Moreover, for low DOFs, the computational efficiency of the complete analysis method is higher than that of the MPCG method. Therefore, a criterion for selecting the reanalysis method under decreased DOFs is proposed. Specifically, when the DOFs are reduced, the MPCG method should be selected. However, if the number of DOFs is less than 50, the complete analysis method should be used for solution.

4.1.4. Selection of the Accurate Solution Methods

Two methods were used for the accurate solution task in this study, namely, the Cholesky factorization (complete analysis) and the UMTF method. The principle of the UMTF method is to record the impact row positions of structural modifications on the stiffness matrix with a location vector and correspondingly update the elements in the decomposition matrix on the respective rows [3]. The computational complexity of this process is closely related to the distribution of elements in the structural stiffness decomposition matrix, making it difficult to accurately calculate the flops. For simplicity, it can be assumed that all rows in the decomposition matrix corresponding to the row of the first non-zero element in the incremental stiffness matrix, and all subsequent rows need to be updated (if the first row of the incremental stiffness matrix has a non-zero element, then the computational complexity of this method is the same as that of the complete analysis). Obviously, the later the row in which the first non-zero element of the incremental stiffness matrix appears, the more computational effort the method saves. As the calculations of the incremental stiffness matrix and location vector will also take time, a selection criterion is set for the accurate solution method after comprehensive consideration, i.e., the UMTF method is used for the accurate solution when the row number where the first non-zero element appears in the incremental stiffness matrix is greater than 30% of the structural DOFs. For example, if the first non-zero element in the incremental stiffness matrix of a 100-DOF structure appears after the 30th row, the UMTF method will be used for the accurate solution. At the same time, considering the requirement for accuracy in the structural optimization phase, it is stipulated that the complete analysis will be used for solution in the last 10 iterations.

4.2. Selection Criterion of Reanalysis Methods

After the above analysis, the reanalysis method selection criterion for structural optimization process is established in this section (Figure 7).

When evaluating a newly generated structure, it is necessary to first determine the stage of the optimization process. If the optimization process enters the last 10 iterations, the structural response is directly calculated using the complete analysis. Otherwise, the corresponding reanalysis method is selected according to the type of structural modifications and the number of DOFs. When the calculation is completed, it is necessary to check whether the calculation accuracy meets the error requirement. If not, the complete analysis should be used for solution, and the final structural response is outputted.

