

Article

Bayesian Estimation of Spatial Lagged Panel Quantile Regression Model

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Abstract

This paper proposes a Bayesian estimation method for spatial lagged panel quantile models. The proposed model simultaneously considers spatial lag effects of the dependent variable and the quantile regression framework, enabling effective capture of spatial dependence and conditional distribution heterogeneity. The research constructs a Bayesian estimation framework based on the asymmetric Laplace distribution by decomposing the random disturbance term into a combination of normal and exponential distributions, successfully developing a probabilistic model with both thick tail robustness and computational efficiency. On this basis, the study derives the full conditional posterior probability distributions of model parameters and designs a hybrid Markov Chain Monte Carlo (MCMC) sampling algorithm integrating Gibbs sampling and Metropolis–Hastings algorithm for parameter estimation. Numerical simulation experiments demonstrate that, compared with traditional estimation methods, the proposed Bayesian estimation approach exhibits superior estimation accuracy and robustness across different quantiles, with particularly pronounced advantages in small sample and heavy-tailed distribution scenarios. This methodology provides a more reliable theoretical tool for analyzing panel data with spatial dependencies. This method can not only accurately quantify the spatial spillover effect, but also identify the different effects of the same influencing factor at different emission levels, which provides a strong methodological support for formulating differentiated and precise emission reduction policies.

Keywords: spatial lagged panel quantile model; quantile regression; Bayesian estimation; asymmetric Laplace distribution; MCMC sampling



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1. Introduction

In Statistical Science and Econometrics, panel data (PD) and Spatial Data analysis are two pivotal methodological tools for handling data with temporal and spatial dimensions. Panel data, characterized by its dual advantages of integrating time series and cross-sectional information, has been extensively utilized for inter-temporal comparisons and dynamic analyses. However, traditional panel data approaches neglect spatial effects, and Lee [1] critically highlighted that disregarding spatial lag effects potentially induces substantial estimation bias, thereby motivating the extension towards spatial lagged panel data methodologies that acknowledge spatial interdependencies and explore spatial spillover mechanisms.

Anselin [2] pointed out that there is a significant spatial dependence between spatial units. Traditional spatial regression models include Spatial Error Model (SEM), Spatial Autoregressive Model (SAR), and Spatial Durbin Model (SDM). In the field of econometrics, the limitations of traditional models have prompted scholars to turn to more explanatory spatial econometric models. Since the 1980s, the basic estimation problem of spatial econometrics has become the focus of research. Scholars such as Besag [3], Ord [4], and Mardia [5] have systematically discussed the Spatial Autoregressive Model. At the level of estimation methodology, in addition to maximum likelihood estimation, the estimation method based on moment conditions has also achieved significant development. The instrumental variable method proposed by Kelejian and Prucha [6] and the generalized moment estimation proposed by Kelejian and Robinson [7] effectively alleviate the strict dependence of traditional methods on distribution assumptions.

The traditional spatial regression model focuses on the marginal influence mechanism of explanatory variables on the mean value of response variables. Although it can effectively deal with the problem of spatial correlation, its inherent defects in the tail characteristics of conditional distribution and the sensitivity of outliers limit comprehensive analysis in the context of the nonlinear relationship. The nonlinear characteristics of spatial lag effects such as distance attenuation or threshold effect further aggravate the complexity of the model. In order to break through the bottleneck of this methodology, Koenkar and Bassett [8] creatively proposed the theoretical framework of quantile regression in 1978. By expanding the estimation target from mean shift to median and more general quantile positions, they pioneered the asymptotic unbiased characterization of the distribution of response variables by covariates.

Although the existing research has discussed the expansion of the spatial lag model and quantile regression respectively, the combination of the two is still a frontier field. On the one hand, the robustness of quantile regression to singular values and outliers can enhance the stability of spatial models; on the other hand, the spatial lag mechanism provides a new explanation framework for the spatial heterogeneity of quantile coefficients. Therefore, introducing the spatial lag term into quantile regression and combining the dynamic characteristics of time series with the spatial correlation mechanism can reveal the influence path of spatial-temporal interaction on different quantiles more comprehensively.

With the increasing application of quantile regression technology, its theoretical limitations gradually appear: when the objective function involves non-smooth loss terms or presents a loss of convexity, the numerical optimization process can easily fall into the local optimal trap; under the condition of finite samples, the asymptotic normality of traditional quantile estimators often cannot be strictly guaranteed. In this regard, the academic community actively explores the path of methodological innovation, in which the integration of Bayesian statistics and quantile regression shows significant potential. Yu and Moyeed [9] successfully overcome the problem of parameter uncertainty quantification by constructing a Bayesian quantile regression model with asymmetric Laplace distribution as a working prior. In recent years, the technology system has been successfully extended to the frontier fields such as longitudinal data modeling [10], high-dimensional penalty estimation [11], and spatial econometric analysis [12].

Under the framework of the Bayesian method, methodological research has been advanced along multiple paths. Kostov [13] innovatively proposed a modeling method based on spatial adjacency matrix. On this basis, Reich and Montserrat [14] successfully constructed the theoretical basis of the Bayesian spatial quantile regression model by introducing the Bernstein polynomial basis function. Subsequently, in the space-time dimension, Yue and Rue [15] incorporated the space-time effect into the linear model framework, and used the integrated nested Laplace approximation algorithm to achieve efficient estimation.

Reich [16] developed an innovative spatiotemporal quantile regression framework by setting the time series change of the quantile level and the constraint of the piecewise Gaussian basis function. Sun and Wang [17] further introduced the adaptive Lasso penalty term, which effectively solved the problem of variable selection under space–time dependence. On the other hand, Yang [18] used the likelihood function of asymmetric Laplace distribution to construct an analytical framework for the unified processing of complete and missing data. Based on its position-scale hybrid representation, Luo et al. [10] developed a hierarchical Bayesian model that can handle both fixed and random effects. Alhamzawi [19] introduced the mixed representation of double exponential distribution and established an efficient Gibbs sampling algorithm for ordered longitudinal data. Alhamzawi et al. [20] simplified the MCMC architecture through the normal–exponential hybrid representation of ALD, which significantly improved the computational efficiency of single-index quantile estimation for ordered data.

Compared with the international research frontier, the research of domestic scholars in the field of Bayesian quantile regression started late but developed rapidly: Zeng [21] used the Bayes-M-H sampling method to conduct in-depth discussion on the quantile autoregressive model (AR), and systematically expounded the theoretical characteristics of its parameter estimation. Mei Bo and Tian Maozai [22] constructed a spatiotemporal quantile regression model based on the thin plate regression spline method to reveal the nonlinear influence mechanism of meteorological factors on the spatial distribution of PM2.5 concentration. The Bayesian spatial lag quantile regression model developed by Fang et al. [23] verified its estimation efficiency and robustness under small sample conditions through empirical research.

Therefore, this paper proposes a Bayesian estimation method for spatial lag panel quantile regression [24] (SLPQR). The spatial lag model is extended to include both time and cross-sectional dimensions, and the quantile model is introduced. Combining the advantages of spatial effect [25], panel data, and quantile regression [8], the corresponding Markov Chain Monte Carlo sampling algorithm (MCMC) is designed for parameter estimation. The model combines spatial statistical analysis, panel data characteristics, and quantile regression methods, and becomes an innovative tool for processing spatiotemporal complex data.

It is worth noting that although mature methods have been developed in the fields of spatial econometrics, panel data analysis, and quantile regression, models and inference schemes that can simultaneously deal with spatial dependence, individual heterogeneity, conditional distribution, and parameter uncertainty quantification within a unified framework are still undetermined. In order to clearly locate the innovative contribution of this study, Table 1 systematically compares this paper with several pioneering and closely related previous studies from the perspective of core methodological characteristics.

As shown in Table 1, existing studies usually focus only on one or several aspects of the above features. For example, the classical literature has laid the foundation for spatial models, quantile regression, or Bayesian inference, respectively. Although recent progress has tried to combine two or three of these elements, such as spatial quantiles, panel quantiles, or Bayesian spatial models, there is no work that integrates the four core dimensions of spatial lag, panel fixed effect, quantile regression, and Bayesian inference into a single model.

The SLPQR model and its supporting hybrid MCMC algorithm proposed in this paper are used to fill this methodological gap. It not only provides a powerful tool for full distribution inference of spatially dependent panel data, but also naturally realizes parameter estimation and uncertainty quantification through the Bayesian framework,

which is especially suitable for small sample, non-normal, and complex data environments with outliers.

Table 1. Methodological features of this study compared with key related literature.

Literature	Spatial Lag	Panel/Fixed Effects	Quantile Regression	Bayesian Inference	Main Methods
Anselin [26]	Yes	No	No	No	Classical cross-sectional spatial regression model, based on ML or GMM.
Koenker & Bassett [8]	No	No	Yes	No	The classical framework of quantile regression is proposed and solved based on linear programming.
Yu & Moyeed [9]	No	No	Yes	Yes	Pioneered Bayesian QR, based on asymmetric Laplace likelihood and MCMC.
Galvao [24]	No	Yes (dynamic panel QR)	Yes	No	Panel data quantile regression, dealing with individual effects and endogeneity.
Lum & Gelfand [12]	Yes (spatial QR)	No	Yes	Yes	Bayesian spatial quantile regression, but limited to cross-sectional data.
SLPQR	Yes	Yes (static panel with fixed effects)	Yes	Yes	First integration: spatial lag panel + fixed effects + quantile regression + Bayesian MCMC inference.

2. Model

2.1. Construction of Spatial Lag Panel Fixed Effect Quantile Model

2.1.1. Spatial Lag Panel Fixed Effect Quantile Model

The spatial lag model was proposed by Anselin [26]. Yu et al. [27] proposed a fixed effect spatial dynamic panel data model. The individual dimension is $i = 1, 2, \dots, N$ and the time dimension is $t = 1, 2, \dots, T$. The spatial lag term of this model is set to Wy_t ; that is, only the dependencies between all spatial units in the same time section are considered. This is different from the dynamic spatial panel model proposed by Yu et al. [27], which contains both time lag and spatial lag. This paper adopts a static setting. On the one hand, it is derived from the theoretical basis of the classical spatial econometric model. On the other hand, it is convenient to construct an efficient Bayesian estimation algorithm under the quantile regression framework to focus on the heterogeneity of spatial dependence at different quantiles of conditional distribution. Under the condition of spatial effect, the spatial lag panel fixed effect quantile model is as follows:

$$y_{it} = \rho_\tau \sum_{j=1}^N w_{ij} y_{jt} + X_{it}^T \beta_\tau + \alpha_i + \varepsilon_{it}, i = 1, 2, \dots, N, t = 1, 2, \dots, T \quad (1)$$

In (1), y_{it} is the dependent variable, X_{it} is the column variable of the K -dimensional explanatory variable, α_i is the individual effect, ρ_τ is the spatial autoregressive coefficient under the quantile τ (In the framework of quantile regression, all regression coefficients are functions of quantile τ , so we use symbols such as ρ_τ and β_τ to distinguish them.), N is the number of cross-sectional data samples, $W = (w_{ij})$ is the spatial weighting matrix, T is the

sample time dimension, u_i is the individual fixed effect, and β_τ is the regression coefficient sequence variable before the explanatory variable X_{it} .

