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Stress Wave Propagation in a Semi-Infinite Rayleigh–Love Rod under the Collinear Impact of a Striker Rod with Different General Impedances

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Abstract: This paper studies elastic stress wave propagation generated by an impact in a system consisting of a moving striker rod and an initially stationary semi-infinite rod. This research emphasizes the role of different general impedances in affecting the response during the wave propagation process. The Rayleigh–Love rod theory is used in this research to consider lateral inertia and Poisson’s effects on longitudinal waves in rods, as these factors lead to greater stress results compared to the traditional wave equation. To solve the complex wave equations in a Rayleigh–Love rod with different general impedances, the numerical inversion of Laplace transformation is applied and verified using the results of previous research. This study demonstrates that variations in general impedances cause different wave reflection and transmission behaviors at the interface. As the wave interacts with this discontinuity of impedance, it may be amplified or attenuated, and changes in impedance can significantly affect wave propagation behaviors.

Keywords: stress wave propagation; Rayleigh–Love rod; lateral inertia and Poisson’s effect; different general impedances; numerical inversion of Laplace transformation



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1. Introduction

The study of stress wave propagation is a fundamental aspect of materials science and engineering, with implications for the design, analysis, and application of mechanical systems subjected to dynamic loading conditions [1–9]. In particular, the interaction of stress waves at the interfaces of materials with differing impedances is critical for understanding the behavior of waves under impact loading. The traditional models often consider idealized conditions, neglecting factors such as lateral inertia and the Poisson effect, which can significantly influence the accuracy of the predictions in real-world situations [10,11]. The Rayleigh–Love rod theory [12–14], an advanced theoretical framework, incorporates these aspects to provide a more comprehensive understanding of one-dimensional stress wave propagation behavior.

This paper delves into the complex dynamics of stress wave propagation in a system composed of a moving striker rod and an initially stationary semi-infinite Rayleigh–Love rod. The focus is primarily on the role of different general impedances in modifying the wave’s response during its propagation through the system. The general impedance $Z = A\rho c$, where A is the cross-sectional area, ρ is the mass density, and c is the wave velocity of a rod, is a parameter that embodies the mechanical resistance against wave transmission at an interface, and it is a crucial factor in determining the wave’s behavior upon encountering a discontinuity of material properties. This research utilizes the Rayleigh–Love rod theory, which accounts for lateral inertia and the Poisson effect, to provide a more detailed insight into stress wave propagation in rods.

The novelty of this paper lies in addressing the challenges associated with the inversion of Laplace transforms for analyzing stress wave propagation, taking into account lateral

inertia and Poisson's effects. This represents a significant advancement over previous research [13,14], which only discussed impacts between a striker rod and a semi-infinite rod with the same impedance. To solve complex wave equations derived from the Rayleigh–Love rod theory with varying general impedances, we employ the approximate formula for numerical inversion of Laplace transforms introduced by Valsa et al. [15]. The precision achieved by this method can be very high, even in cases in which the transformations are irrational or transcendental functions of the complex variable. The algorithms can be easily and efficiently implemented using programming packages such as MATLAB or Mathematica. Based on the results obtained from the proposed method, this study demonstrates that alterations in general impedances at the interface between the striker rod and the semi-infinite rod lead to distinct reflection and transmission behaviors of the stress waves. These interactions can result in the amplification or attenuation of the wave, underscoring the significant impact that impedance variations can have on the propagation of stress waves.

Through a thorough analysis and comparison with the results of previous research, this paper confirms the accuracy of the findings and contributes to the body of knowledge by providing detailed insights into the effects of different general impedances on stress wave propagation in rods. The implications of this research are vast, offering valuable guidance for the design and evaluation of mechanical systems and materials subjected to dynamic impacts, ensuring their integrity and performance in practical applications.

2. Materials and Methods

The research model in this paper is presented in Figure 1. The governing equation of the Rayleigh–Love theory [12] for wave propagation in rods is as follows:

$$\frac{\partial^2 u_1}{\partial t^2} = c_{01}^2 \frac{\partial^2 u_1}{\partial x^2} + v_1^2 \kappa_1^2 \frac{\partial^4 u_1}{\partial x^2 \partial t^2}, \quad -L \leq x \leq 0, \quad (1)$$

$$\frac{\partial^2 u_2}{\partial t^2} = c_{02}^2 \frac{\partial^2 u_2}{\partial x^2} + v_2^2 \kappa_2^2 \frac{\partial^4 u_2}{\partial x^2 \partial t^2}, \quad 0 \leq x < \infty, \quad (2)$$

where $c_{01} = \sqrt{E_1/\rho_1}$ and $c_{02} = \sqrt{E_2/\rho_2}$ are the constant wave velocities without considering Poisson's effect in the striker rod and the semi-infinite rod, respectively. E_1 , E_2 are the Young's moduli values of the striker rod and the semi-infinite rod, respectively. ρ_1 and ρ_2 are the mass densities; v_1 and v_2 are the Poisson ratios; and κ_1 , κ_2 are the radii of gyration of the cross-section of the striker rod and the semi-infinite rod, respectively. The impedances are $Z_1 = A_1 \rho_1 c_1$ and $Z_2 = A_2 \rho_2 c_2$ for the striker rod and the semi-infinite rod, respectively. c_1 and c_2 are the constant wave velocities considering Poisson's effect [16] in the striker rod and the semi-infinite rod, respectively, and are determined as follows:

$$c_1 = \sqrt{\frac{(1-v_1)E_1}{(1+v_1)(1-2v_1)\rho_1}}, \quad (3)$$

$$c_2 = \sqrt{\frac{(1-v_2)E_2}{(1+v_2)(1-2v_2)\rho_2}}. \quad (4)$$

The initial conditions in terms of displacement u can be written in a piecewise manner:

$$u_1(x, 0) = 0, \quad \frac{\partial u_1(x, 0)}{\partial t} = 2v_0, \quad -L \leq x \leq 0, \quad (5)$$

$$u_2(x, 0) = 0, \quad \frac{\partial u_2(x, 0)}{\partial t} = 0, \quad 0 \leq x < \infty, \quad (6)$$

The boundary conditions and continuity conditions at $x = -L$, $x = 0$, and $x = \infty$ are given by:

$$E_1 A_1 \frac{\partial u_1}{\partial x}(-L, t) = 0, \quad x = -L, \quad (7)$$

$$u_1(0^-, t) = u_2(0^+, t), \quad x = 0, \quad (8)$$

$$E_1 A_1 \frac{\partial u_1(0^-, t)}{\partial x} = E_2 A_2 \frac{\partial u_2(0^+, t)}{\partial x}, \quad x = 0, \quad (9)$$

$$\lim_{x \rightarrow \infty} u_2(x, t) = 0, \quad x = \infty, \quad t \geq 0. \quad (10)$$

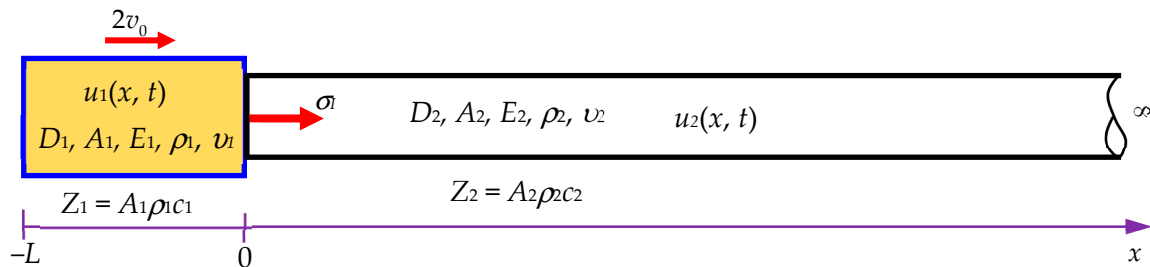


Figure 1. Stress wave propagation in a semi-infinite rod under the collinear impact of the striker rod with different impedances.

To make the Laplace transform process more convenient, we can convert the quantities to dimensionless quantities for substitution into Equations (1) and (2). This is achieved by introducing the following dimensionless variables and parameters:

$$\begin{aligned} \bar{u}_1 &= \frac{u_1}{D_2}, \quad \bar{u}_2 = \frac{u_2}{D_2}, \quad \bar{x} = \frac{x}{D_2}, \quad \bar{t} = \bar{t}_2 = \frac{c_2 t}{D_2}, \quad \bar{t}_1 = \frac{c_1 t}{D_2} = \frac{c_1}{c_2} \bar{t}, \\ \bar{L} &= \frac{L}{D_2}, \quad \bar{\kappa}_1 = \frac{\kappa_1}{D_2}, \quad \bar{\kappa}_2 = \frac{\kappa_2}{D_2}, \quad b_1 = v_1 \bar{\kappa}_1, \quad b_2 = v_2 \bar{\kappa}_2, \quad \bar{v}_0 = \frac{v_0}{c_1}, \end{aligned} \quad (11)$$

where D_2 is the diameter of the semi-infinite rod.

Substituting Equation (11) into Equations (1) and (2), we obtain:

$$\frac{\partial^2 \bar{u}_1}{\partial \bar{t}^2} = m^2 \frac{\partial^2 \bar{u}_1}{\partial \bar{x}^2} + b_1^2 \frac{\partial^4 \bar{u}_1}{\partial \bar{x}^2 \partial \bar{t}^2}, \quad -\bar{L} \leq \bar{x} \leq 0 \quad (12)$$

$$\frac{\partial^2 \bar{u}_2}{\partial \bar{t}^2} = \frac{\partial^2 \bar{u}_2}{\partial \bar{x}^2} + b_2^2 \frac{\partial^4 \bar{u}_2}{\partial \bar{x}^2 \partial \bar{t}^2}, \quad 0 \leq \bar{x} < \infty \quad (13)$$

where $m = \frac{c_1}{c_2}$.

Apply Laplace transformation to find the displacement $U_1(\bar{x}, s)$ and $U_2(\bar{x}, s)$ in s domain (see in Appendix A) as follows:

$$U_1(\bar{x}, s) = \frac{-2\bar{v}_0 E_2 A_2 \sqrt{m^2 + b_1^2 s^2} \left(e^{\frac{\bar{x}s}{\sqrt{m^2 + b_1^2 s^2}}} + e^{\frac{-(\bar{x} + 2\bar{L})s}{\sqrt{m^2 + b_1^2 s^2}}} \right)}{s^2 \left((E_2 A_2 \sqrt{m^2 + b_1^2 s^2} - E_1 A_1 \sqrt{1 + b_2^2 s^2}) e^{\frac{-2\bar{L}s}{\sqrt{m^2 + b_1^2 s^2}}} + E_2 A_2 \sqrt{m^2 + b_1^2 s^2} + E_1 A_1 \sqrt{1 + b_2^2 s^2} \right)} + \frac{2\bar{v}_0}{s^2}, \quad -\bar{L} \leq \bar{x} \leq 0 \quad (14)$$

$$U_2(\bar{x}, s) = \frac{2\bar{v}_0 \left(1 - e^{\frac{-2\bar{L}s}{\sqrt{m^2 + b_1^2 s^2}}} \right) E_1 A_1 \sqrt{1 + b_2^2 s^2}}{s^2 \left((E_2 A_2 \sqrt{m^2 + b_1^2 s^2} - E_1 A_1 \sqrt{1 + b_2^2 s^2}) e^{\frac{-2\bar{L}s}{\sqrt{m^2 + b_1^2 s^2}}} + E_2 A_2 \sqrt{m^2 + b_1^2 s^2} + E_1 A_1 \sqrt{1 + b_2^2 s^2} \right)} e^{\frac{-\bar{x}s}{\sqrt{1 + b_2^2 s^2}}}, \quad 0 \leq \bar{x} < \infty \quad (15)$$

When $E_1 = E_2$, $A_1 = A_2$, $\rho_1 = \rho_2$, $v_1 = v_2$, $c_1 = c_2$, $m = 1$, and $b_1 = b_2 = b$, Equations (14) and (15) are respectively reduced to:

$$U_1(\bar{x}, s) = \frac{\bar{v}_0}{s^2} \left(2 - e^{\frac{\bar{x}s}{\sqrt{1 + b^2 s^2}}} - e^{\frac{-(\bar{x} + 2\bar{L})s}{\sqrt{1 + b^2 s^2}}} \right), \quad -\bar{L} \leq \bar{x} \leq 0 \quad (16)$$

$$U_2(\bar{x}, s) = \frac{\bar{v}_0}{s^2} \left(e^{\frac{-\bar{x}s}{\sqrt{1 + b^2 s^2}}} - e^{\frac{-(\bar{x} + 2\bar{L})s}{\sqrt{1 + b^2 s^2}}} \right), \quad 0 \leq \bar{x} < \infty \quad (17)$$

Equations (16) and (17) are the transformed functions of a semi-infinite rod under the impact of a striker rod of the same material and cross-sectional area, as provided by previous research [14]. This verifies the correctness of Equations (14) and (15).

3. Inverse Laplace Transform

Inverse Laplace transformation of Equations (14) and (15) is challenging, so a numerical method was utilized to perform this task. By using the inverse Laplace transform method provided by Valsa et al. [15], we can compute the inverse Laplace transforms directly and effectively. The derivation of the inverse Laplace transform by Valsa et al. was carried out from the basic transform, as follows:

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st} ds, \quad (18)$$

where the integration variable is $s = \gamma + i\omega$.

The transform $F(s)$ is supposed to fulfil the following assumptions:

1. $F(s)$ is regular for $\text{Re}\{s\} > 0$.
2. $\lim_{|s| \rightarrow \infty} F(s) = 0$.
3. $F^*(s) = F(s^*)$, where the asterisk denotes the complex conjugate value.

To simplify the evaluation of the integral, we approximate the exponential function as follows:

$$e^{st} \approx E_c(st, a) = \frac{e^a}{2 \cosh(a - st)} = \frac{e^a}{e^{-a}e^{st} + e^a e^{-st}} = \frac{e^{st}}{1 + e^{-2a}e^{2st}}, \quad (19)$$

where $E_c(st, a)$ is an approximate expression of e^{st} using the hyperbolic cosine function $\cosh(x)$, with $x = a - st$, and a is a real constant.

