A Novel Method of Time-Varying Formation Control Based on a Directed Graph for Multiple Autonomous Underwater Vehicles

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Abstract: Currently, autonomous underwater vehicles (AUVs) are facing various challenges, rendering multiple-AUV (multi-AUV) formation control a pivotal research direction. The issues surrounding formation control for a multi-AUV system to establish time-varying formations must be investigated. This paper discusses the formation protocol of multi-AUV systems in order to establish the defined time-varying formations. First, when these systems establish formations, the speed of each AUV can be equivalent. After that, consensus-based methods are used to solve the time-varying formation-control problem. The necessary and sufficient process of multi-AUV in achieving time-varying formations is proved. Furthermore, the formula for the time-varying formation center function is provided. Further, we present a protocol law for multi-AUVs to establish time-varying formations. Finally, the theoretical results of a simulation are presented, which validate the formation protocol.

Keywords: autonomous underwater vehicle (AUV); formation control; multiple agents system; time-varying formation

1. Introduction

The autonomous underwater vehicle (AUV) is a highly sophisticated, intricate system that integrates various cutting-edge technologies, such as vehicle control, anomaly detection, information decision-making, and deep-sea navigation. This convergence of advanced technologies enables the AUV to operate efficiently in challenging underwater environments. One significant advantage of the AUV is its ability to cover a larger search area. By utilizing its autonomous capabilities, the AUV can navigate through vast expanses of the ocean floor with precision and speed. This expanded coverage allows for more comprehensive data collection and exploration of previously uncharted territories.

Moreover, by employing multiple AUVs in coordinated missions, the fault tolerance of the entire system can be significantly improved. When one AUV encounters technical issues or malfunctions during an operation, other vehicles within the multiple-AUV (multi-AUV) fleet can seamlessly take over its tasks without disrupting overall mission objectives. This redundancy ensures continuous data acquisition and minimizes potential downtime due to equipment failures. Furthermore, advancements in communication systems have enhanced collaboration among multiple AUVs operating together. These vehicles can exchange real-time information regarding their respective positions, collected data, and environmental conditions. Such seamless communication facilitates efficient coordination between individual units within a multi-AUV team while optimizing resource allocation for maximum productivity.

In formation control, the leader–follower method of multiple agents is the most common structure. In this method, each follower aircraft is controlled to maintain its velocity...
and position to a predefined leader aircraft along the trajectory. This structure rapidly reconfigures and extends to new objects, so it can be quickly copied by other multi-AUVs [1]. Therefore, the leader-follower method is widely employed in real-world engineering problems. However, the formation system is too reliant on the leader. One team [2] studied multi-agent systems with input delays and found that parameters were too wide and difficult to adjust in the real marine environment. Others investigated the formation-control problem of leader–follower-structured AUVs through state-prediction estimation in the scenario of an undependable underwater acoustic channel [3]. Ultimately, the team conducted a comparison of the suggested formation-control strategy with the traditional method, and they proved the effectiveness of the former. Targeting the control problem of AUV formation with delays and interruption in the communication, another team [4] signed a feedback linearization controller and the PD control method, and they transformed the strongly coupled nonlinear vehicle model into a second-order model. Another team [5] proposed a basic formation-control system that could minimize the modifications of standardization among an autonomous surface vehicle and AUVs, and they tested it in a sea environment. Furthermore, researchers examined the leader–follower coordination control problem that arises in a continuous-time multi-AUV formation containing two independent topologies and time-varying delays [6,7]. Using the gradient-descent strategy to approximate the communication delay, a time-lag estimator was designed [8]. They found that the time lag of the leader AUV state packets reduced the actual delay, and that the formulation of the estimator led to obstacle avoidance. Additionally, researchers studied the leader–follower consistency problem of a multi-intelligent body system with input delays [2]. Others [9] studied synchronous state information, proposed a leader–follower consensus control protocol with time-varying delays, and performed a controllability analysis using Lyapunov–Krasovskii functionals. Expanding the research on the leader–follower consensus, one team [10] evaluated the issues of the immeasurable states of intelligent agents and the inherent time delays in the signal-transmission process. They subsequently created an unknown input observer to estimate time delays and established a consensus control protocol for multi-agent systems according to the observer’s estimated values. Moreover, one team [11] examined a novel optimal control method that can achieve the consensus of multiple AUVs and avoid obstacles with minimal control effort. Another team [12] focused on collision-free formation-tracking AUVs under the influence of compound disturbances in intricate ocean environments. To that end, they proposed a novel finite-time extended state observer (FTESO)-based distributed dual closed-loop model-predictive control scheme. In [13], the authors designed a control protocol based on a distributed robust model-predictive control to better coordinate cooperation between AUVs. In [14], the researchers established a robust model-predictive control scheme based on the active disturbance-rejection control strategy for the Untracking task. Of course, extending the use of these systems to multi-AUV systems can be challenging, as it involves orchestrating the control behavior of each subsystem and making sure that the local MPC (Model Predictive Control) optimization problem maintains its closed-loop stability under system constraints. In [15], the authors presented an unprecedented solution to the issue posed by the need to control the nonplanar multi-AUV formation reconfiguration. The proposed method utilizes the graph rigidity and affine transformation (GR-AT) approach, which can tackle this complex problem in a unique manner. In [16], the researchers established the distance rigid graph-based formation-control strategy for vehicles modeled with Euler–Lagrange-like (E-L) equations. The approach differs from linear systems due to the nonholonomic constraint in the former method. Furthermore, researchers [17] suggested a time-varying strategy based on optimization for networked uncertain Euler–Lagrange mobile agents. This strategy adaptively and flexibly considers objective functions that can change in real time, enabling its versatile application. Others [18] investigated the distance-based formation-control problem for multi-agent systems in arbitrary dimensional space. The agents are treated as point masses with dynamics governed by single and double integrators. In addressing rigid graph-based formation acquisition control problems, one team [19]
focused on a class of double-integrator multiagent systems operating in two independent layers. This research illuminated how to acquire formations efficiently within such systems. Also, the work presented in [20] studied the formation coordination control of discrete-time distributed leaderless multi-AUV systems with two independent position–velocity communication topology and control inputs on a nonconvex set. Its focus was how to effectively coordinate multiple AUVs without relying on centralized leadership or global information exchange. In [21], the authors addressed the formation-control problem to deploy a group of AUVs by considering vehicle nonholonomic constraints and obstacle avoidance in the environment. In [22], the authors suggested a fixed-time-based method involving a multi-AUV leader–follower formation-control, achieving the finite-time convergence of formation-tracking error. In [23], the researchers created a two-loop controller to regulate multi-AUV formation, as well as a neural-network based controller and a conditional integrator to tackle model uncertainty and disturbances. In [24], the researchers proposed a new distributed control protocol for solving the guaranteed cost formation tracking control of multi-AUVs with switching topology communication. The protocol realizes formation tracking control of a multi-AUV system, and considers the characteristics of switching topology communication between individuals. In [25], the researchers designed a method of formation control that combines CNN-LSTM (Convolutional Neural Networks-Long Short-Term Memory) prediction and backsteps sliding mode control to overcome hydroacoustic communication constraints in the multi-AUV leader–follower formation. In [26], the researchers transformed the nonlinear AUV model into a second-order integral model via feedback linearization. They designed finite-time sliding-mode disturbance observers for unknown disturbances in the ocean and then estimated the unknown disturbances in finite time.

