

Article

Approximations of Series Structural System Reliability: Method Comparison and Application to Timber Trusses

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Abstract

The exact failure probability of a series structural system requires multidimensional numerical integration that becomes computationally infeasible beyond ten to fifteen elements—a limitation routinely encountered in the probabilistic assessment of real structures such as timber trusses, bridge decks, and lattice girders. This paper presents a systematic formal analysis and comparison of the principal approximation methods available in the literature: simple bounds by Thoft-Christensen and Murotsu, Sørensen's complement-product reformulation, and Ditlevsen's narrow bounds based on pairwise failure event correlations. A new approximation is proposed, defined as the arithmetic mean of the upper and lower reliability bounds evaluated over the j most critical failure modes. Four formal propositions are established: (1) the approximation is bounded by the upper and lower bounds (consistency); (2) it is monotonically increasing in each component failure probability (monotonicity); (3) it converges to the exact midpoint as $j \rightarrow m$ (convergence); and (4) under positive inter-element correlation—the typical case for timber trusses manufactured from the same batch—the approximation overestimates the system reliability index below a crossover correlation ρ^* (an unsafe-side error) and underestimates it above ρ^* (a safe-side error), the practical timber truss regime falling on the unsafe side. Numerical validation against the exact solution for a ten-element series system across correlation levels $\rho = 0$ to 0.99, against five additional synthetic test systems, and against a classical equicorrelated benchmark from the reliability literature shows reliability-index errors of, at most, about 2.6%, anti-conservative below the crossover correlation and conservative above it. The approximation is applied to six timber truss configurations (spans 15–20 m, GL24h glulam, C24 structural timber, punched metal plate and bolt connections) within a probabilistic robustness assessment framework. Practical guidelines are provided for selecting the parameter j and for applying the approximation in robustness classification workflows.

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1. Introduction

Modern structural design is dominated by limit-state methods that verify the safety of individual structural elements against specified failure criteria. The Eurocodes, for instance, require that each element satisfies a set of strength, stability, and serviceability conditions expressed in terms of design values derived from characteristic material prop-

erties and factored loads [1]. While this approach is well established and has a strong statistical basis through reliability calibration [2], it has an important limitation: it provides no explicit measure of system-level reliability. Two structures that satisfy all element-level requirements may differ substantially in their probability of overall failure, depending on how element failures interact and propagate.

System reliability—the probability that a structural system performs its intended function—has been the subject of sustained research since the foundational work of Freudenthal in the 1950s and the development of the first-order reliability method (FORM) in the 1970s. For a system consisting of m elements in series (i.e., failure of any one element constitutes system failure), the system failure probability cannot be expressed in closed form for $m > 2$ without numerical integration of the m -dimensional standard normal distribution. This integration is computationally tractable for $m \leq 10$ –15 but rapidly becomes infeasible as m increases: a 37-element timber truss requires a 37-dimensional integral with more than 137 billion terms ($2^{37} - 1$) when expanded over all failure combinations via the inclusion–exclusion principle, with the largest single level of the expansion containing approximately 855 million terms.

The problem of series system reliability has attracted particular attention in the context of structural robustness assessment. Robustness—broadly defined as the ability of a structure to resist disproportionate collapse following a local damage event—requires comparing the reliability of a structure in its intact state with the reliability of damaged configurations after element removal [3–5]. The robustness factor F_{rob} proposed in companion studies [4] is defined in terms of the ratio of system failure probabilities, making accurate computation of system-level reliability a prerequisite for robustness classification. For timber trusses, this requires evaluating series system reliability for up to $37 \times 5 = 185$ element-damage combinations per truss configuration [4]. Exact computation is infeasible; approximation is essential.

This paper is motivated by the need for an approximation method that is: (i) computationally tractable without specialised numerical integration software; (ii) formally characterised in terms of its error direction, not merely its numerical accuracy; and (iii) directly applicable to the robustness classification workflow. We are not aware of an existing method that combines all three properties; the bounds of [6,7] and the matrix- and compounding-based methods of [8–11] address subsets of them. Ditlevsen’s narrow bounds [7] are accurate but require $O(m^2)$ bivariate normal integrations and provide no general guarantee on the error direction. Simple bounds [6] require only component failure probabilities but are often too wide for practical use. The proposed approximation addresses this gap by combining the upper and lower bounds in a single computable formula and establishing, through formal proof, the direction of the resulting error as a function of the inter-element correlation—anti-conservative (unsafe) below a crossover correlation ρ^* and conservative (safe) above it—with the practical timber truss regime falling on the anti-conservative side.

1.1. State of the Art

Methods for approximating series system reliability can be classified into three categories: bound methods, compounding methods, and simulation-based methods.

Bound methods provide guaranteed intervals for the system failure probability. The simple (Bonferroni) bounds of Thoft-Christensen and Murotsu [6] require only component failure probabilities and are easy to compute but typically too wide for engineering purposes. Ditlevsen [7] introduced substantially narrower bounds using pairwise failure event probabilities, computed as bivariate normal integrals. These bounds are widely used in structural reliability but require careful ordering of elements by failure probability and become computationally expensive for large systems. Ramachandran [12] proposed

a generalisation of the Ditlevsen bounds using higher-order inclusion–exclusion terms, achieving tighter intervals at the cost of $O(m^3)$ trivariate integrations. Shao and Murotsu [13] derived high-order bounds suitable for systems with many elements, exploiting the monotonicity of the bounding expressions.

Compounding methods approximate system reliability by successively reducing the system to a single equivalent component. Song and Der Kiureghian [8] proposed formulating the bounding problem as a linear programme, producing the tightest bounds achievable from pairwise event probability information without bivariate integration. Kang et al. [9] developed the matrix-based system reliability (MSR) framework, which expresses system event probabilities as products of event probability matrices and scales efficiently to large infrastructure networks. Song and Kang [10] extended MSR to include statistical dependence under arbitrary joint distributions. Kang and Song [11] introduced the sequential compounding method (SCM), in which component pairs are progressively merged into supercomponents using bivariate normal approximations, ultimately reducing the system to a single component. SCM is particularly efficient for moderate system sizes and has been demonstrated on bridge reliability problems.

