

Optimal Design and Analysis of Nonlinear Tuned Mass Damper System

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Featured Application: The derived nonlinear design formulas can make nonlinear tuned mass dampers (NTMDs) get a better vibration reduction optimization effect in different cases, which has certain reference significance for the subsequent research of NTMDs and also has certain engineering significance.

Abstract: The tuned mass damper (TMD), as a representative passive control device, has been widely used in various fields such as mechanical vibration and civil engineering. In practical engineering applications, nonlinear characteristics of the TMD inevitably occur due to the installation of limiting devices or large displacements. Neglecting such nonlinearity may adversely affect the control performance of the TMD. Based on this situation, the parameters of the nonlinear tuned mass damper (NTMD) were analyzed, considering the linear and cubic nonlinear stiffness of the TMD, in order to improve its control performance. In this paper, a numerical model of a single-degree-of-freedom controlled structure using an NTMD was established, and the complex variable averaging method was employed to derive the response expression of the structure under the 1:1 resonance condition. Subsequently, the nonlinear design equation for the NTMD frequency was obtained. Based on this nonlinear design formula, the design accuracy was analyzed under 1:1 resonance, undamped and softened stiffness conditions. The results indicate that the NTMD based on a nonlinear design exhibits good control performance.

Keywords: tuned mass dampers; nonlinear design; passive control device; approximate analysis method

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1. Introduction

Building structures are often subjected to complex and unpredictable forces due to natural disasters such as earthquakes [1]. In order to mitigate the adverse effects of these vibrations on structures, various techniques for reducing structural vibrations have been developed [2]. Among the vibration control devices, smart and passive control devices are the most widely used [3–5]; smart dampers are further divided into two types of active and semi-active control systems. In passive control devices, the tuned mass damper (TMD), as a common damping device, is widely used in building, machinery and vehicle systems [6]. It is usually installed on the top layer of the main structure and forms a resonance mechanism when the TMD is tuned to the frequency of the main structure. When the main structure is subjected to external forces or excitations, the mass of the TMD moves in synchronization with the vibration of the structure and undergoes tuned motion due to the action of the spring. This results in the dissipation of energy through the TMD, effectively reducing the amplitude of the main structure's vibration. By improving the vibration characteristics of the building structure or other mechanical systems, the TMD enhances the stability and safety of the structure or system [7,8].

The TMD originated in 1911 when Frahm [9] invented the undamped dynamic vibration absorber (DVA). However, due to its lack of damping, the control effect of the DVA in frequency bands other than tuning was even inferior to that of the uncontrolled structure. In 1956, Den Hartog et al. [10] improved the DVA by introducing damping, thus enhancing its vibration-damping effect and gradually forming the concept of a TMD. Den Hartog further optimized the frequency ratio and damping ratio of the TMD and proposed a specific optimization formula, which served as the basis for the optimal parameter design method of the TMD [11]. Setareh [12] used the integral modal method they discovered to obtain the optimal damping of the TMD. Over the past decades, researchers have conducted numerous studies on TMDs. Li et al. [13] conducted a systematic theoretical and experimental study on the optimal values of structural parameters for TMDs. Qin-Li et al. [14] investigated the control effects of TMDs when different methods were used to determine the TMD design parameters for seismic control. Additionally, researchers have made efforts to develop more general and effective TMDs. Clark extended the single-degree-of-freedom TMD to a multi-degree-of-freedom TMD and proposed the concept of a multi-tuned mass damper (MTMD) [15]. Alamzan proposed a bi-directional homogeneous TMD (BH-TMD) [16]. Zhan et al. [17] utilized a distributed TMD with configuration and parameter optimization to reduce vibrations in multimodal cable-stayed bridges.

Theoretical studies of the TMD were carried out based on a linear TMD at an early stage of development [18–20]. However, linear systems are special cases, and the vast majority of vibrations are nonlinear vibrations. The vibrations of linear systems are only ideal cases obtained by neglecting some parameters. The study of NTMDs, as a practical direction of nonlinear vibrations, has been widely and deeply investigated by more and more researchers. In 1952, Robertson [21] introduced the concept of bandwidth and defined it as the region between two peaks of the amplitude–frequency curve. He found that the bandwidth of an NTMD is longer than that of a linear TMD and considered that the NTMD has more advantages than the linear version. He verified this conclusion by studying the structure of mass dampers with the introduction of three-time hardening stiffness. In 1992, Natsiavas [22] obtained the approximate steady-state solution of the three-time stiffness structure using the averaging method and proved the validity of the approximation method. Jiang et al. [23] analyzed the stability of the structure of a mass damper system with three-time nonlinear stiffness using both experimental and approximate analysis methods, and they verified the accuracy of the approximate analysis method. Manevitch et al. [24] used the approximate analysis method to investigate the stability of a mass damper system with the bifurcation of triple nonlinear stiffness and verified its validity through numerical solutions. Subsequently, Gianluca Gattia et al. [25] analyzed a system of nonlinearly tuned mass dampers with stiffness due to geometric position and investigated the closed separated resonant section of the frequency–amplitude response curve. Gendelman et al. [26] proposed that the control frequency range of a nonlinear energy sink (NES) is wider than that of linear TMDs. Quinn et al. [27] found that an NES can distribute the input energy of a structure from low to high modes. Djemal et al. [28] demonstrated the existence of jumps in nonlinear TMDs. Alexander and Schilder [29] argued that the only way to make the steady-state response of a system better than that under linear TMD control is to eliminate or reduce the high-amplitude periodic solutions generated by nonlinear TMDs. Li and Zhang [30] used an approximate analytical method to obtain an optimal expression for the TMD frequency of a cubic stiffness nonlinear mass-damped system and verified its applicability and effectiveness through numerical methods. Zhang et al. [31] derived a design method for solving the optimal mass ratio of the TMD using the complex variable averaging method and the multiscale method and demonstrated that the method possesses better performance. Li et al. [32] used the harmonic balance method (HBM) to determine the steady-state amplitudes of structures controlled by nonlinear TMDs, obtained the formula for the optimal frequency ratio of TMDs and compared the analytical results with those obtained from numerical calculations to prove the correctness and validity of the formula. Hu et al. [33] used the complex

variable averaging method to derive the approximate analytical solution of the system amplitude and obtained the theoretical optimized design parameters of the TMD, which proved that the optimized parameters of the TMD obtained by a nonlinear design have a good vibration reduction effect before and after the nonlinear characteristics of the TMD are generated. In recent years, there have been significant developments in engineering applications of tuned mass dampers, leading to the introduction and implementation of various new types of tuned mass dampers that have been improved and innovated from the original design. Wang et al. proposed an adaptive-passive eddy current pendulum tuned mass damper (APEC-PTMD), which can retune the frequency by changing the length of the pendulum and adjusting the damping ratio. The paper [34] verifies the function of frequency and damping retuning. Subsequently, Wang et al. applied the APEC-PTMD to a 40-story high-rise building under four different soil conditions. The APEC-PTMD was able to identify the optimal TMD frequency for different soil types and reset its damping ratio, resulting in better seismic performance [35]. Another development is the semi-active eddy current pendulum tuned mass damper (SAEC-PTMD) proposed by Wang et al. This type of TMD can adjust the frequency and damping ratio in real time to ensure the stability of the controlled structure [36]. Wang et al. also applied the adaptive tuned mass damper to the vibration control of pedestrian bridges, achieving excellent results in vibration control [37]. With their higher reliability, wider applicability and increased intelligence, these new types of tuned mass dampers provide more effective solutions for engineering applications. They will play an important role in mechanical and vibration-damping fields and make a positive contribution to enhancing structural safety.

In reality, a linear tuned mass damper (TMD) represents an ideal case. However, due to the occurrence of large displacements and the incorporation of limiters, TMDs inevitably exhibit nonlinear characteristics. Neglecting these characteristics can have a detrimental effect on the control performance of the TMD [38]. In order to investigate the nonlinear characteristics that arise when using TMDs, as well as the control performance exhibited by nonlinear tuned mass dampers (NTMDs) under different conditions, this paper analyzes a main structure controlled by a nonlinearly tuned mass damper. The expression for the structure's response under 1:1 resonance conditions is derived using the complex variable averaging method. This expression is then further analyzed to obtain the nonlinear design equation for the NTMD frequency. The validity of the complex variable averaging method, as well as the accuracy and correctness of the obtained nonlinear design formulas, is verified by comparison with the numerical method. Based on the derived nonlinear design formulation, a comprehensive analysis of the nonlinear tuned mass damper system under damping, 1:1 resonance and softening stiffness conditions is conducted. The obtained analytical results can provide valuable references for future studies on TMD nonlinear characteristics and potentially be applied in practical engineering to enhance the control performance of TMDs in real-world applications.

The previous section provides an introduction to the history of damper development, theoretical research and the current applications. Section 2 presents the system model used in this study, including the equations of motion and their dimensionless form. It also derives an approximate analytical solution for the system's equations of motion. In Section 3, the obtained approximate analytical solution is further analyzed to derive the nonlinear design formulation for the frequency of the nonlinear tuned mass damper. The correctness and accuracy of the complex variable averaging method employed in this paper to analyze the system equations, as well as the validity of the nonlinear design formulation, are verified using numerical methods. Section 4 focuses on the analysis and design of the control performance of the nonlinear tuned mass damper under different conditions, namely a damping condition, a 1:1 resonance condition and a softened stiffness condition. The results demonstrate that employing different design methods for each condition can effectively enhance the control performance of the nonlinear tuned mass damper. In Section 5, the main findings and conclusions of this study are summarized. Additionally, the limitations of the study and suggestions for future research are discussed.

2. Models of Systems, Equations of Motion and Their Dimensionless Forms

The nonlinear system composed of a controlled structure with an ideal single degree of freedom and a tuned mass damper considering stiffness nonlinearity can be simplified to the kinematic structure shown in Figure 1. As can be seen from the figure, the mass of the controlled structure is set to m_1 , the linear spring stiffness is k_1 , and the linear damping is c_1 ; the equivalent mass of the tuned mass damper is set to m_2 , the linear spring stiffness between the controlled structure and the tuned mass damper is k_2 , the nonlinear spring stiffness between the controlled structure and the tuned mass damper is k_3 , and the linear damping between the controlled structure and the tuned mass damper is c_2 .