When $a > \gamma t$, then $|e^{-2a}e^{2st}| < 1$, and the fraction $\frac{1}{1+e^{-2a}e^{2st}}$ can be expanded as a convergent Maclaurin series as follows:

$$E_c(st, a) = e^{st} + \sum_{n=1}^{\infty} (-1)^n e^{-2na} e^{(2n+1)st}. \quad (20)$$

The sum in the expression represents the error of approximation of the exponential function that can be arbitrarily suppressed by a choice of the parameter a . Then, Equation (18) becomes:

$$\begin{aligned} f(t) &\approx f_c(t, a) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s) E_c(st, a) ds = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s) e^{st} ds + \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s) \sum_{n=1}^{\infty} (-1)^n e^{-2na} e^{(2n+1)st} ds \\ &= f(t) + \sum_{n=1}^{\infty} (-1)^n e^{-2na} f(2n+1)t = f(t) - e^{-2a} f(3t) + e^{-4a} f(5t) - \dots = f(t) + \varepsilon_c(t, a) \end{aligned} \quad (21)$$

where $f_c(t, a)$ is the approximate function of $f(t)$.

Moreover, the error $\varepsilon_c(t, a)$ in Equation (21) of the inverse transform can be kept at low values by appropriate choice of the parameter a . The function $E_c(st, a)$ can be expressed as an infinite sum of rational functions in st , since (see [17]):

$$\frac{1}{\cosh z} = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (n + \frac{1}{2})}{(n + \frac{1}{2})^2 \pi^2 + z^2}. \quad (22)$$

Combining Equation (19), Equation (21), and Equation (22), we have:

$$E_c(st, a) = \pi e^a \sum_{n=0}^{\infty} \frac{(-1)^n (n + \frac{1}{2})}{(n + \frac{1}{2})^2 \pi^2 + (a - st)^2}. \quad (23)$$

Substituting Equation (23) into Equation (21), we obtain:

$$f_c(t, a) = \frac{e^a}{2i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s) \sum_{n=0}^{\infty} \frac{(-1)^n (n + \frac{1}{2})}{(n + \frac{1}{2})^2 \pi^2 + (a - st)^2} ds. \quad (24)$$

By interchanging the sequence of summation and integration, we obtain:

$$f_c(t, a) = \frac{e^a}{2i} \sum_{n=0}^{\infty} (-1)^n (n + \frac{1}{2}) I_n, \quad (25)$$

$$\text{where } I_n = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(s)}{(n + \frac{1}{2})^2 \pi^2 + (a - st)^2} ds = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(s)}{G_n(s)} ds = \int_{\gamma-i\infty}^{\gamma+i\infty} H_n(s) ds. \quad (26)$$

To evaluate the integral, we completed the integration path using an arc contour with an infinite radius. Due to Assumption 2 of Laplace transform, the function $F(s)$ equals zero along this semicircle. The integral I_n is then equal to the sum of the residuals of $H_n(s)$ in the poles lying inside the integration contour (see Figure 2), multiplied by $2\pi i$. The poles of $H_n(s)$, in turn, are the poles of $F(s)$ (these are concentrated in $\text{Re}\{s\} \leq 0$ plus the roots of the equation $G_n(s) = 0$, as follows:

$$s_{1,2} = \frac{a \pm i(n + \frac{1}{2})\pi}{t}, \quad (27)$$

which lie in the right half-plane, to the right of the original path of integration $\text{Re}\{s\} = \gamma$. It is therefore advantageous to close the path of integration here and to calculate I_n as follows:

$$I_n = -2\pi i \left[\text{res} H_n(s)|_{s=s_1} + \text{res} H_n(s)|_{s=s_2} \right] = -2\pi i \left[\frac{F(s_1)}{G'_n(s_1)} + \frac{F(s_2)}{G'_n(s_2)} \right], \quad (28)$$

where $G'(s) = \frac{dG(s)}{ds}$.

The negative sign results from the negative sense of integration. Obviously, $G'_n(s) = -2t(a - st)$; therefore,

$$\begin{aligned} I_n &= -2\pi i \left[\frac{F(s_1)}{2\pi i(n + \frac{1}{2})} - \frac{F(s_2)}{2\pi i(n + \frac{1}{2})} \right] = \frac{F(s_1) - F(s_2)}{t(n + \frac{1}{2})} = \frac{F^*(s_1) - F(s_1)}{t(n + \frac{1}{2})} \\ &= \frac{-2\text{Im}\{F(s_1)\}}{t(n + \frac{1}{2})} = \frac{-2\text{Im}\{F[a + i(n + \frac{1}{2})\pi/t]\}}{t(n + \frac{1}{2})}. \end{aligned} \quad (29)$$

Finally, we obtain approximate formulae for the numerical inversion of Laplace transformation, as follows:

$$f_c(t, a) = -\frac{e^a}{t} \sum_{n=0}^{\infty} (-1)^n \text{Im} \left\{ F \left[\frac{a}{t} + i(n + \frac{1}{2}) \frac{\pi}{t} \right] \right\}. \quad (30)$$

In the programming process, as the value of n cannot start from 0 but must begin with 1, Equation (30) is adjusted as follows:

$$f_c(t, a) = -\frac{e^a}{t} \sum_{n=1}^{n_{end}} (-1)^n \operatorname{Im} \left\{ F \left[\frac{a}{t} + i(n-1) \frac{\pi}{t} \right] \right\}, \quad (31)$$

where $f_c(t, a)$ denotes the calculated inverse Laplace transformation of the function $F(s)$, dependent on time t and the convergence parameter a . $F[a/t + i(n-1)\pi/t]$ is Laplace transform function $F(s)$, evaluated at specific points in the complex plane. The argument of F is a complex number, with a/t being the real part and $i(n-1)\pi/t$ being the imaginary part; the term i is the imaginary unit; n_{end} is the final step of the loop.

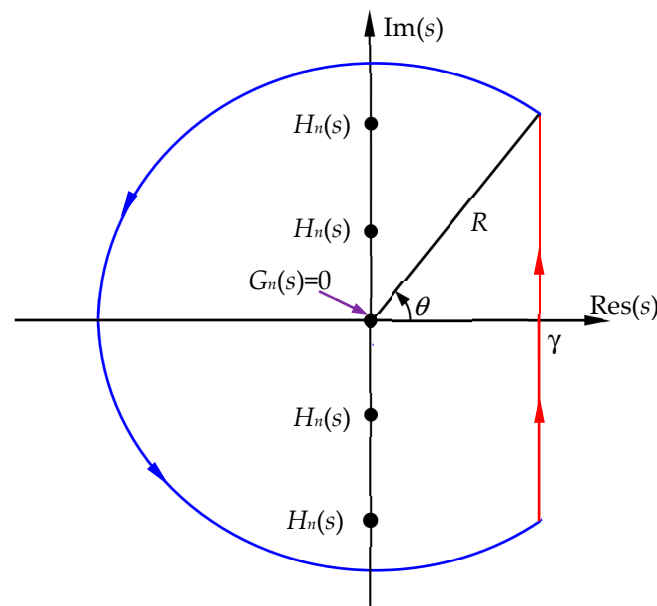


Figure 2. Contour integration path with residues in the complex plane.

The process of numerically inverting Laplace transforms is essential in dissecting complex systems, and MATLAB offers a key function, denoted as $f(t) = \text{INV LAP}\{F(s)\}$ [18]. To accomplish this task, we utilized the inverse Laplace transform technique offered by Valsa et al. [15], which enabled us to carry out the calculation of inverse Laplace transforms directly and efficiently. The flexibility of the $F(s)$ function within INV LAP is notable; it adeptly handles a diverse array of complex variable functions where s is raised to an exponent a . This exponent can be any real value, including both integers and fractions, providing INV LAP with the ability to solve fractional problems and invert functions that consist of both rational and irrational components. One of the primary benefits of utilizing INV LAP is its ability to bypass the often-complex step of calculating the poles or zeros of $F(s)$. The function instead evaluates $F(s)$ at chosen complex values of s . Based on Equation (31), we wrote a code for the inverse Laplace transformation of Equations (14) and (15) to find the displacement functions $\bar{u}_1(\bar{x}, \bar{t})$ and $\bar{u}_2(\bar{x}, \bar{t})$.

Then, the velocities in the striker rod and the semi-infinite rod are determined:

$$\bar{v}_1(\bar{x}, \bar{t}) = \frac{\partial \bar{u}_1}{\partial \bar{t}}, \quad -\bar{L} \leq \bar{x} \leq 0, \quad (32)$$

$$\bar{v}_2(\bar{x}, \bar{t}) = \frac{\partial \bar{u}_2}{\partial \bar{t}}, \quad 0 \leq \bar{x} < \infty. \quad (33)$$

The stress wave propagation in the striker rod and the semi-infinite rod is determined as follows:

$$\bar{\sigma}_1(\bar{x}, \bar{t}) = \frac{\partial \bar{u}_1}{\partial \bar{x}} + \frac{b_1^2}{m^2} \frac{\partial^3 \bar{u}_1}{\partial \bar{x} \partial \bar{t}^2}, \quad -\bar{L} \leq \bar{x} \leq 0, \quad (34)$$

$$\bar{\sigma}_2(\bar{x}, \bar{t}) = \frac{\partial \bar{u}_2}{\partial \bar{x}} + b_2^2 \frac{\partial^3 \bar{u}_2}{\partial \bar{x} \partial \bar{t}^2}, \quad 0 \leq \bar{x} < \infty. \quad (35)$$

The algorithm schematic for the numerical inversion Laplace transform process is illustrated in Figure 3. Applying the numerical inversion Laplace transform to Equations (14) and (15), we obtain the displacement functions $\bar{u}_1(\bar{x}, \bar{t})$ and $\bar{u}_2(\bar{x}, \bar{t})$, respectively. From the obtained results, Equations (32) and (33) are used to calculate the corresponding velocities $\bar{v}_1(\bar{x}, \bar{t})$ and $\bar{v}_2(\bar{x}, \bar{t})$, respectively. Furthermore, stress functions $\bar{\sigma}_1(\bar{x}, \bar{t})$ and $\bar{\sigma}_2(\bar{x}, \bar{t})$ are derived by applying Equations (34) and (35), respectively. Finally, these non-dimensional quantities are converted to dimensional forms using Equation (11), resulting in the physical measures of displacements $u_1(x, t)$ and $u_2(x, t)$, velocities $v_1(x, t)$ and $v_2(x, t)$, and stresses $\sigma_1(x, t)$ and $\sigma_2(x, t)$ for the striker and the semi-infinite rod, respectively. The obtained results should be compared to previous research to confirm the accuracy of the proposed method.

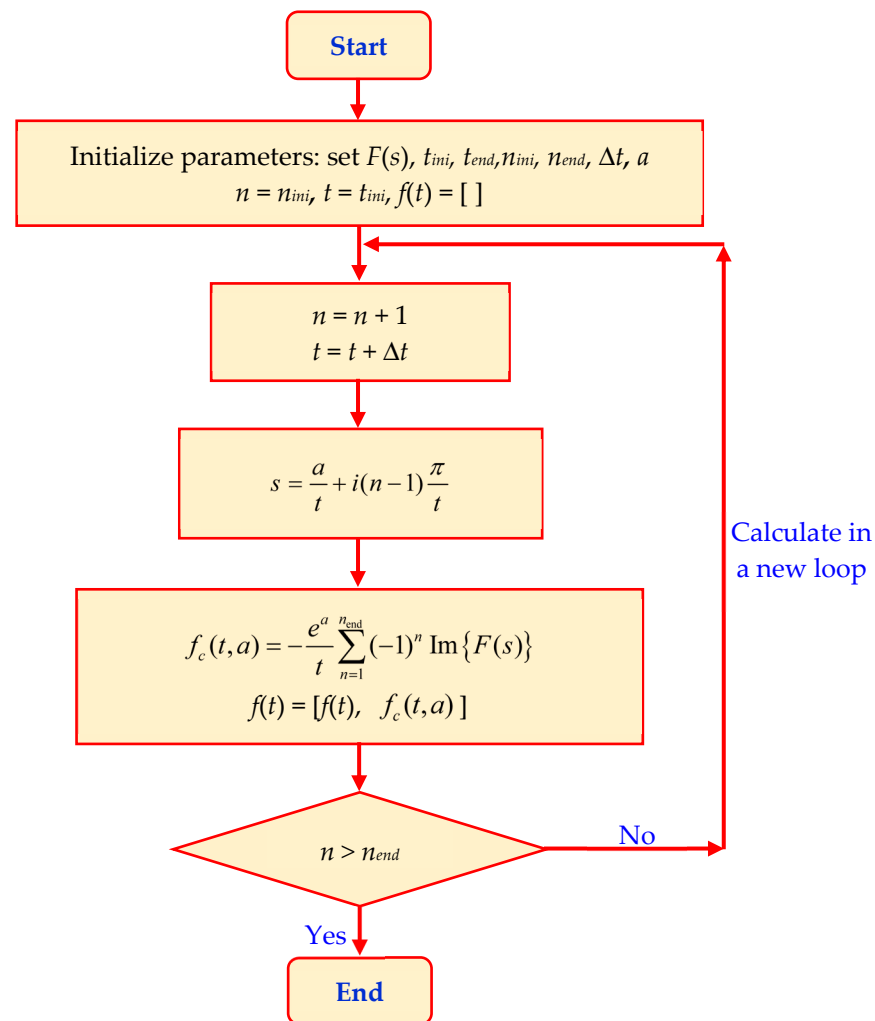


Figure 3. Flowchart of numerical inversion of Laplace transformation.

4. Validation of the Proposed Numerical Algorithm for the Inversion of Laplace Transformation

4.1. Example 1

To confirm the accuracy of the proposed method and the computational analysis code, consider the initial value problem as follows

$$f''(t) + 9f(t) = 2 \sin 3t, \quad (36)$$

given the initial conditions are:

$$f(0) = 1, f'(0) = 0. \quad (37)$$

Using Laplace transform to find $f(t)$, take Laplace transform of both sides of Equation (36):

$$L\{f''(t)\} + L\{9f(t)\} = L\{2\sin 3t\}, \quad (38)$$

$$s^2F(s) - sf(0) - f'(0) + 9F(s) = \frac{6}{s^2 + 9}. \quad (39)$$

Substituting Equation (37) into Equation (39), we obtain $F(s)$ as follows:

$$F(s) = \frac{6}{(s^2 + 9)^2} + \frac{s}{s^2 + 9}. \quad (40)$$

Take the inverse Laplace transform Equation (40) to find $f(t)$ as follows:

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{6}{(s^2 + 9)^2}\right\} + L^{-1}\left\{\frac{s}{s^2 + 9}\right\}. \quad (41)$$

The inverse Laplace transform of the second term is:

$$L^{-1}\left\{\frac{s}{s^2 + 9}\right\} = \cos 3t. \quad (42)$$

Since the inverse Laplace transform of the first term is more complicated, we can use the convolution theorem [19] to find the Laplace transform of the first term:

$$L^{-1}\left\{\frac{6}{(s^2 + 9)^2}\right\} = L^{-1}\left\{\frac{2}{3} \frac{3}{(s^2 + 9)} \frac{3}{(s^2 + 9)}\right\} = \frac{2}{3}(f * g)(t), \quad (43)$$

where $f(t) = g(t) = \sin 3t$, the convolution of Equation (43) is determined as:

$$L^{-1}\left\{\frac{6}{(s^2 + 9)^2}\right\} = \frac{2}{3} \int_0^t \sin 3\tau \sin 3(t - \tau) d\tau. \quad (44)$$

Evaluating the integral of Equation (44), we obtain:

$$\frac{2}{3} \int_0^t \sin 3\tau \sin 3(t - \tau) d\tau = \frac{1}{9} \sin 3t - \frac{1}{3} t \cos 3t. \quad (45)$$

Thus, the exact solution of Equation (36) is:

$$f(t) = \frac{1}{9} \sin 3t - \frac{1}{3} t \cos 3t + \cos 3t. \quad (46)$$

Instead of using the convolution theorem [19], we can apply the numerical inversion of the Laplace transform for $F(s)$ as presented in Section 3. The results between the two methods are shown in Figure 4, which demonstrates a high degree of agreement between the exact and numerical solutions, thereby validating the effectiveness and accuracy of the numerical approach for the numerical inversion of the Laplace transform.