In [27], a fixed-time terminal sliding mode control method was designed for a class of second-order nonlinear systems with unknown dynamics and perturbations. The results of this study were subsequently applied in [28] to achieve consistency control. To enhance convergence speed, Ni et al. [29] improved the fixed-time stable system proposed in [27] by deriving a smaller upper bound for convergence time. They then applied the result to the second-order system discussed in [30], considering the input delay problem, and designed a fixed-time control strategy by using the leader–following mechanism. Building upon the abovementioned work, ref. [31] further improved the convergence speed and investigated a solution for handling the singularity problem of the sliding mode surface. References [30,32,33] transform the delay error system into a second-order system without delay for double integral and first-order linear multi-agent systems with determined input delay, and they use the fixed-time reaching law to achieve fixed-time consistency. In [34], the focus shifts to predicting the future time states of time-varying input delay single integral systems while carefully considering the uncertainties inherent in the transformed systems. Here, nonlinear fixed-time formation protocols for multi-robot systems in both directed and undirected topologies are proposed. In addition, in [33], drawing inspiration from the integral predictor in [35], a predictor was designed to predict the future state for a first-order nonlinear multi-agent system characterized by uniformly fixed input delay and uncertainties. Ref. [36] investigated the cooperative trajectory tracking (CTT) control problem of multiple autonomous underwater vehicles (AUVs), and designed a neural network-based data-driven control algorithm and used the radial basis function neural network to estimate the primary pseudo parameters. Ref. [37] developed a new MRMP (Multi-Robot Motion Planning) algorithm. The fitness for these variants was also measured for simulations where different target motion models were used when calculating the fitness function, highlighting the improved performance when using actual target motion models. Ref. [38] studied a novel reference filter for the Formation Keeping (FK) of multiple AUVs; the FK objectives were treated as servo-constraints. The reference values for the motion control can be directly computed using the Udwadia–Kalaba equation. A low-level Model Predictive Control (MPC) is assumed to be embedded to each agent. The existing limitations in their communication are also considered. Results verified the trajectory
tracking and disturbance rejection, with low constraint violations. Ref. [39] studied a group of heterogeneous AUV systems with intermittent communication links being considered, as well as a finite-time trajectory-tracking control strategy; a distributed trajectory-tracking controller was designed using the states estimated by the intermittent communication network, even without velocity measurements. A homogeneous technique was utilized to prove that all followers can track the leader in a finite time. Finally, the effectiveness of the developed finite-time tracking control strategy is illustrated by numerical simulations. Ref. [40] addresses significant control challenges, including external disturbances, noise, model uncertainties, actuator faults, stochastic switching topologies, time-varying communication delays, and positional information between agents. Stochastic switching topologies are assumed to follow a Markov chain. The proposed framework demonstrates its effectiveness in managing the high nonlinearity and coupled dynamics of AUVs.

This paper investigates the problem of achieving time-varying formations based on the consensus for multi-AUVs. First, the focus is on time-varying formations, which are essential in practical applications where certain formations require agents to have different speeds, such as rotation formations. Only time-varying formations can effectively address this scenario. Second, we present an analysis of AUV formation and the design protocol for multi-AUV systems, aiming to establish time-varying formations. To do so, we propose the necessary and sufficient conditions for these formations. Third, the explicit formula of the AUV formation center functions is provided. Lastly, the theoretical results of simulations are presented.

The rest of this paper is organized as follows. In Section 2, basic concepts and the problem are presented. In Section 3, the main theoretical analysis is presented. In Section 4, three typical working conditions verified by a simulation are presented. Furthermore, Section 5 gives the conclusions and discussion.

2. Background Description

Here, the model of AUV and the basic concepts about graph theory are introduced.

2.1. Basic Concepts

A directed graph $G = (Q, \epsilon, W)$ consists of a set of nodes $Q = \{q_1, q_2, \ldots, q_n\}$, a set of edges $\epsilon \subseteq \{(q_i, q_j) : q_i, q_j \in Q\}$, and a weighted adjacency matrix $W = [w_{ij}] \in \mathbb{R}^{N \times N}$ with nonnegative elements $w_{ij}$. An edge of $G$ is denoted by $e_{ij} = (q_i, q_j)$. Then, if and only if $e_{ij} \in \epsilon$ and $w_{ij} = 0$ for all $i \in \{1, \ldots, N\}$, the $w_{ij} > 0$. The set of neighbors of node $q_i$ is denoted by $N_i = \{q_j \in Q : (q_i, q_j) \in \epsilon\}$. The in-degree of node $q_i$ is defined as $\deg_{\text{in}}(q_i) = \sum_{j=1}^{N} w_{ij}$. The degree matrix of $G$ is denoted by $D = \text{diag}\{\deg_{\text{in}}(q_i), i = 1, 2, \ldots, N\}$. The Laplacian matrix of $G$ is $L = D - W$.

