Article

An Assessment Scheme for Road Network Capacity under Demand Uncertainty

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Abstract: Network capacity is a vital index with which to assess the operation of traffic networks. The majority of existing traffic network capacity models are formed as bi-level programming, which maximizes the traffic flows under equilibrium constraints and is extremely dependent on the current origin–destination (O–D) travel demand. However, an accurate O–D matrix is not easy to obtain in practice. This article aims to provide an assessment method for traffic network capacity under demand uncertainty. To consider the variation in real demand in traffic networks, the current travel demand is treated as unknown parameters that are defined inside a restricted set. Based on the hypothesis of different probability distributions, the road network capacity is calculated by repeated capacity loading experiments with a random sampling of uncertain parameters. To improve the efficiency of the repeated calculations of the traffic assignment model, a sensitivity-analysis-based (SAB) approximation method was developed to avoid the double calculation of the network capacity model for each random O–D matrix. The SAB method significantly improved the calculation efficiency while ensuring accuracy. Using simulation experiments, we researched the reliability of road network capacity and the probabilities of high congestion for each link under uncertain demand.

Keywords: transportation network capacity; demand uncertainty; sensitivity-analysis-based

1. Introduction

The uncertainty of demand and supply determines that the traffic demand and the road network face a dynamic contradiction. They influence and restrict each other in the traffic system. Urban traffic planning aims to find a road network design solution by coupling traffic supply and demand. Only by taking the capacity as the indicator can the traffic system carrying capacity be effectively and reasonably brought into play and the resources of the urban traffic network be fully utilized. Therefore, the urban traffic network should have the ability to handle or absorb the changes in traffic demand, which would indicate that the traffic system can adapt to changes in travel demand. The objective of this study is to determine the maximum capacity of the traffic network in response to varying categories of traffic demand under uncertain demand conditions. This will provide a methodology for a road network design that can adapt to future regional or urban development.

The most prevalent formulation for traffic network capacity is bi-level programming, where the upper-level problem seeks to maximize traffic flow and the lower-level problem incorporates equilibrium constraints. The concept of reserve capacity was proposed by Wong and Yang [1]. Reserve capacity is the greatest multiplier that can be applied to a specific O–D demand matrix without exceeding the capacity constraints. This model has received extensive attention since it was proposed and has been used for the signal resolution control of road network optimization problems [2,3] and subsequently extended.
to traffic network performance evaluations and traffic design [4–7]. The reserve capacity method considers the impact of traffic demand spatial distribution on the overall network capacity. It is more reasonable than the max-flow min-cut theorem. However, this method assumes that traffic demands in all directions grow at the same rate and the demand structure remains unchanged. Consequently, the fixed O–D matrix has a significant impact on the solution of capacity.

With the development of the research on network capacity, an increasing number of scholars have realized that the spatial structure of traffic demand is variable and that changes in demand pattern have a significant impact on network capacity [8]. To reflect this influence, it is assumed that changes in traffic demand patterns, which may result from changes in both the total volume of traffic demand and the spatial structure, are allowed. Accordingly, the concepts of ultimate capacity and practical capacity were proposed, which provide improvements and extensions to the reserve capacity model [9]. The concept of variable demand type for road network capacity has been developed and applied [10–13]. However, in practical application, changes in the external data may cause changes in initial demand or newly increased demand distribution parameter values, and the network capacity result will no longer be a fixed value when considering these changes. Therefore, it is essential to further analyze the uncertainty of demand parameters in the study of road network capacity and identify a suitable index for evaluating the traffic network’s capacity under all possible uncertain parameters.

It is generally believed that the uncertainty in the transportation network mainly derives from three aspects: travel demand, network supply, and travel choice behavior [14]. This study focuses on how demand uncertainty affects the entire transportation network capacity. Demand uncertainty is defined as travel demand fluctuations caused by time factors, special activities, population characteristics, traffic information, etc., which may lead to the periodic congestion of transportation systems [15]. The exact O–D demand data are difficult to accurately obtain for traffic planning and management. It is typically important to assume that O–D demand is a known fixed value while researching traffic network issues. However, the deterministic demand obtained through surveys or forecasts may not be consistent with the real situation. Ignoring demand uncertainty can result in overestimations of network performance in transportation network models. Therefore, an increasing number of scholars are introducing demand uncertainty factors into research on traditional transportation problems. In the aspects of traffic network design and optimization, it includes the traffic equilibrium model [16], bus route optimization [17], multimodal transportation route optimization [18], and hub-and-spoke airline network design [19]. In addition, much logistics network research has considered the demand uncertainty, such as the location of urban–rural logistics integration design [20], urban subway underground logistics system layout problem [21], and multimodal logistics network design problem [22].