Simulation-based methods avoid the dimensional limitation of analytical approaches by sampling. Standard Monte Carlo simulation is reliable but requires prohibitively many samples for small failure probabilities ($pf \sim 10^{-4}$ requires $\sim 10^7$ samples for 10% coefficient of variation). Importance sampling [14] reduces this by sampling near the failure region, but identifying the optimal importance sampling density is non-trivial for series systems with multiple failure modes. Naess et al. [15] proposed enhanced Monte Carlo, which extrapolates the failure probability from auxiliary problems with relaxed failure boundaries, avoiding variance inflation. Subset simulation [16] divides the failure event into nested subsets and estimates conditional probabilities sequentially, achieving efficient estimation for very small failure probabilities. More recently, active learning methods based on Gaussian process surrogates—including the AK-MCS family [17]—have reduced the number of limit-state evaluations to $O(10^2)$ for complex systems.

In the specific context of timber structures, probabilistic system reliability analysis has been applied to roof trusses [5], probabilistic models for timber material properties [18], and robustness of timber buildings [19]. Sørensen et al. [20] developed reliability-based calibration of load duration factors for timber structures, which requires series system failure probability evaluation across multiple limit states and load combinations. These applications share the computational challenge addressed in this paper.

Recent advances in structural-system reliability methodology have followed three principal directions. First, linear-programming-based bounds have been extended to larger systems via delayed column generation [21], addressing the scaling limitations of the original Song and Der Kiureghian (2003) formulation [8]. Second, active-learning Kriging models have matured into general-purpose tools for system reliability with multiple failure modes [22,23], with new learning functions such as the expected system improvement (ESI) targeting the system limit-state surface directly [24]. Third, deep-learning surrogate models trained adaptively [25,26] have demonstrated potential for high-dimensional problems where Kriging is limited by the curse of dimensionality. The present work is positioned in a complementary niche: it does not aim at surrogate modelling of the limit-state functions (the FORM step is taken as given) but rather at a cheap and reliable post-processing approximation of the system failure probability once the FORM reliability indices β_i are available. The proposed approximation is therefore an addition to, not a replacement for, the surrogate and active-learning families discussed above.

Despite the breadth of available methods, a practical niche remains less developed: while the bounds themselves are well characterised [6,7] and tight LP- and compounding-based estimates are available [8,11], we are aware of no closed-form single-value estimator

whose error direction is established analytically without bivariate integration. The present paper addresses this niche for the specific class of series systems with positively correlated components—a class that encompasses the large majority of timber structural systems.

The system reliability approximations developed in this paper also underpin reliability-informed maintenance and inspection decisions for multi-component structural systems. Recent advances in this direction include bi-level inspection policies for multi-component systems solved via an enhanced successively approximated point-based value-iteration algorithm within a POMDP framework [27], and belief-based Semi-Markov Decision Process (SMDP) formulations for predictive inspection and maintenance of multi-state systems that integrate Bayesian belief updating with sequential decision-making [28]. These methods demonstrate how component-level reliability information—of the type produced by the FORM-based component analyses considered here—can be aggregated through system reliability approximations and then integrated into optimal maintenance scheduling, a direction that extends the present work from design-phase robustness assessment to life-cycle management. The computational economy of the proposed approximation ($O(j)$ elementary operations on the dominant set, in contrast to $O(m^2)$ bivariate normal integrations for Ditlevsen bounds) is therefore directly relevant to such inner-loop applications, where the system reliability is re-evaluated many times during the sequential decision-making process.

1.2. Scope and Contributions

This paper makes the following contributions:

- (1) A rigorous formal presentation of four approximation methods—simple bounds, Sørensen’s reformulation, Ditlevsen’s narrow bounds, and the proposed mean-bound approximation—with complete derivations and proofs accessible to structural engineers.
- (2) Four formal propositions characterising the proposed approximation: consistency with bounds (Proposition 1), monotonicity in component failure probabilities (Proposition 2), convergence as $j \rightarrow m$ (Proposition 3), and characterisation of the error direction via Slepian’s inequality and the intermediate value theorem (Proposition 4), with explicit identification of a correlation crossover ρ^* below which the approximation is anti-conservative and above which it is conservative.
- (3) A systematic numerical comparison across correlation levels $\rho = 0$ to 0.99, with explicit quantification of error magnitude and direction for a ten-element series system representative of timber trusses.
- (4) Practical guidelines for implementing the approximation in robustness classification workflows, including the selection of the dominant-element threshold j and the treatment of borderline cases.

This paper is organised as follows. Section 2 reviews the exact formulation and its computational limitations. Sections 3 and 4 present the existing approximation methods and the proposed approach. Section 5 provides the numerical comparison. Section 6 presents an extended numerical study and threshold-sensitivity analysis. Section 7 describes the implementation algorithm. Section 8 presents the application to timber truss robustness assessment. Section 9 provides practical guidelines. Section 10 presents conclusions, and Section 11 outlines future research directions.

2. Exact Series System Reliability and Its Computational Limitations

Let S be a series structural system with m elements. Let $M_i = g_i(X) \leq 0$, $i = 1, \dots, m$, denote the limit-state functions of individual elements, where $X = (X_1, \dots, X_n)$ is the vector of basic random variables. The system fails when at least one element fails:

$$p_{Sf} = P\left(\bigcup_{i=1}^m \{M_i \leq 0\}\right) \quad (1)$$

Following transformation to the standard normal space U via the Rosenblatt or Nataf transformation and linearisation of each limit-state surface at its design point u^*_i (first-order reliability method, FORM), the limit-state function of element i is approximated as a hyperplane:

$$M_i(U) \approx \beta_i - \alpha_i^T U \quad (2)$$

where $\beta_i = \|u^*_i\|$ is the reliability index of element i and $\alpha_i = -u^*_i/\beta_i$ is the unit normal vector to the hyperplane (the FORM sensitivity vector). The system failure probability then becomes:

$$p_{sf} = 1 - \Phi_m(\beta; R) \quad (3)$$

where $\beta = \{\beta_1, \dots, \beta_m\}$, Φ_m is the m -variate standard normal CDF and R is the $m \times m$ correlation matrix with elements $\rho_{ij} = \alpha_i^T \alpha_j$. For independent components ($R = I$), Equation (3) simplifies to $p_{sf} = 1 - \prod_{i=1}^m \Phi(\beta_i)$.