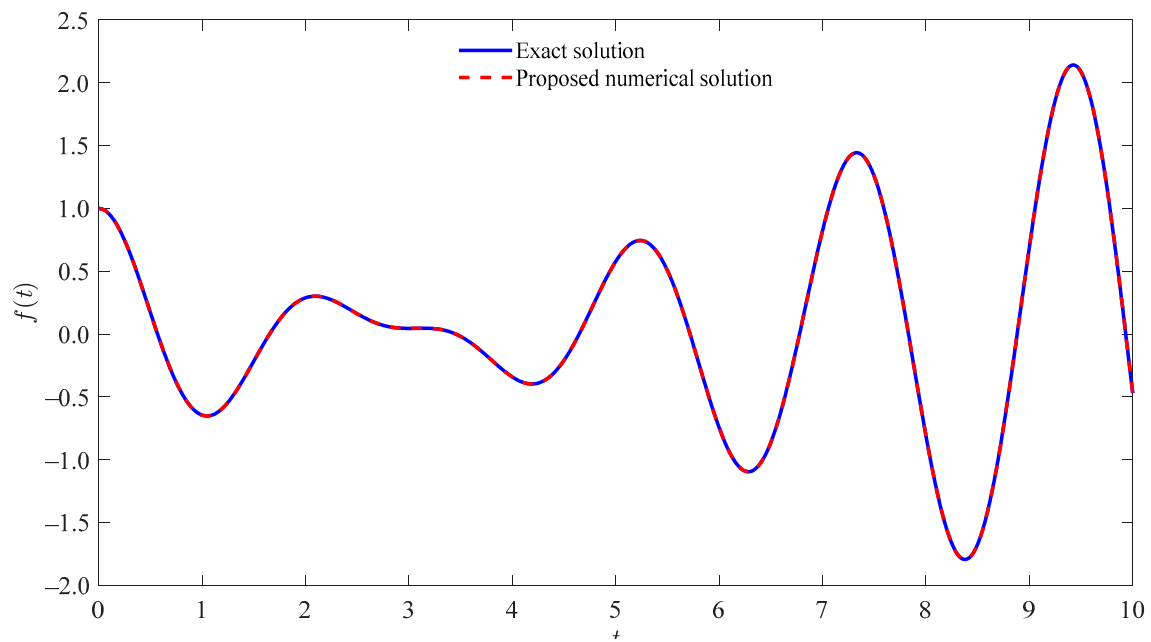


Figure 4. Comparison of the exact solution and the proposed numerical solution of Equation (36).

4.2. Example 2

Example 2 demonstrates the application of the proposed method for solving the Rayleigh–Love wave equation, as modeled in Figure 5. A comparison was performed between the results obtained from approximate formulae for numerical inversion of Laplace transforms and Yang’s solution as presented in his research paper [14]. The parameters used in the model are shown in Table 1. The wave velocity (c) considering Poisson’s effect [16] in the striker rod and the semi-infinite rod is determined as follows

$$c = \sqrt{\frac{(1 - \nu)E}{(1 + \nu)(1 - 2\nu)\rho}}, \quad (47)$$

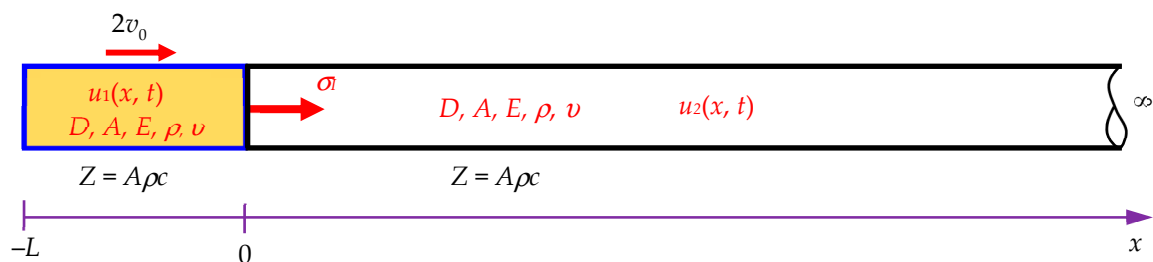


Figure 5. Stress wave propagation in the rods under the impact of a striker rod with the same impedances.

Table 1. Parameters of material properties and geometries of the striker rod and the semi-infinite rod under an initial impact velocity of $2v_0$.

Parameters	Striker Rod	Semi-Infinite Rod
Diameter, D (mm)	30 ($A_1/A_2 = 1$)	30
Young’s modulus, E (GPa)	195 ($E_1/E_2 = 1$)	195
Mass density, ρ (kg/m ³)	7850 ($\rho_1/\rho_2 = 1$)	7850
Poisson’s ratio, ν	0.3	0.3
Rod length, L (m)	0.2	∞
Initial velocity, $2v_0$ (m/s)	11.6	0
Wave speed, c (m/s)	5782.69	5782.69
Impedance, $Z = A\rho c$ (kg/s)	32,087.204	32,087.204

A detailed comparison of the stress propagation over time at varying positions, with the results obtained from both methods, is illustrated in Figures 6–8. These figures illustrate a comparison between the results from the proposed numerical solution and Yang’s analytical solution [14]. These results validate the reliability and accuracy of the numerical approach for analyzing stress wave propagation in the semi-infinite Rayleigh–Love rod. Figures 6–8 illustrate the dynamic behavior of displacement, velocity, and stress at different positions in the striker rod and the semi-infinite Rayleigh–Love rod due to the impact of an initial velocity $2v_0$. This demonstrates that the results from the proposed numerical method adequately match those of Yang’s analytical solution. This agreement highlights the validity of the results obtained from the numerical analysis algorithm and confirms its correctness in predicting the behavior of stress waves in Rayleigh–Love rods. As shown in Figure 8, the number of oscillations of the stress decreases as x increases, due to the effects of lateral inertia and Poisson’s ratio. This results in the dispersion of the signal during the propagation in the x -axis because of the effect of the term $v^2\kappa^2\frac{\partial^4 u}{\partial x^2\partial t^2}$ in the governing equation of Rayleigh–Love rod theory.

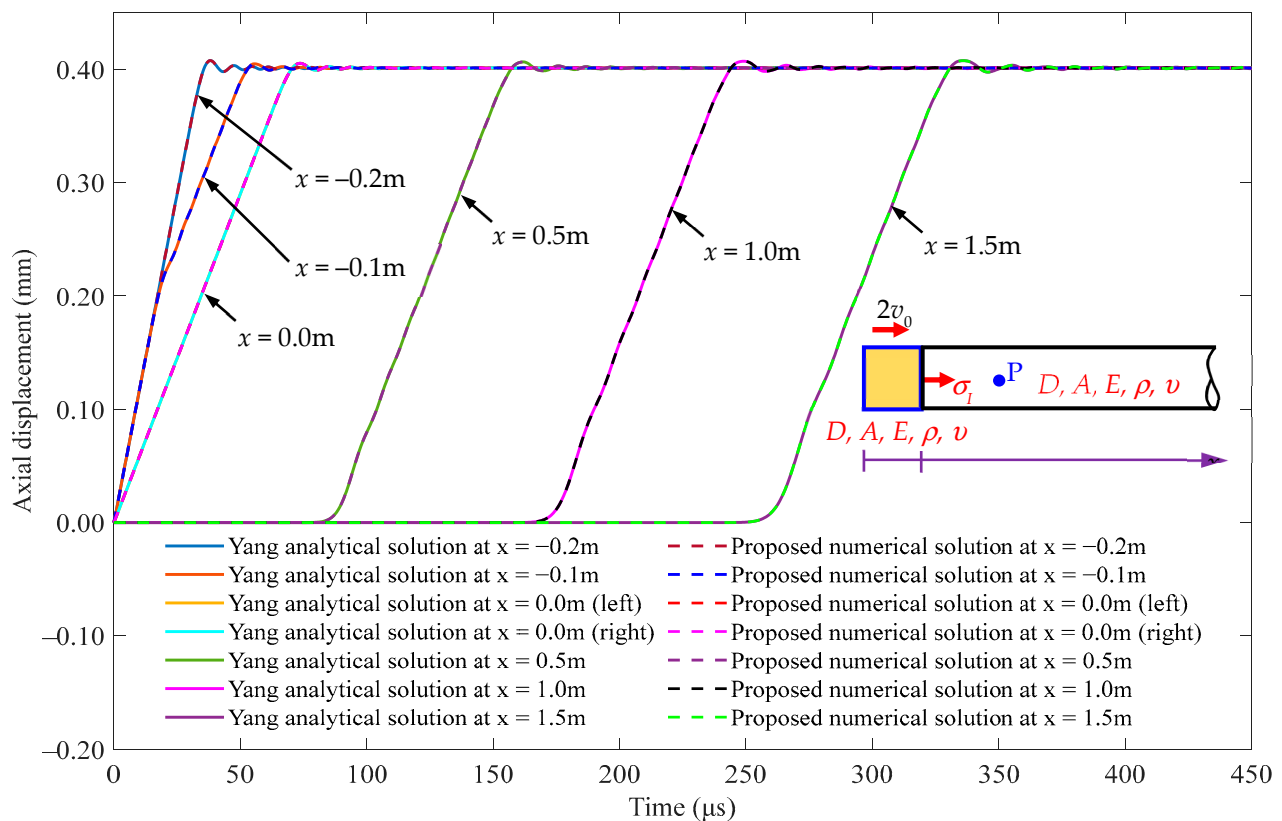


Figure 6. Comparison of axial displacement history between the proposed numerical solution and Yang’s analytical solution at various locations in the rods.

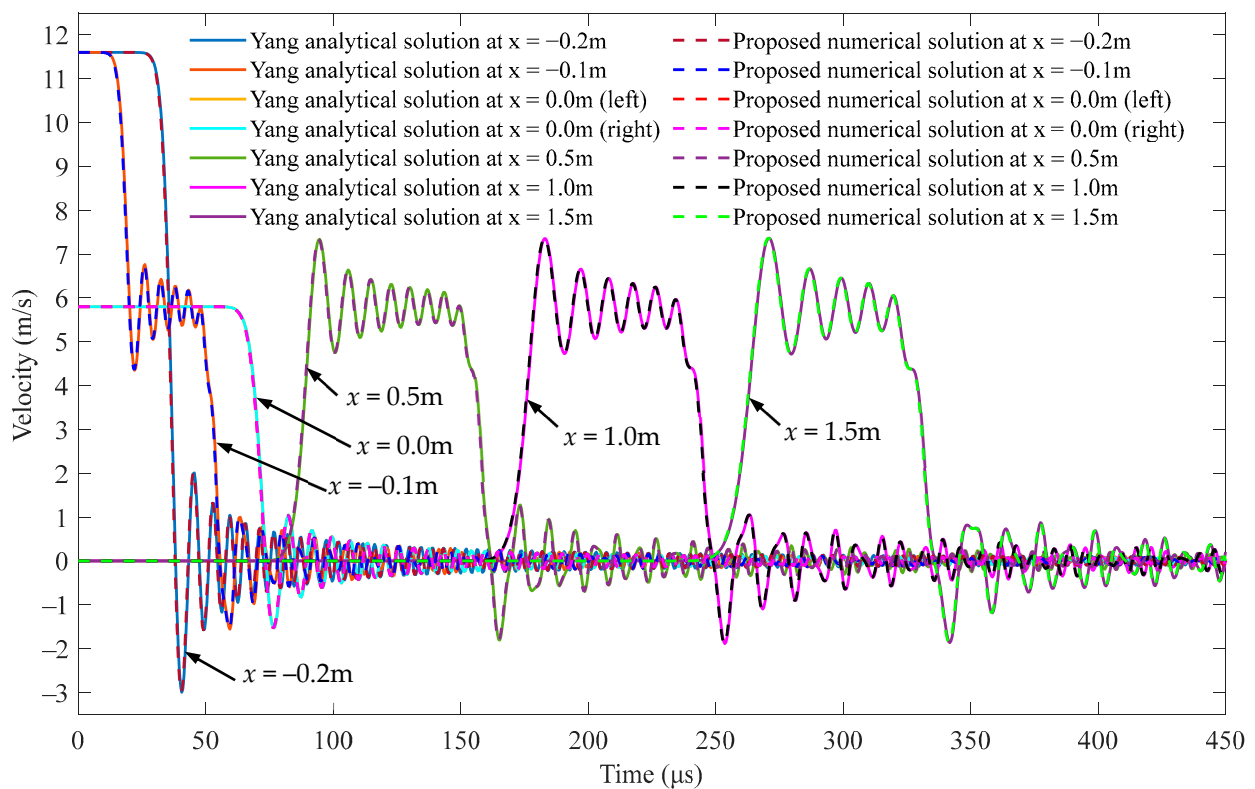


Figure 7. Comparison of velocity history between the proposed numerical solution and Yang's analytical solution at various locations in the rods.