To consider the demand uncertainty in a traffic optimization model, we must introduce demand variables into the modeling process. Current uncertainty optimization theory consists of three main approaches: stochastic programming, robust optimization, and fuzzy programming [23]. To ensure that the system performance is within an acceptable range, these three methods consider all possible values of the uncertain parameters while seeking the optimal solution under the condition of constraints. In this study, an approximate method based on a sensitivity analysis is proposed as a solution to the inefficiency of the repetition capacity loading method. The sensitivity-analysis-based (SAB) approximation method is a heuristic solution strategy. The greatest benefit of the heuristic method is that it is efficient and can approximate the problem’s solution.

In this study, we propose an evaluation technique that uses the extant O–D travel demand as an unreliable parameter for the network capacity problem. It is presumed that the current demand between each O–D pair varies between its upper- and lower-limit intervals. In addition, three distinct probability distributions with interval constraints are presented to supply a constrained set to the travel demand. A sensitivity-analysis-based (SAB) solving framework is used to avoid double-calculation of the network capacity
model for possible travel demand patterns and the numerical experiment. On the basis of this method, we analyze the road network capacity under demand uncertainty and demonstrate that the link flow form of the solution to the capacity problem can disclose the high probability of congestion on each link. The results are useful for enhancing the reliability of network capacity and accurately reflecting network capacity based on demand structure. Therefore, the method is presented to enhance the theory of traffic planning.

The remainder of this paper is organized as follows. The next section analyses the uncertainty of travel demands and introduces the concept of reserve network capacity with uncertain demand. Section 3 presents an approximate algorithm based on a sensitivity analysis to efficiently solve the traffic network capacity model under uncertain demand. Section 4 provides a numerical example to explore the effect of demand uncertainty on a road network. On an example network, the effectiveness of the SAB method is demonstrated. The performance of the SAB method for network capacity assessment is also examined. The conclusion is presented in Section 5.

2. Network Capacity Problem under Demand Uncertainty
2.1. The Uncertainty of Network Capacity

In this section, an easy-to-understand numerical example is offered to demonstrate the effect of travel demand uncertainty on network capacity assessment results. Given a road network (illustrated in Figure 1) consisting of seven road links and six nodes, the initial demands for O–D pairs (1, 3) and (2, 4) are 30 and 20, respectively. In this numerical example, the following BPR-type link performance function is assumed for the link:

$$t_a(v_a) = t_0^a (1 + 0.15(v_a/C_a)^4),$$

where $t_0^a$ and $C_a$ are the free flow time and capacity of link $a$, respectively. Table 1 shows the free flow time and capacity of each link.

![The example network.](image)

**Legend:**
- **Node number**
- **Link number**

**Table 1.** Link information for the example network is shown in Figure 1.

<table>
<thead>
<tr>
<th>Link</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_0^a$</td>
<td>10</td>
<td>4</td>
<td>12</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>$C_a$</td>
<td>100</td>
<td>80</td>
<td>80</td>
<td>50</td>
<td>120</td>
<td>50</td>
<td>50</td>
</tr>
</tbody>
</table>
Figure 2 shows the variation in network capacity with different O-D demand patterns and illustrates fluctuations in the value of the capacity. The capacity of the network is also shown to be uncertain with uncertain O-D travel demands.

2.2. Travel Demand Uncertainty Analysis

While traffic demand forecasts are underway, it is necessary to divide the road network into traffic zones and conduct a large number of macro-O-D surveys. The division of traffic zones and O-D distribution of traffic zones are greatly affected by land use, population, and socio-economic activities. With the continuous expansion of the urban scale, the changes in land-use patterns and urban economic activities will lead to continuous changes in O-D demand. In reality, the demand in the transportation system is constantly changing over time and will even vary from hour to hour. A large number of simultaneous data surveys are required to predict the existing travel demand on the traffic network; there may be discrepancies between the survey data and the actual data, and the discrepancies in these survey data will impact the precision of the existing O-D distribution matrix. Figure 3 depicts the origins and spread of O-D demand uncertainty.

![Figure 2](image-url)  
**Figure 2.** The test network capacity under different travel demand patterns.

**Figure 3.** Illustration of sources and propagation of O-D demand uncertainty.
Therefore, travel demand uncertainty means that the existing travel demand is often difficult to obtain in the practice traffic project and is also not easy to express with permanent values. In the reserve capacity model, the current O–D matrix is typically used as the reference demand matrix, and its patterns have a significant impact on the outputs. In this study, we treat travel demand as the variable and select appropriate methods to measure the uncertainty of existing travel demands.