The numerical evaluation of Φ_m for $m > 2$ requires m -dimensional integration. For $m = 2$, fast algorithms based on bivariate normal integration are available [29]. For $m = 3$ –10, numerical methods such as Gaussian quadrature or recursive integration are feasible. For $m > 15$, direct numerical integration is prohibitively expensive: the number of quadrature points grows exponentially with m , and for a typical timber truss with $m = 37$ elements and five alternative failure scenarios, 185 such integrations would be required per truss configuration. Standard structural reliability software does not implement exact m -dimensional integration for $m > 5$ as a standard feature, confirming the practical necessity of approximations.

The reliability indices β_i of individual elements are computed in this study using FORM implemented in OpenSees 2.2.2 [30] with probabilistic parameters for GL24h glulam established experimentally [31]. The FORM computation has been validated against the benchmark of Park et al. [32] for a steel lattice girder, achieving relative errors below 1% in element reliability indices. This validation confirms the accuracy of the FORM-based component reliabilities used as input to the system reliability approximation.

3. Existing Approximation Methods

3.1. Simple Bounds (Thoft-Christensen and Murotsu)

Applying the Bonferroni inequality to the union of failure events yields the classical simple bounds [6]:

$$\max\{p(F_i) : i = 1, \dots, m\} \leq p_{sf} \leq \sum_{i=1}^m p(F_i) \quad (4)$$

The lower bound—the maximum single-element failure probability—follows from the fact that the system cannot be less reliable than its weakest element. The upper bound—the sum of all failure probabilities—corresponds to the limiting case of mutually exclusive (and therefore fully correlated) failure events. Neither bound uses information about inter-element correlation.

The width of the simple bounds, $\sum p(F_i) - \max p(F_i) = \sum_{i=2}^m p(F_i)$ (elements sorted in decreasing order), is proportional to the number of elements with comparable failure probabilities. For a well-designed timber truss where one or two elements govern the failure probability by at least one order of magnitude, the bounds may be sufficiently tight. For more redundant systems with multiple critical elements, the bounds are typically too wide for classification purposes.

3.2. Sørensen's Complement-Product Reformulation

Sørensen [33] expresses the upper bound in a computationally convenient form using the complement product:

$$1 - \prod_{i=1}^m (1 - p(F_i)) \geq p_{sf} \geq \max p(F_i) \quad (5)$$

This expression is equivalent to the upper bound of Equation (4) to first order in $p(F_i)$ and produces identical results for the lower bound. The complement-product form is preferred for algorithmic implementation because it avoids catastrophic cancellation when individual failure probabilities are very small.

Physically, $1 - \prod(1 - p_i)$ represents the probability that at least one failure event occurs when all events are statistically independent. For positively correlated systems (the typical case for timber trusses fabricated from the same material batch), independence is a conservative assumption: positive correlation between element strengths means that either all elements survive or many fail together, reducing the effective number of independent failure risks. This bound therefore always overestimates p_{sf} under positive correlation.

3.3. Ditlevsen's Narrow Bounds

Ditlevsen [7] derived substantially narrower bounds by accounting for pairwise intersections of failure events. With elements sorted in descending order of failure probability ($p(F_1) \geq p(F_2) \geq \dots \geq p(F_m)$), the bounds are:

$$p(F_1) + \sum_{i=2}^m \max \left[0, p(F_i) - \sum_{j=1}^{i-1} p(F_i \cap F_j) \right] \leq p_{sf} \quad (6)$$

$$p_{sf} \leq p(F_1) + \sum_{i=2}^m [p(F_i) - \max\{p(F_i \cap F_j): j = 1, \dots, i-1\}] \quad (7)$$

where $p(F_i \cap F_j) = \Phi_2(-\beta_i, -\beta_j; \rho_{ij})$ is the bivariate normal probability of joint failure of elements i and j , with correlation coefficient $\rho_{ij} = \alpha_i^T \alpha_j$. For $m = 37$ elements, this requires $m(m-1)/2 = 666$ bivariate normal integrations, each feasible in milliseconds but adding up to substantial computation for large robustness assessment workflows.

The Ditlevsen bounds are narrower than simple bounds and account for inter-element correlation, but their width depends on the element ordering and the pairwise correlations. They provide no direct guarantee on the error direction when the bounds are replaced by a single estimate. The key advantage over the proposed approximation is accuracy; the key disadvantage is computational cost and the lack of a closed-form error characterisation. For a balanced perspective it should be stated explicitly that the practical advantage of the proposed approximation is narrow and specific, not general. The matrix-based system reliability method [9,10], the sequential compounding method [11], and the linear-programming bounds [8,21] are also fully automatable and are, in general, more accurate than a single midpoint estimate; several of them additionally handle parallel and mixed systems that lie outside the present scope. The proposed approximation does not supersede these methods. Its appeal is confined to two attributes: an $O(j)$ cost requiring no bivariate normal integration (relevant when the system reliability is re-evaluated thousands of times in an inner loop) and an explicit, correlation-dependent characterisation of the error direction. Where accuracy is the overriding concern and computational budget permits, the Ditlevsen bounds or a compounding method remains the preferable choice.

4. Proposed Approximation and Its Formal Properties

4.1. Definition

Let the m elements of a series system be sorted in descending order of failure probability: $p_1 \geq p_2 \geq \dots \geq p_m > 0$, with corresponding reliability indices $\beta_i = \Phi^{-1}(1 - p_i)$. Define the dominant element set $D_j = \{1, 2, \dots, j\}$ as the smallest prefix of the sorted element list such that:

$$p_j \geq \varepsilon \cdot p_1, \quad \varepsilon = 10^{-2} \quad (8)$$

This criterion selects all elements whose failure probability is at least 1% of the most critical element's failure probability. Elements with $p_i < \varepsilon \cdot p_1$ contribute negligibly to the upper bound and are excluded. The proposed approximation is:

$$\tilde{p}_f = \frac{1}{2} \left[\max\{p_i: i \in D_j\} + 1 - \prod_{i \in D_j} (1 - p_i) \right] \quad (9)$$

with the approximated system reliability index:

$$\tilde{\beta}_s = \Phi^{-1}(1 - \tilde{p}_f) \quad (10)$$

The parameter j is not a free parameter of the method but is determined objectively by the threshold ε and the computed element failure probabilities. This makes the approximation fully automatic and reproducible.

