

Article

Actuator Fault-Tolerant Control for Mechatronic Systems and Output Regulation with Unknown Reference Signals

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Abstract

Today, mechatronic systems are required to operate reliably and safely. However, actuators can fail, causing the system to malfunction or, in the worst case, resulting in an accident. A clear example of this is the motors of unmanned aerial vehicles. If any of them fail, the vehicle loses control, resulting in a catastrophe and potentially leading to the partial or total loss of the system. Therefore, there is a need to design robust control strategies that allow the system to continue operating even with the loss of one of its actuators. Based on the above, this work presents a controller capable of performing output regulation while tolerating actuator faults in actuated robotic platforms. In contrast to traditional output regulation theory, where a known exosystem provides the reference signal, the proposed approach employs a High-Gain Observer (HGO) to estimate and generate the reference signal from an unknown exosystem. Additionally, an Unknown Input (UI) observer is used to estimate actuator faults, enabling the computation of a fault-tolerant control. The methodology is tested in simulation and real-time experiments on the well-known Furuta pendulum system to illustrate the effectiveness of the proposed approach.

Keywords: Francis equation; high-gain observer; actuator fault; LMIs; fault-tolerant regulator



Academic Editors: Javier Moreno-Valenzuela, Luis Rodolfo García Carrillo and Alejandro Enrique Dzul López

Received: 6 June 2025

Revised: 16 July 2025

Accepted: 23 July 2025

Published: 1 August 2025

Citation: Amador-Macias, M.; Hernández-Cortés, T.; Estrada-Manzo, V.; González-Sierra, J.; Tapia-Herrera, R. Actuator Fault-Tolerant Control for Mechatronic Systems and Output Regulation with Unknown Reference Signals. *Appl. Sci.* **2025**, *15*, 8551. <https://doi.org/10.3390/app15158551>

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1. Introduction

In control theory, the problem of regulation is commonly addressed due to recurring issues in robotics, electrical, and mechatronic systems, among others. Its origins date back to 1976 when linear time-invariant systems were studied [1,2]. Later, in 1990, regulation theory was extended to a nonlinear framework [3]. Regulation theory establishes the conditions for the controller, ensuring that the output of the closed-loop system tracks reference signals and rejects disturbances. For this purpose, a known dynamical exogenous system (also called exosystem) must generate the reference/disturbance signals [4]. Nevertheless, in some cases, such dynamics are unavailable, for example, if an aerial vehicle follows an unknown terrestrial vehicle. Hence, a need arises to develop control algorithms to track unknown but measurable signals.

Some works that have studied the HGO can be found in [5–7]. Specifically, in [5], the author surveyed different results of implementing HGO in nonlinear feedback control, while in [6], the authors proposed a cascade HGO to address the peaking phenomenon.

In [7], the authors have proved that the HGO can be employed in the hybrid linear output regulation theory for minimum phase linear systems with a relative degree greater than one. In addition to the above, a methodology similar to HGO that has gained some interest among researchers is the Active Disturbance Rejection Control approach, developed by Professor Han [8]. This approach consists of designing an extended state observer that allows for estimating system disturbances (both external and internal). The estimated disturbance is then added to the control strategy to cancel these disturbances in the closed-loop system. This technique has experienced significant growth in recent years due to its ease of working with linear and nonlinear systems, as well as the numerous applications found in various systems [9–11].

Other control approaches have been developed to address the problem involving non-smooth and non-periodic exogenous signals. For instance, in [12], an adaptive observer is designed under a relaxed Uniform Persistence of Excitation (UPE) condition to estimate both the state x and the initial value ω_0 from the output e . This approach interprets the exogenous excitation (EE) as equivalent to a non-resonant condition. In [13], a nonlinear Luenberger observer combined with adaptive online gain tuning is used to select the best parameter based on an optimization criterion. A specific least-squares identifier is also proposed for general operating conditions.

On the other hand, fault diagnosis allows the avoidance of physical disrepair or malfunction in the system [14]; such faults may occur in sensors, actuators, or the process [15,16]. In [17], a UI observer is designed to estimate the state of the system and detect faults in actuators and sensors using the linear approximation at the origin; the design conditions are in terms of LMIs [18], due to its broad applicability and flexibility in achieving such performances. Once the faults are estimated, these signals can be used by the controller to form a fault-tolerant control scheme [19]. Similarly, in [20], the output regulation problem is addressed for cooperative heterogeneous linear multi-agent systems under communication and actuator faults. An adaptive dual event-triggered compensator is introduced to estimate the exosystem's state under such faults, while also reducing chattering and avoiding Zeno behavior.

In output regulation theory, the avoidance of such perturbations is considered. However, such perturbations must be modeled, or in the case of regulation of unmodeled signals [21–23], at least be measurable. The problem arises when incipient faults appear in actuators and, unfortunately, are not measurably available for estimation within the scope of unmodeled signal regulation.

To address these problems, this article focuses on designing conditions that enable the closed-loop system to perform output regulation and tolerate actuator faults. The proposed approach is presented through UI observers, whose gain is computed using Linear Matrix Inequalities (LMIs). The approach is based on the linear approximation of nonlinear systems; the stabilizer gain and the fault estimator are computed through LMIs, and a pair of matrix equations calculates the rest of the controller scheme. The observation algorithm is tested through simulations and real-time experiments on the Furuta pendulum system to verify the viability of the proposed approach. It is worth noting that the developments in this article are based on the theory presented in [24], where an HGO provides an estimation of the reference signal, as well as a mathematical model for the exosystem derived from its structure. Additionally, the reason behind using LMIs is that they allow the designer to naturally include performance specifications such as speed convergence (the faster the observer converges, the faster its estimations can be fed to the control signal). Of course, this includes taking care of the transient behavior (similar for the controller, as it stabilizes the system so it can track signals) [18]. Other performance

specifications are related to bounds on the input/output. In our case, bounding the input signals prevents damage to the actuators.

The article is organized as follows: Section 2 presents the general problem to be solved, while Section 3 outlines the methodology employed throughout the work. Then, Section 4 presents the main results, which include the LMI conditions. Section 5 presents the performance of the proposal using the Furuta pendulum system in both simulation and real-time settings. Finally, Section 6 presents the paper's conclusions and outlines some potential future work.

2. Problem Statement

Consider the following nonlinear system

$$\dot{x}(t) = f(x, u, w, f_a), \quad y(t) = h(x), \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the output vector, $f_a \in \mathbb{R}^m$ is the actuator fault vector, and $w \in \mathbb{R}^{p+d}$ is the reference/perturbation signal vector; $f(\cdot, \cdot, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{p+d} \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $h(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^p$ are sufficiently smooth vector fields such that $f(0, 0, 0, 0) = 0$ and $h(0) = 0$.

The task is to find gains $K \in \mathbb{R}^{m \times n}$, $\Pi \in \mathbb{R}^{n \times (p+d)}$, $\Gamma \in \mathbb{R}^{m \times (p+d)}$ and signal $\hat{f}_a(t) \in \mathbb{R}^m$ to constitute the control law

$$u(t) = K(x(t) - \Pi w(t)) + \Gamma w(t) - \hat{f}_a(t), \quad (2)$$

such that the output $y(t)$ is driven towards a prescribed reference $w(t)$ generated by an exosystem (to be estimated) despite the disturbances and the actuator faults $f_a(t)$. In what follows, conditions are given to solve the problem of actuator fault-tolerant output regulation.

For this purpose, the methodology begins by computing a linearization at the origin of the nonlinear system (1), that is,

$$\dot{x}(t) = Ax(t) + B(u(t) + f_a(t)) + Dw(t), \quad y(t) = Cx(t), \quad (3)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{n \times (p+d)}$, and $C \in \mathbb{R}^{p \times n}$ are known matrices.