Further information about graph theory can be obtained from [41].

For a given $\Psi \in \mathbb{C}$, $\text{Re}(\Psi)$ and $\text{Im}(\Psi)$ represent the real and the imaginary part of $\Psi$, respectively. The superscript $H$ denotes the Hermitian adjoint of matrices.

The following lemma is useful for multi-AUV systems.

Lemma 1. Define $L \in \mathbb{R}^{N \times N}$ is the Laplacian matrix of a directed graph $G$ [42]:

1. $L$ has at least one zero eigenvalue, and $1_N$ is the associated eigenvector; that is, $L1_N = 0$;
2. If $G$ has a spanning tree, then $0$ is a simple eigenvalue of $L$, and all the other $N - 1$ eigenvalues have positive real parts;
3. $1_N$ means a column vector of size $N$ with $1$ as its elements, and $I_n$ means an identity matrix with $n$ dimensions;
4. $0$ means zero matrices of an appropriate size, with zero vectors and zero number as special cases.

2.2. Mathematical Model of the AUV

The AUV design in this study is a six-DOF (Degree of Freedom) navigation in water; moreover, we also took into account the actual application situation and the type of sensor equipped, as well as the fact that the AUV actually navigates at the same depth. Thus, we
simplified the entire control model such that the AUV formation moves on the horizontal plane (Figure 1). The designed AUV has two horizontal channels to realize the change in the head direction, so the AUV in the plane space is fully driven, and the kinematics and dynamics formulas [25] are as follows:

\[
\begin{align*}
\dot{\eta} &= J(\psi)\upsilon \\
M\dot{\upsilon} &= \Gamma(\eta, \upsilon) + \tau + d - \Delta_f
\end{align*}
\]  

(1)

where \( \eta \) is a state vector for the AUV, and \( \eta = [x, y, \psi]^T \), where \( x \) is the portrait position, \( y \) is the lateral position, and \( \psi \) is the bow direction angle. Additionally, \( \upsilon = [u, v, r]^T \), where \( u \) is the speed along \( X \), \( v \) is the speed along \( Y \), and \( r \) is the bow angular velocity. Finally, \( \tau = [\tau_1, \tau_2, \tau_3]^T \), where \( \tau \) is the control input vector to be designed, \( \tau_1 \) is the forward force, \( \tau_2 \) is the transverse force, and \( \tau_3 \) is the yawing moment.

Figure 1. The autonomous underwater vehicle (AUV) coordinate diagram.

\( J(\varphi) \) is the bow’s rotation matrix, defined as follows:

\[
J(\varphi) = \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  

(2)

\[
\dot{J}(\varphi) = J(\varphi)S(r)
\]  

(3)

\[
J^T(\varphi)S(r)J(\varphi) = J(\varphi)S(r)J^T(\varphi) = S(r)
\]  

(4)

\[
S(r) = \begin{bmatrix}
0 & -r & 0 \\
r & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  

(5)

Let

\[
d = MJ^T\delta,
\]  

(6)

\[
\delta = [\delta_1, \delta_2, \delta_3],
\]  

(7)
\[ \theta = [\theta_1, \theta_2, \theta_3] \] is uncertainty of the model

\[ \Gamma(\eta, v) = -C(v)v - Dv, \]  

where \( C(v) \) is the Coriolis force and centripetal force matrix, \( D \) is the damping coefficient matrix, and \( M \) is the inertia coefficient matrix. The port and starboard structure of the AUV is symmetrical, so \( Y_r = N_j \), and the geometric center of the AUV coincides with its center of gravity; thus, \( x_g = 0 \):

\[
M = \begin{bmatrix} m - X_u & 0 & 0 \\ 0 & m - Y_v & mx_g - Y_r \\ 0 & mx_g - Y_v & I_z - N_r \end{bmatrix} = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix}, \]

\[
C(v) = \begin{bmatrix} 0 & 0 & -m_{22}v - m_{23}r \\ 0 & 0 & m_{11}u \\ m_{22}v + m_{23}r & -m_{11}u & 0 \end{bmatrix}, \]

\[
D = -\begin{bmatrix} X_u & 0 & 0 \\ 0 & Y_v & Y_r \\ 0 & N_v & N_r \end{bmatrix} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{bmatrix}, \]

where \( m \) is the AUV mass, \( I_z \) is the moment of inertia, \( x_g \) is the distance between the AUV geometric center \( D \) and the center of gravity \( G \), and the symbols \( X_s, Y_s, \) and \( Z_s \) represent the corresponding hydrodynamic derivatives.

A multi-AUV system has \( N \) AUVs. The interaction topology of the multi-AUV system can be detailed by a directed graph \( G \). The \( i \)th AUV can be represented by node \( q_i \) in \( G \), and the interaction channel from the \( i \)th AUV to the \( j \)th AUV can be expressed as \( e_{ij} \), \( i, j \in \{i = 1, 2, \ldots N\} \). Thus, the directed graph \( G \) obtains a spanning tree.

For these AUVs, the inner loop can decide the attitude, and the outer loop can decide position with desired velocity of the system (Figure 2).

![Figure 2. Two-loop formation-control scheme for multiple-AUV (multi-AUV) systems.](image)

The assumed operation scenario carried out in this study is shown in Figure 3. The AUVs in the formation all move in the fixed-depth layer, so the coordination of AUVs in the depth direction is not considered.

The multi-AUV system can be designed as follows:

\[
\begin{cases}
\dot{x}_i(t) = v_i(t) \\
\dot{v}_i(t) = u_i(t),
\end{cases}
\]

where \( i = 1, 2, \ldots N \), \( x_i(t) \in \mathbb{R}^n \), and \( v_i(t) \in \mathbb{R}^n \) refer to the position and velocity vectors of the \( i \)th AUV, respectively, and \( u_i(t) \in \mathbb{R}^n \) refers to the control inputs.
Define $\theta_i(t) = [x_i(t), v_i(t)]^T$, $B_1 = [1, 0]^T$, and $B_2 = [0, 1]^T$. Then, the multi-AUV system in Equation (12) can be rewritten as follows:

$$\dot{\theta}_i(t) = B_1 B_2^T \theta_i(t) + B_2 u_i(t),$$  \hspace{1cm} (13)

- **Figure 3.** All AUVs operate at the same height (distance from the ocean floor).

