

Article

Extended Field Interactions in Poisson's Equation Revision

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Abstract: This investigation introduces a new variational approach to refining Poisson's equation, enabling the inclusion of a broader spectrum of physical phenomena, particularly in the emerging fields of spintronics and the analysis of resonant structures. The innovative formulation extends the traditional capabilities of Poisson's equation, offering a nonlocal extension to classical theories of gravitation and opening new directions for energy conversion and enhanced communication technologies. By introducing a novel geometric structure, $\tilde{\omega}$, into the equation, a deeper understanding of electrostatic potentials is achieved, and the intricate dynamics of the gravitational potential in systems characterized by radial vorticity fluctuations are illuminated. Furthermore, the research elucidates the generation of longitudinal electromagnetic waves and resonant phenomena within dusty plasma media, thereby contributing to the methodological advances in the study of nonequilibrium systems. These theoretical advances have the potential to transform the understanding of complex physical systems and open up opportunities for significant technological achievements across a range of scientific sectors.

Keywords: Poisson equation; variational method; resonant structures; nonlocal effects; nonlocal gravity; electrostatic potential



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1. Introduction

The Poisson equation, fundamental in mathematical physics, provides critical insights across various domains, from electrostatics to gravitational fields [1,2]. This work introduces a novel formulation of the Poisson equation, based on a variational approach [3]. The new method balances entropy maximization with energy minimization, producing differential equations that expand our understanding of physical systems. Beyond traditional applications, the updated equation offers critical insights into resonant structures in electromagnetic, heat transmission, and acoustic systems [4–7]. Serving as a foundational tool in various scientific domains, this adaptable second-order partial differential equation can be solved with a range of analytical and numerical approaches (see, e.g., Refs. [8–11]).

The significance of the modified Poisson equation in understanding complex physical phenomena is highlighted through its numerous applications. The implications for nonlocal gravity theories [12,13], spintronics [14–16], and resonant structures [17–19] are discussed, demonstrating potential improvements in comprehending the complex structures relevant to technological and scientific fields.

An overview of the manuscript follows: Section 2 builds upon prior research introducing the concept of topological torsion current in non-equilibrium physical systems [20,21], focusing on its role and implications for the dynamics of the electric field through variational methods, and incorporating a novel geometric structure, $\tilde{\omega}$. Section 3 investigates the various applications of the modified Poisson equation, covering its impact on understanding Yukawa's potential in atomic physics, detailing the intergalactic gravitational potential, and exploring modifications in spintronic current density. These discussions emphasize the equation's adaptability and its capacity to model complex physical phenomena. The manuscript concludes in Section 4 by summarizing the key findings of the study and proposing directions for future research.

2. Investigating Non-Equilibrium Systems: A Variational Approach to Equilibrium Dynamics

Previous works [20,21] introduced the concept of topological torsion current and its significant implications in spacecraft dynamics and particle physics. Building upon this foundation, the role of topological torsion current in nonequilibrium physical systems is further explored, particularly through variational methods. Central to this approach is the introduction of a new geometric structure represented by ω , which plays a pivotal role in the theoretical framework.

The electric field in such a system can be described by

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} - \tilde{\omega} \Phi \quad (1)$$

Here, the term $\tilde{\omega}$ is defined as

$$\tilde{\omega} = \frac{\boldsymbol{\omega} \times [\mathbf{v}]}{c^2} \quad (2)$$

In this formulation, Φ represents the electric potential, $\mathbf{A}$ is the vector potential, the symbol c represents the speed of light in vacuum, and $\boldsymbol{\omega}$, embodying the newly introduced geometric structure, signifies the angular velocity. This angular velocity component is not just a physical quantity but also encapsulates a spin connection, indicating the presence of a torsion field within the theoretical construct. The cross product $[\boldsymbol{\omega} \times \mathbf{A}]$ is conceptualized as the expression of a torsion field inside the electromagnetic field structure, suggesting that $\tilde{\omega}$ has a geometrical meaning beyond its conventional physical interpretation. This broadens the comprehension of electromagnetic phenomena by incorporating the concept of torsion, opening up novel possibilities for investigating the interaction between geometric structures and physical forces. Even though the symbol $\boldsymbol{\omega}$ represents angular velocity, it also has a geometrical meaning that goes beyond its conventional physical meaning. This geometrical component, which is essential to our new methodology, presents torsion as an essential attribute of the field. In this situation, torsion is a physical representation of the inherent geometrical features of the field rather than just a mathematical abstraction.

Thus, when a charged particle q is taken into consideration, its Liénard–Wiechert potentials can be expressed as follows: $\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{(r - \mathbf{r} \cdot \mathbf{v}/c)}$. Here, $\mathbf{r}$ is the vector from the retarded position to the field point r , the magnitude of the distance vector $\mathbf{r}$ from the charge's retarded position to the observation point in space, and $\mathbf{v}$ is the velocity of the charge at the retarded time. Thus, we express $\mathbf{A}(\mathbf{r}, t)$ as $\frac{v}{c^2}\phi(\mathbf{r}, t)$. Consequently, $[\boldsymbol{\omega} \times \mathbf{A}] = \tilde{\omega}\phi$.

By applying these concepts to the Poisson equation, which characterizes the electrostatic potential due to a charge distribution as $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$, we derive a new form of Poisson's equation. This formulation not only incorporates the standard electrostatic potential, but also includes the influence of the novel geometric structure encapsulated in $\tilde{\omega}$. This advancement opens the door to exploring resonant phenomena, particularly those observed in dusty plasma media, through a lens that integrates both physical and geometric insights. Hence, we have

$$\nabla \cdot \mathbf{E} = -\nabla^2\Phi - \partial_t \nabla \cdot \mathbf{A} - \nabla \cdot \tilde{\omega} \Phi. \quad (3)$$

It is noted that the Coulomb gauge imposes the condition $\nabla \cdot \mathbf{A} = 0$, and the Lorenz gauge, $\nabla \cdot \mathbf{A} = -\partial_t \Phi$. Hence, in the Lorenz gauge, the Laplace equation must be written into the inhomogeneous wave equation in the new form

$$\nabla^2\Phi - \partial_{tt}^2\Phi = -\frac{\rho}{\epsilon_0} - (\nabla \cdot \tilde{\omega})\Phi - (\nabla\Phi \cdot \tilde{\omega}). \quad (4)$$

Exploring the dynamics of charge density oscillations and Coriolis-like terms, it appears that there are now three distinct source terms: (i) the charge density ρ and the

effects of its oscillations, including the well-known plasma oscillations at a high frequency and the magnetohydrodynamic waves at a low frequency; (ii) the divergence of the Coriolis-like source term, $\nabla \cdot \tilde{\omega} \neq 0$; and (iii) a measure of the degree of alignment between the electric field and the Coriolis-like term, $(\nabla \Phi \cdot \tilde{\omega})$.

3. Potential Applications

On the basis of the original equation, the revised version of the Poisson equation has a number of benefits. Since it was designed for spinning systems, it is broader and may therefore be used to mimic a wider range of physical systems, including galaxies and stars. In addition, it is more precise and has a better ability to simulate resonant structures. The new form of the problem may be solved using a variety of numerical techniques and is also computationally efficient. As a result, the equation may be solved quickly and precisely, enabling the modeling of complicated physical systems in a relatively short amount of time.

3.1. Some Insight on Yukawa's Potential

The Yukawa potential, which is an important rotational-invariant scattering potential used in atomic and nuclear physics, can be derived from this formalism. By imposing the constraint $\nabla \cdot \tilde{\omega} = 0$, this potential can be derived from the equation, and it is also possible to derive the Debye–Hückel equation for electrolytes. When considering a solenoidal vortex field with $\nabla \cdot \tilde{\omega} = 0$, and in the stationary case with $\nabla \Phi \cdot \tilde{\omega} = 0$, the well-known Yukawa equation for a particle of charge $\rho = Ze$ can be derived:

$$\nabla^2 \Phi + \kappa^2 \Phi = -\frac{Ze}{\epsilon_0} \delta(\mathbf{r} - \mathbf{r}'). \quad (5)$$

Here, $\kappa = \frac{\tilde{\omega}}{c^2}$ and the solution is

$$G_\omega(\mathbf{r}, \mathbf{r}') = \frac{Ze}{4\pi\epsilon_0} \frac{e^{i\kappa|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}. \quad (6)$$

In Equations (5) and (6), the term $\mathbf{r}'$ represents the position vector of the source of the charge distribution, which, in this case, is associated with a point charge Ze . It specifies the location in space where the charge Ze is situated. On the other hand, $\mathbf{r}$ refers to the observation point in space where the potential Φ and the Green's function G_ω are being evaluated.