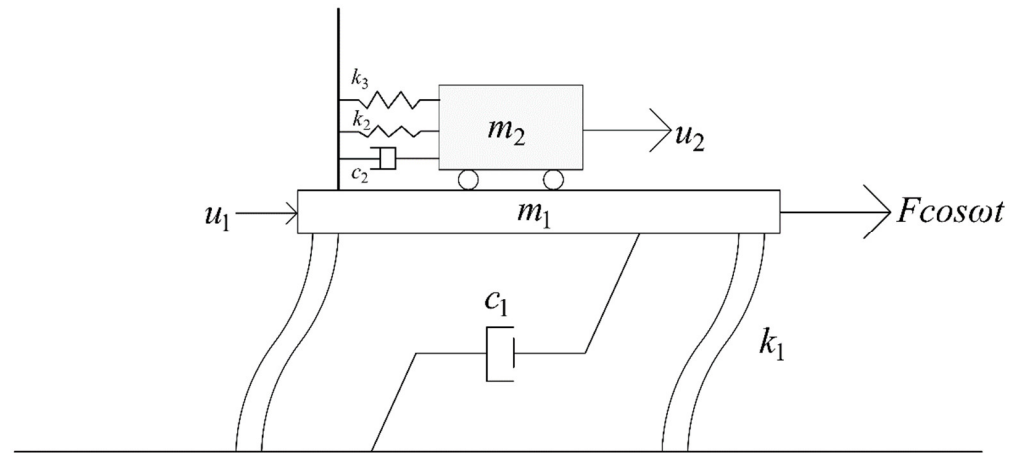


Figure 1. Nonlinear tuned mass damper and controlled structural system.

The kinetic equations of the system can be obtained from Figure 1:

$$m_1 \ddot{u}_1 + c_1 \dot{u}_1 + k_1 u_1 + m_2 \ddot{u}_2 = F \cos \omega t \quad (1)$$

$$m_2 \ddot{u}_2 + c_2 (\dot{u}_2 - \dot{u}_1) + k_2 (u_2 - u_1) + k_3 (u_2 - u_1)^3 = 0 \quad (2)$$

where u_1 is the displacement produced by the controlled structure under the action of external force F , $\dot{u}_1$ is the first-order derivative of u_1 with respect to time t , and $\ddot{u}_1$ is the second-order derivative of u_1 with respect to time t . u_2 is the displacement produced by the nonlinear tuned mass damper under the action of external force F on the controlled structure, $\dot{u}_2$ is the first-order derivative of u_2 with respect to time t , and $\ddot{u}_2$ is the second-order derivative of u_2 with respect to time t . The relative displacement between the controlled structure and the nonlinear tuned mass damper is $u_1 - u_2$, its first-order derivative is $(\dot{u}_1 - \dot{u}_2)$ with respect to time t , and ω is the excitation frequency of the harmonic excitation received.

In order to facilitate the calculation and to reflect the more important parameters in the equation that need to be studied in this paper in the subsequent derivation process, dimensionless coefficients are introduced to remove the units of all parameters involving physical quantities in the equation by replacing the original parameters in the equation with suitable variables. Before dimensionless transformation, a unit length L is first set, which serves only for dimensionless transformation. The frequency of the controlled structure is denoted as $k_1/m_1 = \omega_1^2$, and the frequency of the tuned mass damper is denoted as $k_2/m_2 = \omega_2^2$. To study the effect of the mass ratio between the nonlinear tuned mass damper and the controlled structure, denoted as the mass ratio $m_2/m_1 = \varepsilon \alpha_1$, this paper employs the product of dimensionless parameters ε and α_1 . Additionally, a small and specific dimensionless parameter ε is introduced for subsequent approxi-

mate analytical solutions. A new dimensionless time variable $\tau = \omega_1 t$ is adopted, replacing the original time variable t and transforming all time derivatives to derivatives with respect to the new time variable τ . The original coordinates of the structure are transformed into a dimensionless form. The displacement generated by the controlled structure is denoted as $x = u_1 / L$, and $y = (u_1 - u_2) / L$ represents the relative displacement between the controlled structure and the nonlinear tuned mass damper. The dimensionless damping coefficient $\varepsilon \lambda_1$ replaces the linear damping coefficient c_1 in the controlled structure, while λ_2 replaces the linear damping coefficient c_2 between the controlled structure and the tuned mass damper. The dimensionless coefficient α_2 indicates the nonlinear stiffness of the nonlinear tuned mass damper. The dimensionless quantity Ω_2 represents the ratio of the nonlinear tuned mass damper frequency to the controlled structure frequency, and Ω represents the ratio of the excitation frequency to the controlled structure frequency. The specific dimensionless coefficients are defined as follows:

$$\begin{aligned} \frac{m_2}{m_1} &= \varepsilon \alpha_1, x = \frac{u_1}{L}, y = \frac{u_1 - u_2}{L}, \frac{k_1}{m_1} = \omega_1^2, \frac{k_2}{m_2} = \omega_2^2, \frac{c_1}{m_1 \omega_1} = \varepsilon \lambda_1 \\ \frac{F}{L k_1} &= \varepsilon f, \frac{c_2}{m_2 \omega_1} = \lambda_2, \frac{k_3 L^2}{k_2} = \alpha_2, \frac{\omega}{\omega_1} = \Omega, \frac{\omega_2}{\omega_1} = \Omega_2, \omega_1 t = \tau \end{aligned} \quad (3)$$

After the dimensionless transformation, Equations (1) and (2) become of the following form:

$$(1 + \varepsilon \alpha_1) \ddot{x} + \varepsilon \lambda_1 \dot{x} + x - \varepsilon \alpha_1 \ddot{y} = \varepsilon f \cos \Omega t \quad (4)$$

$$\ddot{y} - \ddot{x} + \lambda_2 \dot{y} + \Omega_2^2 y + \alpha_2 \Omega_2^2 y^3 = 0 \quad (5)$$

Equations (4) and (5) both contain both $\ddot{x}$ and $\ddot{y}$ terms, which is not easy to solve. Combining the two equations yields Equation (6) with only $\ddot{x}$ and Equation (7) with only $\ddot{y}$:

$$\frac{\ddot{x}}{\alpha_1} + \varepsilon \lambda_2 \dot{y} + \varepsilon \Omega_2^2 y + \frac{\varepsilon \lambda_1}{\alpha_1} \dot{x} + \frac{x}{\alpha_1} + \frac{\varepsilon \alpha_2 \Omega_2^2}{\alpha_1} x^3 = \frac{\varepsilon f}{\alpha_1} \cos \Omega t \quad (6)$$

$$\frac{1}{1 + \varepsilon \alpha_1} \ddot{y} + \lambda_2 \dot{y} + \Omega_2^2 y + \alpha_2 \Omega_2^2 y^3 + \frac{\varepsilon \lambda_1 \dot{x}}{1 + \varepsilon \alpha_1} + \frac{x}{1 + \varepsilon \alpha_1} = \frac{\varepsilon f \cos \Omega t}{1 + \varepsilon \alpha_1} \quad (7)$$

For Equations (6) and (7), a direct analytical solution is more difficult, and the original equations are treated by Taylor expansion. Equations (6) and (7) are treated as functions of small parameters, expanded at $r(0 + \varepsilon)$ and retained up to the primary term. Since ε itself is small, the first-order expansion is sufficient for the subsequent analysis:

$$r(\varepsilon) = r(0) + \varepsilon r'(0) \quad (8)$$

Expanding Equation (6) in the form of Equation (8) yields $r(0)$ and $r'(0)$ of Equation (6):

$$\begin{cases} r(0) = \frac{\ddot{x} + x}{\alpha_1} \\ r'(0) = \lambda_2 \dot{y} + \Omega_2^2 y + \alpha_2 \Omega_2^2 y^3 + \frac{\lambda_1}{\alpha_1} \dot{x} - \frac{f}{\alpha_1} \cos \omega t \end{cases} \quad (9)$$

In the same way, Equation (7) is expanded in the same way to obtain $r(0)$ and $r'(0)$ of Equation (7):

$$\begin{cases} r(0) = \ddot{y} + \lambda_2 \dot{y} + \Omega_2^2 y + \alpha_2 \Omega_2^2 y^3 + x \\ r'(0) = -\alpha_1 \ddot{y} + \lambda_2 \dot{y} + \lambda_1 \dot{x} - x - f \cos \Omega t \end{cases} \quad (10)$$

With Equations (9) and (10), Equations (6) and (7) can be transformed into a simpler way with the following:

$$\frac{\ddot{x} + x}{\alpha_1} + \varepsilon \left(\lambda_2 \dot{y} + \Omega_2^2 y + \alpha_3 \Omega_2^2 y^3 + \frac{\lambda_1}{\alpha_1} \dot{x} - \frac{f}{\alpha_1} \cos \Omega t \right) = 0 \quad (11)$$

$$\ddot{y} + y + \varepsilon \delta \left(\lambda_2 \dot{y} + \Omega_2^2 y + \alpha_2 \Omega_2^2 y^3 + x - y \right) = 0 \quad (12)$$

Equation (12) is in the form after omitting the small quantities, where $\varepsilon \delta = 1$. The system of nonlinear equations, which was dimensionless and Taylor-expanded, is already in a form suitable for subsequent solutions using the complex variable averaging method. Such a transformation is performed mainly to transform the system of nonlinear differential equations of higher order into lower order, which is beneficial for subsequent analysis.

