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High-Precision Calculation Using the Method of Analytical Regularization for the Cut-Off Wave Numbers for Waveguides of Arbitrary Cross Sections with Inner Conductors

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Abstract: A method for the accurate calculation of the cut-off wavenumbers of a waveguide with an arbitrary cross section and a number of inner conductors is demonstrated. Concepts of integral and infinite-matrix (summation) operator-valued functions depending nonlinearly on the frequency spectral parameter provide a secure basis for formulating the spectral problem, and the Method of Analytical Regularization is employed to implement an effective algorithm. The algorithm is based on a mathematically rigorous solution of the homogeneous Dirichlet problem for the Helmholtz equation in the interior of the waveguide, excluding the regions occupied by the inner conductor boundaries. A highly efficient method of calculating the cut-off wavenumbers and the corresponding non-trivial solutions representing the modal distribution is developed. The mathematical correctness of the problem statement, the method, and the ability to calculate the cut-off wavenumbers with any prescribed and proven accuracy provide a secure basis for treating these as “benchmark solutions”. In this paper, we use this new approach to validate previously obtained results against our benchmark solutions. Furthermore, we demonstrate its universality in solving some new problems, which are barely accessible by existing methods.

Keywords: Method of Analytical Regularization; cut-off wavenumbers; waveguide with inner conductors

MSC: 78A45 (Scattering and Diffraction); 45F15 (Systems of Singular Linear Integral Equations)



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1. Introduction

The determination of the cut-off frequencies (or cut-off wave numbers) in waveguides with (or possibly without) inner conductors is closely related to the solution of a purely mathematical problem: the finding of non-trivial solutions of a homogeneous boundary value problem (BVP) for the Helmholtz equation. This problem, in turn, can be reduced to a homogeneous boundary integral equation (IE); the latter is treated following [1,2] as a problem of characteristic numbers (CNs) for an integral operator-valued function (OVF) of the (complex) frequency spectral parameter defined in terms of the boundary IE and for the equivalent infinite-matrix (summation) OVF.

The case of TM modes corresponds to the BVP with Dirichlet boundary conditions. Thus, a successful comprehensive analysis of cut-off frequencies directly depends on highly effective solution techniques for the corresponding BVP. Our notion of effectiveness means the fulfillment of three requirements.

The first is an absence of limitations on problem parameters in finding the cut-off wave numbers with any predetermined accuracy. We wish to avoid those methods that

justify restricted accuracy by arguing “they are good enough for practice”; in other words, it is highly desirable for the method to have the means of providing an internal estimate of its own accuracy. The second requirement concerns the “resolving power” of the method: it should not miss any modes of the spectrum, even when their cut-off frequencies lie extremely close to each other on the frequency axis. The third requirement is the ability to generate an algorithm which unerringly recognizes and correctly labels the modes corresponding to their cut-off wave numbers. This last requirement requires some explanation. In general, it is impossible to label the eigenmodes in a waveguide with inner conductors in the manner that is characteristic for empty waveguides of canonical shape; the presence of the inner conductors may produce a severe distortion of the regular electromagnetic field (EMF) oscillations of the empty waveguide eigenmodes observed in the mutually perpendicular directions of the transverse plane. But it is desirable that the perturbed oscillations and their link with the non-perturbed oscillations in the empty waveguide should be traceable.

All these requirements can be fully realized if a correct mathematical treatment is performed of the boundary IE as a problem on CNs for an integral OVF of the (complex) spectral parameter and for equivalent infinite-matrix (summation) OVF constructed using the Method of Analytical Regularization (MAR). Although the MAR has been extensively used to investigate diffraction problems for variously shaped metallic screens, its implementation for the analysis of associated spectral problems has been minimal to date. Of course, this problem has been investigated by various methods already existing in computational electromagnetics (CEM). The application of the MAR approach in this paper is justified by its new capability to study the spectrum of cut-off wave numbers k_c in waveguides with inner conductors. A complete list of CEM methods and a comparative analysis of their effectiveness is a matter for special reviews. Nevertheless, we briefly mention some which were applied for studies of the spectrum of the cut-off wave numbers in waveguides containing inner conductors.

Extensive examination has been carried out for the class of canonically shaped waveguides and inner conductors with circular, elliptical and rectangular cross sections. For this class, the apparatus of the classical method of separation of variables for the solution of the Helmholtz equation, together with the use of the addition theorems (see, for example, [3–6]), is a natural, although not unique, way of analyzing the problem. In [7], the generalized spectral-domain (GSD) technique is used to analyze rectangular waveguides with metal inserts of rectangular and circular cross section. In [8], the authors described a method which appears to be suitable to estimate the frequency parameters of any arbitrarily shaped waveguide, provided that the transformation function needed to map the section into a circular region can be obtained. The application of the finite-difference method in the time domain for obtaining the cut-off frequencies of TE and TM modes in waveguides of arbitrary cross section is described in [9]. Polygonal metal waveguides are analyzed analytically and numerically in [10], where the authors used finite difference time domain models of microwave waveguide junctions. Conformal transformation is one of the elegant mathematical tools widely applied in the analysis of waveguides of circular [11–13] and rectangular cross section [14,15].

It is pertinent to make the two following observations. There is no formal distinction between the calculated cut-off wave numbers for propagating modes in waveguides and the eigenvalues calculated for standing waves in two-dimensional cavity resonators. Also, despite the differing physical nature of sound and electromagnetic waves, the problems in acoustics for waveguides with sound-soft walls are mathematically equivalent to those concerning transverse magnetic (TM) polarized waves in waveguides of the same cross section with ideally conducting walls. From this point of view, the results obtained in acoustics (see, for example [16–20]) are directly applicable to electromagnetic studies in waveguides of the same shape, a fact that seems somewhat under-exploited by both acoustical and radio-engineering communities.

Looking at the refereed papers in the References, one observes that, in most of them, the calculation of cut-off wave numbers has been performed with accuracy not exceeding 3 to 4 significant decimal digits. More importantly, there is no proof of the correctness of the calculated cut-off wavenumbers; the only validation is via comparison with previously published results obtained by alternative methods. This seems to be largely true of the scientific literature in the field: the paper [21] (employing a Method of Moments approach) is perhaps typical. From this perspective, the paper [5] is distinctive because of its scrupulous approach to the estimation of computational accuracy.

The three requirements outlined above are important in practical applications. An example occurs in the publication [22] in which the design of a class of ring-resonator-loaded waveguide notch filters depends upon an accurate determination of the cut-off wavenumbers in the waveguide. Recognizing the practical significance of the calculations, appearing in numerous publications, of cut-off wavenumbers for waveguides with variously shaped outer and inner conductors, there is a need to develop a universal and accurate instrument for providing the ultimate estimate of the correctness of such calculations. This means obtaining a benchmark solution which may be used as an external and independent estimate for other methods. Of more consequence is the goal of calculating the cut-off frequencies of complex waveguide structures with guaranteed accuracy; as a by-product, estimates of the accuracy of calculation by other methods immediately become available.

In this paper, we developed an approach to this category of problems. The new solution employs the basic strategy of the rigorous Method of Analytical Regularization (MAR). It was originally developed for rigorous solution of acoustic and electromagnetic wave diffraction problems for open structures of canonical shape with sharp edges (see, for example [23]). Although the solid mathematical basis for generalizing the MAR to diffraction problems concerning arbitrary open two-dimensional structures (2D) was constructed nearly 30 years ago (see, in particular [24,25]), the practical realization of the appropriate algorithms started only recently. From the list of recent applications of the generalized MAR we extract two papers which are of direct relevance to the present paper: the first [26], which demonstrates the fruitfulness of this approach to impedance calculations for transmission lines with an adjustable inner conductor; and the second [27] which contains the detailed deduction of the solution to the problem of electromagnetic plane wave scattering by arbitrary two-dimensional cavities. This paper [27] and the more recently published work [28], which addresses finding complex eigenvalues of slotted cylindrical cavities of arbitrary shape, contain a description of the necessary mathematical tools, numerical schemes, and related material which we use in the present paper. In fact, we obtain the solution to the more general problem when some or all of the conductors are slotted cylinders of the arbitrary cross section. They may be placed at any arbitrary point of two-dimensional space; the configuration is defined by the relative location of the conductors. In particular, the elements of this general solution may be found in [29].

The process of obtaining the general solution is labour-intensive and lengthy, so we omit the full details of its deduction and draw on previous publications [27–29] for an abbreviated outline of the solution algorithm in order to focus on numerical results and their discussion. Keeping that in mind, in the overwhelming majority of practical cases, the bounding contours of the waveguide (outer conductor) and the inner conductors are described by closed and smooth contours, it is more instructive to describe the solution for this case first. The general solution for finding the complex eigenvalues for slotted waveguides with inner metallic inserts will be presented in a subsequent paper. It should be pointed out that the solution obtained and the proof of its mathematical correctness are comprehensively discussed in previous publications related to slotted cylinders of arbitrary cross section, including references [24,25]. The present paper deals with the less complex problem of closed contours. The mathematical background of this problem does not require special mathematical skills and its solution will be readily accessible to most of the electro-engineering community.

2. Theoretical Background

The core problem is the solution for a hollow waveguide of arbitrary cross section, which we tackle with the MAR. When this problem is solved, the insertion of metallic conductors in the waveguide, although presenting a new BVP, does not complicate the mathematical issue of the approach, the core of which is still defined by the MAR-based solution for a hollow waveguide. The addition of terms describing the mutual electromagnetic coupling between the conductors introduces smooth perturbations that are readily accommodated by the solution method for the hollow waveguide problem. In this section, we first describe the scheme for obtaining the solution for a hollow waveguide of the arbitrary cross section. Then, we obtain the solution when the waveguide is filled by a number of conductors, each of arbitrary cross section.

2.1. Hollow Waveguide of Arbitrary Cross Section

Consider a uniform waveguide which has perfectly electrically conducting (PEC) walls, has an axis aligned with the z -axis, and has a cross-sectional contour L_0 in the x - y plane which is smooth and of arbitrary shape. The cut-off wave numbers k_c of the modes are determined to be those wavenumbers k where non-trivial solutions of the homogeneous Dirichlet BVP for the Helmholtz equation

$$\Delta U(q) + k^2 U(q) = 0$$

occur in the interior domain D with boundary L_0 satisfying the boundary condition

$$U(q)|_{q \in L_0} = 0.$$

Here, $U = E_z$ denotes the z -component of the electric field in the guide (see Figure 1) corresponding to the “cut-off” values of generally complex wavenumbers k , which are eigenvalues of the formulated Dirichlet BVP. (Note that the spectrum of the Dirichlet boundary eigenvalue problem considered with respect to the spectral parameter k in a domain bounded by a closed piecewise smooth contour is a set of isolated positive eigenvalues accumulating at infinity. However, the development of efficient techniques for calculating the spectrum, which is actually an objective of the present study, remains an urgent task.) We seek the solution in the form of a single-layer potential

$$U(q) = \int_{L_0} G_2(k|p_0 - q|) Z_0(p_0) dl_{p_0} \quad (1)$$

where the unknown density function $Z_0(p_0)$ at each p_0 on L_0 is directly associated with the normal component of the electric field

$$Z_0(p_0) \sim \frac{\partial E_z}{\partial n_{p_0}}. \quad (2)$$

Here, the free-space Green’s function G_2 is related to the Hankel function of the first kind and order 0,

$$G_2(k|p_0 - q|) = -\frac{i}{4} H_0^{(1)}(k|p_0 - q|), \quad (3)$$

depending on the distance $|p_0 - q|$ between observation point q and point p_0 on the contour L_0 . The choice of a first-kind Hankel function is consistent with an assumed time-harmonic variation of the electromagnetic field of the form $\exp(-i\omega t)$ where $\omega = kc$, c denoting the speed of light. This time factor is suppressed throughout.

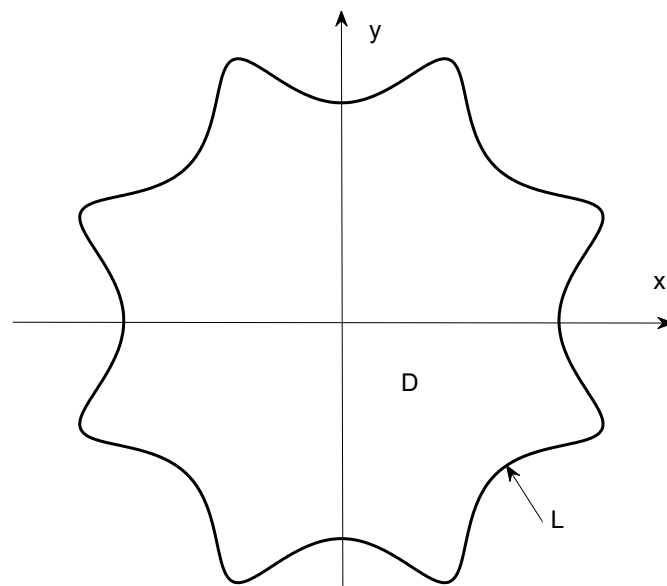


Figure 1. Hollow waveguide of arbitrary cross section.