We assume that the random perturbation term ε_{it} obeys the asymmetric Laplace distribution as $\varepsilon_{it} \sim ALD(0, \sigma, \tau)$, and the probability density function of the random perturbation term is as follows:

$$f(\varepsilon_{it}|\mu, \sigma, \tau) = \frac{\tau(\tau-1)}{\sigma} \exp\left\{-\lambda_\tau\left(\frac{\varepsilon_{it}-\mu}{\sigma}\right)\right\} \quad (2)$$

In this paper, the asymmetric Laplace distribution is selected as the distribution assumption of the random disturbance term, mainly based on the following considerations: Firstly, Yu and Moyeed [9] proved that under this assumption, the maximum a posteriori estimation of Bayesian quantile regression is equivalent to the classical quantile regression estimation, which lays a theoretical foundation for this assumption. Secondly, Kozumi and Kobayashi [28] proved that ALD has a normal-exponential mixed representation, which provides a key computational convenience for constructing an efficient Gibbs sampling algorithm under the hierarchical Bayesian framework. Finally, compared with the Gaussian hypothesis, the thick tail characteristics of ALD make it more robust to outliers and more suitable for capturing the heterogeneity of different quantiles of conditional distribution.

In (2), $0 < \tau < 1$ is the skewness parameter, σ is the scale parameter, and $-\infty < \mu < \infty$ is the position parameter. The asymmetric Laplace distribution can be parameterized into a linear superposition of two independent random variables: one is the exponential distribution with scale parameter adjustment, which mainly captures the characteristics of thick tail; the other is the normal distribution dominated by the mean parameter, which is responsible for depicting the central trend. Let e obey exponential distribution and u obey standard normal distribution. According to Kozumi and Kobayashi [28], we have: $\varepsilon = \sigma k_1 e + k_2 \sqrt{\sigma} e u$, where $k_1 = \frac{1-2\tau}{\tau(1-\tau)}$, $k_2 = \sqrt{\frac{2}{\tau(1-\tau)}}$. Let $v = \sigma e$, which can be re-expressed as: $\varepsilon = k_1 v + k_2 \sqrt{\sigma} v u$.

According to the asymmetric Laplacian distribution (2), model (1) can be written as:

$$y_{it} = \rho_\tau \sum_{j=1}^N w_{ij} y_{jt} + X_{it}^T \beta_\tau + \alpha_i + k_1 v_{it} + k_2 \sqrt{\sigma} v_{it} u_{it} \quad (3)$$

Note: $Y = (Y_{11}, \dots, Y_{N1}, \dots, Y_{1T}, \dots, Y_{NT})^T$, $X_k = (X_{11}, \dots, X_{N1}, \dots, X_{1T}, \dots, X_{NT})^T$, $X = (X_1, X_2, \dots, X_K)$, $\beta = (\beta_1, \beta_2, \dots, \beta_K)^T$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)^T$, $M = I_T \otimes I_N$, $\Lambda = \text{diag}(\sigma v_{11}, \dots, \sigma v_{N1}, \dots, \sigma v_{1T}, \dots, \sigma v_{NT})$, $v = (v_{11}, \dots, v_{N1}, \dots, v_{1T}, \dots, v_{NT})^T$, $u = (u_{11}, \dots, u_{N1}, \dots, u_{1T}, \dots, u_{NT})^T$, $A_\rho = I_{NT} - \rho_\tau (I_T \otimes W)$; $\otimes$ is the tensor product of the matrix. Model (3) can be written in matrix form:

$$Y = A_\rho^{-1} M \alpha + A_\rho^{-1} X \beta_\tau + k_1 A_\rho^{-1} v + k_2^2 \Lambda^{\frac{1}{2}} u \quad (4)$$

2.1.2. Spatial Lag Fixed Effect Quantile Model Likelihood Function

(1) Bayesian quantile regression model

The quantile regression model proposed by Koenker and Bassett [8] is as follows: Let Y be a random variable and $F(y) = P(Y \leq y)$ be its distribution function. For any $\tau \in (0, 1)$, the τ quantile of the random variable Y is defined as $Q_\tau(Y) = \text{arginf}\{y \in \mathbb{R}, F(y) \geq \tau\}$, and the loss function is the following function:

$$\lambda_\tau(u) = \begin{cases} (\tau-1)u, & u < 0. \\ \tau u, & u \geq 0. \end{cases} \quad (5)$$

Quantile regression is used to estimate the regression coefficient by minimizing the sum of the absolute values of the weighted residuals; that is, the parameters are solved in the following ways:

$$\hat{\beta}(\tau) = \underset{\beta(\tau)}{\operatorname{argmin}} \sum_{i=1}^n \lambda_{\tau}(y_i - x_i^T \beta(\tau)) \quad (6)$$

Yu and Moyeed [9] pointed out that minimizing the absolute value of the weighted residuals is equivalent to maximizing the likelihood function based on the asymmetric Laplace distribution, thus establishing the Bayesian quantile regression model, using its density function to construct the likelihood function, and giving the sampling algorithm for parameter posterior inference. That is to say: for a set of random independent samples $Y = (Y_1, \dots, Y_n)^T$ of Y , if $Y_i \sim \text{ALD}(\mu_i, \sigma, \tau)$ is assumed, let $\mu_i = x_i^T \beta_{\tau}$, then the likelihood function that β_{τ} can be transformed into the maximum sample is solved by the minimization Formula (6) as follows:

$$L(\beta, \sigma; Y, \tau) \propto \frac{1}{\sigma^n} \exp \left\{ - \sum_{i=1}^n \lambda_{\tau} \left(\frac{y_i - x_i^T \beta_{\tau}}{\sigma} \right) \right\} \quad (7)$$

(2) Model likelihood function

According to the probability density (2) and the model maximum likelihood function transformation (7), the likelihood function of the model (3) can be written as:

$$L(Y|\rho_{\tau}, \beta_{\tau}, \sigma, v, \alpha) = (2\pi)^{-\frac{NT}{2}} |I_{NT} - \rho_{\tau}(I_T \otimes W)| v_{it}^{-\frac{1}{2}} \times \exp \left\{ - \frac{1}{2k_2^2 \sigma v_{it}} (y_{it} - \rho_{\tau} \sum_{j=1}^N w_{ij} y_{jt} - x_{it} \beta_{\tau} - \alpha_i - k_1 v_{it})^2 \right\} \quad (8)$$

The model likelihood function (8) is written in matrix form:

$$L(Y|\rho_{\tau}, \beta_{\tau}, \sigma, v, \alpha) = (2\pi)^{-\frac{NT}{2}} |A_{\rho}| k_2^{-1} \Lambda^{-\frac{1}{2}} \times \exp \left\{ - \frac{1}{2} (Y - A_{\rho}^{-1} M \alpha - A_{\rho}^{-1} X \beta_{\tau} - k_1 A_{\rho}^{-1} v)^T (A_{\rho}^{-1} k_2^2 \Lambda (A_{\rho}^{-1})^T)^{-1} \times (Y - A_{\rho}^{-1} M \alpha - A_{\rho}^{-1} X \beta_{\tau} - k_1 A_{\rho}^{-1} v) \right\} \quad (9)$$

2.2. Bayesian Estimation and MCMC Sampling Algorithm Design

2.2.1. Bayesian Estimation

In order to establish the Bayesian estimation of the likelihood functions (8) and (9), it is necessary to make a priori assumption on the important parameters in the model. The prior of parameter ρ_{τ} is set as uniform distribution $U(\lambda_{\min}^{-1}, \lambda_{\max}^{-1})$, where $\lambda_{\min}^{-1}$ and $\lambda_{\max}^{-1}$ are the maximum and minimum eigenvalues of the spatial weight matrix, respectively. The prior distribution of parameter β_{τ} is set as multivariate normal distribution $N_k(\beta_{\tau 0}, \Sigma_{\tau 0})$; the prior distribution of parameter α is set as multivariate normal distribution $N(0, \sigma_{\alpha}^2 I_N)$; the prior distribution of parameter u is set as multivariate normal distribution $N(0, I_{NT})$; the prior distribution of parameter σ is set as inverse gamma distribution $IG(\frac{n_0}{2}, \frac{s_0}{2})$; the prior distribution of parameter v_{it} is set to exponential distribution $\exp\{\sigma\}$. The hierarchical Bayesian spatial lag panel quantile model can be expressed as follows:

$$Y = \rho_{\tau}(I_T \otimes W)Y + (I_T \otimes I_N)\alpha + X\beta_{\tau} + k_1 v + k_2^2 \Lambda^{\frac{1}{2}} u$$

$$\begin{aligned} \beta_{\tau} &\sim N_k(\beta_{\tau 0}, \Sigma_{\tau 0}) \\ \rho_{\tau} &\sim U(\lambda_{\min}^{-1}, \lambda_{\max}^{-1}) \\ \sigma &\sim IG(\frac{n_0}{2}, \frac{s_0}{2}) \\ u &\sim N(0, I_{NT}) \\ v_{it} &\sim \exp\{\sigma\} \\ \alpha &\sim N(0, \sigma_{\alpha}^2 I_N) \end{aligned} \quad (10)$$

It should be pointed out that Bayesian estimation relies on prior assumptions, which may introduce certain subjectivity in practical applications. However, the priors used in this paper are all weak information priors, and large variances are set to minimize the influence of priors on the posterior. In addition, users can adjust priors according to specific problems, such as those based on historical data or theoretical constraints. In order to evaluate the robustness, it is recommended to conduct sensitivity analysis in practice and compare the results under different prior settings. Although the Bayesian method sacrifices the ‘objectivity’ and computational simplicity of the frequency, it provides more flexible uncertainty quantification and adaptability to small sample data, which is particularly valuable for the quantile analysis of complex spatial panel data.

According to the prior distribution of the parameters in the likelihood functions (8) (9) and model (10), the conditional posterior probability density of each parameter to be estimated in the model (4) is derived.

(1) The conditional posterior probability density of parameter β_τ is

$$\begin{aligned} \pi(\beta_\tau | Y, \rho_\tau, \sigma, v, \alpha) &\propto L(Y | \rho_\tau, \beta_\tau, \sigma, v, \alpha) \pi(\beta_\tau) \\ &\propto \exp\left\{-\frac{1}{2}(\beta_\tau - \mu(\beta_\tau))^T \Sigma(\beta_\tau)^{-1}(\beta_\tau - \mu(\beta_\tau))\right\} \end{aligned} \quad (11)$$

Since the conditional distribution of y_{it} is a normal distribution and the prior of β_τ is also a normal distribution, according to the theorem of Lindley and Smith [29], it can be determined that the conditional posterior of parameter β_τ obeys a normal distribution with a mean of $\mu(\beta_\tau)$ and a variance of $\Sigma(\beta_\tau)^{-1}$; that is, $\beta_\tau \sim N(\mu(\beta_\tau), \Sigma(\beta_\tau))$, where $\mu(\beta_\tau) = \Sigma(\beta_\tau)(X^T(k_2^2\Lambda)^{-1}(A_\rho Y - M\alpha - k_1v) + \Sigma_0^{-1}\beta_{\tau_0}), \Sigma(\beta_\tau) = (X^T(k_2^2\Lambda)^{-1}X + \Sigma_0^{-1})^{-1}$.

(2) The conditional posterior probability density of parameter α is

$$\begin{aligned} \pi(\alpha | Y, \rho_\tau, \beta_\tau, \sigma, v) &\propto L(Y | \rho_\tau, \beta_\tau, \sigma, v, \alpha) \pi(\alpha) \\ &\propto \exp\left\{-\frac{1}{2}(\alpha - \mu(\alpha))^T \Sigma(\alpha)^{-1}(\alpha - \mu(\alpha))\right\} \end{aligned} \quad (12)$$

Similarly, according to the theorem of Lindley and Smith [29], it can be determined that the conditional posterior of parameter α obeys the normal distribution with the mean value of $\mu(\alpha)$ and the variance of $\Sigma(\alpha)^{-1}$; that is, $\alpha \sim N(\mu(\alpha), \Sigma(\alpha))$, where $\mu(\alpha) = \Sigma(\alpha)[M^T(k_2^2\Lambda)^{-1}(A_\rho Y - X\beta_\tau - k_1v)], \Sigma(\alpha) = (M^T k_2^2 \Lambda^{-1} M + \frac{1}{\sigma_\alpha^2} I_N)^{-1}$.

