

Article

Dynamic Analysis with Three Beam Theories for a Rotating FGM Micro-Beam Based on Meshless Methods

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Abstract

This paper investigates the dynamic characteristics of rotating functionally graded material (FGM) micro-beams based on Euler–Bernoulli beam theory, Euler–Bernoulli beam theory incorporating shear deformation, and Timoshenko theory. The deformation field of the micro-beam is described within a floating coordinate system using the meshless point interpolation method (PIM/RPIM). The couple stress tensor and curvature tensor, which capture the size effect, are incorporated into the potential energy formulation. Employing Lagrange’s equations of the second kind, a higher-order rigid-flexible coupled dynamic model for rotating FGM micro-beams is established under various beam theories. Simulation results obtained from the Euler–Bernoulli theory with shear correction and the Timoshenko model are compared with those from the classical beam model and previous literature. The influences of material gradient index, material characteristic length parameter, and rotational speed profiles on the transient dynamic response and steady-state free vibration of rotating micro-beams are systematically examined. The results show that increasing the material gradient index reduces the structural stiffness, resulting in lower natural frequencies and larger vibration amplitudes, whereas increasing the characteristic length parameter enhances the size effect and improves system stiffness.

Keywords: Euler beam with shear correction; Timoshenko beam; FGM micro-beam; size effect; rigid-flexible coupled dynamics

1. Introduction

Micro-Electro-Mechanical-System (MEMS) [1] is widely applied in various fields such as micro rotorcraft, micro gas turbine generators, and MEMS accelerometers, where many structures can be simplified as rigid-flexible coupled structures with micro-beams mounted on a rigid body. The core components of MEMS can mostly be simplified to micro-beam units on the micrometer or submicrometer dimension. In practical applications [2,3], the ratio of cantilever beam length to beam height for micro accelerometer designs, ranging from 44 to 100, needs to be smaller than the ratio used in previous studies [4]. To ensure that the dynamic characteristics obtained from simulation results align with real data, it is necessary to establish a high-precision dynamic model that comprehensively considers key factors such as size effect and shear effect. This model also provides

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support for the development of performance optimization and reliability design for MEMS.

Functionally Graded Materials (FGMs) have garnered significant attention since their introduction by researchers in 1900, owing to their notable advantages such as heat resistance, high load-bearing capacity, and improved stress concentration [5]. They have been successfully applied in various fields including aerospace, nuclear industry and biology. Most existing research simplifies to Euler–Bernoulli beam theory [6–9] for beam structures, known as the classical beam theory. However, this theory does not account for shear effects; significant errors in system response and natural frequencies may arise when the beam height is large [10–12]. For situations involving thick beams, Timoshenko beam theory is typically adopted, which no longer assumes that the cross-section remains perpendicular to the beam axis after deformation, and belongs to the first-order shear deformation theory [13–15]. Some scholars have also adopted the Euler beam with shear theory [16,17], which adds lateral deformation caused by shear to the Euler–Bernoulli beam theory. Zou Jianqi et al. [18] were the first to derive precise dynamic equations for rotating Timoshenko beams and studied the effects of different influencing factors on the natural frequencies of the beams. Huang et al. [19] employed a rotating Timoshenko beam model to investigate the free vibration phenomena of axially FGM beams rotating at a constant angular velocity around their longitudinal axis, demonstrating that axial FGM has a significant impact on the rotational frequency, critical speed, and mode shape of the beam. They concluded that selecting appropriate material gradient parameters can achieve higher rotational frequencies and critical speeds than those of homogeneous beams. Zhou et al. [20] studied the three-dimensional dynamic characteristics of rotating FGM cantilever beams through the Timoshenko beam theory, exploring the effects of dimensionless angular velocity, FGM index, and slenderness ratio on structural vibration. Additionally, for high-frequency external excitation, using Chebyshev polynomials as test functions proved to be more stable and effective than other polynomials. Taima et al. [21] established three dynamic models of rotating FGM conical beams using classical beam theory, Timoshenko theory, and Reddy theory, and investigated the effects of key parameters such as material distribution, taper ratio, aspect ratio, hub radius, and rotational speed on the natural frequencies of the beams.

In earlier years, scholars focused more on the research of macroscopic beam dynamics. It was not until recent years that they began to focus on the multi-body dynamic response of FGM and Timoshenko beam theory to micro-structures. Ansari et al. [22] studied the bending, buckling, and free vibration characteristics of Timoshenko micro-beams made of FGM. The critical buckling load and natural frequency predicted by the beam models established based on the most general strain gradient theory and classical theory correspond to the maximum and minimum values; as the length size parameter increases, the deflection curve of the FGM micro-beam gradually approaches the curve predicted by classical theory. Shanab et al. [23] proposed a study on the size-dependent static and dynamic characteristics of FG Timoshenko nanobeam embedded in an elastic medium, presenting selected numerical results to explore the effects of material length size parameters, surface parameters, gradient index, elastic medium, and thickness on the static and dynamic responses of FG-Timoshenko nanobeams. Zhang et al. [24] derived the modified Lagrangian function of functionally graded material perforated cylindrical plates using first-order shear deformation theory and studied their transient dynamic response under rapid thermal shock.

In the field of flexible multi-body system dynamics, research on deformation field discretization methods is both a hot topic and a technical challenge. Currently, the assumed mode method (AMM) [25] and the finite element method (FEM) [26] are the two most widely used discretization methods for describing the deformation field of flexible

bodies. Previous methods have their unique advantages and limitations. Nowadays, with the demand for calculating more complex structures such as micro-structures and requiring more precise results, the meshless method [27] has emerged. This method only requires node information without the need for element division, and has the advantages of high solution accuracy, fast convergence, and convenient post-processing. In recent years, it has been widely applied in studying problems such as linear elasticity, large deformation, and elastoplasticity [28]. However, its practical application in flexible multi-body system dynamics still requires further research. Most existing Timoshenko beam formulations combined with meshless methods are mainly developed for homogeneous or macroscale beam problems. They do not simultaneously include large overall rotation, FGM gradation, microscale size effects, and the comparison of different beam assumptions.

Although significant progress has been made in the modeling and analysis of FGM micro-beams, several research gaps still exist. Previous studies have primarily focused on free vibration, size-dependent effects, and static characteristics of rotating FGM micro-beams. Most rigid-flexible coupling dynamic models for rotating FGM beams are established based on the classical Euler–Bernoulli beam theory, which neglects the influence of shear deformation and may lead to reduced accuracy for thick micro-beams. In addition, systematic comparisons among Euler–Bernoulli beam theory, Euler beam theory with shear deformation, and Timoshenko beam theory under a unified rigid-flexible coupling framework are still lacking.

To address these gaps, this paper develops higher-order rigid-flexible coupled dynamic models of rotating FGM micro-beams using three beam theories within a floating frame of reference. The modified couple stress theory is employed to incorporate size effects, and meshless PIM and RPIM formulations are applied to discretize the deformation field. The study systematically investigates the influences of material gradient index, characteristic length parameters, and rotational motion on the transient responses and free vibration characteristics of FGM micro-beams.

2. Dynamic Model

2.1. Physical Model

The physical model is shown in Figure 1 [29]. The hub rotates around the Z axis in the inertial coordinate system OXY , with a rotational inertia of J_{ch} and a radius of a . The FGM micro-beam is fixedly connected to the hub. A floating coordinate system oxy is established on the micro-beam, where τ represents the applied driving torque, R is the vector from the rigid body's mass center to the floating base point o , ρ_0 is the undeformed position vector of point relative to the floating base, u its deformation displacement, and θ the angle between the ox -axis and the OX -axis. The geometric parameters of the micro-beam are: length L , height h , width b , and the material characteristic length parameter is l .

Considering the two types of FGM micro-beam material compositions shown in Figure 2 [29], the upper and lower surfaces of distribution A are composed of pure ceramic and pure metal materials, respectively, with the material composition gradually transitioning from top to bottom along the thickness direction. Both surfaces of distribution B are made of ceramic materials, transitioning to a metal core in the middle layer along the thickness direction.

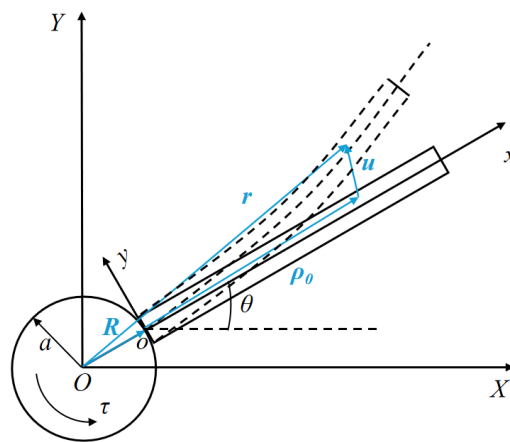


Figure 1. The hub-FGM micro-beam system with rotation motion.

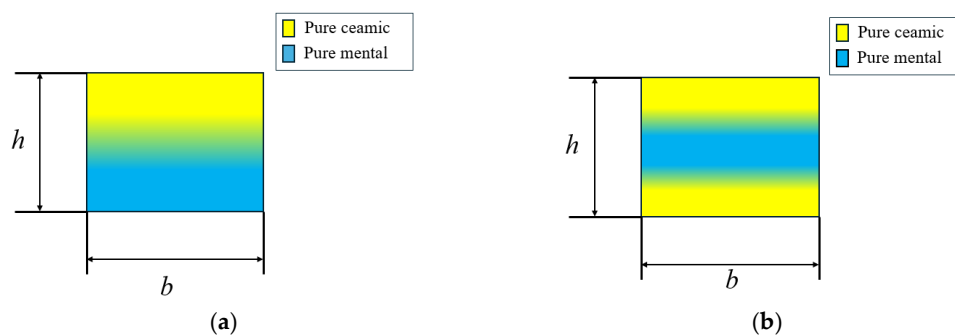


Figure 2. The cross section shape of the FGM micro-beam. (a) Distribution A, (b) Distribution B.

The physical properties of FGM micro-beams are distributed in a gradient manner along the thickness direction according to specific patterns, which are consistent with those reported in the literature, specifically:

Distribution A [30]:

$$\begin{cases} P(y) = (P_c - P_m)V_c(y) + P_m \\ V_c(y) = \left(\frac{2y+h}{2h}\right)^N \\ V_m(y) = 1 - V_c(y) \end{cases}, \quad \left(-\frac{h}{2} \leq y \leq \frac{h}{2}\right) \quad (1)$$

Distribution B [31]:

$$\begin{cases} P(y) = (P_c - P_m)V_c(y) + P_m \\ V_c(y) = \left|\frac{2y}{h}\right|^N \\ V_m(y) = 1 - V_c(y) \end{cases}, \quad \left(-\frac{h}{2} \leq y \leq \frac{h}{2}\right) \quad (2)$$

In the above formula, subscript c corresponds to ceramic, m to metal, $P(y)$ represents the Young's modulus $E(y)$ and density $\rho(y)$ on the FGM micro-beam, while $V_c(y)$ and $V_m(y)$ represent the volume fractions of metal and ceramic materials along the height direction on the FGM micro-beam, respectively. N is the functional gradient index. Furthermore, this paper assumes that the FGM micro-beam is made of Al and Al_2O_3 , and the performance parameters of these two materials are taken from reference [32]: $\rho_c = 2700 \text{ kg/m}^3$, $\rho_m = 3800 \text{ kg/m}^3$, $E_c = 70 \text{ GPa}$, $E_m = 380 \text{ GPa}$.

2.2. System Dynamics Equation

The radius vector $\mathbf{r}$ of any point P on the beam in Figure 1 after deformation is represented as follows:

$$\mathbf{r} = \boldsymbol{\Theta}(R + \boldsymbol{\rho}_0 + \mathbf{u}) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \left[\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_x \\ u_y \end{pmatrix} \right] \quad (3)$$

$\boldsymbol{\Theta}$ is the direction cosine transformation matrix of the coordinate system. The deformation vector $\mathbf{u}$ of any point P on the FGM micro-beam after deformation can be expressed in the floating coordinate system as follows:

$$\begin{cases} u_x^C = w_1 + w_c - y \frac{\partial w_2}{\partial x} \\ u_y^C = w_2 \end{cases} \quad (4)$$

$$\begin{cases} u_x^S = w_1 + w_c - y \frac{\partial w_b}{\partial x} \\ u_y^S = w_b + w_s \end{cases} \quad (5)$$

$$\begin{cases} u_x^T = w_1 + w_c - y\varphi \\ u_y^T = w_2 \end{cases} \quad (6)$$

The superscripts $()^C$, $()^S$, and $()^T$ represent models established based on the Euler–Bernoulli beam theory (referred to as the classical beam theory, abbreviated as the shear-free model), the Euler beam with shear theory (referred to as the shear-considered Euler–Bernoulli theory, abbreviated as the shear model), and the Timoshenko beam theory (abbreviated as the Tim model), respectively. w_1 and w_2 represent the axial and lateral deformations of the FGM micro-beam, respectively. For the Euler beam with shear, w_b represents the lateral deformation caused by bending of the FGM micro-beam, and w_s represents the lateral deformation caused by shear action in the FGM. φ represents the rotation angle in the Timoshenko beam model. w_c represents the second-order nonlinear coupled deformation, and its expression is:

$$w_c^{C,T} = -\frac{1}{2} \int_0^x \left(\frac{\partial w_2}{\partial \xi} \right)^2 d\xi \quad (7)$$

$$w_c^S = -\frac{1}{2} \int_0^x \left(\frac{\partial w_b}{\partial \xi} \right)^2 d\xi \quad (8)$$

To illustrate the differences among these three theories in detail, Figure 3 shows the cross-sections of the three beam theories.

From the deformation radius vector $\mathbf{r}$ and deformation vector $\mathbf{u}$, we can derive the expression for the system's kinetic energy:

$$T_K = \frac{1}{2} \int_V \rho(y) \dot{\mathbf{r}}^T \dot{\mathbf{r}} dV + \frac{1}{2} J_{ch} \dot{\theta}^2 \quad (9)$$

where J_{ch} represents the moment of inertia of the rigid body around the hub of rotation. The kinetic energy consists of two main parts. The first term represents the inertia contribution of the deformable FGM micro-beam, including the translational inertia associated with the flexible deformation and the rigid-flexible coupling motion induced by the overall rotation. The second term denotes the rotational kinetic energy of the rigid hub.

Ignoring the torsion of the beam, according to the modified couple stress theory, the strain potential energy of the system under different beam theories after considering the size effect can be expressed as:

$$U^C = \frac{1}{2} \int_0^L \int_A (\sigma_{ij} \varepsilon_{ij} + m_{ij} \chi_{ij}) dA dx \quad (10)$$

$$U^{S,T} = \frac{1}{2} \int_V \frac{G(y)}{k} \gamma^2 dV + \frac{1}{2} \int_0^L \int_A (\sigma_{ij} \varepsilon_{ij} + m_{ij} \chi_{ij}) dA dx \quad (11)$$

In Equations (10) and (11), the classical strain energy terms describe the elastic energy caused by axial stretching and bending deformation. For the shear model and the Tim model, the additional shear energy term represents the contribution of transverse shear deformation. The curvature-related terms introduced by the modified couple stress theory describe the size-dependent strain energy at the microscale. Where $i, j = x, y, z$. ε_{ij} and σ_{ij} represent the components of the strain tensor and stress tensor, respectively, while χ_{ij} and m_{ij} represent the components of the curvature tensor and the deviatoric components of the couple stress tensor, respectively. The constitutive relationship, as well as the parameters such as Lamé coefficients $\lambda(y)$ and $\mu(y)$, adopted in this paper are consistent with those in reference [33]. Meanwhile, it is assumed that Poisson's ratio ν and material characteristic length parameter l remain constant along the thickness direction. The shear model and

Tim model have an additional term $\frac{1}{2} \int_V \frac{G(y)}{k} \gamma^2 dV$ based on the shear-free model, where $G(y)$ is the shear modulus of the FGM micro-beam and k is the shear correction coefficient, defined as the reciprocal of the conventional shear correction factor. It is introduced to account for the non-uniform distribution of transverse shear stress over the beam cross-section.