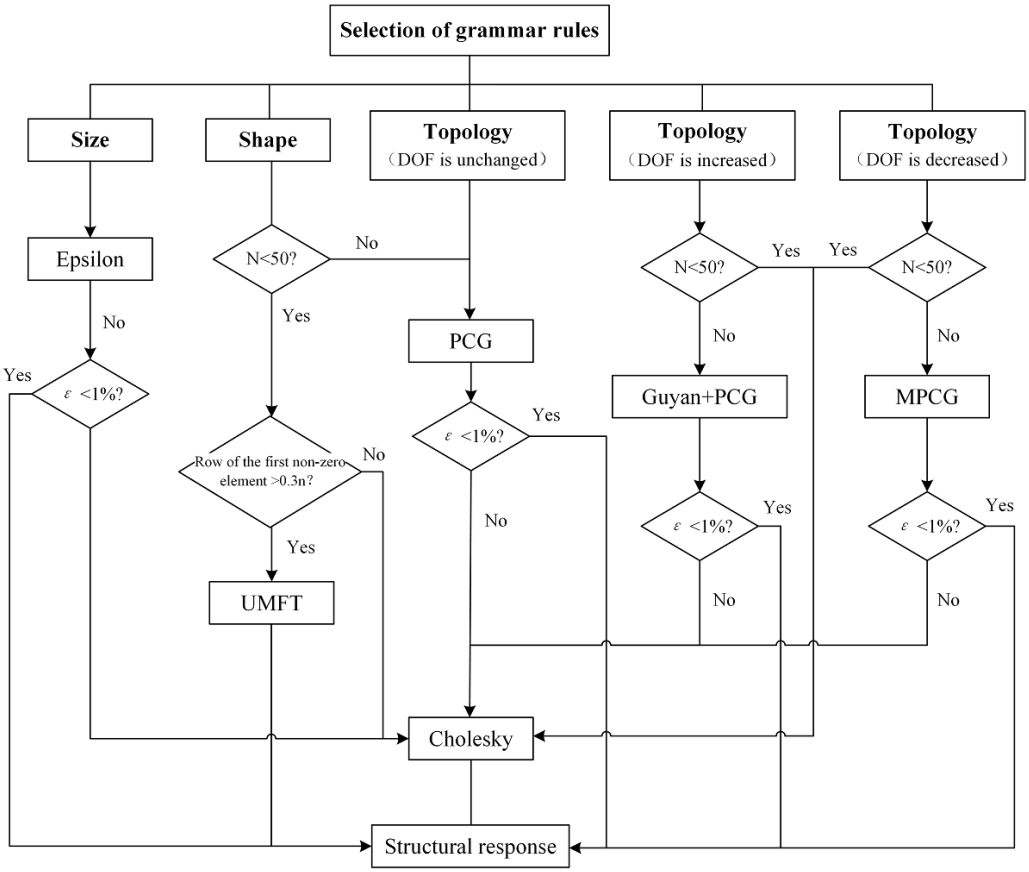


Figure 7. Selection process of reanalysis methods.

5. Numerical Examples

To validate the effectiveness of the proposed selection criterion, two types of numerical case studies were conducted in this section. One type is an example of a truss design based on the STSA-P system [32]. As the DOFs for this type of structural design are generally low, another example type of truss structural optimization with higher DOFs was designed. These examples were implemented using the MATLAB programming language and executed on a computer with Intel(R) Core(TM) i7-6700k CPU @ 4.00GHz.

5.1. Truss Design Calculation Based on User Preferences

The STSA-P method [32] can convert the user's design preferences for structural appearance into design objectives, thus achieving the effect of guiding structural optimization based on user preferences. This method was applied to design a two-hinged arch truss structure with a span of 20 m. Figure 8 shows the initial structure and load conditions. The point load F was 150 kN, and the distributed load q was 100 N/cm. The members were solid and circular, with a material density of 7.85 g/cm^3 and an elastic modulus of 210 GPa. The maximum tensile and compressive stresses were both 31 kN/cm^2. The design objectives of this example included user preferences, structural mass, and maximum node displacement.

Based on the above information, a user was invited to conduct the truss design using the STSA-P method, and the design results are presented in Figure 9 and Table 6.

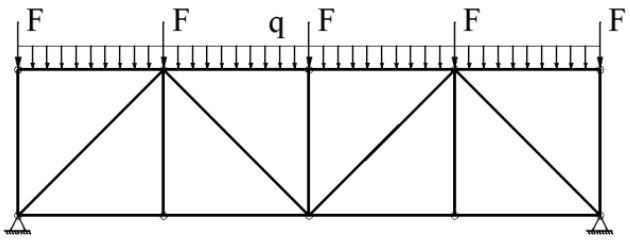


Figure 8. Initial layout of the structure.

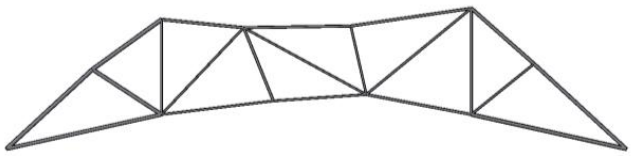


Figure 9. Design results based on STSA-P.

Table 6. Information on design objectives.

Design Objectives	User Preference	Structural Weight	Node Displacement
Value	1.665	1974.152 kg	1.424 cm
Range	"Tolerable"	"Tolerable"	"Desirable"

The design results consisted of 12 nodes, 21 member elements, and 20 DOFs. The design objective values were all within the "tolerable" and "desirable" range, and the user was also satisfied with the design results. In terms of computational efficiency for structural analysis, the total optimization time, the time of using complete analysis (Cholesky factorization) for calculation, and the time of using the selection criterion for calculation were all recorded. Based on these data, the time proportion of complete analysis was calculated to show the time contribution of structural analysis to the entire optimization process. In addition, the efficiency improvement ratio was calculated to demonstrate the efficiency improvement of structural analysis by using the selection criterion. The results are summarized in Table 7.

