

Article

Research on Nonlinear Dynamic Characteristics of Double-Deck Vibrating Flip-Flow Screen Based on Multi-Scale Method

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Abstract

The double-deck vibrating flip-flow screen (DDVFFS) resolves the limitation of the traditional single-deck vibrating flip-flow screen, which cannot perform multi-granularity screening. The nonlinear characteristics of the springs in the DDVFFS lead to issues of unstable vibration and significant deviations between the operational and design amplitudes. Consequently, a dynamic equation incorporating cubic nonlinear stiffness is formulated. These equations are initially expressed in physical coordinates and subsequently transformed into modal coordinates for analyzing the system's nonlinear vibration characteristics under resonance using the multi-scale method. The study then explores the impact of system parameters on this nonlinear behavior. Increasing damping or reducing harmonic forces proves effective in mitigating multiple solutions and vibration jumps. The operational frequency is chosen within the distant single-valued resonance region to minimize the influence of nonlinear stiffness, ensuring stable operation of the DDVFFS. The calculated amplitudes are as follows: the main frame shows an amplitude of 7 mm, whereas the upper and lower floating frames display amplitudes of 9.4 mm and 5.6 mm, respectively. A 180° phase difference between the main frame and the upper or lower frames facilitates deck stretching and slackening.

Keywords: double-deck vibrating flip-flow screen; multi-scale method; internal resonance; nonlinear vibrations

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1. Introduction

The vibrating flip-flow screen is an innovative screening equipment that employs mechanical [1] vibration for material screening and classification. In contrast to conventional vibrating screens [2–5], the flip-flow screen incorporates not only the fundamental vibration of the main frame but also the additional vibration of the floating frame. Conventional vibrating screens often exhibit poor performance when screening wet fine coal materials. In contrast, the vibrating flip-flow screen excels in effectively sieving wet fine coal materials [6–8]. During operation, the floating and main frames move periodically in

opposite directions. This motion causes the elastic deck, mounted on both frames, to alternately stretch and slacken, ensuring that materials continuously bounce on the screen surface. This dynamic process effectively sorts loose materials [9–11]. The vibrating flip-flow screen is extensively utilized in coal preparation plants due to its high screening efficiency and resistance to clogging, attracting significant attention from researchers [12,13].

To design and optimize the structure of the vibrating flip-flow screen, it is essential to develop a dynamic model that describes its operational mechanism and internal interactions. Initially, the mechanical structure is represented as a physical model, where the relationships between forces and constraints among components are analyzed to establish a mechanical model. Subsequently, this mechanical model is translated into a mathematical framework to predict the dynamic response of the vibration system through mathematical solution techniques [14–17]. Cao and Zhao [9,18] developed linear mechanical models for single-deck and double-deck vibrating flip-flow screens. Yu et al. [19] highlighted the critical influence of screen stability on operational performance, proposing an optimal operational frequency range for stable operation. These studies have advanced the theoretical understanding of vibrating flip-flow screens and provided practical production guidance. However, most research has employed linear mathematical models, despite the use of rubber isolation and shear springs with nonlinear mechanical properties in these screens [9,11,17,20]. When neglecting the nonlinear characteristics of the spring, the vibration system becomes unstable at specific frequencies. Additionally, discrepancies exist between the theoretical solution's dynamic response of the system and its actual dynamic response during operation [21–25].

The nonlinear variation in spring stiffness manifests as either soft or hard spring characteristics. This nonlinear dynamic behavior is described by the Duffing equation, which is applied to nonlinear dynamic equation for the vibrating flip-flow screen [26,27]. While numerical methods are typically used to solve equations containing nonlinear terms, they often lack intuitive insights into the model's physical behavior and characteristics. Currently, the multi-scale method, widely employed in rotating shaft systems, automobile suspension systems, and bridge cable-stayed systems, simplifies the solution process for nonlinear systems. It forms a theoretical foundation for approximating analytical solutions to equations describing nonlinear systems [28–31].

Although the coupled Duffing oscillator generally exhibits complex dynamic behaviors including intermittent chaos and ultra-long transient transition in numerical simulations [32,33], the practical engineering operation of DDVFFS mainly focuses on the stable primary resonance region. Despite its effectiveness in predicting vibration jump, the proposed analytical method has certain limitations in characterizing chaotic effects induced by strong nonlinearity.

The double-deck vibrating flip-flow screen (DDVFFS) is a new type of screening equipment. The nonlinear dynamic characteristics caused by the shear spring directly affect the dynamic behavior of the vibrating flip-flow screen [34,35]. Therefore, it is of great significance to study the nonlinear characteristics of the DDVFFS for accurately predicting the vibration response.

The structure of this paper is organized as follows: Section 2 establishes the nonlinear mechanical and mathematical models of the DDVFFS, and conducts perturbation analysis using the multi-scale method. In Section 3, the influence of system parameters on the nonlinear characteristics is investigated through the control variable method, based on the perturbed solution of the amplitude-frequency tuning parameter response equation. Additionally, the vibration response of the DDVFFS at its normal operating frequency is computed. Section 4 presents the conclusions drawn from the study.

2. Perturbation Analysis of DDVFFS

2.1. Mathematical Model Description

In the DDVFFS structure depicted in Figure 1, the main frame is supported by support springs, while shear springs are installed between the upper and lower floating frames and the main frame. When the vibration exciter installed on the main frame operates, the upper and lower floating frames move relative to the main frame. The masses of these components form a three-rigid-body system [18,34,36]. The mechanical model of the DDVFFS is established using the concentrated mass method, illustrated in Figure 2. In the Cartesian coordinate system XOY , the X -direction parallels the deck of the DDVFFS, and the Y -direction is perpendicular to it. As vibrations in the Y -direction do not affect the reverse motion of the main frame and floating frames, dynamic responses in the Y -direction are disregarded. Furthermore, due to the system's small damping coefficient, solutions considering only linear damping are precise [9]. Given the symmetry of rubber springs and by neglecting the influence of normal preload on tangential stiffness, the restoring force of the spring can be described by a cubic function of elastic deformation [27]. The vibration equation in matrix form is established using the stiffness method, as depicted in Equation (1).

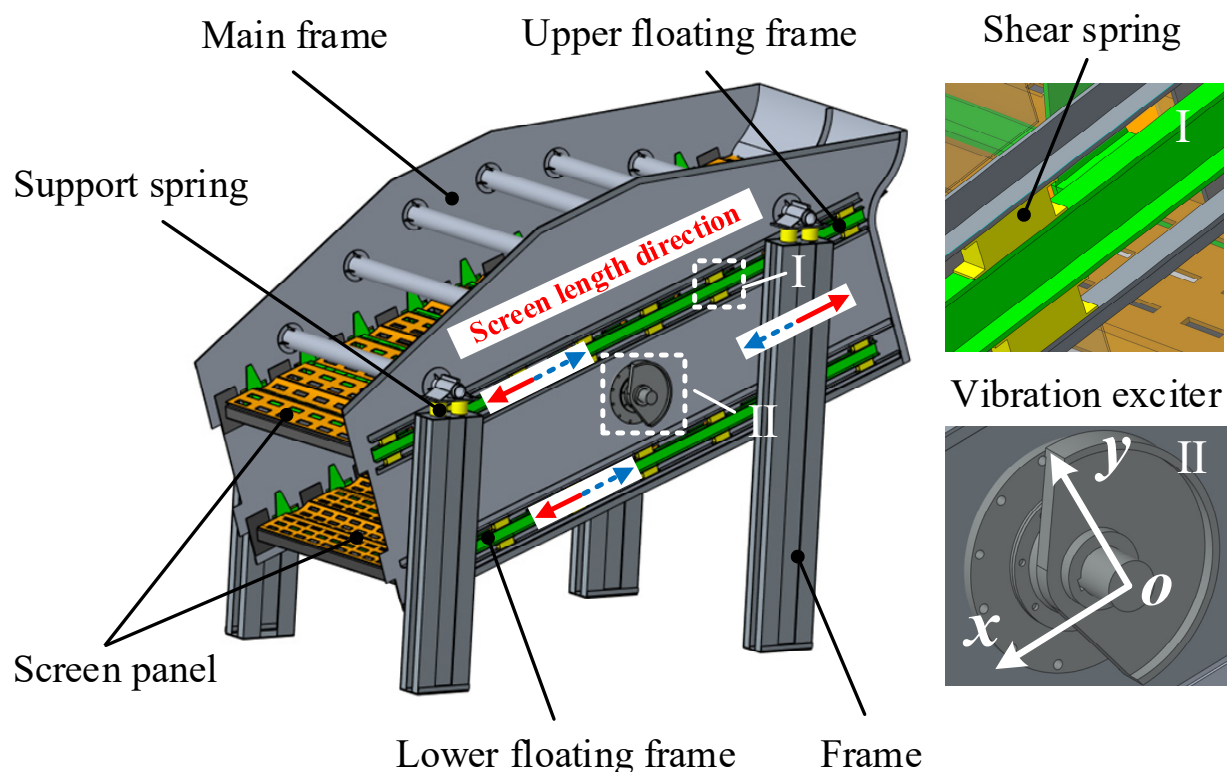


Figure 1. The structure of DDVFFS.

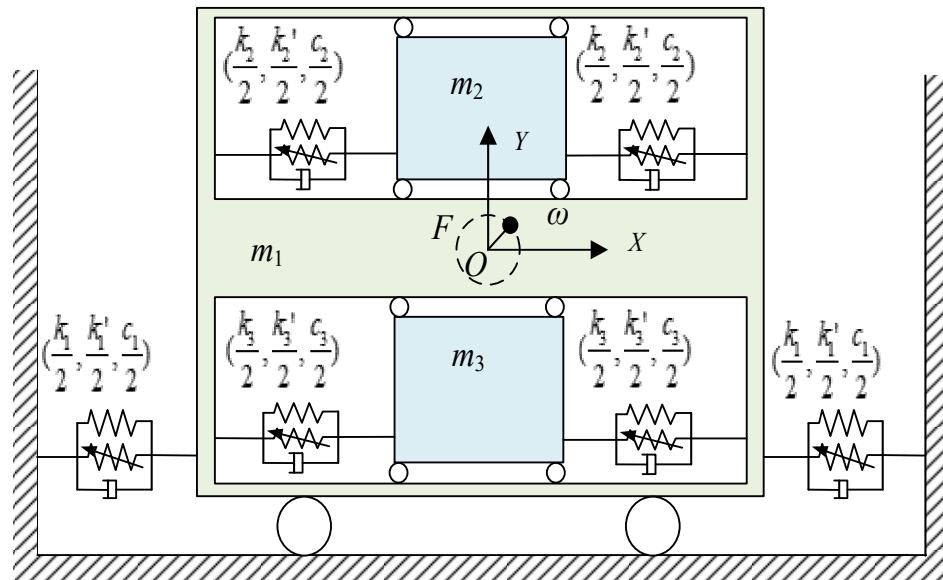


Figure 2. The mechanical model of DDVFFS.

$$M\ddot{X} + C\dot{X} + KX + K' = F, \quad (1)$$

where

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 + c_2 + c_3 & -c_2 & -c_3 \\ -c_2 & c_2 & 0 \\ -c_3 & 0 & c_3 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 + k_3 & -k_2 & -k_3 \\ -k_2 & k_2 & 0 \\ -k_3 & 0 & k_3 \end{bmatrix},$$

$$K' = \begin{bmatrix} k_3'(x_1 - x_3)^3 + k_2'(x_1 - x_2)^3 + k_1'x_1^3 \\ -k_2'(x_1 - x_2)^3 \\ -k_3'(x_1 - x_3)^3 \end{bmatrix}, \quad F = \begin{bmatrix} F \cos \omega t \\ 0 \\ 0 \end{bmatrix}, \quad \ddot{X} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}, \quad \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix},$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where m_1, m_2, m_3 represent the masses of the main frame, upper floating frame, and lower floating frame, respectively. The parameters c_i ($i = 1, 2, 3$) denote the linear damping coefficients, while k_i stands for the linear stiffness coefficients. k_i' represents the nonlinear stiffness coefficient. The variables $x_1, \dot{x}_1$ and $\ddot{x}_1$ denote the displacement, velocity, and acceleration of the main frame, respectively, whereas $x_2, \dot{x}_2$ and $\ddot{x}_2$ represent the displacement, velocity, and acceleration of the upper floating frame. Similarly, $x_3, \dot{x}_3$ and $\ddot{x}_3$ denote the displacement, velocity, and acceleration of the lower floating frame. F represents the amplitude of the exciting force, and ω denotes the operating frequency.