Enforcement of the boundary condition on the contour L_0 ,

$$U(q) = 0, q \in L_0, \quad (4)$$

leads to the homogeneous surface integral equation

$$\int_{L_0} G_2(k|p_0 - q|) Z_0(p_0) dl_{p_0} = 0, q \in L_0. \quad (5)$$

Now parameterize the points $q(\theta_0) = (x_0(\theta_0), y_0(\theta_0))$ on the contour L_0 by smooth periodic functions x_0, y_0 so that $x_0(-\pi) = x_0(\pi), y_0(-\pi) = y_0(\pi)$. With this parameterization, the differential of arc length is

$$l_0(\tau_0) = \sqrt{(x'_0(\tau_0))^2 + (y'_0(\tau_0))^2}, \quad (6)$$

and we may reformulate the electric surface integral Equation (5) in the form

$$G(k)z_0 = 0, \quad (7)$$

where the integral operator-valued function (OVF) $G(k)$ of the (in general, complex) spectral parameter k is defined by

$$G(k)z_0(\tau_0) \equiv -\frac{i}{4} \int_{-\pi}^{\pi} H_0^{(1)}(kR_{00}(\theta_0, \tau_0))z_0(\tau_0)d\tau_0,$$

with the kernel $H_0^{(1)}(kR_{00}(\theta_0, \tau_0))$, the function $z_0(\tau_0) = l_0(\tau_0)Z_0(\tau_0)$ and a distance function

$$R_{00}(\theta_0, \tau_0) = \sqrt{(x_0(\theta_0) - x_0(\tau_0))^2 + (y_0(\theta_0) - y_0(\tau_0))^2}.$$

The homogeneous Equation (7) is the problem of the characteristic numbers (CNs) for the OVF $G(k)$. Although the CNs for the *closed* structures studied in this paper are real, the CNs for *open* structures have a non-zero imaginary part.

Let us now briefly summarise the relevant results of [1] concerning the spectral theory of OVFs. To this end, introduce the following integral operator $\mathcal{L} : X \rightarrow Y$, with a logarithmic singularity of the kernel and integration over a smooth closed contour L_0 ,

$$\mathcal{L}\phi = \mathcal{L}(L_0)\phi = \int_{L_0} \frac{2}{\pi} \ln(|p_0 - q_0|) \phi(p_0) dl_{p_0}, \quad p_0, q_0 \in L_0,$$

where X denotes the Banach space $X = X(L_0) = H_\beta(L_0)$ of Hölder-continuous functions of index $\beta \in (0, 1)$ and $Y = Y(L_0) = H_\beta^1(L_0)$ the space of continuously differentiable functions with Hölder-continuous derivatives of index β on L_0 .

Using the smooth parametrization of closed smooth curve L_0 introduced above, one can separate the logarithmic singularity of the integral operator $\mathcal{L}(L_0)$ in the form

$$\mathcal{L}(L_0)\phi = S\phi + S_1(L_0)\phi = \frac{2}{\pi} \int_{-\pi}^{\pi} \ln\left(\left|\sin \frac{\theta_0 - \tau_0}{2}\right|\right) \phi(\tau_0) d\tau_0 + \int_{-\pi}^{\pi} S_1(\theta_0, \tau_0) \phi(\tau_0) d\tau_0$$

obtained with the help of polar coordinates, or alternatively as

$$\mathcal{L}\phi = \mathcal{L}_0\phi + \mathcal{L}_1(L_0)\phi = \frac{2}{\pi} \int_{-\pi}^{\pi} \ln|\theta_0 - \tau_0| \phi(\tau_0) d\tau_0 + \int_{-\pi}^{\pi} L_1(\theta_0, \tau_0) \phi(\tau_0) d\tau_0.$$

Here, $S_1(\theta_0, \tau_0)$ is a continuously differentiable function of its arguments and integral operators $\mathcal{L}_0$ and S are independent of L_0 and the spectral parameter k .

The following theorem is the main theoretical result, which provides a secure basis for the algorithm developed by the MAR.

Theorem 1. *The operator $S : X \rightarrow Y$ is continuously invertible with continuous inverse $S^{-1} : X \rightarrow Y$. The norm S^{-1} is a constant independent of L_0 and the spectral parameter k ; in particular, for the map $S^{-1} : W_2^1 \rightarrow L_2$ (where L_2 and W_2^1 are standard notations for the spaces of square-integrable functions and functions with the square-integrable derivative), the norm S^{-1} equals 2. If contour L_0 is such that the homogeneous equation $\mathcal{L}\phi = 0$, $\phi \in X$, has only the trivial solution then $\mathcal{L} : X \rightarrow Y$ is continuously invertible and the continuous inverse $\mathcal{L}^{-1} : Y \rightarrow X$ is constructed explicitly in [30].*

The key idea of the MAR is the splitting of the kernel of integral Equation (7) into a singular part and a regular (“smooth”) part $H_{00}(k; \theta_0, \tau_0)$ via

$$H_0^{(1)}(kR_{00}(\theta_0, \tau_0)) = \frac{2i}{\pi} \ln\left(2\left|\sin \frac{\theta_0 - \tau_0}{2}\right|\right) + H_{00}(k; \theta_0, \tau_0). \quad (8)$$

It will be supposed that contour L_0 is such that $H_{00}(k; \theta_0, \tau_0)$ is smooth and continuously differentiable with respect to θ_0 and τ_0 . Then, the OVF $G(k)$ can be represented in the form

$$G(k) = S + H_{00}(k)$$

and classified therefore as a Fredholm integral OVF with a logarithmically singular kernel. The homogeneous Equation (7) can be reduced to the equivalent form

$$\hat{G}(k)z_0 = 0, \quad \hat{G}(k) = I + \hat{H}_{00}(k), \quad \hat{H}_{00}(k) = S^{-1}H_{00}(k),$$

involving a canonical Fredholm operator of the second kind. Here, $\hat{H}_{00}(k) : Y \rightarrow Y$ is a completely-continuous integral OVF [30], which admits analytical continuation to the Riemann surface $\mathcal{R}$ of the logarithmic function $\ln(k)$ with a branch point in the origin. One can show, following [30], that the resolvent set of $\hat{G}(k)$ is not empty, which implies that $\hat{G}(k)$ is a Fredholm OVF and the (spectral) set $\Sigma_{\hat{G}}$ of its characteristic numbers (CNs) is a countable set of isolated points in $\mathcal{R}$.

The properties of the kernel of the IE (7) allow the expansion of $H_{00}(k; \theta_0, \tau_0)$ in a double Fourier series,

$$H_{00}(k; \theta_0, \tau_0) = -\frac{i}{\pi} \sum_{p=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h_{np}^{00}(k) e^{i(n\theta_0 + p\tau_0)}, \quad \theta_0, \tau_0 \in [-\pi, \pi], \quad (9)$$

where $h^{00}(k) = \{h_{np}^{00}(k)\}_{n,p=-\infty}^{\infty}$ may be considered as an infinite matrix with the entries depending on the spectral parameter k (considered on $\mathcal{R}$) and satisfying the condition

$$\sum_{p=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (1+p^2)(1+n^2) |h_{np}^{00}(k)|^2 < \infty. \quad (10)$$

(The factor $-\frac{i}{\pi}$ has been chosen to simplify further transforms.)

Finally, we represent the singular part of the kernel (8) and unknown density function $z_0(\tau_0)$ in the Fourier series

$$\ln \left(2 \left| \sin \frac{\theta_0 - \tau_0}{2} \right| \right) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{|n|} e^{in(\theta_0 - \tau_0)}, \quad \theta_0, \tau_0 \in [-\pi, \pi] \quad (11)$$

and

$$z_0(\tau_0) = \sum_{l=-\infty}^{\infty} \xi_l^{(0)} e^{il\tau_0} \quad (12)$$

where the prime over the summation sign in (11) excludes the term with $n = 0$. We assume that the infinite vector of Fourier coefficients $\xi^0 = \{\xi_l^{(0)}\}_{l=-\infty}^{\infty}$ to be found lies in $l_2 = \left\{ \xi_l^{(0)} : \sum_{l=-\infty}^{\infty} |\xi_l^{(0)}|^2 < \infty \right\}$.

After the replacement of all functions in (7) by their Fourier expansions and performing the integrations, employing the completeness and orthogonality of the exponential functions $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ on the interval $[-\pi, \pi]$ expressed by the identity

$$\int_{-\pi}^{\pi} e^{i(l-n)\tau_0} d\tau_0 = 2\pi \delta_{ln} = 2\pi \begin{cases} 1, & l = n \\ 0, & l \neq n \end{cases} \quad (13)$$

we arrive at the summation-type equations

$$\sum_{n=-\infty}^{\infty} \frac{\xi_n^{(0)}}{|n|} e^{in\theta_0} + \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} h_{np}^{00}(k) \xi_{-p}^{(0)} e^{in\theta_0} = 0, \quad \theta_0, \tau_0 \in [-\pi, \pi]. \quad (14)$$

This is an infinite system of linear algebraic equations (ISLAE) to be solved for the vector $\xi^0 \in l_2$.

Multiplying (14) by $e^{im\theta_0}$ (for a fixed integer m) and using (13) produces the ISLAEs

$$h_{00}^{00}(k) \xi_0^{(0)} + \sum_{p=-\infty}^{\infty} h_{0p}^{00}(k) \xi_{-p}^{(0)} = 0, \quad m = 0 \quad (15)$$

and

$$h_{m0}^{00}(k) \xi_0^{(0)} + \frac{\xi_m^{(0)}}{|m|} + \sum_{p=-\infty}^{\infty} h_{mp}^{00}(k) \xi_{-p}^{(0)} = 0, \quad m \neq 0. \quad (16)$$

At this stage of the derivation of an equivalent form of the infinite-matrix (summation) equation to be solved, we wish to stress that the analytical semi-inversion is achieved by using only the (inverse) exponential Fourier transform in the process of inversion of the singular part of the initial boundary IE. Note that Equations (15) and (16) are not in the final regularised form (i.e., in the form of the second-kind ISLAE) because the coefficients of the terms $\xi_m^{(0)}$ tend to zero as $|m| \rightarrow \infty$.

Further transform is effected by introducing new variables x_p, y_p and rescaled variables X_p, Y_p defined for positive integers p by

$$x_p = \frac{1}{2}(\xi_p^{(0)} + \xi_{-p}^{(0)}), y_p = \frac{1}{2}(\xi_p^{(0)} - \xi_{-p}^{(0)}); X_p = \frac{x_p}{\sqrt{p}}, Y_p = \frac{y_p}{\sqrt{p}}. \quad (17)$$

Note that the introduction of new variables corresponds to re-writing summation Equation (14) in the form with a trigonometric kernel by making the replacement $e^{in\theta_0} = \cos n\theta_0 + i \sin n\theta_0$. Further algebraic transformations (entirely based on readily repeatable simple algebra and omitted for the sake of brevity) produce two coupled ISLAEs of the second kind with respect to the two infinite sequences of unknown Fourier coefficients $\{X_m\}_{m=1}^{\infty}$ and $\{Y_m\}_{m=1}^{\infty}$:

$$\frac{1}{2}K_m^{(+,+)}\xi_0^{(0)} + X_m + \sum_{p=1}^{\infty} (K_{m,p}^{(+,+)}X_p - K_{m,p}^{(+,-)}Y_p) = 0, \quad m = 1, 2, 3, \dots \quad (18)$$

$$\frac{1}{2}K_m^{(-,+)}\xi_0^{(0)} + Y_m + \sum_{p=1}^{\infty} (K_{m,p}^{(-,+)}X_p - K_{m,p}^{(-,-)}Y_p) = 0, \quad m = 1, 2, 3, \dots \quad (19)$$

The equation corresponding to index $m = 0$ is

$$\frac{1}{2}\kappa_{0,0}^{(+,+)}\xi_0^{(0)} + \sum_{p=1}^{\infty} (K_p^{(+,+)}X_p - K_p^{(+,-)}Y_p) = 0 \quad (20)$$

Here, we used the notations

$$\begin{aligned} K_{m,p}^{(+,+)} &= \sqrt{mp} \times \kappa_{m,p}^{(+,+)}, \quad K_m^{(+,+)} = \sqrt{m} \times \kappa_{m,0}^{(+,+)}, \\ \text{where } \kappa_{m,p}^{(+,+)} &= \frac{1}{2} \left[(h_{m,p}^{00} + h_{m,-p}^{00}) + (h_{-m,p}^{00} + h_{-m,-p}^{00}) \right]; \\ K_{m,p}^{(+,-)} &= \sqrt{mp} \times \kappa_{m,p}^{(+,-)}, \quad K_m^{(+,-)} = \sqrt{m} \times \kappa_{m,0}^{(+,-)}, \\ \text{where } \kappa_{m,p}^{(+,-)} &= \frac{1}{2} \left[(h_{m,p}^{00} - h_{m,-p}^{00}) + (h_{-m,p}^{00} - h_{-m,-p}^{00}) \right]; \\ K_{m,p}^{(-,+)} &= \sqrt{mp} \times \kappa_{m,p}^{(-,+)}, \quad K_p^{(+,+)} = \sqrt{p} \times \kappa_{0,p}^{(+,+)}, \\ \text{where } \kappa_{m,p}^{(-,+)} &= \frac{1}{2} \left[(h_{m,p}^{00} + h_{m,-p}^{00}) - (h_{-m,p}^{00} + h_{-m,-p}^{00}) \right]; \\ K_{m,p}^{(-,-)} &= \sqrt{mp} \times \kappa_{m,p}^{(-,-)}, \quad K_p^{(+,-)} = \sqrt{p} \times \kappa_{0,p}^{(+,-)}, \\ \text{where } \kappa_{m,p}^{(-,-)} &= \frac{1}{2} \left[(h_{m,p}^{00} - h_{m,-p}^{00}) - (h_{-m,p}^{00} - h_{-m,-p}^{00}) \right], \end{aligned} \quad (21)$$

and where the quantities $K_{m,p}^{(\pm,\pm)} = K_{m,p}^{(\pm,\pm)}(k)$ depend on the spectral parameter $k \in \mathcal{R}$.