For constructing the control law (2), it is necessary to have the reference signal $w(t)$ and to estimate the actuator fault $f_a(t)$. In contrast to traditional approaches [4], $w(t)$ originates from an unknown exosystem, so an estimation of these signals must be performed; this step is accomplished via HGO. On the other hand, the actuator fault signal is recovered utilizing a UI observer [25]. It is worth clarifying that our results are based on the linearization at the origin of system (1). A possible limitation of the proposed approach arises if the reference signals drive the system trajectories "far" from the equilibrium point. Therefore, the possible trajectories must be in the operational space.

3. Fault-Tolerant Output Regulation Scheme

The output regulation for a nonlinear system (1) or its linear approximation (3) with $f_a(t) = 0$ requires the existence of an exosystem for the generation of references/perturbations $w(t)$; traditionally, such system is known [4]. An interesting case arises when this exosystem is unknown; according to [24], the tracking signals can be represented by $\Psi(t) \in \mathbb{R}^{p+d}$, as long as they are available for measurement, smooth and continuously differentiable, i.e.,

$$\Psi(t) = \begin{bmatrix} \psi_1 & \psi_2 & \dots & \psi_p & \dots & \psi_{p+d} \end{bmatrix}^\top, \quad (4)$$

where $\psi_1, \psi_2, \dots, \psi_p$ are reference signals to be tracked and $\psi_{p+1}, \dots, \psi_{p+d}$ are the disturbance signals to be rejected. Thus, following [24], the solution to the output regulation problem for unknown reference/disturbance signals for (3) consists in constructing the exosystem by estimating the states and outputs of $\Psi(t)$ through the following HGO

$$\dot{w}(t) = Sw(t) + S_H \Psi(t), \quad y_w(t) = \begin{bmatrix} y_{ref}(t) \\ y_{dis}(t) \end{bmatrix} = Q_w w(t), \quad (5)$$

where $Q_w = \begin{bmatrix} Q_{ref} \\ Q_{dis} \end{bmatrix}$, $y_{ref} \in \mathbb{R}^p$ are the p references to track and $y_{dis} \in \mathbb{R}^d$ are the d disturbances to reject; matrices $S \in \mathbb{R}^{\rho(p+d) \times \rho(p+d)}$, $S_H \in \mathbb{R}^{\rho(p+d) \times (p+d)}$, and $Q_{ref} \in \mathbb{R}^{p \times \rho(p+d)}$, $Q_{dis} \in \mathbb{R}^{d \times \rho(p+d)}$ have the following structure

$$S = \begin{bmatrix} S_1 & 0 & 0 & \dots & \dots & 0 \\ 0 & S_2 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & S_p & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \dots & S_{p+d} \end{bmatrix}, \quad S_H = \begin{bmatrix} H_1 & 0 & 0 & \dots & \dots & 0 \\ 0 & H_2 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & H_p & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \dots & H_{p+d} \end{bmatrix},$$

$$Q_{ref} = \begin{bmatrix} Q_1 & 0 & 0 & \dots & \dots & 0 \\ 0 & Q_2 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \dots & 0 \\ 0 & 0 & \dots & Q_p & \dots & 0 \end{bmatrix}, \quad Q_{dis} = \begin{bmatrix} 0 & \dots & Q_{p+1} & 0 & \dots & 0 \\ 0 & \dots & 0 & Q_{p+2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & \dots & Q_{p+d} \end{bmatrix},$$

with $S_i \in \mathbb{R}^{\rho \times \rho}$, $H_i \in \mathbb{R}^{\rho \times 1}$ and $Q_i \in \mathbb{R}^{1 \times \rho}$ are block matrices defined as:

$$S_i = \begin{bmatrix} -\frac{\alpha_1}{\epsilon} & 1 & 0 & \dots & 0 \\ -\frac{\alpha_2}{\epsilon^2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\alpha_{\rho-1}}{\epsilon^{\rho-1}} & 0 & 0 & \dots & 1 \\ -\frac{\alpha_\rho}{\epsilon^\rho} & 0 & 0 & \dots & 0 \end{bmatrix}, \quad H_i = \begin{bmatrix} \alpha_1/\epsilon & \alpha_2/\epsilon^2 & \dots & \alpha_\rho/\epsilon^\rho \end{bmatrix}^\top,$$

$$Q_i = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad i = 1, 2, \dots, p+d.$$

Additionally, ϵ is related with the convergence rate and α_i is chosen such as $s^\rho + \alpha_1 s^{\rho-1} + \dots + \alpha_{\rho-1} s + \alpha_\rho$ be Hurwitz.

One of the inconveniences of using HGOs is the high gain required when the dimension of ρ is high, which can cause a peaking phenomenon during the transient states and potentially can destabilize the system. To address this issue, in this work, the order of ρ is set to 2, resulting in a maximum power of ϵ^2 that reduces the peak effect. Additionally, the selection of ϵ to be small and an adequate α_ρ can improve the convergence of the unknown signal.

Therefore, if (4) is available from measurements for all $t \geq 0$ and considering the exosystem (5), the output regulation problem for unknown signals consists in computing a control law:

$$u(t) = K(x(t) - \Pi w(t)) + \Gamma w(t), \quad (6)$$

where $K \in \mathbb{R}^{m \times n}$ is such that $A + BK$ is Hurwitz, and the mappings $\Pi \in \mathbb{R}^{n \times (p+d)}$ and $\Gamma \in \mathbb{R}^{m \times (p+d)}$ such that

$$\begin{aligned} 0 &= A\Pi + B\Gamma + D - \Pi S - \Pi S_H Q_w, \\ 0 &= C\Pi - Q_{ref}, \end{aligned} \quad (7)$$

are satisfied; these are variations of the so-called Francis equations [2]. Hence, the closed-loop system guarantees that the tracking error

$$e(t) = y(t) - y_{ref}(t) = Cx(t) - Q_{ref}w(t), \quad (8)$$

goes asymptotically to zero.

Several approaches exist for finding K , such as ensuring that $(A + BK)$ is Hurwitz. The LMI framework allows adding performances such as decay rate, input, and output constraints; to do this, consider the Lyapunov function candidate $V(x) = x^T Px$, $P = P^T > 0$, which leads to LMIs

$$X > 0, \quad AX + BM + XA^T + MB^T + 2\varphi X < 0, \quad (9)$$

where $M = KP$, $X = P^{-1}$ and $\varphi > 0$ is related to the maximum speed convergence. The previous LMIs guarantee that $\dot{V} \leq -2\varphi V$. In order to avoid actuator saturation, let us impose a control input bound $\|u(t)\| \leq \beta$, with $\beta > 0$, through the following LMIs

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & X \end{bmatrix} \geq 0, \text{ and } \begin{bmatrix} X & M^T \\ M & \beta^2 \end{bmatrix} \geq 0, \quad (10)$$

where $x(0)$ is the initial condition. If LMIs (9) and (10) are feasible, then the stabilizing gain is computed as $K = MX$ [18].