The most common measures of uncertainty at present are random numbers and interval numbers. Supposing the travel demand satisfies a known random distribution, the random number method can generate a series of random numbers to handle the demand uncertainty problem. If the fluctuation of travel demand is not limited, this will necessitate a capacity of infinity to accommodate all conceivable demand patterns. Thus, the transit demand uncertainty set is defined as an interval [24]. The interval may represent the confidence interval of an estimated demand derived from a survey or O–D matrix estimation. This study combines these advantages by using random numbers and the interval constraint to measure the demand uncertainty.

2.3. Road Network Capacity Model with Uncertain Demand

In this subsection, we extend the classical reserve capacity model by treating the initial/current O–D demand as an uncertain parameter. The following two reasons are given for using the reserve capacity model to assess capacity of the network with uncertain demand:

1. Network capacity is represented by a uniformly increased or decreased multiplier; therefore, the reserve capacity model is straightforward to solve.
2. Given that the initial/current O–D matrix is regarded as an uncertain parameter in the reserve capacity model, the demand distribution is no longer fixed but variable within a given range.

Firstly, the deterministic reserve capacity model is given as follows.

\[
\begin{align*}
\max & \quad \mu \\
\text{s.t.} & \quad v_a(\mu) \leq C_a, \forall a \in A,
\end{align*}
\]

where \( v_a(\mu) \) represents the flows on link \( a \), which is an implicit function of the multiplier \( \mu \), the relationships of which are described in the PSL model [25].

\[
\min \int_0^{t_a} t_a(x)dx - \frac{1}{\theta_k} \sum_{ij} \sum_k f_{ij}^k \ln \alpha_{ij}^k + \frac{1}{\theta_k} \sum_{ij} \sum_k f_{ij}^k \ln f_{ij}^k,
\]

\[
\text{s.t.} \quad \sum_{k \in K_{ij}} f_{ij}^k = \mu q_{ij}, \forall i \in I, j \in J,
\]

\[
v_a = \sum_i \sum_j \sum_{k \in K_{ij}} f_{ij}^k \cdot \delta_{ijk}, \forall a \in A,
\]

\[
f_{ij}^k \geq 0, \forall i \in I, j \in J, k \in K_{ij},
\]

where \( C_a \) represents the capacity of road link \( a \). \( f_{ij}^k \) represents the flow between O–D pairs \((i, j)\) on route \( k \). \( \theta_k \) represents the dispersion parameter related to route selection. \( \delta_{ijk} \) represents the link/route incidence parameter and equals 1 if route \( k \) (from \( i \) to \( j \)) travels through link \( a \) and 0 otherwise. \( t_a(v_a) \) represents the travel time function of the link \( a \). \( \alpha_{ij}^k \in (0, 1) \) represents the path-size factor, which aims to deal with the route-overlapping problem. The typical form is given by Ben-Akiva and Bielaire [25]:

\[
\alpha_{ij}^k = \sum_{a \in A} L_{ij}^a \frac{1}{L_k^a \sum_{l \in K_{ij}} \delta_{ijk}^l}, \forall k \in K_{ij}, i \in I, j \in J,
\]
where \( L_a \) represents the length of link \( a \), \( L_k^{ij} \) represents the length of route \( k \) connecting O-D pairs \((i, j)\), and \( A_k \) represents the collection of all links of route \( k \) between O-D pairs \((i, j)\).

In the above network capacity model, the objective function Equation (2) for the upper-level problem aims to maximize the multiplier of O-D demand. Constraint (3) indicates that the flow of each link cannot exceed its physical volume constraints. The objective function Equation (4) for the traffic assignment considers both route choice and route overlap problem, in which the first term is the classical Beckmann transform term, the second term considers the path-size factor for dealing with route overlap, and the third term is connected to the entropy maximization for stochastic equilibrium. Equations (5) and (6) are defined as demand conservation and flow conservation, respectively. Equation (7) is the nonnegative constraint.

Further, according to the above reserve capacity network model, we introduce the uncertain demand variable \( q \in Q \), where \( Q = \{ q_{ij} \mid q_{ij} \leq q_{ij}^{U} \} \) is an uncertain demand set based on existing demand. Figure 4 compares the exact solution approach and the approximate approach. Clearly, it is very expensive to solve the network capacity with all possible O-D demand patterns because solving the traffic assignment model requires a large number of iterations. Hence, an approximate solution algorithm is proposed in the next section aiming to address the inefficiency of exact solution approach in solving network capacity with uncertain travel demand.