Equation (1) represents a three-degree-of-freedom vibration equation with coupled parameters. To investigate the nonlinear dynamic characteristics of the system, these equations require decoupling and transformation into a set of independent equations using modal coordinates and parameters. In this study, the approximate linear decoupling is performed, with the detailed process outlined in Appendix A where all coefficients are derived. Equation (2) presents the vibration equation in modal coordinates.

$$\begin{aligned}
& \begin{Bmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \\ \ddot{\xi}_3 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} = - \begin{bmatrix} 2\hat{\eta}_{11} & 2\hat{\eta}_{12} & 2\hat{\eta}_{13} \\ 2\hat{\eta}_{21} & 2\hat{\eta}_{22} & 2\hat{\eta}_{23} \\ 2\hat{\eta}_{31} & 2\hat{\eta}_{32} & 2\hat{\eta}_{33} \end{bmatrix} \begin{Bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{Bmatrix} \\
& - \begin{Bmatrix} \hat{a}_1\xi_1^3 + \hat{a}_2\xi_2^3 + \hat{a}_3\xi_3^3 + \hat{a}_4\xi_1^2\xi_2 + \hat{a}_5\xi_1\xi_2^2 + \hat{a}_6\xi_1^2\xi_3 \\ + \hat{a}_7\xi_1\xi_3^2 + \hat{a}_8\xi_2^2\xi_3 + \hat{a}_9\xi_2\xi_3^2 + \hat{a}_{10}\xi_1\xi_2\xi_3 \\ \hat{b}_1\xi_1^3 + \hat{b}_2\xi_2^3 + \hat{b}_3\xi_3^3 + \hat{b}_4\xi_1^2\xi_2 + \hat{b}_5\xi_1\xi_2^2 + \hat{b}_6\xi_1^2\xi_3 \\ + \hat{b}_7\xi_1\xi_3^2 + \hat{b}_8\xi_2^2\xi_3 + \hat{b}_9\xi_2\xi_3^2 + \hat{b}_{10}\xi_1\xi_2\xi_3 \\ \hat{d}_1\xi_1^3 + \hat{d}_2\xi_2^3 + \hat{d}_3\xi_3^3 + \hat{d}_4\xi_1^2\xi_2 + \hat{d}_5\xi_1\xi_2^2 + \hat{d}_6\xi_1^2\xi_3 \\ + \hat{d}_7\xi_1\xi_3^2 + \hat{d}_8\xi_2^2\xi_3 + \hat{d}_9\xi_2\xi_3^2 + \hat{d}_{10}\xi_1\xi_2\xi_3 \end{Bmatrix} + \begin{Bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{Bmatrix}, \quad (2)
\end{aligned}$$

In the i -th modal coordinate, ξ_i , $\dot{\xi}_i$ and $\ddot{\xi}_i$ represent the displacement, velocity, and acceleration, respectively. ω_i denotes the natural frequency in the i -th modal coordinate, while $\hat{f}_i$ stands for the force amplitude in the same coordinate.

2.2. Perturbation Analysis

The system exhibits forced vibration with weak nonlinear stiffness [37], where the excitation force on the second and third degrees of freedom is minimal. Assuming $\omega_1 < \omega_2 < \omega_3$ and $0 < \varepsilon < 1$, $\hat{\eta}_{ij} = \varepsilon \eta_{ij}$, $\hat{a}_k = \varepsilon a_k$, $\hat{b}_k = \varepsilon b_k$, $\hat{d}_k = \varepsilon d_k$, $\hat{f}_1 = f_1$, $\hat{f}_{2,3} = \varepsilon f_{2,3}$ results in Equation (3).

$$\begin{aligned}
& \begin{Bmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \\ \ddot{\xi}_3 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} = -\varepsilon \begin{bmatrix} 2\eta_{11} & 2\eta_{12} & 2\eta_{13} \\ 2\eta_{21} & 2\eta_{22} & 2\eta_{23} \\ 2\eta_{31} & 2\eta_{32} & 2\eta_{33} \end{bmatrix} \begin{Bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{Bmatrix} \\
& - \varepsilon \begin{Bmatrix} a_1\xi_1^3 + a_2\xi_2^3 + a_3\xi_3^3 + a_4\xi_1^2\xi_2 + a_5\xi_1\xi_2^2 + a_6\xi_1^2\xi_3 \\ + a_7\xi_1\xi_3^2 + a_8\xi_2^2\xi_3 + a_9\xi_2\xi_3^2 + a_{10}\xi_1\xi_2\xi_3 \\ b_1\xi_1^3 + b_2\xi_2^3 + b_3\xi_3^3 + b_4\xi_1^2\xi_2 + b_5\xi_1\xi_2^2 + b_6\xi_1^2\xi_3 \\ + b_7\xi_1\xi_3^2 + b_8\xi_2^2\xi_3 + b_9\xi_2\xi_3^2 + b_{10}\xi_1\xi_2\xi_3 \\ d_1\xi_1^3 + d_2\xi_2^3 + d_3\xi_3^3 + d_4\xi_1^2\xi_2 + d_5\xi_1\xi_2^2 + d_6\xi_1^2\xi_3 \\ + d_7\xi_1\xi_3^2 + d_8\xi_2^2\xi_3 + d_9\xi_2\xi_3^2 + d_{10}\xi_1\xi_2\xi_3 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ \varepsilon f_2 \\ \varepsilon f_3 \end{Bmatrix} \quad (3)
\end{aligned}$$

The multiscale method is employed to solve the aforementioned equation, yielding the approximate solution:

$$\xi_i = \xi_{i,0}(T_0, T_1) + \varepsilon \xi_{i,1}(T_0, T_1), \quad (4)$$

where $T_0 = \tau$, $T_1 = \varepsilon \tau$.

There are the following operators:

$$\begin{cases} \frac{d}{d\tau} = \frac{\partial}{\partial T_0} \frac{\partial T_0}{\partial \tau} + \frac{\partial}{\partial T_1} \frac{\partial T_1}{\partial \tau} \approx D_0 + \varepsilon D_1 \\ \frac{d^2}{d\tau^2} = (D_0 + \varepsilon D_1)^2 \approx D_0^2 + 2\varepsilon D_0 D_1 \end{cases} \quad (5)$$

Substitute Equations (4) and (5) into Equation (3), and combine the coefficients of ε^0 and ε^1 , yielding Equations (6) and (7), respectively.

$$\varepsilon^0 : \begin{cases} D_0^2 \xi_{10} + \omega_1^2 \xi_{10} = f_1 \\ D_0^2 \xi_{20} + \omega_2^2 \xi_{20} = 0 \\ D_0^2 \xi_{30} + \omega_3^2 \xi_{30} = 0 \end{cases} \quad (6)$$

$$\varepsilon^1 : \begin{cases} D_0^2 \xi_{11} + \omega_1^2 \xi_{11} = -2D_0 D_1 \xi_{10} - 2\eta_{11} D_0 \xi_{10} - 2\eta_{12} D_0 \xi_{20} - 2\eta_{13} D_0 \xi_{30} - a_1 \xi_{10}^3 \\ -a_2 \xi_{20}^3 - a_3 \xi_{30}^3 - a_4 \xi_{10}^2 \xi_{20} - a_5 \xi_{10} \xi_{20}^2 - a_6 \xi_{10}^2 \xi_{30} - a_7 \xi_{10} \xi_{30}^2 - a_8 \xi_{20}^2 \xi_{30} \\ -a_9 \xi_{20} \xi_{30}^2 - a_{10} \xi_{10} \xi_{20} \xi_{30} \\ D_0^2 \xi_{21} + \omega_2^2 \xi_{21} = -2D_0 D_1 \xi_{20} - 2\eta_{21} D_0 \xi_{10} - 2\eta_{22} D_0 \xi_{20} - 2\eta_{23} D_0 \xi_{30} - b_1 \xi_{10}^3 \\ -b_2 \xi_{20}^3 - b_3 \xi_{30}^3 - b_4 \xi_{10}^2 \xi_{20} - b_5 \xi_{10} \xi_{20}^2 - b_6 \xi_{10}^2 \xi_{30} - b_7 \xi_{10} \xi_{30}^2 - b_8 \xi_{20}^2 \xi_{30} \\ -b_9 \xi_{20} \xi_{30}^2 - b_{10} \xi_{10} \xi_{20} \xi_{30} \\ D_0^2 \xi_{31} + \omega_3^2 \xi_{31} = -2D_0 D_1 \xi_{30} - 2\eta_{31} D_0 \xi_{10} - 2\eta_{32} D_0 \xi_{20} - 2\eta_{33} D_0 \xi_{30} - d_1 \xi_{10}^3 \\ -d_2 \xi_{20}^3 - d_3 \xi_{30}^3 - d_4 \xi_{10}^2 \xi_{20} - d_5 \xi_{10} \xi_{20}^2 - d_6 \xi_{10}^2 \xi_{30} - d_7 \xi_{10} \xi_{30}^2 - d_8 \xi_{20}^2 \xi_{30} \\ -d_9 \xi_{20} \xi_{30}^2 - d_{10} \xi_{10} \xi_{20} \xi_{30} \end{cases} \quad (7)$$