Actuator Fault Estimation

Let us estimate the actuator fault for system (3) with $w(t) = 0$. The faults are assumed to hold $f_a^{(\sigma)}(t) \approx 0$, where σ is the derivative order (this assumption might seem restrictive in theory, but in practice, sinusoidal signals can also be considered; σ is related to the order of their Taylor polynomial approximation). Thus, by defining $z_1(t) = f_a(t)$, $z_2(t) = \dot{f}_a(t)$, $\dots$, $z_\sigma(t) = f_a^{(\sigma-1)}(t)$, the following system can be written

$$\dot{z}(t) = Fz(t), \text{ with } F = \begin{bmatrix} 0 & I_m & 0 & \dots & 0 \\ 0 & 0 & I_m & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_\sigma \end{bmatrix}, \quad (11)$$

where I_m is an identity matrix of $m \times m$; in general I and 0 are identity and zero matrices of proper sizes. Note that the previous expression is a chain of integrators; this idea can be traced back to [26]. Once coupled with the faulty system as in [17], an augmented system yields

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t), \quad y(t) = \bar{C}\bar{x}(t), \quad (12)$$

where

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & B \begin{bmatrix} I_m & 0_{m \times m(\sigma-1)} \end{bmatrix} \\ 0 & F \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \text{and} \quad \bar{C} = [C \ 0_{o \times (n+\sigma m)}].$$

The estimation of $x(t)$ and $f_a(t)$ is done through a UI observer of the form (this observer is also named as an unknown input proportional multi-integral observer [25])

$$\dot{\hat{x}}(t) = \bar{A}\hat{x}(t) + \bar{B}u(t) + \bar{L}(y(t) - \hat{y}(t)), \quad (13)$$

with $\hat{x}(t) = [\hat{x}^\top(t) \hat{z}^\top(t)]^\top$, $(\bar{A}, \bar{B}, \bar{C})$ are defined as in (12); while $\bar{L} = [L_P^\top L_I^\top]^\top$, $L_P \in \mathbb{R}^{o \times n}$, $L_I \in \mathbb{R}^{o \times (n+\sigma m)}$ is the observer gain to be designed such that the observation error

$$\bar{e}(t) = \bar{x}(t) - \hat{x}(t) = \begin{bmatrix} x(t) - \hat{x}(t) \\ z(t) - \hat{z}(t) \end{bmatrix} = \begin{bmatrix} e_x(t) \\ e_z(t) \end{bmatrix}, \quad (14)$$

holds

$$\lim_{t \rightarrow \infty} \bar{e}(t) = 0.$$

The dynamics of the augmented error system are computed from (12) and (14), that is

$$\dot{\bar{e}}(t) = (\bar{A} - \bar{L}\bar{C})\bar{e}(t). \quad (15)$$

Like the previous section, LMI constraints can be used to find the observer gain $\bar{L}$, guaranteeing a decay rate $\gamma > 0$. To this end, consider $V(\bar{e}(t)) = \bar{e}(t)^\top \bar{P} \bar{e}(t)$, $\bar{P} = \bar{P}^\top \in \mathbb{R}^{(n+\sigma m) \times (n+\sigma m)}$ as Lyapunov function candidate, thus LMIs

$$\bar{P} > 0, \quad \bar{P}\bar{A} - \bar{N}\bar{C} + \bar{A}^\top \bar{P} - \bar{C}^\top \bar{N}^\top + 2\gamma \bar{P} < 0, \quad (16)$$

where $\bar{P}$ and $\bar{N} \in \mathbb{R}^{p \times (n+\sigma m)}$ are decision variables. If the previous LMIs are found feasible, thus $\dot{V}(\bar{e}(t)) \leq -2\gamma V(\bar{e}(t))$ and the observer gain is $\bar{L} = \bar{P}^{-1} \bar{N}$.

Thus, both the high-gain observer and the unknown input one are implemented together, so the control law (2) is constructed.

Recall that solving the output regulation problem for unmodeled signals requires finding a gain K such that $(A + BK)$ is Hurwitz; that is, the system is asymptotically stable when $w(t) = 0$; nevertheless, this analysis does not consider actuator faults. Thus, if $f_a(t) \neq 0$ in (3), then the control should take into account the fault as suggested by [27,28]. Therefore, the control law is

$$u(t) = Kx(t) - \hat{f}_a(t), \quad (17)$$

where $\hat{f}_a(t)$ comes from the UI observer (13). If the UI observer is designed to be fast enough, the estimated fault converges towards the real one $\hat{f}_a(t) \rightarrow f_a(t)$, and the system

$$\dot{x}(t) = Ax + B(Kx(t) - \hat{f}_a(t) + f_a(t)), \quad (18)$$

becomes $\dot{x}(t) = (A + BK)x(t)$ and thus $\lim_{t \rightarrow \infty} x(t) = 0$.

Let us summarize the conditions for solving the fault-tolerant output regulation problem for unmodeled signals. The output of the nonlinear system (1) tracks an unknown desired signal $y_{ref}(t)$ in (5) despite the presence of actuator faults $f_a(t)$ in a neighborhood of the origin $x = 0$ if there exists a control law (2) with gains K , Π and Γ and the signal $\hat{f}_a(t)$; as a consequence, the steady-state values are $x_{ss} = \Pi w$ and $u_{ss} = \Gamma w$. As mentioned before, K is computed from LMIs (9) and (10); Π and Γ are obtained from the HGO observer and the solution of the Equation (7), and finally, $\hat{f}_a(t)$ from the UI observer (13). This algorithm is summarized in Figure 1.

Remark 1. Notice that if output regulation takes place, then $x_{ss} = \Pi w$ and $u_{ss} = \Gamma w$, the steady-state of the input $u(t)$ holds when the $f_a(t)$ is incipient, that is, it does not appear abruptly and it might vanish in time [29].

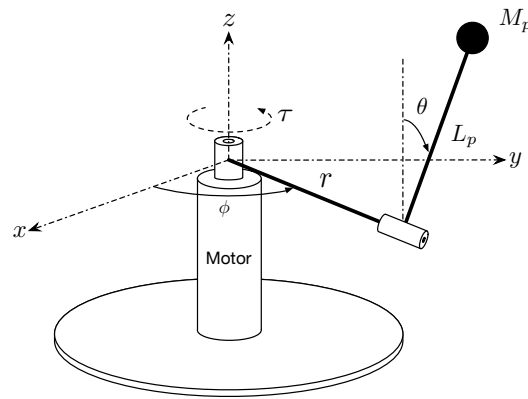


Figure 2. Furuta pendulum.

Table 1. Parameters for the Furuta pendulum.

Variable	Description	Value
M_p	Pendulum mass	0.027 Kg
L_p	Large pendulum	0.153 m
r	Large arm	0.0826 m
J_p	Pendulum inertia	0.000177 m ⁴
J_{arm}	Arm inertia	0.00018374 m ⁴
k_t	Armor constant	0.0333
K_m	Motor constant	0.0333
R_m	Motor resistance	8.7 Ω
g	Gravity	9.81 $\frac{m}{s^2}$

Consequently and by assuming $\rho = 2$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\epsilon = 0.05$, the block matrices for (5), are:

$$S = \begin{bmatrix} -200 & 1 \\ -40000 & 0 \end{bmatrix}, \quad S_H = \begin{bmatrix} 200 \\ 40000 \end{bmatrix}, \quad (22)$$

$$Q_w = Q_{ref} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Then, remembering the linear central manifolds mappings $x_{ss} = \Pi\omega$ and $u_{ss} = \Gamma\omega$, its gains are computed by (7) resulting in

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 & 0.0333 \end{bmatrix}.$$

Therefore, for the stabilization by the LMIs (9) and (10), $\varphi = 5.9$, $\beta = 22$ and $x(0) = \begin{bmatrix} 5^\circ & 3^\circ & 0^\circ/s & 0^\circ/s \end{bmatrix}^\top$ are set, respectively, resulting in

$$K = \begin{bmatrix} 8.8650 & -90.7956 & 3.3963 & -12.2266 \end{bmatrix}. \quad (23)$$

Next, by using the LMI (16) a fault actuator observer of the fifth order was designed, which means that $p_a = 5$ ($f_a^{(5)}(t) \approx 0$), including a decay rate of 1, resulting in the increased observer gain

$$\bar{L} = \begin{bmatrix} 35.8 \\ 293.8 \\ 1286.2 \\ 2707.9 \\ -416.6 \\ -1008.2 \\ -1543.4 \\ -1349.1 \\ -512.2 \end{bmatrix}.$$

It is worth mentioning that the higher the order of the derivative of $f_a(t)$, the better the approximation is, at the cost of increasing the computational burden [17]. In this case, we found that setting $\sigma = 5$ provides a good estimate and performance of the closed-loop system compared to $\sigma = 4$ and $\sigma = 6$. Of course, an appropriate value for σ will depend on the nature of $f_a(t)$.