3. Sensitivity-Analysis-Based Approximation Method

To avoid repeatedly solving the capacity model, we present a sensitivity-analysis-based (SAB) approximate algorithm to solve the network capacity model with uncertain demand. It is crucial for the SAB algorithm to analyze the sensitivity of the PSL model. In this study, the sensitivity analysis method for nonlinear programming developed by Fiacco [26] can be used to derive the analytical derivative relationship.

The three key steps of the SAB algorithm are as follows.
1. The solution to the network capacity problem for a given nominal O–D demand;
2. Perturbation analysis of the travel demand \( q \);
3. The linear approximation of the reaction function.

3.1. Solving the Capacity Model with Nominal Demand

The capacity model with the nominal O–D demand is solved as the first stage of the SAB approximation method. The Algorithm 1 is a description of the process [27].

**Algorithm 1:** Solving the capacity model with the nominal O–D demand.

Step 0: Initialization. Determine the initial values of the O–D demand, \( q_0 \in Q \), and initial value \( \mu(0) \). Set \( n = 0 \).

Step 1: Solving the PSL model. Solve the PSL model based on \( \mu(n) \), and obtain the equilibrium solution \( v_a \).

Step 2: Sensitivity analysis. Use the sensitivity analysis approach to obtain the partial derivatives \( \nabla_{\mu} v(\mu(n)) \) for the PSL model.

Step 3: Local linear approximation. By employing the derivatives to create local linear approximations of the upper-level capacity restrictions, the approximate linear programming problem can then be solved to create an auxiliary multiplier \( \hat{\mu}(n) \).

Step 4: Updating solution. Let \( \mu(n+1) = \mu(n) + \alpha(\hat{\mu}(n) - \mu(n)) \), where \( \alpha \) is a predetermined step size, e.g., \( \alpha = 1/n \).

Step 5: Convergence criterion. Stop if the convergence requirement has been met; if not, return to Step 1 and set \( n := n + 1 \).

See Appendix A for a derivation of the sensitivity analysis of the PSL model.

3.2. Sensitivity Analysis under Demand Uncertainty

The sensitivity analysis of the demand matrix multiplier and the perturbation analysis of the demand variables serve as the foundation of the SAB approximation method with uncertain demand. In Appendix A, a sensitivity analysis of the demand matrix multiplier is provided. We present an analytical derivative relationship between the decision factors in PSL model and the demand variable \( q \) in the subsection [26].

The Lagrange function of the PSL model (formulated in Equations (4)–(7)) involving the perturbed O–D demand \( q \) can be written as follows.

\[
L\left(f_{ij}^{ij}, \lambda_{ij}, q\right) = \sum_{a} f_{ij}^{a} t_{a}(x) dx - \frac{1}{\theta} \sum_{ij} f_{ij}^{ij} \ln \omega_{ij}^{ij} + \frac{1}{\theta} \sum_{ij} f_{ij}^{ij} \ln f_{ij}^{ij} + \sum_{ij} \lambda_{ij} (\mu \cdot q_{ij} - \sum_{ij} f_{ij}^{ij}),
\]

The \( f_{ij}^{ij} \) and Lagrange multiplier \( \lambda_{ij} \) are derived with regard to the route flow as follows.

\[
\frac{\partial L\left(f_{ij}^{ij}, \lambda_{ij}, q\right)}{\partial f_{ij}^{ij}} = \sum_{a} t_{a}(v_{a}) \delta_{ij}^{a} \theta - \frac{1}{\theta} \ln \omega_{ij}^{ij} + \frac{1}{\theta} (\ln f_{ij}^{ij} - 1) - \lambda_{ij} = 0,
\]

\[
\frac{\partial L\left(f_{ij}^{ij}, \lambda_{ij}, q\right)}{\partial \lambda_{ij}} = \mu \cdot q_{ij} - \sum_{ij} f_{ij}^{ij} = 0,
\]

Let \( X = (f_{ij}^{ij}, \lambda_{ij}) \); thus, when the perturbed variable is the O–D demand \( q \), the derivative of \( X \) with regard to \( q \) is as follows.

\[
\begin{bmatrix}
\nabla_{q} f_{ij}^{ij} \\

\nabla_{q} \lambda_{ij}
\end{bmatrix}
= 
\begin{bmatrix}
\nabla^{2} L & -A^{T} \\
-A & O
\end{bmatrix}^{-1}
\begin{bmatrix}
O \\