The differential equations governing the system under the control of nonlinear tuned mass dampers are coupled nonlinear equations. In this paper, the complex variable averaging method is chosen to solve these equations. The complex variable averaging method primarily replaces the displacement and velocity variables of the system with complex variables in order to reduce the order of the original system equations. This simplification allows for a more straightforward representation using techniques such as Fourier series expansion or Taylor expansion. By employing the complex variable averaging method, the analysis can narrow its focus to the 1:1 resonance of the system studied in this paper and facilitate the understanding of the relationship between the nonlinear coefficients of the system and the damping effect of the tuned mass dampers. The averaging method treats the amplitude and phase of the solution as time-dependent parameters and averages the derivatives corresponding to these parameters. Similarly, the complex variable averaging method performs the same procedure, except that it represents the solution in the form of complex numbers, which offers greater convenience in solving. The complex variable averaging method assumes an oscillation in the form of a complex harmonic with the same frequency as the external excitation, similar to the derivation of the averaging method. Therefore, in this analysis, the steady-state response of the nonlinear tuned mass damper is approximated as a first-order harmonic with the same frequency as the equivalent excitation. The first-order approximate solutions for Equations (11) and (12) are set as follows:

$$x = \frac{1}{2i\Omega} A_1 e^{i\Omega t} + cc \quad (13)$$

$$y = \frac{1}{2i\Omega} A_2 e^{i\Omega t} + cc \quad (14)$$

where A_1 is the complex variable that consists of the steady-state amplitude and phase of the controlled structure, A_2 represents the complex variable consisting of the steady-state amplitude and phase of the nonlinear tuned mass damper, and cc is the conjugate complex representing the first half of the solution. By taking the first- and second-order derivatives of the time variable t on both sides of Equations (13) and (14), we obtain the expressions for $\dot{x}$, $\dot{y}$, $\ddot{x}$ and $\ddot{y}$ as follows:

$$\dot{x} = \frac{1}{2} A_1 e^{i\Omega t} + cc \quad (15)$$

$$\dot{y} = \frac{1}{2} \dot{A}_2 e^{i\Omega t} + c c \quad (16)$$

$$\ddot{x} = \dot{A}_1 e^{i\Omega t} + \frac{1}{2} i \Omega A_1 e^{i\Omega t} + c c \quad (17)$$

$$\dot{y} = \dot{A}_2 e^{i\Omega t} + \frac{1}{2} i \Omega A_2 e^{i\Omega t} + c c \quad (18)$$

The results obtained through the complex variable averaging method are simpler compared to the traditional averaging method, and its complex form is shorter than the trigonometric form, which facilitates analysis. By substituting the set solutions, i.e., Equations (13)–(18), into Equations (11) and (12) and considering only the ε^0 and ε terms while omitting higher-order terms and all other terms except $e^{i\Omega t}$, we obtain a system of two coupled first-order differential equations. It is important to note here that the term $e^{i\Omega t}$ in the equation, referred to as the long-term term, keeps increasing over time and renders the approximation method invalid. Thus, the long-term term needs to be eliminated in the process of using the complex averaging method. Only the coefficients before this term are considered to sum to zero, and therefore, the coefficients before the long-term term are retained to obtain the following equation:

$$\begin{aligned} & \frac{1}{\alpha_1} \left(\dot{A}_1 + \frac{1}{2} i \Omega A_1 + \frac{1}{2i\Omega} A_1 \right) \\ & + \varepsilon \left(\frac{\lambda_2}{2} A_2 + \frac{\Omega_2^2}{2i} A_2 + \frac{3\alpha_2 \Omega_2^2}{8i} |A_2|^2 A_2 + \frac{\lambda_1}{2\alpha_1} A_1 - \frac{f}{2\alpha_1} \right) = 0 \end{aligned} \quad (19)$$

$$\begin{aligned} & \left(\dot{A}_2 + \frac{1}{2} i \Omega A_2 + \frac{1}{2i\Omega} A_2 \right) \\ & + \varepsilon \delta \left(\frac{\lambda_2}{2} A_2 + \frac{\Omega_2^2}{2i} A_2 + \frac{3\alpha_2 \Omega_2^2}{8i} |A_2|^2 A_2 + \frac{1}{2i} A_1 - \frac{1}{2i} A_2 \right) = 0 \end{aligned} \quad (20)$$

Influenced by the nonlinearity, the response mechanism of the excitation frequency near the frequency of the controlled structure also becomes very complex. Considering the engineering reality that the mass of the tuned mass damper is much smaller than the mass of the controlled structure, vibration reduction is significant in such cases. Therefore, the vibration reduction effect of the nonlinear mass damper is taken into account when the maximum mass ratio is set to 0.2. Additionally, the case of 1:1 resonance between the excitation frequency and the controlled structure frequency is considered, and the difference between the external excitation frequency and the controlled structure frequency is controlled within the same order of ε^1 , ensuring that Ω satisfies the relationship described by $\Omega = 1 + \varepsilon \sigma$.

In the field of structural vibration control, the introduction of nonlinearity through the implementation of tuned mass dampers using nonlinear stiffness devices can directly exert forces on the structure, leading to stability problems. As a result, the dynamics exhibit a more complex response. In comparison to linear tuned mass dampers, the performance of nonlinear tuned mass dampers is primarily determined by the design parameters Ω_2 and α_2 . This study focuses on investigating the steady-state response of the structure, where the first-order derivatives in Equations (19) and (20) are set to zero. Through transformation, the equations can be simplified to the following form:

$$i\sigma A_1 + \alpha_1 \lambda_2 A_2 - i\alpha_1 \Omega_2^2 A_2 - i\frac{3}{4} \alpha_1 \alpha_2 \Omega_2^2 |A_2|^2 A_2 + \lambda_1 A_1 - f = 0 \quad (21)$$

$$-i\Omega_2^2 A_2 - i\frac{3}{4}\alpha_2\Omega_2^2 |A_2|^2 A_2 + \lambda_2 A_2 - iA_1 + iA_2 = 0 \quad (22)$$

It can be observed that A_1 can be expressed by the equation provided in (22). By combining Equations (21) and (22), the expressions of A_1 and A_2 can be decoupled, resulting in the following:

$$A_1 = -\Omega_2^2 A_2 - \frac{3}{4}\alpha_2\Omega_2^2 |A_2|^2 A_2 - i\lambda_2 A_2 + A_2 \quad (23)$$

$$\begin{aligned} & -i\sigma\Omega_2^2 A_2 - i\sigma\frac{3}{4}\alpha_2\Omega_2^2 |A_2|^2 A_2 + \sigma\lambda_2 A_2 \\ & + i\sigma A_2 - \lambda_1\Omega_2^2 A_2 - \lambda_1\frac{3}{4}\alpha_2\Omega_2^2 |A_2|^2 A_2 \\ & - i\lambda_1\lambda_2 A_2 + \lambda_1 A_2 + \alpha_1\lambda_2 A_2 - i\alpha_1\Omega_2^2 A_2 - i\frac{3}{4}\alpha_1\alpha_2\Omega_2^2 |A_2|^2 A_2 - f = 0 \end{aligned} \quad (24)$$

Expand the expressions of A_1 and A_2 into the plural form:

$$A_1 = a_1 (\cos b_1 + i \sin b_1) \quad ; \quad A_2 = a_2 (\cos b_2 + i \sin b_2) \quad (25)$$

where a_1 and a_2 denote the steady-state amplitudes obtained by the complex variable averaging method, and b_1 and b_2 are the steady-state phases. Substituting the expression of Equation (25) into Equations (23) and (24), and separating the real and imaginary parts, we obtain:

$$a_1 \cos(b_1 - b_2) = -\Omega_2^2 a_2 - \frac{3}{4}\alpha_2\Omega_2^2 a_2^3 + a_2 \quad (26)$$

$$a_1 \sin(b_1 - b_2) = -\lambda_2 a_2 \quad (27)$$

$$\sigma\lambda_2 a_2 - \lambda_1\Omega_2^2 a_2 - \lambda_1\frac{3}{4}\alpha_2\Omega_2^2 a_2^3 + \lambda_1 a_2 + \alpha_1\lambda_2 a_2 = f \cos b \quad (28)$$

$$\alpha\Omega_2^2 a_2 + \sigma\frac{3}{4}\alpha_2\Omega_2^2 a_2^3 - \sigma a_2 + \lambda_1\lambda_2 a_2 + \alpha_1\Omega_2^2 a_2 + \frac{3}{4}\alpha_1\alpha_2\Omega_2^2 a_2^3 = f \sin b \quad (29)$$

3. Frequency Design Formula and Approximate Analytical Solution for NTMD

In the previous section, the nonlinear tuned mass dampers are solved to obtain the modulation–demodulation equations for the primary structure amplitude and the NTMD amplitude. In this section, further analysis based on the modulation–demodulation equations will be performed to obtain the nonlinear design equation of the NTMD. The correctness and accuracy of the complex variable averaging method are demonstrated by comparing the calculation results with those of the conventional numerical methods, and the correctness of the nonlinear design formulation is also demonstrated.

3.1. Nonlinear Design Formula of NTMD

For Equations (26) and (27), the two equations are simultaneously squared and added together, and trigonometric operations are performed to eliminate the phase term to obtain the following equation:

$$a_1^2 = \left(-\Omega_2^2 a_2 - \frac{3}{4}\alpha_2\Omega_2^2 a_2^3 + a_2 \right)^2 + \lambda_2^2 a_2^2 \quad (30)$$

The ratio of the steady-state amplitude of the primary structure and the steady-state amplitude of the NTMD can be set to $E = a_1^2 / a_2^2$, and let $E_2 = a_2^2$. Then the above equation can be reduced to the following:

$$E = \left(-\Omega_2^2 - \frac{3}{4} \alpha_2 \Omega_2^2 E_2 + 1 \right)^2 + \lambda_2^2 \quad (31)$$

If we consider E as a function of the mass damper frequency ω_2^2 , we can obtain a very small value of the amplitude ratio E by taking the derivative of ω_2^2 and setting it to zero. This allows us to find the value of Ω_2^2 corresponding to E approaching a very small value, denoted as $\dot{E} = \partial E / \partial (\Omega_2^2) = 0$. By taking the derivative of both sides of Equation (31) with respect to Ω_2^2 , we can obtain the following expression:

$$\left(-\frac{3\alpha_2\Omega_2^2}{4} E (\Omega_2^2) + 1 - \Omega_2^2 \right) \left(-\frac{3\alpha_2}{4} E (\Omega_2^2) - \frac{3\alpha_2\omega_2^2}{4} \dot{E} (\Omega_2^2) - 1 \right) = 0 \quad (32)$$

The conditions under which Equation (32) holds after verification are

$$-\frac{3\alpha_2\Omega_2^2}{4} E_2 (\Omega_2^2) + 1 - \Omega_2^2 = 0 \quad (33)$$

This condition holds because the second bracket in Equation (32), $\dot{E}_2(\omega_2^2)$, tends to zero when the energy ratio takes a very small value. For the other two terms of the second bracket of Equation (32), it is impossible to equal zero with respect to the hardened elastic element, so Equation (32) holds provided that the value in the first bracket is zero.

