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Data-Driven Containment Control for a Class of Nonlinear Multi-Agent Systems: A Model Free Adaptive Control Approach

Ye Ren ¹ , Honghai Ji ^{1,*}, Deli Li ², Yongqiang Xie ¹, Shuangshuang Xiong ³ and Li Wang ¹

¹ School of Electrical and Control Engineering, North China University of Technology, Beijing 100144, China; yeren@ncut.edu.cn (Y.R.); yqxie66@163.com (Y.X.); li.wang@ncut.edu.cn (L.W.)

² Division of Optical Communications, China Mobile Group Design Institute Co., Ltd., Beijing 100144, China; lideli@cmdi.chinamobile.com

³ School of Automation, Beijing Information Science and Technology University, Beijing 100192, China; shuangshx@bistu.edu.cn

* Correspondence: jihonghai@hotmail.com

Abstract: This paper studies the containment control problem of heterogeneous multi-agent systems (MASs) with multiple leaders. The follower agent dynamics are assumed to be unknown and nonlinear. First, each follower is transformed into an incremental data description based on the dynamic linearization technique. Then, a distributed model-free adaptive containment control law is proposed such that all followers will be driven into the convex hull of the leaders. Furthermore, the algorithm is extended to the time-switching and dynamic leaders case. As a data-driven approach, the proposed controller design uses only the received input and output (I/O) data of these agents rather than agent mathematical models. Finally, to test the potential in real applications, three representative examples considering various environment factors, including external disturbances, are simulated to show the effectiveness and resilience of this method.

Keywords: data-driven control; model free adaptive control; multi-agent systems; containment control



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1. Introduction

Inspired by the coordinated behavior in nature, such as the flocking of birds or migrating of herds, the distributed control of multi-agent systems has been studied with increasing attention [1]. Due to the distributed and flexible structure, the MASs based methods have great potential in solving the coordination problem for complex systems. The applications have been reported in various fields, including vehicle cooperation [2], multi-robot systems [3], and traffic signal control [4]. Until now, numerous algorithms have been designed concerning three types of coordinated tasks, i.e., consensus control, formation control, and containment control [5]. Unlike the single leader setting in consensus or formation control, containment control is regarded as more challenging and requires all the followers to converge to a convex hull spanned by multiple leaders. Containment control also has many potential applications. For instance, a group of robots try to travel across an unsafe tunnel, but only a portion of them are equipped with sensors [6]. Then, the ones with sensors can act as leaders and the others will ensure safety by staying in the area constructed by leaders.

According to the dynamics of agents, the results of containment control can be roughly classified into four categories: (1) the first-order integral system [7,8]; (2) the second-order or high-order integral system [6,9–12]; (3) the general linear system [13–15]; (4) the nonlinear system [16–20]. The pioneering work for a collection of single-integrator agents is investigated in [7], where a “stop-go” rule based hybrid control scheme is designed for leaders, whereas consensus-like local interaction rules are applied for followers. However, the result of [7] is only discussed under a fixed undirected topology. To explore the effect of dynamic communication, the containment problem of single-integrator agents under

switching directed network topologies is considered in [8]. In [9], a type of feedback control law with velocity measurements is proposed for double-integrator MASs and an experimental platform consisting of five wheeled mobile robots is used to validate the result. To reduce the communication burden, [10] developed a finite-time containment control algorithm for double-integrator MASs using only position measurements. In [11], the effects of time-varying delays are taken into consideration, where the linear matrix inequality method and the Lyapunov functional method are jointly used to ensure the containment. In [12], the noise in agent information transmission is formulated and corresponding time-varying state feedback containment control laws are proposed for both first-order and second-order integral MASs. As an extension of [12], ref. [6] considers both measurement noise and polynomial disturbance in high-order integral MASs, where a containment controller is designed with a proportional term and the n th-order integral terms.

Recent work begins to focus on the general linear or nonlinear MASs. In [13], based on the relative outputs of neighboring agents, dynamic observer-type containment controllers are built for both continuous-time and discrete-time general linear MASs. The results of time-varying uncertainties and constant time delays for general linear MASs are shown in [14,15], respectively. As a special kind of nonlinear agents, the rigid body MASs formed by Euler–Lagrange expression is studied in [16], in which a sliding-mode estimator based containment controller is given. Considering the parametric uncertainties of Euler–Lagrange MASs, [17] develops a modified sliding mode containment controller based on the one in [16] without using velocity information. Some results of more general nonlinear agents have also been reported. In [18], the fuzzy logic is utilized to approximate the dynamics of Lipschitz nonlinear MASs and an observer-based containment controller is designed. In [19], sliding mode controllers are proposed to address the finite-time containment problem of MASs with Lipschitz-like nonlinearity and external disturbances. Based on neural networks (NNs) and backstepping techniques, a finite-time containment controller is applied to the high-order nonlinear MASs with dynamic uncertainties in [20].

Although massive achievements have been obtained in [6–20], some limitations still exist for the current algorithms. The containment control schemes used in [6–15] are designed for known linear agents. However, in coordination tasks, the practical MASs always suffer from the effects of unmodelled dynamics and nonlinearities. Thus, the control laws in [6–15] may face performance deterioration in real implementation. The methods presented in [6–20] require that the controlled MASs should be the affine nonlinear system, i.e., the control input term is assumed to be known and linear. Furthermore, the NNs based controller needs extra training processes [20], and the selection of fuzzy membership functions in [18] heavily depends on the experience of users. Therefore, the containment control design for unknown nonlinear MASs is still an open and challenging problem.

Model-free adaptive control (MFAC), initially proposed in [21], is an effective approach to deal with unknown nonlinear systems. With 20 years of development, MFAC has now provided a systematic control framework with a wide range of engineering applications [22–33]. The partial-form dynamic linearization based MFAC controller is designed for a class of single-input and single-output unknown nonlinear system in [22], and it is extended to the multi-input and multi-output system in [23]. Based on the view of an ideal controller, the controller dynamic linearization based MFAC is proposed in [24] and then enhanced by radial basis function NNs in [25]. To address the repetitive tasks over a finite time interval, model-free adaptive iterative learning control (MFAILC) is presented in [26]. It is reported in [27] that MFAC has been applied to over 150 yields. The latest applications and advances can be found in [28–32], including weld penetration [28], steam-water heat exchanger [29], traffic systems [30,31], and the stability analysis of full-form dynamic linearization based MFAC [32]. For the brief review of MFAC, readers are referred to [27,33].

MFAC has also been employed in the coordination of MASs [34–42]. MFAC is first used to address the consensus problem for a group of unknown heterogeneous nonlinear MASs in [34]. Based on the controller dynamic linearization technique and Newton-type

optimization method, a data-driven ideal leader–follower consensus tracking algorithm is designed in [35]. Then, the consensus tracking issue under repetitive environment is solved by using MFILC in [36]. In [37], a MFAC based predictive consensus scheme is presented for MASs with communication constraints. In [38], the consensus problem for the MASs under time delays is solved by the Smith estimator enhanced MFAC scheme. For the formation control of MASs, [39] gives a data-driven formation control scheme based on MFILC. Considering the iterative-varying disturbance, the algorithm in [39] is extended to a robust form formation control law in [40]. In [41], an event-triggered MFAC method is used to perform the containment task where the MASs are suffering from denial-of-service attacks. A model-free adaptive containment control scheme is investigated for a class of multi-input multi-output nonlinear MASs in [42]. However, the MASs in [34–40] only contain one single leader, while no exploration has been conducted for multileaders case. The work in [41,42] only found the result that the defined local containment error is bounded or converged, which is not exactly equivalent to the containment goal. Inspired by above observations, a novel distributed model-free adaptive containment control scheme is proposed in this paper. Compared with existing work, the main contributions of this work are threefold:

- (1) A novel distributed model-free adaptive containment controller is developed. Compared with previous work [34–40], the MASs extend to the multiple-leader case. By virtue of the MFAC framework, this controller is endowed with the data-driven feature, i.e., the design process only utilizes I/O data of the agents rather than their mathematical models. Moreover, this method does not need the extra training process that is required in the NNs based algorithm [20].
- (2) The global containment error is proposed to overcome the analysis nonequivalent problem in [41,42]. Then, the Lyapunov function based stability analysis is conducted to guarantee that the containment goal is achieved, which is different from the conventional contraction mapping principle used in MFAC based protocols (e.g., ref. [35]). In addition, the follower dynamics considered in this paper can contain the nonaffine nonlinear property, whereas the follower dynamics appearing in [6–20] are either affine or linear.
- (3) The control scheme is further extended to a time-switching topology and dynamic leaders case, which makes the controller more applicable and flexible in real implementation.

The remaining part of the paper is organized as follows. Section 2 presents some preliminary knowledge and gives problem formulation. Section 3 shows the main results, including the controller design and stability analysis. Section 4 extends the existing result to a time-switching topology and dynamic leaders case. In Section 5, three numerical examples are provided. Finally, the conclusion is discussed in Section 6.