Let $m_i(t) = [m_{ix}(t), m_{iy}(t)]^T$ be piecewise continuously differentiable vectors ($i = 1, 2, \ldots, N$) and $m(t) = [m_{1x}(t), m_{2x}(t), \ldots, m_{Nx}(t)]^T \in \mathbb{R}^{2N}$.

**Condition 1:** A vector $m(t)$ specifies the time-varying formation. If the multi-AUV in Equation (13) establishes time-varying formation $m(t)$ for the appropriate initial states, then a vector-valued function $c(t) \in \mathbb{R}^2$ satisfies $\lim_{t \to \infty} (\theta_i(t) - m_i(t) - c(t)) = 0$, where $c(t)$ means the formation center function.

To certify the Condition 1, suppose we have an affine pentagon structure of a multi-AUV system with five AUVs moving in the plane. Let $x_{iX}(t) \in \mathbb{R}$, $m_{ix}(t) \in \mathbb{R}$, and $x_{iY}(t) \in \mathbb{R}$; $m_{iy}(t) \in \mathbb{R}$; $c_{ix}(t) \in \mathbb{R}$ and $c_{iy}(t) \in \mathbb{R}$ refer to the position, the formation, and the formation center function of the $i$th AUV along the x- and y-axes in the X–Y plane, respectively.

$$\begin{aligned}
\theta_{ix}(t) &= [x_{iX}(t), x_{iY}(t)]^T \\
\theta_{iy}(t) &= [x_{iX}(t), x_{iY}(t)]^T \\
c_{ix}(t) &= [c_{ix}(t), c_{iy}(t)]^T
\end{aligned} \hspace{1cm} (14)$$

If $\theta_{ix}(t) - m_{ix}(t) - c_{ix}(t) \to 0$ as $t \to \infty$ for all AUVs $i \leq 5$ in Figure 4, the two pentagons indicated by $m_{ix}(t)$ and $\theta_{ix}(t)$ ($i \leq 5$) are congruent.

**Condition 2:** If a vector function $s(t) \in \mathbb{R}^2$ makes $\lim_{t \to \infty} (\theta_i(t) - s(t)) = 0$ ($i = 1, 2, \ldots, N$), then the multi-AUV system in Equation (12) can attain consensus for any bounded initial states, so a consensus function is described as $s(t)$.

From conditions 1 and 2, if $m(t) \equiv 0$, the multi-AUV system in Equation (13) establishes the formation, and then the consensus also can be established. Thus, the time-varying formation center function and the consensus function are critical.

The time-varying formation protocol can be defined as follows:

$$u_i(t) = K_1 (\theta_i(t) - m_i(t)) + K_2 \sum_{j \in N_i} w_{ij} (\theta_j(t) - m_j(t) - (\theta_i(t) - m_i(t))) + m_{ip}(t) \hspace{1cm} (15)$$

where $i = 1, 2, \ldots, N$, $K_1 = [k_{11}, k_{12}]$, and $K_2 = [k_{21}, k_{22}]$.

$K_1$ is the motion mode of the formation center, and $K_2$ is the appropriate value that can lead all AUVs to establish the desired formation.
Figure 4. AUV formation on the X–Y plane with $N = 5$.

Define the following:

$$\begin{align*}
\theta(t) &= [\theta^T_1(t), \theta^T_2(t), \ldots, \theta^T_N(t)]^T, \\
m_{1x}(t) &= [m^T_{1x}(t), m^T_{2x}(t), \ldots, m^T_{Nx}(t)]^T, \\
m_{0x}(t) &= [m^T_{10}(t), m^T_{20}(t), \ldots, m^T_{N0}(t)]^T
\end{align*}$$

(16)

So, multi-AUV system (12) can be written in a compact form as Equation (15).

$$\dot{\theta}(t) = \left(I_N \otimes (B_2K_1 + B_1B_2^\top) - L \otimes (B_2K_2)\right)\dot{\theta}(t) - (I_N \otimes (B_2K_1) - L \otimes (B_2K_2))m(t) + (I_N \otimes B_2)m(t)$$

(17)

This paper focuses on two problems for multi-AUV system in Equation (17):

1. When the time-varying formation is established.
2. If the system obtains the time-varying formation, what is the protocol in Equation (17)?

3. Theoretical Analysis

In this section, the time-varying formation for the multi-AUV system problem in Equation (14) becomes a consensus problem. Then, the necessary and sufficient conditions to establish the time-varying formation $m(t)$ are proved, and the explicit formula of the time-varying formation center function is obtained. Then, a procedure is proposed to establish the gain matrices in Equation (17).

3.1. Formation Analysis

Define $\lambda_i (i = 1, 2, \ldots, N)$ as the eigenvalues of $L$ corresponding to $G$, where $\lambda_1 = 0$ with the associated eigenvector $\pi_1 = 1_N$ and $0 < \text{Re}(\lambda_2) \leq \ldots \leq \text{Re}(\lambda_N)$.

Define the following:

$$U^{-1}LU = J,$$

(18)

where $U = [\pi_1, \pi_2, \ldots, \pi_N]$, $U^{-1} = [\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N]^H$, and the $J$ is the form of Jordan canonical of $L$, $\pi_i \in \mathbb{C}^N$, and $\tilde{u}_i \in \mathbb{C}^N (i = 1, 2, \ldots, N)$. 
Let
\[
\bar{\theta}_i(t) = \theta_i(t) - m_i(t) (i = 1, 2, \ldots, N),
\]
\[
\bar{\theta}(t) = \begin{bmatrix} \bar{\theta}_1^T(t), \bar{\theta}_2^T(t), \ldots, \bar{\theta}_N^T(t) \end{bmatrix}^T,
\]
Equation (17) to obtain m

The following lemma holds directly.