Table 7. Computational efficiency information based on STSA-P.

Indicator	Value
Total optimization time	26.279 s
Cholesky time	3.105 s
Selection criterion time	2.983 s
Time proportion of Cholesky	11.82%
Efficiency improvement ratio	3.93%

Based on the information presented in Table 7, it can be concluded that the total time spent on complete analysis in the design process of the STSA-P system was only 11.82%. This is due to the low structural DOFs, which resulted in shorter calculation times for structure analysis. Additionally, during the optimization process, tasks such as selecting grammar rules and calculating cost functions further decreased the proportion of time spent on structural calculation.

After applying the selection criterion, the efficiency of structural calculation increased by approximately 4%. However, the improvement ratio was limited due to the low DOFs of the structure, which were less than 50. According to the selection criterion, the complete analysis is used for shape modifications and topology modifications that involve changes

in structural DOFs. As a result, the difference between using complete analysis and the selection criterion is further minimized.

In summary, the selection criterion has limited effectiveness in improving efficiency when the number of structural DOFs is low.

5.2. Multi-DOF Truss Calculation

This section included two cases. The first case is a two-span truss structure, and the second is a four-span truss structure. The initial structures for both cases are shown in Figure 10, with a total length of 80 m and 160 m and a height of 4 m. The two-span structure contained 83 nodes with 154 DOFs, and the four-span structure contained 163 nodes with 306 DOFs. Both structures were subject to a 900 kN point load on the upper chord. The material information for both structures was the same as that of the example shown in Section 5.1.

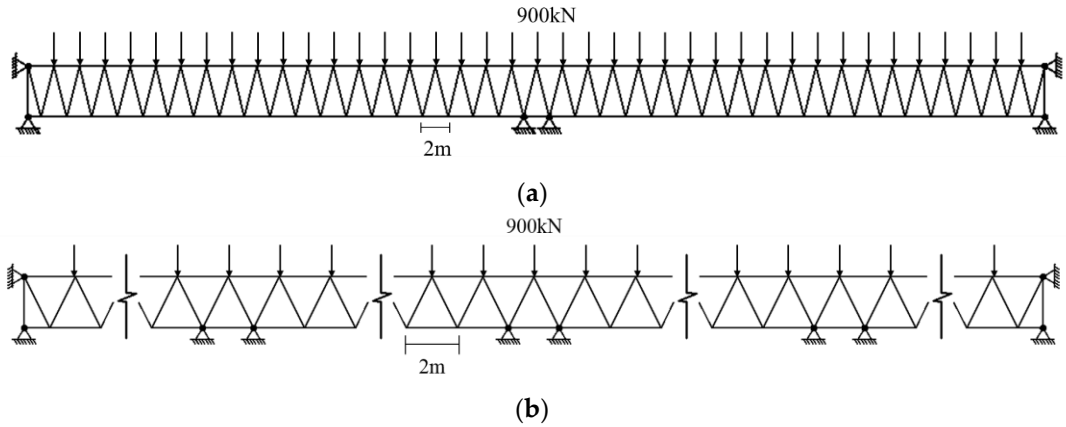


Figure 10. Initial layouts of the multi-span trusses: (**a**) case 1; (**b**) case 2.

The two cases both employed the STSA algorithm for optimization calculations, with the only optimization objective being structural mass. Throughout the optimization process, the load application points were fixed and not subjected to movement. The optimization results information and figures are shown in Table 8 and Figure 11.

Table 8. Information of design results.