When Equation (32) holds, for Equation (29), we have

$$\sigma \lambda_2 a_2 - \lambda_1 a_2 \left(\frac{3}{4} \alpha_2 \Omega_2^2 a_2^2 - 1 + \Omega_2^2 \right) + \alpha_1 \lambda_2 a_2 = f \cos b \quad (34)$$

For Equation (28), there are

$$\begin{aligned} \sigma a_2 \left(\Omega_2^2 + \frac{3}{4} \alpha_2 \Omega_2^2 a_2^2 - 1 \right) + \lambda_1 \lambda_2 a_2 + \\ \alpha_1 a_2 \left(\Omega_2^2 + \frac{3}{4} \alpha_2 \Omega_2^2 a_2^2 - 1 \right) + \alpha_1 a_2 = f \sin b \end{aligned} \quad (35)$$

Let the values in the bracketed terms of Equations (34) and (35) both be zero, then we have:

$$\sigma \lambda_2 a_2 + \alpha_1 \lambda_2 a_2 = f \cos b \quad (36)$$

$$\lambda_1 \lambda_2 a_2 + \alpha_1 a_2 = f \sin b \quad (37)$$

Adding Equations (36) and (37) squared together while using the trigonometric formula, a simple expression for a_2^2 can be obtained by eliminating the phase term:

$$a_2^2 = \frac{f^2}{(\sigma \lambda_2 + \alpha_1 \lambda_2)^2 + (\lambda_1 \lambda_2 + \alpha_1)^2} \quad (38)$$

Substituting Equation (38) into Equation (33) yields

$$\Omega_2 = \sqrt{1 - \frac{3\alpha_2 f^2}{3\alpha_2 f^2 + 4[(\sigma \lambda_2 + \alpha_1 \lambda_2)^2 + (\lambda_1 \lambda_2 + \alpha_1)^2]}} \quad (39)$$

Equation (39) is the nonlinear design equation for the nonlinear tuned mass damper frequency derived in this paper. A similar equation was obtained in reference [19], but the equation obtained in the literature only considers the frequency design for the 1:1 reso-

nance condition, while the Equation (39) obtained in this paper is based on the offset parameter σ and yields an improved frequency design equation that yields the optimal design frequency for the 1:1 resonance and the case near the 1:1 resonance. This equation indirectly shows that the value of ω_2 for the optimal design is less than the controlled structural frequency when the nonlinear coefficient is positive, i.e., when the stiffness is hardened three times. When the nonlinear coefficient is zero, the optimal design is when the controlled structural frequency is equal to the mass damper frequency, which is also consistent with the conventional linear design.

3.2. Validation of the Complex Variable Averaging Method

To verify the correctness and accuracy of the approximate analytical solutions obtained by the complex variable averaging method, Equations (6) and (7) are calculated using the conventional numerical method, and the modulation–demodulation Equations (26)–(29) obtained by the complex variable averaging method are calculated. A two-degree-of-freedom system consisting of a single-degree-of-freedom primary structure and an NTMD with cubic nonlinear stiffness is used as the study object, and the structure of the study object is shown in Figure 1, where the parameters of the primary structure and the NTMD are set as follows:

$$\begin{aligned} m_1 &= 1\text{ kg} ; \quad c_1 = 0.1\text{ N} \cdot \text{s/m} ; \quad k_1 = 1\text{ N/m} \\ m_2 &= 0.02\text{ kg} ; \quad c_2 = 0.0031\text{ N} \cdot \text{s/m} ; \quad k_2 = 0.0161\text{ N/m} \end{aligned}$$

The corresponding dimensionless parameters can also be obtained as

$$\lambda_1 = 5 ; \quad \lambda_2 = 0.155 ; \quad f = 15$$

The set external excitation frequency is

$$\omega = 1\text{ rad/s}$$

It is worth mentioning that the analysis of the study focuses on the 1:1 resonance case, so in the subsequent analysis, the analysis is performed by default based on $\omega = 1\text{ rad/s}$, except for the specially stated cases and the graphs with changing excitation frequencies.

Based on the above parameters, a frequency amplitude diagram related to the NTMD frequency can be obtained.

Figure 2 shows the frequency–amplitude diagram of the structure. The horizontal coordinates are the frequencies of the NTMD, and the vertical coordinates are the amplitudes of the primary structure and the NTMD. The blue dashed line is the result obtained by the numerical method, and the gray solid line is the result obtained by the complex variable averaging method. It can be seen from the figure that the results obtained by the numerical method and the complex variable averaging method are in good agreement with almost no error. This proves the correctness of the modulation–demodulation equation obtained by the complex variable averaging method, and it also proves that the modulation–demodulation equation is highly accurate and meets the requirements of the subsequent analysis.

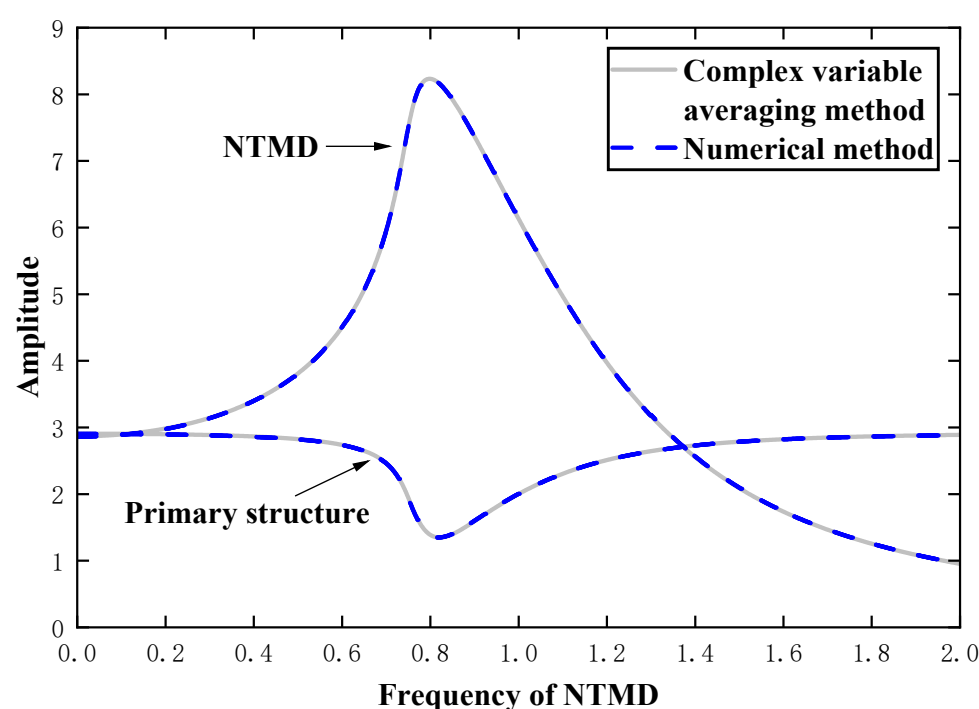


Figure 2. Frequency amplitude plots obtained using the complex variable averaging method and numerical method.

3.3. Damping-Based Design of NTMD Parameters

Having verified the correctness and accuracy of the complex variable averaging method, we now demonstrate the accuracy of the nonlinear design formulation for the NTMD frequency derived from the modulation–demodulation equation. By considering the nonlinear stiffness as α_2 and the structural mass ratio as α_1 , we compare the optimal TMD design frequencies obtained by the numerical method and the complex variable averaging method for different parameter combinations, as shown in Table 1.

Table 1. Optimal frequency of tuned mass damper, comparison between numerical method and analytical method.

α_1	α_2	Numerical Method	Approximate Analysis Method
1	0.01	0.817	0.816
1	0.02	0.707	0.706
2	0.01	0.916	0.910
2	0.02	0.846	0.840
3	0.01	0.958	0.948
3	0.02	0.914	0.903
4	0.01	0.978	0.966
4	0.02	0.947	0.936
5	0.01	0.987	0.977
5	0.02	0.965	0.955

The results indicate that the nonlinear design formulas for NTMD frequencies, derived using the approximate analytical method, closely match the results obtained by the numerical method with only minimal errors. This ensures the correctness of the nonlinear design formulas and demonstrates the good accuracy of the calculation results, thus confirming their suitability for subsequent analysis.

4. Analysis of Nonlinear Tuned Mass Damper Systems Based on Different Conditions

The previous section demonstrated the correctness of the complex variable averaging method. In this section, we investigate the control performance of the NTMD in the damping condition, 1:1 resonance condition and softening stiffness condition and try to enhance the control performance of the NTMD in these three conditions by design.

4.1. Analysis of NTMD Control Performance Based on Damping Conditions

For the undamped tuned mass damper system, the analysis is performed by considering the case where the excitation frequency is equal to the frequency of the controlled structure. Setting the damping of the tuned mass damper to zero and the excitation frequency equal to the controlled structure frequency, we have $\sigma = 0$ and $\lambda_2 = 0$, and then Equation (39) can be reduced to

$$\Omega_2 = \sqrt{\frac{4\alpha_1^2}{3\alpha_2 f^2 + 4\alpha_1^2}} \quad (40)$$

Equation (40) represents the undamped frequency design equation of the nonlinear tuned mass damper derived in this paper. Upon observing Equation (40), we can see that when $\alpha_2 = 0$, there is no nonlinearity in the structure, resulting in $\omega_2 = \omega_1$, and the TMD exhibits an optimal damping effect. This finding aligns with the conclusions drawn by Den Hartog in his previous work.