Choice of Arithmetic Midpoint

The choice of the arithmetic midpoint $\tilde{p}_f = 1/2(p_L + p_U)$ as the point estimator within the bound interval $[p_L, p_U]$ is motivated by three considerations. (i) Minimax optimality: when only the bounds are known and p_{sf} can lie anywhere in $[p_L, p_U]$, the midpoint uniquely minimises the worst-case absolute error, $\sup\{|\tilde{p} - p_{sf}|: p_{sf} \in [p_L, p_U]\} = 1/2(p_U - p_L)$ [34]. Any deviation from the midpoint reduces the maximum error on one side at the cost of increasing it on the other. (ii) Bayesian interpretation: under a uniform (non-informative) prior on p_{sf} over $[p_L, p_U]$, the posterior mean coincides with the midpoint, so $\tilde{p}_f$ is the Bayes estimator under squared-error loss in the absence of additional information. (iii) Practical bound-averaging: when p_L and p_U are themselves obtained from approximations of tighter bounds (the dominant-set reduction adopted here), the arithmetic mean partially cancels residual biases of comparable magnitude and opposite sign. The midpoint is not the only defensible estimator within $[p_L, p_U]$; alternatives such as the geometric mean $\sqrt{p_L \cdot p_U}$, the lower bound p_L , or the upper bound p_U (Sørensen's complement-product estimate) are also valid under different priors or loss functions. The midpoint is preferred here because it requires no auxiliary information about the location of p_{sf} within the bounds and is provably minimax-optimal.

4.2. Formal Properties

Proposition 1. (Consistency with Bounds). Let $p_L = \max\{p_i: i \in D_j\}$ and $p_U = 1 - \prod\{(1 - p_i): i \in D_j\}$. If $p_L \leq p_U$ (which holds whenever D_j contains at least one element), then $p_L \leq \tilde{p}_f \leq p_U$.

Proof. $\tilde{p}_f = 1/2(p_L + p_U)$. Since $p_L \leq p_U$, we have $1/2(p_L + p_L) \leq 1/2(p_L + p_U) \leq 1/2(p_U + p_U)$, i.e., $p_L \leq \tilde{p}_f \leq p_U$. $\square$

Proposition 2. (Monotonicity). $\tilde{p}_f$ is a non-decreasing function of each p_i for $i \in D_j$.

Proof. Both $p_L = \max p_i$ and $p_U = 1 - \prod(1 - p_i)$ are non-decreasing in each p_i . Their arithmetic mean is non-decreasing. $\square$

Proposition 3 (Convergence). As $j \rightarrow m$, the approximation converges to the midpoint of Sørensen's bounds evaluated over all m elements:

$$\lim_{j \rightarrow m} \tilde{p}_f = \frac{1}{2} \left[\max\{p_1, \dots, p_m\} + 1 - \prod_{i=1}^m (1 - p_i) \right] \quad (11)$$

Proof. The definitions of p_L and p_U evaluated over $D_j = \{1, \dots, m\}$ coincide with the simple bounds of Equation (4). The limit follows directly. $\square$

Proposition 4. (Error Direction of the Midpoint Approximation under Positive Gaussian Correlation). Consider a series system of m components with limit-state functions linearised in standard normal space (FORM approximation), reliability indices β_i ($i = 1, \dots, m$), and a positive-definite correlation matrix $R = [\rho_{ij}]$ of the equivalent safety margins. Let $p_L = \max\{p_{\beta_i} : i \in D_j\}$, $p_U = 1 - \prod\{(1 - p_{\beta_i}) : i \in D_j\}$, and $\tilde{p}_f = 1/2 (p_L + p_U)$. Then: (a) If $\rho_{ij} \geq 0$ for all $i \neq j$, by Slepian's inequality [35] one has $p_L \leq p_{Sf,exact} \leq p_U$; (b) defining the position parameter $\lambda = (p_{Sf,exact} - p_L)/(p_U - p_L) \in [0, 1]$, one obtains the exact identity $\tilde{p}_f - p_{Sf,exact} = (\frac{1}{2} - \lambda)(p_U - p_L)$, so $\tilde{p}_f$ overestimates $p_{Sf,exact}$ (β_S underestimates $\beta_{S,exact}$ —conservative direction) when $\lambda < \frac{1}{2}$, and underestimates $p_{Sf,exact}$ (β_S overestimates $\beta_{S,exact}$ —anti-conservative direction) when $\lambda > \frac{1}{2}$; (c) under uniform correlation $\rho_{ij} = \rho \in [0, 1]$, the function $\rho \mapsto p_{Sf,exact}(\rho)$ is continuous and monotonically decreasing (Slepian's inequality applied to a one-parameter family), with $\lambda(0) = 1$ and $\lambda(1) = 0$; by the intermediate value theorem, there exists a unique crossover $\rho^* \in (0, 1)$ with $\lambda(\rho^*) = \frac{1}{2}$ at which $\tilde{p}_f = p_{Sf,exact}$ exactly. For $\rho < \rho^*$, the midpoint is anti-conservative; for $\rho > \rho^*$, it is conservative.