\textbf{Lemma 3.} Let \( c_1 \in \mathbb{R}^2 \) and \( c_2 \in \mathbb{R}^2 \) be linearly independent vectors.

\[
p_j = \Pi_i \otimes c_q (j = 2(i-1)+q; i = 1, 2, \ldots, N; q = 1, 2),
\]

By Lemma 1, one obtains \( J = \text{diag}\{0, T\} \), where \( T \) consists of Jordan blocks corresponding to \( \lambda_i (i = 2, 3, \ldots, N) \).

Let
\[
\begin{align*}
\tilde{U} &= \begin{bmatrix} \tilde{u}_2, \tilde{u}_3, \ldots, \tilde{u}_N \end{bmatrix}^H \\
\xi(t) &= (\tilde{u}_1^H \otimes I_2) \tilde{\theta}(t), \\
\zeta(t) &= (\tilde{U} \otimes I_2) \theta(t)
\end{align*}
\]

The multi-AUV system in Equation (22) thus becomes the following:
\[
\begin{align*}
\dot{\xi}(t) &= (B_2K_1 + B_1B_2^T) \xi(t) + (\tilde{u}_1^H \otimes B_1) (m_v(t) - \dot{m}_x(t)), \\
\dot{\zeta}(t) &= (I_{N-1} \otimes (B_2K_1 + B_1B_2^T) - \tilde{T} \otimes (B_2K_2) \xi(t) + (\tilde{U} \otimes B_1) (m_v(t) - \dot{m}_x(t))
\end{align*}
\]

\textbf{Lemma 4.} Based on [43] \( \phi(t) = Y \varphi(t) \), we know that \( Y \) is a \( 2 \times 2 \) complex matrix with the characteristic polynomial \( f(q) = q^2 + a_1q + a_2 \), so the system is asymptotically stable when the only condition is the following:
\[
\begin{align*}
\Re(a_1) &> 0 \\
\Re(a_1) \Re(a_1, \Pi_2) - \Im(a_2)^2 &> 0
\end{align*}
\]

We can thus prove the necessary and sufficient conditions for the multi-AUV system in Equation (17) to obtain \( m(t) \) by the following theorem.
The multi-AUV system in Equation (17) establishes the time-varying formation $m(t)$ if, and only if, the following conditions hold simultaneously:

1. \[
\lim_{t \to \infty} (m_{iu}(t) - m_{ju}(t)) - (\dot{m}_{iu}(t) - \dot{m}_{ju}(t)) = 0, \quad (i \in \{1, 2, \ldots, N\}, j \in N_i), \quad (28)
\]

2. \[
\text{if } i \in \{1, 2, \ldots, N\} \text{ and } \psi_i = k_{12}k_{11} - \text{Re}(\lambda_i)(k_{12}k_{21} + k_{11}k_{22}) + \left(\text{Re}(\lambda_i)^2 + \text{Im}(\lambda_i)^2\right)k_{21}k_{22}, \quad \text{then:}
\]
\[
\begin{align*}
-k_{12} + \text{Re}(\lambda_i)k_{22} &> 0, \\
(-k_{12} + \text{Re}(\lambda_i)k_{22})\psi_i - \text{Im}(\lambda_i)^2k_{21}^2 &> 0,
\end{align*}
\]

(29) \hspace{1cm} (30)

The proof process is as follows:

Define the following:

\[
\begin{align*}
\tilde{\theta}_{\mathcal{C}}(t) &= (U \otimes I_2) [\bar{\xi}^H(t), 0]^H, \\
\tilde{\theta}_{\mathcal{C}}(t) &= (U \otimes I_2) [0, \xi^H(t)]^H, \\
e_i &\in \mathbb{R}^N
\end{align*}
\]

(31)

where $e_i \in \mathbb{R}^N$ is a vector with 1 as its $i$th component and 0 as other components.

Note that $c_1$ and $c_2$ refer to the linearly independent vectors. $a_1(t), a_2(t),$ and $a_{2k+j}(t)(k = 1, 2, \ldots, N - 1; j = 1, 2)$ must exist to meet $\bar{\xi}(t) = a_1(t)c_1 + a_2(t)c_2$ and $\xi(t) = [a_3(t)c_1^H + a_4(t)c_2^H, \ldots, a_{2N-1}(t)c_1^H + a_{2N}(t)c_2^H]^H$.

Because $[\bar{\xi}^H(t), 0]^H = e_1 \otimes \bar{\xi}(t)$,

\[
\tilde{\theta}_{\mathcal{C}}(t) = (U \otimes I_2)(e_1 \otimes \bar{\xi}(t)) = \pi_1 \otimes \bar{\xi}(t) = a_1(t)p_1 + a_2(t)p_2 \in \mathbb{C}(U)
\]

(32)

Because $p_j(j = 3, 4, \ldots, 2N)$, Equation (32) can be expressed as follows:

\[
\tilde{\theta}_{\mathcal{C}}(t) = \sum_{i=2}^{N} (a_{2i-1}(t)(\pi_i \otimes c_1) + a_{2i}(t)(\pi_i \otimes c_2)) = \sum_{j=3}^{2N} (a_j(t)p_j) \in \mathbb{C}(U)
\]

(33)

\[
[\bar{\xi}(t), \xi(t)]^H = (U^{-1} \otimes I_2)\tilde{b}(t)\text{ then } \tilde{b}(t) = \tilde{\theta}_{\mathcal{C}}(t) + \tilde{\theta}_{\mathcal{C}}(t).
\]

According to Lemmas 2 and 3, the only condition is that when $\lim_{t \to \infty} \tilde{\theta}_{\mathcal{C}}(t) = 0$, the multi-AUV system in Equation (17) obtains $m(t)$, so:

\[
\lim_{t \to \infty} \bar{\xi}(t) = 0,
\]

(34)

\[
\dot{\bar{\xi}}(t) = I_{N-1} \otimes \left( B_2K_1 + B_1B_2^T \right) - \bar{T} \otimes (B_2K_2)\bar{\xi}(t),
\]

(35)

Now, the multi-AUV system in Equation (17) can establish formation $m(t)$ because Equation (25) is under the condition from Equations (26) and (34).