Then, two simulations are carried out using MATLAB®-Simulink® 2018a software (The MathWorks Inc., Natick, MA, USA): the first one (low frequency) with the unmodeled signal $\Psi_1(t) = 15 \sin(36^\circ t) - 15 \cos(60^\circ t)$, and the second with the unmodeled signal $\Psi_2(t) = 15 \sin(120^\circ t) + 15 \cos(60^\circ t)$ (high frequency). Both with the following conditions for $x(0) = [0^\circ \ 2^\circ \ 0^\circ/\text{s} \ 0^\circ/\text{s}]^\top$, $w(0) = [3^\circ \ 0^\circ]^\top$, $\hat{x}(0) = x(0)$, $\hat{z}(0) = [0 \ 0 \ 0 \ 0]^\top$ Volts, and the actuator faults as $f_a(t) = 20.05 \sin(42.97^\circ t)$.

Remark 2. One can note that there are two sets of initial conditions for $x(0)$. The first set, given by $x(0) = [5^\circ \ 3^\circ \ 0^\circ/\text{s} \ 0^\circ/\text{s}]^\top$ is used to obtain the gains in (23). On the other hand, the second set given by $x(0) = [0^\circ \ 2^\circ \ 0^\circ/\text{s} \ 0^\circ/\text{s}]^\top$ is utilized for the implementation. In the case of pure stabilization at the origin, the farther the initialization is, the larger the control signal will be. Therefore, the initial conditions for simulations have to be closer than those for computations. A problem can arise if $x(0)$ for simulation is farther than that in LMIs (10).

4.1. Low Frequency Signal

Figure 3 depicts the behavior of the states when the controller is applied to the system; it can be seen that the system fulfills the restriction to maintain the vertical position of the second link and the tracking of the joint of the first link. Additionally, it can be seen in Figure 4 that the input control signal accomplished the set bounds established in the LMI.

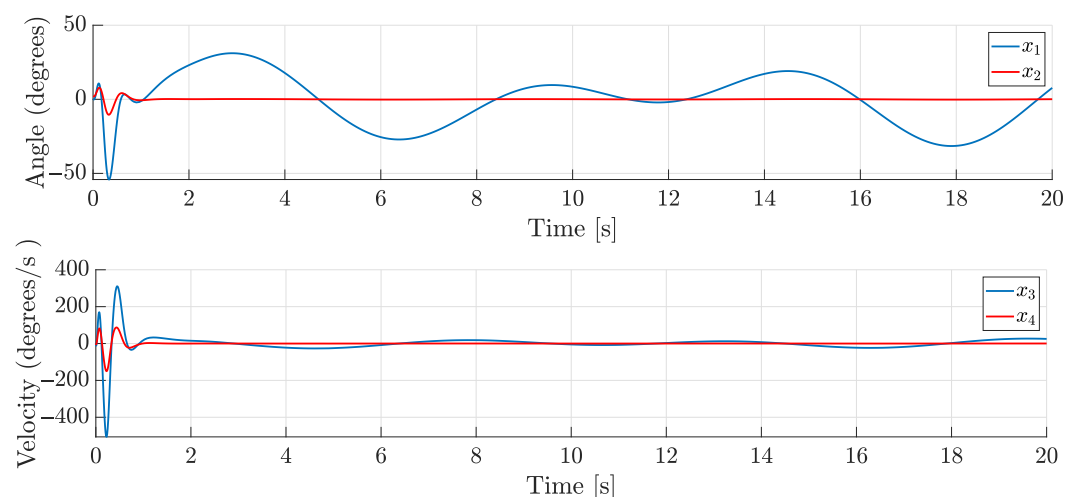


Figure 3. Evolution of x_1 , x_2 , x_3 and x_4 over time when using low frequency signal.

Figure 5 shows how the output x_1 tends to follow the signal w_1 generated by the output of the HGO. Also, note that w_1 estimates the unknown reference signal Ψ_1 .

Regarding the performance of the failure actuator estimation, the state error and virtual states error are depicted in Figure 6. One can observe that the condition in (14), as time passes, is achieved, and the defect that causes failure in the actuator decreases.

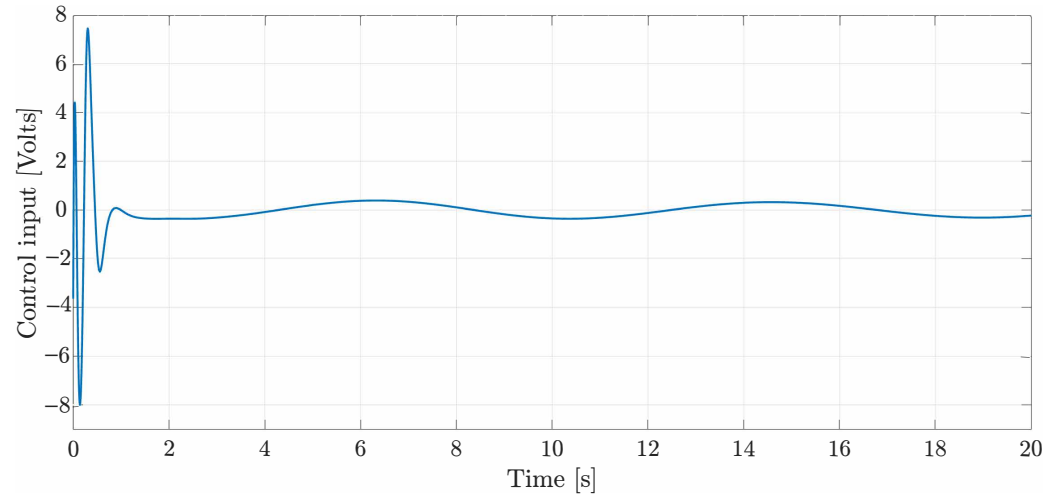


Figure 4. Control signal applied to the Furuta pendulum when using low frequency signal.

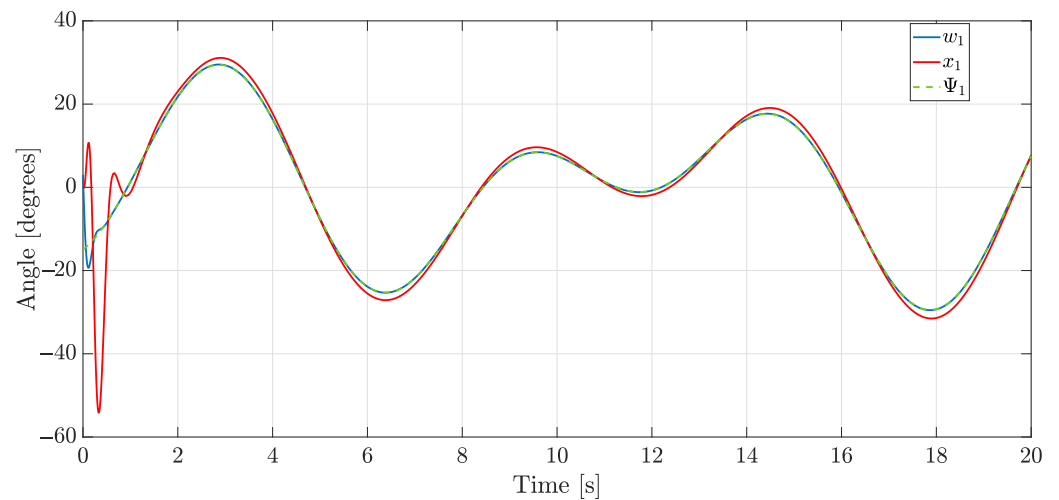


Figure 5. Comparison between the unknown reference signal, the estimated signal by the HGO, and the state x_1 .