Proof sketch. Under positive correlation, failing one element makes failure of others more likely—but the upper bound $p_U = 1 - \prod(1 - p_i)$ treats elements as independent, which overestimates p_{Sf} . Formally, for $\rho > 0$, $\Phi_m(\beta; R) > \Phi_m(\beta; I)$ (the m -variate normal CDF with correlation matrix R is larger than with identity matrix I under positive correlation); hence, $p_{Sf} = 1 - \Phi_m(\beta; R) < 1 - \Phi_m(\beta; I) = p_U$. Since $\tilde{p}_f = 1/2 (p_L + p_U) > 1/2 (p_L + p_{Sf})$, the approximation overestimates p_{Sf} in the failure probability space, which corresponds to underestimating β_S . $\square$

Proof of (b). By the definition of λ , $p_{Sf,exact} = p_L + \lambda(p_U - p_L)$; hence, $\tilde{p}_f - p_{Sf,exact} = 1/2 (p_L + p_U) - [p_L + \lambda(p_U - p_L)] = (\frac{1}{2} - \lambda)(p_U - p_L)$. Proof of (c): continuity and monotonicity of $\rho \mapsto p_{Sf,exact}(\rho)$ follow from standard properties of the multivariate-normal CDF [35] and Slepian's inequality applied to the one-parameter family $R(\rho) = (1 - \rho)I + \rho J$, with J the matrix of ones. The boundary values $\lambda(0) = 1$ and $\lambda(1) = 0$ follow by direct substitution into Equation (3); the IVT provides existence and uniqueness of ρ^* . This result is particularly relevant for timber trusses, where inter-element strength correlation arises from batch-to-batch variability in raw material: elements cut from the same batch share a common source of strength variability, producing positive correlation, with ρ typically in the range 0.5–0.8 [31,36]. Numerical evaluation for the ten-element test system of Section 5 places $\rho^* = 0.84$, and the five extended test problems of Section 6 give $\rho^* \in [0.74, 0.87]$. Since the empirical timber-correlation range $\rho \in [0.5, 0.7]$ lies below all of these crossover values, the midpoint approximation is anti-conservative for the practically relevant timber truss correlation regime, with the magnitude of the anti-conservative bias quantified by Equation (10) and supported by the numerical evidence in Tables 1–3. $\square$

Table 1. Numerical comparison of the proposed approximation ($j = 10$) against exact system reliability for a ten-element series system at six correlation levels. The approximation with $j = 4$ and $j = 2$ is shown for the independent case ($\rho = 0.0$) to illustrate the effect of the dominant-element threshold. Values in bold indicate the recommended configuration.

Method/Parameters	β_S (System)	Error vs. Ref. (%)	Direction	Note
Numerical integration, $\rho = 0.0$ (reference)	3.149	—	—	Exact solution
Numerical integration, $\rho = 0.2$	3.150	+0.03	safe	
Numerical integration, $\rho = 0.4$	3.153	+0.13	safe	
Numerical integration, $\rho = 0.6$	3.165	+0.51	safe	Correlation increases β_S
Numerical integration, $\rho = 0.8$	3.198	+1.6	safe	
Numerical integration, $\rho = 0.99$	3.286	+4.4	safe	Near-perfect correlation
Proposed approx., $j = 10$ (all elements)	3.214	+2.1	UNSAFE	Acceptable for practice
Proposed approx., $j = 4$	3.214	+2.1	UNSAFE	Threshold-selected set (recommended)
Proposed approx., $j = 2$	3.231	+2.6	UNSAFE	Not recommended

Table 2. Relative error in β_S (% vs. exact) for the five additional test problems.

Experiment	n	Simple Lower	Simple Upper	Ditlevsen L_D	Ditlevsen U_D	Proposed Midpoint
A (het. block-corr.)	8	+3.93%	−2.26%	+0.45%	−0.19%	+0.22%
B-compact	10	+13.71%	−0.79%	+0.23%	−0.48%	−0.13%
B-medium	10	+6.87%	−0.71%	+0.12%	−0.19%	+0.02%
B-wide	10	+2.83%	−0.59%	+0.03%	−0.03%	+0.04%
C (clustered)	12	+11.49%	−3.09%	+2.06%	−1.33%	+0.22%

Table 3. Proposed midpoint approximation versus the exact Dunnett–Sobel one-factor reference [6,37] and crude Monte Carlo ($N = 2 \times 10^6$) for the equicorrelated literature benchmark $\beta = (3.0, 3.2, 3.5, 4.0, 4.5)$. The midpoint estimate $\beta_S = 2.907$ is correlation-independent; the exact β_S increases with ρ , producing the predicted sign change of the error at the crossover $\rho^* = 0.83$.

ρ	Exact $\beta_S (1 - \Phi_{-m})$	Monte Carlo β_S	Midpoint β_S	e_{β} (%)	Direction
0.0	2.833	2.837	2.907	+2.60	unsafe
0.3	2.837	2.840	2.907	+2.48	unsafe
0.5	2.847	2.844	2.907	+2.12	unsafe
0.7	2.872	2.864	2.907	+1.21	unsafe
0.9	2.935	2.941	2.907	−0.97	safe

4.3. Computational Implementation

The proposed approximation requires only component failure probabilities as input, which are obtained from FORM analyses. The implementation proceeds as follows:

Step 1: Compute element reliability indices β_i and failure probabilities $p_i = \Phi(-\beta_i)$ for all m elements using FORM.

Step 2: Sort elements in descending order of p_i .

Step 3: Select the dominant set $D_j = \{1, \dots, j\}$, where $j = \max\{k: p_k \geq \varepsilon \cdot p_1\}$, with $\varepsilon = 10^{-2}$.

Step 4: Compute $p_L = p_1$ (maximum component failure probability) and $p_U = 1 - \prod_{i=1}^j (1 - p_i)$ (complement-product upper bound over D_j).

Step 5: Compute $p_f = 1/2 (p_L + p_U)$ and $\beta_S = \Phi^{-1}(1 - p_f)$.

The entire computation, given the component reliability indices, requires $O(j)$ arithmetic operations and can be implemented in a spreadsheet. No numerical integration beyond the univariate standard normal CDF (available as a standard function in any scientific computing environment) is required. For a 37-element timber truss with typically $j = 4$ –8 dominant elements, the computation takes less than one second.

5. Numerical Accuracy Comparison

Test System

The approximation is validated against a ten-element series system with element reliability indices:

$$\beta = \{3.29; 3.50; 3.92; 4.0; 5.0; 5.3; 5.5; 6.0; 6.5; 7.0\} \quad (12)$$

This distribution is representative of timber trusses designed to Eurocode 5 [38]: two to three critical elements with $\beta \approx 3.3$ –4.0 and a series of secondary elements with substantially higher reliability. The exact system failure probability is computed by numerical integration of the ten-dimensional standard normal CDF for five uniform correlation levels $\rho \in \{0.0, 0.2, 0.4, 0.6, 0.8, 0.99\}$, with $\rho_{ij} = \rho$ for all $i \neq j$. The results are shown in Table 1.