So:

\[
\lim_{t \to \infty} (\tilde{U} \otimes B_1)(m_{iv}(t) - \dot{m}_{iv}(t)) = 0,
\]

(36)

Based on Equation (28), then:

\[
\lim_{t \to \infty} (L \otimes B_1)(m_{iv}(t) - \dot{m}_{iv}(t)) = 0,
\]

(37)
Based on Equation (16), then:

$$L = UU^{-1},$$  \hspace{1cm} (38)

Now multiplying both sides of Equation (37) with $U^{-1} \otimes I,$

$$\lim_{t \to \infty} \left( \hat{U} \otimes B_1 \right) (m_v(t) - \hat{m}_x(t)) = 0,$$  \hspace{1cm} (39)

Note that $\hat{U}$ is nonsingular by Lemma 1.

Next, multiplying both sides of Equation (39) with $\hat{U}^{-1} \otimes I_2$ results in Equation (36). It shows Equation (12) being sufficient for Equation (36).

Then, define $\hat{U} = [\hat{U}, \hat{u}], \hat{U} \in \mathbb{C}^{(N-1) \times (N-1)},$ where $\hat{u} \in \mathbb{C}^{(N-1) \times 1}$ means that $\hat{u}$ is the last column vector of $\hat{U}$. Then, we rank $\left( \hat{U} \right) = N - 1$ and rank $\left( \hat{U} \right) = N - 1$.

From Equation (36), we obtain the following:

$$\lim_{t \to \infty} \left( [\hat{U}, \hat{u}] \otimes B_1 \right) (m_v(t) - \hat{m}_x(t)) = 0,$$  \hspace{1cm} (40)

Note that $\hat{U}1_N = 0$, and then:

$$\hat{u} = -\hat{U}1_{N-1}.$$  \hspace{1cm} (41)

Define the following:

$$\begin{cases} 
\overline{m}_x(t) = \begin{bmatrix} m_{1x}(t), m_{2x}(t), \ldots, m_{(N-1)x}(t) \end{bmatrix}^T \\
\overline{m}_v(t) = \begin{bmatrix} m_{1v}(t), m_{2v}(t), \ldots, m_{(N-1)v}(t) \end{bmatrix}^T
\end{cases},$$  \hspace{1cm} (42)

Based on Equations (40) and (41), we obtain the following:

$$\lim_{t \to \infty} \left( \hat{U} \otimes I_2 \right) \left( (I_{N-1} \otimes B_1) \left( \overline{m}_v(t) - \hat{m}_x(t) \right) - ((1_{N-1} \otimes B_1) \times) \right) \left( m_{Nv}(t) - m_{Nx}(t) \right) = 0$$  \hspace{1cm} (43)

Repeat the same procedure by $\hat{U}^{-1} \otimes I_2$, and then for $\forall i \in \{1, \ldots, N-1\},$ as follows:

$$\lim_{t \to \infty} \left( \left( m_{iv}(t) - m_{Nv}(t) \right) - \left( \hat{m}_x(t) - \hat{m}_x(t) \right) \right) = 0,$$  \hspace{1cm} (44)

Equation (12) is held by Equation (44). Equations (12) and (36) are equivalent under Theorem 1. So, based on the structure of $\hat{U}$, the stability of Equation (35) is equivalent to the $N - 1$ subsystem, as follows:

$$\dot{\tilde{z}}_i(t) = \left( B_2(K_1 - \lambda_i K_2) + B_1 B_2^T \right) \tilde{z}_i(t) (i = 2, 3, \ldots, N),$$  \hspace{1cm} (45)

and

$$B_2(K_1 - \lambda_i K_2) + B_1 B_2^T = \begin{bmatrix} 0 & 1 \\
0 & \begin{bmatrix} k_{11} - \lambda_i k_{21} & k_{12} - \lambda_i k_{22} \end{bmatrix} + \begin{bmatrix} 1 \\
0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\
0 & k_{11} - \lambda_i k_{21} \end{bmatrix}$$  \hspace{1cm} (46)

Now, the characteristic polynomial of the state matrix of the $N - 1$ subsystem in Equation (45) can be obtained by $f_i(s) = s^2 - (k_{12} - \lambda_i k_{22}) s - (k_{11} - \lambda_i k_{21}) (i = 2, 3, \ldots, N),$ where $s$ is a complex variable.

Thus, Equation (34) is asymptotically stable when Equation (13) holds.
Now, according to the above, there is an achievable formation constraint from Equation (28), and the internal stability of the subsystem in Equation (24) can be ensured by Equations (29) and (30), and the formation error converges to 0.

If the multi-AUV in Equation (17) obtains \( m(t) \), it means that \( \tilde{\theta}_i(t) - \tilde{\zeta}(t) \to 0 \) as \( t \to \infty \) under Theorem 1. Thus, we can establish the formation center function through Equation (25). The explicit expression of the formation center function is shown as follows:

**Lemma 5.** Multi-AUV in Equation (17) obtains \( m(t) \) with the function \( c(t) \) of the formation center:

\[
\lim_{t \to \infty} (c(t) - c_0(t) - c_b(t)) = 0, \tag{47}
\]

where

\[
c_0(t) = e^{(B_2K_1 + B_1B_2^T)\tau} \left( \bar{u}_1^H \otimes I_2 \right) \theta(0), \tag{48}
\]

\[
c_0(t) = \int_0^t e^{(B_2K_1 + B_1B_2^T)(t-\tau)} \left( \bar{u}_1^H \otimes I_2 \right) \left( m_\theta(\tau) - k_{12}m_\omega(\tau) - k_{13}m_\chi(\tau) \right) d\tau - \left( \bar{u}_1^H \otimes I_2 \right) m(t) \tag{49}
\]

\( c_0(t) \) is consensus function and time-varying formation center function for the multi-AUV system in Equation (17) when \( m(t) \equiv 0 \). \( c_b(t) \) shows the effect of \( m(t) \). If \( m(t) \equiv 0 \), the multi-AUV system’s explicit formula of consensus function is \( c(t) \). According to Theorem 2, the motion modes of the formation center informing the design of \( K_1 \) can be selected by the eigenvalues of \( B_2K_1 + B_1B_2^T \).