The aforementioned results are verified through the numerical simulations of Equations (1) and (2). The parameters are set as follows:

$$\alpha_1 = 1, \alpha_2 = 0.001, f = 15$$

The optimal parameter $\Omega_2 = 0.925$ is obtained from Equation (40), and the initial conditions are set as follows: $x|_{t=0} = 0$, $\dot{x}|_{t=0} = 0$, $y|_{t=0} = 0$, and $\dot{y}|_{t=0} = 0$. The time-history diagram obtained is shown in Figure 3. It is observed that when the damping coefficient is zero but nonlinearity exists, the steady-state amplitude of the controlled structure obtained from the design equation is approximately 0.6% of the transient amplitude, indicating a non-zero value. This discrepancy is attributed to inaccuracies in the calculation of the approximate solution expansion. Figure 3 illustrates that at the resonant frequency, the NTMD significantly absorbs the energy of the structure over time and dissipates it through damping. Figure 4 demonstrates that using the design Equation (40) for optimization can substantially reduce the steady-state amplitude of the structure. Initially, the amplitude increases and then gradually decreases until eventually stabilizing, with the NTMD absorbing most of the energy generated during vibration in the steady-state phase. Figure 5 presents the time curve of the steady-state amplitude of the controlled structure, which is magnified. It can be observed that despite the small steady-state amplitude (approximately 0.6% of the transient amplitude), the optimization effect is non-zero, confirming the meaningfulness of these results. Figure 6 shows that both steady-state and transient amplitudes of the controlled structure, controlled with undamped nonlinear tuned mass dampers, are reduced. This reflects the excellent vibration reduction effect achieved by the nonlinear TMD with a damped condition design at the 1:1 resonance.

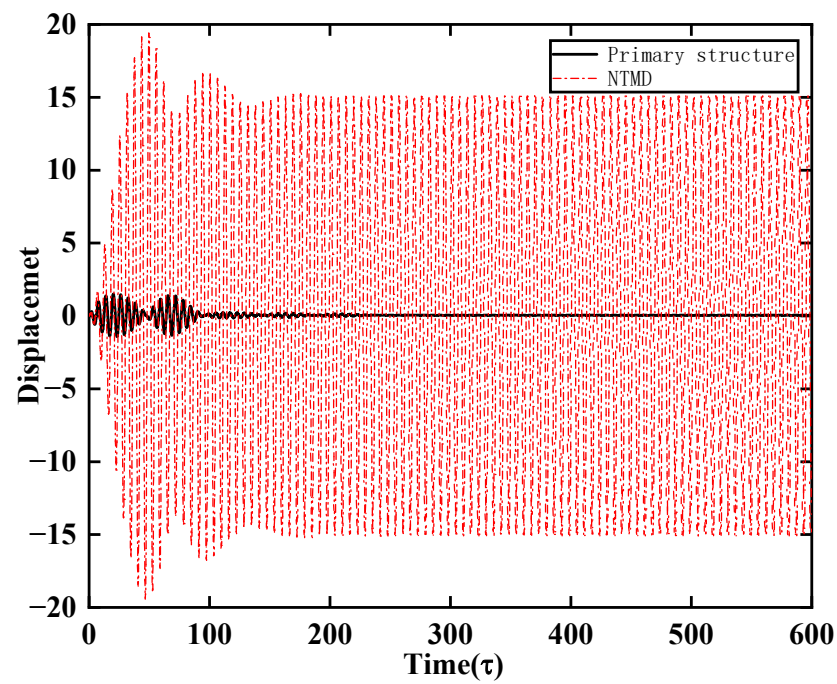


Figure 3. Displacement time course curves of the controlled structure and TMD mass block with nonlinear TMD installed.

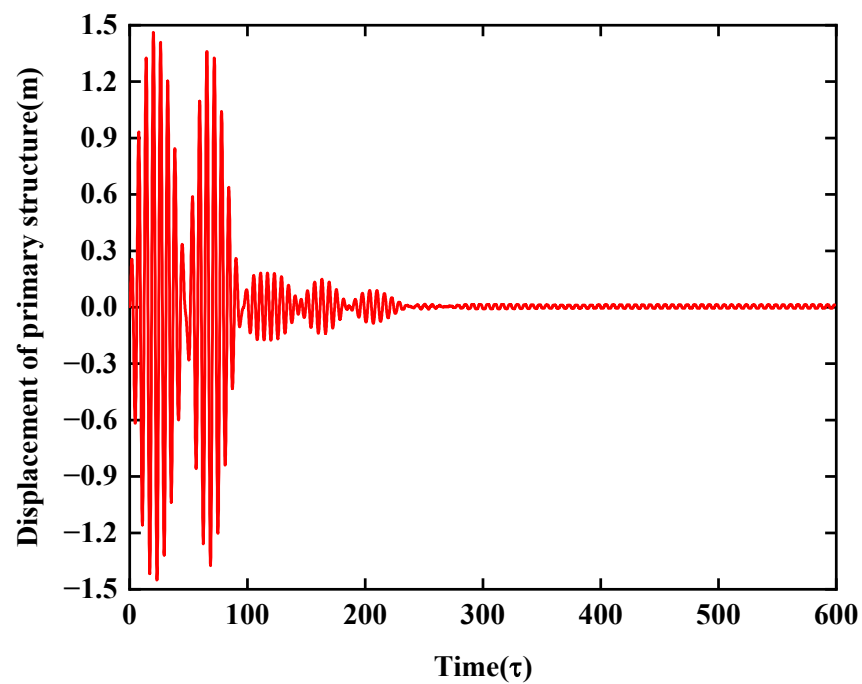


Figure 4. Displacement time curve of the controlled structure with nonlinear TMD installed.

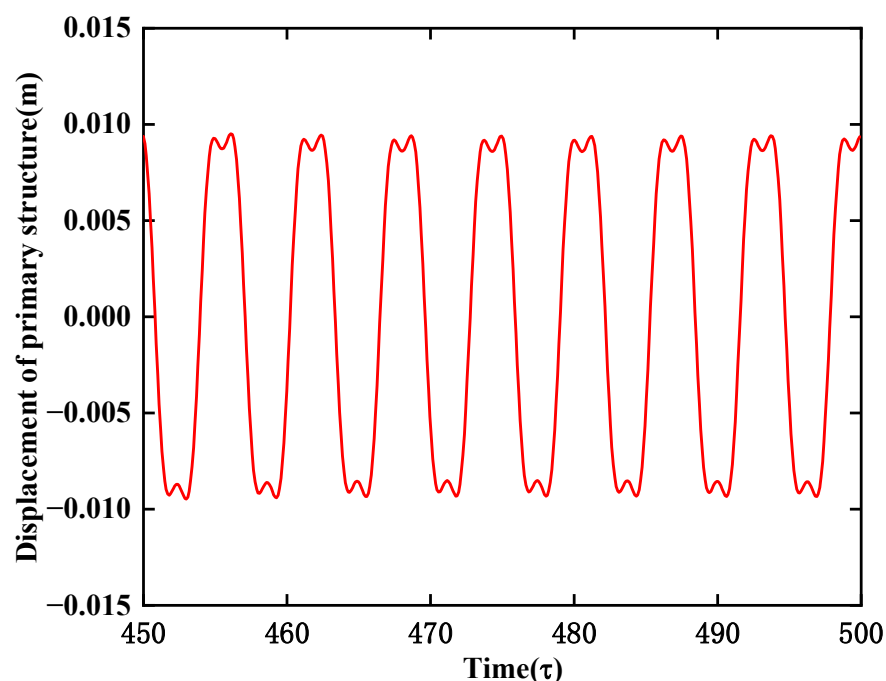


Figure 5. Displacement time curve of controlled structure with intercepted periodically stable solution.

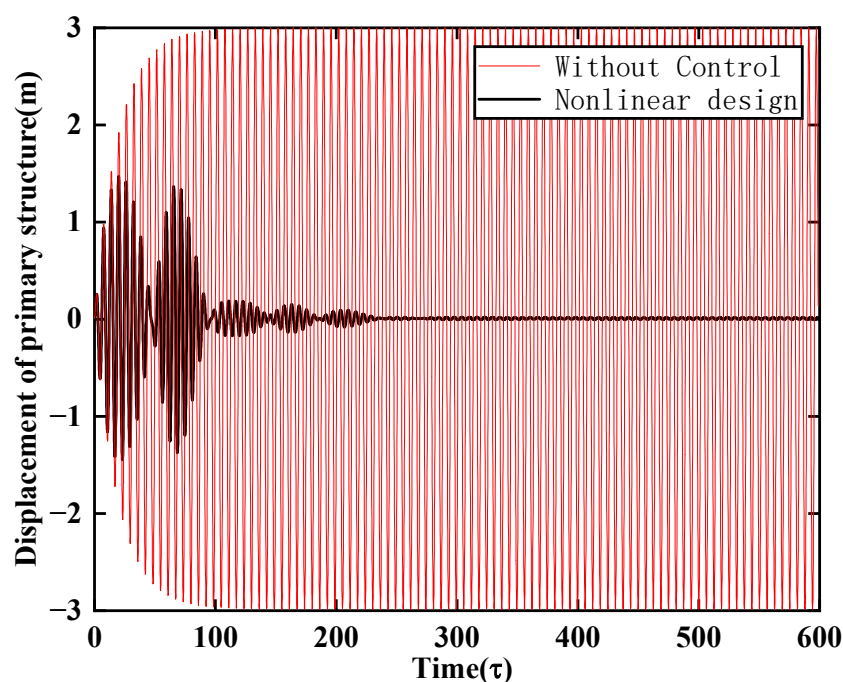


Figure 6. Displacement time curves of the main structure with nonlinear mass damper installed and the uncontrolled main structure.

Figure 7 illustrates the amplitude–frequency response curves of the primary structure. The solid black line in the figure represents the curve without damping, i.e., when the damping coefficient $\lambda_2 = 0$, and the red dashed line corresponds to the curve with the damping coefficient $\lambda_2 = 0.16$. The results obtained using numerical solution methods indicate that the control performance under the undamped nonlinear tuned mass damper is superior when the excitation frequency is near the structural frequency. However, when the excitation frequency slightly deviates from the structural frequency, the amplitude becomes larger compared to the damped nonlinear tuned mass damper. Therefore, when the excitation frequency is uncertain, it is advisable to design the damping such that it

exhibits good vibration reduction effects within a certain range of frequencies. When the excitation frequency is equal to or extremely close to the primary structure frequency, the structure can be damped using non-damped nonlinear dampers, which will effectively reduce the steady-state amplitude to approximately 0.6% of the transient amplitude.