2. Preliminaries and Problem Formulation

2.1. Symbol Notations and Graph Theory

Let $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote the set of real numbers, $n \times 1$ real vectors, and $n \times m$ real matrices, respectively. The diagonal matrix and identity matrix are denoted by $\text{diag}(\cdot)$ and I ; $\mathbf{1}_n = [1, 1, \dots, 1]^\top \in \mathbb{R}^n$ is a vector with all elements are 1; $\|a\|$ denotes the Euclidean norm for a vector $a \in \mathbb{R}^n$; $\text{card}(S)$ stands for the number of elements in the set S .

The follower and leader agents are presented by nodes in the set $\mathcal{F} = \{1, \dots, N\}$ and $\mathcal{W} = \{N + 1, \dots, N + M\}$, respectively. The communication between these agents is described by a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ whose node set is $\mathcal{V} = \mathcal{F} \cup \mathcal{W}$. The edges in the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ represent the communication links between agents. The expression $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{(N+M) \times (N+M)}$ ($i, j \in \mathcal{V}$) is defined as an adjacency matrix with elements $a_{ij} = 1$ if and only if the directed edge $(j, i) \in \mathcal{E}$, otherwise $a_{ij} = 0$. The in-degree matrix of a graph $\mathcal{G}$ is defined by a diagonal matrix $\mathcal{D} = \text{diag}(d_1, \dots, d_{N+M})$ with diagonal entries $d_i = \sum_{j=1}^{N+M} a_{ij}$. The neighbors of the i th agent are denoted by the set $\mathcal{N}_i = \{j | (j, i) \in \mathcal{E}\}$. A directed path from node i to node j is defined as a sequence of consecutive edges which

begin from node i and end in node j . A directed graph is strongly connected if a path exists between any pair of nodes. The Laplacian matrix is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$. Due to the fact that leaders do not receive any information from other agents, the Laplacian matrix $\mathcal{L}$ can be partitioned as the following block matrix [8]:

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathbf{0}_{M \times N} & \mathbf{0}_{M \times M} \end{bmatrix} \quad (1)$$

where $\mathcal{L}_1 \in \mathbb{R}^{N \times N}$ and $\mathcal{L}_2 \in \mathbb{R}^{N \times M}$.

In order to give the readers a better understanding of this paper, the major notations are listed in Notations part.

2.2. Problem Formulation

The MASs consist of N follower agents and the dynamics of the i th agent is described as:

$$y_i(k+1) = f_i(y_i(k), u_i(k)) \quad (2)$$

where $y_i(k) \in \mathbb{R}$, $u_i(k) \in \mathbb{R}$ are the agent output and control input of the i th agent at time instant k , respectively, and $f_i(\cdot)$ is an unknown nonlinear function.

Denote the output of the leader as $w_l(k) \in \mathbb{R}$, $l = N+1, \dots, N+M$, which can only be received by a subset of follower agents.

Definition 1 (Convex Set and Convex Hull [43]). A set $\mathcal{C} \subseteq \mathbb{R}^n$ is said to be convex if $(1-\alpha)x + \alpha y \in \mathcal{C}$, for any $\alpha \in [0, 1]$ and $x, y \in \mathcal{C}$. For a finite set of points $\mathcal{W} = \{w_1, w_2, \dots, w_q\} \in \mathbb{R}^n$, the convex hull $\text{Co}(\mathcal{W})$ is the minimal convex set containing all points in $\mathcal{W}$, that is, $\text{Co}(\mathcal{W}) = \{\sum_{i=1}^q \alpha_i w_i | w_i \in \mathcal{W}, \alpha_i \geq 0, \sum_{i=1}^q \alpha_i = 1\}$. In particular, when $\mathcal{W} \subseteq \mathbb{R}$, $\text{Co}(\mathcal{W}) = [\min_i w_i, \max_i w_i]$.

Definition 2 (Containment Control [10]). The MASs achieve containment if all followers are driven by a distributed control law and move into the convex hull spanned by the leader set $\mathcal{W}$, that is, $\forall i \in 1, \dots, N$, there exist an element $\bar{w}_i \in \text{Co}(\mathcal{W})$, such that

$$\lim_{k \rightarrow \infty} \|y_i(k) - \bar{w}_i\| = 0.$$

where $\bar{w}_i$ is the convergence value for the i th follower agent.

The following assumptions are given before further discussions.

Assumption 1. The partial derivative of $f_i(\cdot)$ with respect to $u_i(k)$ is continuous.

Assumption 2. For all k with finite exceptions, if $\|\Delta u_i(k)\| \neq 0$, the follower agent dynamics (2) satisfies the following generalized Lipschitz condition, that is

$$|\Delta y_i(k+1)| \leq b |\Delta u_i(k)|$$

where $\Delta y_i(k+1) = y_i(k+1) - y_i(k)$, $\Delta u_i(k) = u_i(k) - u_i(k-1)$ and b is a positive constant.

Remark 1. Assumption 1 is a general condition for controller design. Assumption 2 gives a bound of the change rate of the agent's output, i.e., the agent's output change should be finite if the control input change is bounded. A wide range of real agent systems satisfy the assumptions. For instance, the velocity change of an autonomous car must be finite with a bounded change of the accelerator. Similar assumptions can also be found in [22–29,32,33].

Under Assumptions 1 and 2, the unknown agent dynamics (2) can be transformed into the following dynamic linearization model and then the distributed control law will be designed based on it.

Lemma 1 ([44]). Consider follower agents with dynamics (2) satisfying Assumptions 1 and 2. If $\|\Delta u_i(k)\| \neq 0$ holds, then agent (2) can be transformed into a compact form dynamic linearization (CFDL) data model:

$$\Delta y_i(k+1) = \phi_i(k) \Delta u_i(k) \quad (3)$$

where the time-varying variable $\phi_i(k) \in \mathbb{R}$ is named as the pseudo partial derivative (PPD), which is bounded for any time instant k .

Remark 2. By virtual of CFDL technique, the unknown nonlinear agent dynamics (2) are now transformed into an incremental form data description (3) in every operation point with a bounded PPD. In implementation, with proper selected algorithms, PPD can be estimated by using the I/O data of the agents and the estimated value can also be proved to be bounded, which is shown in Theorem 1. Here, the CFDL data model is derived under the condition $\|\Delta u_i(k)\| \neq 0$. In fact, if the case $\|\Delta u_i(k)\| = 0$ happens at some sampling time, a new CFDL data model can also be established after shifting $\gamma_i \in \mathbb{Z}^+$ time steps until $u_i(k + \gamma_i) \neq u_i(k)$ holds. In the next section, the data model (3) will be used to design the distributed control law and derive the stability analysis result.

3. Main Results

In this section, the data driven containment control law is first designed under the stationary leaders and fixed topology case.

Let $\xi_i(k)$ denote the local containment error:

$$\xi_i(k) = \sum_{j=1}^N a_{ij}(y_j(k) - y_i(k)) + \sum_{l=N+1}^{N+M} a_{il}(w_l(k) - y_i(k)) \quad (4)$$

where a_{ij} ($i, j \in \mathcal{V}$) is the entry in the adjacency matrix, and $y_i(k)$ and $w_l(k)$ denote the outputs of the follower agent i and leader agent l , respectively. From this definition, $\xi_j(k)$ can be regarded as the tracking error of the i th agent with all its neighboring agents at time instant k .

The distributed model free adaptive containment control (MFACC) is designed as follows:

Distributed PPD updating algorithm:

$$\hat{\phi}_i(k) = \hat{\phi}_i(k-1) + \frac{\eta \Delta u_i(k-1)}{\mu + |\Delta u_i(k-1)|^2} \times (\Delta y_i(k) - \hat{\phi}_i(k-1) \Delta u_i(k-1)) \quad (5)$$

PPD reset mechanism:

$$\begin{aligned} \hat{\phi}_i(k) &= \hat{\phi}_i(1), \text{ if } |\hat{\phi}_i(k)| \leq \varepsilon \\ &\text{or } |\Delta u_i(k-1)| \leq \varepsilon \\ &\text{or } \text{sign}(\hat{\phi}_i(k)) \neq \text{sign}(\hat{\phi}_i(1)) \end{aligned} \quad (6)$$

Distributed containment input law:

$$u_i(k) = u_i(k-1) + \frac{\rho_i \hat{\phi}_i(k)}{\lambda + |\hat{\phi}_i(k)|^2} \xi_i(k) \quad (7)$$

where $\rho_i \in (0, 1]$, $\eta \in (0, 2]$, $\mu > 0$ and $\lambda > 0$ are adjustable parameters, $\hat{\phi}_i(k)$ is the estimated value of $\phi_i(k)$, $\hat{\phi}_i(1)$ is the initial value of $\hat{\phi}_i(k)$, and $\varepsilon > 0$ is a preset small constant.