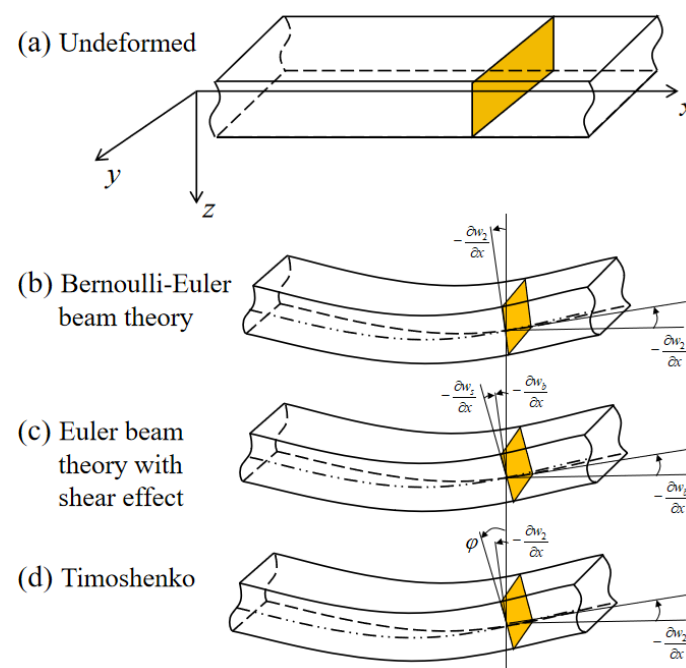


Figure 3. Cross sectional deformation of Euler–Bernoulli beam, Euler beam with shear and Timoshenko beam theory.

The specific expression for shear strain γ is:

$$\gamma = \frac{\partial u_y}{\partial x} - \phi \quad (12)$$

Based on the relevant theoretical formulations of couple stress theory and modified couple stress theory reported in Ref. [34], the non-zero strain components and curvature components for the three beam models can be derived as follows:

$$\begin{aligned}\varepsilon_{xx}^C &= \frac{\partial w_1}{\partial x} - y \frac{\partial^2 w_2}{\partial x^2} \\ \chi_{xz}^C &= \chi_{zx}^C = \frac{1}{2} \frac{\partial^2 w_2}{\partial x^2}\end{aligned}\quad (13)$$

$$\begin{aligned}\varepsilon_{xx}^S &= \frac{\partial w_1}{\partial x} - y \frac{\partial^2 w_b}{\partial x^2} \\ \varepsilon_{xy}^S &= \varepsilon_{yx}^S = \frac{1}{2} \frac{\partial w_s}{\partial x} \\ \chi_{xz}^S &= \chi_{zx}^S = \frac{1}{2} \left(\frac{\partial^2 w_b}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w_s}{\partial x^2} \right)\end{aligned}\quad (14)$$

$$\begin{aligned}\varepsilon_{xx}^T &= \frac{\partial w_1}{\partial x} - y \frac{\partial \varphi}{\partial x} \\ \varepsilon_{xy}^T &= \varepsilon_{yx}^T = \frac{1}{2} \left(\frac{\partial w_2}{\partial x} - \varphi \right) \\ \chi_{xz}^T &= \chi_{zx}^T = \frac{1}{4} \left(\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial \varphi}{\partial x} \right)\end{aligned}\quad (15)$$

According to the von-Kármán nonlinear strain-displacement theory, the stress components and the non-zero components of the higher-order stress tensor, the couple stress, can be obtained as follows:

$$\begin{aligned}\sigma_{xx}^C &= E(y) \varepsilon_{xx}^C = E(y) \left(\frac{\partial w_1}{\partial x} - y \frac{\partial^2 w_2}{\partial x^2} \right) \\ m_{xz}^C &= m_{zx}^C = \mu(y) l^2 \frac{\partial^2 w_2}{\partial x^2}\end{aligned}\quad (16)$$

$$\begin{aligned}\sigma_{xx}^S &= E(y) \varepsilon_{xx}^S = E(y) \left(\frac{\partial w_1}{\partial x} - y \frac{\partial^2 w_b}{\partial x^2} \right) \\ \sigma_{xy}^S &= \sigma_{yx}^S = E(y) \varepsilon_{xy}^S = E(y) \varepsilon_{yx}^S = E(y) \frac{1}{2} \frac{\partial w_s}{\partial x} \\ m_{xz}^S &= m_{zx}^S = \mu(y) l^2 \left(\frac{\partial^2 w_b}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w_s}{\partial x^2} \right)\end{aligned}\quad (17)$$

$$\begin{aligned}\sigma_{xx}^T &= E(y) \varepsilon_{xx}^T = E(y) \left(\frac{\partial w_1}{\partial x} - y \frac{\partial \varphi}{\partial x} \right) \\ \sigma_{xy}^T &= \sigma_{yx}^T = E(y) \varepsilon_{xy}^T = E(y) \varepsilon_{yx}^T = E(y) \frac{1}{2} \left(\frac{\partial w_2}{\partial x} - \varphi \right) \\ m_{xz}^T &= m_{zx}^T = \mu(y) l^2 \frac{1}{2} \left(\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial \varphi}{\partial x} \right)\end{aligned}\quad (18)$$

Further derive the expression for potential energy U :

$$\begin{aligned}U^C &= \frac{1}{2} \int_0^L \int_A \left(\sigma_{xx}^C \varepsilon_{xx}^C + m_{xz}^C \chi_{xz}^C + m_{zx}^C \chi_{zx}^C \right) dA dx \\ &= \frac{1}{2} \int_0^L E_0 b \left(\frac{\partial w_1}{\partial x} \right)^2 dx - \int_0^L E_1 b \frac{\partial w_1}{\partial x} \frac{\partial^2 w_2}{\partial x^2} dx + \frac{1}{2} \int_0^L \left[E_2 + E_0 \frac{l^2}{2(1+\nu)} \right] b \left(\frac{\partial^2 w_2}{\partial x^2} \right)^2 dx\end{aligned}\quad (19)$$

$$\begin{aligned}
U^S &= \frac{1}{2} \int_0^L \int_A (\sigma_{xx}^S \varepsilon_{xx}^S + \sigma_{xy}^S \varepsilon_{xy}^S + \sigma_{yx}^S \varepsilon_{yx}^S + m_{xz}^S \chi_{xz}^S + m_{zx}^S \chi_{zx}^S) dA dx + \frac{1}{2} \int_V \frac{G(y)}{k} \gamma^2 dV \\
&= \frac{1}{2} \int_V \frac{G(y)}{k} \left(\frac{\partial w_s}{\partial x} \right)^2 dV + \frac{1}{2} \int_0^L E_0 b \left(\frac{\partial w_1}{\partial x} \right)^2 dx - \int_0^L E_1 b \frac{\partial w_1}{\partial x} \frac{\partial^2 w_b}{\partial x^2} dx \\
&\quad + \frac{1}{2} \int_0^L \left[E_2 + E_0 \frac{l^2}{2(1+\nu)} \right] b \left(\frac{\partial^2 w_b}{\partial x^2} \right)^2 dx + \frac{1}{4} \int_0^L E_0 b \left(\frac{\partial w_s}{\partial x} \right)^2 dx \\
&\quad + \frac{1}{2} \int_0^L E_0 b \frac{l^2}{2(1+\nu)} \left[\frac{\partial^2 w_b}{\partial x^2} \frac{\partial^2 w_s}{\partial x^2} + \frac{1}{4} \left(\frac{\partial^2 w_s}{\partial x^2} \right)^2 \right] dx
\end{aligned} \tag{20}$$

$$\begin{aligned}
U^T &= \frac{1}{2} \int_0^L \int_A (\sigma_{xx}^T \varepsilon_{xx}^T + \sigma_{xy}^T \varepsilon_{xy}^T + \sigma_{yx}^T \varepsilon_{yx}^T + m_{xz}^T \chi_{xz}^T + m_{zx}^T \chi_{zx}^T) dA dx + \frac{1}{2} \int_V \frac{G(y)}{k} \gamma^2 dV \\
&= \frac{1}{2} \int_V \frac{G(y)}{k} \left(\frac{\partial w_2}{\partial x} - \varphi \right)^2 dV + \frac{1}{2} \int_0^L E_0 b \left(\frac{\partial w_1}{\partial x} \right)^2 dx - \int_0^L E_1 b \frac{\partial w_1}{\partial x} \frac{\partial \varphi}{\partial x} dx + \frac{1}{2} \int_0^L E_2 b \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \\
&\quad + \frac{1}{4} \int_0^L E_0 b \left[\left(\frac{\partial w_2}{\partial x} \right)^2 - 2\varphi \frac{\partial w_2}{\partial x} + \varphi^2 \right] dx + \frac{1}{8} \int_0^L E_0 b \frac{l^2}{2(1+\nu)} \left[\left(\frac{\partial w_2}{\partial x} \right)^2 + 2 \frac{\partial^2 w_2}{\partial x^2} \frac{\partial \varphi}{\partial x} + \left(\frac{\partial \varphi}{\partial x} \right)^2 \right] dx
\end{aligned} \tag{21}$$

Based on the three beam theories, the axial and lateral displacement functions are set as follows:

$$C: \begin{cases} w_1(x, t) = \Phi_x(x) A(t) \\ w_2(x, t) = \Phi_y(x) B(t) \end{cases} \tag{22}$$

$$S: \begin{cases} w_1(x, t) = \Phi_1(x) A(t) \\ w_b(x, t) = \Phi_2(x) B(t) \\ w_s(x, t) = \Phi_3(x) C(t) \end{cases} \tag{23}$$

$$T: \begin{cases} w_1(x, t) = \Phi_1(x) A(t) \\ w_2(x, t) = \Phi_2(x) B(t) \\ \varphi(x, t) = \Phi_3(x) C(t) \end{cases} \tag{24}$$

In the formula, $\Phi_x(x)$ and $\Phi_y(x)$ in the shear-free model correspond to the shape functions of axial and lateral displacement; $\Phi_1(x)$, $\Phi_2(x)$, and $\Phi_3(x)$ in the shear model are the shape functions of axial and lateral displacement in sequence; $\Phi_1(x)$, $\Phi_2(x)$, and $\Phi_3(x)$ in the Tim model are the shape functions of axial, lateral, and angle deformation, respectively. PIM and RPIM are both used for the spatial discretization of the deformation field. After the displacement approximation is introduced, the final dynamic equations keep the same generalized matrix form. The specific expressions of the shape functions for different discretization methods are detailed in the literature [35], which will not be repeated in this paper. In addition, $A(t)$, $B(t)$, and $C(t)$ are the column vectors of nodes varying with time.

The second-order nonlinear coupling deformation caused by lateral deformation can be expressed as:

$$w_c = -\frac{1}{2} \mathbf{B}^T \mathbf{H}(x) \mathbf{B} \tag{25}$$

In the formula, $\mathbf{H}(x)$ represents the coupling shape function, and its expression is as follows:

$$\mathbf{H}^C(x) = \int_0^x \Phi_y'^T(\xi) \Phi_y'(\xi) d\xi \tag{26}$$

$$\mathbf{H}^{S,T}(x) = \int_0^x \Phi_2'^T(\xi) \Phi_2'(\xi) d\xi \tag{27}$$

Among them, $\Phi'_y(\xi)$ represents the first derivative of $\Phi_y(\xi)$ with respect to ξ , $\Phi'_2(\xi)$ represents the first derivative of $\Phi_2(\xi)$ with respect to ξ .

Substituting the above expression into the expressions for u_x and u_y , we obtain:

$$\begin{cases} u_x^C = \Phi_x A - \frac{1}{2} B^T H(x) B - y \Phi'_y B \\ u_y^C = \Phi_y B \end{cases} \quad (28)$$

$$\begin{cases} u_x^S = \Phi_1 A - \frac{1}{2} B^T H(x) B - y \Phi'_2 B \\ u_y^S = \Phi_2 B + \Phi_3 C \end{cases} \quad (29)$$

$$\begin{cases} u_x^T = \Phi_1 A - \frac{1}{2} B^T H(x) B - y \Phi'_3 C \\ u_y^T = \Phi_2 B \end{cases} \quad (30)$$

Taking the first derivative of Equations (28)–(30) with respect to time t , we can obtain the corresponding velocity expressions:

$$\begin{cases} \dot{u}_x^C = \Phi_x \dot{A} - B^T H(x) \dot{B} - y \Phi'_y \dot{B} \\ \dot{u}_y^C = \Phi_y \dot{B} \end{cases} \quad (31)$$

$$\begin{cases} \dot{u}_x^S = \Phi_1 \dot{A} - B^T H(x) \dot{B} - y \Phi'_2 \dot{B} \\ \dot{u}_y^S = \Phi_2 \dot{B} + \Phi_3 \dot{C} \end{cases} \quad (32)$$

$$\begin{cases} \dot{u}_x^T = \Phi_1 \dot{A} - B^T H(x) \dot{B} - y \Phi'_3 \dot{C} \\ \dot{u}_y^T = \Phi_2 \dot{B} \end{cases} \quad (33)$$

Substitute the expressions for u_x , u_y , and $\dot{u}_x$, $\dot{u}_y$ into the system kinetic energy T_k ; and substitute the expressions for w_1 and w_2 into the potential energy U of the FGM micro-beam. Through complex derivation, the kinetic and potential energies of the FGM micro-beam are obtained.

Taking the generalized coordinates $q = (\theta, A^T, B^T)^T$ and generalized forces $F_q = (\tau, 0, 0)^T$ of the shear-free model, as well as the generalized coordinates $q = (\theta, A^T, B^T, C^T)^T$ and generalized forces $F_q = (\tau, 0, 0, 0)^T$ of the shear model and Tim model, we can substitute them into the second kind of Lagrange's equation to obtain:

$$\frac{d}{dt} \left(\frac{\partial T_K}{\partial \dot{q}} \right) - \frac{\partial T_K}{\partial q} = - \frac{\partial U}{\partial q} + F_q \quad (34)$$

After complex derivation, the dynamic equations of the hub-FGM micro-beam system under different beam theories can be obtained. The specific expressions are provided in Appendices A–C.

2.3. Static Mechanics Verification

Taking the collection of concentrated forces at the free end of the cantilever beam in Figure 4 as an example, the accuracy of the shear model and Tim model established in this paper is verified, as well as the corresponding discrete method shape function. The parameters of the cantilever beam are: beam length $L = 1$ m, elastic modulus $E = 7.0 \times 10^{10}$ Pa, beam cross-sectional dimensions $b = 0.02$ m, $h = 0.02$ m, Poisson's ratio $\mu = 0.3$, and shear correction factor $k = 6/5$.

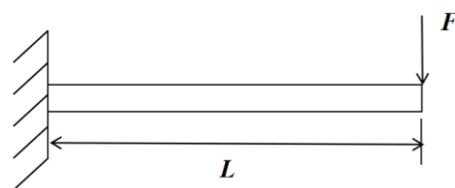
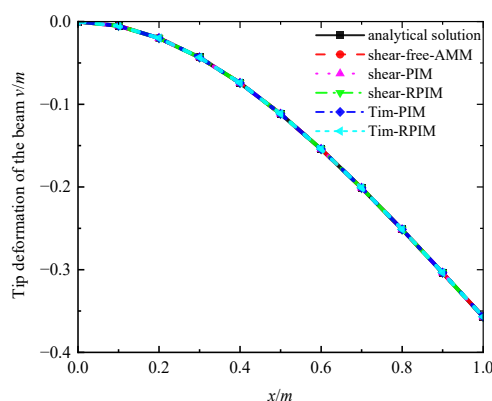


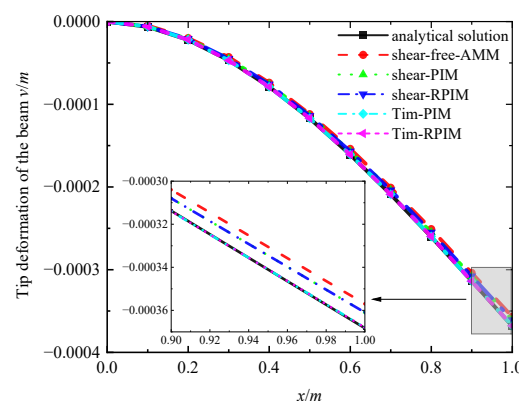
Figure 4. Schematic diagram of concentrated force.