Assuming a radially-dependent electric potential, Equation (5) simplifies to

$$\begin{aligned} \partial_{rr}^2 \Phi + \frac{2}{r} \partial_r \Phi + \left[\frac{1}{r^2} \partial_r (r^2 \tilde{\omega}_r) \right. \\ \left. + \frac{1}{r \sin \phi} \partial_\theta \tilde{\omega}_\theta \right] \Phi + \tilde{\omega}_r \partial_r \Phi = -\frac{Ze}{\epsilon_0}. \end{aligned} \quad (7)$$

Consider simplifying to basic radial dependencies, such as $\tilde{\omega} = \tilde{\omega}(r)$. Then,

$$\partial_{rr}^2 \Phi + \left(\frac{2}{r} + \tilde{\omega}_r \right) \partial_r \Phi + \left[\frac{1}{r^2} \partial_r (r^2 \tilde{\omega}_r) \right] \Phi = -\frac{Ze}{\epsilon_0}. \quad (8)$$

For the particular case $\tilde{\omega} = 1$, Figure 1 shows the analytical solution of Equation (8). Consider the fact that the potential might be either attractive or repulsive.

Analytical solutions to Equation (8) are essential in understanding the behavior of physical systems. Equation (8) is a special case of the general equation and its analytical solution is represented in Figure 1. The potential can be either repulsive or attractive.

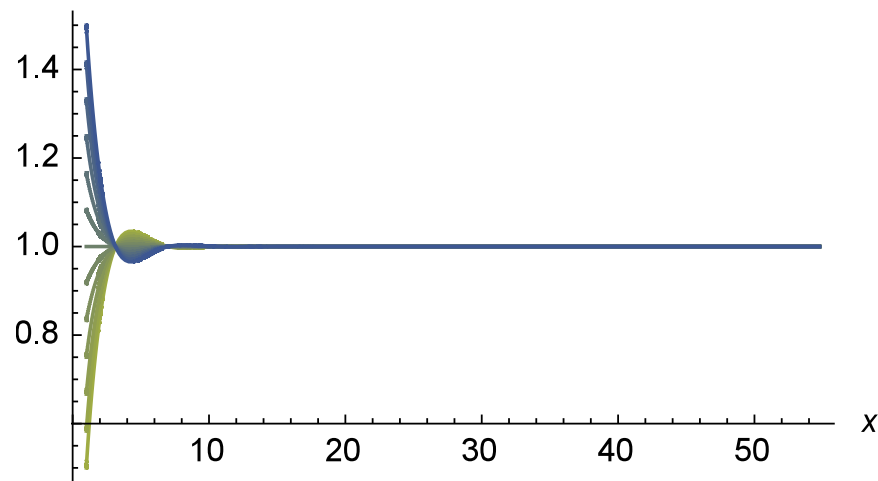


Figure 1. Family of potential curves as a function of distance r . Each curve represents a different case under consideration of basic radial dependencies, such as $\tilde{\omega} = \tilde{\omega}(r)$, illustrating how variations in $\tilde{\omega}_r$ influence the potential. Arbitrary units.

It is very similar to the Lennard–Jones potential. The Lennard–Jones potential is a mathematical model used to describe the interactions between atoms or molecules. It is a simple, effective model to describe the weak intermolecular forces that occur in nature. The Lennard–Jones potential is a combination of two terms: a repulsive term and an attractive term. The repulsive term is a function of the distance between the atoms, and the attractive term is a function of the inverse of the distance between the atoms. However, while it is similar to the Lennard–Jones potential, it introduces a novel aspect by incorporating the radial dependence of $\tilde{\omega}(r)$. This inclusion differentiates it from the standard Lennard–Jones model, which typically focuses on the interplay of repulsive and attractive forces without a specific radial dependence component.

The study of Yukawa potentials in dusty plasmas has significant implications for real-world applications. For instance, it offers critical insights into the behavior of plasma under magnetic influences, which is crucial for advancements in controlled nuclear fusion technology [22]. Additionally, it contributes to our understanding of plasma thermodynamics, potentially aiding in the development of more efficient energy systems [23] or shedding light on phase transitions in plasma, a key aspect that could drive innovations in materials science [24].

3.2. Intergalactic Gravitational Potential

The behavior of gravitational potential in a system where the radial component of the vorticity varies inversely with distance from the center is explored in this Section. The assumption that the radial component is proportional to $1/r$ leads to a solution of the potential in terms of three terms: a term proportional to $1/r$, a term proportional to $\ln(r)/r$, and a term proportional to r^2 . This result sheds light on the behavior of the gravitational potential in idealized and real fluids and is consistent with theoretical predictions and astronomical observations. The obtained intergalactic gravitational potential is consistent with the one caused by dark matter and presents the Navarro–Frenk–White (NFW) profile equation for describing the density distribution of dark matter halos in cosmological simulations.

If the source has a gravitational origin, Equation (8) must be written in the form

$$\partial_{rr}^2 \Phi + \left(\frac{2}{r} + \tilde{\omega}_r \right) \partial_r \Phi + \left[\frac{1}{r^2} \partial_r (r^2 \tilde{\omega}_r) \right] \Phi = -4\pi G \rho. \quad (9)$$

Assuming $\tilde{\omega}_r = 1/r$, the solution for the potential then follows as

$$\Phi(r) = \frac{c_1}{r} + \frac{c_2 \ln(r)}{r} + \frac{r^2}{9} \quad (10)$$

and its representation in Figure 2.

Based on this outcome, it can be concluded that the gravitational potential $\Phi(r)$ is expressible as a composite of three terms: the classical term proportional to $1/r$, a term proportional to $\ln(r)/r$, and a term proportional to r^2 . The coefficients of these terms are given by c_1 , c_2 , and $1/9$, respectively. The assumption that $\tilde{\omega}_r = 1/r$ allowed us to solve the potential $\Phi(r)$ using the equations governing the behavior of gravitational fields. This result provides insight into the behavior of the gravitational potential in a system where the radial component of the vorticity is assumed to vary inversely with the distance from the center of the system. A similar result was obtained in the work of Farkhat Zaripov [25] in the context of dark matter models, where the gravitational potential's characteristics were explored within the framework of modified gravity theories. Zaripov's research, akin to this study, underscores the intricate relationship between field oscillations and gravitational effects, highlighting the adaptability and comprehensiveness of these theoretical frameworks within astrophysical settings.

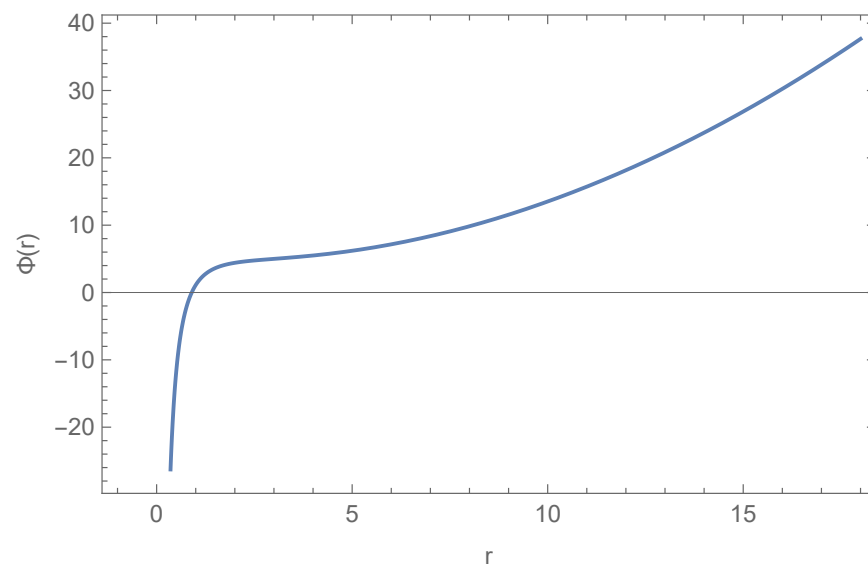


Figure 2. Intergalactic gravitational potential $\Phi(r)$ as a function of the radial distance r (in arbitrary units).

This is consistent with the theoretical predictions and astronomical observations [26]. In an idealized inviscid fluid, the circulation of a vortex is conserved and does not decay with distance. In a real fluid with viscosity, the circulation can decay with distance due to viscosity effects. In 2D viscous flows, the decay rate of circulation with distance can be approximated as $1/r$. In 3D flows, the decay rate can be more complex and can depend on the specific flow conditions and properties.