Remark 3. The PPD is estimated by the distributed updating algorithm (5) and (6), which is the well-known projection algorithm [22–33] and derived by minimizing the following performance index $J(\hat{\phi}_i(k)) = |\Delta y_i(k) - \hat{\phi}_i(k) \Delta u_i(k-1)|^2 - \mu |\hat{\phi}_i(k) - \hat{\phi}_i(k-1)|^2$. With this distributed

algorithm, the PPDs of follower agents are estimated by using their own I/O data. Theoretical results show these estimated values will be bounded if some parameters are appropriately chosen, which is presented in the next section. Furthermore, due to the heterogeneity of the agent dynamics, the trajectories of the estimated PPD $\hat{\phi}_i(k)$ may vary from different follower agents.

Remark 4. The distributed containment control law (7) is derived by minimizing the following index $J(u_i(k)) = |\sum_{j=1}^N a_{ij}(y_j(k) - y_i(k+1)) + \sum_{l=N+1}^{N+M} a_{il}(w_l(k) - y_i(k+1))|^2 + \lambda |\Delta u_i(k)|^2$. As we can see, this performance index tries to decrease the errors between the i th agent and its neighbors. It will be proved in the following part that the MASs will finally achieve the containment control goal. Observing the control scheme (5)–(7), one will find it only uses the local I/O data to estimate the PPD and calculate the input for each follower agent, i.e., no model information of $f_i(\cdot)$ is involved; that is why it is called a data-driven method. The control gain in (7) is also adaptively tuned by $\hat{\phi}_i(k)$, which is continuously updated by the new received I/O data. In addition, the protocols (5)–(7) will regress to the consensus control law in [34] when there is only one leader.

The following assumptions and lemma are prepared before the main result.

Assumption 3. For all follower agents, the sign of the PPD $\phi_i(k)$ ($i = 1, \dots, N$) is unchanged, i.e., $\phi_i(k) > \bar{\varepsilon} > 0$ (or $\phi_i(k) < -\bar{\varepsilon} < 0$), where $\bar{\varepsilon}$ is an arbitrarily small positive constant.

Without loss of generality, it is assumed that $\phi_i(k) > \bar{\varepsilon} > 0$ in the discussion.

Remark 5. Assumption 3 indicates that the follower agent's output incremental direction caused by its input do not change during the containment control process. Similar assumptions can also be found in model-based control theory.

Assumption 4. The communication graph among followers is strongly connected, and each follower can receive the information through a direct path from at least one leader. Assumption 4 guarantees that there are no isolated followers and the containment control problem is well-posed.

Lemma 2 ([45]). For a dynamic system $x(k+1) = f(x(k))$, $V(k) = g(x(k))$ is a positive definite and radially unbounded Lyapunov function. If $\Delta V(k) \leq 0$ and $\Delta V(k) \neq 0$ for any $x(k) \neq 0$, then the equilibrium $x_e = 0$ is global asymptotic stable.

In the stationary leader case, the trajectory of the l th leader is written as $w_l(k) = w_l$ ($l \in \mathcal{W}$), where w_l is a constant. The main result is summarized as Theorem 1.

Theorem 1. Assume that the MASs dynamics satisfies Assumptions 1–3 and the communication topology satisfies Assumption 4. In the stationary leaders case, that is, $w_l(k) = w_l$ ($l \in \mathcal{W}$), let the distributed MFACC (5)–(7) be applied. If the parameter ρ_i ($i = 1, \dots, N$) is selected to satisfy the following convergence condition

$$0 < \rho_i < \frac{1}{d_i}$$

then there exists a $\lambda_{\min} > b^2/4$, and the MASs will achieve the containment control objective defined in Definition 2 when $\lambda > \lambda_{\min}$ and $k \rightarrow \infty$.

Proof. The proof consists of three steps. Step one is to prove that the PPD estimation is bounded. Step two is to derive the convergence condition. Step three is to prove the convergence of the global containment error.

Step 1 The boundedness of $\hat{\phi}_i(k)$ is straightforward if $\|\hat{\phi}_i(k)\| \leq \varepsilon$ or $\|\Delta u_i(k)\| \leq \varepsilon$ or $\text{sign}(\hat{\phi}_i(k)) \neq \text{sign}(\hat{\phi}_i(1))$.

In the other case, the PPD estimation error is defined as $\tilde{\phi}_i(k) = \hat{\phi}_i(k) - \phi_i(k)$. Subtracting $\phi_i(k)$ from both sides of the PPD estimation algorithm (5) yields

$$\begin{aligned}\tilde{\phi}_i(k) &= \tilde{\phi}_i(k-1) - (\phi_i(k) - \phi_i(k-1)) \\ &\quad + \frac{\eta \Delta u_i(k-1)}{\mu + |\Delta u_i(k-1)|^2} \times (\Delta y_i(k-1) \\ &\quad - \hat{\phi}_i(k-1) \Delta u_i(k-1))\end{aligned}\quad (8)$$

Define $\Delta \phi_i(k) = \phi_i(k) - \phi_i(k-1)$. Then, substituting CFDL data description (3) into (8) gives

$$\begin{aligned}\tilde{\phi}_i(k) &= \left(1 - \frac{\eta |\Delta u_i(k-1)|^2}{\mu + |\Delta u_i(k-1)|^2}\right) \\ &\quad \times \tilde{\phi}_i(k-1) - \Delta \phi_i(k)\end{aligned}\quad (9)$$

From Lemma 1, it can be known that $\phi_i(k)$ is bounded, and assume the bound of it as $\bar{b}$. Taking the norm on both sides of (9) leads to

$$\begin{aligned}|\tilde{\phi}_i(k)| &\leq \left| \left(1 - \frac{\eta |\Delta u_i(k-1)|^2}{\mu + |\Delta u_i(k-1)|^2}\right) \tilde{\phi}_i(k-1) \right| + |\Delta \phi_i(k)| \\ &\leq \left| \left(1 - \frac{\eta |\Delta u_i(k-1)|^2}{\mu + |\Delta u_i(k-1)|^2}\right) \tilde{\phi}_i(k-1) \right| + 2\bar{b}\end{aligned}\quad (10)$$

Note the fact that the term $\eta |\Delta u_i(k-1)|^2 / (\mu + |\Delta u_i(k-1)|^2)$ is monotonically increasing with respect to $|\Delta u_i(k-1)|$. Thus, there must exist a constant δ_1 to satisfy the following inequality when $0 < \eta < 2$ and $\mu > 0$:

$$0 \leq \left(1 - \frac{\eta |\Delta u_i(k-1)|^2}{\mu + |\Delta u_i(k-1)|^2}\right) \leq 1 - \frac{\eta \varepsilon^2}{\mu + \varepsilon^2} = \delta_1 < 1 \quad (11)$$

Substituting (11) into (10) gives

$$\begin{aligned}|\tilde{\phi}_i(k)| &\leq \delta_1 |\tilde{\phi}_i(k-1)| + 2\bar{b} \\ &\leq \delta_1^2 |\tilde{\phi}_i(k-2)| + 2\delta_1 \bar{b} + 2\bar{b} \\ &\leq \dots \\ &\leq \delta_1^{k-1} |\tilde{\phi}_i(1)| + \frac{2\bar{b}(1 - \delta_1^{k-1})}{1 - \delta_1}\end{aligned}\quad (12)$$

which means $\tilde{\phi}_i(k)$ is bounded. Since the boundedness of $\phi_i(k)$ is guaranteed by Lemma 1, $\hat{\phi}_i(k)$ is also bounded.

Step 2 Define the collective stack vectors as follows:

$$\begin{aligned}\mathbf{y}(k) &= [y_1(k), y_2(k), \dots, y_N(k)]^T \\ \mathbf{u}(k) &= [u_1(k), u_2(k), \dots, u_N(k)]^T \\ \boldsymbol{\xi}(k) &= [\xi_1(k), \xi_2(k), \dots, \xi_N(k)]^T \\ \mathbf{w}_s &= [w_1, w_2, \dots, w_N]^T\end{aligned}$$

Now, (4) is rewritten as the collective form:

$$\boldsymbol{\xi}(k) = -\mathcal{L}_1 \mathbf{y}(k) - \mathcal{L}_2 \mathbf{w}(k) \quad (13)$$

Using the collective stack vectors, (7) is rewritten as the following compact form:

$$\mathbf{u}(k) = \mathbf{u}(k-1) + \boldsymbol{\rho} \mathbf{H}(k) \boldsymbol{\xi}(k) \quad (14)$$

where

$$\boldsymbol{\rho} = \text{diag}(\rho_1, \dots, \rho_N)$$

$$\mathbf{H}(k) = \text{diag}\left(\frac{\hat{\phi}_1(k)}{\lambda + |\hat{\phi}_1(k)|^2}, \dots, \frac{\hat{\phi}_N(k)}{\lambda + |\hat{\phi}_N(k)|^2}\right)$$

Similarly, the CFDL model (3) is also transformed as the collective form

$$\mathbf{y}(k+1) = \mathbf{y}(k) + \boldsymbol{\Phi}(k) \Delta \mathbf{u}(k) \quad (15)$$

where $\Delta \mathbf{u}(k) = \mathbf{u}(k) - \mathbf{u}(k-1)$, and $\boldsymbol{\Phi}(k) = \text{diag}(\phi_1(k) \dots \phi_N(k))$.