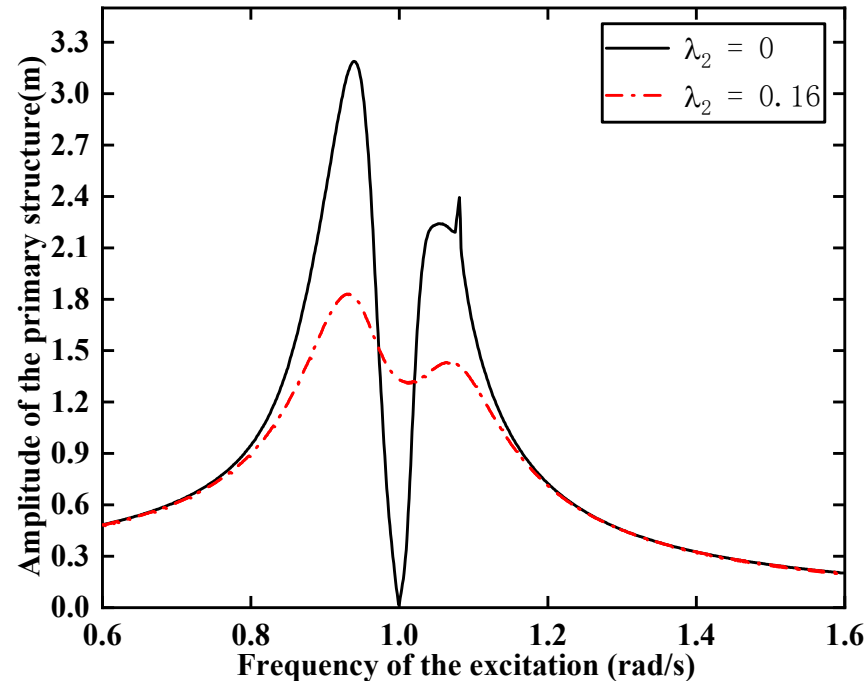


Figure 7. Damping parameters are $\lambda_2 = 0.16$, $\lambda_2 = 0$ amplitude frequency response curve.

4.2. NTMD Control Performance Analysis Based on 1:1 Resonance Conditions

The control performance of the NTMD with damping conditions was studied and analyzed in the previous subsection, and now, the control performance of the nonlinear tuned mass damper will be analyzed based on the 1:1 resonance condition. To study the effects of different mass ratios on the optimized frequency and vibration reduction effect of the NTMD, the nonlinear coefficient $\alpha_2 = 0.3$ is used, and the mass ratios of 0.02, 0.03, 0.04 and 0.05 are considered to obtain the frequency amplitude curves of the primary structure as shown in Figure 8. Mass ratios in engineering are generally very small, so Figure 8 only shows the graph with a maximum mass ratio of 0.05, indicating that the mass of the tuned mass damper is ten percent of the mass of the controlled structure, which is already a fairly large mass ratio in engineering. As shown in the graph, the steady-state amplitude of the controlled structure decreases continuously as the mass ratio increases at the position of the 1:1 resonance. It can be observed that the mass ratio is a favorable factor affecting the amplitude of the controlled primary structure at the 1:1 resonance, and within a certain range, the larger the mass ratio, the better the vibration reduction effect.

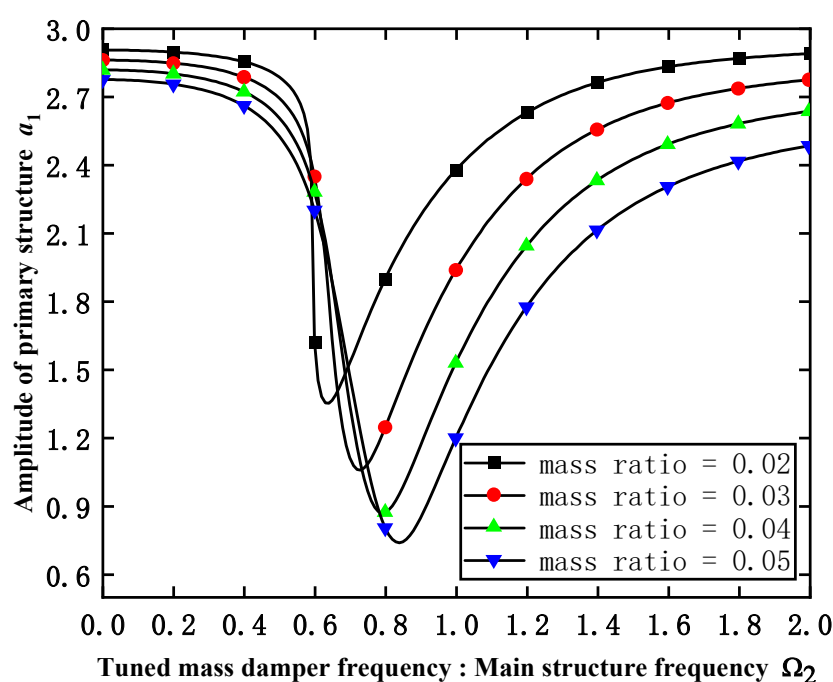


Figure 8. Frequency ratio-controlled structure amplitude curves for different mass ratios.

As shown in Figure 9, the horizontal axis represents the excitation frequency, and the vertical axis represents the steady-state amplitude of the main structure. The original set of differential equations is solved using the Runge–Kutta method in the numerical solution method to obtain the corresponding amplitude–frequency response curves. The control bandwidth of the tuned mass damper increases as the mass ratio increases, while other conditions remain constant. By comparing the maximum steady-state amplitude of the controlled structure at a mass ratio of 0.01 and a mass ratio of 0.02, it can be observed that increasing the mass ratio leads to a decrease in the maximum amplitude. However, the maximum steady-state amplitude of the structure remains the same for mass ratios of 0.02 and 0.03. This suggests that within a certain range, increasing the mass ratio can improve the damping effect of the NTMD without altering other parameters. When the mass ratio reaches a certain degree, the bandwidth of the amplitude–frequency curve expands, but the improvement in the damping effect becomes minimal. Therefore, selecting an appropriate mass ratio can effectively reduce the steady-state amplitude of the structure while controlling the design cost. Based on the obtained results, the mass ratio in engineering generally does not exceed 0.05. The recommended range for the mass ratio is between 0.02 and 0.05, provided that the NTMD control effect is ensured.

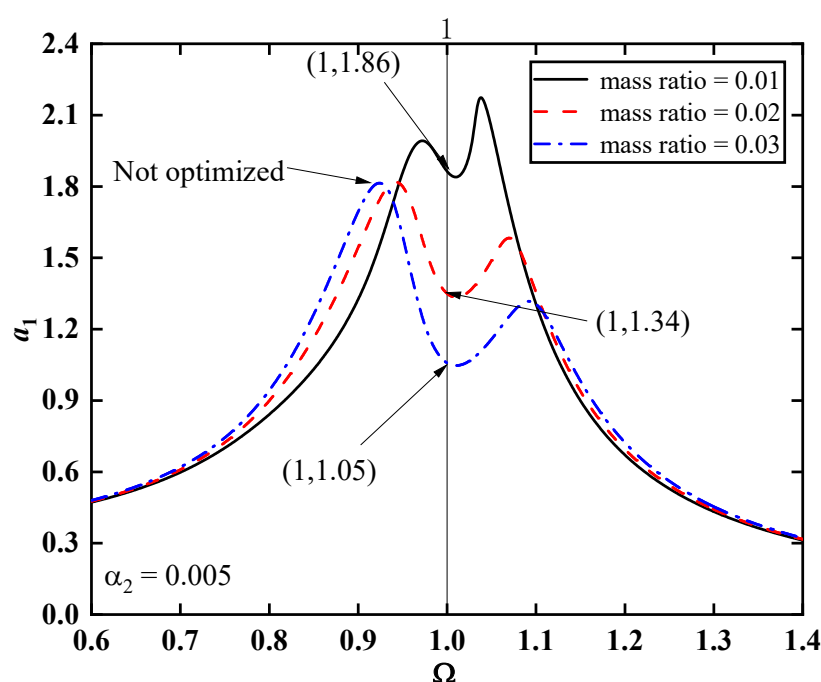


Figure 9. Amplitude–frequency response curves obtained by the Runge–Kutta method.

Figure 10 shows the amplitude–frequency response curve of the structure. The blue dashed line indicates the optimized frequency obtained by the undamped frequency design Equation (40) when the mass ratio is 0.03 and $\sigma = 0$ and the amplitude response curve of the excitation frequency-controlled structure based on this optimized frequency design; the black solid line indicates the amplitude response curve of the controlled structure under the uncontrolled condition without the installation of mass dampers; the red dotted line indicates the amplitude–frequency response curve obtained by using the linear design frequency after installing the nonlinear mass damper. In comparison, it is found that both the tuned mass dampers with linear and nonlinear designs installed in the structure have vibration reduction effects. However, the results of the amplitude–frequency response curves obtained based on the design equations are better than those obtained by the linear design. It is also clear from the figure that the control performance of the linear-based NTMD is better than that of the nonlinear-based NTMD at certain frequency ratios. However, when the frequency ratio is in the range of 0.85–1.05, the control performance is weaker than that of the nonlinear-based NTMD. Therefore, the control effect of the NTMD based on the nonlinear design, although weaker than that of the NTMD with a linear design in some frequency ratio cases, is better overall in terms of the amplitude–frequency response curve control effect. The amplitude will not change to a larger extent, and the stability of the amplitude–frequency response curve is better.