A modified version of the gravitational potential equation that accounts for the spread of dark matter in the cosmos may be used to determine the intergalactic gravitational potential caused by dark matter. A useful equation for describing the density distribution of dark matter halos in cosmological simulations is the Navarro–Frenk–White (NFW) profile. The source of the NFW profile is as follows:

$$\rho(r) = \frac{\rho_0}{(r/r_s)(1 + r/r_s)^2}, \quad (11)$$

where $\rho(r)$ is the density of dark matter at a distance r from the center of the halo, ρ_0 is a characteristic density, and r_s is a scale radius.

Through the Poisson equation, the gravitational potential can be related to the dark matter density as

$$\nabla^2 \Phi = 4\pi G \rho,$$

where Φ is the gravitational potential and G is the gravitational constant.

By inserting the NFW profile into this equation and resolving for Φ , the following expression for the intergalactic gravitational potential attributable to dark matter is obtained:

$$\Phi(r) = -\frac{4\pi G \rho_0 r_s^3}{r} \ln\left(1 + \frac{r}{r_s}\right). \quad (12)$$

The second term obtained in Equation (10) bears resemblance to that of Equation (12).

3.3. A Modification of the Spintronic Current Density

Spintronics is an emerging field of technology that focuses on the manipulation and control of electrons' spin in solid-state devices [27]. The spintronic current density can be optimized to increase performance while using less energy and to better integrate with cutting-edge technology like quantum computers and neural networks. The total current density is derived from Equation (1) and the general constitutive relation $\mathbf{J} = \sigma_c \mathbf{E}$:

$$\mathbf{J} = \sigma_c \mathbf{E}_0 - \sigma_c \frac{\partial \mathbf{A}}{\partial t} - \sigma_c \tilde{\omega} \Phi. \quad (13)$$

In the context of spintronics, Equation (13) suggests that the current density $\mathbf{J}$ can be decomposed into three terms: the first term $\sigma_c \mathbf{E}_0$ is the ordinary Ohmic current, the second term $-\sigma_c \frac{\partial \mathbf{A}}{\partial t}$ represents the contribution from the electromagnetic induction due to time-varying magnetic fields, and the third term $-\sigma_c \tilde{\omega} \Phi$ represents the contribution from the spin accumulation induced by the spin-orbit interaction.

One possible manipulation of Equation (13) is to take its curl, which can give us insight into the behavior of the spintronic current density under certain conditions. By calculating the curl of both sides of Equation (13), the following is obtained:

$$\nabla \times \mathbf{J} = \sigma_c \nabla \times \mathbf{E}_0 - \sigma_c \frac{\partial}{\partial t} (\nabla \times \mathbf{A}) - \sigma_c \nabla \times (\tilde{\omega} \Phi). \quad (14)$$

By applying the Maxwell–Faraday equation $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, the second term on the right-hand side can be simplified:

$$\nabla \times \mathbf{J} = \sigma_c \nabla \times \mathbf{E}_0 + \sigma_c \frac{\partial \mathbf{B}}{\partial t} - \sigma_c \nabla \times (\tilde{\omega} \Phi). \quad (15)$$

If it is assumed that the magnetic field $\mathbf{B}$ is negligible in comparison with the electric field $\mathbf{E}_0$, the second term can be omitted for further simplification:

$$\nabla \times \mathbf{J} = \sigma_c \nabla \times \mathbf{E}_0 - \sigma_c \nabla \times (\tilde{\omega} \Phi). \quad (16)$$

This formula demonstrates how the axial vector term involving spin $\tilde{\omega}$ and the curl of the electric field relate to the curl of the spintronic current density. This connection may be leveraged to comprehend the dynamics of spin currents in intricate geometries and to develop new spintronic devices.

Examining the behavior of the axial vector term $\tilde{\omega} \Phi$ in Equation (13) when other spin textures or geometries are present is one method that could be used. Consider, for instance, the situation of a magnetic domain wall in which the spin direction fluctuates from side to side. This may result in the spin current density having a nonzero value, which could

have potential spintronics applications. This leads to a deeper understanding of spintronic current density and its applications [28–32].

3.4. Applications in Nonlinear Optics and Metamaterials

The ability to generate and control longitudinal electromagnetic waves is still in its inception. Longitudinal electromagnetic waves have the potential to revolutionize communication and data storage. They could be used to transmit data over long distances without loss of quality or speed. These waves could also be used to store large amounts of data, making it possible to store and access large amounts of information in a relatively small space. Classical Maxwell's equations predict the existence of only transversal waves in a vacuum since, without material support, such as in a plasma medium where Langmuir waves are observed, there is no support for their propagation. Theoretical evidence for the existence of longitudinal waves in a vacuum has been documented from several sources, including [33–35], at the observational level [36–38], and in the proposal of two different apparatuses that are configured to transmit and/or receive scalar-longitudinal waves. More experimental evidence is needed to confirm their existence (see [39–44]). Attenuation is the gradual loss of energy that occurs as waves propagate through a medium. However, unlike transverse electromagnetic waves, longitudinal waves are not subject to attenuation effects, since they can travel through a vacuum without a medium. This unique characteristic allows them to transmit data over long distances without loss of quality or speed, which could revolutionize the field of communication. With the ability to transmit large amounts of data over vast distances, global communication and information exchange could be greatly enhanced.

Upon calculating the gradient of Equation (4) and substituting $\square u = \Delta u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$, it becomes apparent that

$$\square \mathbf{E} = \frac{\nabla \rho}{\epsilon_0} + (\nabla \cdot \tilde{\omega}) \mathbf{E} + \Phi(\nabla \nabla \cdot \tilde{\omega}) + (\mathbf{E} \cdot \nabla) \tilde{\omega} + (\tilde{\omega} \cdot \nabla) \mathbf{E} + \mathbf{E} \times [\nabla \times \tilde{\omega}] + \tilde{\omega} \times [\nabla \times \mathbf{E}]. \quad (17)$$

This equation describes the behavior of the electromagnetic field, specifically the electric field $\mathbf{E}$, in the presence of a source charge density ρ and vorticity field $\tilde{\omega}$. The left-hand side of the equation represents the wave equation for the electric field, while the first term on the right-hand side describes the contribution of the source charge density to the electric field; the second term on the right-hand side involves the divergence of the vorticity field, which represents the tendency of the vorticity to cause stretching and rotation of the electric field lines; the third term involves the gradient of the vorticity divergence, which represents the effect of the vorticity on the scalar potential of the electric field; the next two terms involve the dot product of the electric field with the gradient and the vorticity, respectively, and describe the interaction between the electric field and the vorticity; the sixth term involves the cross-product between the electric field and the curl of the vorticity and describes the tendency of the vorticity to induce circulation in the electric field. Finally, the last term involves the cross-product between the vorticity and the curl of the electric field and describes the tendency of the electric field to induce a rotation in the vorticity field. Overall, this equation describes the complex interplay between the electric field and the vorticity field and provides insight into the behavior of electromagnetic waves in various physical systems. The equation resulting from Equation (17) presents a complex scenario for analysis. However, by focusing on the first and last term on the right-hand side, the following can be deduced:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{1}{\epsilon_0} \nabla \rho - \left[\tilde{\omega} \times \frac{\partial \mathbf{B}}{\partial t} \right]. \quad (18)$$

Let us consider each term: (i) $\frac{1}{\epsilon_0} \nabla \rho$ is classical, associated with Langmuir waves, or electron plasma waves, for example; (ii) $\left[\tilde{\omega} \times \frac{\partial \mathbf{B}}{\partial t} \right]$ represents an interaction between a vector field (potentially a velocity or rotational field) and the time derivative of the magnetic field. It represents a new term force.

Drawing from the interaction dynamics between rotating electromechanical systems and temporally varying magnetic fields, an experimental setup is proposed, characterized by a dual-modality operational framework. At the core of this device is either a lattice designed with interlaced wire-medium metamaterial (possible longitudinal EM wave generator) [45] or a rotational plasma arrangement (some methodologies are discussed in Refs. [46,47]). These rotational systems are aligned such that their vectorial representations are orthogonal to the vector field of an externally imposed, pulsed magnetic field, thereby optimizing the conditions for maximal cross-product interaction. Specifically, the metamaterial lattice is designed to exhibit spatial anisotropy due to its cubic lattices and the way these lattices are interlaced to manipulate the wave vector and dielectric permittivity characteristics to support longitudinal wave propagation, while the plasma setup facilitates angular momentum in ionized gases [46,48–50]. The pulsating magnetic field, applied in quadrature to these rotational vectors, serves as a dynamic perturbation, poised to elicit a rich spectrum of wave phenomena, both longitudinal and transverse. An illustrative example of possible experiment is proposed in Figure 3.