Substituting (14) into (15), one has

$$\mathbf{y}(k+1) = \mathbf{y}(k) + \boldsymbol{\rho} \boldsymbol{\Phi}(k) \mathbf{H}(k) \boldsymbol{\xi}(k) \quad (16)$$

Combining (13) and (16) yields

$$\begin{aligned} \mathbf{y}(k+1) &= (\mathbf{I} - \boldsymbol{\rho} \boldsymbol{\Phi}(k) \mathbf{H}(k) \mathcal{L}_1) \mathbf{y}(k) \\ &\quad - \boldsymbol{\rho} \boldsymbol{\Phi}(k) \mathbf{H}(k) \mathcal{L}_2 \mathbf{w}_s \end{aligned} \quad (17)$$

Denote $\boldsymbol{\Omega}(k) = [\mathbf{y}(k)^T \mathbf{w}_s^T]^T$ and

$$\boldsymbol{\Gamma}(k) = \begin{bmatrix} \mathbf{I} - \boldsymbol{\rho} \boldsymbol{\Phi}(k) \mathbf{H}(k) \mathcal{L}_1 & -\boldsymbol{\rho} \boldsymbol{\Phi}(k) \mathbf{H}(k) \mathcal{L}_2 \\ \mathbf{0}_{N \times N} & \mathbf{I} \end{bmatrix}$$

Then, (17) can be rewritten as

$$\boldsymbol{\Omega}(k+1) = \boldsymbol{\Gamma}(k) \boldsymbol{\Omega}(k) \quad (18)$$

The i th ($i \in \mathcal{F}$) diagonal entry $\gamma_{ii}(k)$ of $\boldsymbol{\Gamma}(k)$ can be expressed as

$$\gamma_{ii}(k) = 1 - d_i \cdot \rho_i \cdot \frac{\phi_i(k) \hat{\phi}_i(k)}{\lambda + |\hat{\phi}_i(k)|^2} = 1 - \frac{d_i \rho_i \phi_i(k) \hat{\phi}_i(k)}{\lambda + |\hat{\phi}_i(k)|^2} \quad (19)$$

If the condition $0 < \rho_i < \frac{1}{d_i}$ holds, one has $0 < d_i \rho_i < 1$. According to Assumption 3 and reset condition (6), $\phi_i(k) \hat{\phi}_i(k) > 0$, which means $\frac{\phi_i(k) \hat{\phi}_i(k)}{\lambda + |\hat{\phi}_i(k)|^2} > 0$ when the parameters satisfy $\lambda > 0$. Since $\hat{\phi}_i(k)$ and $\phi_i(k)$ are bounded, we choose $\lambda > \lambda_{\min}$ ($\lambda_{\min} > b^2/4$) to satisfy the following inequality:

$$0 < \frac{\phi_i(k) \hat{\phi}_i(k)}{\lambda + |\hat{\phi}_i(k)|^2} \leq \frac{b |\hat{\phi}_i(k)|}{2\sqrt{\lambda} |\hat{\phi}_i(k)|} < \frac{b}{2\sqrt{\lambda_{\min}}} < 1 \quad (20)$$

which means $0 < \gamma_{ii}(k) < 1$ when $0 < \rho_i < \frac{1}{d_i}$ and $\lambda > \lambda_{\min}$.

By definition of the Laplacian matrix, one has $\mathcal{L}_1 \mathbf{1}_N + \mathcal{L}_2 \mathbf{1}_M = 0$. Note the fact that $\boldsymbol{\Gamma}(k) \geq 0$ and $\boldsymbol{\Gamma}(k) \mathbf{1}_{N+M} = \mathbf{1}_{N+M}$, i.e., $\boldsymbol{\Gamma}(k)$ is a stochastic matrix.

Step 3 Prove the convergence of the global containment error.

Define the upper bounds of the followers, leaders, and the MASs as follows:

$$\begin{aligned}y^+(k) &= \max_i y_i(k) \\w^+ &= \max_l w_l \\\Omega^+(k) &= \max\{y^+(k), w^+\}\end{aligned}$$

Note that one can then define $y^-(k) = \min_i y_i(k)$, $w^- = \min_l w_l$, and $\Omega^-(k) = \min\{y^-(k), w^-\}$ for the lower bound.

For the upper bound, $\Omega^+(k) = w^+$ means all the followers goes into the convex hull of the leaders. Therefore, define the upper global containment error as follows:

$$e^+(k) = \Omega^+(k) - w^+$$

(Define $e^-(k) = w^- - \Omega^-(k)$ for the lower global containment error). Then, one considers the following Lyapunov function $V_1(k) = e^+(k)^2$ for further analysis. The time difference can be written as

$$\begin{aligned}\Delta V_1(k+1) &= V_1(k+1) - V_1(k) \\&= (e^+(k+1) - e^+(k))(e^+(k+1) + e^+(k)) \\&= (\Omega^+(k+1) - \Omega^+(k))(\Omega^+(k+1) + \Omega^+(k) - 2w^+) \quad (21)\end{aligned}$$

Rewrite $y_i(k+1)$ as

$$\begin{aligned}y_i(k+1) &= \sum_{j:y_j(k)=y^+(k)} \gamma_{ij}(k)y_j(k) + \sum_{j:y_j(k)<y^+(k)} \gamma_{ij}(k)y_j(k) \\&\quad + \sum_{l=N+1}^{N+M} \gamma_{il}(k)w_l \quad (22)\end{aligned}$$

where $\gamma_{ij}(k)$ is the entries of $\Gamma(k)$.

According to Lemma 2, two parts need to be proved: (1) $e^+(k) = 0$ is an equilibrium point; (2) For any $e^+(k) \neq 0$, $\Delta V_1(k) \leq 0$ and $\Delta V_1(k) \neq 0$.

(1) $e^+(k) = 0$, i.e., $y^+(k) \leq w^+$.

Substituting $y_j(k) \leq w^+$ and $w_l \leq w^+$ into (22), one derives

$$y_i(k+1) \leq \sum_{j=1}^{N+M} \gamma_{ij}(k)w^+ = w^+ \quad (23)$$

Therefore, $\Omega^+(k+1) = \max\{y^+(k+1), w^+\} = w^+ = \Omega^+(k)$, i.e., $\Delta V_1(k+1) = 0$ in this case. In other words, the $e^+(k) = 0$ is the equilibrium point.

(2) $e^+(k) \neq 0$, i.e., $y^+(k) > w^+$.

Denote $s_i(k) = \sum_{j:y_j(k)=y^+(k)} \gamma_{ij}(k)$. For the i th follower, $s_i(k) = 1$ means the outputs of the i th follower and its neighbors are all equal to $y^+(k)$, which also means only the first term in the right side of (22) exists. Substituting $y_j(k) = y^+(k)$ into (22) gives

$$y_i(k+1) = s_i(k)y^+(k) = y^+(k) \quad \text{if } s_i(k) = 1 \quad (24)$$

Otherwise, if $s_i(k) \neq 1$, the following results can be concluded from Equation (22):

$$\begin{aligned} y_i(k+1) &< \sum_{j:y_j(k)=y^+(k)} \gamma_{ij}(k)y^+(k) + \sum_{j:y_j(k)<y^+(k)} \gamma_{ij}(k)y^+(k) \\ &+ \sum_{l=N+1}^{N+M} \gamma_{il}(k)y^+(k) \\ &= s_i(k)y^+(k) + (1-s_i(k))y^+(k) \\ &= y^+(k) \quad \text{if } s_i(k) \neq 1 \end{aligned} \quad (25)$$

Combining (24) and (25) gives

$$\Omega^+(k+1) - \Omega^+(k) \leq \max\{y^+(k), w^+\} - \Omega^+(k) = 0 \quad (26)$$

It is clear that $\Omega^+(k+1) + \Omega^+(k) - 2w^+ > 0$. Therefore, $\Delta V_1(k+1) \leq 0$ holds. Now, one only needs to prove that $\Delta V_1(k+1) \neq 0$.

Assume that there exists a time instant K_1 such that $\Delta V_1(k+1) \equiv 0$ when $k > K_1$.

Define $\mathcal{S}_1(h) = \{i | y_i(K_1+h) = y^+(K_1)\}$ and $N_1 = \text{card}(\mathcal{S}_1(0))$. Since $\Delta V_1(k+1) \equiv 0$, it means $y^+(K_1+h) = y^+(K_1)$ for all $h > 0$, i.e., $\text{card}(\mathcal{S}_1(h)) > 0$ must hold for all $h > 0$.