Equations (18) and (19) can be made more compact by using (20) to eliminate the coefficient $\xi_0^{(0)}$ via the rearrangement

$$\xi_0^{(0)} = -\frac{2}{\kappa_{0,0}^{(+,+)}} \sum_{p=1}^{\infty} (K_p^{(+,+)}X_p - K_p^{(+,-)}Y_p) = 0. \quad (22)$$

After substituting (22) in (18) and (19), we arrive at the final form of the equations to be solved, being a coupled ISLAE where the index m runs over $m = 1, 2, 3, \dots$:

$$X_m + \sum_{p=1}^{\infty} \left(K_{m,p}^{(+,+)} - \frac{K_m^{(+,+)}}{\kappa_{0,0}^{(+,+)}} K_p^{(+,+)} \right) X_p - \sum_{p=1}^{\infty} \left(K_{m,p}^{(+,-)} - \frac{K_m^{(+,-)}}{\kappa_{0,0}^{(+,+)}} K_p^{(+,-)} \right) Y_p = 0, \quad (23)$$

$$Y_m - \sum_{p=1}^{\infty} \left(K_{m,p}^{(-,-)} - \frac{K_m^{(-,-)}}{\kappa_{0,0}^{(+,+)}} K_p^{(+,-)} \right) Y_p + \sum_{p=1}^{\infty} \left(K_{m,p}^{(-,+)} - \frac{K_m^{(-,+)}}{\kappa_{0,0}^{(+,+)}} K_p^{(+,+)} \right) X_p = 0. \quad (24)$$

This is a Fredholm matrix equation of second kind. More precisely, one can show using the definitions and results of [1,2] that the left-hand sides of ISLAE (23), (24) define an

infinite-matrix (summation) OVF $H(k) : l_2^2 \rightarrow l_2^2$, $k \in \mathcal{R}$, where l_2^2 is a Cartesian product of two copies of spaces l_2 , and Equations (23) and (24) form a Fredholm infinite-matrix (summation) equation system of the second kind equivalent to the initial homogeneous boundary IE (7). This system, written as $H(k) = 0$, constitutes, in turn, the problem on CNs for OVF $H(k)$. Since $H(k)$ is a Fredholm OVF considered in the Hilbert space of sequences l_2^2 and its resolvent set is nonempty, its “spectral” set $\sigma_H \subset \mathcal{R}$ of the sought CNs is discrete, i.e., it consists of isolated points on the complex plane with the only accumulation point at infinity. Based on the equivalence, one can verify that every $k \in \sigma_H$ is a CN of the initial integral OVF $G(k)$ defined by the initial IE (7) and is a sought eigenvalue of the initial Dirichlet BVP.

2.2. Waveguides of Arbitrary Cross Section Filled by M Metallic Conductors

In this section, we consider the more complex case, when a positive number M of metallic conductors is placed inside the waveguide. As before, the PEC waveguide is the outer boundary that is the closed smooth contour L_0 of arbitrary shape. It encloses M inner conductors, which also have PEC boundaries that are arbitrarily variably shaped smooth contours $L_1, L_2, \dots, L_M$ (see Figure 2); the contours are mutually non-intersecting. Using the superposition principle and classical potential theory, we seek the solution for the z -component of the electric field $E_z(q)$ in the domain D as a sum of individual contributions $E_z^{(m)}(q)$, $m = 0, 1, 2, \dots, M$ from each conductor (including the outer conductor $m = 0$) in the form of single-layer potentials

$$U(q) = \sum_{m=0}^M E_z^{(m)}(q) = \sum_{m=0}^M \int_{L_m} G_2(k|p_m - q|) Z_m(p_m) dl_{p_m} \quad (25)$$

where $Z_m(p_m)$, $m = 0, 1, 2, \dots, M$, are unknown density functions, and the argument of the Green’s function G_2 depends on the distance $|p_m - q|$ between observation point q and points p_m on the contour L_m :

$$G_2(k|p_m - q|) = -\frac{i}{4} H_0^{(1)}(k|p_m - q|). \quad (26)$$

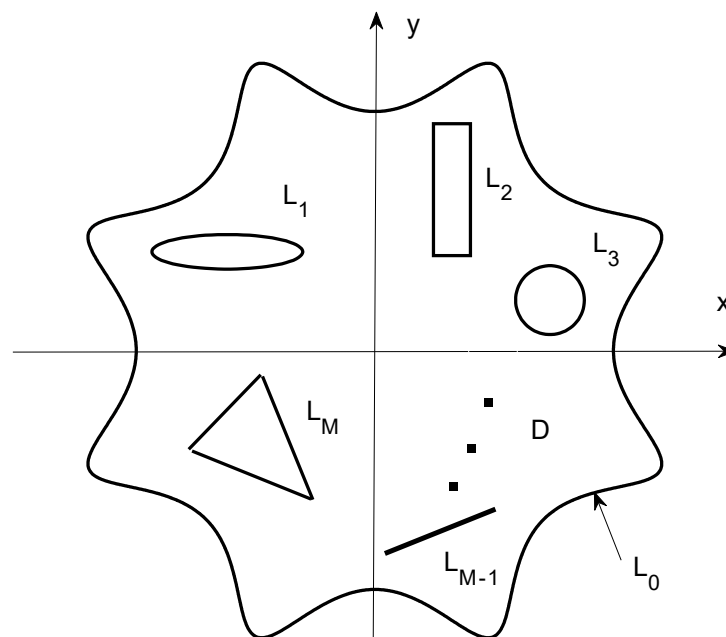


Figure 2. Waveguide of arbitrary cross section with M metallic inserts.

We now enforce the Dirichlet boundary condition on each of the $M + 1$ contours:

$$U(q)|_{q \in L_s} = 0, \quad s = 0, 1, 2, \dots, M. \quad (27)$$

This produces $M + 1$ coupled boundary integral equations

$$\sum_{m=0}^M \int_{L_m} G_2(k|p_m - q_s|) Z_m(p_m) dl_{p_m} = 0, \quad q_s \in L_s \quad (28)$$

where the integral equations are indexed by $s = 0, 1, 2, \dots, M$. Observe that exactly one term in each of the set (28) of integral Equation has a (logarithmically) singular kernel and that the kernels appearing in the remaining terms are non-singular and smooth. Thus, for further consideration, it is convenient to rearrange each of these integral equations in the following form:

$$\int_{L_s} G_2(k|p_s - q_s|) Z_s(p_s) dl_s + \sum_{m=0, m \neq s}^M \int_{L_m} G_2(k|p_m - q_s|) Z_m(p_m) dl_{p_m} = 0, \quad q_s \in L_s \quad (29)$$

If we set $s = 0$ in (29), the sum

$$\sum_{m=1}^M \int_{L_m} G_2(k|p_m - q_0|) Z_m(p_m) dl_{p_m}$$

(in which $q_0 \in L_0$, thus containing terms with non-singular kernels) may be interpreted as a smooth perturbation of Equation (5) describing the hollow waveguide.

Noting that all the contours are mutually non-intersecting, we may treat all the integral equations in (29) in the same manner without particularly distinguishing that corresponding to the outer waveguide. We employ parameterization in a similar manner to that of Section 2.1. On each contour L_m the points $q(\theta_m) = (x_m(\theta_m), y_m(\theta_m))$ are represented by smooth periodic functions x_m, y_m so that $x_m(-\pi) = x_m(\pi)$ and $y_m(-\pi) = y_m(\pi)$, with a corresponding differential of arc length

$$l_m(\tau_m) = \sqrt{(x'_m(\tau_m))^2 + (y'_m(\tau_m))^2}.$$

The coupled system of integral Equation (29) then takes the form

$$\int_{-\pi}^{\pi} H_0^{(1)}(kR_{ss}(\theta_s, \tau_s)) z_s(\tau_s) d\tau_s + \sum_{m=0, m \neq s}^M \int_{-\pi}^{\pi} H_0^{(1)}(kR_{ms}(\theta_s, \tau_m)) z_m(\tau_m) d\tau_m = 0, \quad (30)$$

where $s = 0, 1, 2, \dots, M$; here

$$R_{ss}(\theta_s, \tau_s) = \sqrt{(x_s(\theta_s) - x_s(\tau_s))^2 + (y_s(\theta_s) - y_s(\tau_s))^2}$$

is the distance between two points on the contour L_s parametrized by θ_s and τ_s ; when $m \neq s$, and

$$R_{ms}(\theta_m, \tau_s) = \sqrt{(x_m(\theta_m) - x_s(\tau_s))^2 + (y_m(\theta_m) - y_s(\tau_s))^2}$$

is the distance between two points lying on the non-intersecting contours L_m and L_s —this is always strictly positive, being at least the positive minimum distance between these conductors. In addition, $z_m(\tau_m) = l_m(\tau_m) Z(\tau_m)$ for $m = 0, 1, 2, \dots, M$.

The transformation of the system (30) of first-kind boundary integral equations to a well-conditioned coupled ISLAE of the second kind is performed in a similar manner to that for the isolated waveguide but modified to take account of the additional non-singular coupling terms. Thus, we separate the singular parts from the kernels of the integral equations and then expand their regular parts in their respective Fourier series.

First, the extraction of the singular part is expressed by the decomposition, for each index $s = 0, 1, 2, \dots, M + 1$,

$$H_0^{(1)}(kR_{ss}(\theta_s, \tau_s)) = \frac{2i}{\pi} \ln \left(2 \left| \sin \frac{\theta_s - \tau_s}{2} \right| \right) + H_{ss}(k; \theta_s, \tau_s). \quad (31)$$

As before, the logarithmically singular part has Fourier expansion (see (11))

$$\ln \left(2 \left| \sin \frac{\theta_s - \tau_s}{2} \right| \right) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{|n|} e^{in(\theta_s - \tau_s)}, \quad \theta_s, \tau_s \in [-\pi, \pi]. \quad (32)$$

It will be supposed that each contour L_s is such that $H_{ss}(k; \theta_s, \tau_s)$ is smooth and continuously differentiable with respect to θ_s and τ_s ; this allows its expansion in a double Fourier series.

$$H_{ss}(k; \theta_s, \tau_s) = -\frac{i}{\pi} \sum_{p=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h_{np}^{ss}(k) e^{i(n\theta_s + p\tau_s)}, \quad \theta_s, \tau_s \in [-\pi, \pi] \quad (33)$$

where $\sum_{p=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (1 + p^2)(1 + n^2) |h_{np}^{ss}(k)|^2 < \infty$. The smoothness of the contours and the regularity of the functions $H_0^{(1)}(kR_{ms}(\theta_s, \tau_m))$, for each pair of indices (m, s) with $m \neq s$, allows their expansion in double Fourier series

$$H_0^{(1)}(kR_{ms}(\theta_s, \tau_m)) = -\frac{i}{\pi} \sum_{p=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q_{np}^{ms} e^{i(n\theta_s + p\tau_m)}, \quad \theta_s, \tau_m \in [-\pi, \pi] \quad (34)$$

Finally, the unknown surface current density on the m -th conductor may be expanded in a Fourier series:

$$z_m(\tau_m) = \sum_{l=-\infty}^{\infty} \tilde{\zeta}_l^{(m)} e^{il\tau_m} \quad (35)$$