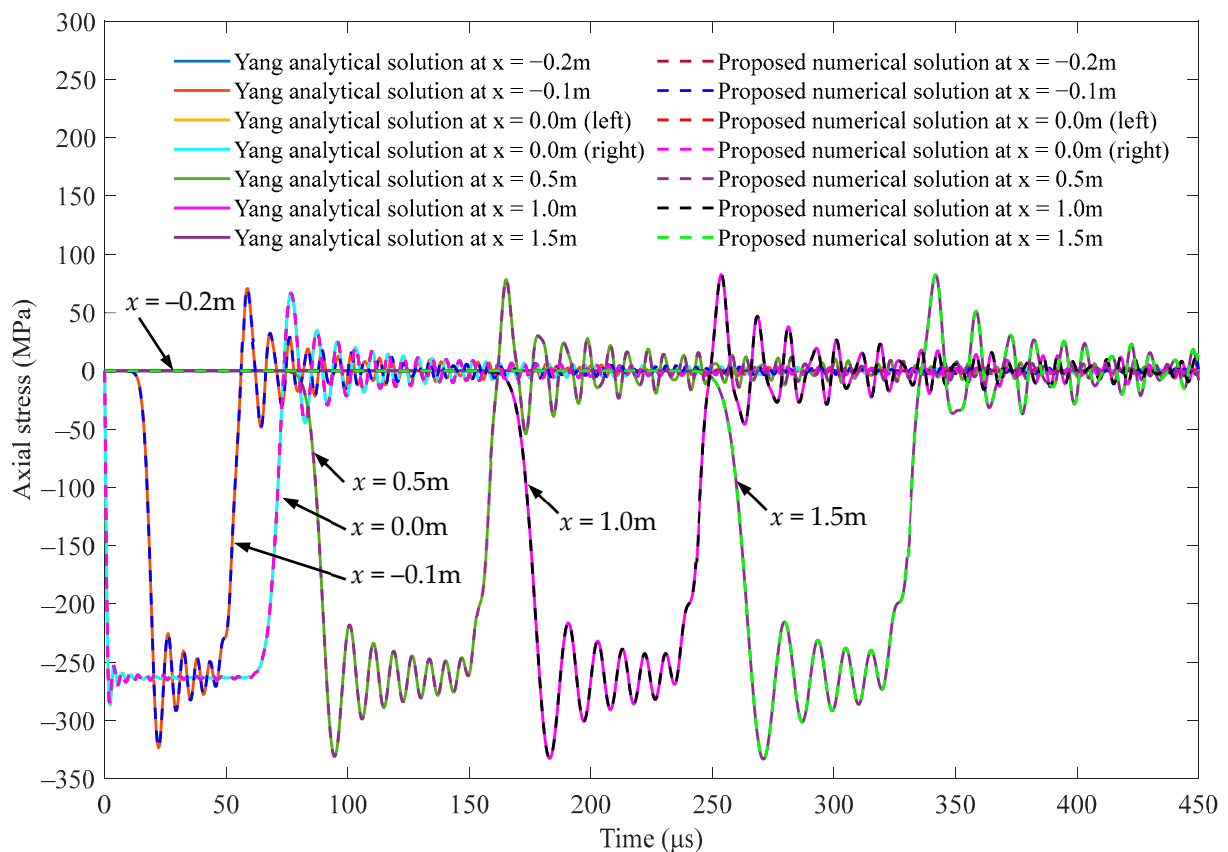


Figure 8. Comparison of stress wave history between the proposed numerical solution and Yang's analytical solution at various locations in the rods.

5. Investigation of Wave Propagation in a Rayleigh–Love Rod under the Collinear Impact of a Striker Rod with Different Impedances

This section verifies stress wave propagation in a rod under the collinear impact of a striker rod with different impedances. The proposed numerical solution is compared to the results provided by Shin et al. [11], which do not include Poisson’s effect, the stress of the striker rod in the k th cycle is:

$$\sigma_{1,k} = \frac{\rho_2 A_2 c_2}{\rho_1 A_1 c_1 + \rho_2 A_2 c_2} \rho_1 c_1 v_{p,k} = \left(\frac{\rho_2 A_2 c_2}{\rho_1 A_1 c_1 + \rho_2 A_2 c_2} \right) \left(\frac{\rho_1 A_1 c_1 - \rho_2 A_2 c_2}{\rho_1 A_1 c_1 + \rho_2 A_2 c_2} \right)^{k-1} \rho_1 c_1 v_i, \quad k = 1, 2, 3, 4, \dots, \quad (48)$$

The stress of the semi-infinite rod in the k th cycle is:

$$\sigma_{2,k} = \left(\frac{\rho_1 A_1 c_1}{\rho_1 A_1 c_1 + \rho_2 A_2 c_2} \right) \rho_2 c_2 v_{p,k} = \left(\frac{\rho_1 A_1 c_1}{\rho_1 A_1 c_1 + \rho_2 A_2 c_2} \right) \left(\frac{\rho_1 A_1 c_1 - \rho_2 A_2 c_2}{\rho_1 A_1 c_1 + \rho_2 A_2 c_2} \right)^{k-1} \rho_2 c_2 v_i, \quad k = 1, 2, 3, 4, \dots \quad (49)$$

where $v_{1,k}$ is the particle velocity of the striker rod in the cycle k th and $v_{2,k}$ is the particle velocity of the semi-infinite rod in the cycle k th.

5.1. Investigation of Stress Wave Propagation in a Semi-Infinite Rod and a Striker Rod at the Interface $x = 0$ m

Based on the stress equation with cyclically varying values without considering the effects of lateral inertia and Poisson’s ratio on the striker rod (Equation (48)) and the semi-infinite rod (Equation (49)), we will now compare the results with those of the proposed numerical solution. This comparison aims to evaluate the results obtained from modifications in the cross-sectional area and material properties of the rods, as indicated in Table 2. In the first case, we investigate the position at $x = 0$, where the interaction between the two rods occurs upon impact. The results obtained are illustrated in Figure 9 and Table 3.

Table 2. Parameters of the striker rod and the semi-infinite rod used in the model under the impact of different impedances.

Parameters	Striker Rod	Semi-Infinite Rod
Diameter, D (mm)	21.213 ($A_1/A_2 = 0.5$) 30 ($A_1/A_2 = 1$) 42.426 ($A_1/A_2 = 2$)	30
Young’s modulus, E (GPa)	195	195
Mass density, ρ (kg/m ³)	7850	7850
Poisson’s ratio, ν	0.3	0.3
Yield stress, σ_0 (MPa)	1275	1275

Table 3. Stress variation in rods without Poisson’s effect (Equations (48) and (49)) over the cycles at $x = 0$ m with different impedances in the striker rod and the semi-infinite rod under the initial impact velocity $2v_0$.

Cycle	$Z_1 = 0.5Z_2$			$Z_1 = Z_2$			$Z_1 = 2Z_2$		
	Time (μs)	σ_1 (MPa)	σ_2 (MPa)	Time (μs)	σ_1 (MPa)	σ_2 (MPa)	Time (μs)	σ_1 (MPa)	σ_2 (MPa)
1	0~69.17	−351.05	−175.52	0~69.17	−263.286	−263.286	0~69.17	−175.52	−351.05
2	69.17~138.34	117.02	58.508	69.17~138.34	0	0	69.17~138.34	−58.51	−117.02
3	138.34~207.52	−39.01	−19.50	138.34~207.52	0	0	138.34~207.52	−19.50	−39.01
4	207.52~276.69	13.00	6.50	207.52~276.69	0	0	207.52~276.69	−6.501	−13.00
5	276.69~345.86	−4.33	−2.17	276.69~345.86	0	0	276.69~345.86	−2.17	−4.33

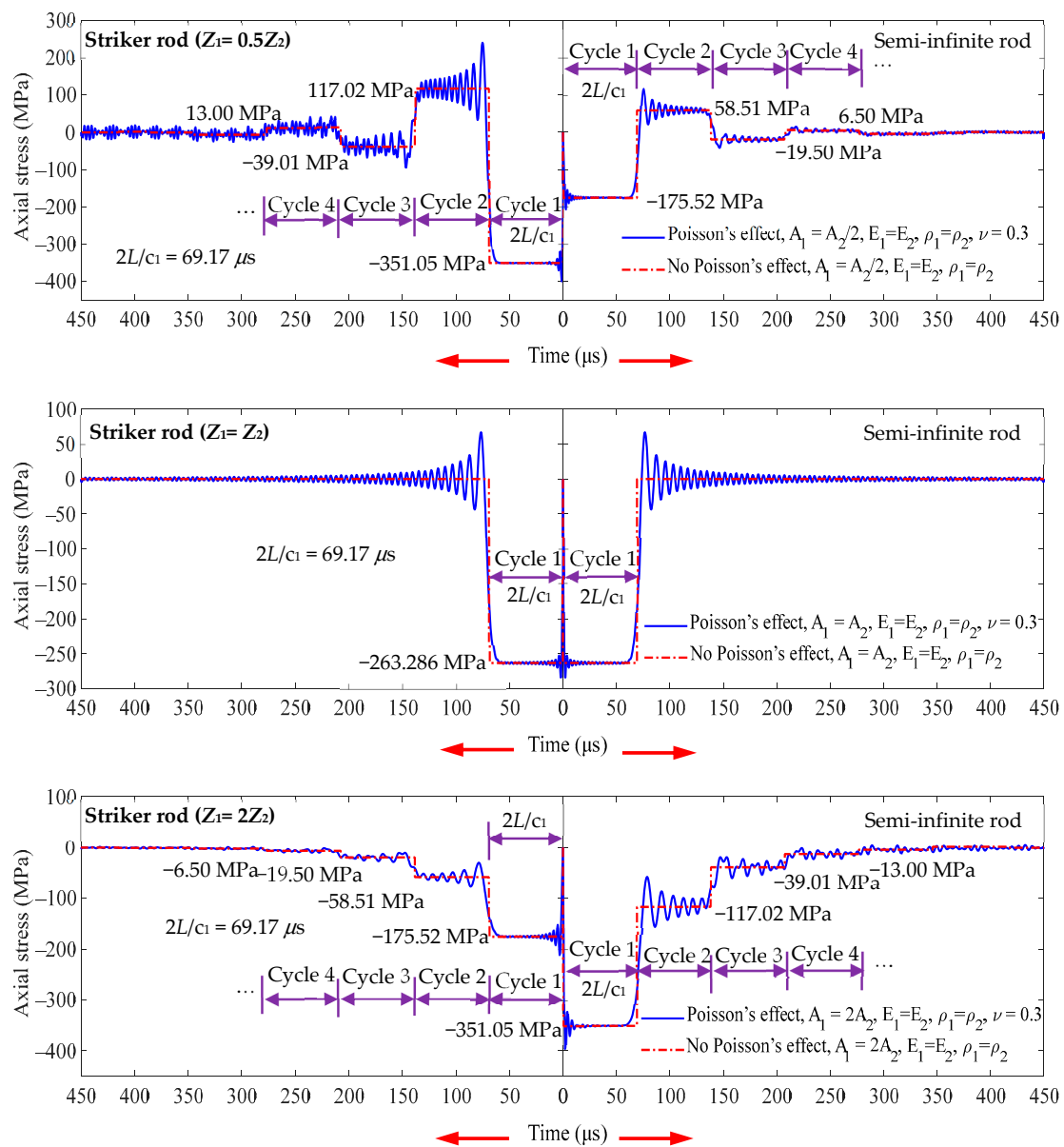


Figure 9. Stress wave propagation was observed at the contact surface ($x = 0$ m) with the different impedances of the striker rod ($A_2 = 7.069 \times 10^{-4} \text{ m}^2$, $E_1 = E_2 = 195 \text{ GPa}$, $\nu_1 = \nu_2 = 0.3$, $\rho_1 = \rho_2 = 7850 \text{ kg/m}^3$, $2v_0 = 11.6 \text{ m/s}$).

In Table 3 and Figure 9, the results of comparing stress wave propagation when varying the cross-sectional area of the striker rod for three cases ($Z_1/Z_2 = 0.5$, $Z_1/Z_2 = 1$, and $Z_1/Z_2 = 2$) are presented. The results obtained from the proposed numerical solution and the solutions with no Poisson's effect reported in Shin and Kim's paper [11] are in good agreement. When considering the influence of Poisson's effect, the stress wave shapes oscillate around the without Poisson's effect line until reaching a steady state, at which point the two results are the same. When comparing the three cases of varying cross-sectional areas, a common observation is that compressive stress is generated in the first cycle for all cases, with a significant variation observed in the second cycle. For $Z_1/Z_2 = 1$, the stresses in both the striker and the semi-infinite rod are equal and amount to -263.286 MPa . The stress returns to zero in the second cycle and continues to oscillate around this value. In the case of $Z_1/Z_2 = 0.5$, the stress in the striker is amplified to -351.05 MPa , while the stress in the semi-infinite rod is attenuated to -175.52 MPa . Tensile stress appears in the second cycle, followed by compressive stress, alternating in this manner and gradually

approaching zero. This indicates that the two rods will separate after impact when $Z_1 < Z_2$. However, the constraint condition expressed by Equation (8) allows the existence of tensile stress at the interface $x = 0$. When $Z_1/Z_2 = 2$, the stress in the striker rod is attenuated to -175.52 MPa, and the stress in the semi-infinite rod is amplified to -351.05 MPa. The stress in each cycle is compressive, decreasing gradually and approaching zero. The results of this research are similar to those reported in Shin and Kim's paper [11]. Notably, the duration of each cycle in all three cases is identical, equal to $2L/c_1 = 69.17 \mu\text{s}$. The oscillation observed in wave propagation is not due to the numerical algorithm effect but is the actual effect of lateral inertia and Poisson's ratio.

5.2. Investigation of Stress Wave Propagation in a Striker Rod at $x = -0.1$ m

This case examines the propagation of stress waves at the position $x = -0.1$ m on the striker rod for three scenarios related to changes in cross-sectional area, Young's modulus, and density, with ratios of 0.5, 1, and 2, respectively, as shown in Figure 10 and Table 4. A remarkable aspect of this case is that a half-cycle is generated with either compressive or tensile stress, and there is no stress during the half-cycle when the rod is released. After the impact in the first half-cycle, compressive stress is generated, and then, depending on the contact surface characteristics between the two rods (cross-sectional area and material properties), the subsequent cycle will produce either compressive or tensile stress. This indicates that the duration of the stress pulse T_i in the striker rod is linearly decaying in each cycle as $T_j = \frac{2L}{c_1} \left(1 + \frac{x}{L}\right)$, where $-L \leq x \leq 0$. The wave velocity (c_1) is calculated as Equation (3).