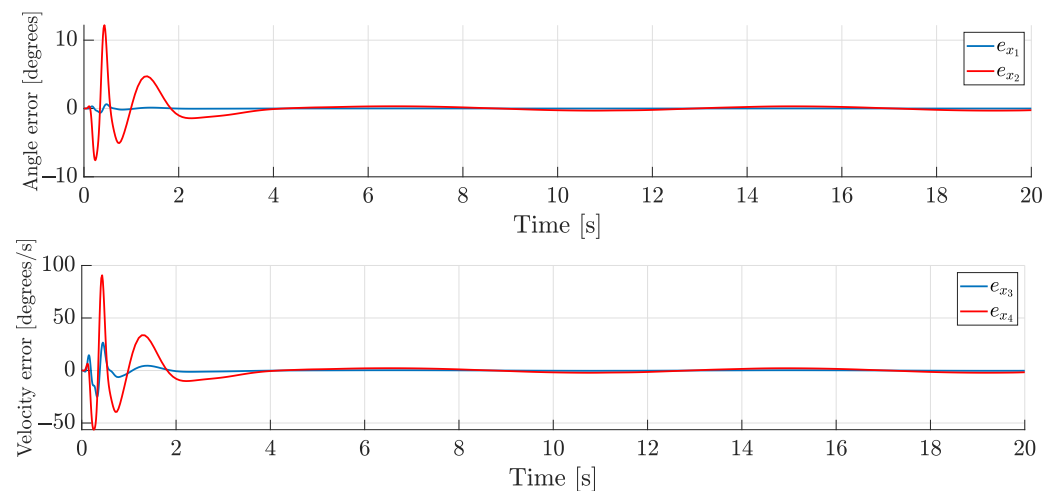


Figure 6. Difference between virtual state and actual state of the system.

4.2. High Frequency Signal

For a quick reference, Figure 7 illustrates the behavior of states when signal control, as shown in Figure 8, is applied to the system. Note that when x_1 is tracking the unknown reference signal, x_2 is oscillating around zero. That means that the second link maintains the vertical position.

Figure 9 shows that the HGO (signal given by w_1) accurately estimates the unknown reference signal Ψ_2 . This does not imply that the system will perform well or accurately track, as this control task depends on the physical configuration and/or limitations of the plant. That is, the state x_1 does not precisely follow the signal w_1 ; this may be related to the mechanical nature of the plant. Additionally, the reference has a high frequency. Furthermore, Figure 10 depicts the fault and its estimation.

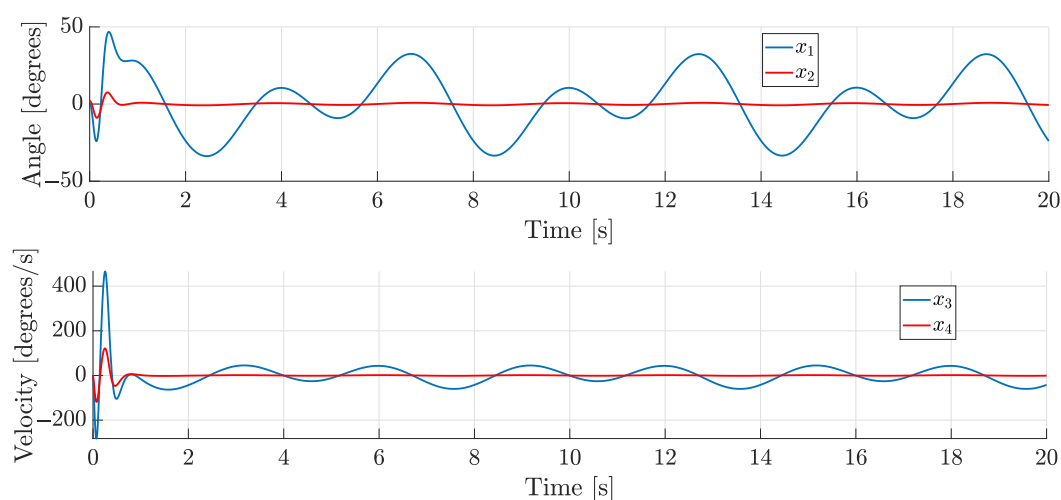


Figure 7. Evolution of x_1 , x_2 , x_3 and x_4 over time when using high frequency signal.

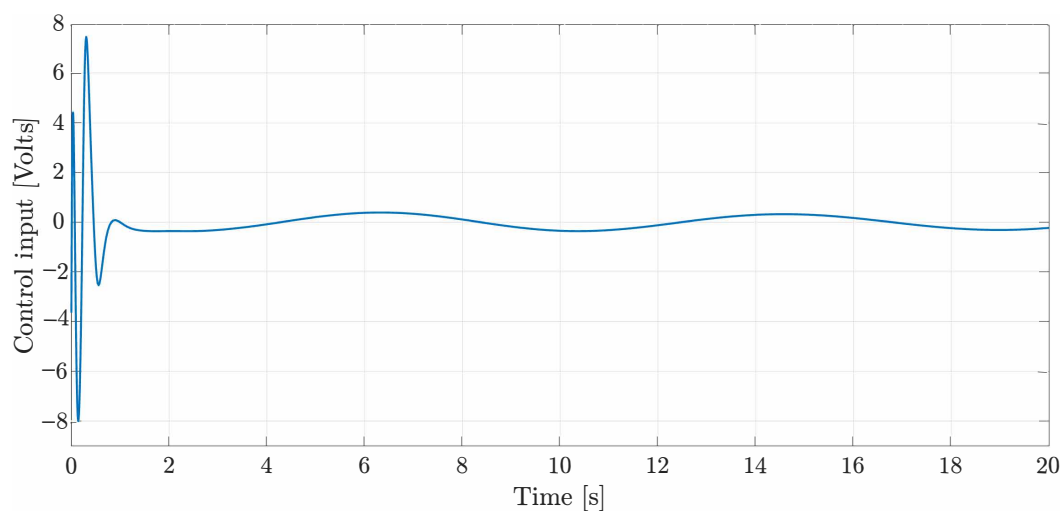


Figure 8. Control signal applied to the Furuta pendulum when using high frequency signal.

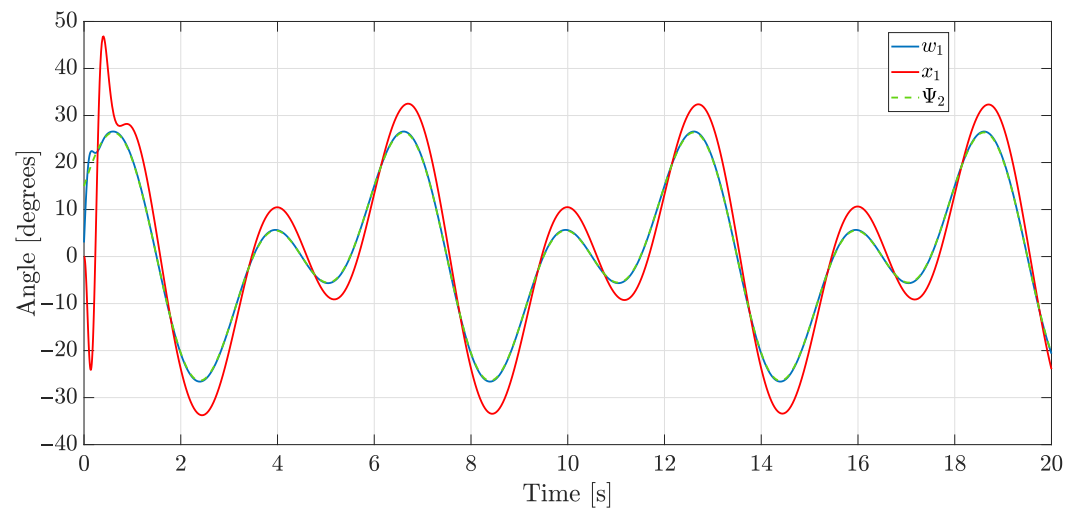


Figure 9. Comparison between the unknown reference signal Ψ_2 , the estimated signal by the HGO and the state x_1 .