Performing the same steps in the solution obtained in Section 2.1, we arrive at the following coupled system of the linear algebraic equations representing a Fredholm matrix system of the second kind, where the indices s and l run over $0, 1, 2, \dots, M$ and $1, 2, \dots, \infty$, respectively:

$$\begin{aligned} X_l^{(s)} + \sqrt{l} \left(\kappa_{l,0}^{(s,s)} \right)^{(+,+)} \tilde{\zeta}_0^{(s)} + \sqrt{l} \sum_{m=0, m \neq s}^M \left(\chi_{l,0}^{(m,s)} \right)^{(+,+)} \tilde{\zeta}_0^{(m)} \\ + \sum_{p=1}^{\infty} \sqrt{lp} \left\{ \left(\kappa_{l,p}^{(s,s)} \right)^{(+,+)} X_p^{(s)} - \left(\kappa_{l,p}^{(s,s)} \right)^{(+,-)} Y_p^{(s)} \right\} \\ + \sum_{m=0, m \neq s}^M \sum_{p=1}^{\infty} \sqrt{lp} \left\{ \left(\chi_{l,p}^{(m,s)} \right)^{(+,+)} X_p^{(m)} - \left(\chi_{l,p}^{(m,s)} \right)^{(+,-)} Y_p^{(m)} \right\} = 0, \end{aligned} \quad (36)$$

$$\begin{aligned} Y_l^{(s)} + \sqrt{l} \left(\kappa_{l,0}^{(s,s)} \right)^{(-,+)} \tilde{\zeta}_0^{(s)} + \sqrt{l} \sum_{m=0, m \neq s}^M \left(\chi_{l,0}^{(m,s)} \right)^{(-,+)} \tilde{\zeta}_0^{(m)} \\ + \sum_{p=1}^{\infty} \sqrt{lp} \left\{ \left(\kappa_{l,p}^{(s,s)} \right)^{(-,+)} X_p^{(s)} - \left(\kappa_{l,p}^{(s,s)} \right)^{(-,-)} Y_p^{(s)} \right\} \\ + \sum_{m=0, m \neq s}^M \sum_{p=1}^{\infty} \sqrt{lp} \left\{ \left(\chi_{l,p}^{(m,s)} \right)^{(-,+)} X_p^{(m)} - \left(\chi_{l,p}^{(m,s)} \right)^{(-,-)} Y_p^{(m)} \right\} = 0, \end{aligned} \quad (37)$$

$$\begin{aligned} \frac{1}{2} \left(\kappa_{l,0}^{(s,s)} \right)^{(+,+)} \tilde{\zeta}_0^{(s)} + \frac{1}{2} \sum_{m=0, m \neq s}^M \left(\chi_{0,0}^{(m,s)} \right)^{(+,+)} \tilde{\zeta}_0^{(m)} \\ + \sum_{p=1}^{\infty} \sqrt{p} \left\{ \left(\kappa_{0,p}^{(s,s)} \right)^{(+,+)} X_p^{(s)} - \left(\kappa_{0,p}^{(s,s)} \right)^{(+,-)} Y_p^{(s)} \right\} \\ + \sum_{m=0, m \neq s}^M \sum_{p=1}^{\infty} \sqrt{p} \left\{ \left(\chi_{0,p}^{(m,s)} \right)^{(+,+)} X_p^{(m)} - \left(\chi_{0,p}^{(m,s)} \right)^{(+,-)} Y_p^{(m)} \right\} = 0, \end{aligned} \quad (38)$$

where we have used the following notations for the matrix elements:

$$\begin{aligned}
 \left(\kappa_{l,p}^{(s,s)}\right)^{(+,+)} &= \frac{1}{2} \left[\left(h_{l,p}^{(s,s)} + h_{l,-p}^{(s,s)}\right) + \left(h_{-l,p}^{(s,s)} + h_{-l,-p}^{(s,s)}\right) \right]; \\
 \left(\chi_{l,p}^{(m,s)}\right)^{(+,+)} &= \frac{1}{2} \left[\left(q_{l,p}^{(m,s)} + q_{l,-p}^{(m,s)}\right) + \left(q_{-l,p}^{(m,s)} + q_{-l,-p}^{(m,s)}\right) \right]; \\
 \left(\kappa_{l,p}^{(s,s)}\right)^{(+,-)} &= \frac{1}{2} \left[\left(h_{l,p}^{(s,s)} - h_{l,-p}^{(s,s)}\right) + \left(h_{-l,p}^{(s,s)} - h_{-l,-p}^{(s,s)}\right) \right]; \\
 \left(\chi_{l,p}^{(m,s)}\right)^{(+,-)} &= \frac{1}{2} \left[\left(q_{l,p}^{(m,s)} - q_{l,-p}^{(m,s)}\right) + \left(q_{-l,p}^{(m,s)} - q_{-l,-p}^{(m,s)}\right) \right]; \\
 \left(\kappa_{l,p}^{(s,s)}\right)^{(-,+)} &= \frac{1}{2} \left[\left(h_{l,p}^{(s,s)} + h_{l,-p}^{(s,s)}\right) - \left(h_{-l,p}^{(s,s)} + h_{-l,-p}^{(s,s)}\right) \right]; \\
 \left(\chi_{l,p}^{(m,s)}\right)^{(-,+)} &= \frac{1}{2} \left[\left(q_{l,p}^{(m,s)} + q_{l,-p}^{(m,s)}\right) - \left(q_{-l,p}^{(m,s)} + q_{-l,-p}^{(m,s)}\right) \right]; \\
 \left(\kappa_{l,p}^{(s,s)}\right)^{(-,-)} &= \frac{1}{2} \left[\left(h_{l,p}^{(s,s)} - h_{l,-p}^{(s,s)}\right) - \left(h_{-l,p}^{(s,s)} - h_{-l,-p}^{(s,s)}\right) \right]; \\
 \left(\chi_{l,p}^{(m,s)}\right)^{(-,-)} &= \frac{1}{2} \left[\left(q_{l,p}^{(m,s)} - q_{l,-p}^{(m,s)}\right) - \left(q_{-l,p}^{(m,s)} - q_{-l,-p}^{(m,s)}\right) \right].
 \end{aligned} \tag{39}$$

The unknown coefficients are

$$\begin{pmatrix} X_l^{(s)} \\ Y_l^{(s)} \end{pmatrix} = \frac{1}{2\sqrt{l}} \begin{pmatrix} \xi_l^{(s)} + \xi_{-l}^{(s)} \\ \xi_l^{(s)} - \xi_{-l}^{(s)} \end{pmatrix}; \quad \begin{pmatrix} X_p^{(m)} \\ Y_p^{(m)} \end{pmatrix} = \frac{1}{2\sqrt{p}} \begin{pmatrix} \xi_p^{(m)} + \xi_{-p}^{(m)} \\ \xi_p^{(m)} - \xi_{-p}^{(m)} \end{pmatrix}. \tag{40}$$

This is a Fredholm matrix system of the second kind. More precisely, in a similar way to that for a single boundary contour, one can show that the left-hand sides of ISLAE (36)–(38) define an infinite-matrix (summation) OVF $H^M(k) : l_2^{2M+2} \rightarrow l_2^{2M+2}$, $k \in \mathcal{R}$, where l_2^{2M+2} is a Cartesian product of $2M+2$ copies of spaces l_2 and Equations (36)–(38) form a Fredholm infinite-matrix (summation) equation system of the second kind equivalent to the initial homogeneous boundary IE (30). This system, written as $H^M(k) = 0$, constitutes, in turn, the problem on CNs for the OVF $H^M(k)$. Since $H^M(k)$ is a Fredholm OVF considered in Hilbert space l_2^{2M+2} and its resolvent set is nonempty, its “spectral” set $\sigma_H^M \subset \mathcal{R}$ of the sought CNs is discrete, i.e., consists of isolated points on the complex plane with the only accumulation point at infinity. Based on the equivalence, one can verify that every $k \in \sigma_H^M$ is a CN of the initial integral OVF defined by the initial IE (30) and is a sought eigenvalue of the initial Dirichlet BVP.

3. Some Aspects of the Numerical Implementation of the MAR and Cross-Validation with Results Obtained by Other Methods

Inhomogeneous well-conditioned systems of the type (18)–(20) and (36)–(38), which arise as a result of employing the MAR, have been discussed in many publications (see, for example, [1,23,26]). Homogeneous systems possessing similar properties to those obtained in the present paper were used in [28], where the complex eigenvalues of a sound-soft elliptic cavity with a variably placed longitudinal slit were calculated. Without repeating all the arguments contained in these and other publications, we wish to discuss a few points.

The first point concerns numerical methods aimed at the calculation of approximate values of CNs of infinite-matrix OVF $H(k)$ and $H^M(k)$ and a justification of this method using the theory of discrete convergence. These two issues are described in detail in [1] and summarized in Theorem 1.17 of Section 1.11. It is proved that: for every CN $k' \in \sigma_H^M$ there exists a sequence $\{k_N^{(0)}\}_{N \geq N_0 > 1}^\infty$ of approximate CNs of the OVF $H(k)$ or $H^M(k)$ such that $k_N^{(0)} \in \sigma_A^{(N)}$ and $|k_N^{(0)} - k'| \rightarrow 0$ as $N \rightarrow \infty$, where $\sigma_A^{(N)}$ denotes the spectrum of finite-dimensional operators with truncated matrix $A_N = A_N(k)$ (a collection of approximate CNs of the OVF $H(k)$ or $H^M(k)$ corresponding to the truncation index N). On the other hand, if $k_N^{(0)} \in \sigma_A^{(N)}$, $N \geq N_0 > 1$, and there exists an infinite subsequence of complex

numbers $\{k_N^{(s,0)}\}_{N \geq N_0 > 1}^\infty \subset \{k_N^{(0)}\}_{N=0}^\infty$ such that $|k_N^{(s,0)} - k'| \rightarrow 0$, $N \rightarrow \infty$ then k' is a CN of the infinite-matrix OVF $H(k)$ or $H^M(k)$.

According to the definition of “approximate” spectrum $\sigma_A^{(N)}$, the numbers $k_N^{(0)} \in \sigma_A^{(N)}$ are zeros of the determinants $\det A_N(k)$. The condition number $\text{cond } A_N(k)$ of the matrix $A_N(k)$ is defined to be the product $\|A_N(k)\| \|A_N^{-1}(k)\|$ of the l_2 -matrix norms of $A_N(k)$ and its inverse; the maximal values of $\text{cond } A_N(k)$ are used to localize and calculate the zeros of the determinants $\det A_N(k)$. The theory elaborated in the preceding paragraph ensures that these zeros do indeed converge (as $N \rightarrow \infty$) to the cut-off wavenumbers sought for the waveguide.

The second point concerns the structure of the matrix A_N , which is formed by the truncation of the system to a finite size. In contrast to the Method of Moments, where the matrix elements are computed by numerical integration, the solution presented here has no need to use such a time-consuming operation since the matrix elements, derived from the Fourier coefficients $h_{n,p}^{(s,s)}$ and $q_{n,p}^{(m,s)}$ ($-\frac{1}{2}N \leq n, p \leq \frac{1}{2}N$) where N is the truncation number, are computed with the use of the Fast Fourier Transform.

The third point in finding the cut-off wavenumbers (i.e., the CNs) is the role played by the spectral dependence of the condition number $\text{cond } A_N(k)$ of the matrix A_N of the truncated algebraic system on the spectral parameter k . If $A_\infty(k)$ denotes the matrix associated with infinite system (36)–(38) and the infinite-matrix (summation) OVF $H^M(k)$, the spectral set $\sigma_H^M \subset \mathcal{R}$ of the sought CNs can be determined as roots of the dispersion equation $\det A_\infty(k) = 0$. (This determinant is well defined because A_∞ is a compact perturbation of the identity matrix and has the property that $\det A_N(k) \rightarrow \det A_\infty(k)$ as $N \rightarrow \infty$.) The algorithm based on the MAR, with the above justification using discrete convergence, provides fast convergence (to the exact spectral values of k) of the solutions (i.e., the approximations to spectral values, or the exact cut-off wavenumbers) of the corresponding characteristic equation $\det A_N(k) = 0$ of the truncated system as the truncation number N increases.

We illustrate these arguments by one simple and instructive example, demonstrating how the spectrum of the cut-off wavenumbers for a hollow elliptical waveguide is transformed as its eccentricity e increases. We consider the circular waveguide ($e = 0$) and three elliptical waveguides of increasing eccentricity: small ($e = 0.1$), intermediate ($e = 0.5$) and high ($e = 0.9$); if a and b denote the major and minor semi-axes of the ellipse, the corresponding aspect ratios b/a are 0.995, 0.886 and 0.436, respectively. We restrict ourselves to consideration of the first three TM-modes: TM_{c01} , TM_{c11} and TM_{s11} .

Figure 3 shows the condition number $\text{cond } A_N(kb)$ as a function of the non-dimensional wave number kb , for these four values of eccentricity and truncation number $N = 256$. As can be seen, the spectral dependence of $\text{cond } A_N(kb)$ is highly sensitive to the roots of the characteristic equation $\det A_N = 0$, indicated by the sharp resonance peaks at those points where kb coincides with one of the roots of this equation. Ideally, in the absence of the heat losses in the walls of the waveguide, the resonance peaks of $\text{cond } A_\infty(kb)$ occur exactly at those values kb , which match precisely the zeros of the characteristic equation $\det A_\infty = 0$ and the resonance maxima of $\text{cond } A_\infty(kb)$ should be of infinite magnitude. In practice, when we employ the truncated equations, the magnitude of the resonance peaks is finite and increases with the growth of the truncation number N ; the accuracy of the calculated cut-off wavenumbers strictly depends on its value. The second kind nature of the Fredholm matrix system obtained by application of the MAR to the waveguide problem ensures that the algorithm for calculating the cut-off wavenumbers with any prescribed (i.e., pre-specified) accuracy is reliable and fast converging.

The scheme for calculating the cut-off wavenumbers is as follows. First, we calculate the spectral dependence $\text{cond } A_N(kb)$ in a fixed interval of kb values locating all (approximate) values of the cut-off wavenumbers: in Figure 3, two or three values are observed. Utilizing a suitable choice of increment Δkb of the wavenumber (usually Δkb lies in the range 0.0001 to 0.001) and increasing the truncation number N as necessary (as a rule

$N = 128$), the resolving power of the method is sufficient to ensure that no modes of the spectrum are missed.