(3) The conditional posterior probability density of parameter v_{it} is

$$\begin{aligned} \pi(v_{it} | y_{it}, \rho_\tau, \beta_\tau, \alpha_i, \sigma) &\propto L(y_{it} | \rho_\tau, \beta_\tau, \sigma, v_{it}, \alpha_i) \pi(v_{it}) \\ &\propto |I_{NT} - \rho_\tau(I_T \otimes W)| v_{it}^{-\frac{1}{2}} \exp\left\{-\frac{1}{2k_2^2\sigma v_{it}}(y_{it} - \rho_\tau \sum_{j=1}^N w_{ij}y_{jt} - x_{it}\beta_\tau - \alpha_i - k_1v_{it})^2\right\} \exp\left\{-\frac{v_{it}}{\sigma}\right\} \\ &\propto v_{it}^{-\frac{1}{2}} \exp\left\{-\frac{1}{2k_2^2\sigma v_{it}}(y_{it} - \rho_\tau \sum_{j=1}^N w_{ij}y_{jt} - x_{it}\beta_\tau - \alpha_i - k_1v_{it})^2\right\} \exp\left\{-\frac{v_{it}}{\sigma}\right\} \\ &\propto v_{it}^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} \frac{(y_{it} - \rho_\tau \sum_{j=1}^N w_{ij}y_{jt} - x_{it}\beta_\tau - \alpha_i)^2}{k_2^2\sigma} v_{it}^{-1} + \left(\frac{k_1^2}{k_2^2\sigma} + \frac{2}{\sigma}\right)v_{it}\right\}. \end{aligned} \quad (13)$$

The conditional posterior of parameter v_{it} obeys the generalized inverse Gaussian distribution, namely $v \sim GIG(\frac{1}{2}, \chi, \psi)$, where $\chi = \frac{(y_{it} - \rho_\tau \sum_{j=1}^N w_{ij}y_{jt} - x_{it}\beta_\tau - \alpha_i)^2}{k_2^2\sigma}$, $\psi = \frac{k_1^2}{k_2^2\sigma} + \frac{2}{\sigma}$.

(4) The conditional posterior probability density of parameter σ is

$$\begin{aligned} \pi(\sigma | Y, \rho_\tau, \beta_\tau, v, \alpha) &\propto L(Y | \rho_\tau, \beta_\tau, \sigma, v, \alpha) \pi(\sigma) \\ &\propto \sigma^{-\frac{s(\sigma)}{2}-1} \exp\left[-\frac{\kappa(\sigma)}{2\sigma}\right]. \end{aligned} \quad (14)$$

The conditional posterior of parameter σ obeys the inverse gamma distribution with shape parameter $\frac{s(\sigma)}{2}$ and scale parameter $\frac{\kappa(\sigma)}{2}$, namely $\sigma \sim IG(\frac{s(\sigma)}{2}, \frac{\kappa(\sigma)}{2})$, where $s(\sigma) = n_0 + NT$, $\kappa(\sigma) = s_0 + (A_\rho Y - M\alpha - X\beta_\tau - k_1 v)^T \times (k_2^2 \text{diag}(v))^{-1} (A_\rho Y - M\alpha - X\beta_\tau - k_1 v)$.

(5) The conditional posterior probability density of parameter ρ_τ is

$$\pi(\rho_\tau | Y, \sigma, \beta_\tau, \sigma, v) \propto L(Y | \rho_\tau, \beta_\tau, \sigma, v, \alpha) \pi(\rho_\tau) \propto |A_\rho| \exp\left\{-\frac{1}{2}(AY - M\alpha - X\beta_\tau - k_1 v)^T (k_2^2 \Lambda)^{-1} (A_\rho Y - M\alpha - X\beta_\tau - k_1 v)\right\} \frac{1}{\lambda_{\max}^{-1} - \lambda_{\min}^{-1}} \quad (15)$$

2.2.2. MCMC Sampling Algorithm Design

Markov Chain Monte Carlo (MCMC) algorithm is a numerical calculation method based on Markov chain. Classical MCMC algorithms include the Metropolis–Hastings algorithm [30] and Gibbs algorithm [31].

According to the complete conditional posterior probability distribution of the model parameters derived above, the MCMC sampling steps are designed. The conditional posterior probability density of parameter ρ_τ is not a common distribution and cannot be directly sampled. Therefore, the M-H algorithm of random walk is used for sampling. The conditional posterior distribution density of other parameters is a common distribution and directly sampled by the Gibbs method. Therefore, the MCMC sampling algorithm of the spatial lag panel fixed effect quantile regression model is as follows:

Assume that $(\beta_\tau^{(p)}, \sigma^{(p)}, \alpha^{(p)}, v_{it}^{(p)}, \rho_\tau^{(p)})$ is the result of the p -th iterative sampling. According to the posterior probability density (11), (12), (13), (14), (15), MCMC sampling algorithm steps are as follows:

Step 1: Given the initial value $(\beta_\tau^{(0)}, \sigma^{(0)}, \alpha^{(0)}, v_{it}^{(0)}, \rho_\tau^{(0)})$.

Step 2: Randomly generate $\beta_\tau^{(p+1)}$ from $\pi(\beta_\tau | Y, \sigma^{(p)}, \alpha^{(p)}, v_{it}^{(p)}, \rho_\tau^{(p)}) \sim N(\mu(\beta_\tau), \Sigma(\beta_\tau))$.

Step 3: Randomly generate $\sigma^{(p+1)}$ from $\pi(\sigma | Y, \beta_\tau^{(p+1)}, \alpha^{(p)}, v_{it}^{(p)}, \rho_\tau^{(p)}) \sim IG(\frac{s(\sigma)}{2}, \frac{\kappa(\sigma)}{2})$.

Step 4: Randomly generate $\alpha^{(p+1)}$ from $\pi(\alpha | Y, \beta_\tau^{(p+1)}, \sigma^{(p+1)}, v_{it}^{(p)}, \rho_\tau^{(p)}) \sim N(\mu(\alpha), \Sigma(\alpha))$.

Step 5: Randomly generate $v_{it}^{(p+1)}$ from $\pi(v_{it} | Y, \beta_\tau^{(p+1)}, \alpha^{(p+1)}, \sigma^{(p+1)}, \rho_\tau^{(p)}) \sim GIG(\frac{1}{2}, \chi, \psi)$.

Step 6: The conditional posterior probability density of parameter ρ_τ is more complex and cannot be written as a common distribution. The M-H sampling algorithm is used to sample it. The standard normal distribution is used as the recommended distribution. Assuming that the current value of ρ_τ is ρ_τ , the preselected value is $\rho_\tau^* = \rho_\tau + c \times s$, and c is the adjustment parameter, where $c = 0.1$, $s \sim N(0, 1)$, then the acceptance probability of ρ_τ^* is:

$$\alpha(\rho_\tau^*, \rho_\tau) = \min\left(1, \frac{\pi(\rho_\tau^* | Y, \alpha, \beta_\tau, v, \sigma)}{\pi(\rho_\tau | Y, \alpha, \beta_\tau, v, \sigma)}\right) \quad (16)$$

Step 7: Let $p = p + 1$; repeat step 2 to step 7 to convergence.

3. Simulation

In this paper, the estimation method of model (10) is tested by numerical simulation, and the simulation results are evaluated.

In the simulation process, the sample points with a sample size of $(N, T) = (10, 10)$ are first generated through model (3), and the construction of the spatial weight matrix W follows the following steps: First, the coordinates of N points are randomly generated in the unit square to represent the spatial unit. Secondly, Tyson polygon segmentation is generated based on these points. Then, the binary symmetric adjacency matrix A is constructed by using the rook adjacency rule; that is, the two polygons share an edge and are judged to be adjacent. Finally, the final W is obtained by standardizing A ; that is,

$W = D^{(-1)}A$, where $D = \text{diag}(A_l)$; l is the unit column vector. This construction process is performed independently in each simulation repetition.

The explanatory variable $x_{it} = (x_{1it}, x_{2it})$ and the model random disturbance term ε_{it} are randomly selected from the standard normal distribution $N(0, 1)$. The real value of parameter $\beta = (\beta_1, \beta_2)$ is set to $\beta = (1, 3)$; the value of ρ is equal to 0.6; the random effect α_i was sampled from the normal distribution $N(0, 1)$. In the case of setting the real value of the parameter and the explanatory variable x_{it} , the random effect α_i , and the random disturbance term ε_{it} , the random data is generated as the sample of the simulation experiment to estimate the model parameters.

In order to provide transparent algorithm implementation details and rigorous convergence diagnostic evidence, enhance the repeatability of the calculation process and the credibility of the results; we performed convergence diagnosis (Table 2), MH step size calibration sensitivity study (Table 3), and posterior summary (Table 4) on three independent MCMC chains.

Table 2. Convergence diagnosis.

Parameter	R-Hat	ESS	ACF
β_1	0.9999	3012.8536	1.3342
β_2	1.0002	4566.9363	0.9569
ρ	1.0011	228.0307	17.9468
δ^2	0.9999	1446.1501	2.8494

Table 3. Sensitivity of Metropolis–Hastings step calibration.

Step Size	Acceptance Rate	Posterior Mean	Posterior SD	Relative Bias
0.01	0.8868	0.4624	0.0491	22.9396
0.05	0.6510	0.4658	0.0639	22.3737
0.1	0.4988	0.4576	0.0766	23.7332
0.2	0.3394	0.4461	0.0908	25.6566

Table 4. A posteriori summary.

Parameter	True Value	Posterior Mean	Posterior SD	X2.5.	X97.5.
β_1	1	0.5800	0.1861	0.2177	0.9501
β_2	3	3.3492	0.1628	3.0268	3.6637
ρ	0.6	0.4670	0.0658	0.3263	0.5843
δ^2	1	0.5733	0.0572	0.4759	0.6994

The above convergence diagnosis shows that the R-hat values of each parameter are close to 1, and ESS is sufficient (more than 1000 except ρ), indicating that the three MCMC chains converge well and the sampling efficiency is reliable. MH step size sensitivity analysis shows that the algorithm has certain robustness to step size selection. There is a certain deviation between the posterior estimation results and the true value, but the posterior interval of each parameter covers the true value or near it, which reflects the rationality of the model estimation. The comprehensive convergence and sensitivity analysis reflect the credibility of the model in this paper.

Moran's I index is divided into global *Moran's I* index and local *Moran's I* index, which is used to analyze the spatial autocorrelation characteristics. The global *Moran's I* index measures the global autocorrelation, while the local *Moran's I* index measures the local autocorrelation. The value of *Moran's I* is positive, indicating a positive correlation; the *Moran's I* value is negative, indicating that the data is negatively correlated, and the absolute value of the value reflects the correlation strength.

There is $N = 10$ in the generated simulation data, and the *Moran's I* index of one group of generated data is calculated. The results are shown in Table 5.

Table 5. Global A-index table.