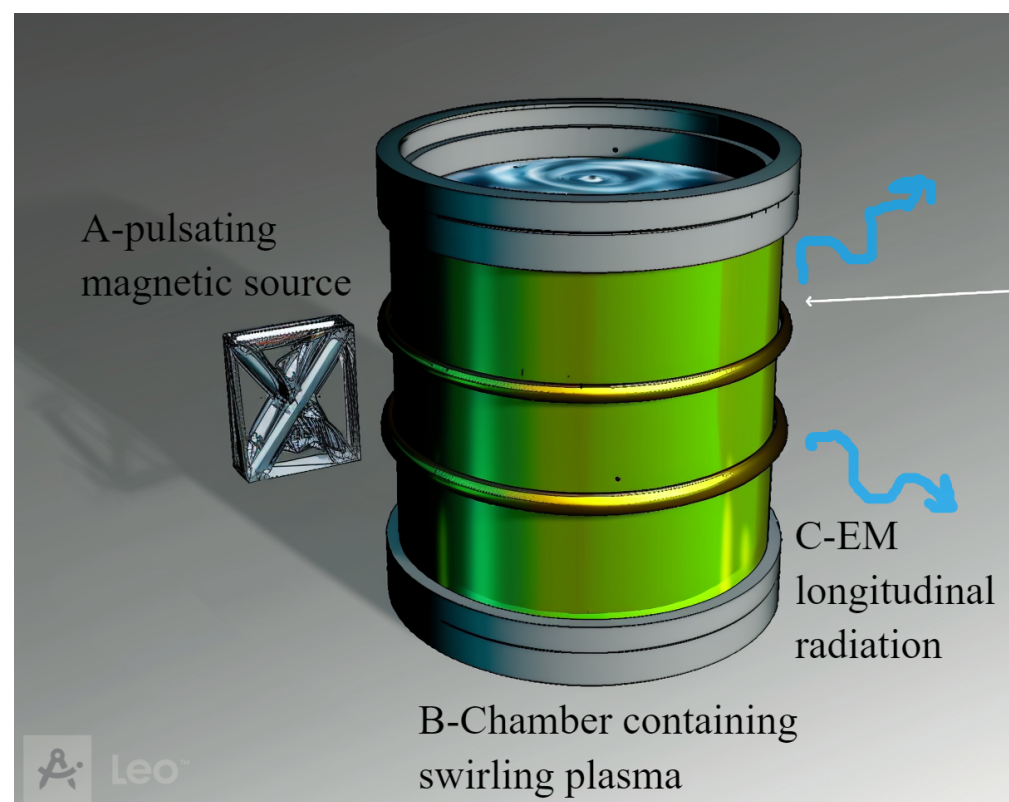


Figure 3. Mockup of an experimental apparatus for generating electromagnetic longitudinal waves. Component A represents the pulsating magnetic field generator. Component B is the cylindrical chamber containing swirling plasma, crucial for the wave generation process. Component C shows the resultant electromagnetic (EM) longitudinal waves (indicated by blue arrows), which are generated as a result of the cross-product interaction between the magnetic field and the swirling plasma. Figure created using <https://www.getleo.ai/> (accessed on 18 January 2024).

A self-propelling device incorporating the previously mentioned principle and producing thrust with a cylindrical setup of radius 3.0 m and height 1.0 m (dimensions chosen for illustrative purpose) can be proposed based on the last term of force. It leverages the synergy of electric and magnetic fields, with a focus on a low-level charge density $\rho(z) = 0.01 \text{ C/m}^3$ (with Z the vertical axis), an angular frequency $\omega(z)$ of the order of 10 kHz (as, for example, observed in Hall Thrusters; see also Ref. [51]) and a dynamic magnetic field $dB/dt(z)$ (with applied frequency source on the order of MHz). Observe that the cross product $\left[\tilde{\omega} \times \frac{\partial \mathbf{B}}{\partial t}\right]$ determines the direction of motion. This approach, based on integrating the differential equation for the electric field term of force $\rho(z)E_z(z, t)$ across the cylinder's volume, suggests the possibility to generate thrust capability exceeding 2000 Newtons (the computational model and code detailing this concept are available in GitHub repository [52]).

3.5. Generation of Electron Beams Carrying Orbital Angular Momentum

The generation of electron beams carrying orbital angular momentum (OAM) [53–69] refers to the creation of electron beams that possess a well-defined helical phase front, similar to a spiral staircase. This helical structure of the electron beam arises due to the presence of OAM, which is a property of a wave that describes the rotation of the wave's phase around its axis of propagation.

One way to generate electron beams carrying OAM is by using electron holography, which is a technique that allows for the measurement and manipulation of electron wavefronts [62–66]. In electron holography, a beam of electrons is split into two paths, with one path being used as a reference wave and the other passing through a sample. The two beams are then recombined, resulting in an interference pattern that contains information about the electron wavefront. By manipulating the electron wavefront using holography, it is possible to introduce a helical phase shift into the electron beam, thereby generating an electron beam carrying OAM. Such electron beams have been used in a variety of applications, such as electron microscopy and nanofabrication, and have the potential to revolutionize fields such as quantum computing and communication. This formalism's application to the issue will be illustrated.

The phase shift associated with an electron beam carrying OAM can be described mathematically using the following equation:

$$\Psi(r, \theta, z) = A(r, \theta, z) \exp(il\theta), \quad (19)$$

where Ψ is the electron wavefunction, A is the amplitude of the wavefunction, r , θ , and z are the cylindrical coordinates, and l is the azimuthal quantum number that determines the amount of OAM carried by the electron beam. The total angular momentum J of an electron beam carrying OAM can be expressed as $J = \hbar l$, where $\hbar$ is the reduced Planck constant. Observing that the axial vector $\tilde{\omega}$ possesses a dimensionality of $1/L$, it prompts the proposal that the axial vector can be articulated as a product of a spherical harmonic function $Y_{lm}(\theta, \phi)$ and a radial unit vector $\mathbf{r}/r$. Based on this postulate, expressions for the divergence of the vector $\Phi(r) \tilde{\omega}$ are derived using the properties of spherical harmonics. Specifically, it uses the expressions for the divergence of $\Phi(r)Y_{l,l+1,m}(\theta, \phi)$, $\Phi(r)Y_{l,l,m}(\theta, \phi)$, and $\Phi(r)Y_{l,l-1,m}(\theta, \phi)$, which are derived using the differential operators for spherical coordinates. These expressions involve derivatives of the radial function $\Phi(r)$ and the spherical harmonic functions, and they provide a way to relate the divergence of $\Phi(r) \tilde{\omega}$ to the function $\Phi(r)$ and its derivatives (see also Ref. [70] with further information on the properties and use of spherical harmonics in physics; and Refs. [71–73] for applications). Hence, the Axial vector $\tilde{\omega}$ is defined by

$$\tilde{\omega} = \frac{\mathbf{r}}{r} Y_{lm}(\theta, \phi). \quad (20)$$

A more extended expression for $\tilde{\omega}$ in terms of vector spherical harmonics is

$$\tilde{\omega} = -\left[\frac{l+1}{2l+1}\right]^{1/2} \mathbf{Y}_{l,l+1,m} + \left[\frac{l}{2l+1}\right]^{1/2} \mathbf{Y}_{l,l-1,m} \quad (21)$$

and for the divergence of $\Phi(r)\tilde{\omega}$, it is obtained that

$$\begin{aligned} \nabla \cdot [\Phi(r)\tilde{\omega}] = & -\left[\frac{l+1}{2l+1}\right]^{1/2} \nabla \cdot [\Phi(r)\mathbf{Y}_{l,l+1,m}(\theta, \phi)] \\ & + \left[\frac{l}{2l+1}\right]^{1/2} \nabla \cdot [\Phi(r)\mathbf{Y}_{l,l-1,m}(\theta, \phi)]. \end{aligned} \quad (22)$$

Since the equations discussed thus far are based on the mathematics of vector spherical harmonics and the characteristics of angular momentum, they offer an effective foundation for explaining helical waves and OAM in electron beams. However, it is important to note that these equations are not necessarily an improvement over existing methods for describing OAM in electromagnetic waves. Instead, they offer an alternative framework for understanding and analyzing electron beams carrying OAM, especially in the context of electron holography and other experimental techniques, but they can be useful for certain applications and may help researchers gain new insights into the behavior of electron beams with OAM. For instance, in electron microscopy [69], these equations can potentially lead to an improvement in the analysis of the electron beam's interaction with samples: when an electron beam with OAM interacts with a sample, the helical phase front can be influenced by the sample's structure and composition, leading to a change in the OAM distribution [74].

Utilizing the derived equations to describe the helical phase front and the divergence of the axial vector enriches the understanding of how electron beams interact with samples. This insight aids in unraveling the impact of the sample's structure on the distribution of the electron beam's orbital angular momentum (OAM), facilitating more precise interpretations of electron microscopy images. Furthermore, these equations could inform the design of innovative electron holography experiments or techniques that leverage electron beams' OAM properties. Manipulating an electron beam's OAM could pioneer new approaches to examine specific structural characteristics of a sample or enhance the resolution of electron microscopy [75–77].