\nabla q (\mu q)
\end{bmatrix},
\]
where $\nabla^2 L = \Delta^T \nabla \phi (0, 0) \Delta + \frac{1}{\sigma^2} \text{diag} \left(1/f_{ij}^d\right)$; furthermore, the derivative of the link flow $v$ with regard to the OD demand $q$ is as follows.

$$\nabla_q v = \Delta \cdot \nabla_q f,$$

(13)

### 3.3. Linear Approximation of the Reaction Function

The proposed model with demand uncertainty can be resolved by shifting to a single-level programming, where the implicit relationship is expressed by a first-order Taylor expansion, as illustrated below:

$$v_a(\mu q) \approx v_a(\mu^* q) + \left[ \frac{\partial v_a(\mu q)}{\partial \mu} \right]_{\mu=\mu^*} (\mu - \mu^*) + \sum_{i \in I} \sum_{j \in J} \left[ \frac{\partial v_a(\mu^* q)}{\partial q} \right]_{q=q_m} \cdot \epsilon_q,$$

(14)

where $\mu^*$ is the max demand multiplier with the nominal demand for the capacity model, and $q_m$ is the demand equilibrium solution of the capacity model under the nominal demand value, which is the product of $\mu^*$ and the nominal demand. $\epsilon_q$ is the perturbation of disruptions to travel demand in the network.

Considering the optimal solution of the overall network capacity model with nominal demand, the reserve capacity under demand uncertainty can be estimated by utilizing the derivatives with regard to both the demand multiplier $\mu$ and travel demand $q$. The sensitivity analysis approach is described in Appendix A and Section 3.2. As a result, the capacity with uncertain demand can be approximated to the following single-level programming.

$$\max \mu,$$

s.t. $v_a(\mu q) \approx v_a(\mu^* q) + \nabla_{\mu} v_a(\mu^* q) (\mu - \mu^*) + \nabla_q v_a(\mu^* q) \cdot \epsilon_q \leq C_a, \forall a \in A,$$

(16)

The procedure of repeated volume-loading experimental are described as Figure 5.

**Figure 5.** Flowchart of sensitivity-based approximate algorithm.
4. Numerical Experiments

4.1. The Random Sampling Distribution of the Travel Demand

This study considers a transportation network with uncertain demand from the Sioux-Falls network, as shown in Figure 6. It contains 24 nodes, 76 links, and 528 O–D pairs. The road network has 1584 available routes serving 528 O–D pairs by the link penalty and link elimination method [28]. The link characteristics (i.e., link capacity, free flow time, length) are equal to those in the original data. The data of the network were obtained via the link https://github.com/bstabler/TransportationNetworks, accessed on 10 May 2022. In this numerical example, the BPR-type link performance function shown in Equation (1) was assumed to be performed. Note that all experiments are performed on Microsoft Windows 10 operating system with Intel(R) Core (TM) i5-4210H CPU @ 2.90 GHz, 4.00 GB RAM. The approximate algorithm is coded in MATLAB 2018a.

In this study, we considered three distributions of uncertain travel demand in the interval constraint: the uniform distribution $[0.5 \bar{q}_{ij}, 1.5 \bar{q}_{ij}]$, the normal distribution $N(\bar{q}_{ij}, 0.25 \bar{q}_{ij}^2)$, and the exponential distribution with parameters $\bar{q}_{ij}$, with the interval $[q_{ij}^L, q_{ij}^U]$, $\forall i, j$. The O–D pair of the network is considered to fluctuate separately, and the boundary of the interval are defined as $q_{ij}^L = 1.5 \bar{q}_{ij}$ and $q_{ij}^U = 0.5 \bar{q}_{ij}$, respectively. All the travel demand samples in the following numerical analysis are randomly sampled from these three travel demand sets. The truncated probability distribution is used to avoid the result of the capacity of the road network caused by the low probability travel demand. Similarly, the failure of the sensitivity analysis method can be avoided by adding interval constraints, which is analyzed in detail in the error analysis section. Figure 7 shows the sample of three O–D pairs.

Figure 6. Sioux-Falls Network.
4.2. Analysis of Network Capacity Reliability

The reliability model of network capacity under uncertain demand is as follows:

$$R(\mu_r q_r) = P(\mu_{\text{max}} q \geq \mu_r q_r),$$

where $R(\mu_r q_r)$ denotes the network capacity reliability when travel demand is $q_r$; $\mu_{\text{max}}$ denotes the network’s largest travel demand multiplier; $\mu_r$ denotes the demand matrix multiplier corresponding to travel demand $q_r$.