Period	<i>Moran's I</i> -Value	z-Value	p-Value
1	0.051	1.284	0.100
2	0.178	2.158	0.015
3	0.124	1.966	0.025
4	0.084	1.826	0.034
5	0.067	1.625	0.052
6	0.101	1.614	0.053
7	0.136	2.012	0.022
8	0.144	1.956	0.025
9	0.030	1.127	0.130
10	0.161	2.060	0.020

It can be seen from Table 5 that except for the first period and the ninth period, the *p*-value of each period is less than 0.05, which passes the significance test. The generated simulation data have obvious spatial agglomeration characteristics, and it is necessary to use the spatial econometric model to measure.

The linear least squares estimation of the generated simulation data is performed, and the estimation results are shown in Table 6.

Table 6. Table of linear regression estimation results of simulated data.

Parameters	Estimated Results	Standard Error	t-Value	p-Value
(Intercept)	0.413	0.354	1.165	0.247
x_1	0.317	0.386	0.821	0.414
x_2	3.250	0.365	8.903	3.13×10^{-14}

It can be seen from Table 6 that the true value of parameter β_1 is 1, and the estimated result is 0.3174. The real value of parameter β_2 is 3, and the estimated result is 3.250. And the *p*-value of the estimated parameters of x_1 , x_2 and constant terms is much larger than 0.05, which does not pass the significance test.

On the basis of not considering the spatial effect, the generated simulation data are estimated by the linear panel data individual fixed effect model. The estimation results are shown in Table 7.

Table 7. Linear regression estimation results of simulated data.

Parameters	Estimated Results	Standard Error	t-Value	p-Value
(Intercept)	−0.533	0.143	−3.72	0.000
x_1	0.965	0.148	6.52	0.000
x_2	3.158	0.147	21.34	0.000

It can be seen from Table 7 that the true value of parameter β_1 is 1, and the estimated result is 0.965. The real value of parameter β_2 is 3, and the estimated result is 3.158. And the *p*-value of the estimated parameters is less than 0.05, which passes the significance test. However, due to the spatial effect of the model, the spatial effect should be included in the model estimation.

In order to estimate the accuracy of the model parameter estimation, the Mean Absolute Error (MAE) and Mean Square Error (MSE) are used to evaluate the model:

$$MAE = \frac{1}{m} \sum \left| \frac{\hat{\beta}_j^i - \beta_{j-true}}{\beta_{j-true}} \right| \quad (17)$$

$$MSE = \frac{1}{m} \sum \left| \hat{\beta}_j^i - \beta_{j-true} \right|^2 \quad (18)$$

In (17) and (18), $\hat{\beta}_j^i$ is the estimator under the i -th sample, β_{j-true} is the true value of β_j , and m is the number of simulation experiments. The smaller the MAE is, the more accurate the estimation is. The smaller the MSE, the more accurate the estimation.

For parameter β , set parameter $\beta_{\tau 0} = 0$, $\Sigma_{\tau 0} = I_K \times 10^4$; for parameter σ , set its hyperparameters $n_0 = 3$, $s_0 = 1$; for parameter α , set its parameter $\sigma_\alpha^2 = 1 \times 10^4$; adjust the parameter $c = 0.5$.

In the simulation, this paper will repeat the simulation time K 200 times. The number of MCMC sampling iterations is designed to be 5000, and 1000 unstable points in the initial period of Burn-in iteration are discarded.

3.1. Simulation 1: Model Estimation Results Under Different Error Distributions

The quantile regression method is more suitable than the mean regression method for processing the data with sharp peaks and thick tails. In order to study whether the distribution of random errors will affect the accuracy of the estimation proposed in this paper, four simulation experiments are designed in this paper. The random disturbance term ε_{it} is extracted from the standard normal distribution $N(0, 1)$, chi-square distribution $\chi^2(1)$, t distribution $t(1)$, and Cauchy distribution $Cauchy(0, 0.5)$, and other parameters remain unchanged. As the error term transitions from normal distribution to chi-square distribution, t distribution, and Cauchy distribution, the generated data set gradually shows a skewed and heavy-tailed distribution trend. Considering the extreme cases, we estimate the five quantiles to be $\tau = (0.1, 0.25, 0.5, 0.75, 0.9)$.

(1) The random disturbance term ε_{it} of the model is randomly selected from the standard normal distribution $N(0, 1)$.

It is assumed that the random disturbance term ε_{it} of the model is randomly selected from the standard normal distribution $N(0, 1)$, and the other parameters are generated in the same way.

When the model error obeys other distributions, the sampling iteration result trajectory of the model parameters is visually close to the trajectory in the case of normal distribution. This similarity is mainly reflected in the convergence form of the posterior mean sequence, rather than the overall characteristics of the posterior distribution. In fact, under different error distributions, there may still be differences in the posterior standard deviation, 95% confidence interval, and MCMC convergence diagnostic indicators, such as ESS and $\hat{R}$, which are reflected in the subsequent tables and graphs. Although the sampling trajectories are intuitively similar, the dispersion degree and tail characteristics of the posterior distribution vary with the error distribution, which also reflects that the model has certain robustness to the distribution hypothesis. Therefore, for the sake of clarity, this section only analyzes and explains the parameter iteration process and its posterior probability density distribution under normal distribution.

When $\tau = 0.5$, the parameter iteration trajectory diagram of each parameter to be estimated to abandon the first 1000 unstable points is drawn as shown in Figure 1.

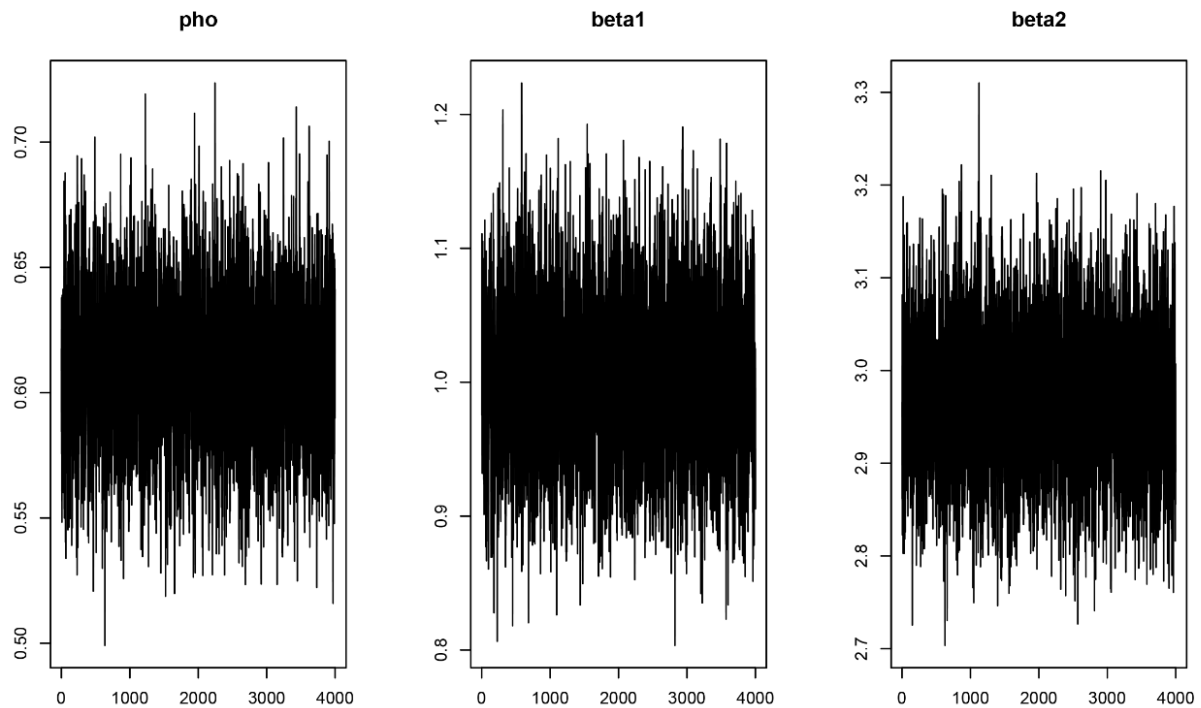


Figure 1. Iteration trajectory of parameters to be estimated.

It can be seen from Figure 1 that the sequence of the first 1000 sampling values of each parameter fluctuate around a constant value, which indicates that the Markov chain composed of sampling values is convergent, and the MCMC simulation process is stable. The parameters show slight fluctuations in the iterative process and always remain near the real value, indicating that the method of parameter estimation using the results after 5000 iterations is effective.

In order to verify the stability and consistency of the model parameter estimation, the distribution of the parameter estimation sequence values in the estimation process is counted, and the posterior probability density maps of the estimated variables β_1 , β_2 , and ρ at the five quantiles $\tau = (0.1, 0.25, 0.5, 0.75, 0.9)$ are drawn as shown in Figure 2.

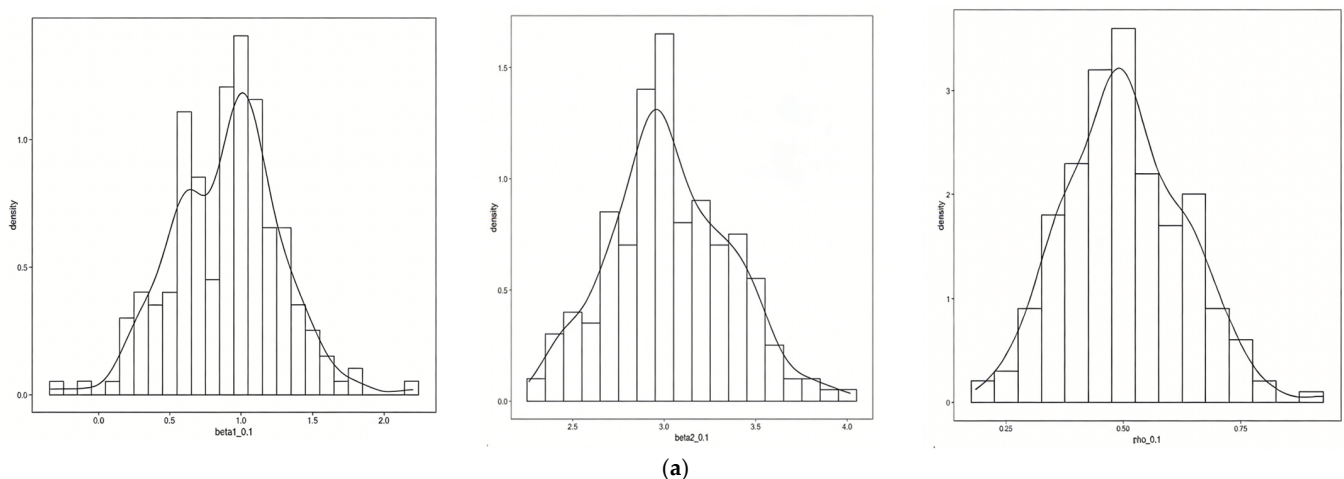


Figure 2. Cont.