In Figure 4, the closed-form analytical solutions derived from classical beam theory are adopted as benchmark solutions to verify the accuracy of the proposed model. The AMM solution for the Euler–Bernoulli beam model is obtained based on the assumed admissible functions reported in Ref. [31]. AMM is used for the Euler–Bernoulli model because its modal function is based on the Euler–Bernoulli beam. For the shear model and the Tim model, the displacement field contains additional shear-related variables. Therefore, PIM and RPIM are adopted to provide a more flexible spatial approximation.

Figure 5 shows the deflection values at various points on the cantilever beam. To quantify the validation accuracy, the relative error is calculated by comparing the present numerical result with the analytical solution. The error is used to evaluate the agreement between the present formulation and the benchmark solution. The Euler–Bernoulli beam model employs AMM, with a modal order of 4; the Euler beam with shear model and the Timoshenko beam model utilize PIM and RPIM, respectively, with 11 nodes each. The beam adopts a uniform node arrangement in the axial direction. As can be seen in Figure 5a, when the beam height $h = 0.02$, it is considered a slender beam, and the effects of shear can be neglected when considering or not. However, in Figure 5b, where the beam height is $h = 0.2$, it is observed that the results of the shear-free model are smaller than those of the shear model, which are in turn smaller than those of the Tim model. Only the results of the Timoshenko beam model align well with the analytical solution, indicating that the influence of shear cannot be ignored and that the Timoshenko beam theory best fits the actual situation.



(a)



(b)

Figure 5. Deflection of each point in the cantilever beam. (a) $h = 0.02$ m, (b) $h = 0.2$ m.

Figure 6 illustrates that when analyzing the response of beam ends using the Timoshenko beam theory, a shear locking phenomenon occurs, where the deflection obtained from the Timoshenko beam model tends towards zero. In this paper, the shear locking issue is avoided by constructing high-order shape functions. When $h = 0.02$ m, the results obtained from the Euler–Bernoulli beam model, the Euler beam with shear model, and the Timoshenko beam model are well coupled with the analytical solutions.

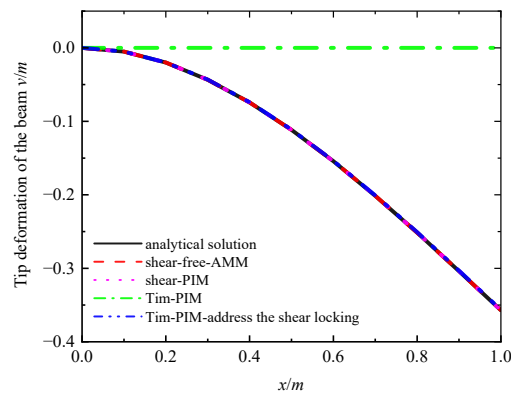


Figure 6. Deflection of each point in the cantilever beam.

3. Natural Frequency Analysis of Lateral Bending

Assuming that the large overall rotation is uniform, the natural frequency of a flexible rotating cantilever beam is primarily dominated by lateral bending vibration. Considering only the lateral vibration of the beam, its free vibration equation is as follows:

$$C : M_{33} \ddot{\mathbf{B}} + \left[\theta^2 (aC_0 + \mathbf{D} - M_{33}) + \mathbf{K}_2 + \frac{l^2}{2(1+\nu)} \mathbf{K}_3 \right] \mathbf{B} = 0 \quad (35)$$

$$S : M_{33} \ddot{\mathbf{B}} + \left[\theta^2 (aD_0 + \mathbf{D}_2 - M_{33}) + \mathbf{K}_2 + \frac{l^2}{2(1+\nu)} \mathbf{K}_3 \right] \mathbf{B} = 0 \quad (36)$$

$$T : M_{33} \ddot{\mathbf{B}} + \left[\theta^2 (aD_0 + \mathbf{D}_2 - M_{33}) + \mathbf{K} \mathbf{K}_0 + \frac{1}{2} \mathbf{K}_2 + \frac{1}{4} \frac{l^2}{2(1+\nu)} \mathbf{K}_4 \right] \mathbf{B} = 0 \quad (37)$$

Normalize the above equation and define the following dimensionless variables:

$$\varsigma = \frac{t}{T}, \quad \zeta = \frac{x}{L}, \quad \delta = \frac{a}{L}, \quad \alpha = \frac{h}{L}, \quad \beta = \frac{h}{l} \quad (38)$$

$$\gamma = T \dot{\theta}, \quad \kappa = \frac{\mathbf{B}}{L}, \quad \phi = \frac{E_c}{E_m}, \quad \vartheta = \frac{\rho_c}{\rho_m} \quad (39)$$

where $T = (12\rho_m L^4 / E_c h^2)^{1/2}$, δ is the dimensionless ratio of the hub radius, β is the dimensionless length parameter, and γ is the dimensionless rotational angular velocity. Substituting the above dimensionless variables into the free vibration equation of micro-beams, the equation can be rewritten as:

$$C : \bar{M}_{33} \ddot{\kappa} + \left[\gamma^2 (\bar{C}_0 + \bar{\mathbf{D}} - \bar{M}_{33}) + \left(\varepsilon + \frac{\eta}{2(1+\nu)\beta^2} \right) \bar{\mathbf{K}}_2 \right] \kappa = 0 \quad (40)$$

$$S : \bar{M}_{33} \ddot{\kappa} + \left[\gamma^2 (\bar{D}_0 + \bar{\mathbf{D}}_2 - \bar{M}_{33}) + \left(\varepsilon + \frac{\eta}{2(1+\nu)\beta^2} \right) \bar{\mathbf{K}}_2 \right] \kappa = 0 \quad (41)$$

$$T : \bar{M}_{33} \ddot{\kappa} + \left[\gamma^2 (\bar{D}_0 + \bar{\mathbf{D}}_2 - \bar{M}_{33}) + \frac{\eta}{2\alpha^2} \frac{1+(1+\nu)k}{(1+\nu)k} \bar{\mathbf{K}}_2 + \frac{\eta}{8(1+\nu)\beta^2} \bar{\mathbf{K}}_4 \right] \kappa = 0 \quad (42)$$

In which:

$$\bar{M}_{33} = \int_0^1 \Phi^T(\zeta) \Phi(\zeta) d\zeta \quad (43)$$

$$\bar{\mathbf{K}}_2 = \int_0^1 \Phi^{*T}(\zeta) \Phi''(\zeta) d\zeta \quad (44)$$

$$\bar{C}_0 = \bar{D}_0 = \int_0^1 \delta H(\zeta) d\zeta \quad (45)$$

$$\bar{D} = \bar{D}_2 = \int_0^1 \zeta H(\zeta) d\zeta \quad (46)$$

$$\bar{H}(\zeta) = \mathbf{R}^{iT} \int_0^\zeta \boldsymbol{\Phi}^{i'T}(\xi) \boldsymbol{\Phi}^{i'}(\xi) d\xi \mathbf{R}^i \quad (47)$$

$$\eta = \frac{12(\mathcal{G} + N)}{\mathcal{G} + N} \quad (48)$$

$$\varepsilon = \begin{cases} \frac{3(N^2 + N + 2)(\phi - 1)}{(N + 2)(N + 3)(\mathcal{G} + N)} + \frac{N + 1}{N + \mathcal{G}} & \text{Distribution A} \\ \frac{(N + 1)(3\phi + N)}{(N + 3)(\mathcal{G} + N)} & \text{Distribution B} \end{cases} \quad (49)$$

Equations (48) and (49) are introduced to provide a unified representation of the dimensionless coefficients appearing in the vibration equations of the three beam models. They are not additional governing equations, but intermediate coefficient definitions obtained after substituting the material-property distribution functions into the normalized formulation.

Solving Equations (40)–(42) obtains the characteristic equation:

$$C : \omega^2 \bar{\mathbf{M}}_{33} \boldsymbol{\psi} = \left[\gamma^2 (\bar{\mathbf{C}}_0 + \bar{\mathbf{D}} - \bar{\mathbf{M}}_{33}) + \left(\varepsilon + \frac{\eta}{2(1+\nu)\beta^2} \right) \bar{\mathbf{K}}_2 \right] \boldsymbol{\psi} \quad (50)$$

$$S : \omega^2 \bar{\mathbf{M}}_{33} \boldsymbol{\psi} = \left[\gamma^2 (\bar{\mathbf{D}}_0 + \bar{\mathbf{D}}_2 - \bar{\mathbf{M}}_{33}) + \left(\varepsilon + \frac{\eta}{2(1+\nu)\beta^2} \right) \bar{\mathbf{K}}_2 \right] \boldsymbol{\psi} \quad (51)$$

$$T : \omega^2 \bar{\mathbf{M}}_{33} \boldsymbol{\psi} = \left[\gamma^2 (\bar{\mathbf{D}}_0 + \bar{\mathbf{D}}_2 - \bar{\mathbf{M}}_{33}) + \frac{\eta}{2\alpha^2} \frac{1 + (1+\nu)k}{(1+\nu)k} \bar{\mathbf{K}}_2 + \frac{\eta}{8(1+\nu)\beta^2} \bar{\mathbf{K}}_4 \right] \boldsymbol{\psi} \quad (52)$$

where ω is the dimensionless natural frequency, and $\boldsymbol{\psi}$ is the eigenvector matrix. By solving this equation, the obtained results are the dimensionless natural frequencies of the lateral bending vibration of the rotating FGM flexible micro-beam.

To verify the correctness of the shear model and Tim model constructed in this paper, the models were degenerated into macro-beam models, and their simulation results were compared with those from reference [34]. Table 1 presents the first three dimensionless natural frequencies when the size effect is not considered, the PIM is used, and the relevant parameters are $\gamma = 10$, $\delta = 0$, $\alpha = 0.004$, $\phi = 151/70$, $\varphi = 3000/2707$, $N = 0.5$, and $k = 6/5$. From Table 1, it can be seen that when the size effect is not considered, the simulation results obtained by the present model are in good agreement with those of the rotating FGM flexible beam under macro-scale conditions reported in Ref. [34], which verifies the correctness of the proposed formulation. Meanwhile, the natural frequencies predicted by the Euler beam with shear effect model and the Timoshenko beam model are generally lower than those predicted by the Euler–Bernoulli beam model. This is because the shear-deformable models introduce additional transverse shear flexibility, which reduces the effective stiffness of the beam. Therefore, the Euler–Bernoulli model may overestimate the natural frequencies when shear deformation is non-negligible.

Table 1. First three dimensionless natural frequencies of beam system.

	ω	Shear-Free Model	Shear Model	Tim Model	Ref. [34]
Distribution A	ω_1	5.8809	5.8803	5.8344	5.901
	ω_2	36.4620	36.4605	36.3048	36.519
	ω_3	88.3645	88.3578	88.3509	88.428
	ω_1	6.1688	6.1682	6.1685	6.186

Distribution	ω_2	38.0717	38.0699	38.0713	38.119
B	ω_3	93.5376	93.5297	93.5260	93.587

Figure 7 depicts the curves comparing the first three dimensionless natural frequencies of the Euler–Bernoulli beam model, the Euler beam with shear model, and the Timoshenko beam model as a function of the dimensionless angular velocity γ , using the meshless method with $\alpha = 0.002$, $\phi = 38/7$, $\varphi = 38/27$, $N = 0.5$, $\beta = 1$, $\delta = 0.2$, and $k = 6/5$. It can be observed from the figure that the results of the several models align well. The parameters chosen for the three models in this case still pertain to slender beams. In the enlarged view, the results of the Euler–Bernoulli beam are slightly higher than those of the Euler beam with shear model and the Timoshenko beam model, indicating that the shear effect is not significant in slender beams.

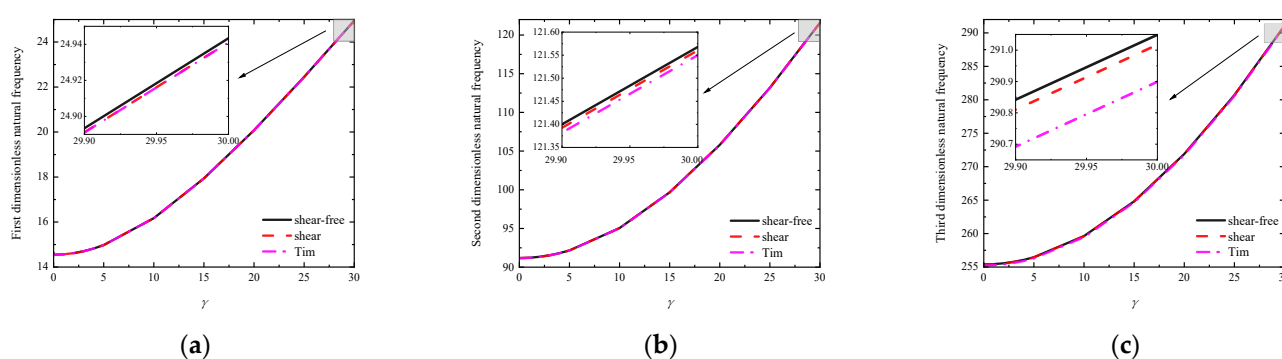


Figure 7. The first three dimensionless natural frequencies of the micro-beam system as angular velocity γ under distribution A. (a) First dimensionless natural frequency, (b) Second dimensionless natural frequency, (c) Third dimensionless natural frequency.

Table 2 presents the first three dimensionless natural frequencies of the Euler beam with shear deformation under different numbers of meshless nodes using the meshless discretization method. It can be observed that, as the number of nodes increases from 9 to 17, the first three dimensionless natural frequencies change only slightly. This indicates that the influence of further nodal refinement on the calculated natural frequencies becomes negligible. The small variation in the frequency results demonstrates that the adopted meshless discretization can provide stable and convergent predictions for the natural frequencies of the rotating FGM micro-beam. Therefore, the number of nodes used in the subsequent simulations is considered sufficient to ensure numerical accuracy while maintaining computational efficiency.

Table 2. The first three dimensionless natural frequencies of the Euler beam with shear deformation under different numbers of meshless nodes.

Number of Nodes	9	11	13	15	17
First dimensionless natural frequency	18.7434	18.7106	18.6926	18.6818	18.6747
Second dimensionless natural frequency	82.6873	82.6208	82.5842	82.5619	82.5474
Third dimensionless natural frequency	86.1165	86.0263	85.9773	85.9477	85.9284

The increase in the dimensionless angular velocity enhances the centrifugal stiffening effect of the rotating micro-beam. As the rotational speed increases, the equivalent axial tensile effect induced by rotation becomes stronger, which increases the effective stiffness of the system and consequently raises the natural frequencies. Therefore, the upward trend of the frequency curves is physically consistent with the rigid-flexible coupling characteristics of rotating beams.