The reliability of the Sioux-Falls network capacity was calculated based on 500 random samples. Figure 8 displays the outcomes of the network capacity reliability investigation. According to the three probability distributions, the reliability of the network capacity with nominal demand ranges from 25% to 35%, making the network capacity unreliable when solely considering deterministic demand. The findings of this study can help avoid grossly overestimating the road network capacity and obtain a more reasonable value.
we used the methodology proposed by Qin et al. [29] to evaluate road network efficiency. This approach to figuring out network efficiency considers the impact of network size (the patterns and the structure of the road network, so as to enhance the efficiency and the demand with exponential distribution. It helps avoid grossly overestimating the road network capacity and obtain a more reasonable reliable when solely considering deterministic demand. The capacity with nominal demand ranges from 25% to 35%, making the network capacity un-
gation. According to the three probability distributions, the reliability of the network ca-
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gation. The efficiency of the road network.

Figure 9 shows the efficiency of the network under uncertain demand. The road net-
work efficiency under nominal demand. Network efficiency and network capacity are positively correlated. Network efficiency under demand uncertainty is 18% higher and 82% lower than that under nominal demand. According to Figure 9, network efficiency under demand uncertainty is 18% higher and 82% lower than that under nominal demand. Network efficiency and network capacity are positively correlated. Therefore, travel demand patterns have a great impact on the efficiency of the road network, meaning that we need to carry out demand management to further couple travel demand patterns and the structure of the road network, so as to enhance the efficiency and the capacity of the road network.

\[ E = \frac{1}{n_A} \sum_{a \in A} \frac{x_a}{l_a(x_a)} \]  

where \( n_A \) represents the number of the links. \( x_a \) donates the flows of the links.

Figure 9 shows the efficiency of the network under uncertain demand. The road network efficiency under demand uncertainty is 18% higher and 82% lower than that under nominal demand. Network efficiency and network capacity are positively correlated. Therefore, travel demand patterns have a great impact on the efficiency of the road network.
was solved for 500 travel demand samples; the highly saturated numbers are shown in Figure 10.

The largest saturation rate is the bottleneck that affects the capacity of the whole network. However, the location of the bottleneck links is also different for different varieties of demand pattern. We must consider the impact of uncertain demand patterns on critical link identification results. The highly saturated number is counted at each link whenever its V/C ratio is greater than 0.9 [23]. As a result, link saturation ratios are calculated as [total saturated number/number of samples].

This subsection only verifies the 500 random samples with normal distribution \( N(q_{ij}, 0.25q_{ij}^2) \) and the interval \([q_{ij}^L, q_{ij}^U]\), \( \forall i, j \). The experiments calculated the V/C of each link under the nominal travel demand in Figure 10a. Then, the high-saturation rate was solved for 500 travel demand samples; the highly saturated numbers are shown in Figure 10b. The numbers in Figure 10 are the same as in Figure 6. Only the V/C of link 10-16 and link 16-10 was greater than 0.9 under the nominal demand. However, there are six links of the road network where V/C is likely to be greater than 0.9 under demand uncertainty. Figure 10 shows that links 10-16 and 16-10 have high saturation rates between 85% and 98%, and link 16-17, link 17-16, link 17-19, and link 19-17 have high saturation rates between 20% and 35%.

![Figure 9](image-url)  
**Figure 9.** The efficiency of the Sioux-Falls network.

### 4.4. Critical Link Identification with Uncertain Demand

Volume-to-capacity (V/C) ratio measures the link’s congestion. The link with the largest saturation rate is the bottleneck that affects the capacity of the whole network. However, the location of the bottleneck links is also different for different varieties of demand pattern. We must consider the impact of uncertain demand patterns on critical link identification results. The highly saturated number is counted at each link whenever its V/C ratio is greater than 0.9 [23]. As a result, link saturation ratios are calculated as [total saturated number/number of samples].

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![Figure 10](image-url)  
**Figure 10.** (a) The V/C rate of each link under nominal demand; (b) the link saturation under uncertain travel demand.
Traffic managers can apply traffic management methods for important links to raise the capacity of links and optimize traffic network design in order to improve the effectiveness of network operation and boost network capacity. In this section, based on the critical link results with uncertain demand obtained from the above solution, the capacity of the six high-saturation links derived in the previous subsection is increased by 10%. The before-and-after comparison of the capacity with nominal demand and uncertain demand is shown in Figure 11. The reliability of the network capacity value of 68,505 increased from 0.8 to 1. Table 2 shows that the 80% reliability of network capacity under normal distribution with uncertain demand increased by 23%, and the improvement under the nominal demand case increased the road network capacity by only 15%, which was due to the traffic managers failing to fully understand the information about the critical links. Thus, critical link analysis under uncertain demand using the high-saturation-link analysis method can aid traffic managers in obtaining more comprehensive information and serve as a basis for improving network performance.