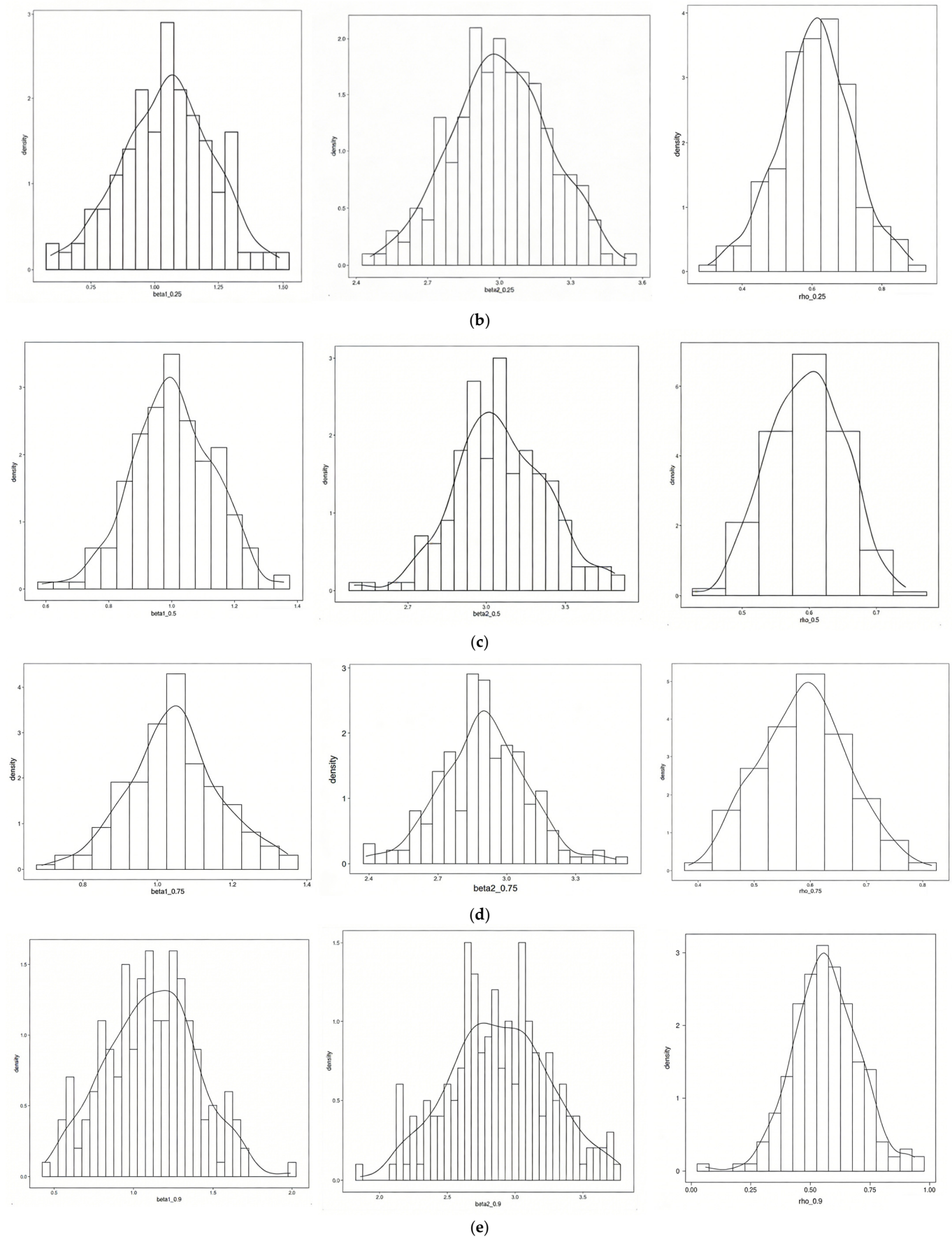


Figure 2. The posterior distribution probability density map of the estimated variables at different quantiles. (a) When $\tau = 0.1$, the posterior distribution probability density diagram of the estimated

variables β_1 , β_2 , and ρ . (b) When $\tau = 0.25$, the posterior distribution probability density diagram of the estimated variables β_1 , β_2 , and ρ . (c) When $\tau = 0.5$, the posterior distribution probability density diagram of the estimated variables β_1 , β_2 , and ρ . (d) When $\tau = 0.75$, the posterior distribution probability density diagram of the estimated variables β_1 , β_2 , and ρ . (e) When $\tau = 0.9$, the posterior distribution probability density diagram of the estimated variables β_1 , β_2 , and ρ .

It can be seen from Figure 2 that at different quantiles, the estimated variables β_1 , β_2 , and ρ are close to the real values, and the posterior distribution of each variable is basically the same, indicating that the model estimation results proposed in this section are more robust.

Additionally, 200 simulation experiments are carried out, and the results are summarized in Table 8.

Table 8. Summary table of simulation results under normal error.

Quantiles	Parameter	True Value	Estimated Value	MAE	MSE
0.1	β_1	1	1.0015	0.0904	0.0983
	β_2	3	3.0020	0.0481	0.1422
	ρ	0.6	0.6026	0.1104	0.0506
0.25	β_1	1	0.9987	0.0563	0.0576
	β_2	3	3.0006	0.0408	0.1258
	ρ	0.6	0.6004	0.0554	0.0349
0.5	β_1	1	0.9975	0.0483	0.0453
	β_2	3	2.9994	0.0346	0.0995
	ρ	0.6	0.6014	0.0416	0.0264
0.75	β_1	1	0.9988	0.0669	0.0441
	β_2	3	2.9770	0.0383	0.1167
	ρ	0.6	0.5970	0.0654	0.0370
0.9	β_1	1	1.0068	0.0822	0.0889
	β_2	3	3.0138	0.0469	0.1252
	ρ	0.6	0.5966	0.0884	0.0511

From the estimation results of Table 8, it can be seen that the deviation between the estimated value and the real value is small, indicating that the model estimation accuracy is high, and the Bayesian method is used to estimate the spatial lag panel fixed effect quantile model. The mean value of all parameters is close to the true value, and the MAE and MSE are small, which indicates that the SLPQR model has better performance when the data satisfies the classical hypothesis. The Bayesian framework effectively integrates the advantages of spatial lag effect and quantile regression.

(2) The random disturbance term ε_{it} of the model is randomly extracted from the chi-square distribution $\chi^2(1)$.

Assuming that the random disturbance term ε_{it} of the model is randomly selected from the chi-square distribution $\chi^2(1)$, and the other parameters are generated in the same way, 200 simulation experiments are carried out. The experimental results are summarized in Table 9.

It can be seen from Table 9 that when the error obeys the chi-square distribution $\chi^2(1)$, the MAE and MSE of the model parameter estimation increase compared with the normal distribution, and the mean value of the parameter estimation is slightly offset, but the MAE is controlled within a reasonable range, and the SLPQR model is still robust.

The MAE and MSE estimated by the model at 0.1 and 0.9 quantiles are increased compared with the normal distribution, but the deviation from the true value is not significant, indicating that the Bayesian estimation method of the spatial lag panel fixed effect quantile model is better for the estimation of extreme values in the case of a slight skewed distribution of the sample.

Table 9. Summary table of simulation results under chi-square error.

Quantiles	Parameter	True Value	Estimated Value	MAE	MSE
0.1	β_1	1	0.9877	0.0904	0.1058
	β_1	3	2.9928	0.0481	0.1505
	ρ	0.6	0.5917	0.1104	0.0713
0.25	β_1	1	1.0011	0.0563	0.0691
	β_2	3	3.0024	0.0408	0.1332
	ρ	0.6	0.5985	0.0554	0.0368
0.5	β_1	1	0.9998	0.0483	0.0490
	β_2	3	3.0017	0.0346	0.1064
	ρ	0.6	0.5973	0.0416	0.0315
0.75	β_1	1	1.0050	0.0669	0.0489
	β_2	3	2.9785	0.0383	0.1107
	ρ	0.6	0.6012	0.0654	0.0376
0.9	β_1	1	1.0107	0.0822	0.1007
	β_2	3	3.0034	0.0469	0.1285
	ρ	0.6	0.6000	0.0884	0.0508

(3) The random disturbance term ε_{it} of the model is randomly extracted from the t distribution $t(1)$.

Assuming that the random disturbance term ε_{it} of the model is randomly selected from the t distribution $t(1)$, and the other parameters are generated in the same way, 200 simulation experiments are carried out. The experimental results are summarized in Table 10.

Table 10. Summary of simulation results under t distribution error.

Quantiles	Parameter	True Value	Estimated Value	MAE	MSE
0.1	β_1	1	1.0421	0.1010	0.1250
	β_2	3	2.9927	0.0489	0.1823
	ρ	0.6	0.6049	0.1013	0.0763
0.25	β_1	1	0.9822	0.0675	0.0832
	β_2	3	2.9719	0.0425	0.1549
	ρ	0.6	0.6055	0.0664	0.0502
0.5	β_1	1	0.9959	0.0474	0.0585
	β_2	3	3.0357	0.0343	0.1310
	ρ	0.6	0.6003	0.0539	0.0397
0.75	β_1	1	1.0500	0.0750	0.0926
	β_2	3	3.0598	0.0401	0.1492
	ρ	0.6	0.5997	0.0692	0.0520
0.9	β_1	1	0.9768	0.0945	0.1161
	β_2	3	3.0711	0.0478	0.1758
	ρ	0.6	0.6033	0.1078	0.0796

From Table 10, it can be seen that when the error obeys the t distribution $t(1)$, the experimental data shows a clear skew tendency. The MAE and MSE of the model parameter estimation increase, and the mean value of the parameter estimation slightly shifts. However, the MAE and MSE are controlled within a reasonable range, and the SLPQR model is still robust, indicating that the Bayesian estimation method of the SLPQR model can better deal with the complex data distribution.

In the extreme case, that is, 0.1 and 0.9 quantiles, the deviation between the estimated value of the model and the real value is not significant, indicating that the SLPQR model has strong robustness. The MAE of parameter β_1 at the 0.9 quantile is 0.0945, and the MSE

is 0.1161. When the chi-square error is distributed, the MAE of parameter β_1 at the 0.9 quantile is 0.0822, and the MSE is 0.1007.

In the simulation experiment, when the error obeys the t distribution, the MAE and MSE of the parameter estimation of the SLPQR model at the 0.1 and 0.9 quantiles increase slightly, indicating that the Bayesian estimation method of the SLPQR model can better deal with the complex data distribution. The accurate parameter estimation results at extreme values can provide an effective decision-making reference for the prediction of extreme values of complex data in actual data applications.

(4) The random disturbance term ε_{it} of the model is randomly extracted from the Cauchy distribution $Cauchy(0, 0.5)$.

Assuming that the random disturbance term ε_{it} of the model is randomly selected from the Cauchy distribution $Cauchy(0, 0.5)$, and the other parameters are generated in the same way, 200 simulation experiments are carried out. The experimental results are summarized in Table 11.

Table 11. Cauchy error simulation experiment results summary table.

Quantiles	Parameter	True Value	Estimated Value	MAE	MSE
0.1	β_1	1	0.9317	0.1213	0.1521
	β_2	3	3.0427	0.0539	0.2012
	ρ	0.6	0.6475	0.1261	0.0927
0.25	β_1	1	1.0434	0.0864	0.1087
	β_2	3	2.9265	0.0562	0.2102
	ρ	0.6	0.5613	0.0842	0.0633
0.5	β_1	1	0.9513	0.0760	0.0954
	β_2	3	2.9590	0.0395	0.1497
	ρ	0.6	0.6218	0.0705	0.0547
0.75	β_1	1	0.9543	0.0622	0.0780
	β_2	3	2.9618	0.0558	0.2165
	ρ	0.6	0.5678	0.0781	0.0589
0.9	β_1	1	0.9164	0.1536	0.1917
	β_2	3	3.0355	0.0524	0.1894
	ρ	0.6	0.6255	0.0917	0.0699

It can be seen from Table 11 that when the error distribution deviates from the normal distribution and obeys the Cauchy distribution $Cauchy(0, 0.5)$, the MAE and MSE of the parameter estimation increase significantly, but in the most serious case of thick tail, the parameter estimation of the SLPQR model is still close to the real value.