Tables 3–5 present the first three dimensionless natural frequencies of the system as the rotational speed varies for different beam length-to-height ratios α . The meshless radial basis function interpolation method is employed, with $\phi = 38/7$, $\varphi = 38/27$, $N = 0.5$, $\beta = 1$, $\delta = 0.4$, and $k = 6/5$. As evident from the tables, when the beam length-to-height ratio α remains consistent, the first three dimensionless natural frequencies of both the shear model and the Tim model for FGM micro-beams are lower than those of the shear-free model. As the beam length-to-height ratio $1/\alpha$ decreases, the difference between the natural frequencies of FGM micro-beams with and without shear effects increases, indicating that the Euler–Bernoulli beam model neglecting shear is no longer suitable for calculating squat beams. Both the shear model and the Tim model are applicable for the calculation of thick beams. For beams with a large length-to-height ratio, the structure behaves as a slender beam, and the transverse shear deformation is weak. In this range, the predictions of the Euler–Bernoulli model, the Euler beam with shear effect model, and the Timoshenko beam model are close to each other, indicating that the Euler–Bernoulli assumption is still acceptable. However, as the length-to-height ratio decreases, the beam becomes relatively thick, and the shear deformation effect becomes more pronounced. In this case, the Euler–Bernoulli model tends to overestimate the natural frequencies because it neglects shear flexibility.

Table 3. The first dimensionless natural frequencies of transverse vibration at different speeds.

	γ	$1/\alpha = 500$			$1/\alpha = 100$			$1/\alpha = 50$		
		Shear-Free	Shear	Tim	Shear-Free	Shear	Tim	Shear-Free	Shear	Tim
Distribution A	1	14.5735	14.5734	14.5731	14.5732	14.5713	14.5711	14.5724	14.5647	14.5648
	2	14.6578	14.6577	14.6575	14.6575	14.6555	14.6552	14.6566	14.6486	14.6478
	5	15.2327	15.2326	15.2323	15.2324	15.2298	15.2290	15.2314	15.2208	15.2206
	10	17.1071	17.1069	17.1065	17.1067	17.1011	17.1003	17.1054	17.0832	17.0828
	20	22.9369	22.9359	22.9357	22.9362	22.9129	22.9119	22.9340	22.8412	22.8409
	30	29.9277	29.9250	29.9250	29.9267	29.8591	29.8579	29.9237	29.6542	29.6541
Distribution B	1	14.9930	14.9929	14.9918	14.9927	14.9906	14.9897	14.9918	14.9835	14.9832
	2	15.0749	15.0749	15.0738	15.0746	15.0725	15.0715	15.0737	15.0650	15.0648
	5	15.6347	15.6346	15.6335	15.6343	15.6315	15.6306	15.6333	15.6219	15.6214
	10	17.4673	17.4671	17.4661	17.4669	17.4611	17.4604	17.4655	17.4424	17.4422
	20	23.2182	23.2172	23.2166	23.2174	23.1938	23.1932	23.2152	23.1209	23.1202
	30	30.1644	30.1617	30.1615	30.1634	30.0953	30.0947	30.1602	29.8888	29.8889

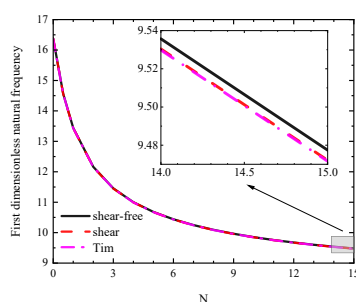
Table 4. The second dimensionless natural frequencies of transverse vibration at different speeds.

	γ	$1/\alpha = 500$			$1/\alpha = 100$			$1/\alpha = 50$		
		Shear-Free	Shear	Tim	Shear-Free	Shear	Tim	Shear-Free	Shear	Tim
Distribution A	1	91.2178	91.2145	91.2067	91.2058	91.1242	91.1156	91.1682	90.8435	90.8304
	2	91.3646	91.3614	91.3536	91.3526	91.2706	91.2621	91.3150	90.9887	90.9753
	5	92.3860	92.3826	92.3750	92.3737	92.2891	92.2767	92.3355	91.9987	91.9877
	10	95.9434	95.9396	95.9327	95.9306	95.8362	95.8178	95.8903	95.5151	95.4778
	20	108.9921	108.9866	108.9815	108.9768	108.8407	108.7440	108.9293	108.3889	108.0343
	30	127.7152	127.7065	127.7031	127.6966	127.4815	127.1485	127.6384	126.7866	125.4696
Distribution B	1	93.8380	93.8344	93.8341	93.8251	93.7362	93.7361	93.7848	93.4313	93.4292
	2	93.9807	93.9772	93.9769	93.9678	93.8785	93.8782	93.9274	93.5724	93.5619
	5	94.9739	94.9702	94.9700	94.9607	94.8688	94.8677	94.9198	94.5540	94.5504
	10	98.4378	98.4337	98.4334	98.4240	98.3221	98.3155	98.3810	97.9757	97.9500
	20	111.1955	111.1897	111.1888	111.1793	111.0346	110.9499	111.1288	110.5546	110.2088
	30	129.6082	129.5991	129.5954	129.5885	129.3634	129.0381	129.5272	128.6360	127.3310

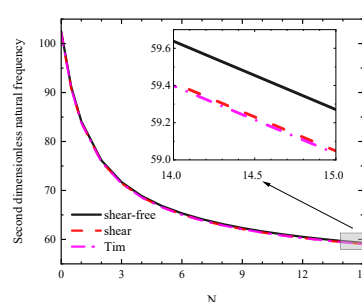
Table 5. The third dimensionless natural frequencies of transverse vibration at different speeds.

	γ	$1/\alpha = 500$			$1/\alpha = 100$			$1/\alpha = 50$		
		Shear-Free	Shear	Tim	Shear-Free	Shear	Tim	Shear-Free	Shear	Tim
Distribution A	1	255.4378	255.4166	255.3342	255.3578	254.8291	254.7266	255.1078	253.0208	252.8235
	2	255.5954	255.5741	255.4918	255.5153	254.9856	254.8834	255.2651	253.1743	252.9817
	5	256.6952	256.6737	256.5921	256.6147	256.0780	255.9651	256.3632	254.2451	254.0575
	10	260.5801	260.5576	260.4786	260.4982	259.9364	259.8049	260.2421	258.0258	257.7522
	20	275.4994	275.4727	275.4027	275.4117	274.7463	274.3545	275.1378	272.5160	271.2074
	30	298.4919	298.4577	298.3984	298.3952	297.5457	296.2750	298.0930	294.7538	289.8700
Distribution B	1	262.7229	262.6998	262.6934	262.6367	262.0615	262.0362	262.3679	260.0994	259.9840
	2	262.8760	262.8529	262.8466	262.7898	262.2136	262.1876	262.5208	260.2483	260.1098
	5	263.9455	263.9221	263.9163	263.8588	263.2755	263.2475	263.5885	261.2881	261.1745
	10	267.7254	267.7009	267.6949	267.6372	267.0282	266.9828	267.3623	264.9610	264.7770
	20	282.2715	282.2427	282.2351	282.1775	281.4621	281.1563	281.8845	279.0678	277.8307
	30	304.7702	304.7339	304.7162	304.6671	303.7634	302.5675	304.3454	300.7958	296.0411

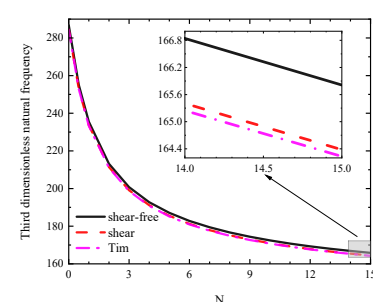
Figure 8 depicts the variation in the first three dimensionless natural frequencies of different beam models with the functional gradient index N , using the meshless method, under the conditions of $\gamma = 1$, $\phi = 38/7$, $\varphi = 38/27$, $\delta = 0.2$, $\alpha = 0.02$, and $\beta = 1$. It can be observed from the figure that the natural frequency of the system decreases as the functional gradient index N increases. This is because as N increases, the proportion of metal in the two distributed material compositions increases, leading to an increase in the system's flexibility and thus a decrease in natural frequency. Comparing distributions A and B, when $N = 0$, the system consists of pure ceramic material, and the first three dimensionless natural frequencies for both distributions are the same. However, as N increases, the decrease rate for distribution A is faster than that for distribution B, due to its higher proportion of metal components. It is also evident from the figure that the natural frequency of the Euler–Bernoulli beam model is always higher than that of the other two beam models. The influence of shear on the first three dimensionless natural frequencies becomes more pronounced, making the shear effect in this FGM micro-beam system non-negligible. Simultaneously, it demonstrates the superiority and wide applicability of the Euler beam with shear model and Timoshenko beam model established in this paper.



(a) First dimensionless natural frequency



(b) Second dimensionless natural frequency



(c) Third dimensionless natural frequency

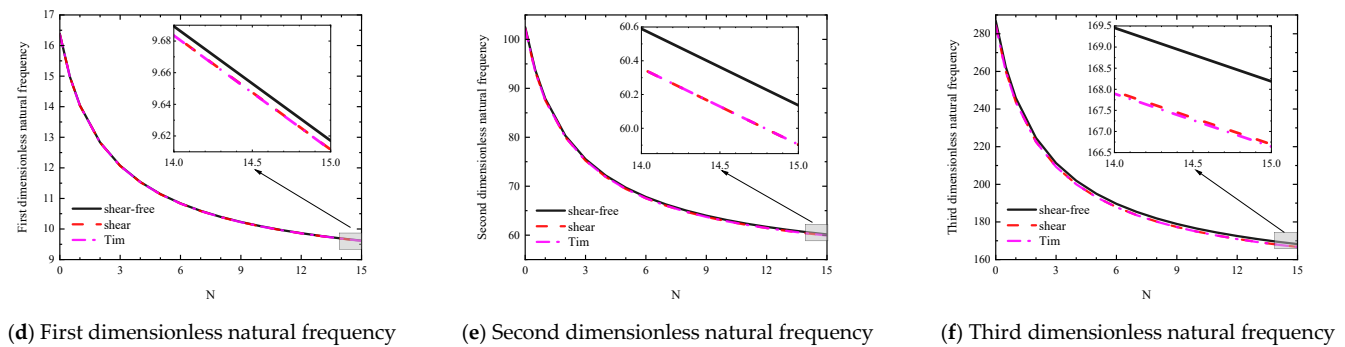


Figure 8. The variation in the first three dimensionless natural frequencies of the micro-beam system with the functional gradient index N ; distribution A is represented by (a–c), while distribution B is represented by (d–f).

Figure 9 presents the curves of the first three natural frequencies in dimensionless form when the dimensionless length parameter β varies, using the RPIM with $\phi = 38/7$, $\varphi = 38/27$, $\gamma = 1$, $\delta = 0.2$, and $N = 1$. The graph illustrates that the first three natural frequencies in dimensionless form of the system decrease as the dimensionless length parameter β increases. When the length parameter is small, the size effect is evident, leading to an increase in system stiffness. As the dimensionless length parameter β gradually increases, the size effect becomes weaker and gradually aligns with the results of macroscopic beam theory. In the enlarged graph, when $\alpha = 0.002$, the FGM micro-beam system is in a slender beam structure, and the first three natural frequencies in dimensionless form of the Euler–Bernoulli beam, Euler beam with shear, and Timoshenko beam models coincide well, further verifying the accuracy of the Euler beam with shear effect and Timoshenko beam models established in this paper.

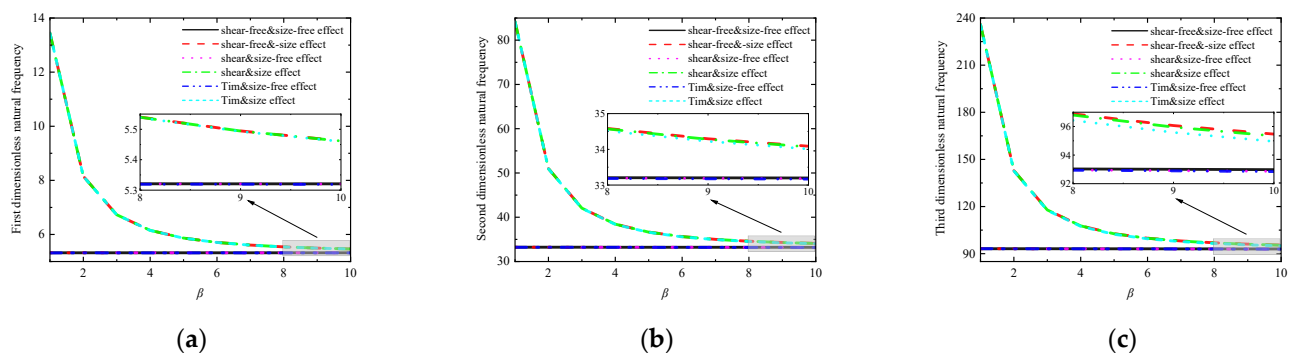


Figure 9. The variation in the first three dimensionless natural frequencies of the micro-beam system with β under distribution A ($\alpha = 0.002$). (a) First dimensionless natural frequency, (b) Second dimensionless natural frequency, (c) Third dimensionless natural frequency.

Figure 10 shows the first three dimensionless natural frequencies of the system when the dimensionless length parameter β varies, using the meshless radial basis point interpolation method with $\phi = 38/7$, $\varphi = 38/27$, $\gamma = 1$, $\delta = 0.2$, and $N = 1$. Comparing Figure 9 and Figure 10, Figure 9 still represents a slender beam structure, while Figure 10 changes α , thus becoming squat beams. It is evident from Figure 10 that the natural frequencies of both the shear model and the Timoshenko beam model are smaller than those of the shear-free model. As β increases, unlike in Figure 9, the results without size effects in the first three dimensionless natural frequencies in Figure 10 no longer remain straight lines but gradually decrease, and the results with size effects still tend to approach those without size effects.

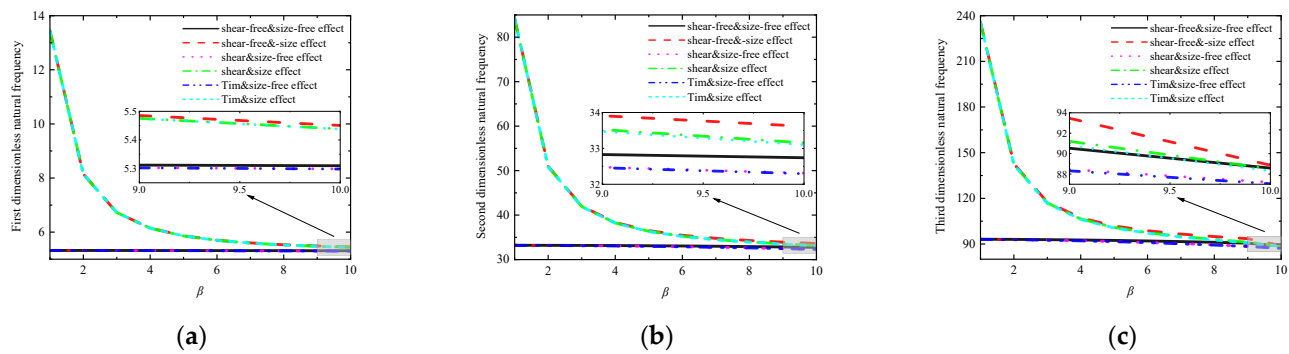


Figure 10. The variation in the first three dimensionless natural frequencies of the micro-beam system with β under distribution A ($\alpha = 0.01$). (a) First dimensionless natural frequency, (b) Second dimensionless natural frequency, (c) Third dimensionless natural frequency.

4. Dynamic Response

In this section, a high-order rigid-flexible coupling model is employed, primarily utilizing the PIM to perform dynamic simulations of rotating FGM micro-beams when large overall motion is known. The main focus is on comparing the differences among three models. The beam parameters are set to be consistent with those in reference [8], adopting distribution B. The FGM micro-beam has a length of $L = 5$ mm, a width of $b = 10$ μm , a height of $h = 10$ μm , a hub radius of $R = 2.5$ mm, a material characteristic length parameter of $l = 10$ μm , and a functional gradient index of $N = 1$.