According to Assumption 4, at least one agent in $\mathcal{S}_1(h)$ whose neighbor's output is less than $y^+(K_1)$, i.e., there exist a certain follower $y_p(K_1+h)$ satisfying $y_p(K_1+h) \in \mathcal{S}_1(h)$ and $s_p(K_1+h) \neq 1$. From (25), one derives $y_p(K_1+h+1) < y^+(K_1)$, which implies

$$\text{card}(\mathcal{S}_1(h+1)) \leq \text{card}(\mathcal{S}_1(h)) - 1 \quad (27)$$

Repeating the process in (27), one gets

$$\begin{aligned} \text{card}(\mathcal{S}_1(N_1)) &\leq \text{card}(\mathcal{S}_1(N_1-1)) - 1 \\ &\leq \text{card}(\mathcal{S}_1(N_1-2)) - 2 \\ &\dots \\ &\leq \text{card}(\mathcal{S}_1(0)) - N_1 = 0 \end{aligned} \quad (28)$$

which contracts to $\text{card}(\mathcal{S}_1(h)) > 0$. Therefore, the assumption is violated.

According to Lemma 2, one concludes that $e^+(k)$ will converge to 0. This completes the proof. $\square$

Remark 6. According to the analysis in Theorem 1, if there is only one leader, all followers will converge to the leader's output. In other words, consensus is a special case of the containment control discussed in this paper.

4. Extension to Switching Topologies and Dynamic Leaders

Due to transmission failure and other factors, the communication topology between agents may change in the containment process. On the other hand, the trajectories of leaders usually change over time in practical agent coordination tasks. Therefore, the proposed containment design is extended to the switching topologies and dynamic leaders situation.

A switching graph $\mathcal{G}(k)$ associated with time instant k is now used to indicate the communication topology among agents. The graph switches from the set $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_\psi\}$, where $\psi \in \mathbb{Z}^+$ is the total number of possible graphs. The time-varying adjacency matrix and the Laplacian matrix are denoted by $\mathcal{A}(k) = [a_{ij}(k)] \in \mathbb{R}^{(N+M) \times (N+M)}$ ($i, j \in \mathcal{V}$) and $\mathcal{L}(k)$, respectively. The two blocks in (1) are also time-varying and written as $\mathcal{L}_1(k)$ and $\mathcal{L}_2(k)$. The in-degree matrix in time instant k is denoted by $\mathcal{D}(k) = \text{diag}(d_1(k), \dots, d_{N+M}(k))$ with diagonal entries $d_i(k) = \sum_{j=1}^{N+M} a_{ij}(k)$. For a specific interaction graph $\mathcal{G}_l$ in the set $\mathcal{G}$, the corresponding in-degree matrix is defined as $\mathcal{D}^l = \text{diag}(d_1^l, \dots, d_{N+M}^l)$. The neighbors of the i th agent in time instant k are denoted by the set $\mathcal{N}_i(k)$.

Assumption 5. For every graph in $\mathcal{G}$, the communication graph among followers is strongly connected and each follower can receive the information via a directed path from at least one leader.

Next, we consider the time varying leaders. Assume that the trajectories of leaders have lower and upper bounds, that is

$$\tilde{w}^- \leq w_l(k) \leq \tilde{w}^+, l = N + 1, \dots, N + M$$

where $\tilde{w}^-$ and $\tilde{w}^+$ are two constants.

Considering the above changes, the local containment error in (4) is modified as

$$\begin{aligned} \bar{\xi}_i(k) = & \sum_{j=1}^N a_{ij}(k)(y_j(k) - y_i(k)) \\ & + \sum_{l=N+1}^{N+M} a_{il}(k)(w_l(k) - y_i(k)) \end{aligned} \quad (29)$$

The control input (7) is extended as

Extended Distributed containment input law:

$$u_i(k) = u_i(k-1) + \frac{\rho_i \hat{\phi}_i(k)}{\lambda + |\hat{\phi}_i(k)|^2} \bar{\xi}_i(k) \quad (30)$$

where $\rho_i \in (0, 1]$, $\eta \in (0, 2]$, $\mu > 0$, $\lambda > 0$ are parameters.

Now, Equations (5), (6), and (30) consist the extended algorithm, and the result is given as the following theorem.

Theorem 2. In the switching topologies and dynamic leaders case, assume that the MASs dynamics satisfy Assumptions 1–3 and the communication topology satisfies Assumption 5. Let the distributed MFACC (5), (6), and (30) be applied. If the parameter ρ_i ($i = 1, \dots, N$) is selected to satisfy the following convergence condition

$$0 < \rho_i < \frac{1}{\max_{l=1,2,\dots,\psi} d_i^l}$$

then there exists a $\lambda_{min} > b^2/4$, and the MASs will converges to $[\tilde{w}^-, \tilde{w}^+]$ when $\lambda > \lambda_{min}$ and $k \rightarrow \infty$.

Proof. Define the time-varying leaders' output collective stack vectors as $\mathbf{w}(k) = [w_1(k), w_2(k), \dots, w_N(k)]^T$.

In this case, Equation (17) becomes

$$\mathbf{y}(k+1) = \tilde{\mathbf{T}}(k) \begin{bmatrix} \mathbf{y}(k) \\ \mathbf{w}(k) \end{bmatrix} \quad (31)$$

where

$$\begin{aligned} \tilde{\mathbf{T}}(k) = & \begin{bmatrix} I - \rho \Phi(k) \mathbf{H}(k) \mathcal{L}_1(k) & -\rho \Phi(k) \mathbf{H}(k) \mathcal{L}_2(k) \end{bmatrix} \\ = & [\tilde{\gamma}_{ij}] \in \mathbb{R}^{N \times (N+M)}. \end{aligned}$$

The $\tilde{\gamma}_{ii}(k)$ is now written as

$$\tilde{\gamma}_{ii}(k) = 1 - \frac{d_i(k) \rho_i \phi_i(k) \hat{\phi}_i(k)}{\lambda + |\hat{\phi}_i(k)|^2} \quad (32)$$

If the condition $0 < \rho_i < \frac{1}{\max_{l=1,2,\dots,\psi} d_i^l}$ holds, one has $0 < d_i(k)\rho_i < 1$. From the analysis in (18)–(20), it is easy to verify that $\tilde{\Gamma}(k)$ also is a stochastic matrix when $\lambda > \lambda_{min}$ ($\lambda_{min} > b^2/4$).

Define $\tilde{\Omega}(k) = \max\{y^+(k), \tilde{w}^+\}$ and $V_2(k) = \tilde{e}^+(k)^2$, where $\tilde{e}^+(k) = \tilde{\Omega}(k) - \tilde{w}^+$.

$$\Delta V_2(k+1) = (\tilde{\Omega}(k+1) - \tilde{\Omega}(k))(\tilde{\Omega}(k+1) + \tilde{\Omega}(k) - 2\tilde{w}^+) \quad (33)$$

Rewrite $y_i(k+1)$ as

$$\begin{aligned} y_i(k+1) = & \sum_{j:y_j(k)=y^+(k)} \tilde{\gamma}_{ij}(k)y_j(k) + \sum_{j:y_j(k)<y^+(k)} \tilde{\gamma}_{ij}(k)y_j(k) \\ & + \sum_{l=N+1}^{N+M} \tilde{\gamma}_{il}(k)w_l(k) \end{aligned} \quad (34)$$

Following the same steps from (23) to (26), one concludes

$$\Delta V_2(k+1) = 0, \text{ if } \tilde{e}^+(k) = 0.$$

$$\Delta V_2(k+1) \leq 0, \text{ if } \tilde{e}^+(k) \neq 0.$$

One only needs to prove that there does not exist a time instant K_2 such that $\Delta V_2(k+1) \equiv 0$.

Assume that $\Delta V_2(k+1) \equiv 0$ when $k > K_2$.

Define $\mathcal{S}_2(h) = \{i | y_i(K_2+h) = y^+(K_2)\}$ and $N_2 = \text{card}(\mathcal{S}_2(0))$. Since $\Delta V_2(k+1) \equiv 0$, it means $y^+(K_2+h) = y^+(K_2)$ for all $h > 0$, i.e., $\text{card}(\mathcal{S}_2(h)) > 0$ must hold for all $h > 0$.

According to Assumption 5, at least one agent in $\mathcal{S}_2(h)$ whose neighbor's output is less than $y^+(K_2)$, i.e., there exists a certain follower $y_p(K_2+h)$ satisfying $y_p(K_2+h) \in \mathcal{S}_2(h)$ and $s_p(K_2+h) \neq 1$. From (27), one gets

$$\begin{aligned} \text{card}(\mathcal{S}_2(N_2)) & \leq \text{card}(\mathcal{S}_2(N_2-1)) - 1 \\ & \leq \text{card}(\mathcal{S}_2(N_2-2)) - 2 \\ & \dots \\ & \leq \text{card}(\mathcal{S}_2(0)) - N_2 = 0 \end{aligned} \quad (35)$$

which contracts to $\text{card}(\mathcal{S}_2(h)) > 0$. Therefore, the assumption is violated. In summary, one concludes that $\tilde{e}^+(k)$ converges to 0. This completes the proof. $\square$

Remark 7. When the leaders become time-invariant after time instant, one has $\tilde{w}^+ = \max_l w_l$ and $\tilde{w}^- = \min_l w_l$, and the result in Theorem 2 will regress to the result in Theorem 1.