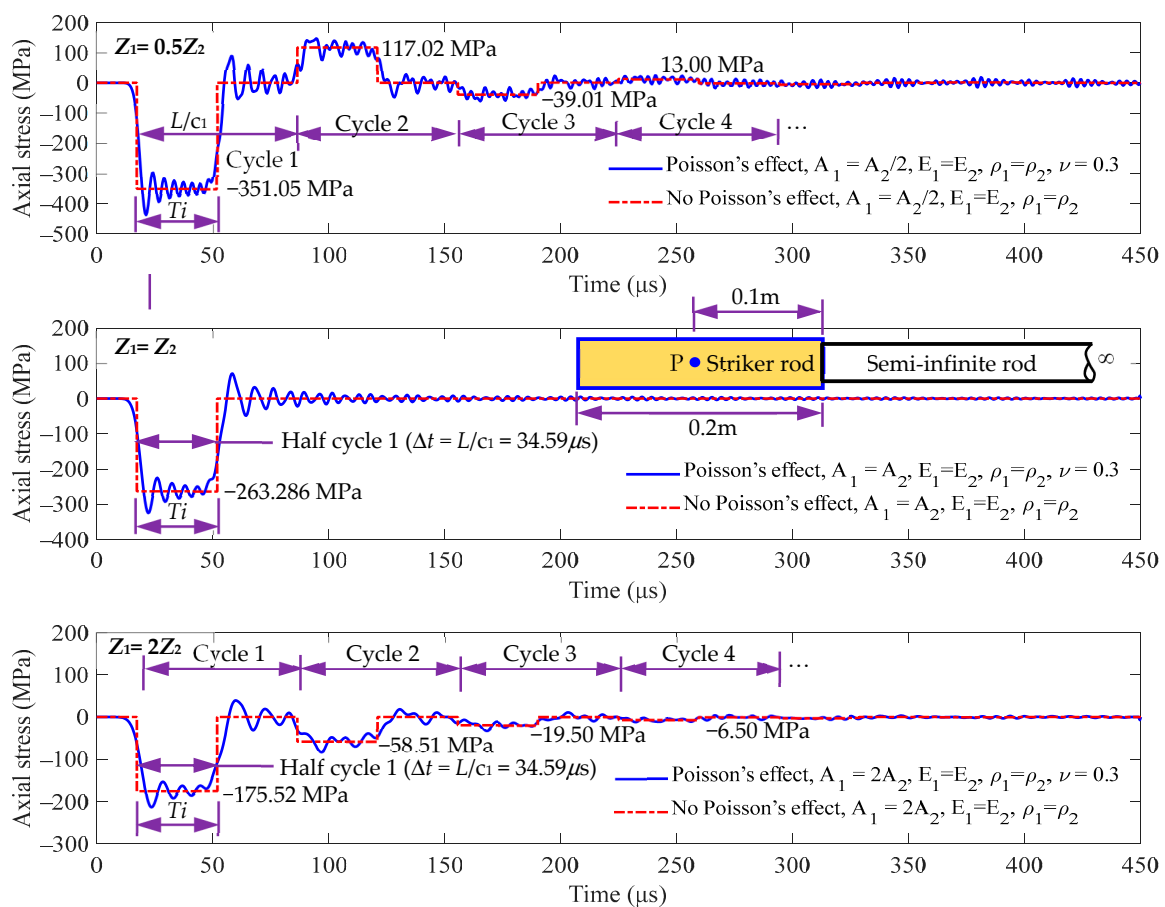


Figure 10. Comparison of stress wave propagation in the striker rod observed at $x = -0.1$ m under the impact of the striker rod with different impedances by varying the cross-sectional area ($Z_2 = 32,087.204 \text{ kg/s}$, $A_2 = 7.069 \times 10^{-4} \text{ m}^2$, $E_1 = E_2 = 195 \text{ GPa}$, $v_1 = v_2 = 0.3$, $\rho_1 = \rho_2 = 7850 \text{ kg/m}^3$, $2v_0 = 11.6 \text{ m/s}$, $L = 0.2 \text{ m}$).

Table 4. Stress variation in rods with $\nu = 0.3$ and the solution without Poisson's effect (Equation (48)) over the cycles at $x = -0.1$ m with different impedances, under the initial impact velocity $2v_0 = 11.6$ m/s, $Z_2 = 32,087.204$ kg/s.

Cycle		$Z_1 = 0.5Z_2$		$Z_1 = Z_2$		$Z_1 = 2Z_2$	
		Time (μ s)	σ_1 (MPa)	Time (μ s)	σ_1 (MPa)	Time (μ s)	σ_1 (MPa)
1	1/2	17.29~51.88	−351.05	17.29~51.88	−263.286	17.29~51.88	−175.52
	1/2	51.88~86.46	0	51.88~86.46	0	51.88~86.46	0
2	1/2	86.46~121.05	117.02	86.46~121.05	0	86.46~121.05	−58.51
	1/2	121.05~155.64	0	121.05~155.64	0	121.05~155.64	0
3	1/2	155.64~190.22	−39.01	155.64~190.22	0	155.64~190.22	−19.5
	1/2	190.22~224.81	0	190.22~224.81	0	190.22~224.81	0
4	1/2	224.81~259.39	13.00	224.81~259.39	0	224.81~259.39	−6.50
	1/2	259.39~293.98	0	259.39~293.98	0	259.39~293.98	0
5	1/2	293.98~328.57	−4.33	293.98~328.57	0	293.98~328.57	−2.17
	1/2	328.57~363.15	0	328.57~363.15	0	328.57~363.15	0

5.3. Investigation of Stress Wave Propagation in a Semi-Infinite Rod at $x = 0.5$ m

This case investigates stress wave propagation at the position $x = 0.5$ m on the semi-infinite rod with different impedances, as shown in Figure 11 and Table 5. All cases initially generate compressive stress, with notable variations in the second cycle. Specifically, the $Z_1 = 0.5Z_2$ case alternates between tensile stress and compressive stress, approaching zero; the $Z_1 = 0.5Z_2$ case sees stress oscillating around zero; and $Z_1 = 2Z_2$ consistently shows decreasing compressive stress towards zero. Particularly, when $Z_1 = Z_2$, the stress wave appears in only one stress compression cycle, after which the stress returns to oscillate around zero. Each cycle's duration remains constant at $T_i = 2L/c_1 = 69.17$ μ s in all cases.

Table 5. Stress variation in rods without Poisson's effect (Equation (49)) over the cycles at $x = 0.5$ m with different impedances in the semi-infinite rod under the initial impact velocity $2v_0$.

Cycle	$Z_1 = 0.5Z_2$		$Z_1 = Z_2$		$Z_1 = 2Z_2$	
	Time (μ s)	σ_2 (MPa)	Time (μ s)	σ_2 (MPa)	Time (μ s)	σ_2 (MPa)
1	86.46~155.64	−175.52	86.46~155.64	−263.286	86.46~155.64	−351.05
2	155.64~224.81	58.51	155.64~224.81	0	155.64~224.81	−117.02
3	224.81~293.98	−19.50	224.81~293.98	0	224.81~293.98	−39.01
4	293.98~363.15	6.50	293.98~363.15	0	293.98~363.15	−13.00
5	363.15~432.32	−2.167	363.15~432.32	0	363.15~432.32	−4.33

To further validate the accuracy of the method we propose, the results obtained are compared to finite element method (FEM) results. This 3D finite element analysis is a kind of numerical experiment to simulate the real performance of the system. The 3D FEM model is constructed as shown in Figure 12. The parameters for material properties and the geometries of the striker rod and the finite rod used in this model are detailed in Table 1; the length of the striker is 0.2 m, and that of the finite rod is 2.5 m. The approximate element size is 0.01 m, with a maximum deviation factor of 0.05 and a minimum element size of 0.001 m. The boundary conditions for the finite rod are set with one end free at $x = 0$ and fixed at $x = L$. In the Abaqus finite element analysis code, the solid element type C3D8R was employed and had 11,193 nodes and 8672 elements in the model, as shown in Figure 12b. In the simulation conducted using the Abaqus/Explicit [20], the system of equations was solved through an explicit time integration method. The time increment selected for the simulation is $\Delta t = 0.5 \times 10^{-7}$ (s). This setting guarantees that the condition $\Delta t \leq L_{\min}/c = 1.729 \times 10^{-7}$ (s), as recommended for stable explicit dynamic

analysis in Abaqus [20], where $L_{\min}$ is the smallest element dimension in the mesh and c is the wave speed in the material of the rod, ensuring that the simulation remains stable and accurate. Since the rods are axisymmetric, resulting in no friction force in the interaction of this model, the contact interaction between bodies is defined as frictionless in the Abaqus contact model. Figure 13 demonstrates the accuracy of the proposed numerical solution in simulating displacement, velocity, and stress in a rod, as shown by its comparison to FEM results. Particularly, in axial displacement, the proposed solution aligns closely with the FEM, indicating high accuracy in predicting displacement behavior over time. Despite minor discrepancies in velocity oscillations, the overall trend closely matches that of the FEM, underscoring the method's reliability. The stress response also shows a strong correlation with the FEM standard, solidifying the proposed method's potential as a valuable tool for dynamic stress analysis. In addition to its comparison with the FEM, the proposed numerical solution demonstrates considerable consistency with Shin and Kim's solution [11], which—for Poisson's ratio—is equal to zero. This agreement also confirms the accuracy of the proposed numerical solution, reinforcing its potential as a reliable method for understanding the impact dynamics of rods with varying cross-sectional areas.

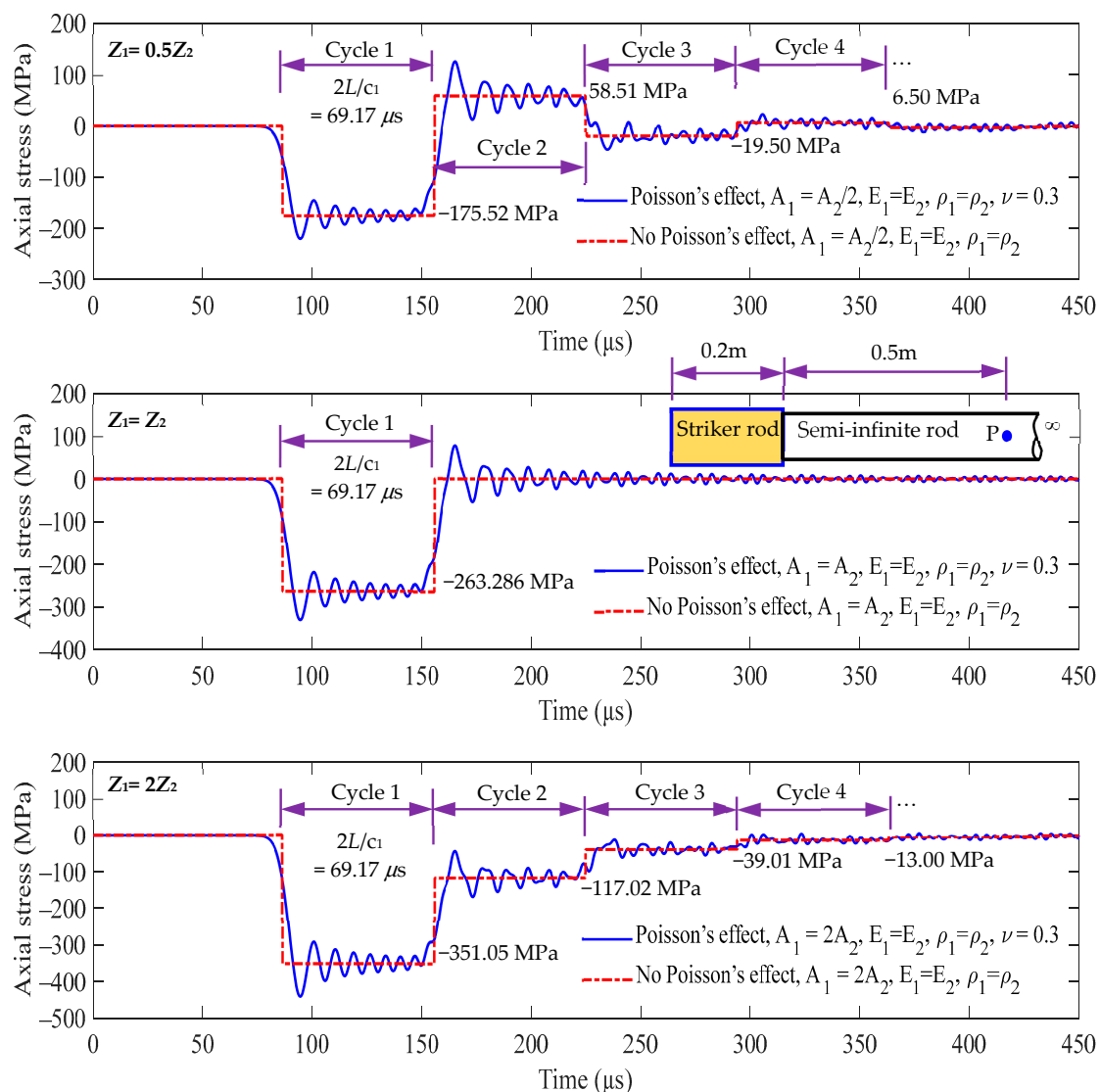


Figure 11. Comparison of stress wave propagation in a semi-infinite rod observed at $x = 0.5$ m under the impact of different impedances of the striker rod ($Z_2 = 32,087.204$ kg/s, $A_2 = 7.069 \times 10^{-4}$ m², $E_1 = E_2 = 195$ GPa, $v_1 = v_2 = 0.3$, $\rho_1 = \rho_2 = 7850$ kg/m³, $2v_0 = 11.6$ m/s, $L = 0.2$ m).

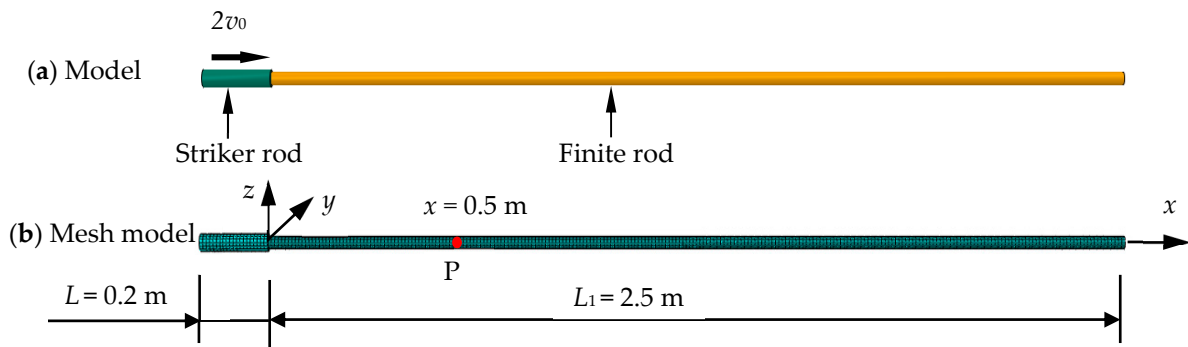


Figure 12. The 3D finite element model under the initial impact velocity $2v_0$ using solid elements in Abaqus.