The general solution of Equation (7) is expressed as:

$$\begin{cases} \xi_{10} = A_1 e^{i\omega_1 T_0} + B e^{i\omega T_0} + cc \\ \xi_{20} = A_2 e^{i\omega_2 T_0} + cc \\ \xi_{30} = A_3 e^{i\omega_3 T_0} + cc \end{cases}, \quad (8)$$

where A_i represents an undetermined complex number, and cc denotes its complex conjugate,

$$B = \frac{f_1}{2(\omega_1^2 - \omega^2)}.$$

Considering primary resonance from the operating frequency and internal resonance among natural frequencies, the assumptions are $\omega = \omega_3 + \varepsilon\sigma_3$, $\omega_1 = \omega_3/3 + \varepsilon\sigma_1$ and $\omega_2 = \omega_3 + \varepsilon\sigma_2$, with σ_1 , σ_2 and σ_3 as tuning parameters. By eliminating secular terms, the resulting differential equations governing slow amplitude and phase are as follows:

$$\begin{cases} -2i\omega_1 D_1 A_1 - 2\eta_{11} i\omega_1 A_1 - 3a_1 A_1^2 \bar{A}_1 - 6a_1 A_1 B^2 - 2a_5 A_2 \bar{A}_2 A_1 - 2a_7 A_3 \bar{A}_3 A_1 - a_6 \bar{A}_1^2 A_3 e^{-i3\sigma_1 T_1} \\ -2a_6 A_1 B \bar{A}_3 e^{i\sigma_3 T_1} - 3a_1 \bar{A}_1^2 B e^{i(\sigma_3 - 3\sigma_1)T_1} - a_4 \bar{A}_1^2 A_2 e^{i(\sigma_2 - 3\sigma_1)T_1} - 2a_4 A_1 B \bar{A}_2 e^{i(\sigma_3 - \sigma_2)T_1} \\ -2a_4 A_1 B A_2 e^{i(\sigma_2 - \sigma_3)T_1} - a_{10} A_1 A_2 \bar{A}_3 e^{i\sigma_2 T_1} - a_{10} A_1 \bar{A}_2 A_3 e^{-i\sigma_2 T_1} = 0 \\ -2i\omega_2 D_1 A_2 - 2\eta_{22} i\omega_2 A_2 - 3b_2 A_2^2 \bar{A}_2 - 2b_4 A_1 \bar{A}_1 A_2 - 2b_4 B^2 A_2 - 2b_9 A_3 \bar{A}_3 A_2 \\ + [-2\eta_{23} i\omega_3 A_3 - 3b_3 A_3^2 \bar{A}_3 - 2b_6 A_1 \bar{A}_1 A_3 - 2b_6 B^2 A_3 - 2b_8 A_2 \bar{A}_2 A_3] e^{-i\sigma_2 T_1} \\ + [f_2 / 2 - 2\eta_{21} i\omega B - 3b_1 B^3 - 6b_1 A_1 \bar{A}_1 B - 2b_5 A_2 \bar{A}_2 B - 2b_7 A_3 \bar{A}_3 B] e^{i(\sigma_3 - \sigma_2)T_1} - b_1 A_1^3 e^{i(3\sigma_1 - \sigma_2)T_1} \\ -b_4 B^2 \bar{A}_2 e^{i(2\sigma_3 - \sigma_2)T_1} - b_5 B A_2^2 e^{i(\sigma_2 - \sigma_3)T_1} - b_6 B^2 \bar{A}_3 e^{i(2\sigma_3 - \sigma_2)T_1} - b_7 A_3^2 B e^{-i(\sigma_2 + \sigma_3)T_1} - b_8 A_2^2 \bar{A}_3 e^{i\sigma_2 T_1} \\ -b_9 A_3^2 \bar{A}_2 e^{-i2\sigma_2 T_1} - b_{10} B A_2 \bar{A}_3 e^{i\sigma_3 T_1} - b_{10} B \bar{A}_2 A_3 e^{i(\sigma_3 - 2\sigma_2)T_1} - b_{10} B A_2 A_3 e^{-i\sigma_3 T_1} = 0 \\ [-2i\omega_3 D_1 A_3 - 2\eta_{33} i\omega_3 A_3 - 3d_3 A_3^2 \bar{A}_3 - 2d_6 A_1 \bar{A}_1 A_3 - 2d_6 B^2 A_3 - 2d_8 A_2 \bar{A}_2 A_3] \\ + [-2\eta_{32} i\omega_2 A_2 - 3d_2 A_2^2 \bar{A}_2 - 2d_4 A_1 \bar{A}_1 A_2 - 2d_4 B^2 A_2 - 2d_9 A_3 \bar{A}_3 A_2] e^{i\sigma_2 T_1} \\ + [f_3 / 2 - 2\eta_{31} i\omega B - 3d_1 B^3 - 6d_1 A_1 \bar{A}_1 B - 2d_5 A_2 \bar{A}_2 B - 2d_7 A_3 \bar{A}_3 B] e^{i\sigma_3 T_1} - d_1 A_1^3 e^{i3\sigma_1 T_1} \\ -d_4 B^2 \bar{A}_2 e^{i(2\sigma_3 - \sigma_2)T_1} - d_5 B A_2^2 e^{i(2\sigma_2 - \sigma_3)T_1} - d_6 B^2 \bar{A}_3 e^{i2\sigma_3 T_1} - d_7 A_3^2 B e^{-i\sigma_3 T_1} - d_8 A_2^2 \bar{A}_3 e^{i2\sigma_2 T_1} \\ -d_9 A_3^2 \bar{A}_2 e^{-i\sigma_2 T_1} - d_{10} B A_2 \bar{A}_3 e^{i(\sigma_2 + \sigma_3)T_1} - d_{10} B \bar{A}_2 A_3 e^{i(\sigma_3 - \sigma_2)T_1} - d_{10} B A_2 A_3 e^{i(\sigma_2 - \sigma_3)T_1} = 0 \end{cases} \quad (9)$$

Substituting polar coordinates $A_1 = 0.5h_1(T_1)e^{i\beta_1(T_1)}$, $A_2 = 0.5h_2(T_1)e^{i\beta_2(T_1)}$, $A_3 = 0.5h_3(T_1)e^{i\beta_3(T_1)}$ into Equation (9) and separating the real and imaginary parts yields $\theta_1 = (\sigma_3 - 3\sigma_1)T_1 - 3\beta_1$, $\theta_2 = (\sigma_3 - \sigma_2)T_1 - \beta_2$, and $\theta_3 = \sigma_3 T_1 - \beta_3$. Equation (10) describes the response of amplitude-frequency tuning parameters as follows:

$$\begin{aligned}
h_1' &= -\frac{1}{\omega_1} h_1 (\eta_{11} \omega_1 + \frac{1}{8} a_6 h_1 h_3 \sin(\theta_1 - \theta_3) + \frac{1}{2} a_6 h_3 B \sin \theta_3 + \frac{3}{4} a_1 h_1 B \sin \theta_1 + \frac{1}{8} a_4 h_1 h_2 \sin(\theta_1 - \theta_2)) \\
\theta_1' &= \sigma_3 - 3\sigma_1 - \frac{1}{\omega_1} (\frac{9}{8} a_1 h_1^2 + 9a_1 B^2 + \frac{3}{4} a_5 h_2^2 + \frac{3}{4} a_7 h_3^2 + \frac{3}{8} a_6 h_1 h_3 \cos(\theta_1 - \theta_3) + \frac{3}{2} a_6 h_3 B \cos \theta_3 \\
&\quad + \frac{9}{4} a_1 h_1 B \cos \theta_1 + \frac{3}{8} a_4 h_1 h_2 \cos(\theta_1 - \theta_2) + 3a_4 h_2 B \cos \theta_2 + \frac{3}{4} a_{10} h_2 h_3 \cos(\theta_3 - \theta_2)) \\
h_2' &= \frac{1}{\omega_2} (-\eta_{22} \omega_2 h_2 + (\frac{1}{8} b_8 h_2^2 h_3 - \frac{3}{8} b_3 h_3^3 - \frac{1}{4} b_6 h_1^2 h_3 - b_6 B^2 h_3 - \frac{1}{4} b_8 h_2^2 h_3) \sin(\theta_2 - \theta_3) + (\frac{1}{2} f_2 \\
&\quad - 3b_1 B^3 - \frac{3}{2} b_1 h_1^2 B - \frac{1}{4} b_5 h_2^2 B - \frac{1}{2} b_7 h_3^2 B) \sin \theta_2 - \frac{1}{8} b_1 h_1^3 \sin(\theta_2 - \theta_1) - \frac{1}{2} b_4 B^2 h_2 \sin(2\theta_2) \\
&\quad - \frac{1}{2} b_6 B^2 h_3 \sin(\theta_2 + \theta_3) - \frac{1}{4} b_7 B h_3^2 \sin(\theta_2 - 2\theta_3) - \frac{1}{8} b_9 h_3^2 h_2 \sin(2\theta_2 - 2\theta_3) \\
&\quad - \frac{1}{4} b_{10} h_3 h_2 B \sin(2\theta_2 - \theta_3) - \eta_{23} \omega_3 h_3 \cos(\theta_2 - \theta_3) - 2\eta_{21} \omega B \cos \theta_2) \\
\theta_2' &= \sigma_3 - \sigma_2 - \frac{1}{\omega_2 h_2} (\frac{3}{8} b_2 h_2^3 + \frac{1}{4} b_4 h_1^2 h_2 + b_4 B^2 h_2 + \frac{1}{4} b_9 h_3^2 h_2 - \eta_{23} \omega_3 h_3 \sin(\theta_2 - \theta_3) + (\frac{3}{8} b_3 h_3^2 + \frac{1}{4} \\
&\quad b_6 h_1^2 + b_6 B^2 + \frac{3}{8} b_8 h_2^2) h_3 \cos(\theta_2 - \theta_3) - 2\eta_{21} \omega B \sin \theta_2 - \frac{1}{2} (f_2 - 6b_1 B^3 - 3b_1 h_1^2 B - \frac{3}{2} b_5 h_2^2 B - b_7 \\
&\quad h_3^2 B) \cos \theta_2 + \frac{1}{8} b_1 h_1^3 \cos(\theta_2 - \theta_1) + \frac{1}{2} (b_4 B^2 h_2 \cos(2\theta_2) + b_6 B^2 h_3 \cos(\theta_2 + \theta_3) + b_{10} h_3 h_2 B \\
&\quad \cos \theta_3) + \frac{1}{4} b_7 B h_3^2 \cos(\theta_2 - 2\theta_3) + \frac{1}{8} b_9 h_3^2 h_2 \cos(2\theta_2 - 2\theta_3) + \frac{1}{4} b_{10} h_3 h_2 B \cos(2\theta_2 - \theta_3)) \\
h_3' &= \frac{1}{\omega_3} (-\eta_{33} \omega_3 h_3 - (\frac{3}{8} d_2 h_2^2 + \frac{1}{4} d_4 h_1^2 + d_4 B^2 + \frac{1}{8} d_9 h_3^2) h_2 \sin(\theta_3 - \theta_2) - \eta_{32} \omega_2 h_2 \cos(\theta_3 - \theta_2) \\
&\quad + (\frac{1}{2} f_3 - 3d_1 B^3 - \frac{3}{2} d_1 B - \frac{1}{2} d_5 h_2^2 B - \frac{1}{4} d_7 h_3^2 B) \sin \theta_3 - 2\eta_{31} \omega B \cos \theta_3 - \frac{1}{8} d_1 h_1^3 \sin(\theta_3 - \theta_1) \\
&\quad - \frac{1}{2} B^2 (d_4 h_2 \sin(\theta_2 + \theta_3) + d_6 h_3 \sin(2\theta_3)) - \frac{1}{4} h_2 B (d_5 h_2 \sin(\theta_3 - 2\theta_2) + d_{10} h_3 \sin(2\theta_3 - \theta_2)) \\
&\quad - \frac{1}{8} d_8 h_2^2 h_3 \sin(2\theta_3 - 2\theta_2)) \\
\theta_3' &= \sigma_3 - \frac{1}{\omega_3 h_3} (\frac{3}{8} d_3 h_3^3 + \frac{1}{4} d_6 h_1^2 h_3 + d_6 B^2 h_3 + \frac{1}{4} d_8 h_2^2 h_3 - \eta_{32} \omega_2 h_2 \sin(\theta_3 - \theta_2) + (\frac{3}{8} d_2 h_2^2 + \frac{1}{4} d_4 \\
&\quad h_1^2 h_2 + d_4 B^2 h_2 + \frac{3}{8} d_9 h_3^2 h_2) \cos(\theta_3 - \theta_2) - 2\eta_{31} \omega B \sin \theta_3 - \frac{1}{2} (f_3 - 6d_1 B^3 - 3d_1 B - d_5 h_2^2 B - \frac{3}{2} \\
&\quad d_7 h_3^2 B) \cos \theta_3 + \frac{1}{8} d_1 h_1^3 \cos(\theta_3 - \theta_1) + \frac{1}{2} d_4 B^2 h_2 \cos(\theta_2 + \theta_3) + \frac{1}{4} d_5 B h_2^2 \cos(\theta_3 - 2\theta_2) + \frac{1}{2} \\
&\quad d_6 B^2 h_3 \cos(2\theta_3) + \frac{1}{8} d_8 h_2^2 h_3 \cos(2\theta_3 - 2\theta_2) + \frac{1}{4} d_{10} h_3 h_2 B \cos(2\theta_3 - \theta_2) + \frac{1}{2} d_{10} h_3 h_2 B \cos \theta_2)
\end{aligned} \tag{10}$$