5. Simulation

This section gives three numerical examples to verify the theoretical results. The example settings and parameter selections are shown in Table 1. It is noteworthy that the agent dynamics presented in the simulation are only used to generate the I/O data and the controller design process contains no model information.

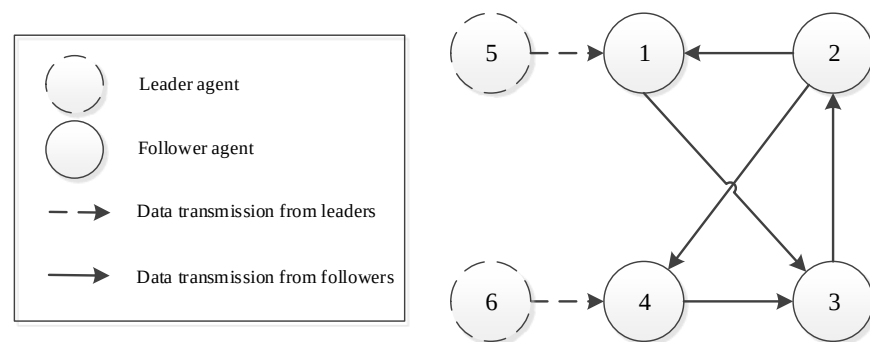
All tests in this paper are conducted in MATLAB R2018a with the Intel CPU Core (TM) i7-11800H @ 2.30 GHz (Intel, Santa Clara, CA, USA). The sample time is chosen as 0.1 s. The time step is set to be 500 for Example 1 and 1000 for Examples 2–3.

Table 1. Example settings and parameter selections.

	Example 1	Example 2	Example 3
Leader number	2	3	3
Leader trajectories	stationary	time-varying	stationary and time-varying
Topology	fixed	time-switching	time-switching
Theorem validation	Theorem 1	Theorem 2	Theorems 1–2
Disturbance	none	none	additive channel disturbance
Parameters	$\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0.3, \lambda \in [0.5 \ 5], \eta = 0.5, \mu = 1, \varepsilon = 0.0001$		

5.1. Example 1

Four follower agents are labeled as agents 1–4 and two leaders are labeled as agents 5–6, i.e., $\mathcal{F} = \{1, 2, 3, 4\}$ and $\mathcal{W} = \{5, 6\}$. The communication network is fixed in this case as shown in Figure 1, in which the topology among followers is strongly connected. Moreover, the information from agents 5–6 can only be received by agent 1 and agent 4, respectively.

**Figure 1.** Communication graph in Example 1.

The dynamics of four followers is described as

$$\text{Agent 1 : } y_1(k+1) = 0.7y_1(k) + u_1(k)$$

$$\text{Agent 2 : } y_2(k+1) = 0.6\cos(y_2(k)) + 0.3y_2(k) + u_2(k)$$

$$\text{Agent 3 : } y_3(k+1) = \frac{y_3(k)}{1 + y_3^2(k)} + u_3(k)$$

$$\text{Agent 4 : } y_4(k+1) = \frac{y_4(k)}{1 + y_4^2(k)} + u_4^3(k)$$

It is worth noting that the followers are heterogeneous and include linear and nonlinear dynamics. Agent 1 is a linear agent, which is widely used in autonomous vehicle formation control [2]. The cosine nonlinear dynamics of agent 2 is taken from [46]. The nonaffine nonlinear descriptions of agents 3–4 are picked from [36]. The heterogeneity and nonlinearity of followers become a great challenge to the controller design.

The trajectories of two leaders are chosen as piece-wise step signals:

$$\text{Agent 5 : } w_5(k) = \begin{cases} 1.4, & \text{if } k < 165 \\ 1.6, & \text{if } 165 \leq k < 330 \\ 1.3, & \text{if } k \geq 330 \end{cases}$$

$$\text{Agent 6 : } w_6(k) = \begin{cases} 0.7, & \text{if } k < 165 \\ 1.2, & \text{if } 165 \leq k < 330 \\ 1.1, & \text{if } k \geq 330 \end{cases}$$

The initial conditions $y_i(0)$ ($i \in \mathcal{F}$) are selected as $y_1(0) = 1.2$, $y_2(0) = 1.8$, $y_3(0) = 0.6$, $y_4(0) = 0.4$. Initial values of the control inputs and estimated PPD are set as $u_i(0) = 0$ ($i \in \mathcal{F}$) and $\hat{\phi}_i(0) = 0.5$ ($i \in \mathcal{F}$). Other parameters are listed in Table 1. Observing the communication network in Figure 1, the in-degree matrix is $\mathcal{D} = \text{diag}(2, 1, 2, 2, 0, 0)$ and the reciprocal of its largest diagonal entry is 0.5. Thus, $\rho_i = 0.3 < 1/d_i$ ($i \in \mathcal{F}$) satisfies the convergence condition in Theorem 1.

The outputs and inputs of agents are shown in Figures 2 and 3 (with $\lambda = 2$), respectively. They show that the follower agents are driven into the convex hull of leaders gradually. At the time instant $k = 165$ and $k = 330$, the convex hull moves to another location due to the change of the leaders' outputs. The followers can still converge to the new convex hull, which verifies the result in Theorem 1. To further study the influence of the key parameter, the tracking performance with different λ (i.e., $\lambda = 5, 2, 1$, and 0.5) is presented in the sub-figures in Figure 2. The results show that, within a certain range, the smaller the λ value, the faster the tracking speed of follower agents. However, one should also note that the overshoots and oscillations may be intensified if the λ value is too small. The PPD estimation (with $\lambda = 2$) is shown in Figure 4. Due to the heterogeneous dynamics of followers, the PPD estimations are quite different from each other even though they have the same initial value.

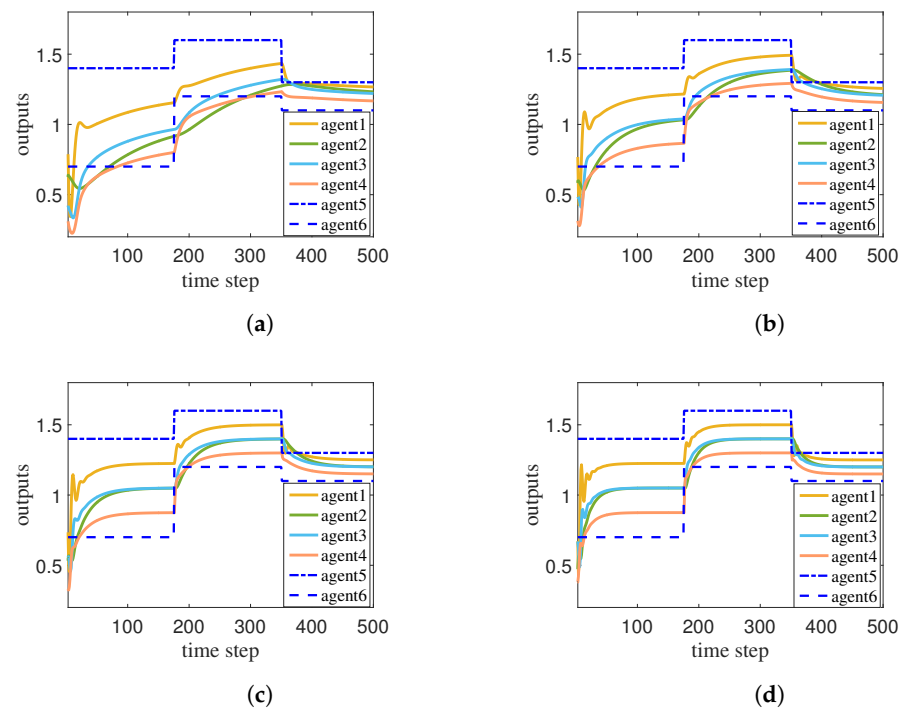


Figure 2. Outputs of agents in Example 1: (a) $\lambda = 5$; (b) $\lambda = 2$; (c) $\lambda = 1$; (d) $\lambda = 0.5$.

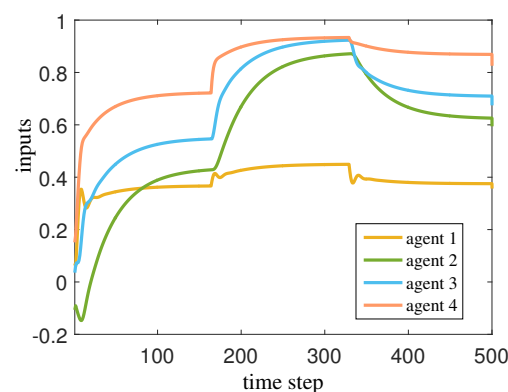


Figure 3. Inputs of agents in Example 1 ($\lambda = 2$).