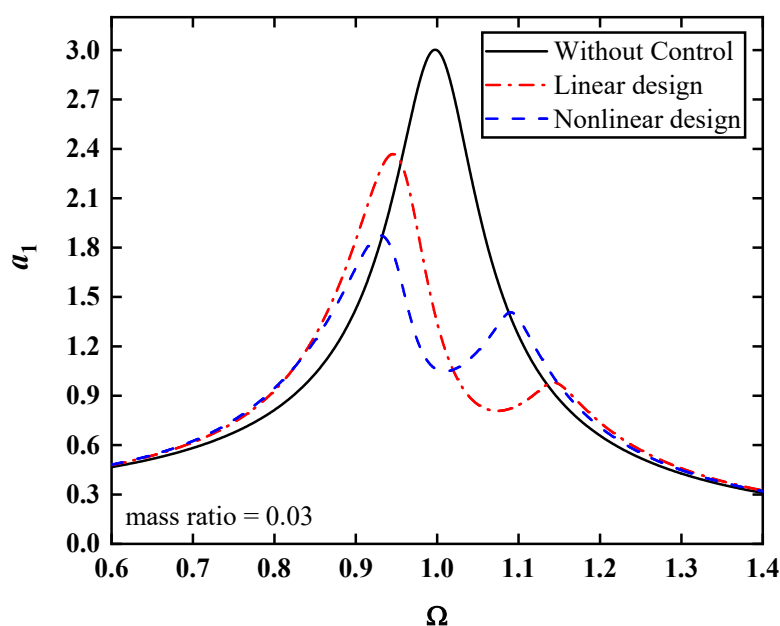


Figure 10. Amplitude and frequency response curves under different designs.

Figure 11 shows the time-history response curves obtained by selecting the frequency ratio at the maximum amplitude of the controlled structure. The blue solid line shows the displacement time-history of the controlled structure without the damper installed, the black dashed line shows the displacement time-history of the controlled structure controlled by the linearly designed TMD, and the red solid line shows the displacement time-history of the controlled structure controlled by the nonlinearly designed NTMD. It is observed that the transient response and steady-state response of the structure with the NTMD installed are smaller than those of the uncontrolled structure. The amplitude at the maximum peak of the amplitude-frequency response curve is reduced by 20% compared to the amplitude at the maximum peak of the uncontrolled structure when the NTMD is designed using the linear design method. Using the nonlinear design formulation for the NTMD, the amplitude of the amplitude response curve at the maximum peak is reduced by 40%.

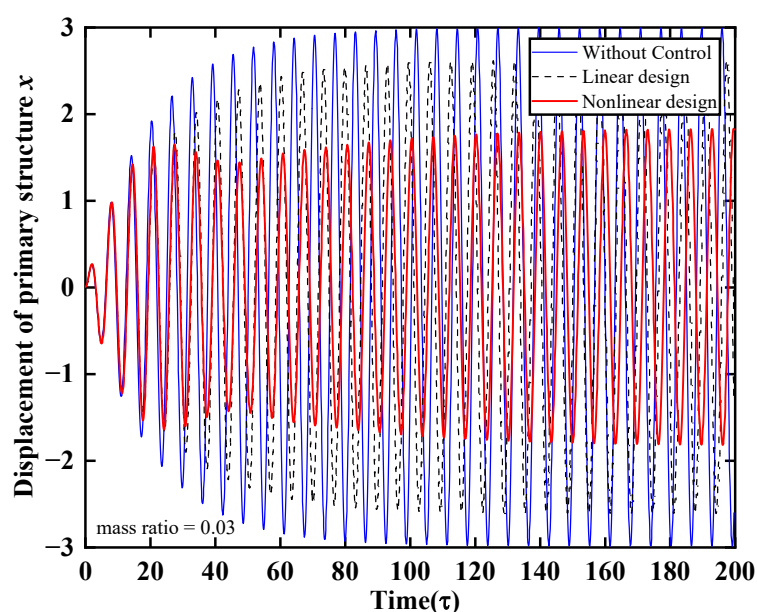


Figure 11. Time range response curves under different designs.

Figure 12a depicts the three-dimensional relationship between the nonlinear coefficients, the frequency ratio and the steady-state amplitude of the primary structure, while Figure 12b illustrates the $\alpha_2 - \Omega_2$ parameter plane. From these two plots, it can be observed that the depression represents the optimal design frequency ratio. As the nonlinear coefficient increases, the optimal design frequency ratio gradually decreases, while the corresponding minimum controlled structural steady-state amplitude remains unchanged in size. Therefore, the nonlinear coefficient can be utilized as a design parameter to adjust the optimal frequency ratio of the structure, assuming other conditions remain unchanged. In the nonlinear design process, the nonlinear coefficients generally cannot be too large or the structure will become unstable [33].

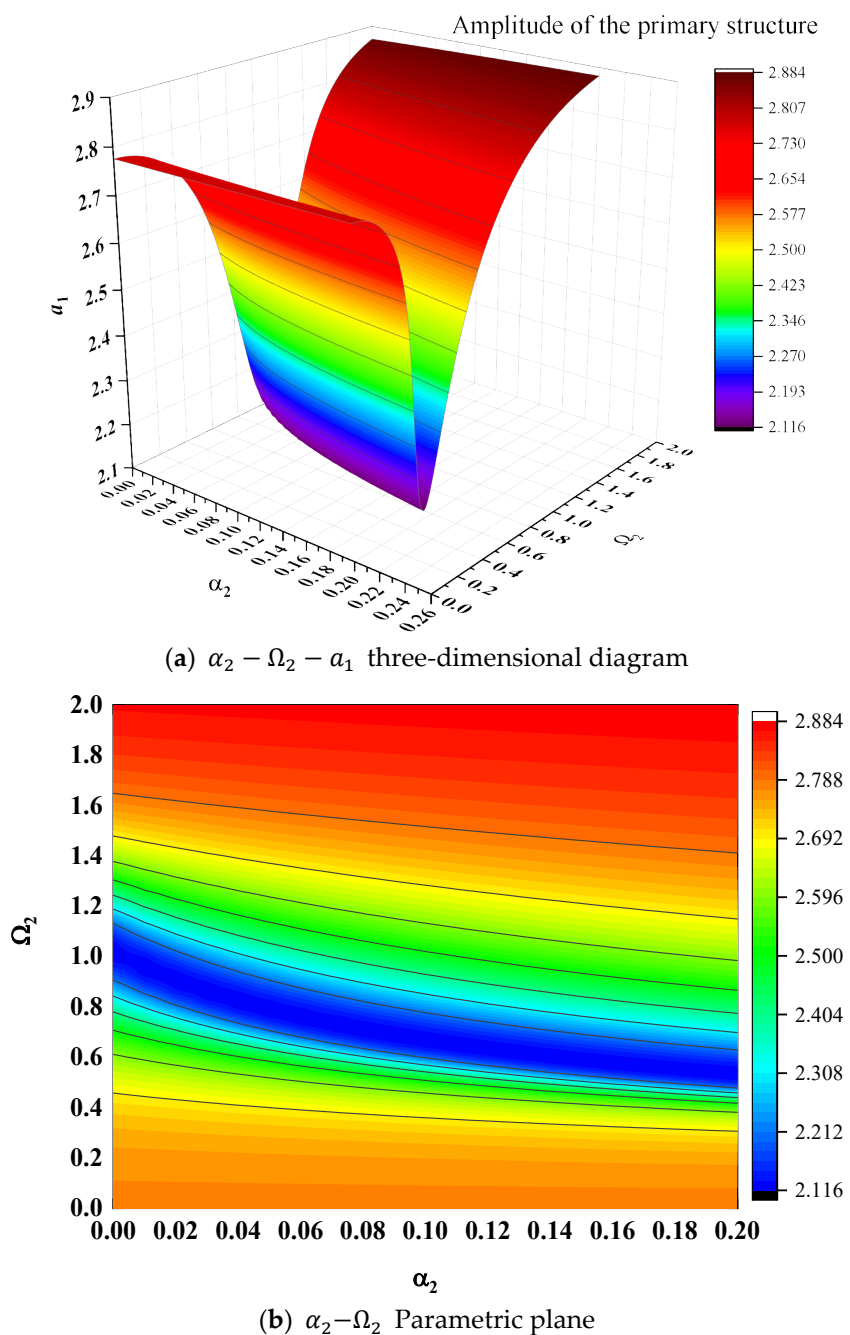


Figure 12. Frequency–mass ratio of mass-damper-controlled structural steady-state amplitude.

4.3. Analysis of NTMD Control Performance Based on Softening Stiffness Conditions

The previous section focused on studying and analyzing the nonlinear mass damper under the 1:1 resonance condition. In this section, we investigate the control performance of the nonlinear tuned mass damper under the softened stiffness condition. Figure 13 illustrates the amplitude–frequency response curves related to the frequency ratio of the excitation frequency and the primary structure at $\sigma \neq 0$. Upon observing Figure 13, it can be observed that the maximum peak of the controlled structure does not occur at the 1:1 resonance; instead, it shifts away from the 1:1 resonance, resulting in double peaks on both sides. In the case of $\sigma \neq 0$, the nonlinear NTMD frequency design formulation corresponding to Equation (39) allows for designing such an offset, aiming to identify optimal design parameters that effectively reduce peak displacement.

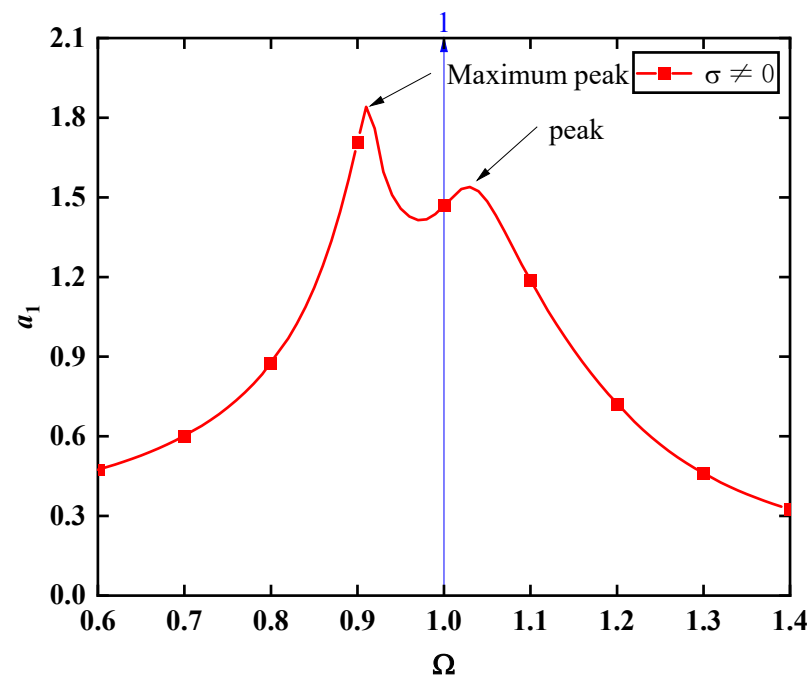


Figure 13. Amplitude–frequency response curve of the controlled structure.