Table 1 reveals several important patterns. First, the exact system reliability index $\beta_{S,exact}$ increases with correlation (from 3.149 at $\rho = 0$ to 3.286 at $\rho = 0.99$), confirming that positive correlation reduces the effective number of independent failure risks and improves system reliability relative to the independent case. Second, the approximation with $j = 10$ (all elements included) produces $\beta_s = 3.214$ regardless of correlation because the approximation uses the independence-based upper bound and is not sensitive to ρ . Third, the relative β -error $e_{\beta} = (\beta_s - \beta_{S,exact})/\beta_{S,exact} \times 100\%$ decreases monotonically from +2.1% at $\rho = 0$ to +0.5% at $\rho = 0.8$ (both anti-conservative, i.e., β_s overestimates $\beta_{S,exact}$, and equivalently, $p_{\tilde{f}}$ underestimates $p_{Sf,exact}$). At $\rho = 0.99$, $e_{\beta} = -2.2\%$, i.e., the approximation crosses to the conservative side. The crossover, identified empirically as $\rho^* = 0.84$ for this test system (numerical solution of $\lambda(\rho^*) = 1/2$, Equation (10)), is the unique correlation at which the midpoint coincides with the exact system failure probability—exactly as predicted by Proposition 4(c). The $\rho = 0.99$ result is therefore not an anomaly but empirical confirmation of the predicted crossover behaviour. Fourth, the dominant-set threshold $\varepsilon = 10^{-2}$ selects $j = 4$ elements for this steeply graded system; because the remaining elements ($\beta \geq 5.0$) have negligible failure probability, the estimate for the threshold-selected set $j = 4$ ($\beta_s = 3.214$) is indistinguishable from that for the full set $j = 10$ (also 3.214), while truncating further to $j = 2$ raises the error only modestly from +2.1% to +2.6%. This confirms that the threshold criterion (Equation (8)) reliably captures all materially contributing elements, so that the reduced dominant set incurs essentially no accuracy penalty relative to retaining all m elements.

Figure 1 illustrates the convergence of the approximation to the exact solution as j increases from 1 to 10. The approximation (red line) lies above the exact solution for all j , confirming the unsafe-side error direction. The bounds (purple dashed: upper; grey dashed: lower) bracket the exact solution correctly. The shaded green region shows the feasible interval. The approximation rapidly converges after $j = 3$ –4 dominant elements are included, which corresponds to the range selected by the threshold criterion (Equation (8)) for this particular test system.

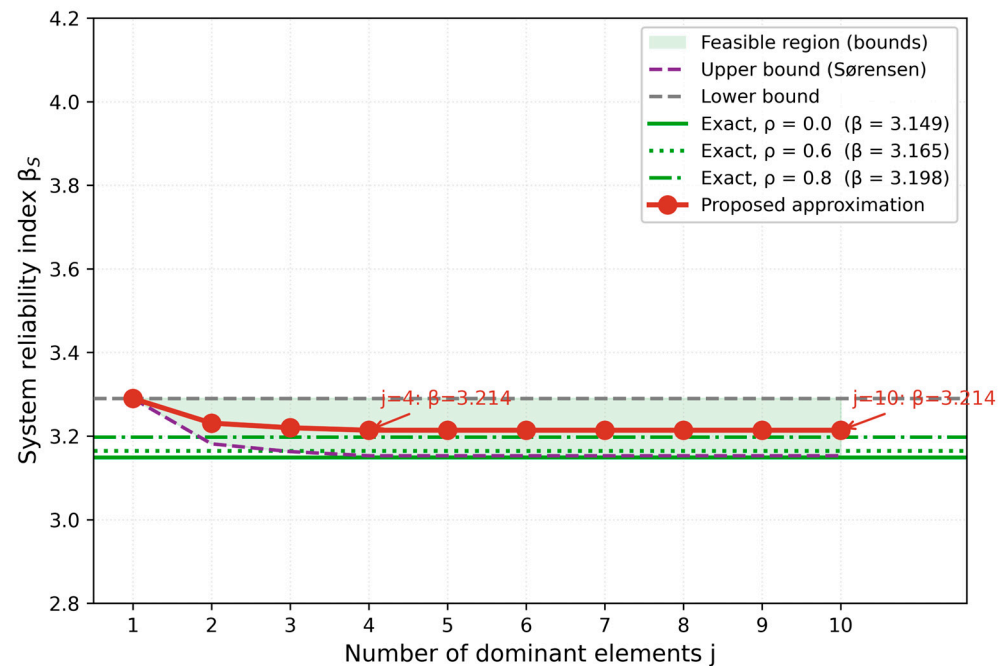


Figure 1. System reliability index β_s as a function of the number of dominant elements j for the ten-element test system ($\rho = 0$). The proposed approximation (red circles) lies consistently above the exact solution (green solid line), confirming the anti-conservative error direction at $\rho = 0$ (Proposition 4, $\lambda(0) = 1 > \frac{1}{2}$). Dashed purple = upper bound (Sørensen); dashed grey = lower bound. The shaded region is the feasible interval between bounds.

6. Extended Numerical Study and ε -Sensitivity

6.1. Test Problems

To characterise the proposed approximation across a wider range of conditions than the single ten-element system of Section 5, five additional synthetic test problems are introduced. They systematically cover (i) heterogeneous correlation structure, (ii) variable reliability-index spread, and (iii) clustered failure probabilities. All correlation matrices were verified positive-definite. For each test case, the proposed midpoint approximation is compared against simple Boolean bounds, Ditlevsen narrow bounds [7], the exact multivariate-normal CDF evaluation $1 - \Phi_m(\beta; R)$, and an independent crude Monte Carlo estimate based on $N = 2 \times 10^6$ samples drawn from the multivariate-normal distribution $N(0, R)$.