When the large overall motion law is known, the angular displacement law of the flexible beam is given; the model established in this paper can be transformed into the following form:

$$C: \begin{pmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \ddot{A} \\ \ddot{B} \end{pmatrix} = \begin{pmatrix} Q_A - M_{21}\ddot{\theta} \\ Q_B - M_{31}\ddot{\theta} \end{pmatrix} \quad (53)$$

$$S, T: \begin{pmatrix} M_{22} & M_{23} & M_{24} \\ M_{32} & M_{33} & M_{34} \\ M_{42} & M_{43} & M_{44} \end{pmatrix} \begin{pmatrix} \ddot{A} \\ \ddot{B} \\ \ddot{C} \end{pmatrix} = \begin{pmatrix} Q_A - M_{21}\ddot{\theta} \\ Q_B - M_{31}\ddot{\theta} \\ Q_C - M_{41}\ddot{\theta} \end{pmatrix} \quad (54)$$

When the large overall motion is known, its rotation law is assumed to be as follows:

$$\dot{\theta} = \begin{cases} \frac{\omega_0}{T}t - \frac{\omega_0}{2\pi} \sin\left(\frac{2\pi}{T}t\right) & 0 < t \leq T \\ \omega_0 & t > T \end{cases} \quad (55)$$

The above formula represents that the hub starts accelerating from rest with an acceleration time of 10 ms, and then rotates at a constant speed with an angular velocity of ω_0 . The transient dynamic equations are solved by the Newmark time-integration method. Unless otherwise specified, the initial displacement and velocity are set to zero. The prescribed rigid-body rotation follows Equation (55). The time step is selected to keep the displacement responses stable and convergent.

Figure 11a shows the axial deformation w_1 at the tip of the FGM micro-beam calculated by the three models when $\omega_0 = 100$ rad/s. Figure 11b shows the axial shortening caused by lateral bending deformation at the beam tip, namely the secondary coupled deformation w_c . In Figure 11, the length-to-height ratio of the micro-beam is 500, indicating that the model is a slender beam. The results obtained from the three models are quite consistent. What can be calculated by the classical beam model can also be calculated by the Euler beam with shear and Timoshenko beam models established in this paper. Figure

11c shows the lateral deformation u_y at the tip of the FGM micro-beam, and Figure 11d is a partial enlarged view during uniform rotation from 10 ms to 15 ms. The amplitudes and frequencies of the Euler–Bernoulli beam, Euler beam with shear, and Timoshenko beam models exhibit high consistency. This demonstrates the correctness and applicability of the Euler beam with shear and Timoshenko beam systems established in this paper.

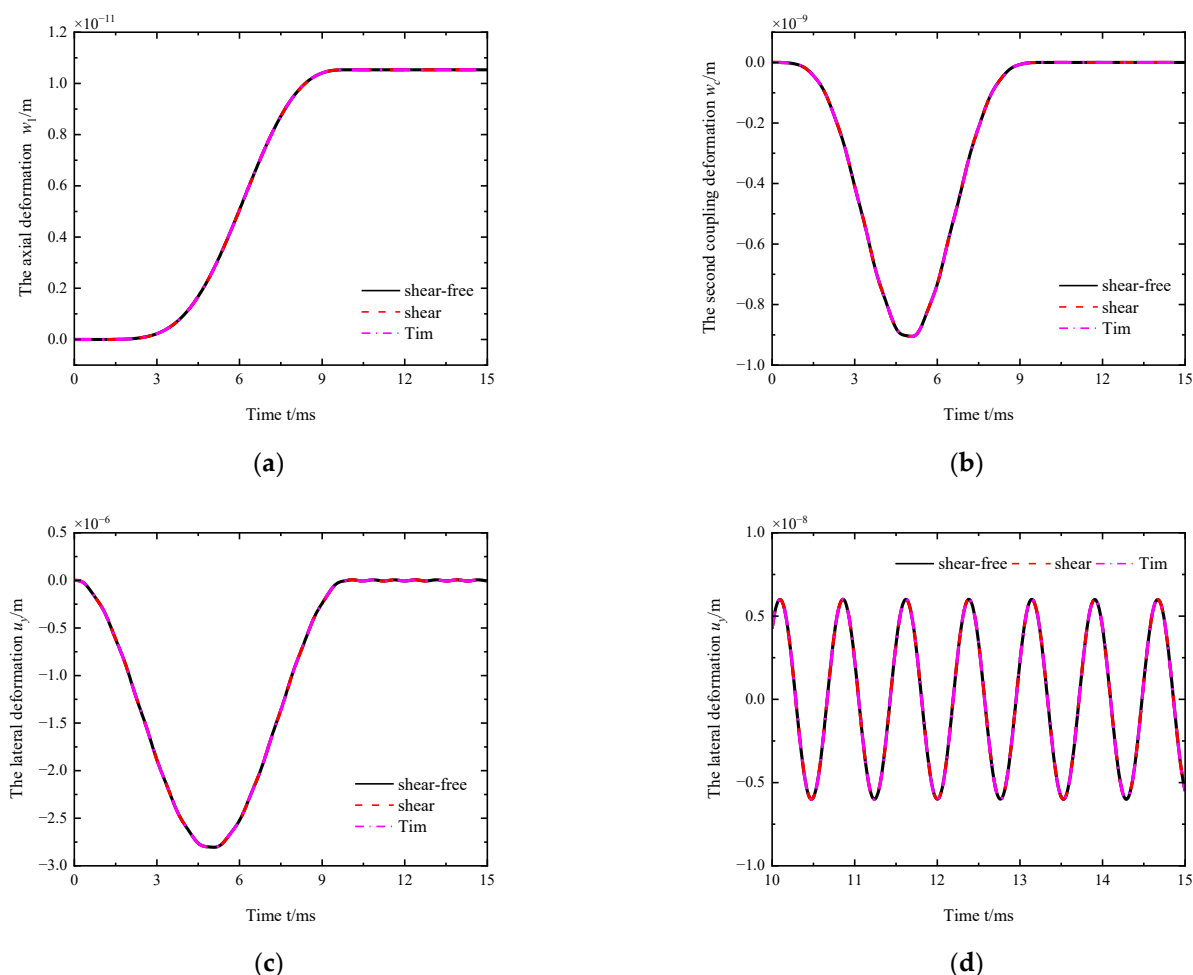


Figure 11. Deformation at the end of micro-beam ($\omega = 100$ rad/s). (a) The axial deformation, (b) the second coupling deformation, (c) the lateral deformation, (d) enlarged 10–15 ms.

Figure 12 illustrates the lateral deformation at the tip of the central beam under different theories as the functional gradient index N varies at a rotational speed of $\omega_0 = 100$ rad/s. In Figure 12a,b, the enlarged views of the local lateral deformation at the tip of the FGM micro-beam with increasing beam height h are clear and evident: the simulation results from the Euler beam with shear theory and Timoshenko beam theory consistently align well, while the results from the Euler–Bernoulli beam theory without considering shear effects deviate to some extent, further confirming the limitations of the Euler–Bernoulli beam theory for thick beam analysis. Figure 12c–h show that the lateral deformation at the beam tip increases with the increase in the functional gradient index N under different theories; the local enlarged views reveal that after extensive motion reaches a steady state, the vibration equilibrium position is located on the beam axis, and as the functional gradient index N increases, the system flexibility enhances, resulting in an increase in amplitude.

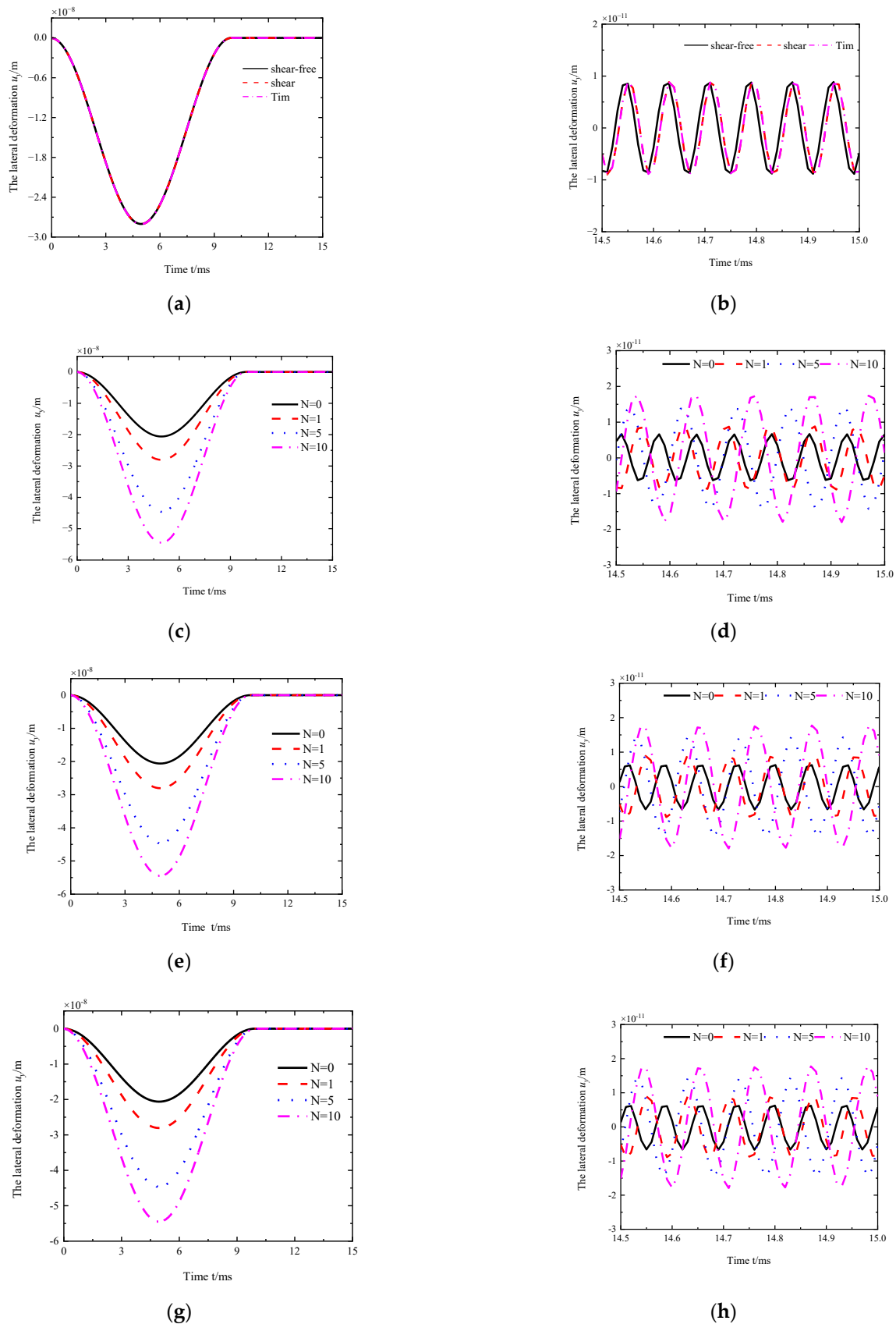


Figure 12. The end response of the FGM micro-beam at different N values ($h = 100 \mu\text{m}$, $l = 100 \mu\text{m}$). (a) The tip response ($N = 1$), (b) enlarge 14.5–15 ms, (c) The tip response under Euler–Bernoulli beam theory, (d) enlarge 14.5–15 ms, (e) The tip response under Euler beam with shear theory, (f) enlarge 14.5–15 ms, (g) The tip response under Timoshenko beam theory, (h) enlarge 14.5–15 ms.

Figure 13 shows the lateral deformation diagrams of the tip of FGM micro-beams when comparing the Euler–Bernoulli beam, the Euler beam with shear, and Timoshenko beam models at $\omega_0 = 100$ rad/s. Figure 13d–f correspond to the cases where size effects are not considered in Figure 13a–c. For slender beams with a smaller beam height h , the shear effect can be neglected. However, as the beam height h increases, the influence of shear effects on FGM micro-beams must be considered. By comparing Figure 13a–c and Figure 13d–f, it can be found that the change in material characteristic length parameter l and size effects has a significant impact on the dynamic response of FGM micro-beams. Since size effects significantly enhance the rigidity of the system, the maximum amplitude of lateral deformation decreases significantly.

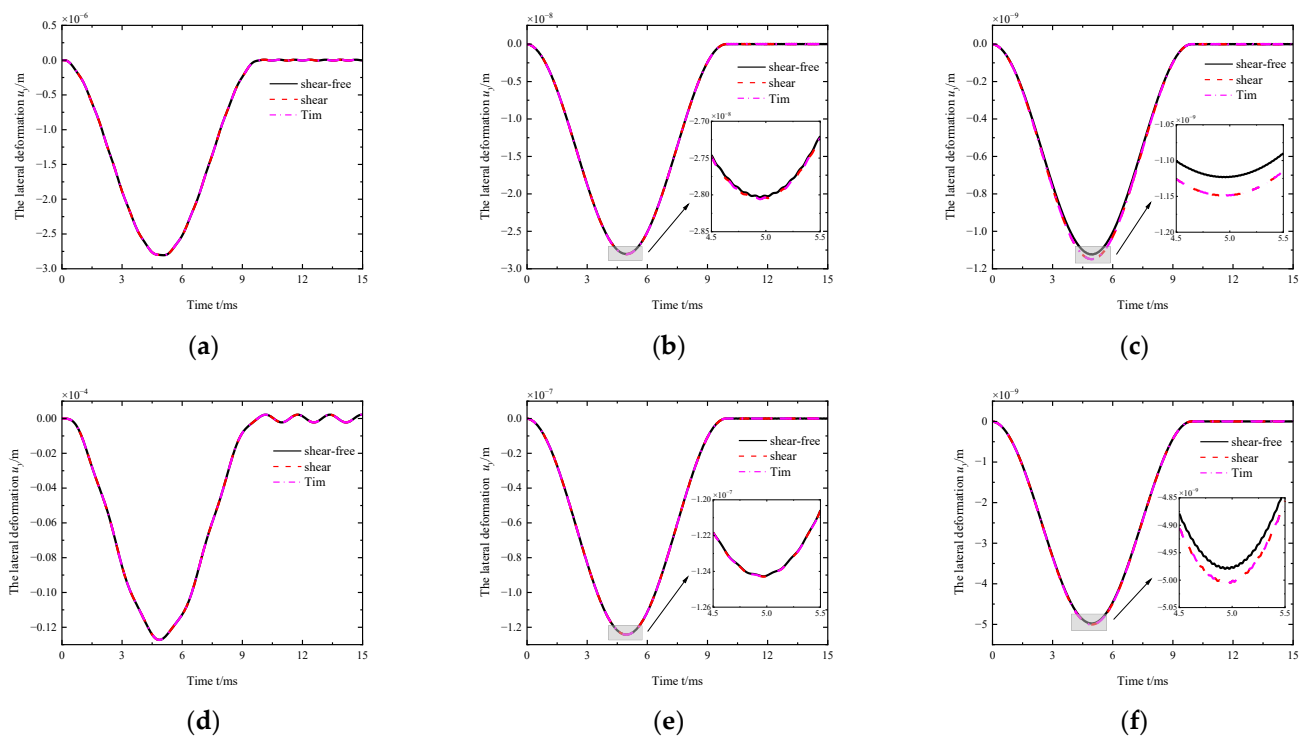


Figure 13. The lateral deformation at the tip of the beam ($\omega = 100$ rad/s). (a) $h = 10 \mu m, l = 10 \mu m$, (b) $h = 100 \mu m, l = 100 \mu m$, (c) $h = 500 \mu m, l = 500 \mu m$, (d) $h = 10 \mu m, l = 0 \mu m$, (e) $h = 100 \mu m, l = 0$, (f) $h = 500 \mu m, l = 0 \mu m$.