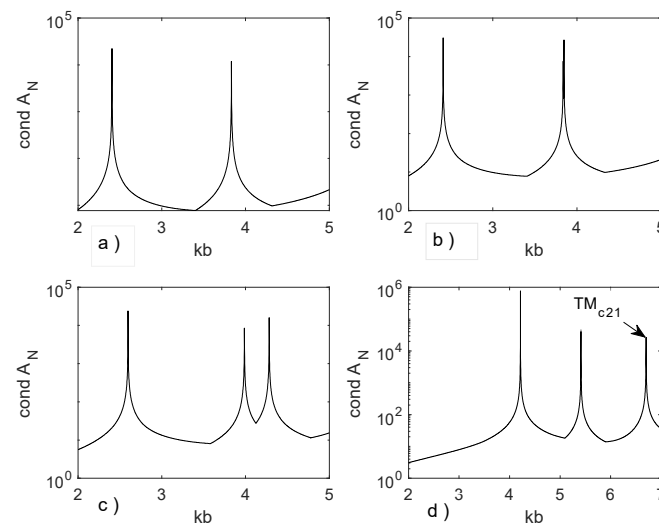


Figure 3. Condition number $\text{cond } A_N(kb)$ versus wave number kb , for eccentricities (a) $e = 0$ (circular) (b) $e = 0.1$; (c) $e = 0.5$; (d) $e = 0.9$. Truncation number N equals 256.

3.1. Cut-Off Wavenumbers of an Elliptical Waveguide

When the eccentricity e of the waveguide cross section increases from 0 (a circular waveguide) to 0.1 (a slightly deformed circular waveguide), the TM_{c11} and TM_{s11} modes, which are degenerate in the circular waveguide, split into two independent modes in the elliptic waveguide. The resonance peaks reveal the approximate values of the cut-off wavenumbers: $kb \approx 3.832$ ($e = 0$) and $kb \approx 3.837, 3.846$ ($e = 0.1$). These and the other approximate values of the cut-off wavenumbers found serve as initial values for finding more precise results in solving the dispersion equation $\det A_N = 0$ as the truncation number N is increased. A typical sequence of approximate solutions $\gamma_{c01}^{(n)}$ of the dispersion equation $\det A_N = 0$ for the wavenumber $kb = \gamma_{c01}$ of the lowest even mode TM_{c01} is shown in Table 1 (γ_{c01} is the eigenvalue (cut-off wave number) of the mode TM_{c01}). It demonstrates convergence as N increases.

Table 1. Convergence of approximations $\gamma_{c01}^{(N)}$ to wavenumber γ_{c01} of the lowest even mode TM_{c01} .

Truncation Number N	Computed Value $\gamma_{c01}^{(N)}$ of γ_{c01}
32	$2.410633039187760 - i \cdot 0.000000066967616$
64	$2.410857871457190 - i \cdot 0.00000001040154$
128	$2.410885878808779 - i \cdot 0.00000000016227$
256	$2.410889376581807 - i \cdot 0.000000000000253$
512	$2.410889813714743 - i \cdot 0.000000000000004$
1024	$2.410889868251958 - i \cdot 0.000000000000000$
2048	$2.410889875181512 - i \cdot 0.000000000000000$
4096	$2.410889876035204 - i \cdot 0.000000000000000$

Application of Aitken's delta-squared process (see, for example, [31]) shows that $\gamma_{c01}^{(N)} \approx \gamma_{c01} + cN^{-p}$ for some constant c independent of N and an exponent p very close to 3; the extrapolated value of 2.410889876, achieved with the values $\gamma_{c01}^{(N)}$ computed for $N = 64, 128$ and 256, agrees, to 9 decimal places, with that achieved in the computation

of $\gamma_{c01}^{(4096)}$ with the much larger truncation number of 4096. This suggests that Aitken extrapolation is an effective tool for achieving high accuracy for modest truncation numbers.

Let us compare the results of the calculation of the first three cut-off wavenumbers γ_{c01} , γ_{c11} and γ_{s11} of the elliptical waveguides having eccentricities $e = 0.1, 0.5$ and 0.9 with those obtained in [28]. In this paper, the eigenmode sequence for an elliptical waveguide with arbitrary eccentricity was validated against the values obtained by direct calculation of the parametric zeros of the modified Mathieu functions of the first kind. The results are presented in terms of cut-off wavelengths λ_c/b , which we have expressed in terms of the cut-off relative wavenumbers $k_cb = 2\pi/(\lambda_c/b)$; the comparison is displayed in Table 2. The results show near-perfect coincidence. There is a maximum difference of one unit in the last (eighth) decimal digit for some values of the cut-off wavenumbers; most probably, the difference is due to a rounding error resulting from the rearrangement of the results in [28].

Table 2. Normalized cut-off wavenumbers k_cb ($\gamma_{c(s)nl}$) of an elliptic waveguide.

	γ_{c01}	γ_{c01}	γ_{c11}	γ_{c11}	γ_{s11}	γ_{s11}
Method	MAR	[28]	MAR	[28]	MAR	[28]
$e = 0.1$	2.41088987	2.41088987	3.83653893	3.83653893	3.84619062	3.84619061
$e = 0.5$	2.59677924	2.59677924	3.98640797	3.98640797	4.28221524	4.28221523
$e = 0.9$	4.21513438	4.21513437	5.41326079	5.41326080	7.76364414	7.76364415

The computations in this and subsequent sections were performed on a standard modern PC with sufficient memory for the inversion of a matrix of order 4096 by the usual process of Gaussian elimination. The programming used the MATLAB (R2023b) environment (including its procedures for Gaussian elimination and the Fast Fourier Transform) with an unoptimized code. For a fixed truncation number N , the computational time did not depend on the eccentricity e of the ellipse nor on the relative wave number ka . As an indication of the speed and efficiency of the algorithm, the times measured at each individual wavenumber (with eccentricity $e = 0.9$) for truncation numbers of 128, 256, 512 and 1024 were, respectively, 0.037, 0.132, 0.590 and 3.252 s.

In the following two short Sections 3.2 and 3.3, we make comparisons between results obtained by the MAR and other methods for cross validation. They concern the cut-off wavenumbers in waveguides comprising a circular outer conductor and eccentric inner conductor (Section 3.2) and a circular–rectangular waveguide (Section 3.3).

3.2. Waveguides with Circular Outer Conductors and Eccentric Circular Inner Conductors

In this section, we consider the structure shown in Figure 4 of a circular waveguide of radius a enclosing an inner circular conductor of radius b , eccentrically placed with its centre offset by a distance d . We compare our results obtained by the MAR with those obtained in [3] by the classical method of separation of variables and the use of addition theorems. From the variety of the results contained in [3], we choose for cross validation the set of parameters $b/a = d/a = 0.25$. The spectral dependence $\text{cond } A_N(ka)$ for this set of parameters is shown in Figure 5. It allows us to find the cut-off wavenumbers with nearly the same accuracy for both symmetric and antisymmetric modes; these are presented in [3], with four significant decimal digits of accuracy. We present magnified views of the spectral dependence in narrow intervals near the cut-off wavenumbers, belonging to a pair of symmetric modes and a pair of antisymmetric modes (Figure 6).

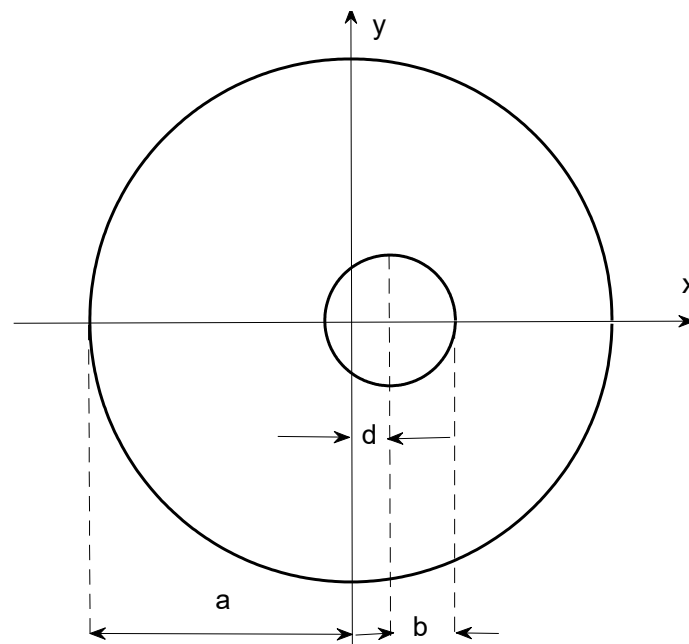


Figure 4. Geometry of a circular waveguide with an eccentrically placed inner circular conductor.

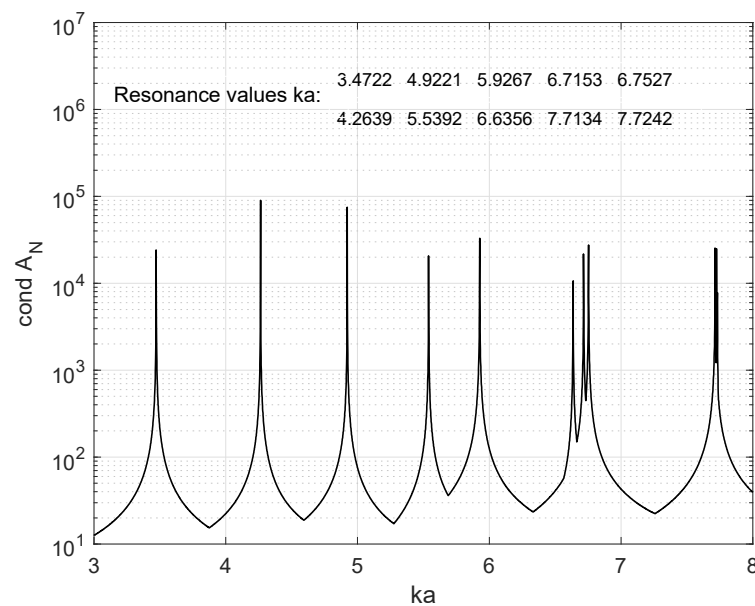


Figure 5. Spectral dependence of the condition number $\text{cond } A_N(ka)$, $N = 128$ ($b/a = d/a = 0.25$).

Taking the approximate resonance values ka as initial values, the root finding code produces the cut-off wavenumbers with a proven accuracy of 8 significant decimal digits (see Table 3). One observes near-perfect coincidence of the results with those of [3]; there is a tiny deviation of less than 0.0001 for both symmetric and antisymmetric modes.

For the computations displayed in Figure 5, the interval $3 \leq ka \leq 8$ was subdivided into 500 intervals with step $\Delta ka = 0.001$. The total computational time was 73.036 s; for each individual value of ka , the time was $73.036/501 = 0.146$ s. Upon doubling the truncation number N to 256, the corresponding times were 296.846 s (total) and 0.592 s (individual).

The convergence of the sequence of approximations $\gamma_{c01}^{(N)}$ to the wavenumber γ_{c01} of the lowest even mode TM_{c01} is displayed in Table 4. The computational time associated with each truncation number is also displayed.

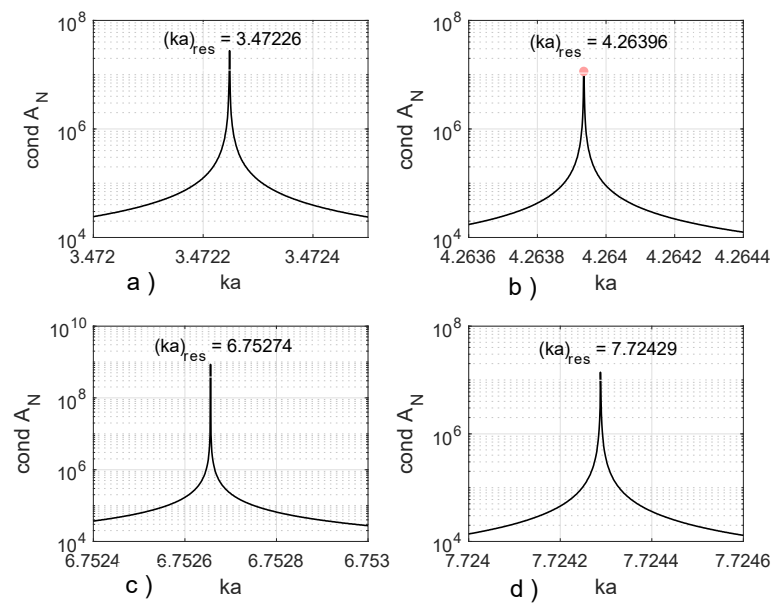


Figure 6. Magnified view of the spectral dependence $\text{cond } A_N(ka)$, $N = 128$ ($b/a = d/a = 0.25$), in the neighbourhood of resonance points: (a) $(ka)_{\text{res}} = 3.47226$; (b) $(ka)_{\text{res}} = 4.26396$; (c) $(ka)_{\text{res}} = 6.75274$; (d) $(ka)_{\text{res}} = 7.72429$.

Table 3. Cut-off wavenumbers for TM -modes in ascending order.

Symmetric Modes		Anti-Symmetric Modes	
L. Zhang et al. [3]	MAR	L. Zhang et al. [3]	MAR
3.4723	3.47226067	4.2640	4.26396049
4.9221	4.92212067	5.5393	5.53930291
5.9268	5.92681068	6.6357	6.63570629
6.7154	6.71544084	7.7135	7.71353358
6.7527	6.75275190	7.7243	7.72430654

Table 4. Convergence of the approximations to the wavenumber γ_{c01} of the lowest even mode TM_{c01} as a function of truncation number.