![Figure 11. The improvement in network capacity under critical link.](image)

### Table 2. The comparison of network capacity.

<table>
<thead>
<tr>
<th></th>
<th>80% Reliability of Network Capacity with Uncertain Demand</th>
<th>Network Capacity with Nominal Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before improvement</td>
<td>68,505</td>
<td>73,202</td>
</tr>
<tr>
<td>After improvement</td>
<td>84,376</td>
<td>84,020</td>
</tr>
<tr>
<td>Improved ratio</td>
<td>23%</td>
<td>15%</td>
</tr>
</tbody>
</table>

### 4.5. Critical OD Pair Identification with Uncertain Demand

In this section, based on the sensitivity analysis of link flow to nominal demand proposed in Section 3.2, we obtain the results of the sensitivity analysis of demand perturbations on key links link 10-16 and link 16-10. The result is shown in Figure 12. We obtain the five O–D pairs which affect the traffic of the critical road section the most and call them the critical OD pairs of the traffic network. O–D (9,16), O–D (10,7), O–D (11,16), O–D (10,16), and O–D (11,18) are the five O–Ds most affected by the disturbance of travel demand on link 10-16. O–D (16,9), O–D (7,9), O–D (16,11), O–D (16,10), and O–D (18,11) are the five O–Ds most affected by the disturbance of travel demand on link 16-10.
According to the result of O–D pair identification, we can take traffic demand management measures to control the change of key O–D pairs reasonably and improve the coupling between the demand distribution and the road network structure, for example, increasing the proportion of bus trips between critical O–D pairs to replace cars, thus reducing the total demand between the critical O–D pairs. This will improve the overall capacity of the road network. This study assumes a 30% reduction in critical O–D demand, an increase in demand matrix multiplier from 0.204 to 0.225, and an 8.72% increase in road network capacity as shown in Figure 13.

![Figure 12](image1.png)

**Figure 12.** (a) Link 10-16 sensitivity analysis results for travel requirements; (b) Link 16-10 sensitivity analysis results for travel requirements.

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![Figure 13](image2.png)

**Figure 13.** The improvement in network capacity under critical O–D pairs.

4.6. **Performance of the Sensitivity-Based Approximate Approach**

4.6.1. Error Analysis

For three different distributions, we randomly sampled 200 travel demand results. The comparison of the error of the exact solution was obtained by repeatedly solving the capacity model, and the estimated solution of the approximate approach is shown in Figure 14. The error of the two solutions under uniform distribution and normal distribution was within 5%. The error of the two solutions under exponential distribution has 199 results within 5%, and only one error value was greater than 5%, at 8.96%. The error analysis of the approximate solutions shows that the results of the SAB approximation approach for solving the network capacity values are reliable.
According to the principle of sensitivity analysis, the error is related to the nominal travel demand. When the nominal travel demand is the mean of the O-D demand distribution, the average error is minimum. The effectiveness of the sensitivity analysis algorithm is also related to the upper and lower bounds of the interval constraints. In this study, five different intervals were selected to validate the results of the model, which were $[0.5\bar{q}_{ij}, 1.5\bar{q}_{ij}]$, $[0.4\bar{q}_{ij}, 1.6\bar{q}_{ij}]$, $[0.3\bar{q}_{ij}, 2\bar{q}_{ij}]$, $[0.2\bar{q}_{ij}, 3\bar{q}_{ij}]$, and $[0.1\bar{q}_{ij}, 4\bar{q}_{ij}]$. $\bar{q}_{ij}$ is the nominal value for travel demand. The success of the algorithm is judged by the result of the travel demand matrix multiplier. The success rate of the algorithm is calculated as [total number of the positive travel demand matrix multiplier/number of samples]. From the results, we can see that when the interval constraint is $[0.4\bar{q}_{ij}, 1.6\bar{q}_{ij}]$, the success rate of the sensitivity-based analysis algorithm is 100%, which shows that the algorithm is effective. However, the success rate of the algorithm will decrease as the travel demand range expands, and we think the algorithm begins to fail. Therefore, the application of the proposed method in the actual project needs to pre-assess the range of travel demand fluctuations.

4.6.2. The Efficiency of SAB Algorithm

By comparing the solution time of the exact solution with the approximate solution for 50 samples (shown in Table 3), we observed that the time required to repeatedly solve the bi-level programming model was 5066.8 s, and the solution time of the SAB approximate algorithm was only 114.7 s. The solution efficiency is improved by 97.7%.