For instance, consider an application in which researchers are working with twisted light beams (optical vortices) carrying OAM in optical communication systems [75–83]. They could use these equations to analyze the effects of different media on the propagation of the twisted light beams and their OAM properties. By understanding the changes in the axial vector and its divergence, researchers could optimize the design of optical communication systems to minimize the loss of OAM information during transmission, thus improving the efficiency and reliability of these systems. Similarly, these equations can be applied in the field of plasmonics, where the interaction of light with metallic nanostructures to confine and manipulate EM waves on the nanoscale is of importance [84–86]. By using these equations, we can analyze the impact of different nanostructures on the OAM properties of the incident EM waves and design nanostructures that can efficiently manipulate the OAM properties for various applications, such as optical tweezers, super-resolution imaging, or quantum information processing [87–91].

3.6. Application of the Axial Vector and Its Divergence to Plasmonics

To illustrate the potential use of equations for the axial vector and its divergence, let us consider an example in the field of plasmonics. The interaction of a twisted light beam with a metallic nanostructure, such as a metallic nanorod, will be analyzed to determine how the nanostructure influences the OAM properties of the incident electromagnetic wave.

Initially, the axial vector $\tilde{\omega}$, as defined in Equation (20), will be utilized. Following this, the expression for the divergence of $\Phi(r)\tilde{\omega}$, presented in Equation (22), will be employed.

The interaction between the twisted light beam and the metallic nanorod, as depicted in Figure 4, will be examined.

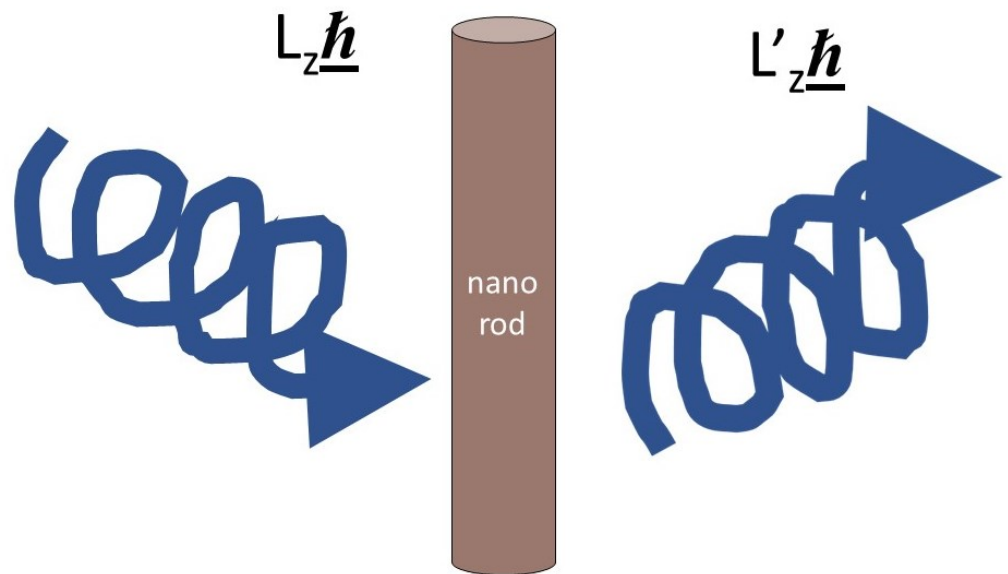


Figure 4. Illustration of the interaction between a twisted light beam and a metallic nanorod, demonstrating the principles of electron holography in generating electron beams with orbital angular momentum (OAM). The figure depicts the helical phase front of the electron beam, induced by the OAM, as it interacts with the nanorod, highlighting the potential for manipulation and analysis of the OAM properties in various applications, including electron microscopy and optical communications.

This interaction modifies the amplitude $\Phi(r)$ of the EM wave. In this case, we can calculate the new amplitude $\Phi'(r)$ after the interaction with the nanorod by solving Maxwell's equations or using numerical methods like finite-difference time-domain (FDTD) simulations.

In this scenario, the new amplitude $\Phi'(r)$ following the interaction with the nanorod can be determined by solving Maxwell's equations or employing numerical methods such as finite-difference time-domain (FDTD) simulations.

With the new amplitude $\Phi'(r)$, the new axial vector $\tilde{\omega}'$ can be computed, as well as its divergence. By comparing the divergence of $\tilde{\omega}$ and $\tilde{\omega}'$, the impact of the metallic nanorod on the OAM properties of the incident EM wave can be analyzed.

A simple model will be further developed, utilizing the axial vector and its divergence, to understand the impact of a metallic nanorod on the OAM properties of an incident electromagnetic wave. The analysis will focus on the changes in the axial vector and its divergence before and after interaction with the nanorod. Let us assume the following: The incident twisted light beam has a helical wavefront described by the wave function $\Psi(r, \theta, z) = A(r, \theta, z) \exp(il\theta)$, where l is the azimuthal quantum number that determines the OAM. The metallic nanorod has a length L and a radius a . It is positioned along the z -axis and interacts with the incident EM wave (see Figure 4).

- Step 1: Calculating the initial axial vector and its divergence for the incident EM wave. The initial axial vector can be calculated using the equations provided for $\tilde{\omega}$ and its divergence $\nabla \cdot (\Phi(r) \tilde{\omega})$ before the interaction with the nanorod.
- Step 2: Defining the interaction between the twisted light beam and the metallic nanorod. For simplicity, it is assumed that the interaction between the twisted light beam and the nanorod can be represented by an interaction factor α . This factor depends on the size, shape, and material properties of the nanorod as well as the wavelength of the incident EM wave. The new amplitude after the interaction can be represented as $\Phi'(r) = \alpha\Phi(r)$.

- Step 3: Calculating the new axial vector and its divergence after the interaction. Using the new amplitude $\Phi'(r)$, the new axial vector $\tilde{\omega}'$ and its divergence $\nabla \cdot (\Phi'(r) \tilde{\omega}')$ after interaction with the nanorod can be calculated.
- Step 4: Comparing the initial and final axial vectors and their divergences. By comparing the axial vectors $\tilde{\omega}$ and $\tilde{\omega}'$, as well as their divergences, the changes in the OAM properties of the incident EM wave due to interaction with the metallic nanorod are analyzed.
- Step 5: Optimizing the design of the metallic nanorod.

On the basis of the analysis of changes in the axial vector and its divergence, researchers can optimize the design parameters of the metallic nanorod (such as length, radius, and material) to efficiently manipulate the OAM properties of the incident EM waves for specific applications. This procedure is illustrated in Figure 5, and the code used for the implementation can be found on Github at [92].

This simple model helps illustrate the use of the axial vector and its divergence in analyzing the impact of a metallic nanorod on the OAM properties of an incident electromagnetic wave. It also provides a starting point for designing nanostructures that can efficiently manipulate the OAM properties for various applications. However, a more realistic model would require solving Maxwell's equations or using numerical methods like FDTD simulations to account for the complex interaction between the twisted light beam and the nanostructure.