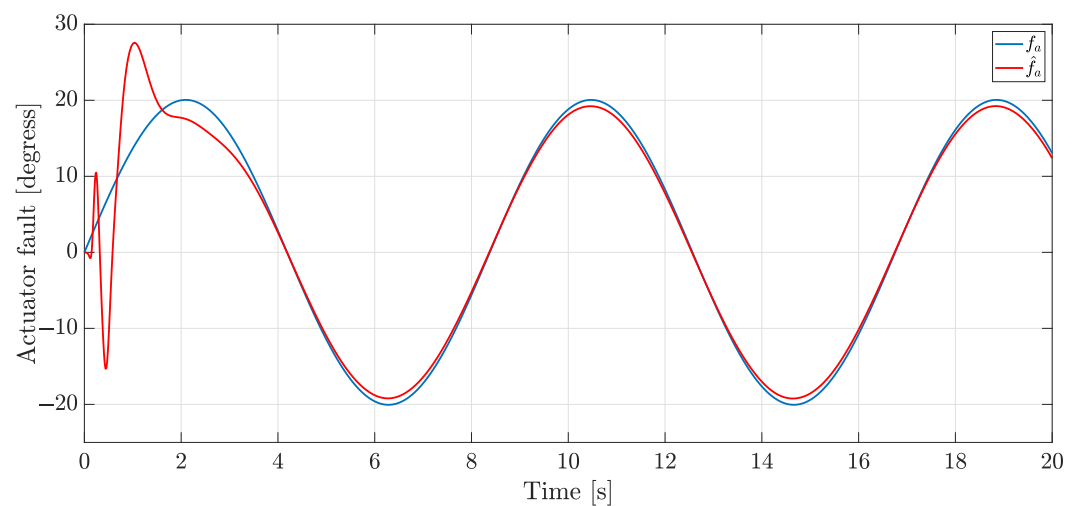


Figure 10. Comparison between the real actuator fault and the estimated fault.

5. Experimental Test

To verify the effectiveness of the proposed controller in real systems, the Furuta pendulum is implemented in real-time. A comparison with the results in [24] is provided to evaluate performance and efficiency. In that sense, the QNET 2.0 Rotary Pendulum board from Quanser (119 Spy Ct, Markham, ON L3R 5H6, Canada), based on the Furuta pendulum mechanism (see Figure 11), was used to verify the proposed control scheme via hardware-in-the-loop experiments.

The two cases in the simulations are considered using the same development for the linearized system (21), the control law (23), and the design of the HGO (22), in such a way that the implementation is based on the same model (Furuta pendulum). Additionally, for the design of the fault observer, a simple modification is provided by the following analysis: it is well known that the observer performs better as more information becomes available. That is to say, for reconstructing the states as well as the actuator fault, we set the output as $\tilde{y}(t) = Cx$, assuming that all state information is available (whole information regulation problem). On the other hand, the output defined in (21) holds. Therefore, since the design of the HGO is independent of the UI observer, an observation gain is designed based on the following output (it gives account of 2 position sensors):

$$\bar{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t),$$

such that $\bar{C}_a = [C_a \ 0_{o \times (n+p_a m)}]$. Thus, by applying the LMI (16) for the fault actuator observer of the fifth order with a decay rate of 4, one can obtain

$$\bar{L} = \begin{bmatrix} 48 & 55 \\ -12 & 23 \\ 1274 & 1883 \\ 365 & 847 \\ 1151 & 1721 \\ 10395 & 15563 \\ 54576 & 81712 \\ 155867 & 233343 \\ 187914 & 281266 \end{bmatrix}.$$

The results in [24] show that, although the simulation and real-time results are sufficiently close to validate the effectiveness of the regulator, there is still room for improvement. Considering the error between the behavior of the physical plant and the reference signal as an actuator fault, it is treated as an estimable signal and is discarded from the regulator.



Figure 11. Inverse rotary pendulum system.

5.1. Low Frequency Signal

First, the results of [24] are compared with those of the current slow signal. Therefore, Figures 12 and 13 illustrate how the output x_1 tends to the reference w_1 . Concretely, the state x_1 has more oscillations in Figure 12, while, in Figure 13, despite the initial oscillation, the synchronization between x_1 and w_1 is more accurate.

In parallel, in Figure 14, it is observed that thanks to the fault-tolerant control, the stabilization action of the inverted pendulum x_2 is improved. One can observe that when using the control given in [24], x_2 exhibits oscillations of greater amplitude than when using the proposed control strategy. Figure 15 shows the states related to x_1 and x_2 of the UI observer, demonstrating the precision with which it estimates the variables of interest in real-time.

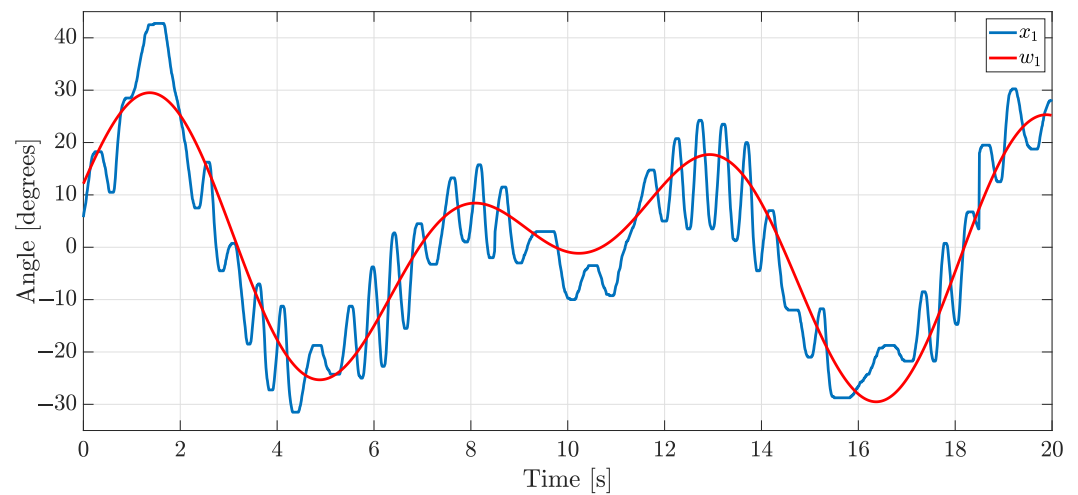


Figure 12. Comparison between x_1 and w_1 when using control law given in [24] for low frequency signal.