Figure 14 illustrates the influence of size effects on the lateral deformation of FGM micro-beams under different dimensionless length parameters $\beta = h/L$ at a rotational speed $\omega_0 = 100$ rad/s. Panels (a) to (c) show the results obtained by varying the h parameter, while (d) to (f) are corresponding enlarged views of local areas. The results indicate that considering size effects significantly enhances the system stiffness, leading to a reduction in the amplitude of lateral deformation at the beam tip and an increase in vibration frequency. As β increases, the size effect gradually diminishes, and the simulation results tend to align with those obtained from macroscopic beam theory.

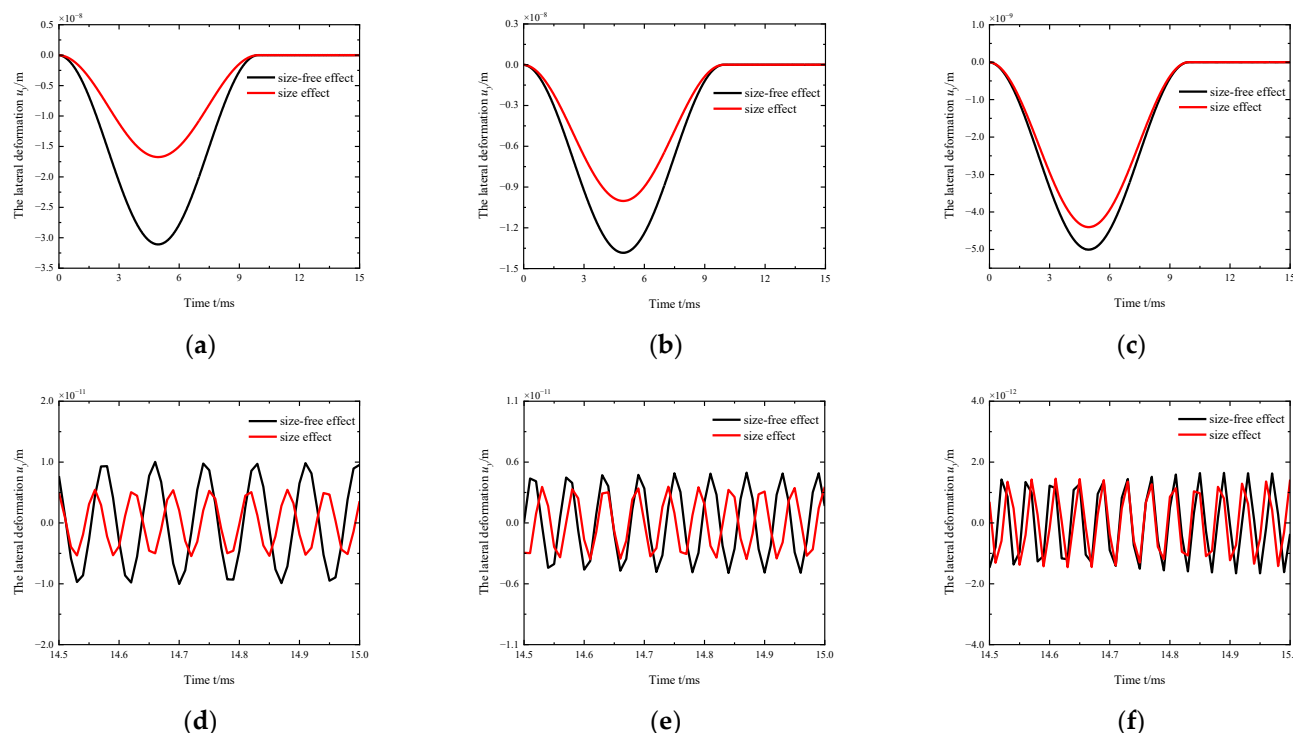


Figure 14. The end response of FGM micro-beams with different β values under Euler beam with shear theory ($\omega = 100$ rad/s, $h = 100$ μm , $l = 100$ μm). (a) $\beta = 2$ rad/s, (b) $\beta = 3$ rad/s, (c) $\beta = 5$ rad/s, (d) enlarge 14.5–15 ms, (e) enlarge 14.5–15 ms, (f) enlarge 14.5–15 ms.

As shown in Figure 15, the numerical stability of the transient response is further checked by changing the number of meshless nodes. The transverse and longitudinal displacements at the free end are used as representative quantities. It can be observed that the response curves obtained with different nodal densities are highly consistent with the original results. The main vibration trends, amplitudes, and phases remain almost unchanged, and no numerical divergence appears. This demonstrates that the present spatial discretization and Newmark time integration scheme can yield stable and reliable dynamic responses for the rotating FGM micro-beam.

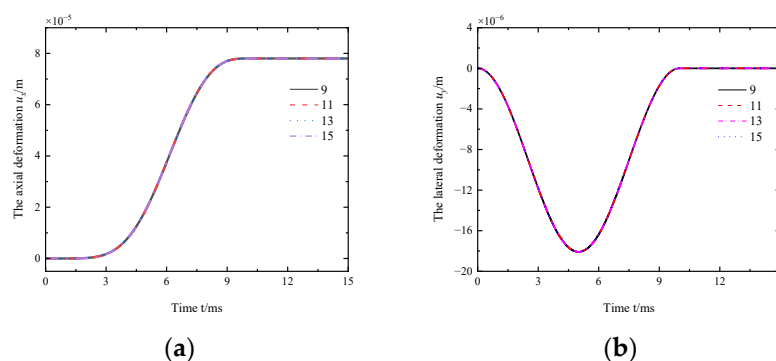


Figure 15. Deformation at the end of the micro-beam (after adding the number of nodes). (a) The axial deformation, (b) The lateral deformation.

5. Conclusions

1. A higher-order rigid-flexible coupled nonlinear dynamic model is established for functionally graded material (FGM) micro-beams undergoing large overall motions with shear deformation considered. The model retains relevant higher-order terms

arising from nonlinear coupling deformation—often neglected in existing literature—within the frameworks of Euler–Bernoulli beam theory, Euler–Bernoulli beam theory with shear correction, and Timoshenko beam theory, based on the modified couple stress theory. The accuracy of the proposed model is validated through static examples, and a method for eliminating shear locking in slender beam analysis using Timoshenko beam theory is also presented.

2. The first three dimensionless natural frequencies of the system increase with rotational speed but decrease with an increasing functional gradient index. As the beam height increases, the shear effect becomes more pronounced, significantly influencing the variations in dimensionless angular velocity, functional gradient index, and size effect. The first three dimensionless natural frequencies predicted by the Euler beam with shear correction and the Timoshenko beam model are consistently lower than those obtained from the classical Euler–Bernoulli beam model, indicating the greater flexibility of the models that account for shear deformation.
3. The size effect significantly enhances the system stiffness, leading to a notable increase in the natural frequencies and a reduction in the deformation at the beam tip. For beams with larger heights, the shear effect cannot be neglected. Variations in the functional gradient index N also alter the system stiffness: as the metallic component increases and the ceramic component decreases, the system becomes more flexible, resulting in lower natural frequencies and larger response amplitudes.

Beyond the numerical findings summarized above, this study provides several important contributions. First, a unified rigid-flexible coupling dynamic framework is established for rotating FGM micro-beams based on three beam theories, enabling a systematic comparison of their modeling capabilities. Second, the modified couple stress theory is incorporated into the formulation to account for microscale size-dependent effects. Third, the PIM and RPIM meshless methods are successfully applied to describe the deformation field of rotating FGM micro-beams with different shear assumptions. Finally, the applicability of the Euler–Bernoulli beam model, the shear-effect Euler beam model, and the Timoshenko beam model under different size-effect conditions is clarified, providing useful guidance for the selection of beam theories in microscale dynamic analyses.

Nevertheless, several limitations remain in the present work. Thermal effects, damping, geometric imperfections, and material nonlinearities are not considered. In addition, the analysis is restricted to rotating FGM micro-beams. Future research will focus on extending the proposed framework to thermomechanical coupling problems, nonlinear dynamic behavior, and more complex microscale structures such as micro-plates, micro-shells, and MEMS flexible systems.

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Appendix A. Dynamic Equations of Euler–Bernoulli Beam Theory

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{A} \\ \ddot{B} \end{bmatrix} = \begin{bmatrix} Q_\theta \\ Q_A \\ Q_B \end{bmatrix} \quad (A1)$$

In the formula:

$$\begin{aligned} M_{11} = & J_{ch} + J_{cb} + J_2 + 2S_x A + 2(S_{y1} - S_{y2})B + A^T M_1 A + B^T M_2 B - 2A^T M_4 B - \underline{B^T(aC_0 + D)B} \\ & + \frac{1}{4} \int_0^L \rho_0 b [\underline{B^T H(x)B \cdot B^T H(x)B}] dx - \int_0^L \rho_0 b [\underline{B^T H(x)B \cdot \Phi_x A}] dx \\ & + \int_0^L \rho_1 b [\underline{B^T H(x)B \cdot \Phi'_y B}] dx \end{aligned} \quad (A2)$$

$$M_{21} = M_{12}^T = -M_3 B - S_{x1}^T \quad (A3)$$

$$M_{22} = M_1 \quad (A4)$$

$$M_{23} = M_{32}^T = -M_4 - \int_0^L \rho_0 b [\underline{\Phi_x^T B^T H(x)}] dx \quad (A5)$$

$$\begin{aligned} M_{31} = M_{13}^T = & S_y^T + M_3^T A + Y^T + (M_5 - M_5^T)B + \underline{C_1 B} + \int_0^L \rho_0 b [\underline{H(x)B \cdot \Phi_y B}] dx \\ & - \frac{1}{2} \int_0^L \rho_0 b [\underline{\Phi_y^T \cdot B^T H(x)B}] dx \end{aligned} \quad (A6)$$

$$M_{33} = M_2 + \int_0^L \rho_0 b [\underline{H(x)B \cdot B^T H(x)}] dx + \int_0^L \rho_1 b [\underline{H(x)B \cdot \Phi'_y + \Phi'_y{}^T \cdot B^T H(x)}] dx \quad (A7)$$

$$\begin{aligned} Q_\theta = & \tau - 2\dot{\theta} [S_x \dot{A} + (S_{y1} - S_{y2})\dot{B} + A^T M_1 \dot{A} + B^T M_2 \dot{B} - \dot{A}^T M_4 B - A^T M_4 \dot{B} - \underline{B^T(aC_0 + D)\dot{B}}] \\ & - \underline{\dot{B}^T C_1 \dot{B}} - \int_0^L \rho_0 b [\underline{\dot{B}^T H(x)\dot{B} \cdot \Phi_y B}] dx - \dot{\theta} \int_0^L \rho_1 b [\underline{2\dot{B}^T H(x)B \cdot \Phi'_y B + B^T H(x)B \cdot \Phi'_y \dot{B}}] dx \\ & + \dot{\theta} \int_0^L \rho_0 b [\underline{2\dot{B}^T H(x)B \cdot \Phi_x A + B^T H(x)B \cdot \Phi_x \dot{A} - \dot{B}^T H(x)B \cdot B^T H(x)B}] dx \end{aligned} \quad (A8)$$

$$\begin{aligned} Q_A = & \dot{\theta}^2 S_x^T + 2\dot{\theta} M_3 \dot{B} + (\dot{\theta}^2 M_1 - K_0)A - (\dot{\theta}^2 M_4 - K_1)B + \int_0^L \rho_0 b [\underline{\Phi_x^T \cdot \dot{B}^T H(x)\dot{B}}] dx \\ & - \frac{1}{2} \dot{\theta}^2 \int_0^L \rho_0 b [\underline{\Phi_x^T \cdot B^T H(x)B}] dx \end{aligned} \quad (A9)$$

$$\begin{aligned} Q_B = & \dot{\theta}^2 (M_2 - \underline{aC_0 - D})B + \dot{\theta}^2 (S_{y1}^T - S_{y2}^T) - (\dot{\theta}^2 M_4^T - K_1^T)A + 2\dot{\theta} (M_5^T - M_5)\dot{B} - 2\dot{\theta} M_3^T \dot{A} \\ & - \left[K_2 + \frac{l^2}{2(1+\nu)} K_3 \right] B + 2\dot{\theta} \int_0^L \rho_0 b [\underline{\Phi_y^T \cdot B^T H(x)\dot{B} - H(x)B \cdot \Phi_y \dot{B}}] dx \\ & + \dot{\theta}^2 \int_0^L \rho_0 b \left[\frac{1}{2} \underline{H(x)B \cdot B^T H(x)B - H(x)B \cdot \Phi_x A} \right] dx - \int_0^L \rho_0 b [\underline{H(x)B \cdot \dot{B}^T H(x)\dot{B}}] dx \\ & - \int_0^L \rho_1 b [\underline{\Phi_y^T \cdot \dot{B}^T H(x)\dot{B}}] dx + \dot{\theta}^2 \int_0^L \rho_1 b \left[\frac{1}{2} \underline{\Phi_y^T \cdot B^T H(x)B + H(x)B \cdot \Phi'_y B} \right] dx \end{aligned} \quad (A10)$$

The term with three underscores in the formula represents an additional term when considering size effects, while the terms with single and double underscores arise from the consideration of nonlinear coupled deformation, denoted as w_c . The aforementioned definitions are applicable to the Euler beam with shear effect and Timoshenko beam models presented in Appendices B and C.

In the aforementioned equation, the corresponding constant matrix is:

$$J_{cb} = \int_0^L \rho_0 b(a+x)^2 dx \quad (A11)$$

$$J_2 = \int_0^L \rho_2 b dx \quad (A12)$$

$$Y = \int_0^L \rho_2 b \Phi'_y(x) dx \quad (A13)$$

$$M_1 = \int_0^L \rho_0 b \Phi_x^T(x) \Phi_x(x) dx \quad (A14)$$

$$M_2 = \int_0^L \rho_0 b \Phi_y^T(x) \Phi_y(x) dx + \int_0^L \rho_2 b \Phi_y'^T(x) \Phi_y'(x) dx \quad (A15)$$

$$M_3 = \int_0^L \rho_0 b \Phi_x^T(x) \Phi_y(x) dx \quad (A16)$$

$$M_4 = \int_0^L \rho_1 b \Phi_x^T(x) \Phi_y'(x) dx \quad (A17)$$

$$M_5 = \int_0^L \rho_1 b \Phi_y'^T(x) \Phi_y'(x) dx \quad (A18)$$

$$S_x = \int_0^L \rho_0 b(a+x) \Phi_x(x) dx \quad (A19)$$

$$S_y = \int_0^L \rho_0 b(a+x) \Phi_y(x) dx \quad (A20)$$

$$S_{x1} = \int_0^L \rho_1 b \Phi_x(x) dx \quad (A21)$$

$$S_{y1} = \int_0^L \rho_1 b \Phi_y(x) dx \quad (A22)$$

$$S_{y2} = \int_0^L \rho_1 b(a+x) \Phi_y'(x) dx \quad (A23)$$

$$C_0 = \int_0^L \rho_0 b H(x) dx \quad (A24)$$

$$C_1 = \int_0^L \rho_1 b H(x) dx \quad (A25)$$

$$D = \int_0^L \rho_0 b x H(x) dx \quad (A26)$$

$$K_0 = \int_0^L E_0 b \Phi_x'^T(x) \Phi_x'(x) dx \quad (A27)$$

$$K_1 = \int_0^L E_1 b \Phi_x'^T(x) \Phi_y''(x) dx \quad (A28)$$

$$K_2 = \int_0^L E_2 b \Phi_y''^T(x) \Phi_y''(x) dx \quad (A29)$$

$$K_3 = \int_0^L E_0 b \Phi_y''^T(x) \Phi_y''(x) dx \quad (A30)$$

The specific expressions for the density ρ_0, ρ_1, ρ_2 and elastic modulus E_0, E_1, E_2 related to the y -coordinate in the thickness direction of the FGM micro-beam are as follows:

$$(\rho_0, \rho_1, \rho_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(y) (1, y, y^2) dy \quad (A31)$$

$$(E_0, E_1, E_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(y) (1, y, y^2) dy \quad (A32)$$

The above formula represents the specific calculation process for distribution A. Distribution B can be derived by modifying the expressions for density and elastic modulus.