Truncation Number N	Computed Cut-Off Wavenumber of the Lowest Symmetric Mode	Time (Seconds)
64	$3.472157345180335 - i \cdot 0.000000009781369$	1.796
128	$3.472247785036224 - i \cdot 0.000000000152295$	3.121
256	$3.472259067569722 - i \cdot 0.000000000002378$	10.515
512	$3.472260477172100 - i \cdot 0.000000000000037$	10.515
1024	$3.472260653349862 - i \cdot 0.000000000000001$	207.292
2048	$3.472260675371377 - i \cdot 0.000000000000000$	1340.219
4096	$3.472260678124045 - i \cdot 0.000000000000000$	

Similar to that observed in Section 3.1, application of Aitken's delta-squared process [31] shows that $\gamma_{c01}^{(N)} \approx \gamma_{c01} + cN^{-p}$ for some constant c and an exponent p very close to 3; the 8 decimal place extrapolated value of 3.47226068, achieved with the values $\gamma_{c01}^{(N)}$ computed for $N = 64, 128$ and 256 , agrees, to 8 decimal places, with that achieved in the computation of $\gamma_{c01}^{(4096)}$ with the much larger truncation number of 4096. This offers yet

again confirmation that Aitken extrapolation is an effective tool in achieving high accuracy for modest truncation numbers.

3.3. Analysis of a Circular–Rectangular Waveguide

In the previous two sections, we demonstrated the capability of the MAR-based solution to provide an independent estimate of the correctness of results obtained by other methods. In other words, we may assert that the MAR-based solution should be recognized and treated as a benchmark solution. We wish to illustrate this assertion with one more example concerning higher-order modal characteristics of circular–rectangular coaxial waveguides [32]. The authors of this paper carried out a rigorous analysis combining the orthogonal expansion method and the Galerkin method for finding the higher-order eigenmodes in a circular–rectangular (C-R) waveguide. Despite a faultless theoretical part, the calculation of the cut-off frequencies seems to have significant errors, which are especially apparent for the higher modes. In contrast to the validated results of the previous sections, the claimed accuracy of the results presented in [32] cannot be validated in full. With the aim of validating results in [32], we modelled the outer conductor by a rectangle with rounded-off corners and the inner conductor by a circle, as shown in Figure 7. We consider the waveguide with a shifted inner conductor ($d \neq 0$); the case examined in [32] is a particular case (namely, $d = 0$) of this more general problem.

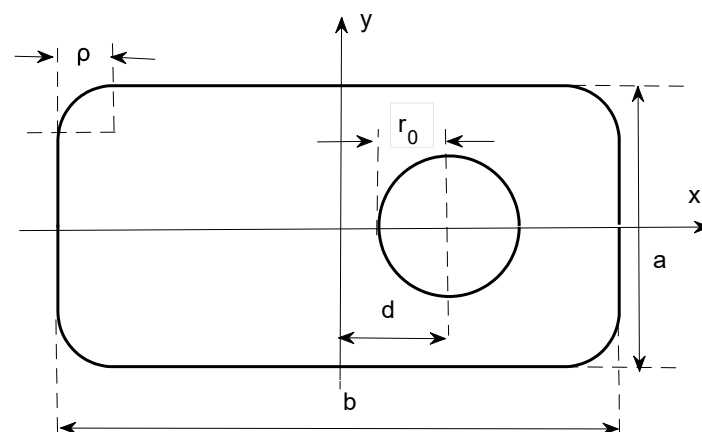


Figure 7. Geometry of a circular–rectangular waveguide with shifted inner conductor.

The cut-off wavenumbers were calculated in [32] for a waveguide with an outer conductor of exactly rectangular cross section. The MAR requires that the contours of the conductors are smooth, so our calculations employed rectangular structures with corners rounded off by circular arcs of a suitably chosen radius ρ . To eliminate the influence of the rounded-off corners, we performed a sequence of calculations as follows. We used the same size structures as [32]: the inner circular conductor having radius $r_0 = 0.636$ cm, the outer rectangular conductor having sides $a = 2.54$ cm, $b = 5.08$ cm; the rounding radius was chosen to be $\rho = 0.1, 0.02$ and 0.005 cm. It was observed that the seventh decimal digit stabilizes when ρ does not exceed 0.02 cm; further reduction in this parameter does not change the cut-off wavenumbers when calculated with an accuracy of 7 significant decimal digits.

The results obtained by the MAR and the method of [32] are presented in Table 5. Notice that when ρ takes the value 0.02 cm and 0.005 cm, all the computed cut-off wavenumbers k_c satisfy $k_c \rho \leq 0.08$ and $k_c \rho \leq 0.02$, i.e., the corner radius is electrically small and even very small in the latter case. The average element size of 0.1016 cm used by authors of [32] in the finite element method for cross validation is five times the size of the round-off radius (0.02 cm). Related studies of the impact of rounding are given in [33], where it is shown that the perturbations at wavenumber k to the surface distribution and scattering pattern of the rectangular corner are $O((k\rho)^{2/3})$ and $O((k\rho)^{4/3})$, respectively, as $k\rho \rightarrow 0$.

Table 5. A comparison of the cut-off wavenumbers k_c (cm^{-1}) of various TM-modes computed using two methods: the MAR with rounded corners of varying radii ϱ and the method of Wang et al. [30].

TM-Mode	MAR, $\varrho = 0.1 \text{ cm}$	MAR, $\varrho = 0.02 \text{ cm}$	MAR, $\varrho = 0.005 \text{ cm}$	H. Wang et al. [30]
TM_{11}	1.9948124	1.9948058	1.9948058	1.9948099
TM_{21}	1.9953724	1.9953659	1.9953659	1.9953650
TM_{12}	2.8694972	2.8694828	2.8694828	2.8694867
TM_{22}	2.8721943	2.8721798	2.87217979	2.8721777
TM_{31}	3.3197479	3.3197356	3.3197356	3.3197216
TM_{41}	3.3265163	3.3265037	3.3265037	3.3264943
TM_{32}	3.7716492	3.7716114	3.7716117	3.7715987
TM_{42}	3.7941610	3.7941210	3.7941210	3.7941012
TM_{51}	3.9730566	3.9730249	3.9730250	3.9728472
TM_{23}	3.9844121	3.9843794	3.9843793	3.9843494

A comparison of the final two columns of Table 5 shows that the claim in [32] of 7 significant decimal digits of accuracy cannot be sustained. It seems likely that the discrepancy is due to an insufficient number of terms taken in the Bessel–Fourier series given in Equations (29) and (30) of [32]. In a parallel way, a discrepancy would arise in using the MAR if an improper choice of the truncation number N were made. The correctness of a given number of decimal digits is verified by stabilization of the last decimal digit under successive increments of the truncation number. The computation of higher modes with larger cut-off wavenumbers requires larger truncation numbers to keep the same number of correct decimal digits as lower modes.

The data collected in Table 6 illustrate this assertion. As indicated by the bold digits, the cut-off wavenumbers for three modes are computed with increasing accuracy as the truncation number is increased. The process stops when computed wavenumbers stabilize at 6 decimal digits. The radius ϱ of curvature of the rounded corner used is 0.02 cm.

Table 6. Computed values of the cut-off wavenumbers k_c (cm^{-1}) for three modes as a function of truncation number.

Truncation Number N	TM_{11}	TM_{22}	TM_{23}
128	1.99478678	2.87210785	3.98418755
512	1.99480566	2.87217868	3.98437637
1024	1.99480584	2.87217966	3.98437902
2048	1.99480588	2.87217979	3.98437935
4096		2.87217980	3.98437939

4. Air-Filled Rectangular Waveguides with Arbitrary Located Inner Conductors: Metallic Strips and Strip-like Elliptic Cylinders

In this section, we investigate the cut-off wavenumbers in which inner conductors of various shapes and locations occupy the waveguide. The geometry is shown in Figure 8. Inner conductors of three different cross sections are considered: (a) an infinitely thin strip; (b) a thick strip, which is rectangular with rounded corners; and (c) an inner conductor of elliptic cross section. Inner conductors may take arbitrary locations inside the outer conductor; their cross section may also be rotated around their symmetry axis relative to the $x - y$ axis, subject to the restriction that the inner conductor does not make electrical contact with the walls of the rectangular waveguide. The sides of the rectangular waveguide are a and b , ($b > a$); its corners are rounded with a radius of curvature ϱ_0 . The centre of

symmetry of the inner conductor is located at (x_0, y_0) , and its angle α ($0 \leq \alpha \leq 2\pi$) of orientation is measured by the anti-clockwise angle of its major axis of symmetry relative to the $x - y$ axis.

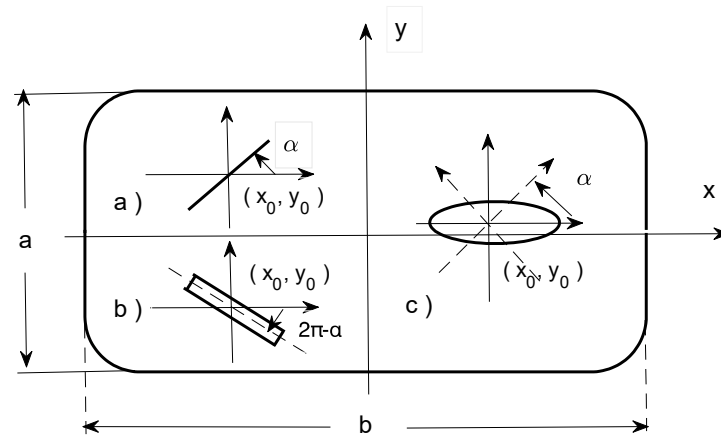


Figure 8. Rectangular waveguide with sides a and b , ($b > a$). 3 types of metallic insertions are considered: (a) an infinitely thin strip; (b) a thick strip with rounded corners; (c) an inner conductor of elliptic cross section. The location of the centre of symmetry of the inner conductor is denoted (x_0, y_0) , and the angle α specifies its orientation.

For each type of inner conductor, a common parameter will be used to describe its width l , semiwidth $w = l/2$ and thickness $2d$; these are defined as follows. The infinitesimally thin strip is characterized by its width l ; its thickness $2d$ is zero. For the conductor of elliptic cross section with minor and major semi-axes a' and b' , respectively, l is defined to be $2b'$, and thickness $2d$ is defined to be $2a'$. The thick strip is characterized as a rectangle of width l and thickness $2d$; its corners are rounded with a radius of curvature q_i . In all calculations, we set $a = 1$ unit so that other parameters may be interpreted as normalized values; for example, the non-dimensional ratio $b/a = 2$ simply becomes $b = 2$; the non-dimensional cut-off relative wavenumbers $k_c a$ become the values k_c .

The idealized problem (a) with the infinitesimally thin strip is not directly soluble by the method described in this paper; it is solved using the so-called “cavity solution” of [28] when the thin strip is regarded as a part of a larger smooth closed contour. The results are discussed in the next section. The cut-off wavenumbers k_c thus obtained are reference values for comparison when the infinitesimally thin strip is replaced by an “elliptic strip” with ratio $d/w = a'/b' \ll 1$ or by a strip of small but finite thickness with ratio $d/w \ll 1$. The cut-off wavenumbers for problems (b) and (c) with strips of non-zero thickness d were obtained using the general solution (36)–(38), where we set $M = 1$ and use a smooth parameterization of the contours L_0 and L_1 of the outer and inner conductors. The infinitesimally thin strip results are discussed in the next section and those for the strips of non-zero thickness are in the following. The ratio b/a of the sides of the outer conductor is fixed at the value 2.

4.1. Waveguide with an Infinitesimally Thin Strip Placed Inside at Varying Locations

In this section, we consider how the cut-off wavenumbers of the isolated waveguide are perturbed by the introduction of an infinitesimally thin PEC strip; the semi-width of the strip will be denoted w , and its centre is the geometric centre of the rectangular waveguide, which is also taken as the origin of the Cartesian coordinates ($x_0 = y_0 = 0$). The strip is parallel to the wider wall of the waveguide; its semi-width varies in the interval $0.001 \leq w \leq 0.25$. The evolution of the cut-off wavenumbers $\gamma_{m,n}$, as the semi-width w of the strip increases, is shown in Figure 9.

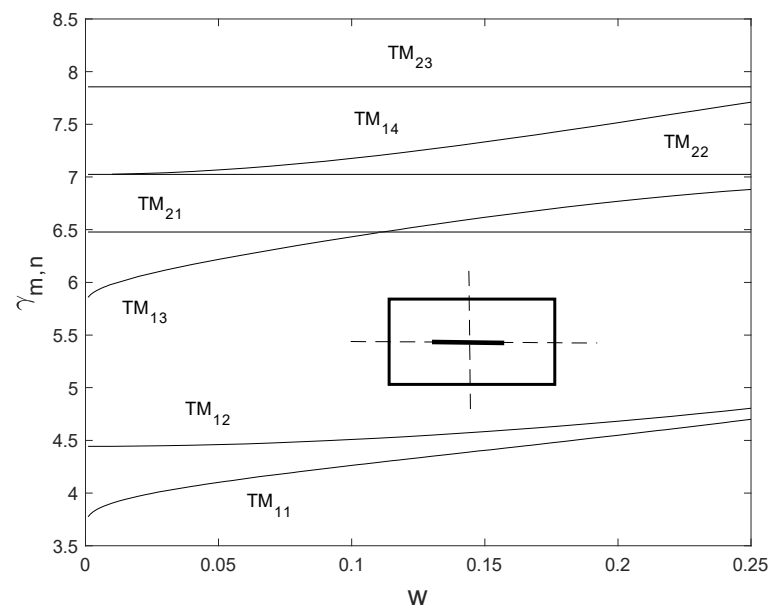


Figure 9. Cut-off wavenumbers $\gamma_{m,n}$ of the infinitesimally thin metallic strip, with centre $(x_0, y_0) = (0, 0)$ and orientation angle $\alpha = 0^\circ$, as a function of semi-width w .