At the 0.1 and 0.9 quantiles, the MAE of the three parameters of the model did not exceed 0.2, the MSE was mostly below 0.2, and the SLPQR model estimation was still robust. At the same time, at the 0.9 quantile, for the estimation of the spatial autocorrelation coefficient in the model, when the error obeys the Cauchy distribution, the MAE estimated by the spatial autocorrelation coefficient is 0.0917 and the MSE is 0.0699.

In the case that the error obeys the t distribution, the MAE estimated by the spatial autocorrelation coefficient is 0.1078 and the MSE is 0.0796. When the error obeys the Cauchy distribution, the MAE of the spatial autocorrelation coefficient estimation is slightly smaller than that of the t distribution, the result is more accurate than that of the t distribution, the MSE increases slightly, and the result is slightly lower than that of the t distribution. The accuracy of the distribution is slightly reduced, indicating that the model has a strong ability to estimate the spatial autocorrelation coefficient under the skewed distribution, and the estimation is less affected by the data distribution.

The model parameter estimation effect is visualized by a box plot, and the estimation of the model under different error distributions is visually displayed. The box plots of the estimated variables β_1 , β_2 , and ρ under different error distributions are drawn in Figure 3.

It can be seen from Figure 3 that as the skewness of the experimental simulation data increases, the MAE and MSE of the parameter estimation increase, and the accuracy of the parameter estimation gradually decreases, but the SLPQR model is still robust. Figure 3b,c show that the mean value of parameter estimation under chi-square distribution and t distribution is slightly shifted, but MAE is controlled within a reasonable range. Figure 3d shows that the parameter estimates of the SLPQR model are still close to the true values under the Cauchy distribution, that is, in the most severe heavy-tailed case.

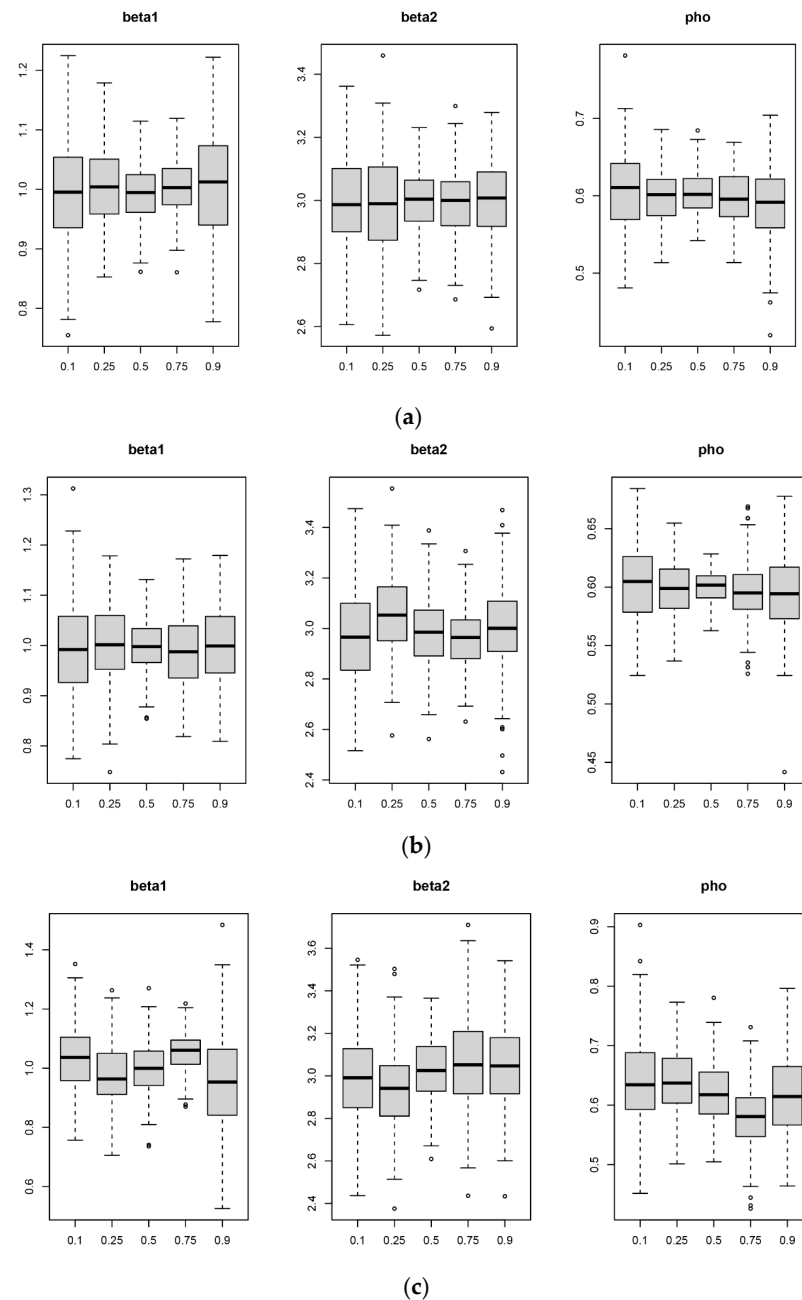


Figure 3. Cont.

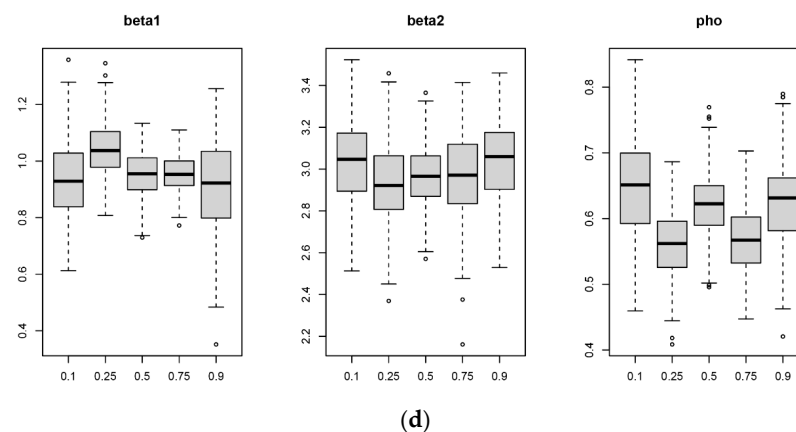


Figure 3. Box plot of parameter estimation results. (a) When $N(0,1)$, the box plot of the estimated results for the estimated variables β_1 , β_2 , and ρ . (b) When $\chi^2(1)$, the box plot of the estimated results for the estimated variables β_1 , β_2 , and ρ . (c) When $t(1)$, the box plot of the estimated results for the estimated variables β_1 , β_2 , and ρ . (d) When $Cauchy(0,0.5)$, the box plot of the estimated results for the estimated variables β_1 , β_2 , and ρ .

It can be seen from the experimental results of simulation 1 that the estimated values of each parameter are close to the real values, and the model estimation effect is good. The model has a small estimation error at 0.1 and 0.9 quantiles, indicating that the spatial lag panel fixed effect Bayesian quantile model has a better estimation of extreme values.

3.2. Simulation 2: Model Estimation Results Under Different Sample Sizes

In the simulation process, the sample points with sample sizes of $(N, T) = (10, 5)$, $(N, T) = (10, 10)$, and $(N, T) = (10, 40)$ are first generated by the model. The random error ε_{it} obeys the normal distribution $N(0, 1)$, and the other conditions remain unchanged.

In this section, the estimation results under different sample sizes at three quantiles $\tau = (0.25, 0.5, 0.75)$ are shown in Tables 12–14.

From Tables 12–14, it can be seen that the SLPQR model can still maintain high accuracy when the sample size is small to $(N, T) = (10, 5)$. The MAE of β_1 at the 0.25 quantile is 0.0703 and the MSE is 0.0868. Compared with the large sample scenario $(N, T) = (10, 40)$, the MAE of β_1 is reduced to 0.0373 and the MSE is reduced to 0.0244. There is no systematic bias in parameter estimation under small samples. This shows that the model is adaptive to small sample data and is suitable for limited observation scenarios in actual regional economic monitoring and social survey.

Table 12. Summary of experimental results when $(N, T) = (10, 5)$.

Quantiles	Parameter	True Value	Estimated Value	MAE	MSE
0.25	β_1	1	0.9446	0.0703	0.0868
	β_2	3	2.9997	0.0447	0.1744
	ρ	0.6	0.5518	0.0828	0.0582
0.5	β_1	1	1.0257	0.0477	0.0592
	β_2	3	3.0430	0.0521	0.1969
	ρ	0.6	0.6026	0.0545	0.0411
0.75	β_1	1	0.9789	0.0385	0.0505
	β_2	3	2.9739	0.0394	0.1491
	ρ	0.6	0.6342	0.0697	0.0516

Table 13. Summary of experimental results when $(N, T) = (10, 10)$.

Quantiles	Parameter	True Value	Estimated Value	MAE	MSE
0.25	β_1	1	0.9987	0.0563	0.0576
	β_1	3	3.0006	0.0408	0.1258
	ρ	0.6	0.6004	0.0554	0.0349
0.5	β_1	1	0.9975	0.0483	0.0453
	β_2	3	2.9994	0.0346	0.0995
	ρ	0.6	0.6014	0.0416	0.0264
0.75	β_1	1	0.9988	0.0669	0.0441
	β_2	3	2.9770	0.0383	0.1167
	ρ	0.6	0.5970	0.0654	0.0370

Table 14. Summary of experimental results when $(N, T) = (10, 40)$.

Quantiles	Parameter	True Value	Estimated Value	MAE	MSE
0.25	β_1	1	0.9744	0.0373	0.0468
	β_2	3	2.9936	0.0101	0.0944
	ρ	0.6	0.5780	0.0438	0.0282
0.5	β_1	1	1.0003	0.0259	0.0355
	β_2	3	2.9816	0.0121	0.0742
	ρ	0.6	0.5995	0.0250	0.0213
0.75	β_1	1	0.9910	0.0299	0.0244
	β_2	3	3.0241	0.0959	0.0879
	ρ	0.6	0.6107	0.0343	0.0261

From Tables 12–14, it can be seen that with the increase in sample size, the estimation accuracy of the SLPQR model is further improved. As the sample size increases, the MAE and MSE of the SLPQR model decrease significantly, which verifies the asymptotic consistency of the model under large samples. In the small sample scenario, the MSE of parameter ρ estimation at the 0.75 quantile is 0.0261. In the large sample scenario $(N, T) = (10, 5)$, the MSE of parameter ρ estimation at the 0.75 quantile is 0.0516. Compared with the large sample scenario, the MSE of the small sample scenario is reduced by 49.4%.

The experimental results of simulation 2 demonstrate the accuracy of the Bayesian estimation method of the spatial lag panel fixed effect quantile model under the condition of small samples.

3.3. Simulation 3: Comparative Analysis of Model Estimation Results

(1) Comparison between SLPQR model and mean model

Fixed quantile $\tau = 0.5$. The random disturbance term ε_{it} is extracted from the standard normal distribution $N(0, 1)$, chi-square distribution $\chi^2(1)$, t distribution $t(1)$, and Cauchy distribution $Cauchy(0, 0.5)$, and the remaining parameter conditions are unchanged. The random data is generated as a sample of the simulation experiment to estimate the model parameters. The spatial lag panel Bayesian quantile model (SLPQR), Ordinary Least Square (OLS), Panel Spatial Lagged Model (PSLR), and Least Absolute Deviation (LAD) proposed in this paper are used to estimate the parameters. The experiment is repeated 200 times. The simulation results are shown in Table 15.