Appendix B. Dynamic Equations of the Euler Beam with Shear Effect Theory

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{A} \\ \ddot{B} \\ \ddot{C} \end{bmatrix} = \begin{bmatrix} Q_\theta \\ Q_A \\ Q_B \\ Q_C \end{bmatrix} \quad (\text{A33})$$

In the formula:

$$\begin{aligned} M_{11} = & J_{ch} + J_{cb} + J_2 + \mathbf{B}^T (\mathbf{M}_2 - a\mathbf{D}_0 - \mathbf{D}_2) \mathbf{B} + \mathbf{A}^T \mathbf{M}_1 \mathbf{A} + \mathbf{C}^T \mathbf{M}_3 \mathbf{C} \\ & + 2(\mathbf{S}_{y1} - \mathbf{S}_{y2}) \mathbf{B} + 2\mathbf{S}_x \mathbf{A} + 2\mathbf{S}_{z1} \mathbf{C} - 2\mathbf{A}^T \mathbf{M}_7 \mathbf{B} + 2\mathbf{B}^T \mathbf{M}_6 \mathbf{C} \\ & + \int_0^L \rho_0 b \left[\frac{1}{4} \mathbf{B}^T \mathbf{H}(x) \mathbf{B} \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} - \mathbf{B}^T \mathbf{H}(x) \mathbf{B} \cdot \Phi_1 \mathbf{A} \right] dx \\ & + \int_0^L \rho_1 b \left[\mathbf{B}^T \mathbf{H}(x) \mathbf{B} \cdot \Phi_2' \mathbf{B} \right] dx \end{aligned} \quad (\text{A34})$$

$$M_{22} = M_1 \quad (\text{A35})$$

$$M_{33} = M_2 + \int_0^L \rho_0 b \left[\mathbf{H}(x) \mathbf{B} \cdot \mathbf{B}^T \mathbf{H}(x) \right] dx + \int_0^L \rho_1 b \left[\mathbf{H}(x) \mathbf{B} \cdot \Phi_2' + \Phi_2'^T \cdot \mathbf{B}^T \mathbf{H}(x) \right] dx \quad (\text{A36})$$

$$M_{44} = M_3 \quad (\text{A37})$$

$$M_{21} = \mathbf{M}_{12}^T = -\mathbf{S}_{x1}^T - \mathbf{M}_4 \mathbf{B} - \mathbf{M}_5 \mathbf{C} \quad (\text{A38})$$

$$\begin{aligned} M_{31} = & \mathbf{M}_{13}^T = \mathbf{S}_y^T + \mathbf{M}_4^T \mathbf{A} + \mathbf{Y}^T + (\mathbf{D}_1 + \mathbf{M}_8 - \mathbf{M}_8^T) \mathbf{B} + \mathbf{M}_9 \mathbf{C} \\ & + \int_0^L \rho_0 b \left[\Phi_2 \mathbf{B} \cdot \mathbf{H}(x) \mathbf{B} + \Phi_3 \mathbf{C} \cdot \mathbf{H}(x) \mathbf{B} - \frac{1}{2} \Phi_2^T \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} \right] dx \end{aligned} \quad (\text{A39})$$

$$M_{41} = \mathbf{M}_{14}^T = \mathbf{S}_z^T + \mathbf{M}_5^T \mathbf{A} - \mathbf{M}_9^T \mathbf{C} - \frac{1}{2} \int_0^L \rho_0 b \left[\Phi_3^T \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} \right] dx \quad (\text{A40})$$

$$M_{32} = \mathbf{M}_{23}^T = -\mathbf{M}_7^T - \int_0^L \rho_0 b \left[\mathbf{H}(x) \mathbf{B} \cdot \Phi_1 \right] dx \quad (\text{A41})$$

$$M_{42} = \mathbf{M}_{24}^T = 0 \quad (\text{A42})$$

$$M_{43} = \mathbf{M}_{34}^T = \mathbf{M}_6^T \quad (\text{A43})$$

$$\begin{aligned} Q_\theta = & \tau - 2\dot{\theta} [\mathbf{S}_x \dot{\mathbf{A}} + \mathbf{B}^T (\mathbf{M}_2 - a\mathbf{D}_0 - \mathbf{D}_2) \dot{\mathbf{B}} + (\mathbf{S}_{y1} - \mathbf{S}_{y2}) \dot{\mathbf{B}} + \mathbf{S}_{z1} \dot{\mathbf{C}} + \mathbf{A}^T \mathbf{M}_1 \dot{\mathbf{A}} \\ & - \dot{\mathbf{A}}^T \mathbf{M}_7 \mathbf{B} - \mathbf{A}^T \mathbf{M}_7 \dot{\mathbf{B}} + \mathbf{C}^T \mathbf{M}_3 \dot{\mathbf{C}} + \dot{\mathbf{B}}^T \mathbf{M}_6 \mathbf{C} + \mathbf{B}^T \mathbf{M}_6 \dot{\mathbf{C}}] - \dot{\mathbf{B}}^T \mathbf{D}_1 \dot{\mathbf{B}} \\ & - \int_0^L \rho_0 b \left[\Phi_2 \mathbf{B} \cdot \dot{\mathbf{B}}^T \mathbf{H}(x) \dot{\mathbf{B}} + \Phi_3 \mathbf{C} \cdot \dot{\mathbf{B}}^T \mathbf{H}(x) \dot{\mathbf{B}} \right] dx \\ & + \dot{\theta} \int_0^L \rho_0 b \left[2\Phi_1 \mathbf{A} \cdot \mathbf{B}^T \mathbf{H}(x) \dot{\mathbf{B}} + \Phi_1 \dot{\mathbf{A}} \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} - \mathbf{B}^T \mathbf{H}(x) \mathbf{B} \cdot \mathbf{B}^T \mathbf{H}(x) \dot{\mathbf{B}} \right] dx \\ & - \dot{\theta} \int_0^L \rho_1 b \left[2\Phi_2' \mathbf{B} \cdot \mathbf{B}^T \mathbf{H}(x) \dot{\mathbf{B}} + \Phi_2' \dot{\mathbf{B}} \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} \right] dx \end{aligned} \quad (\text{A44})$$

$$\begin{aligned} Q_A = & \dot{\theta}^2 \mathbf{S}_x^T + (\dot{\theta}^2 \mathbf{M}_1 - \mathbf{K}_0) \mathbf{A} + (\mathbf{K}_1 - \dot{\theta}^2 \mathbf{M}_7) \mathbf{B} + 2\dot{\theta} \mathbf{M}_4 \dot{\mathbf{B}} + 2\dot{\theta} \mathbf{M}_5 \dot{\mathbf{C}} \\ & + \int_0^L \rho_0 b \left[\Phi_1^T \cdot \dot{\mathbf{B}}^T \mathbf{H}(x) \dot{\mathbf{B}} - \frac{1}{2} \dot{\theta}^2 \cdot \Phi_1^T \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} \right] dx \end{aligned} \quad (\text{A45})$$

$$\begin{aligned} \mathbf{Q}_B = & \dot{\theta}^2 (\mathbf{M}_2 - a\mathbf{D}_0 - \mathbf{D}_2) \mathbf{B} + \dot{\theta}^2 (\mathbf{S}_{y1}^T - \mathbf{S}_{y2}^T) + (\mathbf{K}_1^T - \dot{\theta}^2 \mathbf{M}_7^T) \mathbf{A} + 2\dot{\theta} (\mathbf{M}_8^T - \mathbf{M}_8) \dot{\mathbf{B}} - 2\dot{\theta} \mathbf{M}_4^T \dot{\mathbf{A}} \\ & - \left[\mathbf{K}_2 + \frac{l^2}{2(1+\nu)} \mathbf{K}_3 \right] \mathbf{B} - \frac{1}{2} \frac{l^2}{2(1+\nu)} \mathbf{K}_4 \mathbf{C} + \dot{\theta}^2 \mathbf{M}_6 \mathbf{C} - 2\dot{\theta} \mathbf{M}_9 \dot{\mathbf{C}} \\ & + \dot{\theta}^2 \int_0^L \rho_0 b \left[\frac{1}{2} \mathbf{H}(x) \mathbf{B} \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} - \mathbf{H}(x) \mathbf{B} \cdot \boldsymbol{\Phi}_1 \mathbf{A} \right] dx - \int_0^L \rho_0 b \left[\mathbf{H}(x) \mathbf{B} \cdot \dot{\mathbf{B}}^T \mathbf{H}(x) \dot{\mathbf{B}} \right] dx \end{aligned} \quad (\text{A46})$$

$$\begin{aligned} & + \dot{\theta}^2 \int_0^L \rho_1 b \left[\frac{1}{2} \boldsymbol{\Phi}_2'^T \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} + \mathbf{H}(x) \mathbf{B} \cdot \boldsymbol{\Phi}_2' \mathbf{B} \right] dx - \int_0^L \rho_1 b \left[\boldsymbol{\Phi}_2'^T \cdot \dot{\mathbf{B}}^T \mathbf{H}(x) \dot{\mathbf{B}} \right] dx \\ & + 2\dot{\theta} \int_0^L \rho_0 b \left[\boldsymbol{\Phi}_2^T \cdot \mathbf{B}^T \mathbf{H}(x) \dot{\mathbf{B}} - \mathbf{H}(x) \mathbf{B} \cdot \boldsymbol{\Phi}_2 \dot{\mathbf{B}} - \mathbf{H}(x) \mathbf{B} \cdot \boldsymbol{\Phi}_3 \dot{\mathbf{C}} \right] dx \\ \mathbf{Q}_C = & \left[\dot{\theta}^2 \mathbf{M}_6^T - \frac{1}{2} \frac{l^2}{2(1+\nu)} \mathbf{K}_4^T \right] \mathbf{B} + \left[\dot{\theta}^2 \mathbf{M}_3 - \mathbf{K}_7 - \frac{1}{2} \mathbf{K}_5 - \frac{1}{4} \frac{l^2}{2(1+\nu)} \mathbf{K}_6 \right] \mathbf{C} + \dot{\theta}^2 \mathbf{S}_{z1}^T \\ & - 2\dot{\theta} \mathbf{M}_5^T \dot{\mathbf{A}} + 2\dot{\theta} \mathbf{M}_9^T \dot{\mathbf{B}} + 2\dot{\theta} \int_0^L \rho_0 b \left[\boldsymbol{\Phi}_3^T \cdot \mathbf{B}^T \mathbf{H}(x) \dot{\mathbf{B}} \right] dx \end{aligned} \quad (\text{A47})$$

In the aforementioned equation, the corresponding constant matrix is:

$$J_{cb} = \int_0^L \rho_0 b (a+x)^2 dx \quad (\text{A48})$$

$$J_2 = \int_0^L \rho_2 b dx \quad (\text{A49})$$

$$\mathbf{Y} = \int_0^L \rho_2 b \boldsymbol{\Phi}_2'(x) dx \quad (\text{A50})$$

$$\mathbf{M}_1 = \int_0^L \rho_0 b \boldsymbol{\Phi}_1^T(x) \boldsymbol{\Phi}_1(x) dx \quad (\text{A51})$$

$$\mathbf{M}_2 = \int_0^L \rho_0 b \boldsymbol{\Phi}_2^T(x) \boldsymbol{\Phi}_2(x) dx + \int_0^L \rho_2 b \boldsymbol{\Phi}_2'^T(x) \boldsymbol{\Phi}_2'(x) dx \quad (\text{A52})$$

$$\mathbf{M}_3 = \int_0^L \rho_0 b \boldsymbol{\Phi}_3^T(x) \boldsymbol{\Phi}_3(x) dx \quad (\text{A53})$$

$$\mathbf{M}_4 = \int_0^L \rho_0 b \boldsymbol{\Phi}_1^T(x) \boldsymbol{\Phi}_2(x) dx \quad (\text{A54})$$

$$\mathbf{M}_5 = \int_0^L \rho_0 b \boldsymbol{\Phi}_1^T(x) \boldsymbol{\Phi}_3(x) dx \quad (\text{A55})$$

$$\mathbf{M}_6 = \int_0^L \rho_0 b \boldsymbol{\Phi}_2^T(x) \boldsymbol{\Phi}_3(x) dx \quad (\text{A56})$$

$$\mathbf{M}_7 = \int_0^L \rho_1 b \boldsymbol{\Phi}_1^T(x) \boldsymbol{\Phi}_2'(x) dx \quad (\text{A57})$$

$$\mathbf{M}_8 = \int_0^L \rho_1 b \boldsymbol{\Phi}_2'^T(x) \boldsymbol{\Phi}_2(x) dx \quad (\text{A58})$$

$$\mathbf{M}_9 = \int_0^L \rho_1 b \boldsymbol{\Phi}_2'^T(x) \boldsymbol{\Phi}_3(x) dx \quad (\text{A59})$$

$$\mathbf{S}_x = \int_0^L \rho_0 b (a+x) \boldsymbol{\Phi}_1(x) dx \quad (\text{A60})$$

$$\mathbf{S}_y = \int_0^L \rho_0 b (a+x) \boldsymbol{\Phi}_2(x) dx \quad (\text{A61})$$

$$\mathbf{S}_z = \int_0^L \rho_0 b (a+x) \boldsymbol{\Phi}_3(x) dx \quad (\text{A62})$$

$$\mathbf{S}_{x1} = \int_0^L \rho_1 b \boldsymbol{\Phi}_1(x) dx \quad (\text{A63})$$

$$\mathbf{S}_{y1} = \int_0^L \rho_1 b \boldsymbol{\Phi}_2(x) dx \quad (\text{A64})$$

$$\mathbf{S}_{z1} = \int_0^L \rho_1 b \boldsymbol{\Phi}_3(x) dx \quad (\text{A65})$$

$$\mathbf{S}_{y2} = \int_0^L \rho_1 b (a+x) \boldsymbol{\Phi}_2'(x) dx \quad (\text{A66})$$

$$\mathbf{D}_0 = \int_0^L \rho_0 b \mathbf{H}(x) dx \quad (\text{A67})$$

$$\mathbf{D}_1 = \int_0^L \rho_1 b \mathbf{H}(x) dx \quad (\text{A68})$$

$$\mathbf{D}_2 = \int_0^L \rho_0 b x \mathbf{H}(x) dx \quad (\text{A69})$$

$$\mathbf{K}_0 = \int_0^L E_0 b \boldsymbol{\Phi}_1'^T(x) \boldsymbol{\Phi}_1'(x) dx \quad (\text{A70})$$

$$\mathbf{K}_1 = \int_0^L E_1 b \boldsymbol{\Phi}_1'^T(x) \boldsymbol{\Phi}_2''(x) dx \quad (\text{A71})$$

$$\mathbf{K}_2 = \int_0^L E_2 b \boldsymbol{\Phi}_2''^T(x) \boldsymbol{\Phi}_2''(x) dx \quad (\text{A72})$$

$$\mathbf{K}_3 = \int_0^L E_0 b \boldsymbol{\Phi}_2''^T(x) \boldsymbol{\Phi}_2''(x) dx \quad (\text{A73})$$

$$\mathbf{K}_4 = \int_0^L E_0 b \boldsymbol{\Phi}_2''^T(x) \boldsymbol{\Phi}_3''(x) dx \quad (\text{A74})$$

$$\mathbf{K}_5 = \int_0^L E_0 b \boldsymbol{\Phi}_3'^T(x) \boldsymbol{\Phi}_3'(x) dx \quad (\text{A75})$$

$$\mathbf{K}_6 = \int_0^L E_0 b \boldsymbol{\Phi}_3''^T(x) \boldsymbol{\Phi}_3''(x) dx \quad (\text{A76})$$

$$\mathbf{K}_7 = \int_V \frac{G(y)}{k} \boldsymbol{\Phi}_3'^T(x) \boldsymbol{\Phi}_3'(x) dV \quad (\text{A77})$$

In the above formula (A77), $G(y)$ represents the shear modulus of the FGM micro-beam, and k denotes the shear correction coefficient of the FGM micro-beam. The definitions of density and elastic modulus are provided in Appendix A.