3. Results and Discussion

3.1. Amplitude-Frequency Tuning Parameter Response of the System

The system parameters considered are as follows: $F = 4000$ N, $\varepsilon = 0.1$; masses $m_1 = 214$ kg, $m_2 = 55$ kg, $m_3 = 38$ kg; spring constants $k_1 = 6 \times 10^4$ N/m, $k_2 = 6 \times 10^4$ N/m, $k_3 = 5.6 \times 10^4$ N/m and equivalent spring constants $k_1' = k_2' = 3 \times 10^4$ N/m³, $k_3' = 2.8 \times 10^4$ N/m³; with damping coefficients $c_1 = c_2 = 30$ and $c_3 = 28$. The values of equivalent spring stiffness are determined based on the static loading test results of rubber springs. Parameter fitting of experimental data reveals that the linear spring stiffness is 2.1 times the equivalent spring stiffness. The natural frequencies of the vibration systems are $\omega_1 = 13.6$ rad/s, $\omega_2 = 35.2$ rad/s, and $\omega_3 = 44.3$ rad/s. Given that the vibration screen operates based on system resonance, and the operational frequency closely aligns with the third-order natural frequency, the region around this frequency is chosen for frequency sweeping.

Substituting the above parameters into Equation (10), the amplitude-frequency tuning parameter response curves of each mode are obtained, as shown in Figure 3. The first-order mode depicted in Figure 3a remains unexcited as its natural frequency is significantly different from the sweep frequency, whereas the second- and third-order modes are excited. System nonlinearity induces curvature in the response curves and the appear-

ance of multi-valued regions. The red and blue curves correspond to forward and backward sweeps, respectively. As shown in Figure 3b,c, the response amplitude gradually increases with the forward sweep frequency ω increment. Upon further increase, a jump from point C to D occurs, leading to a sudden decrease in response amplitude, followed by stabilization with increasing ω . Conversely, during the backward sweep (decreasing ω), the response amplitude initially rises slightly. Upon further reduction, a jump from point A to B results in a sudden increase in response amplitude, followed by a decrease as ω decreases, eventually stabilizing. Notably, the backward jump consistently lags behind the forward process, and the region between these jumps (A to C) is characterized by instability.

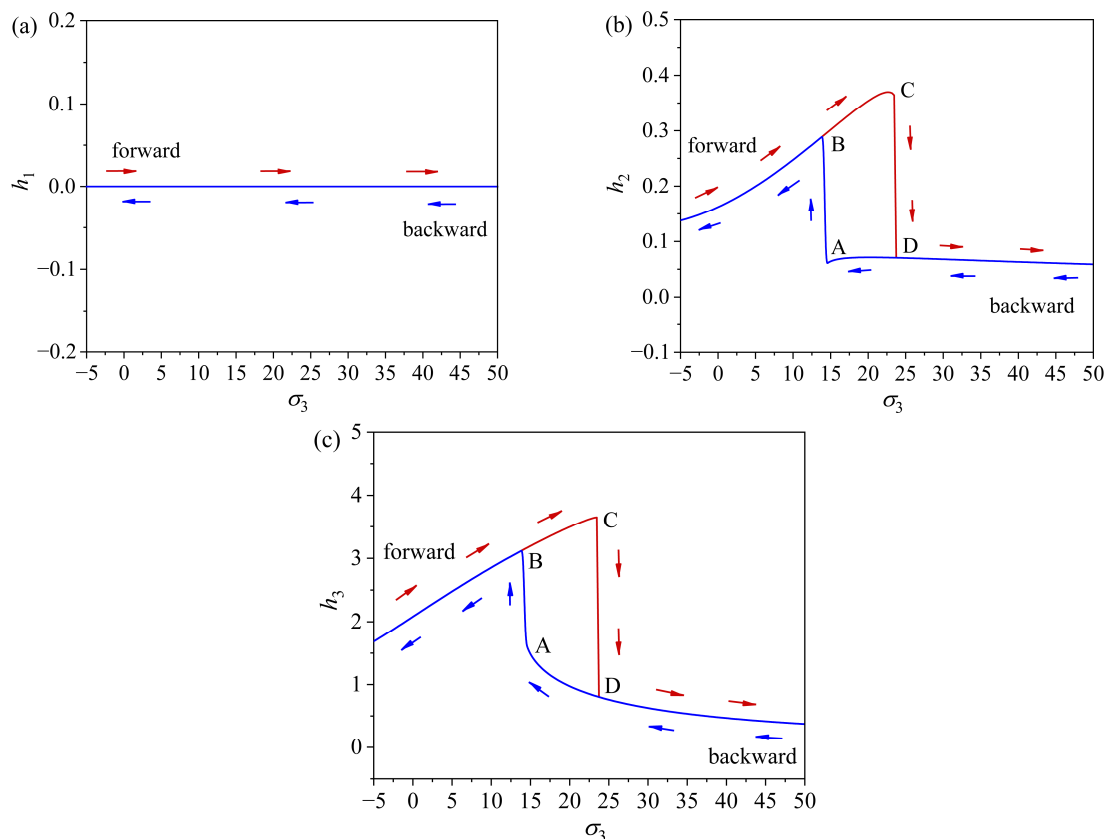


Figure 3. The amplitude-frequency tuning parameter response curve of each modal: (a) The response of h_1 - σ_3 . (b) The response of h_2 - σ_3 . (c) The response of h_3 - σ_3 .

3.2. The Effect of System Parameters on Nonlinear Characteristics of the System

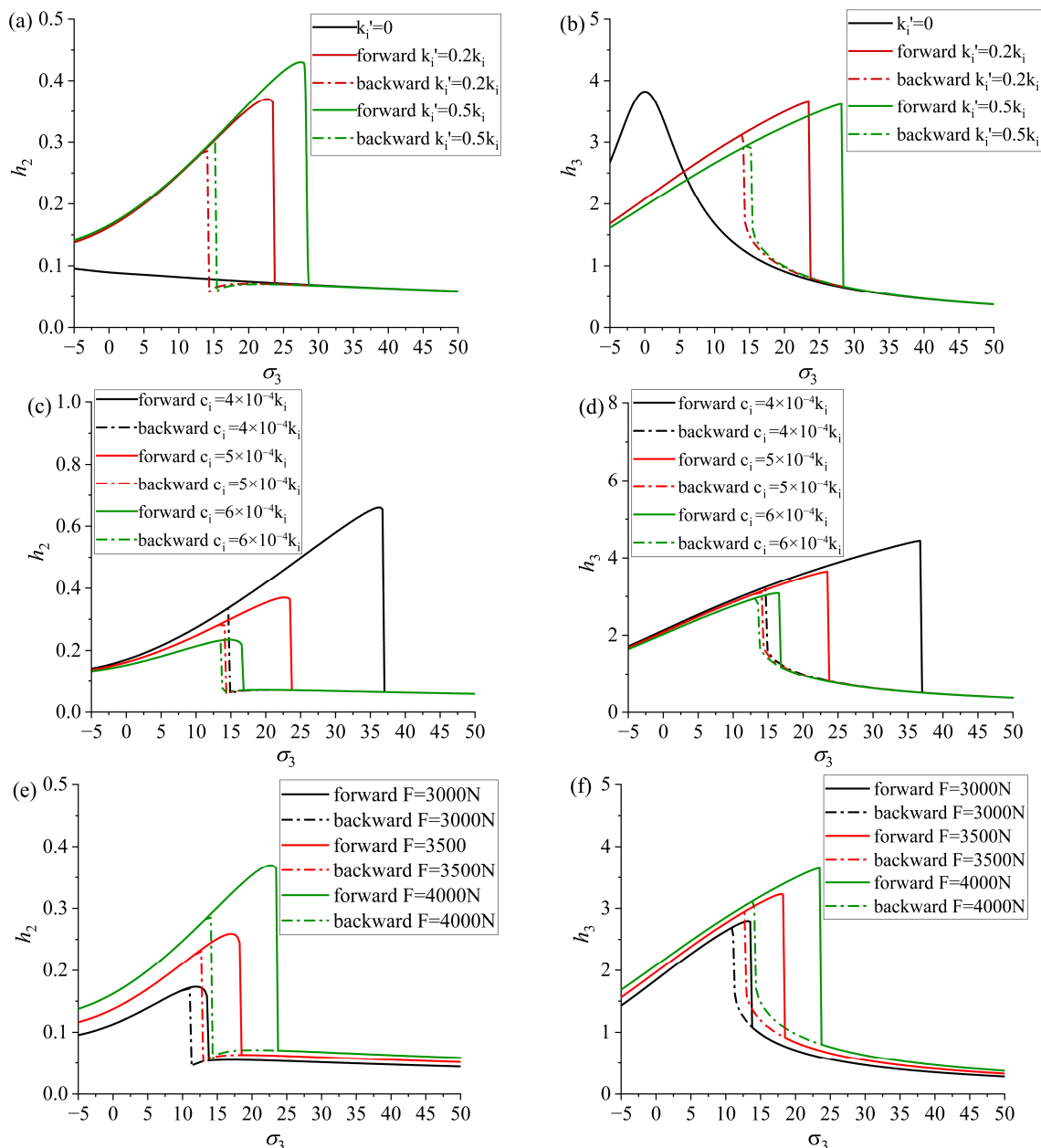
Since the first-order modal is not excited, the results of parameters on the amplitude-frequency tuning parameter response of the second-order and third-order modal are studied.

Figure 4a,b indicate the changes in the second-order and third-order modal amplitude-frequency tuning parameter response of the system with nonlinear parameters k_1' , k_2' and k_3' . When $k_1' = 0$, the system behaves linearly: h_2 remains unchanged near its third-order natural frequency, and the response curve of h_3 shows no bending, maintaining symmetry about the axis $\sigma_3 = 0$. However, for $k_1' > 0$, both h_2 and h_3 exhibit rightward bending in their response curves, indicating gradual hardening characteristics. As k_1' increases, the peak value of h_2 rises, expanding its double-value resonance region. Conversely, the peak value of h_3 decreases with increasing k_1' , also accompanied by an expanded double-value resonance region.

Change the value of the damping coefficient c_1 to study the influence of damping on system nonlinearity, as shown in Figure 4c,d. With an increase in c_1 , the peak values of h_2 and h_3 decrease, with h_2 showing a more pronounced reduction. Simultaneously, the double-value resonance region of h_2 and h_3 also contracts.

The influence of the exciting force on the amplitude-frequency tuning parameter response curve is studied, and the results are shown in Figure 4e,f. With the increase of F , both the peak values of h_2 and h_3 grow larger, with the peak value of h_2 showing a more pronounced increase. Simultaneously, the double-value resonance region of h_2 and h_3 also expands.