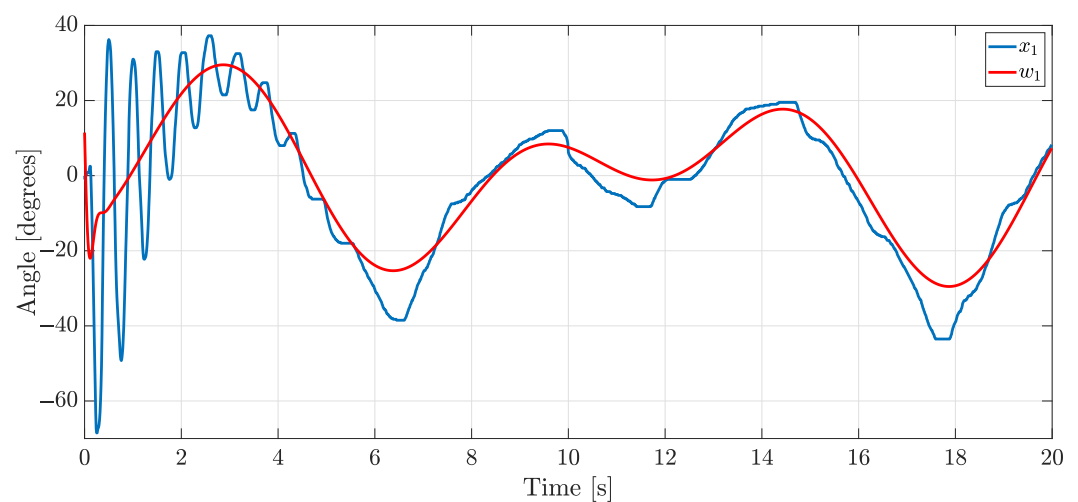


Figure 13. Comparison between x_1 and w_1 when using the proposed control strategy for low frequency signal.

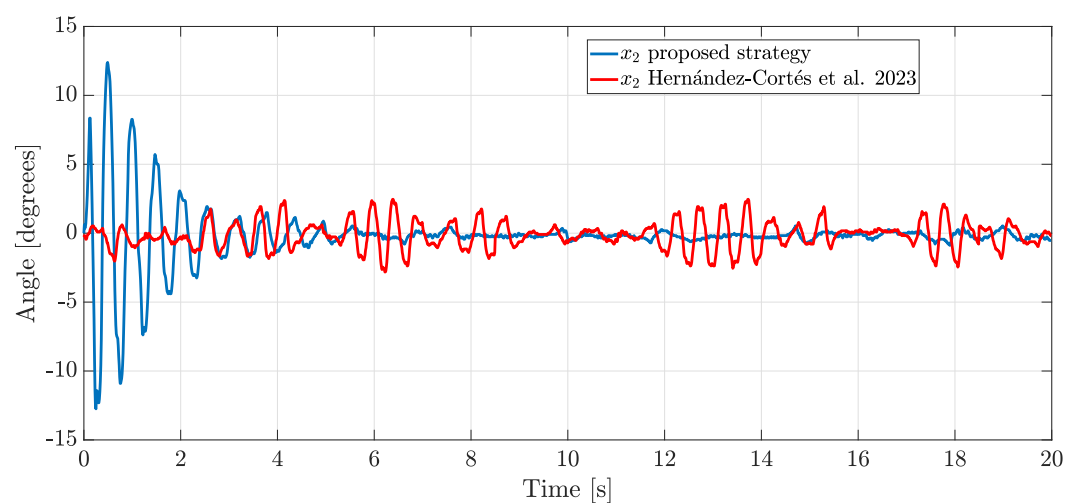


Figure 14. Comparison of x_2 between the control given in [24] and the proposed control strategy for low frequency signal.

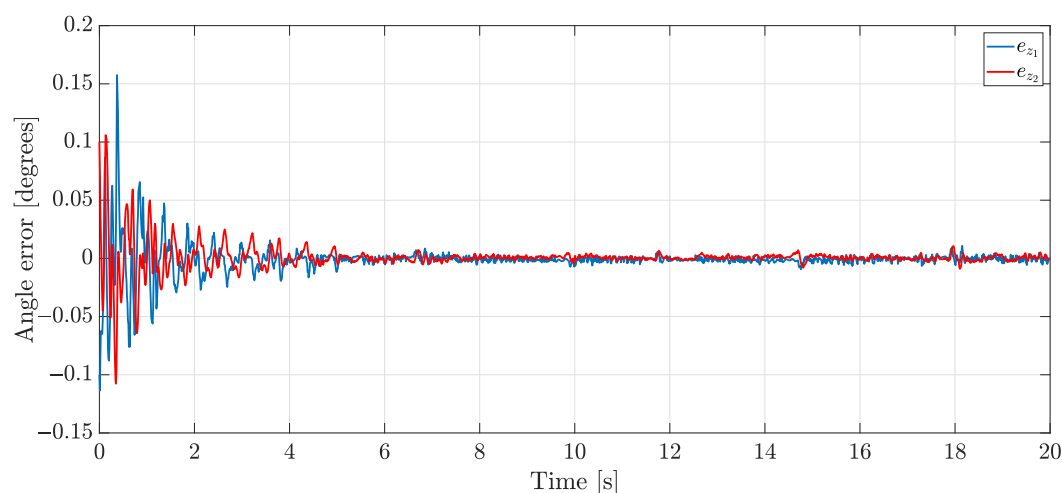


Figure 15. Difference between virtual fault and actual fault of the system.

5.2. High-Frequency Signal

In this case, in order to prove the performance of the proposed approach, a faster reference is applied as, for an unknown reference, it is better appreciated that the error limits are lower for the fault-tolerant regulator than for the normal one, in addition to the fact that the number of oscillations is reduced, as can be seen in Figures 16 and 17. To enable a quantitative comparison, we employ the Mean Square Error (MSE). The results for the normal case [24] are $MSE = 71.9577$ and for the fault-tolerant regulator are $MSE = 29.7233$. Because the MSE is lower than that obtained using the control given in [24], it can be concluded that our proposal has a better performance in following the trajectory and maintaining the vertical position of the link.

As in the previous case, the stabilization action is improved by rejecting the disturbances caused by actuator failures, thereby reducing the oscillations of x_2 , as shown in Figure 18.

Note that, although the control strategy has been designed for the linear approximation, both in numerical simulations (where the nonlinear equations have been used) and in real-time experiments, the Furuta pendulum has been stabilized, and its horizontal beam effectively tracks an unknown reference signal. In contrast, the vertical beam remains in the vertical position.

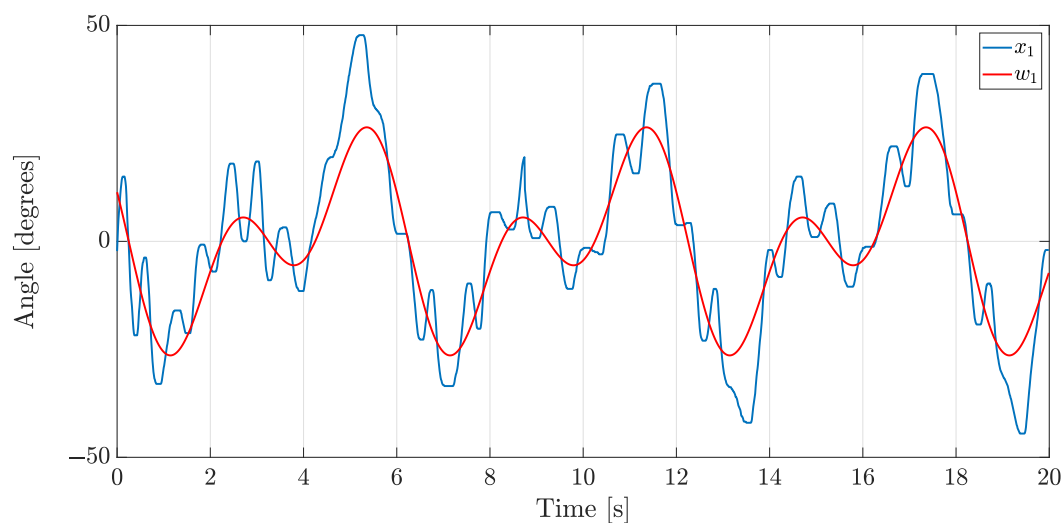


Figure 16. Comparison between x_1 and w_1 when using control law given in [24] for high frequency signal.