Appendix C. Dynamic Equations of Timoshenko Beam Theory

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} & \mathbf{M}_{14} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} & \mathbf{M}_{24} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} & \mathbf{M}_{34} \\ \mathbf{M}_{41} & \mathbf{M}_{42} & \mathbf{M}_{43} & \mathbf{M}_{44} \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{A} \\ \ddot{B} \\ \ddot{C} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_\theta \\ \mathbf{Q}_A \\ \mathbf{Q}_B \\ \mathbf{Q}_C \end{bmatrix} \quad (\text{A78})$$

In the formula:

$$\begin{aligned} \mathbf{M}_{11} = & J_{ch} + J_{cb} + J_2 + \mathbf{B}^T (\mathbf{M}_2 - a \mathbf{D}_0 - \mathbf{D}_2) \mathbf{B} + \mathbf{A}^T \mathbf{M}_1 \mathbf{A} + \mathbf{C}^T \mathbf{M}_3 \mathbf{C} + 2 \mathbf{S}_x \mathbf{A} + 2 \mathbf{S}_{y1} \mathbf{B} \\ & - 2 \mathbf{S}_{z1} \mathbf{C} - 2 \mathbf{A}^T \mathbf{M}_5 \mathbf{C} + \int_0^L \rho_0 b \left[\frac{1}{4} \mathbf{B}^T \mathbf{H}(x) \mathbf{B} \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} - \mathbf{B}^T \mathbf{H}(x) \mathbf{B} \cdot \boldsymbol{\Phi}_1 \mathbf{A} \right] dx \\ & + \int_0^L \rho_1 b \left[\mathbf{B}^T \mathbf{H}(x) \mathbf{B} \cdot \boldsymbol{\Phi}_3 \mathbf{C} \right] dx \end{aligned} \quad (\text{A79})$$

$$\mathbf{M}_{22} = \mathbf{M}_1 \quad (\text{A80})$$

$$\mathbf{M}_{33} = \mathbf{M}_2 + \int_0^L \rho_0 b \left[\mathbf{H}(x) \mathbf{B} \cdot \mathbf{B}^T \mathbf{H}(x) \right] dx \quad (\text{A81})$$

$$\mathbf{M}_{44} = \mathbf{M}_3 \quad (\text{A82})$$

$$\mathbf{M}_{21} = \mathbf{M}_{12}^T = -\mathbf{S}_{x1}^T - \mathbf{M}_4 \mathbf{B} \quad (\text{A83})$$

$$\mathbf{M}_{31} = \mathbf{M}_{13}^T = \mathbf{S}_y^T + \mathbf{M}_4^T \mathbf{A} - \mathbf{M}_6 \mathbf{C} + \underline{\underline{\mathbf{D}_1 \mathbf{B}}} + \int_0^L \rho_0 b \left[\underline{\underline{\Phi_2 \mathbf{B} \cdot \mathbf{H}(x) \mathbf{B} - \frac{1}{2} \Phi_2^T \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B}}} \right] dx \quad (\text{A84})$$

$$\mathbf{M}_{41} = \mathbf{M}_{14}^T = \mathbf{S}_{z1}^T + \mathbf{M}_6^T \mathbf{B} \quad (\text{A85})$$

$$\mathbf{M}_{32} = \mathbf{M}_{23}^T = - \int_0^L \rho_0 b \left[\underline{\underline{\mathbf{H}(x) \mathbf{B} \cdot \Phi_1}} \right] dx \quad (\text{A86})$$

$$\mathbf{M}_{42} = \mathbf{M}_{24}^T = -\mathbf{M}_5^T \quad (\text{A87})$$

$$\mathbf{M}_{43} = \mathbf{M}_{34}^T = \int_0^L \rho_1 b \Phi_3^T \mathbf{B}^T \mathbf{H}(x) dx \quad (\text{A88})$$

$$\begin{aligned} Q_\theta = & \tau - 2\dot{\theta} \left[\mathbf{A}^T \mathbf{M}_1 \dot{\mathbf{A}} + \mathbf{B}^T (\mathbf{M}_2 - a \mathbf{D}_0 - \mathbf{D}_2) \dot{\mathbf{B}} + \mathbf{C}^T \mathbf{M}_3 \dot{\mathbf{C}} + \mathbf{S}_x \dot{\mathbf{A}} + \mathbf{S}_{y1} \dot{\mathbf{B}} - \mathbf{S}_z \dot{\mathbf{C}} - \dot{\mathbf{A}}^T \mathbf{M}_5 \mathbf{C} - \mathbf{A}^T \mathbf{M}_5 \dot{\mathbf{C}} \right] \\ & - \underline{\underline{\dot{\mathbf{B}}^T \mathbf{D}_1 \dot{\mathbf{B}}}} + \dot{\theta} \int_0^L \rho_0 b \left[\underline{\underline{2 \Phi_1 \mathbf{A} \cdot \mathbf{B}^T \mathbf{H}(x) \dot{\mathbf{B}} + \Phi_1 \dot{\mathbf{A}} \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} - \mathbf{B}^T \mathbf{H}(x) \mathbf{B} \cdot \mathbf{B}^T \mathbf{H}(x) \dot{\mathbf{B}}}} \right] dx \end{aligned} \quad (\text{A89})$$

$$\begin{aligned} & - \dot{\theta} \int_0^L \rho_1 b \left[\underline{\underline{\Phi_3 \dot{\mathbf{C}} \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} + 2 \Phi_3 \mathbf{C} \cdot \mathbf{B}^T \mathbf{H}(x) \dot{\mathbf{B}}}} \right] dx - \int_0^L \rho_0 b \left[\underline{\underline{\Phi_2 \mathbf{B} \cdot \dot{\mathbf{B}}^T \mathbf{H}(x) \dot{\mathbf{B}}}} \right] dx \\ Q_A = & \dot{\theta}^2 \mathbf{S}_x^T + (\dot{\theta}^2 \mathbf{M}_1 - \mathbf{K}_0) \mathbf{A} + (\mathbf{K}_1 - \dot{\theta}^2 \mathbf{M}_5) \mathbf{C} + 2\dot{\theta} \mathbf{M}_4 \dot{\mathbf{B}} \\ & + \int_0^L \rho_0 b \left[\underline{\underline{\Phi_1^T \cdot \dot{\mathbf{B}}^T \mathbf{H}(x) \dot{\mathbf{B}} - \frac{1}{2} \dot{\theta}^2 \cdot \Phi_1^T \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B}}} \right] dx \end{aligned} \quad (\text{A90})$$

$$\begin{aligned} Q_B = & \dot{\theta}^2 (\mathbf{M}_2 - a \mathbf{D}_0 - \mathbf{D}_2) \mathbf{B} + \dot{\theta}^2 \mathbf{S}_{y1}^T - 2\dot{\theta} \mathbf{M}_4^T \dot{\mathbf{A}} + 2\dot{\theta} \mathbf{M}_6 \dot{\mathbf{C}} - \mathbf{K} \mathbf{K}_0 \mathbf{B} + \mathbf{K} \mathbf{K}_1 \mathbf{C} \\ & - \frac{1}{2} \mathbf{K}_2 \mathbf{B} + \frac{1}{2} \mathbf{K}_3 \mathbf{C} - \frac{1}{4} \frac{l^2}{2(1+\nu)} \underline{\underline{\mathbf{K}_4 \mathbf{B}}} - \frac{1}{4} \frac{l^2}{2(1+\nu)} \underline{\underline{\mathbf{K}_5 \mathbf{C}}} \\ & + \dot{\theta}^2 \int_0^L \rho_0 b \left[\underline{\underline{\frac{1}{2} \mathbf{H}(x) \mathbf{B} \cdot \mathbf{B}^T \mathbf{H}(x) \mathbf{B} - \mathbf{H}(x) \mathbf{B} \cdot \Phi_1 \mathbf{A}}} \right] dx \end{aligned} \quad (\text{A91})$$

$$\begin{aligned} & + \dot{\theta}^2 \int_0^L \rho_1 b \left[\underline{\underline{\mathbf{H}(x) \mathbf{B} \cdot \Phi_3 \mathbf{C}}} \right] dx - \int_0^L \rho_0 b \left[\underline{\underline{\mathbf{H}(x) \mathbf{B} \cdot \dot{\mathbf{B}}^T \mathbf{H}(x) \dot{\mathbf{B}}}} \right] dx \\ & + 2\dot{\theta} \int_0^L \rho_0 b \left[\underline{\underline{\Phi_2^T \cdot \mathbf{B}^T \mathbf{H}(x) \dot{\mathbf{B}} - \Phi_2 \dot{\mathbf{B}} \cdot \mathbf{H}(x) \mathbf{B}}} \right] dx \\ Q_C = & -\dot{\theta}^2 \mathbf{S}_z^T - \dot{\theta}^2 \mathbf{M}_5^T \mathbf{A} + \dot{\theta}^2 \mathbf{M}_3 \mathbf{C} - 2\dot{\theta} \mathbf{M}_6^T \dot{\mathbf{B}} + \mathbf{K} \mathbf{K}_1^T \mathbf{B} - \mathbf{K} \mathbf{K}_2 \mathbf{C} + \mathbf{K}_1^T \mathbf{A} - \mathbf{K}_6 \mathbf{C} \\ & + \frac{1}{2} \mathbf{K}_3^T \mathbf{B} - \frac{1}{2} \mathbf{K}_7 \mathbf{C} - \frac{1}{4} \frac{l^2}{2(1+\nu)} \underline{\underline{\mathbf{K}_5^T \mathbf{B}}} - \frac{1}{4} \frac{l^2}{2(1+\nu)} \underline{\underline{\mathbf{K}_8 \mathbf{C}}} \\ & + \int_0^L \rho_1 b \left[\underline{\underline{\frac{1}{2} \dot{\theta}^2 \cdot \Phi_3^T \cdot \mathbf{B}^T \mathbf{H}(x) \dot{\mathbf{B}} - \Phi_3^T \cdot \dot{\mathbf{B}}^T \mathbf{H}(x) \dot{\mathbf{B}}}} \right] dx \end{aligned} \quad (\text{A92})$$

In the aforementioned equation, the corresponding constant matrix is:

$$\mathbf{J}_{cb} = \int_0^L \rho_0 b (a+x)^2 dx \quad (\text{A93})$$

$$\mathbf{J}_2 = \int_0^L \rho_2 b dx \quad (\text{A94})$$

$$\mathbf{M}_1 = \int_0^L \rho_0 b \Phi_1^T(x) \Phi_1(x) dx \quad (\text{A95})$$

$$\mathbf{M}_2 = \int_0^L \rho_0 b \Phi_2^T(x) \Phi_2(x) dx \quad (\text{A96})$$

$$\mathbf{M}_3 = \int_0^L \rho_2 b \Phi_3^T(x) \Phi_3(x) dx \quad (\text{A97})$$

$$\mathbf{M}_4 = \int_0^L \rho_0 b \Phi_1^T(x) \Phi_2(x) dx \quad (\text{A98})$$

$$\mathbf{M}_5 = \int_0^L \rho_1 b \boldsymbol{\Phi}_1^T(x) \boldsymbol{\Phi}_3(x) dx \quad (\text{A99})$$

$$\mathbf{M}_6 = \int_0^L \rho_1 b \boldsymbol{\Phi}_2^T(x) \boldsymbol{\Phi}_3(x) dx \quad (\text{A100})$$

$$\mathbf{S}_x = \int_0^L \rho_0 b(a+x) \boldsymbol{\Phi}_1(x) dx \quad (\text{A101})$$

$$\mathbf{S}_y = \int_0^L \rho_0 b(a+x) \boldsymbol{\Phi}_2(x) dx \quad (\text{A102})$$

$$\mathbf{S}_z = \int_0^L \rho_1 b(a+x) \boldsymbol{\Phi}_3(x) dx \quad (\text{A103})$$

$$\mathbf{S}_{x1} = \int_0^L \rho_1 b \boldsymbol{\Phi}_1(x) dx \quad (\text{A104})$$

$$\mathbf{S}_{y1} = \int_0^L \rho_1 b \boldsymbol{\Phi}_2(x) dx \quad (\text{A105})$$

$$\mathbf{S}_{z1} = \int_0^L \rho_2 b \boldsymbol{\Phi}_3(x) dx \quad (\text{A106})$$

$$\mathbf{D}_0 = \int_0^L \rho_0 b \mathbf{H}(x) dx \quad (\text{A107})$$

$$\mathbf{D}_1 = \int_0^L \rho_1 b \mathbf{H}(x) dx \quad (\text{A108})$$

$$\mathbf{D}_2 = \int_0^L \rho_0 b x \mathbf{H}(x) dx \quad (\text{A109})$$

$$\mathbf{K}_0 = \int_0^L E_0 b \boldsymbol{\Phi}_1'^T(x) \boldsymbol{\Phi}_1'(x) dx \quad (\text{A110})$$

$$\mathbf{K}_1 = \int_0^L E_1 b \boldsymbol{\Phi}_1'^T(x) \boldsymbol{\Phi}_3'(x) dx \quad (\text{A111})$$

$$\mathbf{K}_2 = \int_0^L E_0 b \boldsymbol{\Phi}_2'^T(x) \boldsymbol{\Phi}_2'(x) dx \quad (\text{A112})$$

$$\mathbf{K}_3 = \int_0^L E_0 b \boldsymbol{\Phi}_2'^T(x) \boldsymbol{\Phi}_3(x) dx \quad (\text{A113})$$

$$\mathbf{K}_4 = \int_0^L E_0 b \boldsymbol{\Phi}_2''^T(x) \boldsymbol{\Phi}_2''(x) dx \quad (\text{A114})$$

$$\mathbf{K}_5 = \int_0^L E_0 b \boldsymbol{\Phi}_2''^T(x) \boldsymbol{\Phi}_3'(x) dx \quad (\text{A115})$$

$$\mathbf{K}_6 = \int_0^L E_2 b \boldsymbol{\Phi}_3'^T(x) \boldsymbol{\Phi}_3'(x) dx \quad (\text{A116})$$

$$\mathbf{K}_7 = \int_0^L E_0 b \boldsymbol{\Phi}_3^T(x) \boldsymbol{\Phi}_3(x) dx \quad (\text{A117})$$

$$\mathbf{K}_8 = \int_0^L E_0 b \boldsymbol{\Phi}_3'^T(x) \boldsymbol{\Phi}_3'(x) dx \quad (\text{A118})$$

$$\mathbf{K}\mathbf{K}_0 = \int_V \frac{G(y)}{k} \boldsymbol{\Phi}_2'^T(x) \boldsymbol{\Phi}_2'(x) dV \quad (\text{A119})$$

$$\mathbf{K}\mathbf{K}_1 = \int_V \frac{G(y)}{k} \boldsymbol{\Phi}_2'^T(x) \boldsymbol{\Phi}_3(x) dV \quad (\text{A120})$$

$$\mathbf{K}\mathbf{K}_2 = \int_V \frac{G(y)}{k} \boldsymbol{\Phi}_3^T(x) \boldsymbol{\Phi}_3(x) dV \quad (\text{A121})$$

In the above Formulas (A119)–(A121), $G(y)$ represents the shear modulus of the FGM micro-beam, and k denotes the shear correction coefficient of the FGM micro-beam. The definitions of density and elastic modulus are provided in Appendix A.

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