Figure 4g,h present the influence of the upper floating frame ($m_2 = 38$, $m_2 = 46$, $m_2 = 55$) on the frequency tuning parameter response curve. As m_2 increases, the peak value of h_2 increases, and the double-value resonance region expands. Initially, the peak value of h_3 increases, followed by a decrease, while its double-value resonance region also expands.



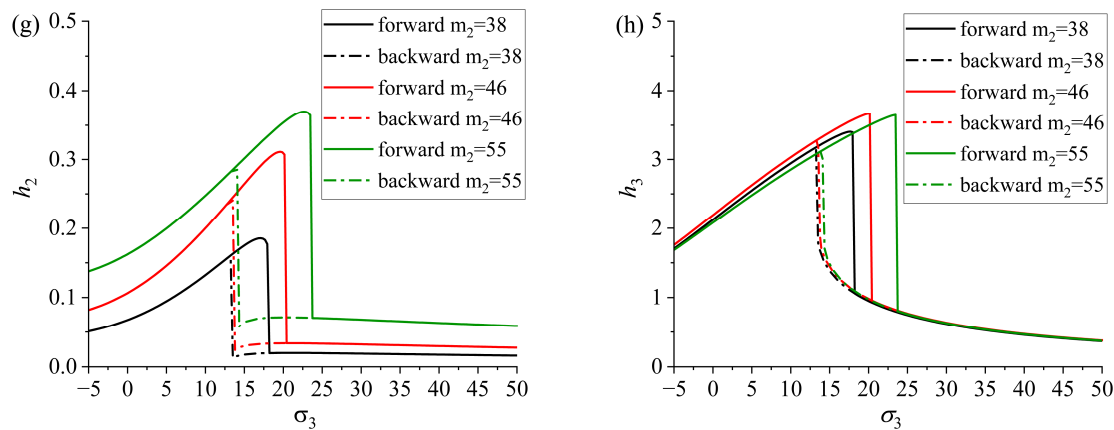
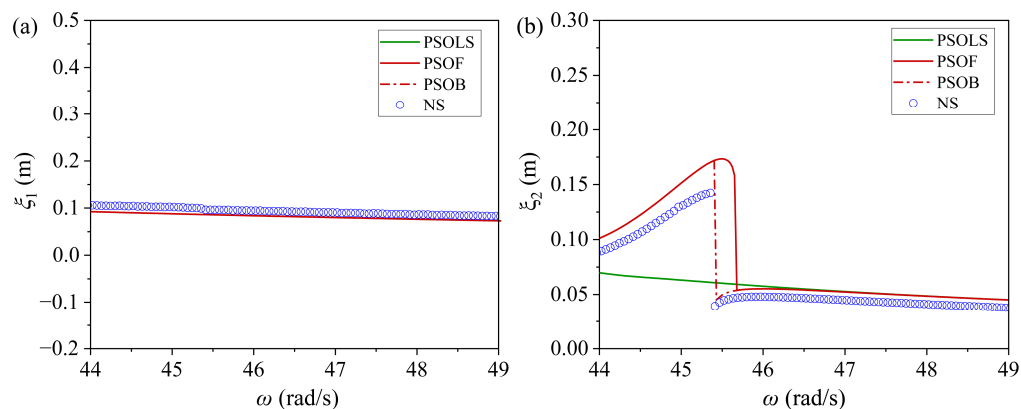


Figure 4. The effect of system parameters on amplitude-frequency tuning parameter response of each modal: (a) The effect of k_i' on h_2 - σ_3 curve. (b) The effect of k_i' on h_3 - σ_3 curve. (c) The effect of c_1 on h_2 - σ_3 curve. (d) The effect of c_1 on h_3 - σ_3 curve. (e) The effect of F on h_2 - σ_3 curve. (f) The effect of F on h_3 - σ_3 curve. (g) The effect of m_2 on h_2 - σ_3 curve. (h) The effect of m_2 on h_3 - σ_3 curve.

3.3. Numerical Verification

According to Equations (8) and (10), the perturbed solution is derived. To obtain the numerical solution, Equation (2) is solved using the Runge–Kutta method. This numerical solution is then compared with the perturbed solution to validate its accuracy. For the analysis, $F = 3000$ N and other parameters from Section 3.1 are utilized. The amplitude-frequency response comparison between approximate and numerical solutions is presented in Figure 5. In the plot, the solid green line represents the perturbed solution of the linear system (PSOLS), the solid red line shows the forward sweep perturbed solution (PSOF), the dot-dash red line depicts the backward sweep perturbed solution (PSOB), and the blue circles denote the numerical solution (NS). Although the first-order perturbation solution produces quantitative discrepancies compared with numerical results at the resonance of the second-order mode, this method is capable of capturing qualitative trends including vibration jump with high analytical efficiency, which significantly reduces the computational cost of the model.



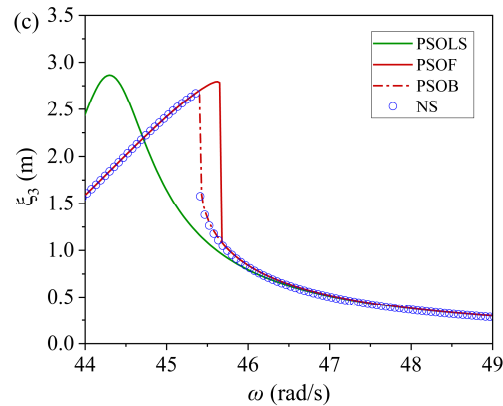


Figure 5. The amplitude-frequency response comparison: (a) First-order modal. (b) Second-order modal. (c) Third-order modal.

Figure 5 illustrates two stability responses of the system within the frequency range. Specifically, $\omega = 45.5$ rad/s is selected to investigate the steady-state solution of the system. In Figure 6, both time-domain and phase curve analyses of the system are presented under varying initial conditions, chosen near the stable response value.

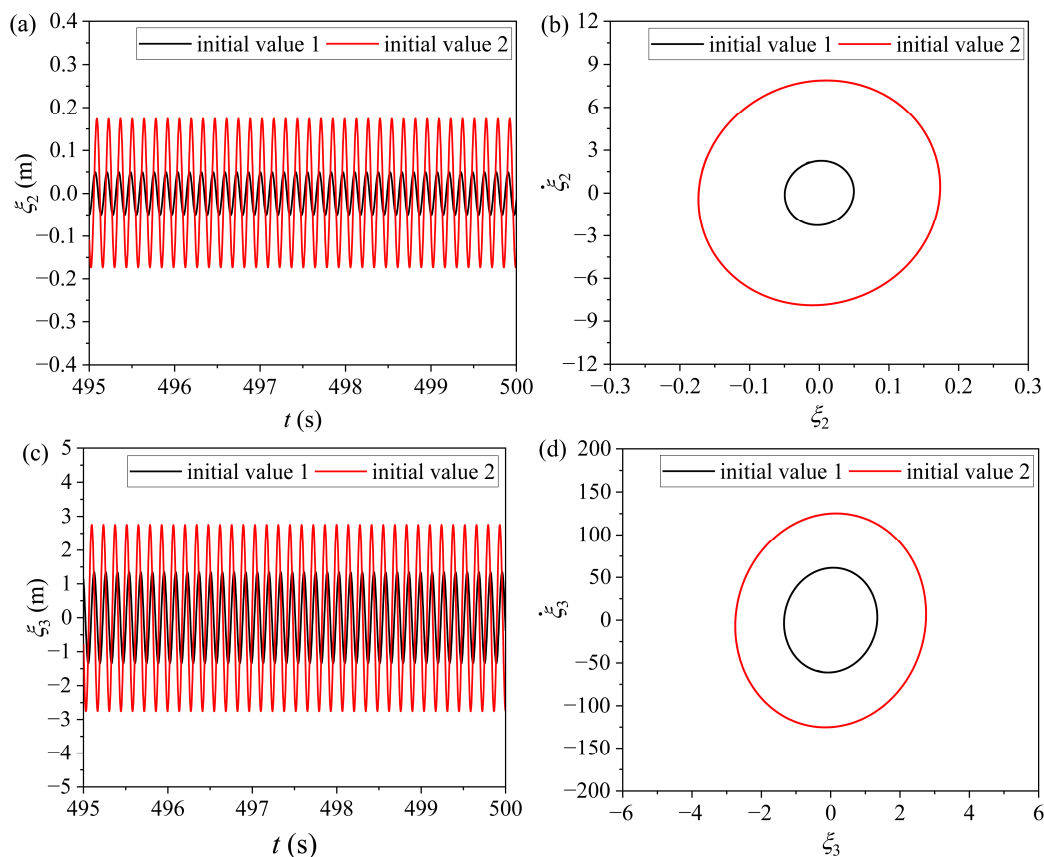


Figure 6. The time-domain curve and phase curve of the system: (a) The time-domain curve of ξ_2 . (b) The phase curve of ξ_2 . (c) The time-domain curve of ξ_3 . (d) The phase curve of ξ_3 .

The multiscale method effectively computes the steady-state and transient responses of nonlinear systems. Figures 7 and 8 depict the initial 1 s transient response and the last 1 s steady-state response of both perturbed and numerical solutions. During the transient

phase, there are quantitative differences between the first-order modal perturbed and numerical solutions, yet their trends are similar. However, both the perturbed and numerical solutions of the second and third-order modals exhibit consistent behaviors. In the steady state, the values of the perturbed and numerical solutions for each modal are nearly identical. Thus, perturbed solutions accurately depict the responses of nonlinear vibration systems in both transient and steady states.