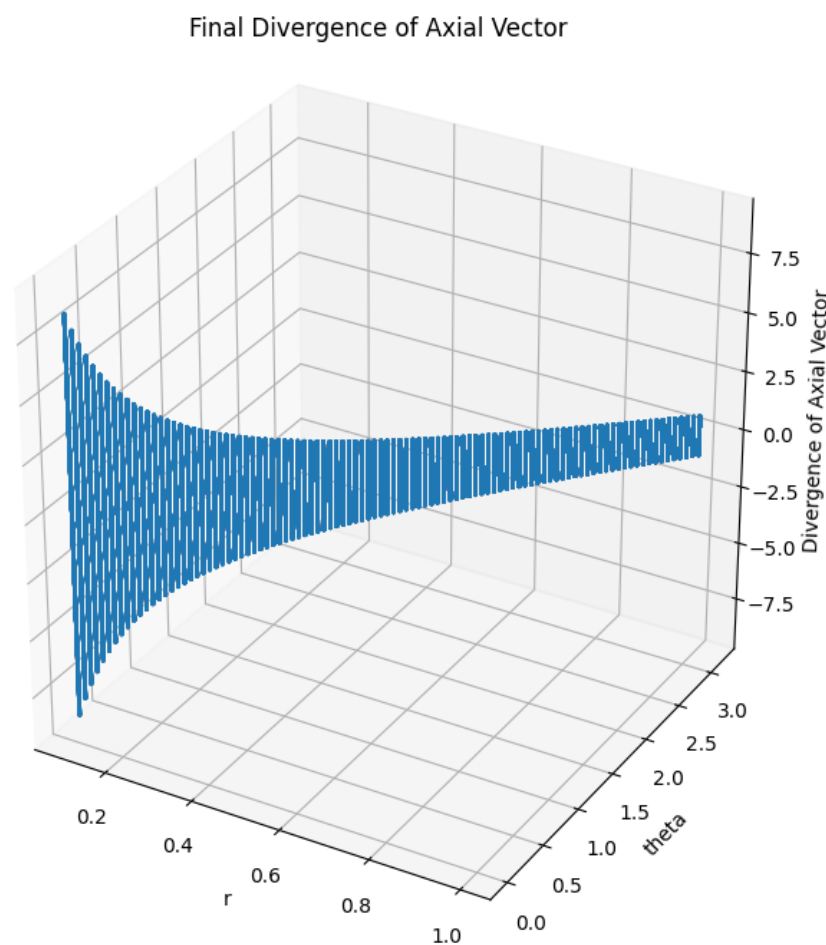


Figure 5. Graphical representation of the calculated differences in the axial vector $\tilde{\omega}$, as derived from the proposed method. This figure quantitatively illustrates the variations in the axial vector resulting from specific interactions or conditions in the study, emphasizing the method's capability to detect and analyze subtle changes in electron beam properties in plasmonic and other advanced applications.

4. Conclusions

The modified Poisson equation provides insight into several physical systems. It is developed from variational concepts, and its potential in spintronics and its application in modeling resonant structures and nonlocal gravity theories enable new research directions, including new sources of propulsion, energy conversion, and communication, although, without any doubt, future research is still needed to fully understand the potential of these applications. Through the use of the axial vector and its divergence in conducting plasmonic research, a novel approach to control the orbital angular momentum of electromagnetic waves as they interact with nanostructured surfaces was proposed. Interesting implications for metamaterials and nonlinear optics are expected, particularly with regard to producing and managing longitudinal electromagnetic waves. The potential for new wave induction techniques is highlighted by the new components of the Poisson equation, in particular the interaction between rotational vector fields and time-varying magnetic fields. Our suggested experimental setup seeks to investigate these interactions further, with the potential to transform data storage and communication technologies by merging spinning plasma settings or metamaterial lattices.

Although this work improves the modeling of complex systems using a modified Poisson equation, it has limitations: (i) Linearity Assumption: Since this approach is based on linear responses, it might not adequately account for nonlinear dynamics in intricate or out-of-equilibrium systems; (ii) Idealized Conditions: Although adding a geometric structure $\tilde{\omega}$ enriches the analysis, simplifications may overlook some geometric effects (the influence of the spatial configuration, curvature, or topology of a system on its physical properties and behaviors); (iii) Computational Demands: Due to computational limitations, the modified equation's practical application may be restricted by the difficulty of solving it, particularly when nonlocal and torsion terms are involved; (iv) Experimental Validation Needed: Further theory development may be necessary to determine the applicability of this method to quantum or relativistic phenomena, especially those involving torsion fields, which await experimental validation.

In summary, although the method demonstrates the potential for investigating physical phenomena, its limitations underscore the need for ongoing research to enhance the approach and expand its applications. Future endeavors will concentrate on overcoming these challenges to deepen our understanding and broaden the application of complex structures.

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References

1. Poisson, C. *Mécanique Analytique*; Paris, France, 1811. Available online: https://openlibrary.org/books/OL24227682M/Trait%C3%A9_de_m%C3%A9canique (accessed on 1 January 2024).
2. Maxwell, J.C. *A Treatise on Electricity and Magnetism*; Clarendon Press: Oxford, UK, 1873.
3. Pinheiro, M.J. A Variational Method in Out-of-Equilibrium Physical Systems. *Sci. Rep.* **2013**, *3*, 3454. [[CrossRef](#)] [[PubMed](#)]
4. Myers, W.D.; Swiatecki, W.J. The nuclear shape transition and its relation to a new class of mesoscopic systems. *Phys. Lett. B* **1998**, *441*, 1–4.
5. Wright, E.L. The intergalactic medium. In *Universe in a Nutshell: The Physics of Everything*; Dresselhaus, M.S., Dresselhaus, G., Eds.; Springer: New York, NY, USA, 2001; pp. 49–57.
6. Zutic, I.; Fabian, J.; Das Sarma, S. Spintronics: Fundamentals and Applications. *Rev. Mod. Phys.* **2004**, *76*, 323. [[CrossRef](#)]
7. Serdyukov, V.; Semchenko, I.; Tretyakov, S.; Sihvola, A. *Electromagnetics of Bi-anisotropic Materials: Theory and Applications*; Gordon and Breach: Amsterdam, The Netherlands, 2001.
8. Gilbarg, D.; Trudinger, N.S. *Elliptic Partial Differential Equations of Second Order*, 2nd ed.; Springer: Berlin, Germany, 1983.