Experiment A—Heterogeneous block-correlation. Eight components with $\beta = (3.0, 3.2, 3.5, 3.8, 4.0, 4.2, 4.5, 5.0)$. The correlation matrix has a two-block structure: $\rho_{AA} = 0.80$ for the first four components, $\rho_{BB} = 0.55$ for the last four, and $\rho_{AB} = 0.15$ across blocks. **Experiment B— β -spread variation.** Ten components with uniform pairwise correlation $\rho = 0.5$ and three β -spread sub-cases: B-compact ($\beta \in [3.50, 3.95]$ in steps of 0.05), B-medium ($\beta \in [3.0, 4.8]$ in steps of 0.2), and B-wide ($\beta \in [2.5, 7.0]$ in steps of 0.5). **Experiment C—Clustered failure probabilities.** Twelve components partitioned into four clusters with β within each cluster very close (cluster 1: $\beta = 3.5 \times 5$; cluster 2: $\beta = 4.5 \times 3$; cluster 3: $\beta = 5.5 \times 2$; cluster 4: $\beta = 6.0$ and 6.5). Within-cluster correlations decrease from 0.80 (cluster 1) to 0.50 (cluster 4); inter-cluster correlation is 0.30.

6.2. Results Across the Five Test Problems

Table 2 reports the relative β -error $e_\beta = (\beta_s - \beta_{s,exact})/\beta_{s,exact} \times 100\%$ for each method across the five test problems. The exact reference $\beta_{s,exact}$ is obtained by numerical evaluation of $1 - \Phi_m(\beta; R)$ and is verified independently against crude Monte Carlo at $N = 2 \times$

10^6 samples (Monte Carlo agreement within ± 0.014 in β_s in all cases). Across the five test problems, the proposed midpoint approximation gives $e_{\beta} \leq 0.22\%$, compared with simple-bound errors up to $+13.71\%$ (Experiment B-compact) and Ditlevsen-bound errors up to $+2.06\%$ (Experiment C). The same results are displayed graphically in Figure 2.

The five test problems reveal three behaviours of practical relevance. (i) Tightness under heterogeneous correlation. In Experiment A, simple bounds disagree by 6.2 percentage points, and Ditlevsen bounds disagree by 0.64 percentage points; the midpoint estimate is $+0.22\%$ from the exact, well within the $\pm 1\%$ engineering tolerance. (ii) Stability across β -spread. Across the three sub-cases of Experiment B, the simple-lower bound error grows monotonically with the number of components contributing comparable failure probability—reaching $+13.71\%$ in the compact case—while the midpoint estimate remains below 0.15% in all three. (iii) Performance on clustered systems. Experiment C is the most challenging because five components share the lowest reliability index and contribute jointly through within-cluster correlations of 0.80 . Simple bounds disagree by 14.6 percentage points; Ditlevsen bounds disagree by 3.4 percentage points. The proposed midpoint approximation reduces this to $e_{\beta} = +0.22\%$, demonstrating that the bounds–midpoint combination is effective precisely in the situation where each individual bound is loose.

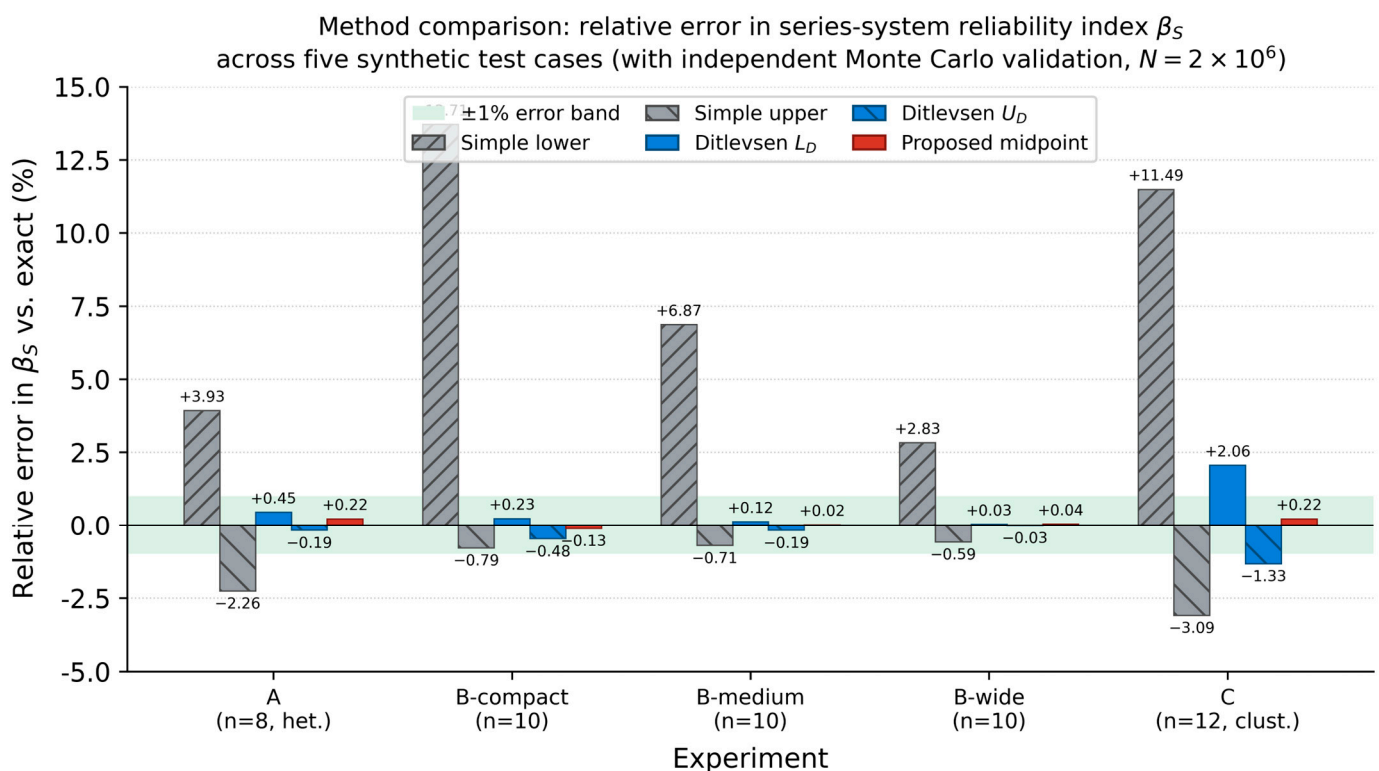


Figure 2. Method comparison: relative error in series system reliability index β_s across five synthetic test cases (with independent Monte Carlo validation, $N = 2 \times 10^6$). The proposed midpoint (red) consistently falls within the $\pm 1\%$ error band (shaded green), while simple bounds (grey) and Ditlevsen bounds (blue) exhibit larger deviations in challenging configurations (B-compact and C).