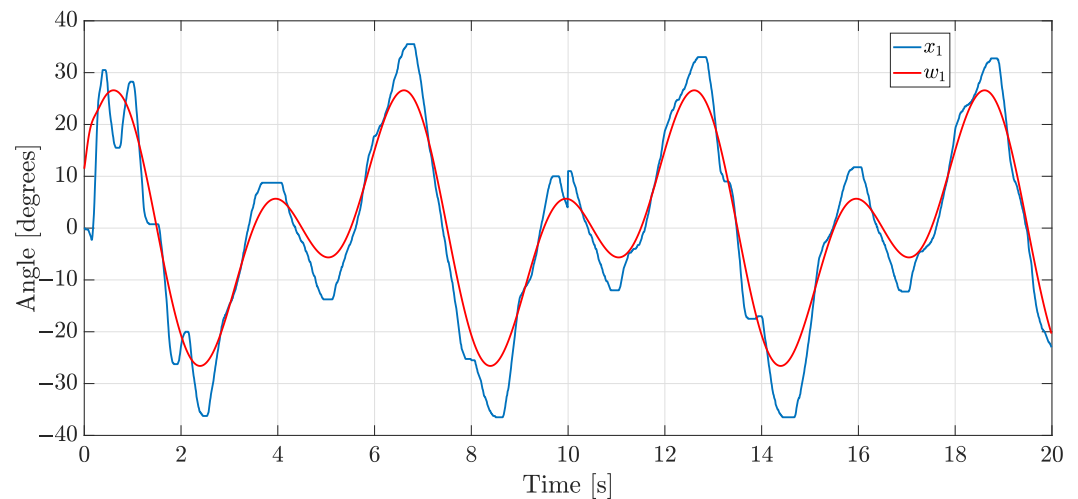


Figure 17. Comparison between x_1 and w_1 when using the proposed control strategy for high frequency signal.

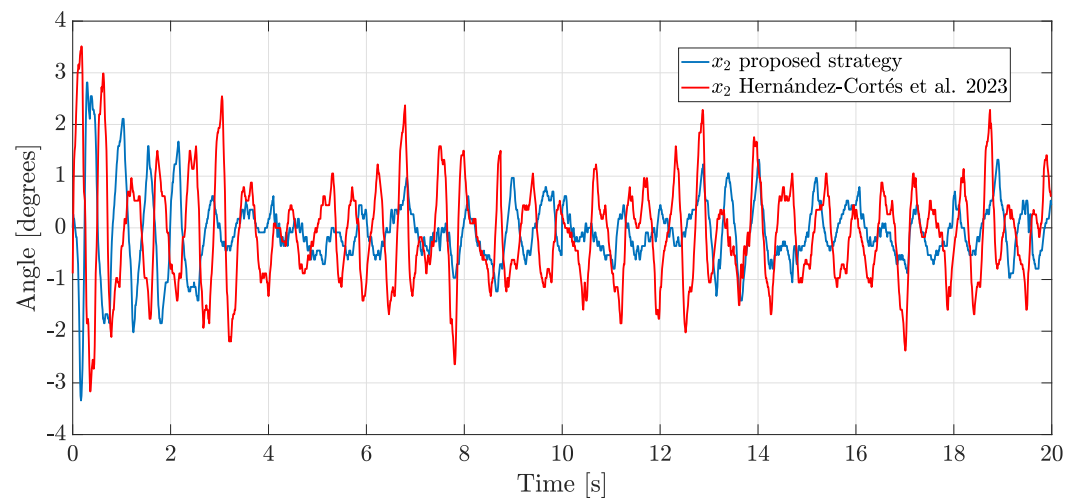


Figure 18. Comparison of x_2 between the control given in [24] and the proposed control strategy for high frequency signal.

Finally, Tables 2 and 3 summarize the results obtained from the simulations and real-time experiments for the low- and high-frequency signals. Based on the results reported in Tables 2 and 3, it can be concluded that the proposed approach achieves better performance compared to [24].

Table 2. Comparison between the simulation and experimental results for low-frequency signals.

Aspect	Low-Frequency Signal	
	Simulation	Experiment
Reference	$\Psi_1(t)$	$\Psi_1(t)$
Tracking performance	Good tracking of w_1 by x_1	Improved synchronization over method in [24]
HGO convergence	Accurate and fast	Effective via w_1 estimation
x_2 stabilization	Minimal oscillations, good control	Reduced oscillation amplitude

Table 2. Cont.

Aspect	Low-Frequency Signal	
	Simulation	Experiment
Convergence time	~ 2 s	Visually similar
Fault estimation accuracy	High accuracy via UI observer	Real-time fault estimation is accurate
Control signal compliance	Within bounds $\beta = 22$	Satisfies LMI constraints
Remarks	Smooth tracking and stable control	Clear improvement versus previous method

Table 3. Comparison between the simulation and experimental results for high-frequency signals.

Aspect	High-Frequency Signal	
	Simulation	Experiment
Reference	$\Psi_2(t)$	$\Psi_2(t)$
Tracking performance	Less precise tracking due to signal frequency	Better tracking, reduced oscillations
HGO convergence	Converges with some degradation	Still effective, captures signal trend
x_2 stabilization	Controlled, with some disturbance	Vertical link more stable than previous method
Convergence time	~ 3 s	Slightly longer, with fewer oscillations
Fault estimation accuracy	Acceptable, some transient error	More precise than prior strategy
Control signal compliance	Within bounds despite demand	Remains compliant with constraints
MSE	–	29.72 (vs. 71.96 in [24])
Remarks	Limited by mechanical response	Strong fault rejection and tracking

6. Conclusions

The present work provides the output regulation theory for the case of unmodeled reference/disturbance signals and non-modeled failures that influence the actuator's operation through the HGO theory and the UI observer. The estimation of faults and unmodeled signals has been carried out using HGO of different dimensions and UI observers, demonstrating that even for simpler faults in the mechanism, the proposed modification enhances the performance of the output regulation. The output regulation theory, which applies to cases of unmodeled reference and disturbance signals with faults in actuators, is an alternative. As a result, it has been possible to mitigate the tracking error further for nonlinear problems observed, not only for exogenous reference/perturbation systems but also for internal model disturbances. This opens up new opportunities for its application on various platforms, including robotics, process control, synchronization, and entertainment. Finally, a Furuta Pendulum system with a different trajectory to follow has been presented in real-time to demonstrate the advantages of the proposed approach.

Author Contributions: Conceptualization, M.A.-M., T.H.-C. and R.T.-H.; data curation, V.E.-M.; formal analysis, M.A.-M., T.H.-C., V.E.-M. and R.T.-H.; funding acquisition, J.G.-S.; investigation, M.A.-M., T.H.-C., V.E.-M., J.G.-S. and R.T.-H.; methodology, M.A.-M., T.H.-C., V.E.-M. and R.T.-H.; project administration, T.H.-C., V.E.-M. and J.G.-S.; resources, T.H.-C., V.E.-M. and J.G.-S.; software, M.A.-M.; supervision, T.H.-C., V.E.-M., J.G.-S. and R.T.-H.; validation, T.H.-C., V.E.-M. and J.G.-S.; visualization, T.H.-C., V.E.-M. and J.G.-S.; writing—original draft, M.A.-M., T.H.-C., V.E.-M., J.G.-S. and R.T.-H.; writing—review and editing, T.H.-C., V.E.-M. and J.G.-S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Instituto Politécnico Nacional—Secretaría de Investigación y Posgrado by project 20253806 and in part by project Universidad Nacional Autónoma de México PAPIIME PE104125. The APC was funded by Instituto Politécnico Nacional.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

HGO	High-Gain Observer
UI	Unknown Input
LMIs	Linear Matrix Inequalities
MSE	Mean Square Error

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