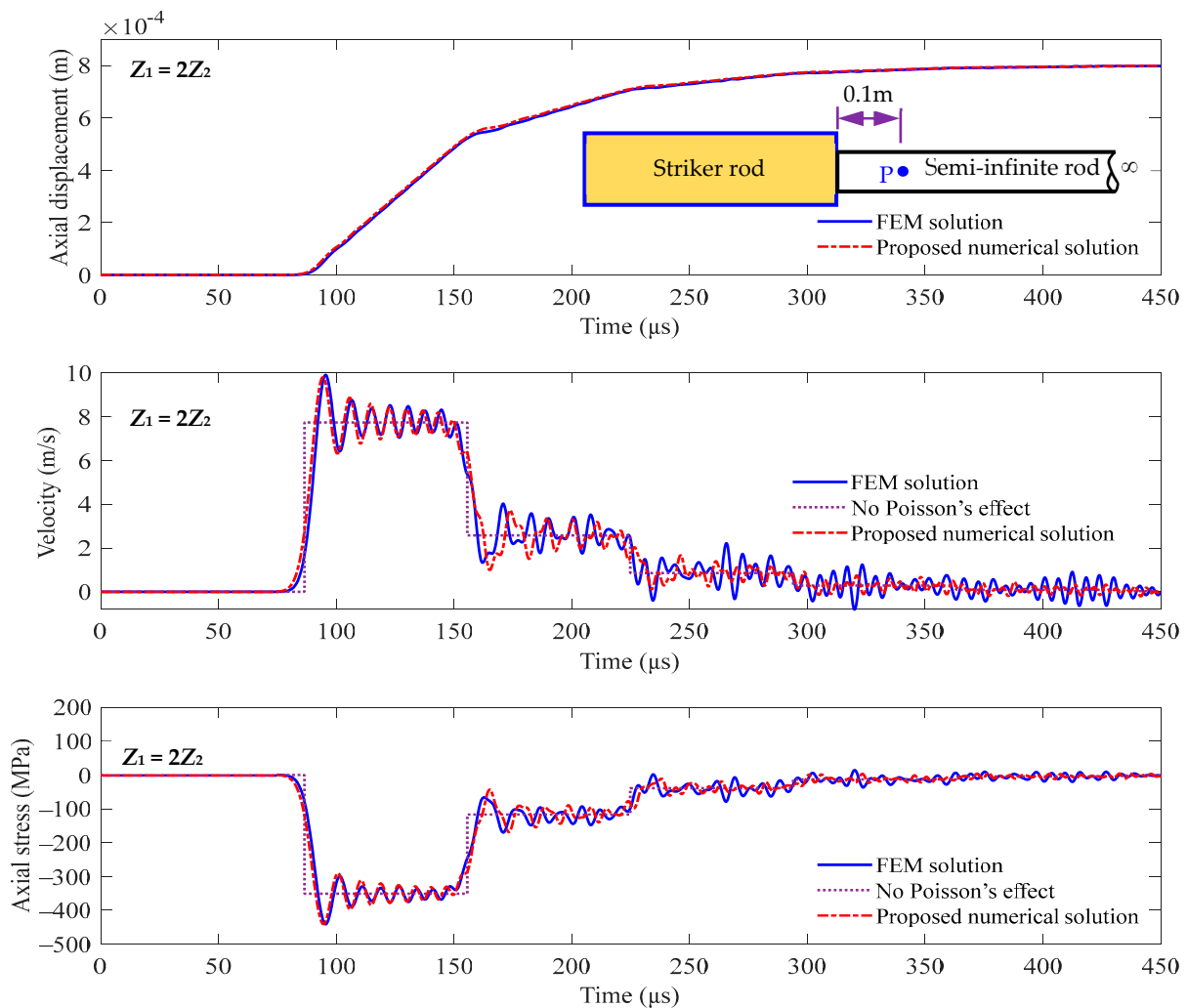


Figure 13. Comparison of stress wave propagation in the semi-infinite rod observed at $x = 0.5$ m under the impact of the striker rod with different impedances by varying the cross-sectional area ($Z_2 = 32,087.204$ kg/s, $A_1 = 2A_2 = 2 \times 7.069 \times 10^{-4}$ m², $E_1 = E_2 = 195$ GPa, $v_1 = v_2 = 0.3$, $\rho_1 = \rho_2 = 7850$ kg/m³, $2v_0 = 11.6$ m/s, $L = 0.2$ m). The results of no Poisson's effect are determined based on the research by Shin and Kim [11].

In Figure 14, when considering the cases with the same impedance but varying cross-sectional areas A_1 and A_2 , as well as differing $\rho_1 c_1$ and $\rho_2 c_2$, we can observe that the generated stress wave undergoes only one cycle, yet the duration of the stress signal varies.

It is evident that as the modulus of elasticity E_1 increases, the duration of the stress signal decreases. When both rods have the same impedance, the generated incident waveforms are simpler compared to those produced by rods with differing impedances, as discussed in Section 5.1. Thus, for practical purposes, to simplify the wave propagation analysis and to obtain a clearer output signal, it is advantageous to use rods with identical impedance. The results obtained in this study are compared with those of Shin et al. [11] to evaluate the accuracy of the proposed method. Shin et al. [11] present the impact of a striker on a bar with different general impedance, leading to various wave behaviors that can cause the striker and bar to separate or bond together. Shin et al. [11] do not consider the effects of lateral inertia and Poisson's ratio, resulting in the transmission of rectangular pulses without oscillation in wave propagation as shown in Figure 14. Our study is more comprehensive; it includes the oscillation in wave propagation. This oscillation leads to higher stress magnitudes, which can potentially cause different structural performances. Therefore, it is crucial to consider the effects of lateral inertia and Poisson's ratio during wave propagation. We verify our results through comparisons with the results in Shin et al. [11] and FEM results. The result of Shin et al. [11] is just a special case of our general analysis model.

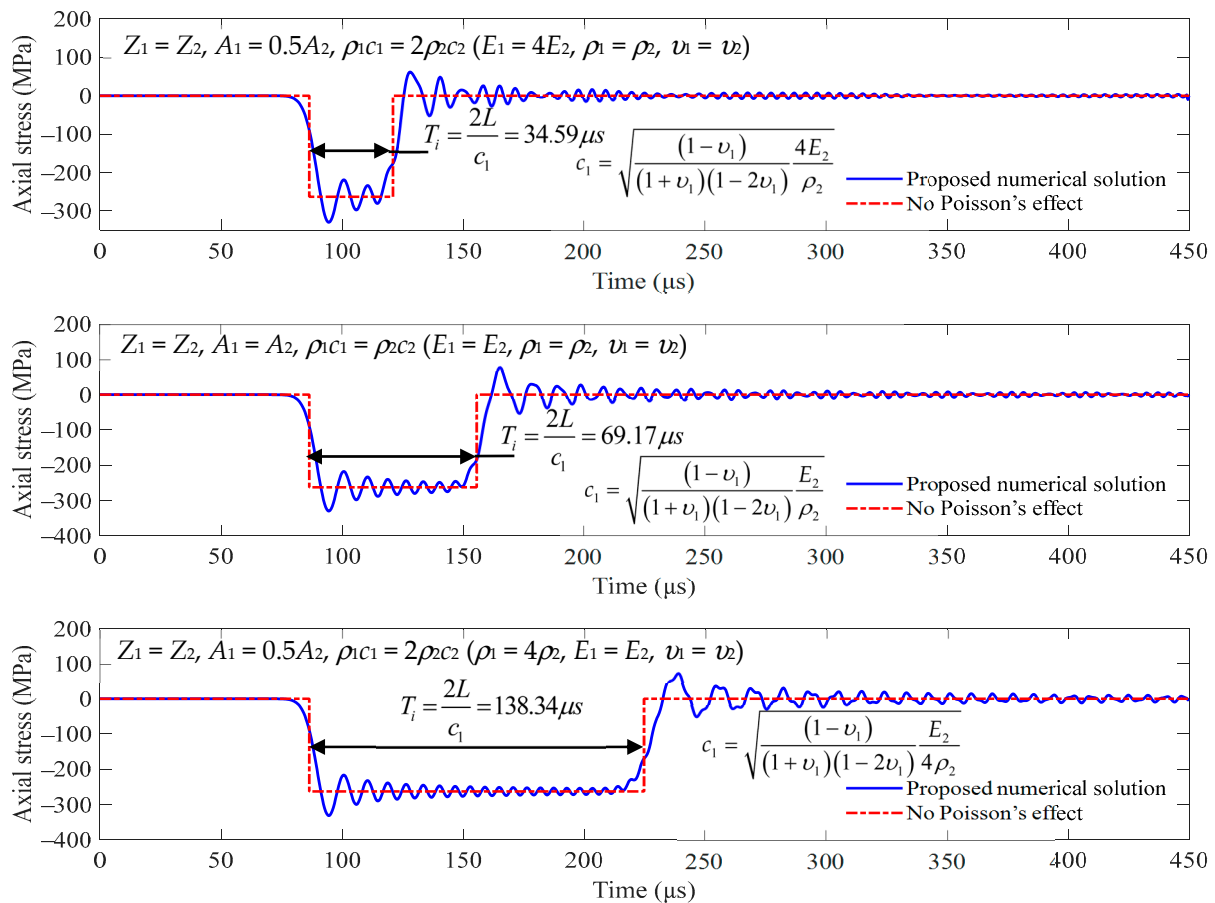


Figure 14. Comparison of stress wave propagation in the semi-infinite rod observed at $x = 0.5$ m under the impact of a striker rod with the same impedance by varying the cross-sectional area and material properties ($Z_1 = Z_2 = 32,087.204$ kg/s, $A_1 = 0.5A_2 = 0.5 \times 7.069 \times 10^{-4}$ m², $E_2 = 195$ GPa, $\nu_1 = \nu_2 = 0.3$, $\rho_2 = 7850$ kg/m³, $2v_0 = 11.6$ m/s, $L = 0.2$ m). The results of no Poisson's effect are determined based on the research by Shin and Kim [11].

6. Travel Time History

In Figures 15–17, points P_1 to P_{5L} are positions located in the striker rod and P_{5R} to P_{10} are positions located in the semi-infinite rod. These illustrate the stress wave travel time histories at positions P_1 to P_{10} of two rods with various cross-sectional areas ($A_1 = A_2$,

$A_1 = 2A_2$, and $A_1 = 0.5A_2$) at the initial impact velocity $2v_0$. These cases facilitate a fine understanding of the interaction between stress waves with varying cross-sectional areas.

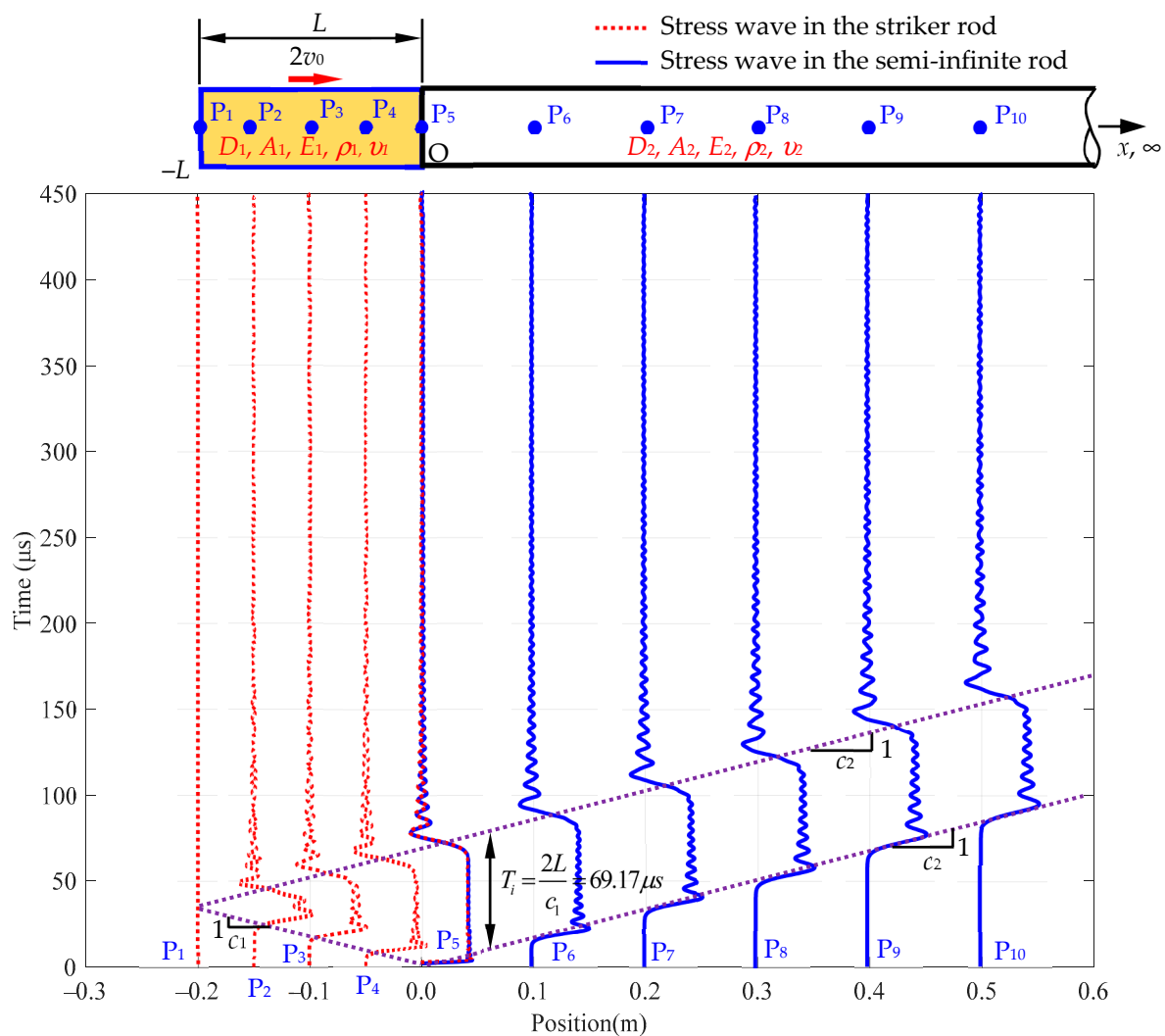


Figure 15. Stress wave propagation histories in the striker rod and the semi-infinite rod at various locations ($Z_1 = Z_2 = 32,087.204 \text{ kg/s}$, $A_1 = A_2 = 7.069 \times 10^{-4} \text{ m}^2$, $E_1 = E_2 = 195 \text{ GPa}$, $v_1 = v_2 = 0.3$, $\rho_1 = \rho_2 = 7850 \text{ kg/m}^3$, $2v_0 = 11.6 \text{ m/s}$, $L = 0.2 \text{ m}$, impact energy $KE = 74.67 \text{ J}$).