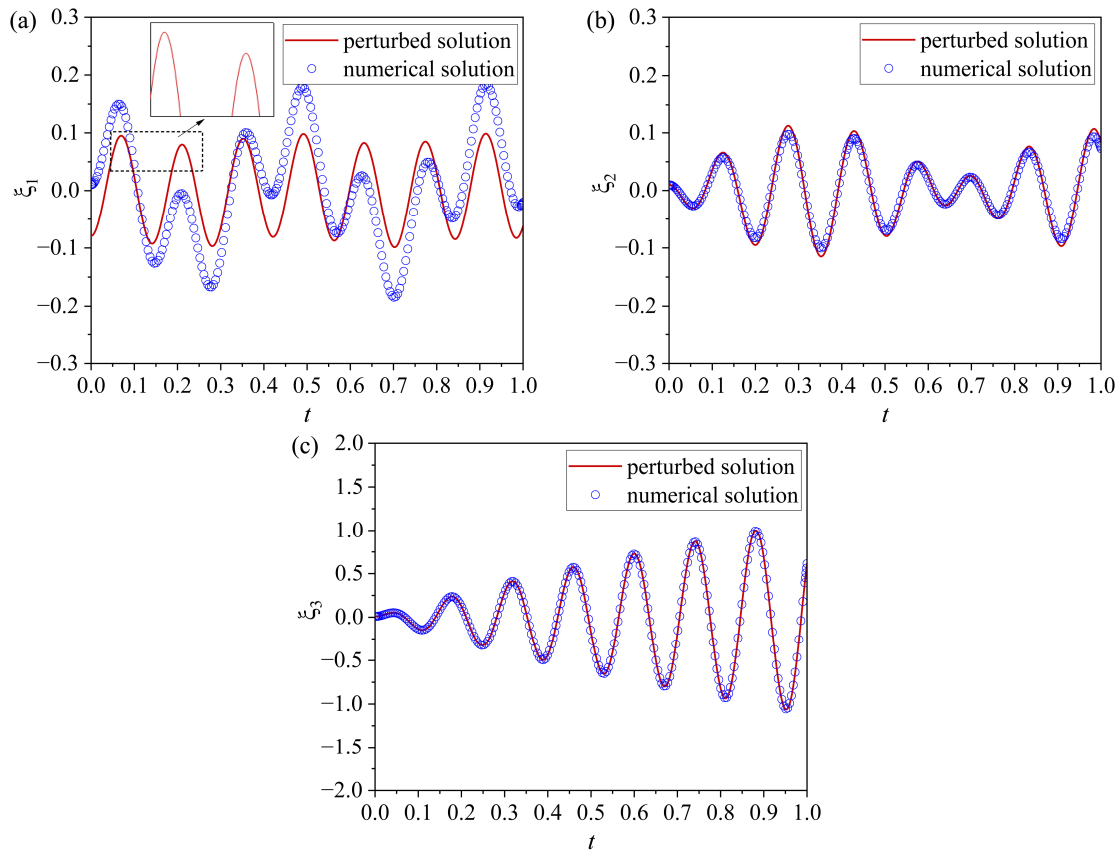
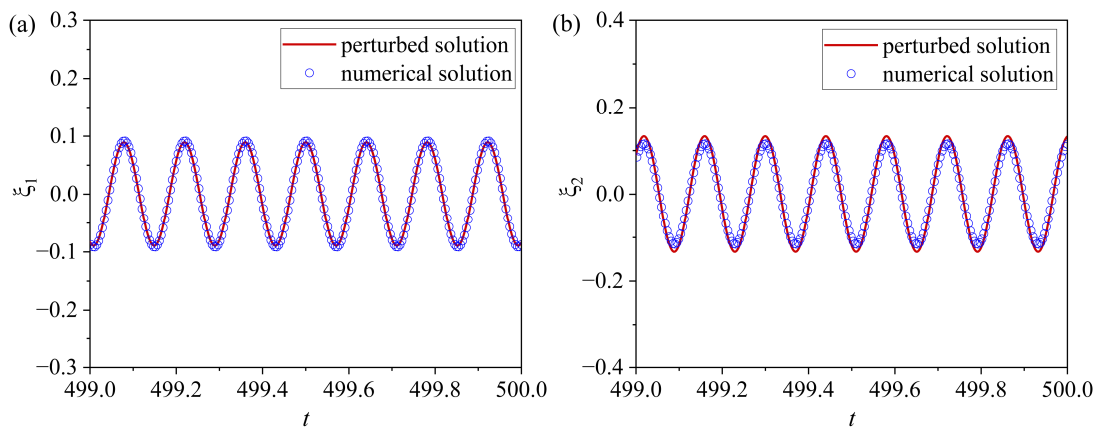


Figure 7. The transient time-domain response of the system: (a) The response of ξ_1 . (b) The response of ξ_2 . (c) The response of ξ_3 .



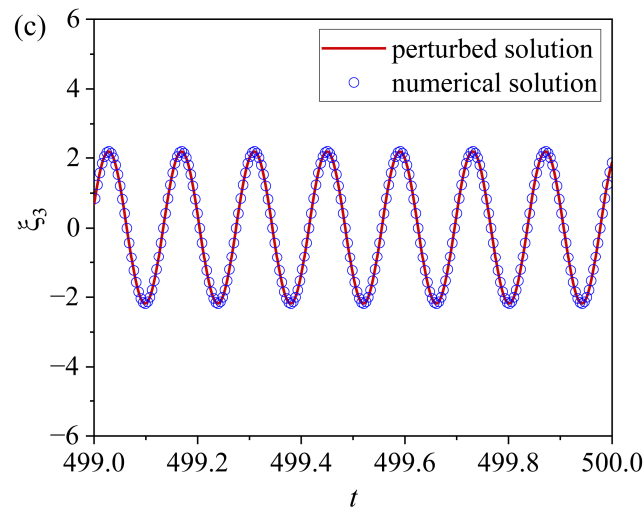


Figure 8. The steady time-domain response of the system: (a) The response of ξ_1 . (b) The response of ξ_2 . (c) The response of ξ_3 .

3.4. The Displacement Response of DDVFFS

As previously discussed, the characteristics of multi-valued responses and the stability of amplitude-frequency response in modal coordinates suggest that these features also apply to the system's response in its original physical coordinates. Figure 5 illustrates that the superposition principle does not hold in the nonlinear system's multi-valued response region near resonance. Conversely, the response curves of both linear and nonlinear systems align in regions where responses are slightly distant and single-valued. Consequently, the superposition principle remains applicable for solving the dynamic responses in the original physical coordinates of the DDVFFS within these slightly distant single-valued response regions.

To achieve deck stretching and slackening while avoiding instability from excessive resonance, parameters $F = 1500$ N and operating frequency $\omega = 50.3$ rad/s are chosen. The dynamic response in modal coordinates is computed using the multi-scale method, substituted into Equation (A8) in Appendix A. Subsequently, time-domain responses of the DDVFFS in physical coordinates are depicted in Figure 9a. The red, blue, and green solid lines represent the displacement responses of the main frame, upper floating frame, and lower floating frame, respectively. Additionally, the numerical solution using the Runge–Kutta method is obtained by solving Equation (1). Figure 9b compares the steady-state time-domain responses between the perturbed solution and the numerical solution. The red, blue, and green circles represent the numerical displacement responses of the main frame, upper floating frame, and lower floating frame, respectively. The error between the perturbed and numerical solutions is found to be minimal. Specifically, the amplitudes of the main frame, upper floating frame, and lower floating frame are 7 mm, 9.4 mm, and 5.6 mm, respectively. Furthermore, the phase difference between the main frame and both the upper and lower floating frames is 180° . This phase relationship indicates that the upper and lower floating frames move in the same direction, while the main frame moves in reverse with them.

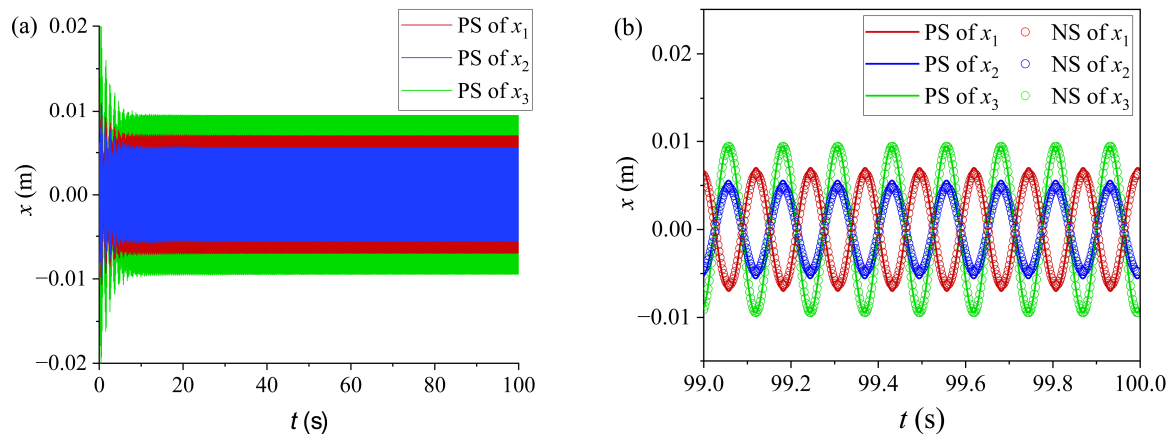


Figure 9. The time-domain response of DDVFFS: (a) Full-time history. (b) Comparison of perturbed and numerical solutions.

4. Conclusions

This paper investigates the nonlinear dynamic characteristics of DDVFFS. Initially, the vibration equations in physical coordinates are transformed into modal coordinates. Subsequently, these equations are approximated using the multi-scale method to derive amplitude-frequency tuning parameter response equations. Secondly, the influence of system parameters on the nonlinear characteristics is analyzed based on the amplitude-frequency tuning parameter response equation. Furthermore, both approximate analytical and numerical solutions are compared across the frequency and time domains. Finally, it is observed that the linear and nonlinear systems exhibit identical dynamic characteristics in the slightly distant single-valued resonance region. The displacement response of the DDVFFS is resolved within the distant single-valued resonance region. The key conclusions drawn from this study are as follows:

- (1) The nonlinearity of the system causes the second-order and third-order modal response curves to bend, leading to the appearance of a multi-valued region. Nonlinear jump behavior can be mitigated by increasing damping and reducing the excitation force.
- (2) In modal coordinates, perturbed solutions can describe the dynamic responses of nonlinear vibration systems in both frequency and time domains.
- (3) At an excitation force of $F = 1500$ N and a working frequency of $\omega = 50.3$ rad/s, the amplitudes of the main frame, upper floating frame, and lower floating frame are 7 mm, 9.4 mm, and 5.6 mm, respectively. The phase difference between the main frame and the upper and lower floating frames is 180° , indicating the stretching and slackening of the deck.

In this work, the influence of pre-compression on the shear stiffness of shear springs was neglected. In practice, shear springs exhibit coupled mechanical properties integrating shear deformation and normal compression, and their effective stiffness is strongly dependent on preload conditions. The quantitative effect of compression displacement on shear stiffness will be systematically explored in our future research.

Furthermore, linear viscous damping is adopted for simplified modeling. Nevertheless, the damping characteristics of rubber components are typically amplitude-dependent. Nonlinear damping can further correct the position of jump points and adjust the width of the multi-solution region. In follow-up research, fractional-order derivatives and nonlinear damping models will be introduced to effectively improve the prediction accuracy of the established model.

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Equation (1) is transformed into the vibration equation in modal coordinates after being decoupled.

From Equation (1), the eigenvalues are described as follows:

$$(K - \lambda M)\phi_i = 0 \quad (A1)$$

where ϕ_i represents the principal coordinate, and λ denotes the eigenvalue.

By substituting K and M into Equation (A1), the characteristic matrix is derived:

$$K - \lambda M = \begin{bmatrix} (k_1 + k_2 + k_3) - \lambda m_1 & -k_2 & -k_3 \\ -k_2 & k_2 - \lambda m_2 & 0 \\ -k_3 & 0 & k_3 - \lambda m_3 \end{bmatrix}. \quad (A2)$$

The characteristic equation can be derived from Equation (A2):

$$-\lambda^3 m_1 m_2 m_3 + \lambda^2 [m_2 m_3 (k_1 + k_2 + k_3) + m_1 (m_2 k_3 + m_3 k_2)] + \lambda [m_2 k_3^2 + m_3 k_2^2 - (k_1 + k_2 + k_3)(m_2 k_3 + m_3 k_2) - m_1 k_2 k_3] + k_2 k_3 (k_1 + k_2 + k_3) - k_3^2 k_2 - k_2^2 k_3 = 0. \quad (A3)$$

Let $a = -m_1 m_2 m_3$, $b = m_2 m_3 (k_1 + k_2 + k_3) + m_1 (m_2 k_3 + m_3 k_2)$, $c = -[m_2 k_3 (k_1 + k_2) + m_3 k_2 (k_1 + k_3) + m_1 k_2 k_3]$, $d = k_1 k_2 k_3$, $p = (3ac - b^2)/3a^2$, $q = (27a^2 d - 9abc + 2b^3)/27a^3$. By solving Equation (A3), the eigenvalues are as follows:

$$\lambda_1 = \frac{(-1 + i\sqrt{3})}{2} \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \frac{(-1 + i\sqrt{3})^2}{2} \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \frac{b}{3a}, \quad (A4)$$

$$\lambda_2 = \frac{(-1 + i\sqrt{3})^2}{2} \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \frac{(-1 + i\sqrt{3})}{2} \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \frac{b}{3a}, \quad (A5)$$

$$\lambda_3 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \frac{b}{3a}. \quad (A6)$$

According to Equations (A4)–(A6), the $\lambda_1 = \omega_1^2$, $\lambda_2 = \omega_2^2$, $\lambda_3 = \omega_3^2$ are introduced, and the undamped linear natural frequency ω_1 , ω_2 , ω_3 are solved.