### Table 3. Comparison of computation time with exact and approximate solutions.

<table>
<thead>
<tr>
<th></th>
<th>Time (s)</th>
<th>Total (50 Samples)</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact solution approach</td>
<td>5066.8</td>
<td></td>
<td>101.34</td>
</tr>
<tr>
<td>SAB approximate approach</td>
<td>114.7</td>
<td></td>
<td>2.29</td>
</tr>
</tbody>
</table>

5. Conclusions

In this study, we developed an approximation method based on sensitivity analysis for a reserve capacity model with uncertain demand. We took into account the impact of the existing travel demand on the capacity of the reserve network, as well as the complexity of the regional and temporal changes in the travel demand. Additionally, this approximation method can mitigate the overly conservative scenario of robust optimization solving for network capacity along with the low efficiency of classic repetitive solving for reserve network capacity under various trip demand patterns. The reserve network capacity of travel demand on a bounded interval constraint can be solved using the sensitivity-based approximation approach since the error of the reserve capacity value is within a reasonable range when comparing the approximate approach and the exact solution approach.
order to increase the overall effectiveness of the traffic network, this study optimized the crucial links and determined the reliability of the reserve network capacity with uncertain demand. As a result, transportation managers and planners could theoretically be guided by this evaluation method for road network capacity. However, this study only considered the impact of different types of travel demand and the lack of travel demand fluctuation factor analysis. In future research, we can carry out this part of the research, such as weather conditions and other factors on the impact of travel demand quantitative analysis. At the same time, this study identifies critical links under uncertain demand and uses the assumed link improvement results to analyze the changes in road network capacity. In the follow-up study, we can further use the method proposed in this paper to identify the key links. In the actual road network, we take practical traffic management measures to study the improvement of road network capacity.

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**Appendix A. Sensitivity Analysis of the PSL Model**

The Lagrange function of the lower-level model (formulated in Equations (4)–(7)) involving the perturbed O–D multiplier $\mu$ can be written as follows.

$$
L(f_{ij}^k, \lambda_{ij}, \mu) = \sum_a \int_a^0 t_a(x)dx - \frac{1}{\theta} \sum_k f_{ij}^k \ln \omega_{ij}^k + \frac{1}{\theta} \sum_k f_{ij}^k \ln f_{ij}^k + \sum_{ij} \lambda_{ij} (\mu \cdot q_{ij} - \sum_{ij} f_{ij}^k),
$$

(A1)

The $f_{ij}^k$ and Lagrange multiplier $\lambda_{ij}$ are derived with regard to the route flow as follows.

$$
\frac{\partial L(f_{ij}^k, \lambda_{ij}, \mu)}{\partial f_{ij}^k} = \sum_a t_a(x) \delta_{ij}^a r - \frac{1}{\theta} \ln \omega_{ij}^k + \frac{1}{\theta} (\ln f_{ij}^k - 1) - \lambda_{ij} = 0, \quad (A2)
$$

$$
\frac{\partial L(f_{ij}^k, \lambda_{ij}, \mu)}{\partial \lambda_{ij}} = \sum_a t_a(x) \delta_{ij}^a r - \frac{1}{\theta} \ln \omega_{ij}^k + \frac{1}{\theta} (\ln f_{ij}^k - 1) - \lambda_{ij} = 0, \quad (A3)
$$

Let $X = (f_{ij}^k, \lambda_{ij})$; the derivative of $X$ with regard to $\mu$ is as follows.

$$
\begin{bmatrix}
\nabla_\mu f_{ij}^k \\
\nabla_\mu \lambda_{ij}
\end{bmatrix}
= \left[ \nabla^2 L \begin{bmatrix} \Lambda^T & 0 \\ -\Lambda & O \end{bmatrix}^{-1} \begin{bmatrix} O \\ \nabla_\mu (\mu q) \end{bmatrix} \right],
$$

(A4)

where $\nabla^2 L = \Delta^T \nabla_\mu t(v^*, 0) \Delta + \frac{1}{\theta} \text{diag} \left( 1/f_{ij}^k \right)$.

Furthermore, the derivative of the link flow $v$ with regard to multiplier $\mu$ can be obtained as:

$$
\nabla_\mu v = \Delta \cdot \nabla_\mu f,
$$

(A5)
References


2. Ceylan, H.; Bell, M.G.H. Reserve capacity for a road network under optimized fixed time traffic signal control. *Intell. Transp. Syst.* 2004, 8, 87–99. [CrossRef]


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