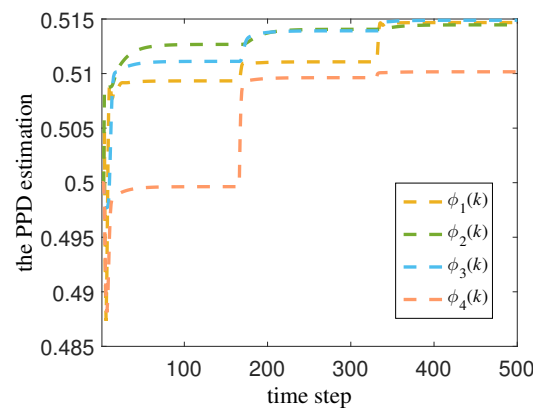


Figure 4. PPD estimations in Example 1 ($\lambda = 2$).

5.2. Example 2

The time-switching topology and dynamic leaders are simulated in this case. The followers dynamics are assumed to be the same with Case 1. The extended algorithm Equations (5), (6), and (30) will be implemented in the MASs. The communication topology is assumed to switch over three graphs, i.e., $S_{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$, which are presented in Figure 5. An extra time-varying signal $\sigma(k)$ is introduced to indicate the switching process. The definition of $\sigma(k)$ is given as follows:

$$\sigma(k) = \begin{cases} 1 & \text{if } \mathcal{G}(k) = \mathcal{G}_1 \\ 2 & \text{if } \mathcal{G}(k) = \mathcal{G}_2 \\ 3 & \text{if } \mathcal{G}(k) = \mathcal{G}_3 \end{cases}$$

The dynamic trajectories of the leaders are selected as

$$\begin{aligned} \text{Agent 5 : } w_5(k) &= (0.5 + 0.001k)\sin(\pi k/150) \\ \text{Agent 6 : } w_6(k) &= (0.5 + 0.001k)\sin(\pi k/150) - 1 \\ \text{Agent 7 : } w_7(k) &= 0.3\sin(\pi k/100) - 0.5 \end{aligned}$$

From the description, one can see the amplitude of leader 5 and 6 increases gradually over time.

All the initial values and the parameters are the same with Case 1 while $\lambda = 0.5$. From Figure 5, the in-degree matrices are $\mathcal{D}^1 = \text{diag}(2, 1, 3, 2, 0, 0)$, $\mathcal{D}^2 = \text{diag}(3, 1, 2, 2, 0, 0)$, and $\mathcal{D}^3 = \text{diag}(2, 1, 2, 3, 0, 0)$ for graphs $\mathcal{G}_1, \mathcal{G}_2$, and $\mathcal{G}_3$, respectively. Thus, it is easy to check $\rho_i = 0.3 < 1/\max_{l=1,2,\dots,\psi} d_i^l = \frac{1}{3}$ also satisfies the convergence condition in Theorem 2.

Figure 6 depicts the changes of indicative signal $\sigma(k)$ from $k = 1$ to $k = 50$, where one can observe that the communication network randomly switches among three graphs in different time instants. Figures 7–9 show the outputs, inputs and estimated PPD of the follower agents, respectively. In Figure 7, the green curves present the envelopes of the leaders' trajectory, i.e., these two curves are the upper and lower boundaries of the convex hull. It can be found all followers are driven into the space within these two curves, which verifies the result in Theorem 2. Furthermore, the trajectories in Figures 7–9 are not smooth, which is caused by the quick switch of the communication topologies and the dynamic changes of the leaders.

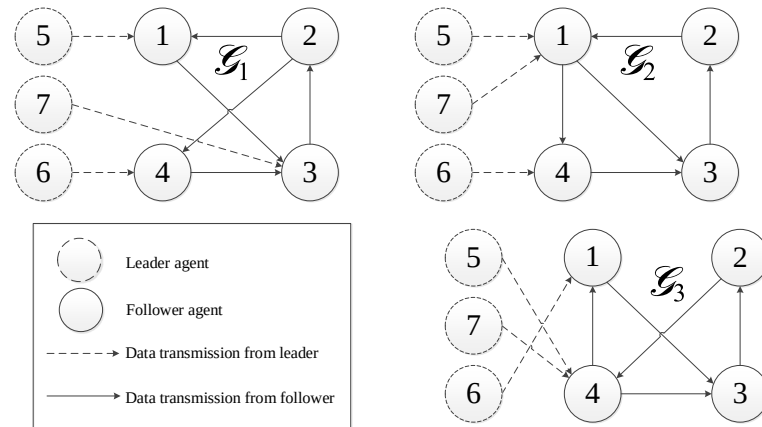


Figure 5. Communication graphs in Example 2.

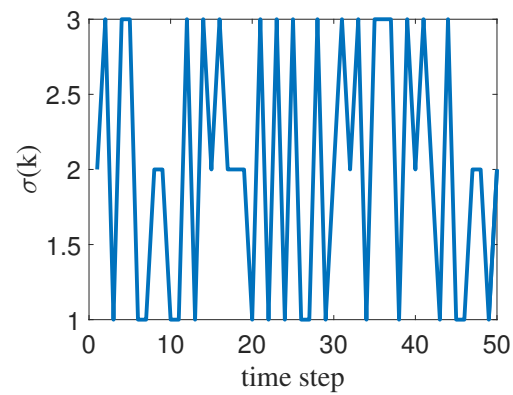


Figure 6. Topology switching over time.

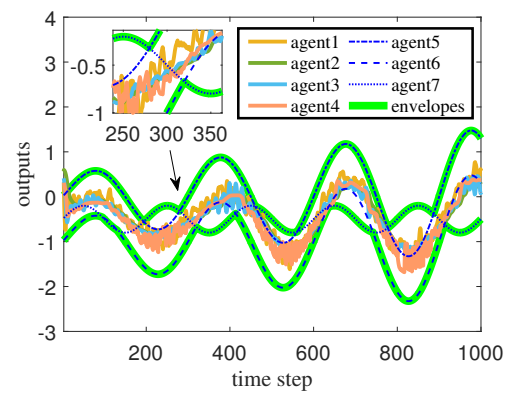


Figure 7. Outputs of agents in Example 2.

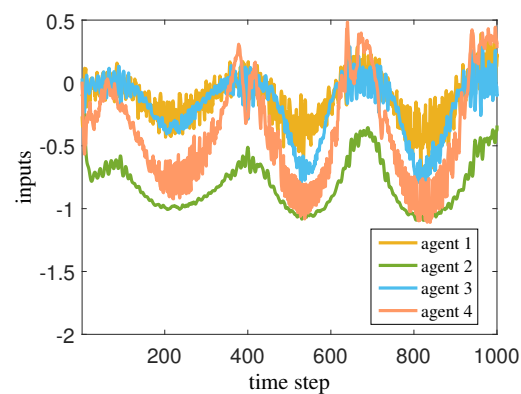


Figure 8. Inputs of agents in Example 2.

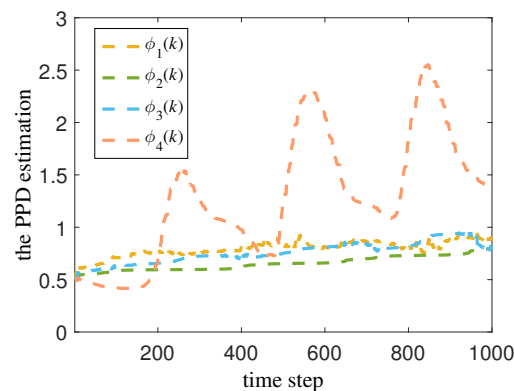


Figure 9. PPD estimations in Example 2.

5.3. Example 3

A more practical simulation with external disturbance is conducted in this case. The followers' dynamics and topologies are the same as Example 2. The leaders' outputs are modified as piece-wise functions to simulate the transition of different types of tasks.

$$\begin{aligned} \text{Agent 5 : } w_5(k) &= \begin{cases} -1.2 & \text{if } 0 < k \leq 250 \\ 0.6 & \text{if } 250 < k \leq 500 \\ (0.5 + 0.001k)\sin(\pi k/150) & \text{if } k > 500 \end{cases} \\ \text{Agent 6 : } w_6(k) &= \begin{cases} -1.6 & \text{if } 0 < k \leq 250 \\ 0.8 & \text{if } 250 < k \leq 500 \\ (0.5 + 0.001k)\sin(\pi k/150) - 1 & \text{if } k > 500 \end{cases} \\ \text{Agent 7 : } w_7(k) &= \begin{cases} -1.8 & \text{if } 0 < k \leq 250 \\ 1 & \text{if } 250 < k \leq 500 \\ 0.3\sin(\pi k/100) - 0.5 & \text{if } k > 500 \end{cases} \end{aligned}$$

The communication channels are assumed to be intervened with external disturbances.

$$\begin{aligned} \hat{w}_l(k) &= w_l(k) + \beta \cdot n_l(k) \quad (l = N + 1, \dots, N + M) \\ \hat{y}_i(k) &= y_i(k) + \beta \cdot n_i(k) \quad (i = 1, \dots, N) \end{aligned}$$

where $\hat{w}_l(k)$ and $\hat{y}_i(k)$ are the real information that the follower receives from leaders and its neighboring followers, respectively, and $n_i(k)$ is the channel disturbance following the normal distribution with intensity β .