Structural Information	Case 1	Case 2
Node number	79	146
Member number	155	289
Structural weight	59,695.90 kg	148,992.50 kg

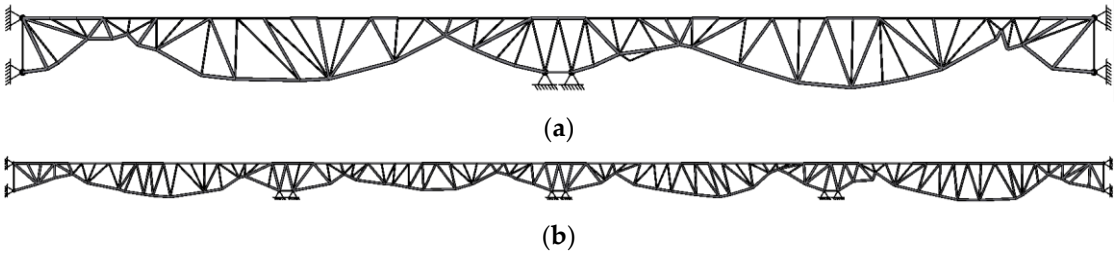


Figure 11. Design results based on STSA: (**a**) case 1; (**b**) case 2.

It can be observed in Figure 11 that both results showed a reverse arch shape at the support and between the supports, which is consistent with mechanical requirements. The computational efficiency information of the structural optimization process is summarized in Table 9.

Table 9. Computational efficiency information based on STSA.

Indicator	Case 1	Case 2
Total optimization time	291.032 s	1394.512 s
Cholesky time	116.104 s	733.956 s
Selection criterion time	100.195 s	510.468 s
Time proportion of Cholesky	39.89%	52.63%
Efficiency improvement ratio	13.70%	30.45%

Based on the results in Table 9, it can be observed that the efficiency of using the selection criterion was improved by 13.7% in case 1 (with final DOFs of 143) compared to the complete analysis, and 30.45% in case 2 (with final DOFs of 272). In addition, as shown in the example in Section 5.1 (with final DOFs of 20), an improvement in computational efficiency could be achieved with increasing structural DOFs. On the other hand, the proportion of structural calculation to the structural optimization process also increased with the increase in structural DOFs.

In summary, the selection criterion of reanalysis methods proposed in this paper was effective in improving computational efficiency, resulting in a significant reduction of computational time required for structural optimization, especially for structures with high DOFs, while ensuring calculation accuracy.

6. Conclusions

This paper presents a selection criterion for reanalysis methods that can improve the efficiency of structural analysis in the optimization of plane trusses. The proposed criterion selects automatically the most suitable reanalysis method according to the structural modification type. The effects of different structural modifications, such as size, shape, and topology, are first analyzed qualitatively to establish the correspondence between different modification types and candidate reanalysis methods. Then, the number of floating point operations (flops) and the storage space of input/output and intermediate variables (MS values) are statistically calculated for all the candidate methods. The flop ratio (*Fr*) and Ms ratio (*Mr*) are defined to investigate the most suitable reanalysis method under different structural modifications in a quantitative manner.

Based on the analysis, this study establishes a selection criterion for reanalysis methods. Specifically, when a structure undergoes size modifications and topology modifications with unchanged degrees of freedom (DOFs), the Epsilon and PCG methods are recommended for computation, respectively. The UMFT method is recommended for shape modification when the row of the first non-zero element that appears in the incremental stiffness matrix is greater than 30% of the structural DOFs. The Guyan + PCG method and the MPCG method are recommended for topology modifications with increased DOFs and decreased DOFs, respectively. It should be noted that the overall error ε of the computation results should not exceed 0.01. Otherwise, an accurate solution method should be applied.

Finally, two types of numerical examples are conducted to confirm the effectiveness of the proposed criterion, in which the selection criterion is combined with the Structural Topology and Shape Annealing (STSA) algorithm.

In future work, the extended study will focus on the exploration of more comprehensive indicators for assessing computational efficiency beyond flops and MS values. The objective is to provide a more accurate reflection of algorithmic performance. Furthermore, the development of space structure generation algorithms will be undertaken to validate the effectiveness of the proposed criterion in the computation of three-dimensional structures.

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