When the softening stiffness nonlinearity coefficient $\alpha_2 = -0.005$, three design methods with frequency offsets $\sigma = -10$, $\sigma = 0$ and $\sigma = 10$ are selected, and the amplitude–frequency response curves are obtained for these three cases, as shown in Figure 14. It can be observed from the figure that the maximum peak amplitude of the primary structure obtained when using an offset for the design is smaller than the peak amplitude at 1:1 resonance. Therefore, it can be concluded that the design based on a small offset yields better optimization results for the NTMD when it exhibits softening stiffness nonlinearity. Furthermore, the difference between the optimization results for $\sigma = -10$ and $\sigma = 10$ is small, and both are more effective than the design at 1:1 resonance.

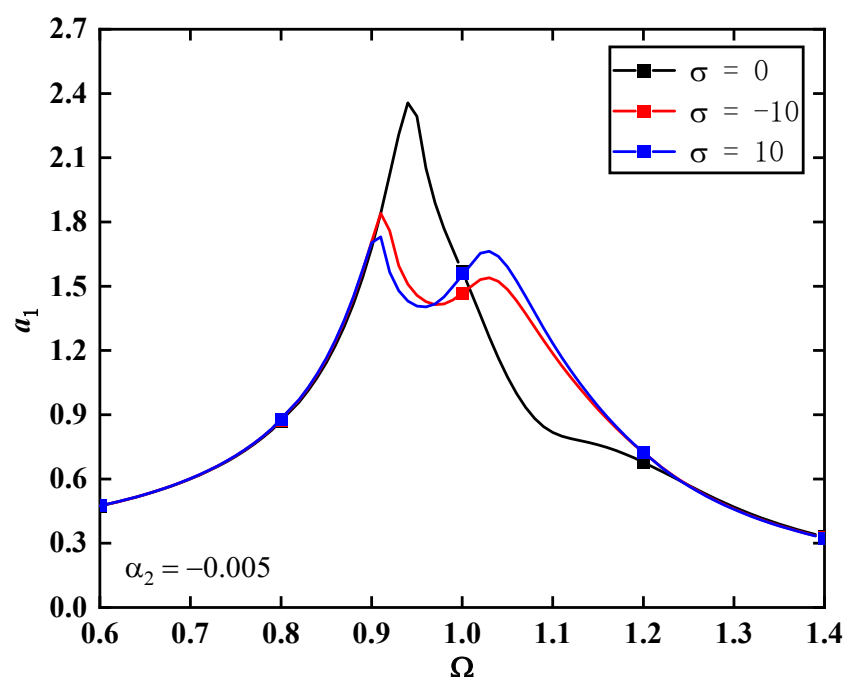


Figure 14. Amplitude–frequency response curve of the controlled structure.

By utilizing Equation (39) and selecting different values of σ and α_2 , we can obtain the design shown in Table 2 for the softening stiffness. The optimal design frequencies of the tuned mass damper under various conditions, as well as the optimal frequencies calculated through numerical methods, are obtained. From Table 2, it can be concluded that the optimal frequency gradually increases with the absolute value of the softening nonlinearity for different frequency offsets in the table. Moreover, if the excitation frequency is unknown, the results obtained by the improved design method based on the offset demonstrate better performance. The optimal NTMD design frequency achieved for an offset of $\sigma = -10$ and $\sigma = 10$ is closest to the numerical solution.

Table 2. Optimal frequency of TMD based on offset and numerical method.

Nonlinear Coefficient	$\sigma = -10$	$\sigma = -5$	$\sigma = 0$	$\sigma = 5$	$\sigma = 10$	Optimal Frequency (Numerical Method)
$\alpha_2 = -0.005$	1.086	1.135	1.156	1.115	1.071	1.084
$\alpha_2 = -0.004$	1.067	1.104	1.119	1.089	1.055	1.062
$\alpha_2 = -0.003$	1.049	1.075	1.085	1.065	1.041	1.036
$\alpha_2 = -0.002$	1.032	1.048	1.054	1.042	1.027	1.013
$\alpha_2 = -0.001$	1.016	1.023	1.026	1.020	1.013	0.991
$\alpha_2 = 0$	1	1	1	1	1	0.969

Figure 15 compares the amplitude–frequency response curves of the uncontrolled structure, the NTMD-controlled structure in the case of 1:1 resonance, the NTMD-controlled structure with an offset of $\sigma = 10$ and the optimal NTMD-controlled structure calculated by the numerical method. It can be observed from the figure that the optimization achieved through the improved design method based on the offset can approach the optimal scenario when the NTMD exhibits the nonlinear characteristic of softening stiffness and the excitation frequency is uncertain, particularly in cases close to 1:1 resonance.

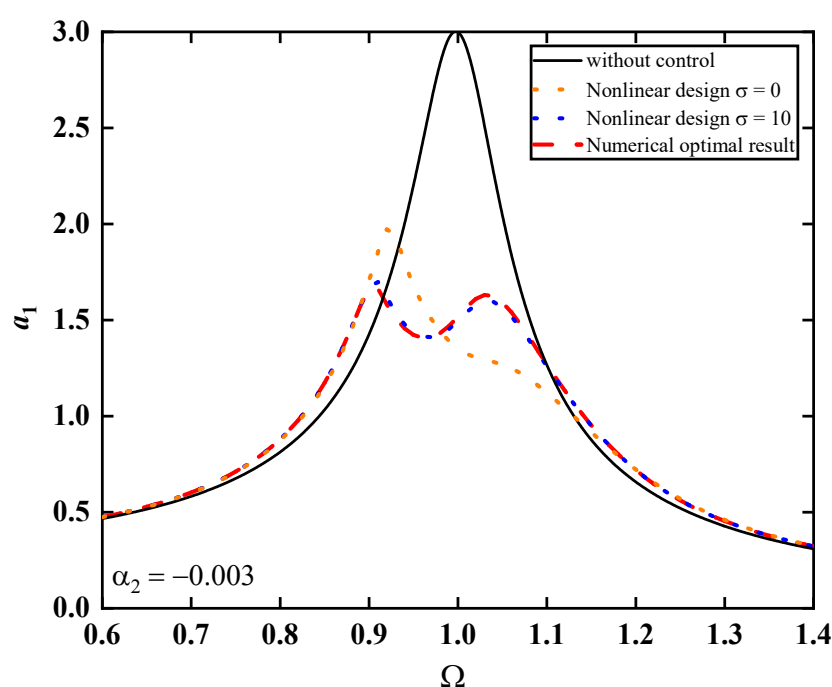


Figure 15. Amplitude and frequency response curves in different cases.

5. Conclusions

This paper investigates the nonlinear characteristics of a tuned mass damper (TMD) resulting from large displacements and the utilization of a limiting device. The system under study is a two-degree-of-freedom system comprising a primary structure subjected to harmonic excitation and a TMD. The approximate analytical solution of the system response is obtained through the complex variable averaging method, and the nonlinear design formulation of the NTMD is derived from it. The main contents and conclusions of this paper can be broadly divided into the following sections:

1. The frequency design formula for the NTMD was derived using the approximate analytical solution obtained through the complex variable averaging method. The results obtained from the approximate analytical solution and the nonlinear design equation were compared with those obtained from the conventional numerical method. The comparison results demonstrate good agreement between the results obtained from the modulation–demodulation equation derived through the complex variable averaging method and the numerical method. Furthermore, the error in the optimal NTMD design frequency obtained from the nonlinear design formula is small compared to the results obtained from the numerical method, indicating that the accuracy meets the analysis requirements.
2. The control performance of the NTMD was investigated based on the damping condition, and the undamped frequency design formula for the NTMD was derived. The results demonstrate that the amplitudes of both the transient and steady-state phases in the NTMD system designed using the undamped frequency design formula are effectively controlled. The damping of the NTMD can be tailored to different excitation frequency conditions, leading to improved control. These findings hold significant reference value for the analysis and design of similar systems.
3. The control performance of the NTMD was investigated under the 1:1 resonance condition. The analysis results show that increasing the mass ratio within a certain range can enhance the control bandwidth and improve the control performance of the NTMD. The control performance of the nonlinear-design-based NTMD is generally superior to that of the linear-design-based NTMD, and the nonlinear coefficients can

be employed as design parameters to adjust the optimal frequency ratio of the structure while keeping the remaining conditions constant. These findings have valuable reference for further studies on the control performance of the NTMD under 1:1 resonance conditions.

4. An investigation was conducted on the NTMD that possesses softening stiffness nonlinearity. The results indicate that, for the NTMD exhibiting softening stiffness nonlinearity, an offset-based nonlinear design formulation is suitable for achieving improved control performance. In scenarios approaching 1:1 resonance with unknown excitation frequency, the improved design method based on the offset can achieve optimal control effectiveness. These findings hold reference significance for future studies on the softening stiffness nonlinearity of the NTMD.

Shortcomings of the Study and Prospects for Future Work

When using Equation (39) for design, the nonlinear coefficients of the NTMD should not be too large. If the nonlinear coefficient is too high, the structure may experience multivalued responses, also known as jumping phenomena, which can have adverse effects on the nonlinear design. Additionally, during the study, it was observed that the system exhibits various nonlinear phenomena in certain cases, and further research is expected to analyze and explain these phenomena and their causes.

Furthermore, as research progresses, excitation with noise is becoming more prevalent in industrial production processes compared to harmonic excitation. Considering the excitation force of Gaussian white noise and studying how the control performance of TMD systems changes due to variations in excitation force would be an interesting avenue for exploration.

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