9. Reddy, J.N. *An Introduction to the Finite Element Method*, 3rd ed.; McGraw-Hill: New York, NY, USA, 2006.
10. Mammoli, A.A.; Kassab, Y.M. *Boundary Element Analysis: Mathematical Aspects and Applications*; CRC Press: Boca Raton, FL, USA, 2018.
11. Trefethen, L.N. Spectral methods. In *Handbook of Numerical Analysis*; Ciarlet, P.G., Lions, J.-L., Eds.; Elsevier: Amsterdam, The Netherlands, 1997; pp. 209–305.
12. Tarasov, V.E. Nonlocal classical theory of gravity: Massiveness of nonlocality and mass shielding by nonlocality. *Eur. Phys. J. Plus* **2022**, *137*, 1336. [\[CrossRef\]](#)
13. Mashhoon, B. *Nonlocal Gravity*; Oxford Academic: Oxford, UK, 2022. [\[CrossRef\]](#)
14. Pawar, S.; Duadi, H.; Fixler, D. Recent Advances in the Spintronic Application of Carbon-Based Nanomaterials. *Nanomaterials* **2023**, *13*, 598. [\[CrossRef\]](#) [\[PubMed\]](#)
15. Lu, W.-t.; Yuan, Z. Progress in ultrafast spintronics research. *Sci. Sin. Phys. Mech. Astron.* **2022**, *52*, 270007. [\[CrossRef\]](#)
16. Dey, C.; Yari, P.; Wu, K. Recent advances in magnetoresistance biosensors: A short review. *Nano Futur.* **2023**, *7*, 012002. [\[CrossRef\]](#)
17. Lenells, J. Poisson structure of a modified Hunter-Saxton equation. *J. Phys. A* **2008**, *41*, 285207. [\[CrossRef\]](#)
18. Burgnies, L.; Vanbésien, O.; Sadaune, V.; Lippens, D.; Nagle, J.; Vinter, B. Resonant tunneling structures with local potential perturbations. *J. Appl. Phys.* **1994**, *75*, 4527–4532. [\[CrossRef\]](#)
19. Krasil'shchik, J.; Verbovetsky, A.; Vitolo, R. *Variational Poisson Structures*; Springer: Berlin/Heidelberg, Germany, 2017; pp. 193–214. [\[CrossRef\]](#)
20. Pinheiro, M.J. Some effects of topological torsion currents on spacecraft dynamics and the flyby anomaly. *Mon. Not. R. Astron. Soc.* **2016**, *461*, 3948–3953. [\[CrossRef\]](#)
21. Pinheiro, M.J. Ergotropic Dynamics: Contribution for an Extended Particle Dynamics. In *Rhythmic Advantages in Big Data and Machine Learning*; Bandyopadhyay, A., Ray, K., Eds.; Springer: Singapore, 2022. [\[CrossRef\]](#)
22. Feng, Y.; Lu, S.; Wang, K.; Lin, W.; Huang, D. *Dynamics and Transport of Magnetized Two-Dimensional Yukawa Liquids*; Springer: Berlin/Heidelberg, Germany, 2019.
23. Castello, F.L.; Tolias, P.; Hansen, J.S.; Dyre, J.C. Isomorph invariance and thermodynamics of repulsive dense bi-Yukawa one-component plasmas. *Phys. Plasmas* **2019**, *26*, 053705. [\[CrossRef\]](#)
24. Maity, S.; Das, A. Molecular dynamics study of crystal formation and structural phase transition in Yukawa system for dusty plasma medium. *Phys. Plasmas* **2019**, *26*, 023703. [\[CrossRef\]](#)
25. Zaripov, F. Dark Matter as a Result of Field Oscillations in the Modified Theory of Induced Gravity. *Symmetry* **2020**, *12*, 41. [\[CrossRef\]](#)
26. Hayashi, E.; Navarro, J.F.; Springel, V. The shape of the gravitational potential in cold dark matter haloes. *MNRAS* **2007**, *377*, 50–62. [\[CrossRef\]](#)
27. Hirohata, A.; Yamada, K.; Nakatani, Y.; Prejbeanu, I.; Diény, B.; Pirro, P.; Hillebrands, B. Review on spintronics: Principles and device applications. *J. Magn. Magn. Mater.* **2020**, *509*, 166711. [\[CrossRef\]](#)
28. Shi, C.; Wen-Jeng, H. High spin current density in gate-tunable spin-valves based on graphene nanoribbons. *Dent. Sci. Rep.* **2023**, *13*, 9234. [\[CrossRef\]](#)
29. Qi, A.; Sébastien, L.; Beux, I.; O'Connor, J.-O.K. Large scale, high density integration of all spin logic. In Proceedings of the 2018 Design, Automation & Test in Europe Conference & Exhibition (DATE), Dresden, Germany, 19–23 March 2018; pp. 131–136. [\[CrossRef\]](#)
30. Shaahin, A.; Zhezhi, H.; Yu, B.; Jie, H.; Mingjie, L.; Ronald, F.; DeMara, D.F. Leveraging Spintronic Devices for Efficient Approximate Logic and Stochastic Neural Networks. In Proceedings of the 2018 on Great Lakes Symposium on VLSI, Chicago, IL, USA, 23–25 May 2018; pp. 397–402. [\[CrossRef\]](#)
31. Jonietz, F.; Mühlbauer, S.; Pfleiderer, C.; Neubauer, A.; Münzer, W.; Bauer, A.; Adams, T.; Georgii, R.; Böni, P.; Rembert, A.; et al. Spin Transfer Torques in MnSi at Ultralow Current Densities. *Science* **2010**, *330*, 1648–1651. [\[CrossRef\]](#)
32. Cockburn, B.F. The emergence of high-density semiconductor-compatible spintronic memory. In Proceedings of the International Conference on MEMS, NANO and Smart Systems, Banff, AB, Canada, 23 July 2003; pp. 321–326. [\[CrossRef\]](#)
33. van Vlaenderen, K.J.; Waser, A. Generalisation of classical electrodynamics to admit a scalar field and longitudinal waves. *Hadron. J.* **2001**, *24*, 609–628.
34. van Vlaenderen, K.J. A generalization of classical electrodynamics for the prediction of scalar field effects. *arXiv* **2003**, arXiv:physics/0305098v1.
35. Khvorostenko, N.P. Longitudinal electromagnetic waves. *Sov. Phys. J.* **1992**, *35*, 223–227. [\[CrossRef\]](#)
36. Mostein, C.; Wesley, J.P. Observation of scalar longitudinal electrodynamic waves. *Europhys. Lett.* **2002**, *59*, 514. [\[CrossRef\]](#)
37. Zaymidoroga, O.A.; Podgainy, D.V. Observation of Electrosalar Radiation during a Solar Eclipse. In *Cosmic Rays for Particle and Astroparticle Physics, Proceedings of the 12th ICATPP Conference, Como, Italy, 7–8 October 2010*; World Scientific: Singapore, 2011; p. 84.
38. Hively, L. Systems, Apparatuses, and Methods for Generating and/or Utilizing Scalar-Longitudinal Waves. U.S. Patent No. 9,306,527 B1. Issued 5 April 2016.
39. Jefimenko, O.D. Causality, electromagnetic potentials, and longitudinal waves. *Am. J. Phys.* **1992**, *60*, 839–844.
40. Kong, J.A. Electromagnetic wave theory of closed structures. *Prog. Electromagn. Res.* **1998**, *18*, 269–346.
41. Lindell, I.V.; Sihvola, A. Perfect electromagnetic conductor. *J. Electromagn. Waves Appl.* **2005**, *19*, 861–869. [\[CrossRef\]](#)

42. He, Q.; Zhao, X.; Mu, Q. Electromagnetic wave in a plasma-filled coaxial cable. *IEEE Trans. Plasma Sci.* **2006**, *34*, 1778–1781.
43. Shukla, P.K.; Eliasson, B. Electromagnetic waves in dusty plasmas. *Phys. Plasmas* **2007**, *14*, 054502.
44. Wu, H.; Li, Q.; Wu, J. Electromagnetic wave propagation in graphene. *J. Phys. Condens. Matter* **2012**, *24*, 164206.
45. Sakhno, D.; Koreshin, E.; Belov, P.A. Longitudinal electromagnetic waves with extremely short wavelength. *Phys. Rev. B* **2021**, *104*, L100304. [\[CrossRef\]](#)
46. Chang, C.C.; Lundgren, T.S. Flow of an Incompressible Fluid in a Hydromagnetic Capacitor. *Phys. Fluids* **1959**, *2*, 627–632. [\[CrossRef\]](#)
47. Zhang, K.; Wang, Y.; Tang, H.; Li, Y.; Wang, B.; York, T.M.; Yang, L. Two-dimensional analytical investigation into energy conversion and efficiency maximization of magnetohydrodynamic swirling flow actuators. *Energy* **2020**, *209*, 118479. [\[CrossRef\]](#)
48. Bos, W.J.T.; Neffaa, S.; Schneider, K. Rapid generation of angular momentum in bounded magnetized plasma. *Phys. Rev. Lett.* **2008**, *101*, 235003. [\[CrossRef\]](#) [\[PubMed\]](#)
49. Durrani, I.R. Photon orbital angular momentum in a plasma vortex. *Bull. Pure Appl. Sci. Phys.* **2012**, *31*, 103–111.
50. Fitzpatrick, R.; Yu, E. Angular momentum injection into a Penning–Malmberg trap. *Phys. Plasmas* **1997**, *4*, 917–930. [\[CrossRef\]](#)
51. Oreshko. Generation of Strong Fields in Plasma. *Doklady Physics* **2001**, *46*, 9–11. [\[CrossRef\]](#)
52. Pinheiro, M.J. CylindricalPlasmaThruster.ipynb. GitHub Repository. 2024. Available online: https://github.com/mjgpinheiro/Physics_models/blob/main/CylindricalPlasmaThruster.ipynb (accessed on 23 January 2024).
53. Sun, H.; Liu, B.; Feng, C. Short-wavelength radiation pulses with time-varying orbital angular momentum from tailored relativistic electron beams. *Opt. Lett.* **2023**, *48*, 3701–3704. [\[CrossRef\]](#)
54. Saitoh, K.; Yonezawa, T.; Nambu, H.; Tanimura, S.; Uchida, M. Orbital Angular Momentum Resolved Convergent-Beam Electron Diffraction by the Post-Selected Injection of Electron Beam. *Microscopy* **2022**, *71*, 374–379. [\[CrossRef\]](#)
55. Yan, J.-A.; Geloni, G. Self-seeded free-electron lasers with orbital angular momentum. *Adv. Photonics Nexus* **2022**, *2*, 036001. [\[CrossRef\]](#)
56. Pastor, I.; Alvarez-Estrada, R.F.; Roso, L.; Castejón, F. Fundamental Studies on Electron Dynamics in Exact Paraxial Beams with Angular Momentum. *Photonics* **2022**, *9*, 693. [\[CrossRef\]](#)
57. Tavabi, A.H.; Rosi, P.; Rotunno, E.; Roncaglia, A.; Belsito, L.; Frabboni, S.; Pozzi, G.; Gazzadi, G.C.; Lu, P.-H.; Nijland, R.; et al. Experimental Demonstration of an Electrostatic Orbital Angular Momentum Sorter for Electron Beams. *Phys. Rev. Lett.* **2021**, *126*, 094802. [\[CrossRef\]](#)
58. Liu, P.; Yan, J.; Afanasev, A.; Benson, S.V.; Hao, H.; Mikhailov, S.; Popov, V.; Wu, Y. Orbital angular momentum beam generation using a free-electron laser oscillator. *arXiv* **2020**, arXiv:2007.15723. [\[CrossRef\]](#)
59. Xu, P.; Zhang, C. Orbital angular momentum microwave generated by free electron beam. In *International Conference on Communications*; Springer: Berlin/Heidelberg, Germany, 2019. [\[CrossRef\]](#)
60. Nguyen, K.X.; Jiang, Y.; Cao, M.C.; Purohit, P.; Yadav, A.K.; García-Fernández, P.; Tate, M.W.; Chang, C.S.; Aguado-Puente, P.; Íñiguez, J.; et al. Transferring Orbital Angular Momentum to an Electron Beam Reveals Toroidal and Chiral Order. *arXiv* **2020**, arXiv:2012.04134. [\[CrossRef\]](#)
61. Eickhoff, K.; Rathje, C.; Köhnke, D.; Kerbstadt, S.; Englert, L.; Bayer, T.; Schäfer, S.; Wollenhaupt, M. Orbital angular momentum superposition states in transmission electron microscopy and bichromatic multiphoton ionization. *New J. Phys.* **2020**, *22*, 103045. [\[CrossRef\]](#)
62. Mafakheri, E.; Tavabi, A.H.; Lu, P.-H.; Balboni, R.; Venturi, F.; Menozzi, C.; Gazzadi, G.C.; Frabboni, S.; Sit, A.; Grillo, V.; et al. Realization of electron vortices with large orbital angular momentum using miniature holograms fabricated by electron beam lithography. *Appl. Phys. Lett.* **2017**, *110*, 093113. [\[CrossRef\]](#)
63. Grillo, V.; Gazzadi, G.C.; Mafakheri, E.; Balboni, R.; Venturi, F.; Menozzi, C.; Gazzadi, G.C.; Frabboni, S.; Sit, A.; Grillo, V.; et al. Realization of electron vortices with large orbital angular momentum using miniature holograms fabricated by electron beam lithography. *arXiv* **2016**, arXiv:1612.00654. [\[CrossRef\]](#)
64. Grillo, V.; Gazzadi, G.C.; Mafakheri, E.; Mafakheri, E.; Balboni, R.; Venturi, F.; Menozzi, C.; Gazzadi, G.C.; Frabboni, S.; Sit, A.; Grillo, V.; et al. Holographic Generation of Highly Twisted Electron Beams. *Phys. Rev. Lett.* **2015**, *114*, 034801. [\[CrossRef\]](#)
65. Shi, Z.; Wan, Z.; Zhan, Z.; Liu, Q.; Fu, X. Super-resolution orbital angular momentum holography. *Nat. Commun.* **2023**, *14*, 1869. [\[CrossRef\]](#)
66. Grillo, V.; Tavabi, A.H.; Venturi, F.; Larocque, H.; Balboni, R.; Gazzadi, G.C.; Frabboni, S.; Lu, P.H.; Mafakheri, E.; Bouchard, F.; et al. Measuring the orbital angular momentum spectrum of an electron beam. *Nat. Commun.* **2017**, *8*, 15536. [\[CrossRef\]](#) [\[PubMed\]](#)
67. Allen, L.; Beijersbergen, M.W.; Spreeuw, R.J.C.; Woerdman, J.P. Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes. *Phys. Rev. A* **1992**, *45*, 8185. [\[CrossRef\]](#)
68. Lichte, H.; Lehmann, M. Electron holography—Basics and applications. *Physics* **2008**, *71*, 016102. [\[CrossRef\]](#)
69. McMorran, B.J.; Agrawal, A.; Anderson, I.M.; Herzing, A.A.; Lezec, H.J.; McClelland, J.J.; Unguris, J. Electron Vortex Beams with High Quanta of Orbital Angular Momentum. *Science* **2011**, *331*, 192. [\[CrossRef\]](#)
70. Edmonds, A.R. *Angular Momentum in Quantum Mechanics*; Princeton University Press: Princeton, NJ, USA, 1974.
71. Uchida, M.; Tonomura, A. Generation of electron beams carrying orbital angular momentum. *Nature* **2010**, *464*, 737–739. [\[CrossRef\]](#)
72. Verbeeck, J.; Tian, H.; Schattschneider, P. Production and application of electron vortex beams. *Nature* **2010**, *467*, 301–304. [\[CrossRef\]](#)