From the results, it is clear that the amplitude of the stress waves differs in each case, with the wave amplitude increasing as the cross-sectional area A_1 increases. This demonstrates the sensitivity of wave transmission to the cross-sectional area; specifically, when A_1 is larger than A_2 , the amplitude of the stress wave across the rods increases, implying that a larger cross-sectional area facilitates the transmission of more impact energy. In contrast, when A_1 is smaller than A_2 , the amplitude decreases, which may indicate increased reflection rather than transmission of wave energy at the interface between the rods. Positions P_1 to P_{5L} in the striker rod have pre-existing wave energy at the interface between the rods. In the initial state, the striker rod moves with a velocity of $2v_0$, creating a stress in the semi-infinite rod, which will cause a reflection back to the striker rod. The waveform shape at points P_{5R} to P_{10} is the same in form, but the timing of the wave initiation differs among the positions, and the amplitude of wave propagation is different in all cases, reflecting the impact of the varying cross-sectional areas on the amplitude of wave propagation at these positions. This phenomenon underscores the critical role of the cross-sectional area in the modulation of wave energy transmission, where a larger A_1 facilitates the conveyance of a greater amount of wave energy across the rods.

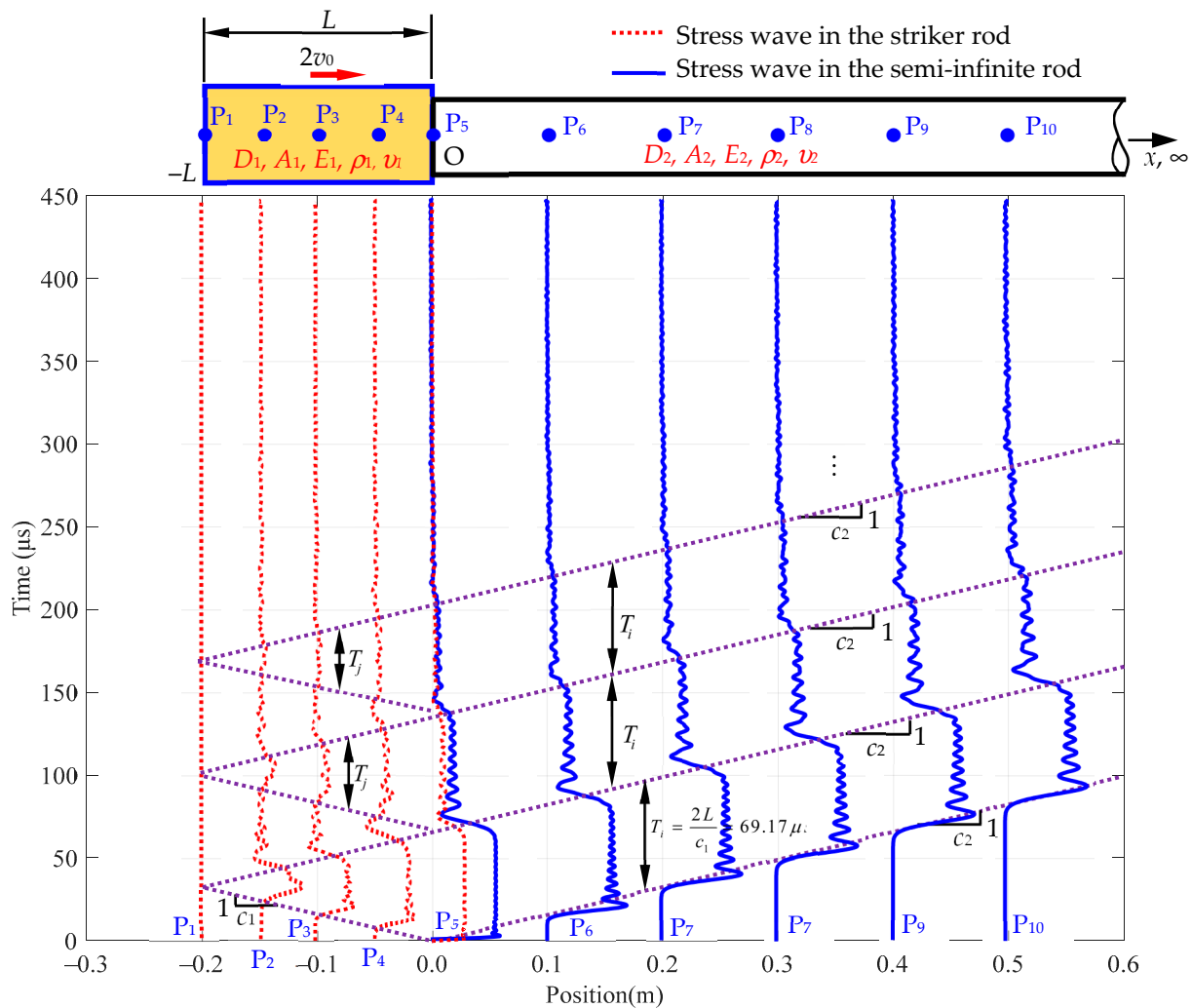


Figure 16. Stress wave propagation histories in the striker rod and the semi-infinite rod at various locations ($Z_1 = 2Z_2 = 64,174.408 \text{ kg/s}$, $A_1 = 2A_2 = 14.138 \times 10^{-4} \text{ m}^2$, $E_1 = E_2 = 195 \text{ GPa}$, $v_1 = v_2 = 0.3$, $\rho_1 = \rho_2 = 7850 \text{ kg/m}^3$, $2v_0 = 11.6 \text{ m/s}$, $L = 0.2 \text{ m}$, impact energy $KE = 149.34 \text{ J}$).

Due to the traction-free boundary condition at the left end, the striker stress is zero at any instance at this point. The stress at each point reaches its maximum value when a stress wave travels through and then gradually decreases to zero. The duration of the stress pulse T_j in the striker rod decays linearly in each cycle as $T_j = \frac{2L}{c_1} \left(1 + \frac{x}{L}\right)$, where $-L \leq x \leq 0$, and the duration of the stress pulse T_i in the semi-infinite rod is $T_i = \frac{2L}{c_1}$, where $x \geq 0$. The wave velocity c_1 in all of the cases presented here is the same, because only the cross-sectional area of the striker rod is changed in the three cases, while all of the material properties are the same. In the case where $Z_1 = Z_2$, as shown in Figure 15, the stress wave propagation occurs in a single cycle, resulting in a clear signal after cycle 1. In the case where $Z_1 \neq Z_2$, as shown in Figures 16 and 17, the wave propagation occurs over multiple consecutive cycles, creating a complex wave signal. This indicates that in order to easily identify the stress wave signal, using the condition $Z_1 = Z_2$ at the material interface is the most appropriate. Specifically, Figure 16 shows that when $Z_1 > Z_2$, a series of compression waves are displayed over consecutive cycles, causing the two rods to remain joined. Conversely, Figure 17 demonstrates that when $Z_1 < Z_2$, a tensile wave appears after the first cycle, leading to the separation of the two rods after the impact.

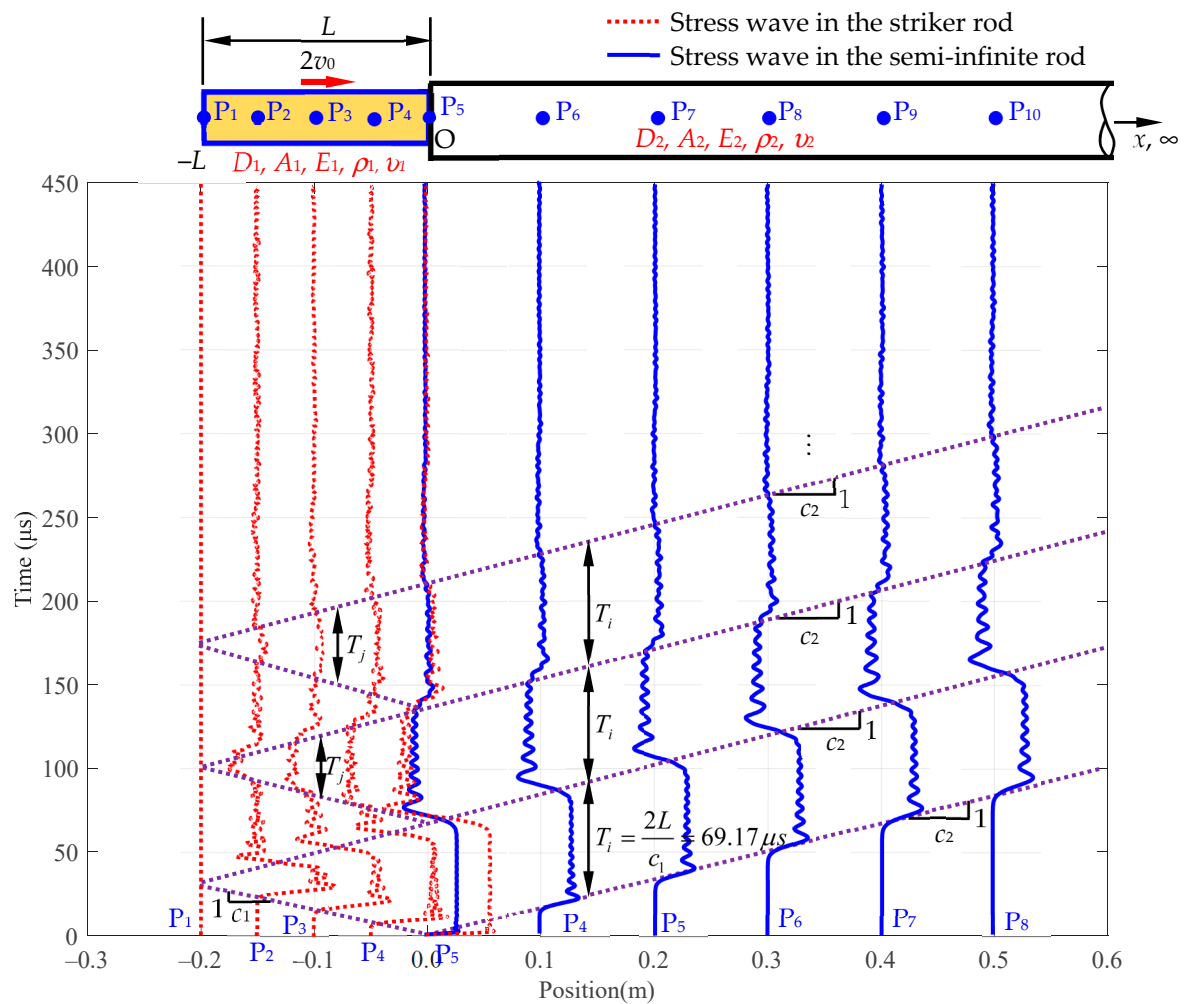


Figure 17. Stress wave propagation histories in the striker rod and the semi-infinite rod at various locations ($Z_1 = 0.5Z_2 = 16,043.602 \text{ kg/s}$, $A_1 = 0.5A_2 = 3.5345 \times 10^{-4} \text{ m}^2$, $E_1 = E_2 = 195 \text{ GPa}$, $v_1 = v_2 = 0.3$, $\rho_1 = \rho_2 = 7850 \text{ kg/m}^3$, $2v_0 = 11.6 \text{ m/s}$, $L = 0.2 \text{ m}$, input energy = 37.34 J).

7. Conclusions

This study presents an in-depth analysis of stress wave propagation in a semi-infinite Rayleigh–Love rod system when impacted by a striker rod with varying impedances. The different impedances between the striker and the semi-infinite rod give rise to distinct wave signals, occurring in one cycle or multiple impact cycles. These observations offer significant insights into the effect of impedance contrast on the characteristics of the waves. The influence of lateral inertia and Poisson’s ratio on longitudinal waves in rods leads to higher stress results compared to scenarios where these factors are not considered.

This research applied approximate formulae for numerically inverting Laplace transformations to handle complex equations. This is an effective method for solving the problem of wave propagation in rods with different impedances; it provides a reliable and accurate tool to analyze stress wave behaviors in rods involving interactions with different impedances. The comparison analyses with Yang’s analytical solution with the same impedance [14], Shin and Kim’s solution [11] without effects of lateral inertia and Poisson’s ratio, and numerical experiment by finite element analysis affirmed the high accuracy of this numerical inversion Laplace transform approach.

This study provides examples showing that changing the material parameters of the striker rod results in different waveforms. These findings enable the development of customized experimental models that can precisely achieve goals in materials testing and analysis.

An interesting finding from this study is that when $Z_1 < Z_2$, a positive reflection wave occurs after the first cycle, causing the two rods to separate; thus, there is no need to continue the analysis after Cycle 1. Conversely, when $Z_1 > Z_2$, multiple cycles of compressive waves occur, leading to the rods adhering together. Notably, with the same impedance, the clarity of the signal is optimal, making it easier to analyze and identify the geometric and material properties of the sample.

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Appendix A. Solution of Laplace Transform of Equations (11) and (12)

This section derives the displacement $U_1(\bar{x}, s)$ and $U_2(\bar{x}, s)$ in the s domain.