It can be seen from Table 15 that the MAE and MSE of the SLPQR model are better than the traditional mean model (OLS model and PSLR model) when the random error obeys the normal distribution. Compared with the traditional mean model, the MAE and MSE of each parameter to be estimated in the SLPQR model are small, indicating that under the standard normal distribution, the estimation accuracy of the least squares estimation and the spatial lag panel model is not as good as that of the SLPQR model.

Table 15. Parameter estimation results of classical mean model under different error distributions.

	Model	Parameter	True Value	Estimated Value	MAE	MSE	
Normal Distribution $N(0,1)$	SLPQR	β_1	1	0.9975	0.0483	0.0453	
		β_2	3	2.9994	0.0346	0.0995	
		ρ	0.6	0.6014	0.0416	0.0264	
	OLS	β_1	1	0.9689	0.3193	0.3865	
		β_2	3	2.9714	0.1059	0.3895	
	LAD	β_1	1	0.9932	0.0952	0.0092	
		β_2	3	3.0346	0.1028	0.0110	
	PSLR	β_1	1	0.9932	0.0737	0.0960	
		β_2	3	3.0346	0.0824	0.1047	
		ρ	0.6	0.6040	0.0439	0.0547	
	SLPQR	β_1	1	0.9998	0.0483	0.0490	
		β_2	3	3.0017	0.0346	0.1064	
ρ		0.6	0.5973	0.0416	0.0315		
Chi-square Distribution $\chi^2(1)$	OLS	β_1	1	0.9552	0.2813	0.3505	
		β_2	3	2.9554	0.1194	0.4391	
		β_1	1	0.9932	0.0964	0.0800	
	LAD	β_2	3	3.0948	0.1052	0.1209	
		β_1	1	0.9932	0.1056	0.1575	
	PSLR	β_2	3	3.0948	0.1057	0.1988	
		ρ	0.6	0.6257	0.0989	0.0996	
		SLPQR	β_1	1	1.0035	0.0483	0.0571
	β_2		3	2.9896	0.0346	0.1166	
	ρ		0.6	0.6062	0.0416	0.0417	
	t distribution $t(1)$	OLS	β_1	1	-0.0852	2.9010	6.1793
			β_2	3	2.8793	0.8959	5.9430
β_1			1	1.1956	0.2041	0.0416	
LAD		β_2	3	2.7949	0.1253	0.0156	
		β_1	1	1.1956	0.2041	0.2452	
PSLR		β_2	3	2.7949	0.3924	0.3680	
		ρ	0.6	0.7284	0.2975	0.2525	
		SLPQR	β_1	1	0.9959	0.0474	0.0585
β_2			3	3.0357	0.0343	0.1310	
ρ			0.6	0.6003	0.0539	0.0397	
Cauchy distribution $Cauchy(0,0.5)$		OLS	β_1	1	0.7794	1.2493	2.8804
			β_2	3	2.5764	0.5429	4.4090
	LAD	β_1	1	0.6943	0.2047	0.0501	
		β_2	3	2.6475	0.3924	0.2057	
	PSLR	β_1	1	0.6943	0.3789	0.5857	
		β_2	3	2.6475	0.6954	1.2657	
		ρ	0.6	0.4245	0.3875	0.8576	

As the error gradually shows a skewed distribution, in the case of the error obeying the Cauchy distribution, the parameter estimation results of the least squares estimation and the spatial lag panel model are too different from the real value. In the thick-tailed scenario, namely the Cauchy distribution, the estimated MAE of β_2 is 0.1310, and the estimation only deviates from the true value by 3.43%, while the OLS model deviation is as high as 54.29%. The SLPQR model effectively suppresses the distortion of the extreme value to the estimation through the adaptive weight distribution under the Bayesian framework. The estimated value of ρ is close to the true value of 0.6, and its MSE is 0.0539, while the deviation of the PSLR model is 38.75%.

Under the normal error, SLPQR ($\tau = 0.5$), OLS, and LAD all show high accuracy, and the MAE difference between the three is less than 5%, which indicates that under

ideal conditions, each method is equally effective in central position estimation. Under the heavy-tailed t distribution, OLS fails seriously, while LAD maintains robustness. The robustness of SLPQR ($\tau = 0.5$) is comparable to or even slightly better than LAD, which indicates that SLPQR inherits the distribution robustness of quantile regression. However, LAD can only provide median estimation, while SLPQR can provide estimation of multiple quantiles at the same time and still maintain good accuracy in extreme quantiles, which proves that the core advantage of SLPQR is to extend the robustness to the analysis of the entire conditional distribution.

The SLPQR model effectively alleviates the impact of heavy-tailed distribution on parameter estimation through the probabilistic calibration mechanism of the Bayesian framework. The spatial lag panel Bayesian quantile model proposed in this paper has the best estimation results under the skewed distribution of parameters.

(2) Comparison between SLPQR model and classical quantile model

Fixed quantile $\tau = (0.25, 0.5, 0.75)$. The random data is generated as the sample of the simulation experiment and then the model parameters are estimated. The spatial lag panel Bayesian quantile model (SLPQR), quantile regression (QR), and panel spatial lagged quantile regression model (SLMQR) proposed in this paper are used to estimate.

The experiment was repeated 200 times. The simulation results of the parameter estimation of the SLPQR model and the classical quantile model are shown in Table 16.

Table 16. Comparison of parameter estimation results between SLPQR and classical quantile model.

Model	Quantiles	Parameter	True Value	Estimated Value	MAE	MSE
SLPQR	0.25	β_1	1	0.9812	0.1747	0.2188
		β_2	3	3.0252	0.0534	0.2004
		ρ	0.6	0.6004	0.0554	0.0349
	0.5	β_1	1	0.9849	0.1638	0.2013
		β_2	3	3.0034	0.0501	0.1834
		ρ	0.6	0.6014	0.0416	0.0264
	0.75	β_1	1	0.9746	0.1675	0.2150
		β_2	3	2.9907	0.0544	0.2032
		ρ	0.6	0.5970	0.0654	0.0370
	0.25	β_1	1	0.8942	0.4247	0.5127
		β_2	3	2.8958	0.1308	0.4923
		ρ	0.6	0.6025	0.1057	0.0549
QR	0.25	β_1	1	0.9300	0.3921	0.4840
		β_2	3	2.8900	0.1259	0.4595
		ρ	0.6	0.6027	0.0324	0.0524
	0.5	β_1	1	0.9003	0.4611	0.5645
		β_2	3	2.8692	0.1326	0.5007
		ρ	0.6	0.5842	0.0544	0.0524
	0.75	β_1	1	0.9492	0.0963	0.0576
		β_2	3	2.8996	0.1208	0.1258
		ρ	0.6	0.6025	0.1057	0.0549
	0.25	β_1	1	0.9686	0.0980	0.0842
		β_2	3	2.9294	0.1224	0.1075
		ρ	0.6	0.6027	0.0324	0.0524
SLMQR	0.25	β_1	1	0.9725	0.0675	0.0675
		β_2	3	2.9552	0.1038	0.0975
		ρ	0.6	0.5842	0.0544	0.0524
	0.5	β_1	1	0.9725	0.0675	0.0675
		β_2	3	2.9552	0.1038	0.0975
		ρ	0.6	0.5842	0.0544	0.0524
	0.75	β_1	1	0.9725	0.0675	0.0675
		β_2	3	2.9552	0.1038	0.0975
		ρ	0.6	0.5842	0.0544	0.0524
	0.25	β_1	1	0.9725	0.0675	0.0675
		β_2	3	2.9552	0.1038	0.0975
		ρ	0.6	0.5842	0.0544	0.0524

It can be seen from Table 16 that under different quantiles, compared with the quantile regression model and the spatial lag panel quantile model, the spatial lag fixed effect Bayesian quantile model proposed in this section can obtain more robust results under the same conditions, and the MAE and MSE of the SLPQR model are smaller.

Simulation 3 shows that under the same conditions, the spatial lag fixed effect Bayesian quantile model (SLPQR model) can obtain more robust results, and its estimation effect

is better than the quantile regression model, least squares estimation model, spatial lag panel model, and spatial lag panel quantile model. The Bayesian estimation method of the spatial lag panel fixed effect quantile model can better deal with the estimation problem of skewed data than the traditional model.

4. Model Application

The Bayesian estimation method of the spatial lag fixed effect quantile model proposed above aims to solve the problem of spatial effect and variable parameter estimation when the data distribution is complex in practical problems. From the existing literature, the problem of carbon emissions is mostly studied by means of mean regression. The huge gap in carbon emissions between provinces in China makes the carbon emission data easily deviate from the normal distribution. The previous mean regression method is not well applied to this non-normal carbon emission data. Based on this, this section applies the Bayesian estimation of the spatial lag fixed effect quantile model to the analysis of the influencing factors of carbon emissions.

4.1. The Calculation Method of Carbon Emissions and the Selection of Impact Indicators

4.1.1. Carbon Emissions Measurement and Data Spatial Correlation

This paper selects the carbon emission panel data of 31 provinces and cities in China from 2010 to 2020 for research. Since there is no carbon dioxide emission data in the statistical yearbook, the carbon emissions of China's provinces are calculated according to the carbon emission measurement method of energy consumption provided by the "2006 IPCC Guidelines for National Greenhouse Gas Inventories":

$$CO_2 = \sum_{i=1}^n EC_i \times NCV_i \times CEF_i \times COF_i \times \frac{44}{12} \quad (19)$$

In Equation (19), EC_i is the i th energy consumption, NCV_i is the average low calorific value of the i th energy consumption, CEF_i is the carbon emission coefficient of the average low calorific value of the i th energy consumption, COF_i is the carbon oxidation factor of the i th energy consumption, and 44 and 12 are the molar amounts of carbon dioxide molecules and carbon elements, respectively.

According to Equation (19), the total carbon emissions and the average carbon emissions of each province from 2010 to 2020 are calculated, and the heat map of the average carbon emissions of each province is drawn as shown in Figure 4.

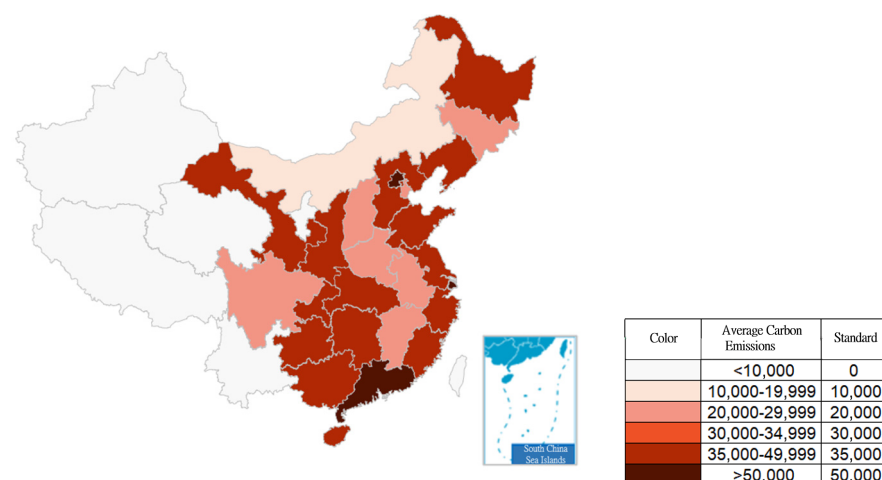


Figure 4. The average heat map of total carbon emissions in each province from 2010 to 2020.

It can be seen from Figure 4 that the provincial carbon emissions in the western region of China are relatively low. The carbon emissions in Xinjiang, Tibet, Qinghai, Ningxia, and Yunnan are less than 10,000. The total carbon emissions in the Inner Mongolia Autonomous Region are low, 11,490.09. Beijing, Shanghai, and Guangdong are regions with higher carbon emissions, all reaching more than 50,000. At the same time, the carbon emissions of other provinces show a high level, and the total carbon emissions of China's provinces have obvious spatial agglomeration and spatial correlation.