It can be seen that the cut-off wavenumbers $\gamma_{m,n}$ of the $TM_{2,n}$ ($n = 1, 2, 3$) modes are nearly independent of the strip width, indeed nearly independent of the strip's presence or absence. These modes have an even number $m = 2$ of oscillations between the wider walls of the waveguide; the distribution of the modal electromagnetic field for modes does not cross the locations of the intensity maxima, and, hence, the strip does not sensibly perturb this distribution. On the other hand, the cut-off wavenumbers $\gamma_{m,n}$ of the $TM_{1,n}$ ($n = 1, 2, 3, 4$) modes are noticeably affected, increasing in value as w increases.

We now consider how the cut-off wavenumbers of the isolated waveguide are perturbed when the centre of the strip is held fixed at $(x_0, y_0) = (0, 0)$, is rotated anticlockwise by an angle α of 30° and 90° . The results are displayed in Figure 10a,b, respectively. In contrast to the case $\alpha = 0^\circ$, the cut-off wavenumbers of *all* modes are increasingly perturbed as the semi-width increases. Common to the cut-off wavenumbers computed at different orientation angles α ($0^\circ < \alpha < 90^\circ$) is the splitting of the $TM_{2,2}$ and $TM_{1,4}$ modes; in the empty waveguide, these modes are degenerate but as the semi-width of strip embedded in rectangular waveguide increases, the computed cut-off wavenumbers of these two modes diverge significantly.

When the strip lies in the vertical position ($\alpha = 90^\circ$) exactly, it is not obvious whether the $TM_{2,2}$ and $TM_{1,4}$ modes split or not. To the degree of resolution shown in Figure 10b), the plots of the cut-off wavenumbers $\gamma_{2,2}(w)$ and $\gamma_{1,4}(w)$, as functions of the semi-width w , do not reveal any signs of mode splitting. However, the MAR enables us to reveal the true situation by means of high-precision calculation of the spectral dependence *cond* $A_N(ka)$ in the neighbourhood of the cut-off wavenumbers $\gamma_{2,2}(0) = \gamma_{1,4}(0) = 7.02481473$ of the $TM_{2,2}$ and $TM_{1,4}$ modes existing in the hollow rectangular waveguide.

When the strip lies in the vertical position ($\alpha = 90^\circ$), it is parallel to the smaller sides of the rectangle. In this orientation, the plots *cond* $A_N(ka)$ presented in Figure 11 demonstrate the evolution of the cut-off wavenumbers $\gamma_{2,2}(w)$ and $\gamma_{1,4}(w)$ as the strip semi-width w increases. Our extensive calculations establish that modes remain degenerate up to a critical semi-width w_0 lying between 0.25 and 0.3. When $w = 0.25$, Figure 11a shows that the resonance value of the function *cond* $A_{256}(ka)$ occurs at a single point $ka = 7.02480$. Beyond the critical value w_0 one observes the emergence of a resonance duplet in *cond* $A_{256}(ka)$ corresponding to modal splitting: when $w = 0.3$, the resonance duplet maxima in the graph of the condition number arise at values $(ka)_1 = 6.98625$ and $(ka)_2 = 6.99220$; when $w = 0.4$, the corresponding values are $(ka)_1 = 7.02256$ and $(ka)_2 = 7.02266$.

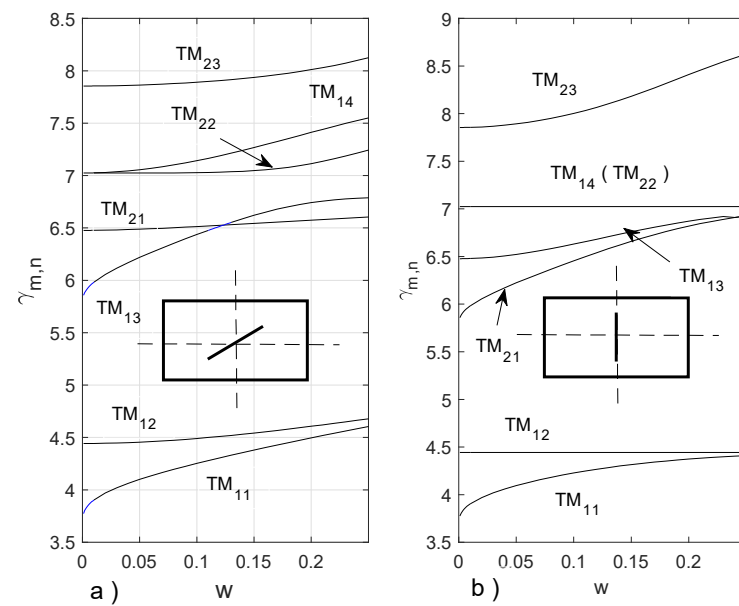


Figure 10. Cut-off wavenumbers $\gamma_{m,n}$ of the infinitesimally thin metallic strip, with centre $(x_0, y_0) = (0, 0)$ and orientation angle α , as a function of semi-width $\gamma_{1,4}(w)$: (a) $\alpha = 30^\circ$, (b) $\alpha = 90^\circ$.

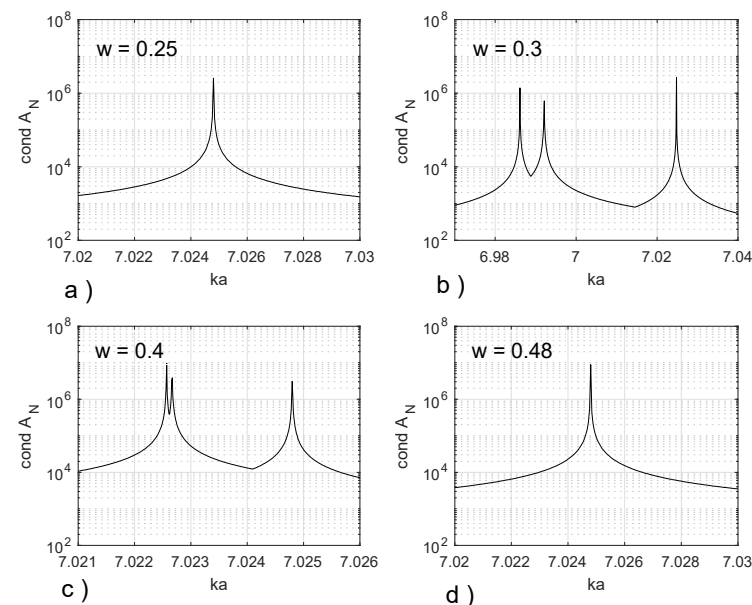


Figure 11. Evolution of the resonance duplet in the condition number $\text{cond } A_{256}(ka)$ as w increases: (a) $w = 0.25$, (b) $w = 0.3$, (c) $w = 0.4$, and (d) $w = 0.48$.

The emergence of a new duplet of cut-off wavenumbers is undoubtedly because of the gradual transformation of the original waveguide, of sides a and b with $b/a = 2$, into a two-sectioned waveguide, comprising two coupled waveguides, which are square of sides a and $b' = b/2$, with $b'/a = 1$. The basic resonance peak of the condition number graph ($\text{cond } A_{256}(ka)$) remains in a stable position (at $ka = 7.02480$) whatever the semi-width w of the strip. The emergence of a new duplet of the resonance peaks in the graph reflects the splitting of the $TM_{1,2}$ - and $TM_{2,1}$ -modes in each of the square waveguides. Upon further extension of the strip, so that its ends approach the waveguide walls, the $TM_{2,2}$ - and $TM_{1,4}$ -modes of the rectangular waveguide ($b/a = 2$), become the degenerate $TM_{1,2}$ - and $TM_{2,1}$ -modes for the square waveguide ($b'/a = 1$ with $b' = b/2$), as seen in Figure 11d. The calculated cut-off wavenumbers for the cases represented in Figure 11 are collected in Table 7. The calculation was performed with an accuracy of 6 significant decimal digits.

Table 7. Fine effects of mode splitting. The calculated cut-off wavenumbers $\gamma_{m,n}$ for the strips of varying width examined in Figure 11a–d. The columns labelled $\text{cond } A_{256}$ record the values ka where the local maxima of $\text{cond } A_{256}(ka)$ occur; the adjacent entries are the calculated cut-off wavenumbers $\gamma_{m,n}$.

$w = 0.25$		$w = 0.3$		$w = 0.4$		$w = 0.48$	
$\text{cond } A_{256}$	$\gamma_{m,n}$	$\text{cond } A_{256}$	$\gamma_{m,n}$	$\text{cond } A_{256}$	$\gamma_{m,n}$	$\text{cond } A_{256}$	$\gamma_{m,n}$
7.02480	7.024814	6.98625	6.986267	7.02256	7.022578	7.02480	7.024814
		6.99220	6.992208	7.02266	7.022682		
		7.02480	7.024814	7.02480	7.024814		

4.2. What Is the Difference in Cut-Off Wavenumbers between a Strip of Small but Non-Zero Thickness and an Infinitesimally Thin Strip?

When its thickness is small compared with the wavelength, the electromagnetic response of an infinitesimally thin strip is a reasonable approximation to that of a strip of small but non-zero thickness; however, in practice, it is highly desirable to have some indication of the deviation of the cut-off wavenumbers calculated for an infinitesimally thin strip from those obtained for a strip of non-zero thickness. In this section, we will examine the following two thin strip-like conductors of non-zero thickness: elliptic strips (elliptic cylinders with small aspect ratios of the minor a' and major b' semi-axes, i.e., $a'/b' \ll 1$); and cylinders of rectangular cross section of width $2w$ and thickness $2d$ with rounded off semi-circular ends, as shown in Figure 12.

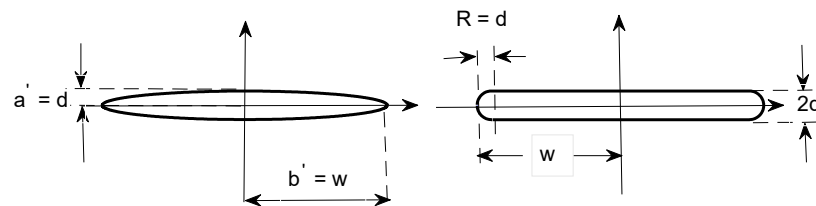


Figure 12. Geometry of the strip-like inner conductors: on the left, an elliptic cylinder ($a'/b' \ll 1$); on the right, a thick strip with rounded ends ($d/w \ll 1$).

Table 8 compares the calculated cut-off wavenumbers for the first seven modes of the rounded rectangular waveguide sides with $a = 1$ and $b = 2$ when the inner conductor is an infinitesimally thin strip, or an elliptic strip of aspect ratio $a'/b' = d/w$ taking the values 0.005, 0.01, 0.02 and 0.05. The strip is located symmetrically inside the waveguide and is oriented at an angle $\alpha = 0^\circ$. The semi-width parameter w for each type of strip is fixed at 0.25. The first row of the table shows the cut-off wavenumbers $\gamma_{m,n}^{(0)}$ for the infinitesimally thin strip; subsequent pair of rows show the perturbed cut-off wavenumbers $\gamma_{m,n} = \gamma_{m,n}^{(0)} + \delta_{m,n}$ (upper row) and the relative perturbation $\Delta_{m,n} = \delta_{m,n} / \gamma_{m,n}^{(0)}$ (lower row) when the thin strip is replaced by an elliptic strip of increasing thickness. Note that the perturbations $\delta_{m,n}$ (and relative perturbations $\Delta_{m,n}$) are always positive.

Table 9 presents a similar comparison when the elliptic strips are replaced by rounded rectangular strips of width $2w$ and thickness $2d$; the rounding radius is $\rho = 0.25d$; the relative thickness d/w at values 0.01, 0.02 and 0.05 are examined. Again, note that the perturbations $\delta_{m,n}$ (and relative perturbations $\Delta_{m,n}$) are always positive.