### 3.2. Protocol Design

Next, we propose a method for determining the gain matrix of the multi-AUV implementations in Equation (13) of the time-varying formations in Equation (17).

**Theorem 2.** In Theorem 1, when Equation (12) is held, let

\[
K_2 = [\text{Re}(\lambda_2)]^{-1} B_2^TP, \tag{50}
\]

The multi-AUV in Equation (13) obtains the time-varying formation under Equation (17), where \( P \) is the positive-definite solution to the algebraic Riccati equation:

\[
P \left( B_2K_1 + B_1B_2^T \right) + \left( B_2K_1 + B_1B_2^T \right)^T P - PB_2^TP + I, \tag{51}
\]

The proof process is as follows:

Define the Lyapunov function candidate by considering the \( N - 1 \) subsystems described in Equation (48):

\[
V_i(t) = \bar{c}_i^H(t)P \bar{c}_i(t)(i = 2, 3, \ldots, N), \tag{52}
\]

Taking the derivative of \( V_i(t) \) with respect to \( t \), then:

\[
\dot{V}_i(t) = -\bar{c}_i^H(t)\bar{c}_i(t) + \left( 1 - 2\text{Re}(\lambda_2)[\text{Re}(\lambda_2)]^{-1} \right) \times \bar{c}_i^H(t)Pb_2B_2^TP \bar{c}_i(t), \tag{53}
\]

Note that \( 0 < \text{Re}(\lambda_2) \leq \ldots \leq \text{Re}(\lambda_N) \), which makes \( \dot{V}_i(t) \leq -\bar{c}_i^H(t)\bar{c}_i(t)(i = 2, 3, \ldots, N) \) under Equation (53).

Now, the \( N - 1 \) subsystems detailed by Equation (43) are asymptotically stable according to the Lyapunov’s second method for stability. Multi-AUV in Equation (13) establishes a time-varying formation through Equation (17), with the proof shown in Theorem 1. Thus, the proof of Theorem 2 has been completed.

So, the procedure of the multi-AUV protocol in Equation (17) is confirmed. First, setting the appropriate \( K_1 \) allows us to design the motion modes of the formation center by
allocating the eigenvalues of $B_2K_1 + B_1B_2^T$ at the desired locations in the complex plane. As $(B_1B_2^T, B_2)$ is controllable, $K_1$ is always present. At last, we design $K_2$ to satisfy Equation (30) by using the conclusion of Theorem 2.

4. Simulation

In this section, we present simulations tested on five AUVs that demonstrate the efficacy of the theoretical results. The multi-AUV’s interaction topology is shown in Figure 5, and had 0–1 weights.

![Figure 5. Directed interaction topology $G$ about the five AUVs.](image)

Define the following:

\[
\begin{align*}
\theta_i(t) &= [x_{iX}(t), v_{iX}(t), x_{iY}(t), v_{iY}(t)]^T \\
m_i(t) &= [m_{iX}(t), m_{iY}(t)]^T
\end{align*}
\]

So, the dynamics of each AUV in the case through Kronecker product with $n = 2$ are shown:

\[
\dot{\theta}(t) = \left(I_N \otimes \left(B_2K_1 + B_1B_2^T\right)\right)\theta_i(t) + (I_N \otimes B_1)u_i(t), \quad (55)
\]

Simulation 1:

Define $m_i(t)$ ($i = 2, 3, \ldots, N$) as follows:

\[
\begin{align*}
g_i(t) &= \text{sign}\left(\sin\left(\omega t / 2 + \pi (i - 1) / 5\right)\right) \\
m_{iX}(t) &= r \cos\left((\omega t + 2\pi (i - 1) / 5) - 1\right)g_i(t) \\
m_{iY}(t) &= -\omega r \sin\left((\omega t + 2\pi (i - 1) / 5) - 1\right)g_i(t) \\
m_{iX}(t) &= r \sin\left(\omega t + 2\pi (i - 1) / 5\right) \\
m_{iY}(t) &= \omega r \cos\left(\omega t + 2\pi (i - 1) / 5\right)
\end{align*}
\]

where $r = 7m$ and $\omega = 0.214\text{rad/s}$. When the multi-AUV system successfully obtained the required time-varying formation configuration, the entire formation exhibited a highly coordinated, dynamic motion pattern. In this formation, a total of five AUVs were engaged in action, and they sailed strictly according to the preset figure-eight pattern. This mode not only required each AUV to have a high degree of autonomous navigation, but also required precise communication and coordination between them to ensure the stable operation of the entire fleet. Note that to ensure that there was no collision between the AUVs within the formation while maintaining sufficient spacing to facilitate the transmission of information and the execution of tasks, these AUVs maintained a phase separation of 0.4 T radians on the trajectory.

This design not only considered the overall stability of the formation, but also the independence and flexibility of the AUV so that the entire formation could effectively achieve various tasks in the intricate underwater environment.

The motion modes of the formation center were chosen at $-0.58 + 1.26j$ and $-0.58 - 1.26j$ with $j^2 = -1$, $K_1 = I_2 \otimes [-2, -1.2]$, meaning that when the multi-AUVs established the time-varying formation, the formation center was stationary.
According to the Theorem 2, we designed $K_2 = I_2 \otimes [0.3416, 0.7330]$. Moreover, the initial states of the five AUVs were as follows:

$$
\begin{align*}
\theta_1(0) &= [0.16, 0.03, -0.07, 0.03]^T \\
\theta_2(0) &= [-4.92, -0.08, 6.38, -0.04]^T \\
\theta_3(0) &= [-12.37, -0.26, 4.08, -0.03]^T \\
\theta_4(0) &= [-12.73, 0.03, -4.56, -0.04]^T \\
\theta_5(0) &= [4.63, 0.05, 6.9, 0.02]^T
\end{align*}
$$

(57)

Figure 6 shows the state trajectories of the multi-AUV system in the simulation and the desired $c(t)$. In this Figure, the AUVs’ and $c(0)$’s initial states are represented by circles, and the five AUVs’ and $c(t)$’s final states are shown as the following different patterns: AUV1 is a square, AUV2 is a diamond, AUV3 is an upside-down triangle, AUV4 is a standard triangle, AUV5 is a left-pointing triangle, and $c(t)$ is pentagram.