73. Bliokh, K.Y.; Rodríguez-Fortuño, F.J.; Nori, F.; Zayats, A.V. Spin—Orbit interactions of light. *Nat. Photonics* **2015**, *9*, 796–808. [CrossRef]
74. Bliokh, K.Y.; Ivanov, I.P.; Guzzinati, G.; Clark, L.; Boxem, R.V.; Beche, A.; Juchtmans, R.; Alonso, M.A.; Schattschneider, P.; Nori, F.; et al. Theory and applications of free-electron vortex states. *Physics* **2017**, *690*, 1–70. [CrossRef]
75. Shangguan, J.-T.; Sun, Q.; Jin, L.; Deng, X.; Cheng, H.; Yu, L. The coupling of multi-channel optical vortices based on angular momentum conservation using a single-layer metal metasurface. *EPL Europhys. Lett.* **2023**, *141*, 35001. [CrossRef]
76. Paroli, B.; Siano, M.; Cremonesi, L.; Potenza, M.N. High data-transfer density using 4-states optical vortices for deep space optical communication links. In Proceedings of the International Conference on Optical Network Design and Modelling, Coimbra, Portugal, 8–11 May 2023.
77. Suci, S.; Bulzan, G. A.; Isdraila, T.-A.; Pălici, A.M.; Ataman, S.; Kusko, C.; Ionicioiu, R. Quantum communication networks with optical vortices. *Phys. Rev. A* **2023**, *108*, 052612. [CrossRef]
78. Luo, H.; Yang, K.; Li, P.-W.; Wen, F.; Gu, Y.; Wu, Z. Generation and verification of optical vortices with controlled phase based on coherent beam combining. *Phys. Scr.* **2022**, *97*, 105103. [CrossRef]
79. Xie, Z. Nondiffractive polarization feature of optical vortices. *Adv. Photonics* **2023**, *5*, 030530. [CrossRef]
80. Kovalev, A.A. Optical vortices with an infinite number of screw dislocations. *Comput. Opt.* **2021**, *45*, 497–505. [CrossRef]
81. Pryamikov, A.D.; Hadzievski, L.; Fedoruk, M.P.; Turitsyn, S.; Acevec, A. Optical vortices in waveguides with discrete and continuous rotational symmetry. *J. Eur. Opt. Soc. Rapid Publ.* **2021**, *17*, 1–28. [CrossRef]
82. Innes, T.; Elliott, O.; Scruggs, S. Optical Networking with Hybrid Optical Vortices. US Patent 10506312. Available online: <https://www.freepatentsonline.com/10506312.html> (accessed on 29 January 2024).
83. Zhao, L.; Jiang, T.; Mao, M.; Zhang, Y.; Wei, Z.; Deng, D.; Luo, A. Improve The Capacity Of Data Transmission In Orbital Angular Momentum Multiplexing By Adjusting Link Structure. *IEEE Photonics J.* **2020**, *12*, 1–11. [CrossRef]
84. Chau, K.K.W. Multiscale models of plasmonic structural colors with nanoscale surface roughness. *Opt. Lett.* **2023**, *48*, 1738–1741. [CrossRef]
85. Jia, H.; Tsoi, C.C.; El Abed, A.I.; Yu, W.; Jian, A.; Sang, S.; Zhang, X. Metallic Plasmonic Nanostructure Arrays for Enhanced Solar Photocatalysis. *Laser Photonics Rev.* **2023**, *2023*, 2200700. [CrossRef]
86. Singh, A.K.; Kumar, A.; Dixit, S.; Kumar, A. *Interaction of Light with Plasmonic Nanostructures Fabricated by Nanosphere Lithography*; Springer: Berlin/Heidelberg, Germany, 2021. [CrossRef]
87. Fisicaro, G.; Genovese, L.; Andreussi, O.; Marzari, N.; Goedecker, S. A generalized Poisson and Poisson-Boltzmann solver for electrostatic environments. *J. Chem. Phys.* **2016**, *144*, 014103. [CrossRef] [PubMed]
88. Chicone, C.; Mashhoon, B. Nonlocal gravity: Modified Poisson’s equation. *J. Math. Phys.* **2012**, *53*, 042501. [CrossRef]
89. John, D.; Anderson James, K.; Campbell John, E.; Ekelund, J.E.; Jordan, J.F. Anomalous orbital-energy changes observed during spacecraft flybys of Earth. *Phys. Rev. Lett.* **2008**, *100*, 091102.
90. Tsoi, M.; Jansen, A.G.M.; Bass, J.; Chiang, W.H. Spin current in magnetic nanostructures. *J. Magn. Magn. Mater.* **1998**, *200*, 479–490.
91. Trier, F.; Vaz, D.; Bruneel, P.; Noël, P.; Fert, A.; Vila, L.; Attanem J.P.; Barthelemy, A.; Gabay, M.; Jaffres, H.; et al. Electric-Field Control of Spin Current Generation and Detection in Ferromagnet-Free SrTiO₃-Based Nanodevices. *Nano Lett.* **2019**, *20*, 395–401. [CrossRef]
92. Pinheiro, M.J. Physics_models: OAM_EM.ipynb. GitHub repository. 2021. Available online: https://github.com/mjgpinheiro/Physics_models/blob/main/OAM_EM.ipynb (accessed on 14 April 2023).

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