The regular modal matrix ψ and regular modal coordinate ξ_i are determined from the eigenmatrix of Equation (A2), following this specific solving process.

The adjoint matrix A^* of Equation (A2):

$$\mathbf{A}^* = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix},$$

where $A_{11} = (k_2 - \omega^2 m_2)(k_3 - \omega^2 m_3)$, $A_{12} = A_{21} = k_2(k_3 - \omega^2 m_3)$, $A_{13} = A_{31} = k_3(k_2 - \omega^2 m_2)$, $A_{22} = (k_1 + k_2 + k_3 - \omega^2 m_1)(k_3 - \omega^2 m_3) - k_3^2$, $A_{23} = A_{32} = k_2 k_3$, $A_{33} = (k_1 + k_2 + k_3 - \omega^2 m_1)(k_2 - \omega^2 m_2) - k_2^2$

Next, the modal matrix Φ is obtained:

$$\Phi = \begin{bmatrix} (k_2 - \omega_1^2 m_2)(k_3 - \omega_1^2 m_3) & (k_2 - \omega_2^2 m_2)(k_3 - \omega_2^2 m_3) & (k_2 - \omega_3^2 m_2)(k_3 - \omega_3^2 m_3) \\ k_2(k_3 - \omega_1^2 m_3) & k_2(k_3 - \omega_2^2 m_3) & k_2(k_3 - \omega_3^2 m_3) \\ k_3(k_2 - \omega_1^2 m_2) & k_3(k_2 - \omega_2^2 m_2) & k_3(k_2 - \omega_3^2 m_2) \end{bmatrix}. \quad (\text{A7})$$

Based on Φ and M , the modal mass matrix M_p and the mode shape matrix Z are determined as follows:

$$M_p = \Phi^T M \Phi, \quad Z = \begin{bmatrix} 1/\sqrt{M_{p11}} & 0 & 0 \\ 0 & 1/\sqrt{M_{p22}} & 0 \\ 0 & 0 & 1/\sqrt{M_{p33}} \end{bmatrix}.$$

Finally, the regular modal matrix ψ is obtained as $\psi = \Phi Z$.

Equation (1) in the content is decoupled using ψ , with detailed steps listed below.

The relationship between physical coordinates and modal coordinates:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} = \begin{Bmatrix} \psi_{11}\xi_1 + \psi_{12}\xi_2 + \psi_{13}\xi_3 \\ \psi_{21}\xi_1 + \psi_{22}\xi_2 + \psi_{23}\xi_3 \\ \psi_{31}\xi_1 + \psi_{32}\xi_2 + \psi_{33}\xi_3 \end{Bmatrix}. \quad (\text{A8})$$

The regularization of the principal mass M :

$$I = \psi^T M \psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{A9})$$

The regularization of the principal stiffness K :

$$A = \psi^T K \psi = \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}. \quad (\text{A10})$$

The regularization of the damping C :

$$\psi^T C \psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{bmatrix}^T \begin{bmatrix} c_1 + c_2 + c_3 & -c_2 & -c_3 \\ -c_2 & c_2 & 0 \\ -c_3 & 0 & c_3 \end{bmatrix} \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{bmatrix} = \begin{bmatrix} 2\hat{\eta}_{11} & 2\hat{\eta}_{12} & 2\hat{\eta}_{13} \\ 2\hat{\eta}_{21} & 2\hat{\eta}_{22} & 2\hat{\eta}_{23} \\ 2\hat{\eta}_{31} & 2\hat{\eta}_{32} & 2\hat{\eta}_{33} \end{bmatrix}, \quad (\text{A11})$$

where $2\hat{\eta}_{11} = [(c_1 + c_2 + c_3)\psi_{11} - c_2\psi_{21} - c_3\psi_{31}]\psi_{11} - (c_2\psi_{11} - c_2\psi_{21})\psi_{21} - (c_3\psi_{11} - c_3\psi_{31})\psi_{31}$,
 $2\hat{\eta}_{21} = [(c_1 + c_2 + c_3)\psi_{12} - c_2\psi_{22} - c_3\psi_{32}]\psi_{11} - (c_2\psi_{12} - c_2\psi_{22})\psi_{21} - (c_3\psi_{12} - c_3\psi_{32})\psi_{31}$,
 $2\hat{\eta}_{31} = [(c_1 + c_2 + c_3)\psi_{13} - c_2\psi_{23} - c_3\psi_{33}]\psi_{11} - (c_2\psi_{13} - c_2\psi_{23})\psi_{21} - (c_3\psi_{13} - c_3\psi_{33})\psi_{31}$,
 $2\hat{\eta}_{12} = [(c_1 + c_2 + c_3)\psi_{11} - c_2\psi_{21} - c_3\psi_{31}]\psi_{12} - (c_2\psi_{11} - c_2\psi_{21})\psi_{22} - (c_3\psi_{11} - c_3\psi_{31})\psi_{32}$,

$$\begin{aligned}
2\hat{\eta}_{22} &= [(c_1 + c_2 + c_3)\psi_{12} - c_2\psi_{22} - c_3\psi_{32}]\psi_{12} - (c_2\psi_{12} - c_2\psi_{22})\psi_{22} - (c_3\psi_{12} - c_3\psi_{32})\psi_{32}, \\
2\hat{\eta}_{32} &= [(c_1 + c_2 + c_3)\psi_{13} - c_2\psi_{23} - c_3\psi_{33}]\psi_{12} - (c_2\psi_{13} - c_2\psi_{23})\psi_{22} - (c_3\psi_{13} - c_3\psi_{33})\psi_{32}, \\
2\hat{\eta}_{13} &= [(c_1 + c_2 + c_3)\psi_{11} - c_2\psi_{21} - c_3\psi_{31}]\psi_{13} - (c_2\psi_{11} - c_2\psi_{21})\psi_{23} - (c_3\psi_{11} - c_3\psi_{31})\psi_{33}, \\
2\hat{\eta}_{23} &= [(c_1 + c_2 + c_3)\psi_{12} - c_2\psi_{22} - c_3\psi_{32}]\psi_{13} - (c_2\psi_{12} - c_2\psi_{22})\psi_{23} - (c_3\psi_{12} - c_3\psi_{32})\psi_{33}, \\
2\hat{\eta}_{33} &= [(c_1 + c_2 + c_3)\psi_{13} - c_2\psi_{23} - c_3\psi_{33}]\psi_{13} - (c_2\psi_{13} - c_2\psi_{23})\psi_{23} - (c_3\psi_{13} - c_3\psi_{33})\psi_{33}.
\end{aligned}$$

The regularization of nonlinear stiffness K' :

$$\psi^T K' \psi \xi_i = \left\{ \begin{aligned} &\hat{a}_1 \xi_1^3 + \hat{a}_2 \xi_2^3 + \hat{a}_3 \xi_3^3 + \hat{a}_4 \xi_1^2 \xi_2 + \hat{a}_5 \xi_1 \xi_2^2 + \hat{a}_6 \xi_1^2 \xi_3 + \hat{a}_7 \xi_1 \xi_3^2 + \hat{a}_8 \xi_2^2 \xi_3 + \hat{a}_9 \xi_2 \xi_3^2 + \hat{a}_{10} \xi_1 \xi_2 \xi_3 \\ &\hat{b}_1 \xi_1^3 + \hat{b}_2 \xi_2^3 + \hat{b}_3 \xi_3^3 + \hat{b}_4 \xi_1^2 \xi_2 + \hat{b}_5 \xi_1 \xi_2^2 + \hat{b}_6 \xi_1^2 \xi_3 + \hat{b}_7 \xi_1 \xi_3^2 + \hat{b}_8 \xi_2^2 \xi_3 + \hat{b}_9 \xi_2 \xi_3^2 + \hat{b}_{10} \xi_1 \xi_2 \xi_3 \\ &\hat{d}_1 \xi_1^3 + \hat{d}_2 \xi_2^3 + \hat{d}_3 \xi_3^3 + \hat{d}_4 \xi_1^2 \xi_2 + \hat{d}_5 \xi_1 \xi_2^2 + \hat{d}_6 \xi_1^2 \xi_3 + \hat{d}_7 \xi_1 \xi_3^2 + \hat{d}_8 \xi_2^2 \xi_3 + \hat{d}_9 \xi_2 \xi_3^2 + \hat{d}_{10} \xi_1 \xi_2 \xi_3 \end{aligned} \right\}, \quad (A12)$$