We use Laplace transformation properties for the derivative as follows

$$L\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{n-2}(0) - f^{n-1}(0). \quad (A1)$$

We can convert the initial and boundary condition terms in Laplace transformation. First, the initial conditions in terms of displacement and velocity can be written as follows:

$$\bar{u}_1(\bar{x}, 0) = 0, \quad \frac{\partial \bar{u}_1(\bar{x}, 0)}{\partial \bar{t}} = 2\bar{v}_0, \quad -\bar{L} \leq \bar{x} < 0, \quad (A2)$$

$$\bar{u}_2(\bar{x}, 0) = 0, \quad \frac{\partial \bar{u}_2(\bar{x}, 0)}{\partial \bar{t}} = 0, \quad 0 \leq \bar{x} < \infty. \quad (A3)$$

The boundary conditions at $\bar{x} = -\bar{L}$, $\bar{x} = 0$, and $\bar{x} = \infty$ are given by:

$$\frac{\partial \bar{u}_1}{\partial \bar{x}}(-\bar{L}, \bar{t}) = 0, \quad (A4)$$

$$\bar{u}_1(0^-, \bar{t}) = \bar{u}_2(0^+, \bar{t}), \quad \frac{\partial \bar{u}_1(0^-, \bar{t})}{\partial \bar{x}} = \frac{E_2 A_2}{E_1 A_1} \frac{\partial \bar{u}_2(0^+, \bar{t})}{\partial \bar{x}}, \quad 0 \leq \bar{t} < \infty, \quad (A5)$$

$$\lim_{\bar{x} \rightarrow \infty} \bar{u}_2(\bar{x}, \bar{t}) = 0, \quad \bar{t} \geq 0. \quad (A6)$$

If we apply Laplace transformation to both sides of Equations (7) and (8), we have:

$$L\left\{\frac{\partial^2 \bar{u}_1}{\partial \bar{t}^2}\right\} = m^2 L\left\{\frac{\partial^2 \bar{u}_1}{\partial \bar{x}^2}\right\} + b_1^2 L\left\{\frac{\partial^4 \bar{u}_1}{\partial \bar{x}^2 \partial \bar{t}^2}\right\}, \quad -\bar{L} \leq \bar{x} \leq 0, \quad (A7)$$

$$L\left\{\frac{\partial^2 \bar{u}_2}{\partial \bar{t}^2}\right\} = L\left\{\frac{\partial^2 \bar{u}_2}{\partial \bar{x}^2}\right\} + b_2^2 L\left\{\frac{\partial^4 \bar{u}_2}{\partial \bar{x}^2 \partial \bar{t}^2}\right\}, \quad 0 \leq \bar{x} < \infty, \quad (A8)$$

where L is the operator of Laplace transformation.

$$L\{\bar{u}_1(\bar{x}, \bar{t})\} = U_1(\bar{x}, s), \quad (A9)$$

$$L\{\bar{u}_2(\bar{x}, \bar{t})\} = U_2(\bar{x}, s). \quad (A10)$$

From Equations (A2) and (A3), we have:

$$L\left\{\frac{\partial^2 \bar{u}_1(\bar{x}, \bar{t})}{\partial \bar{t}^2}\right\} = s^2 U_1(\bar{x}, s) - s \bar{u}_1(\bar{x}, 0) - \frac{\partial \bar{u}_1(\bar{x}, 0)}{\partial \bar{t}} = s^2 U_1(\bar{x}, s) - 2\bar{v}_0, \quad (\text{A11})$$

$$L\left\{\frac{\partial^2 \bar{u}_1}{\partial \bar{x}^2}\right\} = \frac{\partial^2 U_1(\bar{x}, s)}{\partial \bar{x}^2}, \quad (\text{A12})$$

$$L\left\{\frac{\partial^4 \bar{u}_1}{\partial \bar{x}^2 \partial \bar{t}^2}\right\} = L\left\{\frac{\partial^2 (\partial^2 \bar{u}_1 / \partial \bar{t}^2)}{\partial \bar{x}^2}\right\} = L\left\{\frac{\partial^2 (s^2 U_1(\bar{x}, s) - s \bar{u}_1(\bar{x}, 0) - \frac{\partial \bar{u}_1(\bar{x}, 0)}{\partial \bar{t}})}{\partial \bar{x}^2}\right\} = \frac{s^2 \partial^2 U_1(\bar{x}, s)}{\partial \bar{x}^2}. \quad (\text{A13})$$

Then, Equation (A7) can be rewritten as follows:

$$\frac{d^2 U_1}{d\bar{x}^2} - \frac{s^2}{(m^2 + b_1^2 s^2)} U_1 = \frac{-2\bar{v}_0}{(m^2 + b_1^2 s^2)}, \quad -\bar{L} \leq \bar{x} \leq 0. \quad (\text{A14})$$

Similarly, Equation (A8) can be rewritten as follows:

$$\frac{d^2 U_2}{d\bar{x}^2} - \frac{s^2}{(1 + b_2^2 s^2)} U_2 = 0, \quad 0 \leq \bar{x} < \infty. \quad (\text{A15})$$

We now present step-by-step methods to solve Equations (A14) and (A15). First, find the homogeneous solution of Equation (A14). We have the following homogeneous differential equation:

$$\frac{d^2 U_1}{d\bar{x}^2} - \frac{s^2}{(m^2 + b_1^2 s^2)} U_1 = 0. \quad (\text{A16})$$

Assuming $U_1 = e^{r\bar{x}}$, we have:

$$U_1' = r e^{r\bar{x}}, \quad U_1'' = r^2 e^{r\bar{x}}. \quad (\text{A17})$$

Then, substituting Equation (A17) into Equation (A16), we obtain:

$$r^2 e^{r\bar{x}} - \frac{s^2}{m^2 + b_1^2 s^2} e^{r\bar{x}} = \left(r^2 - \frac{s^2}{m^2 + b_1^2 s^2}\right) e^{r\bar{x}} = 0. \quad (\text{A18})$$

Solving the characteristic equation of Equation (A18), we obtain two roots r_1 and r_2 as follows:

$$r_1 = \frac{-s}{\sqrt{m^2 + b_1^2 s^2}}, \quad r_2 = \frac{s}{\sqrt{m^2 + b_1^2 s^2}}. \quad (\text{A19})$$

Therefore, the homogeneous solution is:

$$U_1^h(\bar{x}, s) = B_1 e^{r_1 \bar{x}} + B_2 e^{r_2 \bar{x}} = B_1 e^{\frac{-\bar{x}s}{\sqrt{m^2 + b_1^2 s^2}}} + B_2 e^{\frac{\bar{x}s}{\sqrt{m^2 + b_1^2 s^2}}}, \quad (\text{A20})$$

where B_1 and B_2 are constants.

Now, we need to find a particular solution for Equation (A14). Let us assume that the particular solution is of the following form:

$$U_1^p(\bar{x}, s) = A(s). \quad (\text{A21})$$

where $A(s)$ is a function of s that does not depend on $\bar{x}$. Using the method of undetermined coefficients, we can find:

$$A(s) = \frac{2\bar{v}_0}{s^2}. \quad (\text{A22})$$

Thus, we can obtain the particular solution as follows:

$$U_1^p(\bar{x}, s) = \frac{2\bar{v}_0}{s^2}. \quad (\text{A23})$$

The general solution for Equation (A14) is as follows:

$$U_1(\bar{x}, s) = U_1^h(\bar{x}, s) + U_1^p(\bar{x}, s) = B_1 e^{\frac{-\bar{x}s}{\sqrt{m^2+b_1^2s^2}}} + B_2 e^{\frac{\bar{x}s}{\sqrt{m^2+b_1^2s^2}}} + \frac{2\bar{v}_0}{s^2}, \quad -\bar{L} \leq \bar{x} \leq 0. \quad (\text{A24})$$

For Equation (A15), $\frac{d^2 U_2}{d\bar{x}^2} - \frac{s^2}{(1+b_2^2s^2)} U_2 = 0$, $0 \leq \bar{x} < \infty$.

Similarly, we can find a solution to this homogeneous differential Equation (A15) as follows:

$$U_2(\bar{x}, s) = B_3 e^{\frac{-\bar{x}s}{\sqrt{1+b_2^2s^2}}} + B_4 e^{\frac{\bar{x}s}{\sqrt{1+b_2^2s^2}}}, \quad 0 \leq \bar{x} < \infty \quad (\text{A25})$$

where B_1, B_2, B_3 , and B_4 in Equations (A20) and (A25) are constants to be determined by the transformed boundary conditions.

With the boundary condition $U_1'(-\bar{L}, s) = 0$, from Equation (A20), we have:

$$U_1'(\bar{x}, s) = B_1 e^{\frac{-\bar{x}s}{\sqrt{m^2+b_1^2s^2}}} \left(-\frac{s}{\sqrt{m^2+b_1^2s^2}} \right) + B_2 e^{\frac{\bar{x}s}{\sqrt{m^2+b_1^2s^2}}} \left(\frac{s}{\sqrt{m^2+b_1^2s^2}} \right), \quad (\text{A26})$$

$$\Rightarrow U_1'(-\bar{L}, s) = B_1 e^{\frac{\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} \left(\frac{-s}{\sqrt{m^2+b_1^2s^2}} \right) + B_2 e^{\frac{-\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} \left(\frac{s}{\sqrt{m^2+b_1^2s^2}} \right) = 0, \quad (\text{A27})$$

$$\Rightarrow B_1 = B_2 e^{\frac{-2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}}. \quad (\text{A28})$$

With the boundary condition Equation (10), $U_2(\infty, s) = 0$, from Equation (A25), we have:

$$U_2(+\infty, s) = B_3 e^{\frac{-\infty s}{\sqrt{1+b_2^2s^2}}} + B_4 e^{\frac{+\infty s}{\sqrt{1+b_2^2s^2}}} = 0. \quad (\text{A29})$$

So, $B_4 = 0$ for a nontrivial solution.

Now, $U_2(\bar{x}, s)$ becomes:

$$U_2(\bar{x}, s) = B_3 e^{\frac{-s\bar{x}}{\sqrt{1+b_2^2s^2}}}. \quad (\text{A30})$$

With the boundary condition $U_1(0^-, s) = U_2(0^+, s)$, from Equations (A24) and (A25), we have:

$$B_1 e^{\frac{-0s}{\sqrt{m^2+b_1^2s^2}}} + B_2 e^{\frac{0s}{\sqrt{m^2+b_1^2s^2}}} + \frac{2\bar{v}_0}{s^2} = B_3 e^{\frac{-0s}{\sqrt{1+b_2^2s^2}}} + B_4 e^{\frac{0s}{\sqrt{1+b_2^2s^2}}}, \quad (\text{A31})$$

$$\Rightarrow B_1 + B_2 + \frac{2\bar{v}_0}{s^2} = B_3, \quad (\text{Note } B_4 = 0). \quad (\text{A32})$$

With the boundary condition $\frac{\partial U_1'(0^-, t)}{\partial \bar{x}} = \frac{E_2 A_2}{E_1 A_1} \frac{\partial U_2'(0^+, t)}{\partial \bar{x}}$, from Equations (A24) and (A25), we have:

$$B_1 e^{\frac{-\bar{x}s}{\sqrt{m^2+b_1^2s^2}}} \left(-\frac{s}{\sqrt{m^2+b_1^2s^2}} \right) + B_2 e^{\frac{\bar{x}s}{\sqrt{m^2+b_1^2s^2}}} \left(\frac{s}{\sqrt{m^2+b_1^2s^2}} \right) = \frac{E_2 A_2}{E_1 A_1} \left(B_3 e^{\frac{-\bar{x}s}{\sqrt{1+b_2^2s^2}}} \left(-\frac{s}{\sqrt{1+b_2^2s^2}} \right) + B_4 e^{\frac{\bar{x}s}{\sqrt{1+b_2^2s^2}}} \left(\frac{s}{\sqrt{1+b_2^2s^2}} \right) \right), \quad (\text{A33})$$

$$\Rightarrow B_1 e^{\frac{-0s}{\sqrt{m^2+b_1^2s^2}}} \left(-\frac{s}{\sqrt{m^2+b_1^2s^2}} \right) + B_2 e^{\frac{0s}{\sqrt{m^2+b_1^2s^2}}} \left(\frac{s}{\sqrt{m^2+b_1^2s^2}} \right) = \frac{E_2 A_2}{E_1 A_1} \left(B_3 e^{\frac{-0s}{\sqrt{1+b_2^2s^2}}} \left(-\frac{s}{\sqrt{1+b_2^2s^2}} \right) + B_4 e^{\frac{0s}{\sqrt{1+b_2^2s^2}}} \left(\frac{s}{\sqrt{1+b_2^2s^2}} \right) \right), \quad (\text{A34})$$

$$\Rightarrow (B_1 - B_2) \frac{s}{\sqrt{m^2+b_1^2s^2}} = \frac{E_2 A_2}{E_1 A_1} B_3 \frac{s}{\sqrt{1+b_2^2s^2}}, \quad (\text{A35})$$

$$\Rightarrow B_1 - B_2 = B_3 \frac{E_2 A_2}{E_1 A_1} \frac{\sqrt{m^2+b_1^2s^2}}{\sqrt{1+b_2^2s^2}} \quad (\text{A36})$$

Solving Equations (A28), (A32) and (A36), we obtain B_1 , B_2 , and B_3 as follows:

$$B_1 = \frac{-2\bar{v}_0 E_2 A_2 \sqrt{m^2+b_1^2s^2}}{s^2 \left(E_2 A_2 \sqrt{m^2+b_1^2s^2} - E_1 A_1 \sqrt{1+b_2^2s^2} + e^{\frac{2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} E_2 A_2 \sqrt{m^2+b_1^2s^2} + e^{\frac{2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} E_1 A_1 \sqrt{1+b_2^2s^2} \right)}, \quad (\text{A37})$$

$$B_2 = \frac{-2\bar{v}_0 E_2 A_2 \sqrt{m^2+b_1^2s^2} e^{\frac{2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}}}{s^2 \left(E_2 A_2 \sqrt{m^2+b_1^2s^2} - E_1 A_1 \sqrt{1+b_2^2s^2} + e^{\frac{2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} E_2 A_2 \sqrt{m^2+b_1^2s^2} + e^{\frac{2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} E_1 A_1 \sqrt{1+b_2^2s^2} \right)}, \quad (\text{A38})$$

$$B_3 = \frac{-2\bar{v}_0 \left(1 - e^{\frac{2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} \right) E_1 A_1 \sqrt{1+b_2^2s^2}}{s^2 \left(E_2 A_2 \sqrt{m^2+b_1^2s^2} - E_1 A_1 \sqrt{1+b_2^2s^2} + e^{\frac{2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} E_2 A_2 \sqrt{m^2+b_1^2s^2} + e^{\frac{2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} E_1 A_1 \sqrt{1+b_2^2s^2} \right)}. \quad (\text{A39})$$

Thus, we have the transformed solutions of Equations (A24) and (A25) for the striker rod and the semi-infinite rod, as follows:

$$U_1(\bar{x}, s) = \frac{-2\bar{v}_0 E_2 A_2 \sqrt{m^2+b_1^2s^2} \left(e^{\frac{\bar{x}s}{\sqrt{m^2+b_1^2s^2}}} + e^{\frac{-(\bar{x}+2\bar{L})s}{\sqrt{m^2+b_1^2s^2}}} \right)}{s^2 \left(\left(E_2 A_2 \sqrt{m^2+b_1^2s^2} - E_1 A_1 \sqrt{1+b_2^2s^2} \right) e^{\frac{-2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} + E_2 A_2 \sqrt{m^2+b_1^2s^2} + E_1 A_1 \sqrt{1+b_2^2s^2} \right)} + \frac{2\bar{v}_0}{s^2}, \quad -\bar{L} \leq \bar{x} \leq 0, \quad (\text{A40})$$

$$U_2(\bar{x}, s) = \frac{2\bar{v}_0 \left(1 - e^{\frac{-2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} \right) E_1 A_1 \sqrt{1+b_2^2s^2}}{s^2 \left(\left(E_2 A_2 \sqrt{m^2+b_1^2s^2} - E_1 A_1 \sqrt{1+b_2^2s^2} \right) e^{\frac{-2\bar{L}s}{\sqrt{m^2+b_1^2s^2}}} + E_2 A_2 \sqrt{m^2+b_1^2s^2} + E_1 A_1 \sqrt{1+b_2^2s^2} \right)} e^{\frac{-\bar{x}s}{\sqrt{1+b_2^2s^2}}}, \quad 0 \leq \bar{x} < \infty \quad (\text{A41})$$

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