The index of carbon emissions data from 2010 to 2020 was calculated, and the results are shown in Table 17.

Table 17. Global *Moran's I* index table.

Period	<i>Moran's I</i> -Value	z-Value	p-Value
2010	0.331	3.007	0.001
2011	0.285	2.695	0.004
2012	0.224	2.108	0.015
2013	0.328	3.056	0.001
2014	0.310	2.897	0.002
2015	0.273	2.592	0.005

Table 17 shows that the *p*-values corresponding to the *Moran's I* index values in all years are less than 0.05, indicating that the total carbon emissions of 31 provinces in China have obvious positive autocorrelation in space. The total carbon emissions do not show a completely random state in spatial distribution, but show that the similar values of some provinces tend to agglomerate in space. Therefore, the influence of spatial correlation on total carbon emissions should be considered in the model of analyzing the influencing factors of carbon emissions.

4.1.2. Descriptive Statistical Analysis of Influencing Factor Index Data

This paper intends to select the economic scale index [32–34], urban environmental governance efficiency index [35,36], ecological land carbon sink density index [37,38], industrial pollutant emission index [38,39], and urban consumption structure index [40–42] as the explanatory variables of the carbon emission influencing factor model. The explanatory variable index data are derived from the 2011–2021 “China Statistical Yearbook” and the 2011–2021 provincial statistical yearbooks. The descriptive statistical indicators of each explanatory variable are summarized as Table 18.

Table 18. Descriptive statistics on national variable indicators.

Variables	C	ESI	EGEI	ECSD	IPEI	UCSI
Minimum value	6410.8	95.67	4.2	500	5.58	18.63
Maximum value	62,600	16,191.06	66.8	136,593	560.02	50.05
Average number	31,245.7	3702.52	33.67	19,797.84	155.23	31.68
Standard deviation	13,478.84	3124.85	18.23	18,366.93	114.3	5.44
Coefficient of variation	0.43	0.84	0.54	0.93	0.74	0.17
Skewness	−0.218	1.648	0.045	2.682	0.975	0.527

Table 18 shows that there are significant differences within the indicators. The coefficient of variation for each indicator is calculated by the quotient of the mean and standard

deviation, and the coefficient of variation is very different, and the degree of variation between the indicators is different. By calculating the skewness of each index, it can be seen that except for the urban environmental governance efficiency index, each index basically deviates from the normal distribution. At this time, the mean regression can no longer adapt to this skewed distribution, and there may be a large error between the estimated results and the true values. Quantile regression should be used.

After calculating the average and median of each parameter variable by province, there are obvious differences in the value range and dispersion degree of each parameter variable in different provinces, indicating that provinces may have a certain impact on carbon emissions. Therefore, the research on the influencing factors of carbon emissions needs to consider the differences between provinces, so the panel data model is adopted.

In order to weaken the influence of the index magnitude and the heteroscedasticity in the data on the estimation results, all the indexes are logarithmically processed in the data preprocessing stage. The least square method and quantile regression method are used to estimate the parameters of the carbon emission prediction model. The results are shown in Table 19.

Table 19. The estimation results of the least squares method and quantile regression method.

Variables	The Least Squares Method			Quantile Regression	
	Regression Coefficient	Standard Error	<i>p</i> -Value	Regression Coefficient	Confidence Interval
C	5.753	0.587	0.000	5.889	[4.971, 6.897]
ESI	0.018	0.048	0.715	0.043	[−0.158, 0.105]
EGEI	−0.228	0.034	0.000	−0.298	[−0.391, −0.240]
ECSD	−0.425	0.051	0.000	−0.362	[−0.539, −0.292]
IPEI	−0.176	0.031	0.000	−0.132	[−0.221, −0.076]
UCSI	0.103	0.157	0.511	0.067	[−0.152, 0.241]

Table 19 shows that the estimation results of the least squares method and the quantile regression model are not the same. The *p*-value of the economic scale index in the least squares method is 0.715, which is significantly greater than 0.05. The estimation results of the urban environmental governance index and the ecological land carbon sink density index are quite different.

Based on this, this section intends to use the Bayesian estimation method of the spatial lag fixed effect quantile model to estimate and analyze the influencing factors of carbon emission data.

4.2. Spatial Lag Panel Fixed Effect Quantile Regression Model

In order to overcome the shortcomings of the traditional mean model and the traditional quantile model, it is difficult to accurately measure the partial characteristics of the above variables. This section considers the Bayesian estimation method of the spatial lag panel fixed effect quantile regression model, and establishes the following model for empirical analysis.

$$C_{it}(\tau|\alpha_i, x_{it}) = \rho \sum_{j=1}^N w_{ij} C_{jt} + \beta_{1\tau} ESI_{it} + \beta_{2\tau} EGEI_{it} + \beta_{3\tau} ECSD_{it} + \beta_{4\tau} IPCI_{it} + \beta_{5\tau} CESI_{it} + \alpha_i \quad (20)$$

Define the weight matrix w_{ij} as the reciprocal of the spatial distance:

$$w_{ij} = \begin{cases} \frac{1}{d_{ij}}, i \neq j \\ 0, i = j \end{cases} \quad (21)$$

The parameters to be estimated and the spatial correlation coefficient in model (20) are estimated. The MCMC chain length is designed to be 5000, and the first 1000 unstable points in the MCMC sampling process are discarded in the estimation process. At the same time, three quantiles $\tau = (0.25, 0.5, 0.75)$ are estimated, and the parameter estimation results of model (20) are shown in Table 20.

Table 20. Parameter estimation results of spatial lag panel fixed effect quantile model.

Quantiles	Variables	95% Confidence Interval	Estimated Value	Standard Deviation
0.25	β_1	[−0.0532, 0.2117]	0.0883	0.0421
	β_2	[−0.7393, 0.0701]	−0.3042	0.1457
	β_3	[−0.4811, −0.1876]	−0.3199	0.0518
	β_4	[−0.1263, 0.083]	−0.0063	0.0434
	β_5	[−0.1078, 0.2832]	0.1102	0.0665
	ρ	[0.4189, 0.5152]	0.4669	0.0200
0.5	β_1	[0.0096, 0.1855]	0.0985	0.0310
	β_2	[−0.6325, −0.0147]	−0.3282	0.1233
	β_3	[−0.396, −0.2021]	−0.2989	0.0391
	β_4	[−0.1223, 0.1506]	0.0131	0.0553
	β_5	[−0.0506, 0.1406]	0.0502	0.0392
	ρ	[0.3834, 0.4614]	0.4196	0.0141
0.75	β_1	[−0.0453, 0.1639]	0.0741	0.0358
	β_2	[−0.5147, 0.0762]	−0.2288	0.1079
	β_3	[−0.1369, 0.1095]	−0.0165	0.0461
	β_4	[−0.1283, 0.1311]	0.0092	0.0479
	β_5	[−0.1876, 0.1755]	0.0009	0.0717
	ρ	[0.5285, 0.6285]	0.5748	0.0191

The study found that there is a significant positive spatial spillover effect on carbon emissions, and the spatial correlation is stronger at the high quantile (0.75), indicating that high-emission areas are more affected by neighboring provinces. The economic scale has always promoted carbon emissions, especially at the medium emission level (0.5 quantile). Urban environmental governance and ecological land carbon sinks continue to inhibit carbon emissions, but the inhibitory effect is stronger in low-emission areas. Industrial pollutant emissions have a promoting effect on medium- and high-emission areas, showing an “inverted U” type effect; the impact of urban consumption structure is relatively weak and decreases with the increase in quantile. On the whole, the influence intensity of each factor on carbon emissions varies significantly with different quantiles.

Therefore, in order to promote provincial carbon emission reduction, it is necessary not only to build a multi-level policy system, but also to strengthen total control and dynamic quotas for high-emission areas. It is important to optimize environmental protection investment and improve governance efficiency in medium-developed areas. It is also necessary to promote the realization of the value of ecological products in carbon sink enrichment areas and develop the under-forest economy. At the same time, we will improve the closed-loop of intelligent early warning and control of industrial pollution, guide the green transformation of consumption structure in urbanized areas, and encourage the public to live a low-carbon life.

5. Discussion and Conclusions

This paper constructs a Bayesian estimation framework for the spatial lag panel fixed effect quantile regression model by assuming that the random error term obeys the asymmetric Laplace distribution. This method incorporates the spatial dependence structure as a spatial lag term into the model, and treats the individual heterogeneous features as random effects, so as to realize the refinement of the data heterogeneous structure based on the traditional Bayesian hierarchical modeling. Although the asymmetric Laplace distribution lacks conjugate priors, this paper effectively overcomes the technical difficulties in posterior sampling by combining Gibbs sampling and the Metropolis–Hastings algorithm.

Numerical simulations show that the model has good robustness under various non-standard error structures such as skewed, heavy-tailed, chi-square distribution, *t* distribution, and Cauchy distribution, and the estimated parameters are close to the true value. The model is insensitive to outliers at extreme quantiles and has excellent robustness. At the same time, it has shown high accuracy in the small sample situation, and shows asymptotic consistency with the expansion of the sample size, adapting to a variety of empirical research scenarios.

Compared with the traditional mean model, such as the least squares estimation model [43] and the spatial lag panel model [26], the core advantage of SLPQR is its distribution robustness. When the error obeys the normal distribution, the estimation accuracy of SLPQR is equivalent to that of the traditional method. When the data is skewed or thick-tailed, the estimation accuracy of the traditional mean model is significantly deteriorated and is extremely sensitive to outliers. SLPQR relies on the Bayesian framework to effectively suppress the interference of extreme values, so as to maintain stable and reliable estimation results under complex error distribution. Compared with classical quantile models, such as the quantile regression model [8] and spatial lag panel quantile model [14], SLPQR further improves the accuracy and robustness of estimation. Although all kinds of quantile models aim at characterizing quantiles with different conditional distributions, SLPQR can always obtain smaller estimation errors at each quantile by embedding the Bayesian MCMC algorithm, indicating that it is more sufficient and efficient in using sample information. The value of the SLPQR model proposed in this study lies not only in improving the robustness and efficiency of estimation through Bayesian framework and spatial panel setting, but also in its quantile regression essence, which enables researchers to go beyond the traditional central trend analysis and reveal the heterogeneity pattern of the spatial dependence effect in different positions of conditional distribution. In the traditional robust mean models, such as LAD, this cannot be achieved.

Aiming at the common censoring mechanism and variable missing problems in economic data, we can further develop a more robust estimation system based on the existing Bayesian quantile regression theory, expand the application boundary of the parametric Bayesian quantile regression method, and further explore the effective balance between estimation efficiency and model interpretation ability in high-dimensional data scenarios.

Through quantile analysis, this study finds that there are significant regional differences in the influencing factors of carbon emissions: the spatial spillover effect is positive and the strongest in high-emission areas; economic scale has always promoted carbon emissions, which is most obvious in medium-emission areas. Both environmental governance and ecological carbon sinks can inhibit carbon emissions, but the inhibition effect of the latter decreases with the increase in emission levels. Industrial pollution has a promoting effect on medium- and high-emission areas, showing an “inverted U” type; the urban consumption structure has less impact and plays a stronger role in low-carbon areas. In the future, a more comprehensive index system should be constructed, including energy

intensity, technological progress, and other factors, so as to enhance the reliability of the conclusions and provide accurate data support for carbon emission reduction policies.

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