Moreover, a widely used protocol [12] is chosen as comparison.

$$u_i(k) = a_i^g(k)\bar{\xi}_i(k) \quad (36)$$

where $a_i^g(k)$ is the time-varying gain. It is worth noting that $a_i^g(k)$ is difficult to determine if the agent model is unavailable. In this test, via a trial-and-error method, the gain is tuned similarly to the form in [12] as $a_1^g(k) = 0.3/(1 + 0.001k)$, $a_2^g(k) = 1/(1 + 0.005k)$, $a_3^g(k) = 0.3/(1 + 0.001k)$, $a_4^g(k) = 0.3/(1 + 0.001k)$.

Denote the integral square error (ISE) of global containment errors as the performance index:

$$\text{ISE} = \sum_k \bar{e}^+(k)^2 + \bar{e}^-(k)^2 \quad (37)$$

The received outputs and global containment errors of the MAS with MFACC ($\lambda = 0.5$) are shown in Figure 10. It can be clearly seen that MFACC still works well under communication channel disturbances with different intensities, and the global containment errors converge, which also validates the theoretical results in this paper. The comparison results

when using the protocol in [12] are given in Figure 11. One can observe the controller is stable, but the tracking performance is not satisfying facing unknown heterogeneous follower dynamics. In summary, the ISE results of two methods under various channel environment are listed in Table 2.

Table 2. Performance comparison in Example 3.

	MFACC, $\lambda = 0.5$	MFACC, $\lambda = 1$	MFACC, $\lambda = 2$	Method in [12]
ISE, $\beta = 0.1$	119.8	189.2	327.8	793.5
ISE, $\beta = 0.2$	120.0	188.8	327.4	794.6
ISE, $\beta = 0.5$	120.7	188.2	326.1	800.6
ISE, $\beta = 1$	125.6	188.5	324.9	824.2

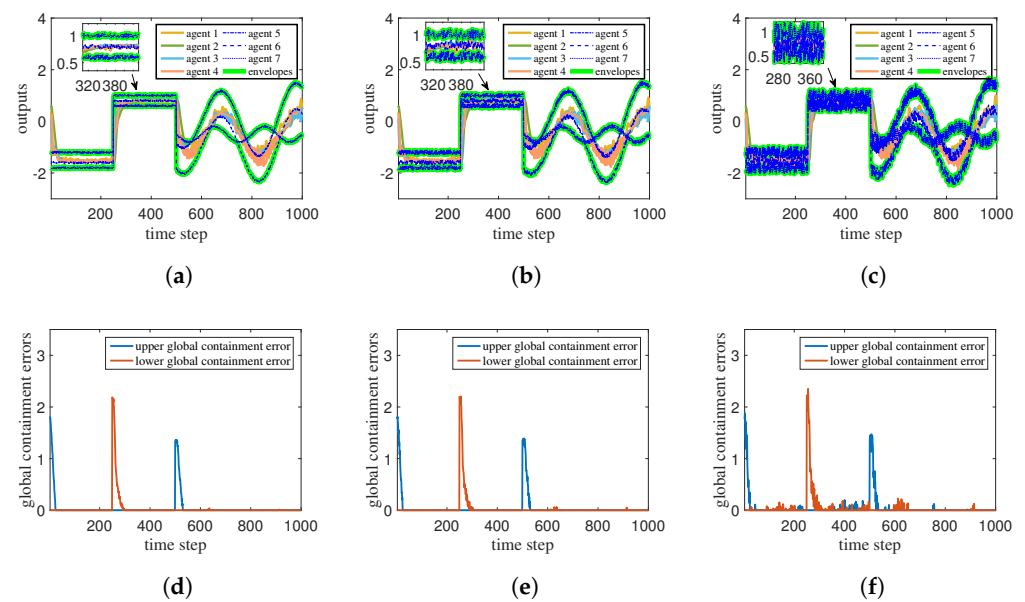


Figure 10. Outputs of agents with MFACC for Example 3: (a) MFACC, $\beta = 0.1$; (b) MFACC, $\beta = 0.2$; (c) MFACC, $\beta = 0.5$; (d) MFACC, $\beta = 0.1$; (e) MFACC, $\beta = 0.2$; (f) MFACC, $\beta = 0.5$.

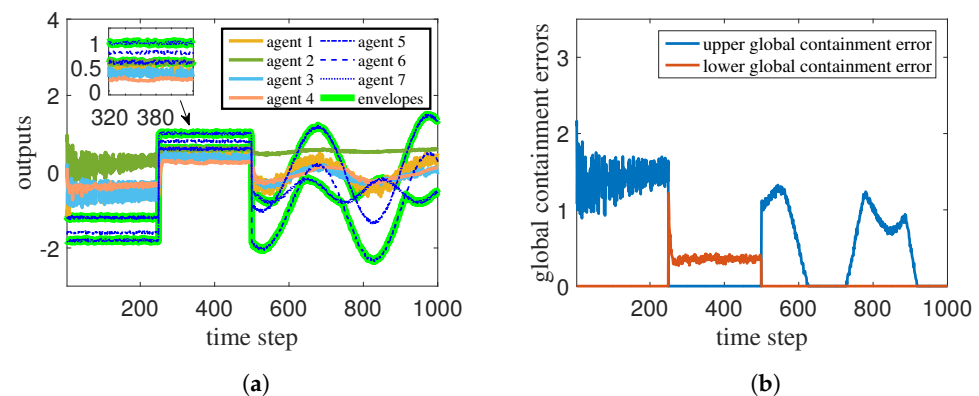


Figure 11. Outputs of agents with the method in [12] for Example 3: (a) the method in [12], $\beta = 0.1$; (b) the method in [12], $\beta = 0.1$.

6. Conclusions

In this paper, the containment control problem of MASs is solved using the MFAC approach. Based on the dynamical linearization technique, the heterogeneous unknown nonlinear agents are transformed into CFDL data models, and then the novel distributed MFAC algorithm was designed to ensure that the followers are driven into the convex hull of the leaders. Further extension includes the results under time-switching topologies and dynamic leaders. The proposed algorithm is data-driven, i.e., only I/O data rather than

systems models are used in the controller design. Three numerical examples are given to validate the effectiveness of this method. This paper brings the discussion of coordinated MASs MFAC schemes from the single leader field to multiple leaders field along with a novel stability analysis method.

Although the proposed method acquires several advantages, it still has limitations facing real applications. In the future, the containment control problem in the event of sensor failure and time delay will be investigated. Moreover, the algorithm should be validated by practical experiments.

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Notations

The following abbreviations are used in this manuscript:

Symbol	Meaning
$\mathcal{F}$	the set of followers
$\mathcal{W}$	the set of leaders
$\mathcal{G}$	the communication topology graph
$\mathcal{A} \setminus \mathcal{A}(k)$	the adjacency matrix with fixed\switching topologies
$\mathcal{D} \setminus \mathcal{D}(k)$	the in-degree matrix with fixed\switching topologies
$\mathcal{L} \setminus \mathcal{L}(k)$	the Laplacian matrix with fixed\switching topologies
$Co(\mathcal{W})$	the convex hull of the set $\mathcal{W}$
$y_i(k) \setminus \hat{y}_i(k)$	the output without\with disturbances of the i th follower agent
$u_i(k)$	the input of the i th follower agent
$f_i(\cdot)$	the unknown follower dynamics
$\bar{w}_i$	the convergence value for the i th follower agent
$w_l(k) \setminus \hat{w}_l(k)$	the output without\with disturbances of the l th leader agent
$\phi_i(k)$	the pseudo partial derivative
$\hat{\phi}_i(k)$	the estimated pseudo partial derivative
$\tilde{\phi}_i(k)$	the pseudo partial derivative estimation error
$\xi_i(k) \setminus \bar{\xi}_i(k)$	the local containment error with fixed\switching topologies
$\rho_i, \eta, \lambda, \mu$	adjustable parameters of controllers
$\mathbf{y}(k)$	the vector collecting all follower agents' outputs
$\mathbf{u}(k)$	the vector collecting all follower agents' inputs
$\xi(k)$	the vector collecting all local containment errors
$\mathbf{w}_s \setminus \mathbf{w}(k)$	the vector collecting all stationary\dynamic leaders' outputs
$w^+ \setminus w^-$	the upper\lower bound of stationary leaders' outputs
$\bar{w}^+ \setminus \bar{w}^-$	the upper\lower bound of dynamic leaders' outputs
$y^+(k) \setminus y^-(k)$	the upper\lower bound of followers' outputs
$\Omega^+(k) \setminus \Omega^-(k)$	the upper\lower bound of the MASs with stationary leaders
$\tilde{\Omega}^+(k) \setminus \tilde{\Omega}^-(k)$	the upper\lower bound of the MASs with dynamic leaders
$e^+(k) \setminus \bar{e}^+(k)$	the upper global containment error with stationary\dynamic leaders

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