where

$$\begin{aligned}
\hat{a}_1 &= \psi_{11}[k_3'(\psi_{11} - \psi_{31})^3 + k_2'(\psi_{11} - \psi_{21})^3 + k_1'\psi_{11}^3] - \psi_{21}k_2'(\psi_{11} - \psi_{21})^3 - \psi_{31}k_3'(\psi_{11} - \psi_{31})^3, \\
\hat{a}_2 &= \psi_{11}[k_3'(\psi_{12} - \psi_{32})^3 + k_2'(\psi_{12} - \psi_{22})^3 + k_1'\psi_{12}^3] - \psi_{21}k_2'(\psi_{12} - \psi_{22})^3 - \psi_{31}k_3'(\psi_{12} - \psi_{32})^3, \\
\hat{a}_3 &= \psi_{11}[k_3'(\psi_{13} - \psi_{33})^3 + k_2'(\psi_{13} - \psi_{23})^3 + k_1'\psi_{13}^3] - \psi_{21}k_2'(\psi_{13} - \psi_{23})^3 - \psi_{31}k_3'(\psi_{13} - \psi_{33})^3, \\
\hat{a}_4 &= 3\psi_{11}[k_3'(\psi_{11} - \psi_{31})^2(\psi_{12} - \psi_{32}) + k_2'(\psi_{11} - \psi_{21})^2(\psi_{12} - \psi_{22}) + k_1'\psi_{11}^2\psi_{12}] - 3\psi_{21}k_2'(\psi_{11} - \psi_{21})^2(\psi_{12} - \psi_{22}) - 3\psi_{31}k_3'(\psi_{11} - \psi_{31})^2(\psi_{12} - \psi_{32}), \\
\hat{a}_5 &= 3\psi_{11}[k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})^2 + k_2'(\psi_{11} - \psi_{21})(\psi_{12} - \psi_{22})^2 + k_1'\psi_{11}\psi_{12}^2] - 3\psi_{21}k_2'(\psi_{11} - \psi_{21})(\psi_{12} - \psi_{22})^2 - 3\psi_{31}k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})^2, \\
\hat{a}_6 &= 3\psi_{11}[k_3'(\psi_{11} - \psi_{31})^2(\psi_{13} - \psi_{33}) + k_2'(\psi_{11} - \psi_{21})^2(\psi_{13} - \psi_{23}) + k_1'\psi_{11}^2\psi_{13}] - 3\psi_{21}k_2'(\psi_{11} - \psi_{21})^2(\psi_{13} - \psi_{23}) - 3\psi_{31}k_3'(\psi_{11} - \psi_{31})^2(\psi_{13} - \psi_{33}), \\
\hat{a}_7 &= 3\psi_{11}[k_3'(\psi_{11} - \psi_{31})(\psi_{13} - \psi_{33})^2 + k_2'(\psi_{11} - \psi_{21})(\psi_{13} - \psi_{23})^2 + k_1'\psi_{11}\psi_{13}^2] - 3\psi_{21}[k_2'(\psi_{11} - \psi_{21})(\psi_{13} - \psi_{23})^2] - 3\psi_{31}[k_3'(\psi_{11} - \psi_{31})(\psi_{13} - \psi_{33})^2], \\
\hat{a}_8 &= 3\psi_{11}[k_3'(\psi_{12} - \psi_{32})^2(\psi_{13} - \psi_{33}) + k_2'(\psi_{12} - \psi_{22})^2(\psi_{13} - \psi_{23}) + k_1'\psi_{12}^2\psi_{13}] - 3\psi_{21}k_2'(\psi_{12} - \psi_{22})^2(\psi_{13} - \psi_{23}) - 3\psi_{31}k_3'(\psi_{12} - \psi_{32})^2(\psi_{13} - \psi_{33}), \\
\hat{a}_9 &= 3\psi_{11}[k_3'(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33})^2 + k_2'(\psi_{12} - \psi_{22})(\psi_{13} - \psi_{23})^2 + k_1'\psi_{12}\psi_{13}^2] - 3\psi_{21}k_2'(\psi_{12} - \psi_{22})(\psi_{13} - \psi_{23})^2 - 3\psi_{31}k_3'(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33})^2, \\
\hat{a}_{10} &= 6\psi_{11}[k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33}) + k_2'(\psi_{11} - \psi_{21})(\psi_{12} - \psi_{22})(\psi_{13} - \psi_{23}) + k_1'\psi_{11}\psi_{12}\psi_{13}] - 6\psi_{21}k_2'(\psi_{11} - \psi_{21})(\psi_{12} - \psi_{22})(\psi_{13} - \psi_{23}) - 6\psi_{31}k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33}), \\
\hat{b}_1 &= \psi_{12}[k_3'(\psi_{11} - \psi_{31})^3 + k_2'(\psi_{11} - \psi_{21})^3 + k_1'\psi_{11}^3] - \psi_{22}k_2'(\psi_{11} - \psi_{21})^3 - \psi_{32}k_3'(\psi_{11} - \psi_{31})^3, \\
\hat{b}_2 &= \psi_{12}[k_3'(\psi_{12} - \psi_{32})^3 + k_2'(\psi_{12} - \psi_{22})^3 + k_1'\psi_{12}^3] - \psi_{22}k_2'(\psi_{12} - \psi_{22})^3 - \psi_{32}k_3'(\psi_{12} - \psi_{32})^3, \\
\hat{b}_3 &= \psi_{12}[k_3'(\psi_{13} - \psi_{33})^3 + k_2'(\psi_{13} - \psi_{23})^3 + k_1'\psi_{13}^3] - \psi_{22}k_2'(\psi_{13} - \psi_{23})^3 - \psi_{32}k_3'(\psi_{13} - \psi_{33})^3, \\
\hat{b}_4 &= 3\psi_{12}[k_3'(\psi_{11} - \psi_{31})^2(\psi_{12} - \psi_{32}) + k_2'(\psi_{11} - \psi_{21})^2(\psi_{12} - \psi_{22}) + k_1'\psi_{11}^2\psi_{12}] - 3\psi_{22}k_2'(\psi_{11} - \psi_{21})^2(\psi_{12} - \psi_{22}) - 3\psi_{32}k_3'(\psi_{11} - \psi_{31})^2(\psi_{12} - \psi_{32}), \\
\hat{b}_5 &= 3\psi_{12}[k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})^2 + k_2'(\psi_{11} - \psi_{21})(\psi_{12} - \psi_{22})^2 + k_1'\psi_{11}\psi_{12}^2] - 3\psi_{22}k_2'(\psi_{11} - \psi_{21})(\psi_{12} - \psi_{22})^2 - 3\psi_{32}k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})^2, \\
\hat{b}_6 &= 3\psi_{12}[k_3'(\psi_{11} - \psi_{31})^2(\psi_{13} - \psi_{33}) + k_2'(\psi_{11} - \psi_{21})^2(\psi_{13} - \psi_{23}) + k_1'\psi_{11}^2\psi_{13}] - 3\psi_{22}k_2'(\psi_{11} - \psi_{21})^2(\psi_{13} - \psi_{23}) - 3\psi_{32}k_3'(\psi_{11} - \psi_{31})^2(\psi_{13} - \psi_{33}), \\
\hat{b}_7 &= 3\psi_{12}[k_3'(\psi_{11} - \psi_{31})(\psi_{13} - \psi_{33})^2 + k_2'(\psi_{11} - \psi_{21})(\psi_{13} - \psi_{23})^2 + k_1'\psi_{11}\psi_{13}^2] - 3\psi_{22}k_2'(\psi_{11} - \psi_{21})(\psi_{13} - \psi_{23})^2 - 3\psi_{32}k_3'(\psi_{11} - \psi_{31})(\psi_{13} - \psi_{33})^2,
\end{aligned}$$

$$\begin{aligned}
\hat{b}_8 &= 3\psi_{12}[k_3'(\psi_{12} - \psi_{32})^2(\psi_{13} - \psi_{33}) + k_2'(\psi_{12} - \psi_{22})^2(\psi_{13} - \psi_{23}) + k_1'\psi_{12}^2\psi_{13}] - 3\psi_{22}k_2'(\psi_{12} - \psi_{22})^2 \\
&\quad (\psi_{13} - \psi_{23}) - 3\psi_{32}k_3'(\psi_{12} - \psi_{32})^2(\psi_{13} - \psi_{33}) \\
\hat{b}_9 &= 3\psi_{12}[k_3'(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33})^2 + k_2'(\psi_{12} - \psi_{22})(\psi_{13} - \psi_{23})^2 + k_1'\psi_{12}\psi_{13}^2] - 3\psi_{22}k_2'(\psi_{12} - \psi_{22}) \\
&\quad (\psi_{13} - \psi_{23})^2 - 3\psi_{32}k_3'(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33})^2 \\
\hat{b}_{10} &= 6\psi_{12}[k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33}) + k_2'(\psi_{11} - \psi_{21})(\psi_{12} - \psi_{22})(\psi_{13} - \psi_{23}) + k_1'\psi_{11}\psi_{12}\psi_{13}] \\
&\quad - 6\psi_{22}k_2'(\psi_{11} - \psi_{21})(\psi_{12} - \psi_{22})(\psi_{13} - \psi_{23}) - 6\psi_{32}k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33}) \\
\hat{d}_1 &= \psi_{13}[k_3'(\psi_{11} - \psi_{31})^3 + k_2'(\psi_{11} - \psi_{21})^3 + k_1'\psi_{11}^3] - \psi_{23}k_2'(\psi_{11} - \psi_{21})^3 - \psi_{33}k_3'(\psi_{11} - \psi_{31})^3 \\
\hat{d}_2 &= \psi_{13}[k_3'(\psi_{12} - \psi_{32})^3 + k_2'(\psi_{12} - \psi_{22})^3 + k_1'\psi_{12}^3] - \psi_{23}k_2'(\psi_{12} - \psi_{22})^3 - \psi_{33}k_3'(\psi_{12} - \psi_{32})^3 \\
\hat{d}_3 &= \psi_{13}[k_3'(\psi_{13} - \psi_{33})^3 + k_2'(\psi_{13} - \psi_{23})^3 + k_1'\psi_{13}^3] - \psi_{23}k_2'(\psi_{13} - \psi_{23})^3 - \psi_{33}k_3'(\psi_{13} - \psi_{33})^3 \\
\hat{d}_4 &= 3\psi_{13}[k_3'(\psi_{11} - \psi_{31})^2(\psi_{12} - \psi_{32}) + k_2'(\psi_{11} - \psi_{21})^2(\psi_{12} - \psi_{22}) + k_1'\psi_{11}^2\psi_{12}] - 3\psi_{23}k_2'(\psi_{11} - \psi_{21})^2 \\
&\quad (\psi_{12} - \psi_{22}) - 3\psi_{33}k_3'(\psi_{11} - \psi_{31})^2(\psi_{12} - \psi_{32}) \\
\hat{d}_5 &= 3\psi_{13}[k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})^2 + k_2'(\psi_{11} - \psi_{21})(\psi_{12} - \psi_{22})^2 + k_1'\psi_{11}\psi_{12}^2] - 3\psi_{23}k_2'(\psi_{11} - \psi_{21}) \\
&\quad (\psi_{12} - \psi_{22})^2 - 3\psi_{33}k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})^2 \\
\hat{d}_6 &= 3\psi_{13}[k_3'(\psi_{11} - \psi_{31})^2(\psi_{13} - \psi_{33}) + k_2'(\psi_{11} - \psi_{21})^2(\psi_{13} - \psi_{23}) + k_1'\psi_{11}^2\psi_{13}] - 3\psi_{23}k_2'(\psi_{11} - \psi_{21})^2 \\
&\quad (\psi_{13} - \psi_{23}) - 3\psi_{33}k_3'(\psi_{11} - \psi_{31})^2(\psi_{13} - \psi_{33}) \\
\hat{d}_7 &= 3\psi_{13}[k_3'(\psi_{11} - \psi_{31})(\psi_{13} - \psi_{33})^2 + k_2'(\psi_{11} - \psi_{21})(\psi_{13} - \psi_{23})^2 + k_1'\psi_{11}\psi_{13}^2] - 3\psi_{23}k_2'(\psi_{11} - \psi_{21}) \\
&\quad (\psi_{13} - \psi_{23})^2 - 3\psi_{33}k_3'(\psi_{11} - \psi_{31})(\psi_{13} - \psi_{33})^2 \\
\hat{d}_8 &= 3\psi_{13}[k_3'(\psi_{12} - \psi_{32})^2(\psi_{13} - \psi_{33}) + k_2'(\psi_{12} - \psi_{22})^2(\psi_{13} - \psi_{23}) + k_1'\psi_{12}^2\psi_{13}] - 3\psi_{23}k_2'(\psi_{12} - \psi_{22})^2 \\
&\quad (\psi_{13} - \psi_{23}) - 3\psi_{33}k_3'(\psi_{12} - \psi_{32})^2(\psi_{13} - \psi_{33}) \\
\hat{d}_9 &= 3\psi_{13}[k_3'(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33})^2 + k_2'(\psi_{12} - \psi_{22})(\psi_{13} - \psi_{23})^2 + k_1'\psi_{12}\psi_{13}^2] - 3\psi_{23}k_2'(\psi_{12} - \psi_{22}) \\
&\quad (\psi_{13} - \psi_{23})^2 - 3\psi_{33}k_3'(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33})^2 \\
\hat{d}_{10} &= 6\psi_{13}[k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33}) + k_2'(\psi_{11} - \psi_{21})(\psi_{12} - \psi_{22})(\psi_{13} - \psi_{23}) + k_1'\psi_{11}\psi_{12}\psi_{13}] \\
&\quad - 6\psi_{23}k_2'(\psi_{11} - \psi_{21})(\psi_{12} - \psi_{22})(\psi_{13} - \psi_{23}) - 6\psi_{33}k_3'(\psi_{11} - \psi_{31})(\psi_{12} - \psi_{32})(\psi_{13} - \psi_{33})
\end{aligned}$$

The regularization of the force F :

$$\boldsymbol{\psi}^T \mathbf{F} = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{bmatrix}^T \begin{Bmatrix} F \cos(\omega t) \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \psi_{11} F \cos(\omega t) \\ \psi_{12} F \cos(\omega t) \\ \psi_{13} F \cos(\omega t) \end{Bmatrix} = \begin{Bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{Bmatrix}. \quad (\text{A13})$$

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