**Figure 6.** The trajectories of five AUVs and $c(t)$ in the simulation. (a) The position trajectories of five AUVs in the simulation; (b) the velocity trajectories of five AUVs in the simulation.
Figure 7 describes the snapshots of the $c(t)$ with the 16 to 21 second positions and velocities of the five AUVs.

Choosing applicable initial states, we realized the desired time-varying formation transformation in the simulation, and the AUVs did not collide in this process.

Simulation 2:

The motion modes of the formation center function in this simulation were assigned to move periodically. Define $m(t)$ as follows:

$$
\begin{align*}
    m_{iX}(t) &= r \sin(\omega t) + d \ast \cos(2\pi(i - 1)/5) \\
    m_{iY}(t) &= \omega r \cos(\omega t) \\
    \end{align*}
$$

where $r = 20m$, $d = 8m$, and $\omega = 0.1\text{rad/s}$. We chose $K_1 = I_2 \otimes [-3, -1.2]$ to assign the motion modes of the formation center at $0.5j$ and $-0.5j$ with $j^2 = -1$.

According to Theorem 2, we defined the $K_2 = I_2 \otimes [1.3416, 2.7330]$.

Let the initial state of the AUV be $\theta_i(0) = [i\Theta_{i1}, i\Theta_{i2}, i\Theta_{i3}, i\Theta_{i4}]^T (i = 1, 2, \ldots, 5)$, where $i\Theta_{in} (n = 1, 2, 3, 4)$ is pseudorandom value with a uniform distribution on the interval $(0, 1)$.

As a result, the multi-AUV system successfully obtained the required formation configuration and entered a state of collaborative work. In this mode, the center of the formation did not remain stationary, but moved periodically. As the center of the formation moved, the five AUVs moved closely around it, forming a dynamic pentagon. This mode of operation is typical for most formation situations. In practice, the formation center rarely remains stationary because such a static state is not conducive to the AUV’s mission in complex underwater environments. Therefore, periodically moving the formation center can make the whole formation more flexible, adaptable, and better able to cope with various unexpected situations and environmental changes.

Figure 8a demonstrates AUVs’ position and $c(t)$ in the simulation. Figure 8b shows the velocity of the AUVs and $c(t)$ in the simulation. The identification in this Figure is the same as that in Simulation 1.

Simulation 3

The formation center function’s motion modes in this simulation were assigned to move periodically, and the formation center varied randomly within a range of 10 m. We defined the time-varying formation $m(t)$ ($i = 2, 3, \ldots, N$) as follows:
\[
\begin{align*}
    m_{ixX}(t) &= r \sin(\omega t + 2\pi (i - 1)/5) \\
    m_{ioX}(t) &= \omega r \cos(\omega t + 2\pi (i - 1)/5) \\
    m_{ixY}(t) &= r \cos(2\omega t + 2\pi (i - 1)/5) \\
    m_{ioY}(t) &= -\omega r \sin(\omega t + 2\pi (i - 1)/5)
\end{align*}
\] (59)

where \( r = 50m \) and \( \omega = 0.1\text{rad/s} \). Let \( K_1 = I_2 \otimes [-0.25, 0] \) to designate the formation center’s motion modes at \( 0.5j \) and \( -0.5j \) with \( j^2 = -1 \). Define the \( K_2 = I_2 \otimes [1.57, 4.32] \). The initial location selection was consistent with that in simulation 2.

Figure 8. State trajectories of five AUVs in the simulation and \( c(t) \). (a) The position trajectories of five AUVs in the simulation; (b) the velocity trajectories of five AUVs in the simulation.
Figure 9a demonstrates the AUVs’ position and $c(t)$ in the simulation. Figure 9b shows the velocity of the AUVs and $c(t)$ in the simulation. The identification in this Figure is the same as that in Simulation 1.

Figure 9. State trajectories of the five AUVs in the simulation and $c(t)$. (a) The position trajectories of the five AUVs in the simulation; (b) the velocity trajectories of the five AUVs in the simulation.

5. Conclusions and Discussion

This paper discusses how to realize a complex, varied time-varying formation of multi-AUVs and the formation-control strategy. In the current ocean exploration and technology
application, an AUV is an efficient and autonomous underwater exploration tool, and its formation control and cooperative work ability are critical in completing complex tasks. Through theoretical analysis and algorithm design, we provided a feasible solution for multi-AUVs to establish the desired time-varying formation.

Consensus-based time-varying formation problems for multi-AUVs systems are investigated and simulation of the formation theories to multi-AUVs are presented. On the formation control level, based on the dynamic characteristics of the AUV, a consensus-based formation protocol for multi-AUV systems to achieve time-varying formations is proposed. Then, formation problems are transformed into consensus problems. Necessary and sufficient conditions for multi-AUV systems to achieve time-varying formation are presented, and an explicit expression of the time-varying formation center function is given. The formations can be any time-varying vectors satisfying condition (12). Different from most cases of collaboration, the velocities of the multi-AUVs studied in this paper are not equal when they reach the cooperative state, and the research content is more general. In addition, the interaction topology of each AUV only needs to have a spanning tree, without the information mentioned previously. Furthermore, a procedure to design the protocol for multi-AUV systems to achieve time-varying formations is proposed. The method was then verified by three typical simulations. These results not only enrich the theoretical system of formation control, but also provide an important theoretical basis for practical application.

However, owing to the complexity and variability of the marine environment, the limitation of sensor accuracy, and the unreliability of underwater communication, it is often difficult to establish the expected effect with the traditional formation-control algorithm. Therefore, how to determine the practicability of AUV formation has become the focus of research.

Future research needs to pay more attention to practical application scenarios and to develop robust formation-control algorithms that can adapt to complex marine environments. In this way, we can ensure that AUVs can perform tasks stably and reliably under various harsh conditions.

In addition, the research content of this study was aimed at isomorphic AUVs, but the formation algorithm of heterogeneous AUVs is more valuable to study. For practicality, the heterogeneous AUV formation structure can make the formation system have stronger flexibility and adaptability to complete more complex tasks.

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