Table 8. Cut-off wavenumbers for $TM_{m,n}$ -modes in the rounded rectangular waveguide of sides $a = 1$ and $b = 2$; rounding radius $\varrho = 0.01$ with various inner conductors with the centre located at $(x_0, y_0) = (0, 0)$ and orientation angle $\alpha = 0^\circ$. Each column shows the cut-off wavenumber for the infinitesimally thin strip of semi-width w , the corresponding cut-off wavenumber and the relative difference $\Delta_{m,n}$ when the strip is replaced by an elliptic strip of semi-width w equal to its semi-major axis and thickness $2d$ equal to its minor axis.

	d/w	$TM_{1,1}$	$TM_{1,2}$	$TM_{1,3}$	$TM_{2,1}$	$TM_{2,2}$	$TM_{1,4}$	$TM_{2,3}$
Thin strip	0.000	4.699720	4.804192	6.475592	6.880896	7.024814	7.710194	7.853981
	0.005	4.702703	4.805858	6.482328	6.888568	7.025587	7.713700	7.857580
	$\Delta_{m,n}$	0.000635	0.000347	0.001040	0.001115	0.000110	0.000455	0.000458
Elliptic strips	0.01	4.705672	4.807522	6.488118	6.896279	7.026372	7.717212	7.861226
	$\Delta_{m,n}$	0.001266	0.000693	0.001934	0.002236	0.000222	0.000910	0.000922
	0.02	4.711575	4.810844	6.499754	6.911824	7.027974	7.724254	7.868661
	$\Delta_{m,n}$	0.002522	0.001385	0.003731	0.004495	0.000450	0.001824	0.001869
	0.05	4.728984	4.820755	6.535065	6.959426	7.033045	7.745534	7.892164
	$\Delta_{m,n}$	0.006227	0.003448	0.009184	0.011413	0.001172	0.004584	0.004862

Table 9. Cut-off wavenumbers for $TM_{m,n}$ -modes in the rounded rectangular waveguide of sides $a = 1$ and $b = 2$; rounding radius $\varrho = 0.01$ with various inner conductors with the centre located at $(x_0, y_0) = (0, 0)$ and orientation angle $\alpha = 0^\circ$. Each column shows the cut-off wavenumber for the infinitesimally thin strip of semi-width w , the corresponding cut-off wavenumber and the relative difference $\Delta_{m,n}$ when the strip is replaced by a rounded rectangular strip of semi-width w and thickness $2d$.

	d/w	$TM_{1,1}$	$TM_{1,2}$	$TM_{1,3}$	$TM_{2,1}$	$TM_{2,2}$	$TM_{1,4}$	$TM_{2,3}$
Thin strip	0.000	4.699720	4.804192	6.475592	6.880896	7.024814	7.710194	7.853981
	0.01	4.730638	4.826366	6.506220	6.928066	7.030347	7.749254	7.870873
	$\Delta_{m,n}$	0.006579	0.004616	0.004730	0.006855	0.000788	0.005066	0.002151
Thick strips	0.02	4.754750	4.843135	6.536790	6.972584	7.036501	7.780812	7.888865
	$\Delta_{m,n}$	0.011709	0.008106	0.009451	0.013325	0.001664	0.009159	0.004442
	0.05	4.816639	4.886328	6.631948	7.058027	7.107963	7.867458	7.950136
	$\Delta_{m,n}$	0.024878	0.017097	0.024145	0.025742	0.011836	0.020397	0.012243

The magnitude of the perturbation to the cut-off wavenumbers in a hollow waveguide by (one or more inner conductors basically depends on two factors: conductor location inside the waveguide and its size. It is convenient to measure the conductor size by the ratio η_S of the cross-sectional area S_c of the conductor and the cross-sectional area S_w of the hollow waveguide, i.e., $\eta_S = S_c/S_w$. The shifts in the cut-off wavenumbers can be directly related to the values η_S . The rectangular waveguide with rounded off corners has cross-sectional area $S_w = ab - (4 - \pi)\rho^2$, where ρ is the rounding off radius. Similarly, the cross-sectional area S_{tc} of the conductor of non-zero thickness d and semi-width w is $S_{tc} = 2wd - (4 - \pi)\rho^2$. The cross-sectional area S_{es} of the elliptic strip is $S_{es} = \pi a'b' = \frac{1}{2}wd$.

Even prior to calculation, it is crystal clear that the shifts $\delta_{m,n}$ will be positive because insertion of an inner conductor into the waveguide, whether infinitesimally thin or of non-zero thickness, reduces the “working” cross-sectional area of the waveguide, thus leading to positive shifts in the spectrum.

Table 8 shows that as the elliptic strip thickens, the increase in the shifts $\Delta_{m,n}$ is proportional to the increase in cross-sectional area S_{es} . In all cases presented in Table 8, the semi-width w of the elliptic strip is fixed (at 0.25), so the cross-sectional area S_{es} is proportional

to the parameter d/w ; for example, as d/w increases from 0.005 to 0.05, the cross-sectional area increases tenfold. The same behaviour is observed in the relative shifts $\Delta_{m,n}$; for example, as d/w increases from 0.005 to 0.05, the relative shift $\Delta_{1,1}$ increases very nearly by tenfold, from 0.000635 to 0.006227. Let us denote by $\Delta_{m,n}(d/w)$ the dependence of the relative shift of the $\text{TM}_{m,n}$ -mode on the parameter d/w . The ratios $\Delta_{m,n}(0.01)/\Delta_{m,n}(0.005)$, $\Delta_{m,n}(0.02)/\Delta_{m,n}(0.005)$ and $\Delta_{m,n}(0.05)/\Delta_{m,n}(0.005)$ were computed for the seven modes displayed in Table 8 and found to be in very good agreement with the ratios of the cross-sectional areas, being 2, 4 and 10, respectively: in fact, the arithmetic means of these ratios over the seven modes were 1.984, 3.967 and 10.02, respectively.

Examining the data presented in Table 9 reveals that thick strips (with rounded-off corners) are associated with much more significant relative shifts $\Delta_{m,n}$ than those shown in Table 8 for elliptic strips, at the same values of the parameter d/w . This may be attributed to a significant difference in the cross sections of the elliptic strip and the thick strip: in the former case, the parameter d/w describes its maximum thickness, whereas in the latter, the parameter describes a thickness that is constant over most of the strip. Ignoring the effect of rounding the corners, the ratio S_{tc}/S_{es} approximately equals $4/\pi (\approx 1.273)$.

The simplistic idea of accounting for the difference by multiplication by this ratio in general fails; the shape of the inner conductor and its location influences the values of the shifts $\Delta_{m,n}$ by more than the increase in the cross-sectional area of the thick strip over the elliptic strip. On the one hand, the arithmetic mean of the ratios $\Delta_{m,n}(0.02)/\Delta_{m,n}(0.01)$ computed over the seven modes displayed in Table 9 equals 1.923 and this roughly corresponds to the doubling of the cross-section area (1.923). However, after increasing the parameter d/w to 0.05, the ratios $\Delta_{m,n}(0.05)/\Delta_{m,n}(0.02)$ produce such a spread in values that it renders senseless the use of the arithmetic mean in a providing a cross-sectional area factor in a way comparable that extracted from the ratios $\Delta_{m,n}(0.02)/\Delta_{m,n}(0.01)$. We conclude that the problems studied in this section are multi-parametric and requires comprehensive investigation, especially in addressing some practical problems.

5. Electrically Large Structures

It is natural to ask whether the MAR is effective for structures that are electrically much larger than those discussed in the preceding sections. In other words, can the cut-off wavenumbers be calculated accurately, and, bearing in mind that the spectrum becomes increasingly dense or crowded, can the separate spectral lines be resolved? With what computational effort?

The following example provides positive answers to these questions. Figure 13 displays the spectral dependence of the condition number $\text{cond } A_N(ka)$ for the circular waveguide with an eccentrically embedded circular rod studied with the same parameters ($b/a = d/a = 0.25$) as in Section 3.2 (see Figure 4). A truncation number of $N = 512$ was employed in this figure. The condition number was computed in the interval $90.8 \leq ka \leq 91.2$ using 500 points with increment $\Delta ka = 0.002$. The computation times were approximately the same as for the computations described in Section 3.2 for the same truncation numbers.

It is apparent that the spectral lines are very much more closely packed than those in Figure 5. Nonetheless, the spectrum can be well resolved—with the same computational effort—by restricting the interval of interest to one containing about the same number of lines and by suitably choosing the increment Δka . Fifteen spectral lines can be discerned. Focusing on the mode in the vicinity of the non-dimensional wavenumber $ka = 91.015$, Table 10 displays the sequence of the approximations to the non-dimensional wavenumber γ as a function of increasing truncation number. Convergence to the 3 decimal place estimate of $\gamma = 91.018$ is obtained with a truncation number of 2048. Increasing the truncation number improves the number of correct digits.

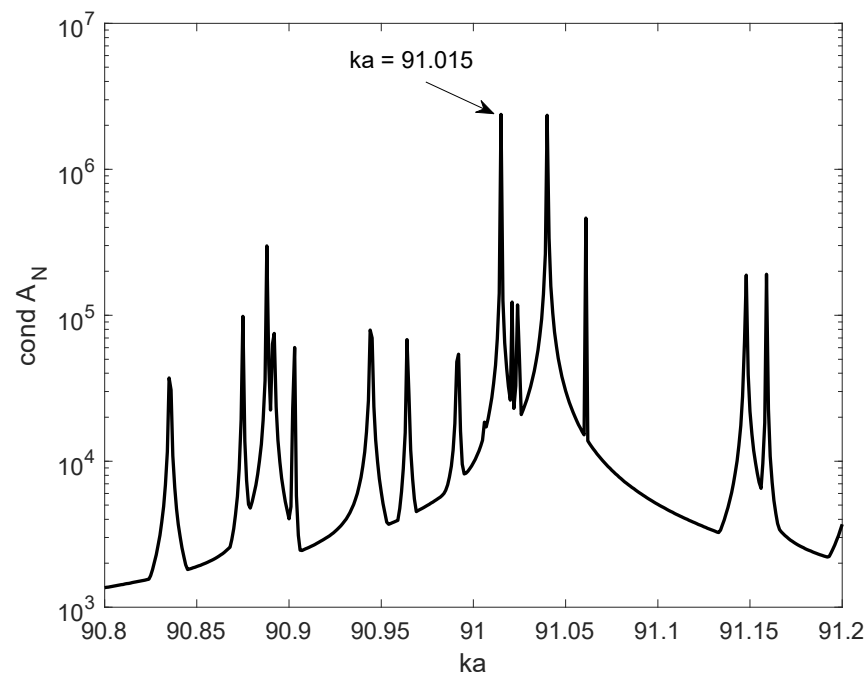


Figure 13. Spectral dependence of the condition number $\text{cond } A_N(ka)$ computed with truncation number $N = 512$ for the circular waveguide with an eccentrically embedded circular rod ($b/a = d/a = 0.25$).

Table 10. Convergence of the approximations to the wavenumber γ of the mode in the vicinity of $ka = 91.015$ as a function of truncation number.

Truncation Number N	Computed Value $\gamma^{(N)}$ of γ
128	$91.026520534035598 - i \cdot 0.000032228513004$
256	$91.022057188386270 - i \cdot 0.001314350150315$
512	$91.020887459853000 - i \cdot 0.000012035120209$
1024	$91.018051214398398 - i \cdot 0.000000181342561$
2048	$91.018431876293164 - i \cdot 0.000000002809210$

We conclude that the MAR is indeed effective for computing cut-off wavenumbers in waveguides that are much larger electrically than those previously discussed. However, it should be noted that, due to the crowded nature of the spectrum, it is no longer possible to make a correspondence between these modes and those of the empty waveguide. Finally, it is worth remarking that the MAR is effective in a range of problems solving in the low-frequency, diffraction and quasi-optics regions. High-frequency scattering by electrically large cavities of electrical size exceeding 100 wavelengths has been investigated in [23].

6. Conclusions

A rigorously correct mathematical approach has been developed and applied to the important radio-engineering problem of high-precision calculation of the cut-off wavenumbers for waveguides of arbitrary cross sections with inner conductors. The initial BVP may be formulated as a boundary IE with a (complex) spectral parameter k , and the problem may be considered as finding the characteristic numbers for an integral operator-valued function (OVF) or its infinite-matrix (summation) OVF.

The Method of Analytical Regularization (MAR) provides an effective tool in this process: its direct application to the ill-conditioned integral equations derived from a single-layer density representation transforms the problem to the determination of non-trivial solutions of a homogeneous well-conditioned system of coupled infinite matrix equations.

The second kind nature of the system guarantees that the solution computed for the system truncated to a system of finite order converges to the exact solution of the infinite system as the truncation number increases indefinitely. In practice, the solutions of the finite (truncated) systems exhibit fast convergence. Calculation of the matrix elements with the required accuracy is realized by use of the Fast Fourier Transform (FFT), making the filling of the matrix an extremely fast and convenient procedure.

A central role in the investigation of the spectrum of cut-off wavenumbers is played by the spectral characteristics of the condition number of the truncated matrices. These computations allow us to find quite accurate initial approximations to the cut-off wavenumbers, which can be subsequently refined to highly accurate values. This is repeatedly exemplified in the results obtained in Sections 3 and 4. Furthermore, the high-resolution power of this spectral characteristic approach ensures that no modes are missed, even though their cut-off wavenumbers are separated by extremely tiny amounts on the wavenumber axis.

Section 3 is devoted to the comparison with, and cross validation of, previously published results obtained by different methods: the hollow elliptical waveguide, the circular waveguide with an inner conductor eccentrically placed, and a circular–rectangular waveguide with one inner circular conductor. In the latter case, the superior accuracy of our approach highlighted the limited accuracy of the previously published results. Because of the second-kind nature of the system of equations obtained by the MAR, we wish to claim that our computed solutions, the accuracy of which are the result of a process with guaranteed convergence, should be regarded as benchmark solutions for the results obtained in this Section.

As an application, Section 4 presented a careful analysis of the cut-off wavenumbers in rectangular waveguides with an inner conductor where an infinitesimally thin strip is replaced by a strip of non-zero thickness. Two classes of thickened strips were considered: those of elliptical cross section and a rectangular thickened strip (with rounded corners). In the first case, it was found that the perturbation to the cut-off wavenumbers could be explained in terms of the area of the elliptical cross section; however, in the latter case, a similar concept failed to provide a consistent explanation for the computed perturbations.

Finally, the MAR is shown to be effective for computing cut-off wavenumbers in electrically large waveguides. In Section 5, the spectrum of a structure that was around ten times larger than those examined in Sections 3 and 4 was successfully resolved for a comparable amount of computational effort.

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