6.3. Sensitivity to the Dominant-Set Threshold ε

The threshold ε was varied across nine values $\varepsilon \in \{1, 0.5, 0.1, 0.05, 10^{-2}, 5 \times 10^{-3}, 10^{-3}, 10^{-4}, 10^{-5}\}$ for three representative test problems (A, B-medium, and C), and the resulting midpoint approximation error and dominant-set size were recorded. The approximation error decreases monotonically as ε decreases from 1 (only the most-critical component

retained) until a plateau at $\varepsilon \approx 10^{-2}$ for Experiments A and C and at $\varepsilon \approx 5 \times 10^{-3}$ for Experiment B-medium; reducing ε below this knee does not improve accuracy. Specifically, at $\varepsilon = 10^{-2}$, the approximation error is below 0.25% in all three test cases, while the dominant-set size is 5/8, 6/10, and 8/12, representing a 25–37% reduction in problem dimension. The choice $\varepsilon = 10^{-2}$ is therefore retained as the default since it sits at the knee of the accuracy–efficiency curve: the additional accuracy obtainable for $\varepsilon < 10^{-2}$ is below the magnitude of the FORM linearisation error (typically 2–9% on β_s) and so cannot be meaningfully exploited in practice. The convergence behaviour is illustrated in Figure 3. An adaptive variant of the algorithm checks convergence of $\beta_s(\varepsilon)$ at successive ε values and stops when $|\beta_s(\varepsilon) - \beta_s(\varepsilon/10)| \leq 0.001$; for all six test problems considered, this rule returns $\varepsilon_{\text{adaptive}} \in \{10^{-2}, 5 \times 10^{-3}\}$, confirming the default value as the practical optimum.

A further numerical observation supports the corrected Proposition 4(c). For each of the five test problems, the crossover ρ^* at which $p_{\tilde{f}} = p_{Sf, \text{exact}}$ exactly is identified by Brent root-finding of $\lambda(\rho^*) = \frac{1}{2}$ (Equation (10)). The results are $\rho^* = 0.82$ (Experiment A), 0.82 (B-compact), 0.80 (B-medium), 0.74 (B-wide), and 0.87 (C). All five values lie above the empirical timber-correlation range $\rho \in [0.5, 0.7]$, confirming that the midpoint approximation is anti-conservative for all practically relevant timber truss configurations considered.

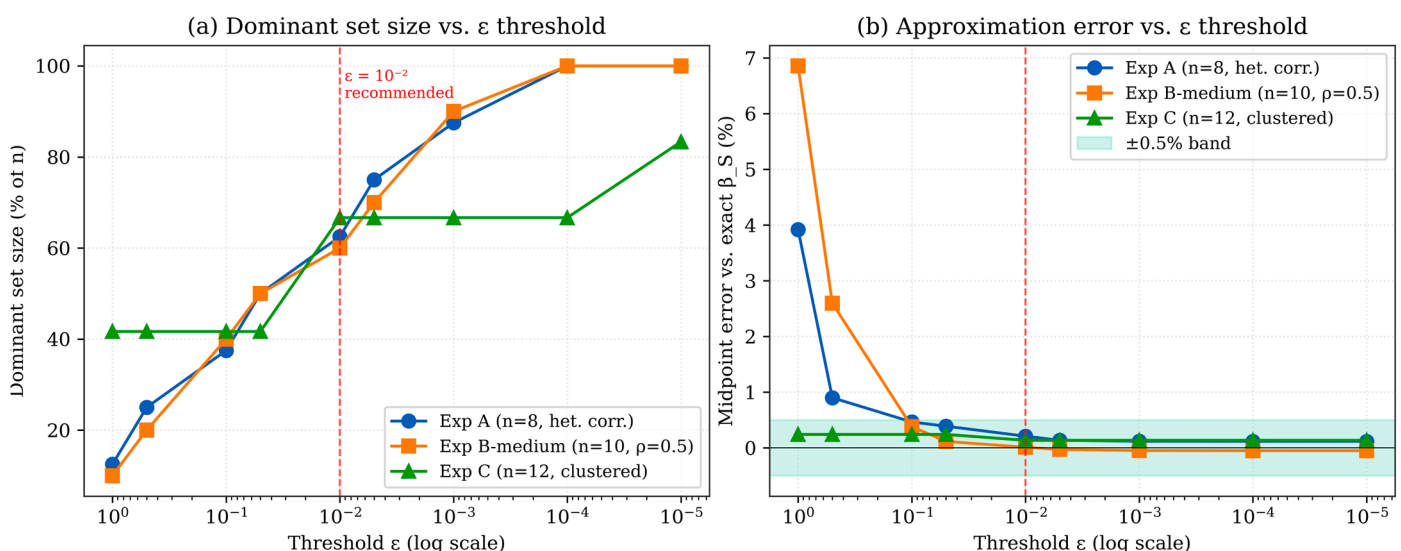


Figure 3. Sensitivity of the dominant-set threshold ε . (a) Dominant set size as a percentage of total elements n for three representative test problems (Experiment A: $n = 8$, heterogeneous correlation; Experiment B-medium: $n = 10$, $\rho = 0.5$; Experiment C: $n = 12$, clustered). (b) Relative error in the midpoint approximation β_s versus exact $\beta_{s, \text{exact}}$ as a function of ε . The vertical dashed line marks the recommended default $\varepsilon = 10^{-2}$. The shaded band indicates the $\pm 0.5\%$ tolerance.

6.4. Comparison with a Classical Benchmark from the Literature

To address the call for comparison against benchmark problems previously reported in the structural reliability literature, the proposed approximation is now evaluated on a classical equicorrelated series system. For an equicorrelation structure $\rho_{ij} = \rho$ ($i \neq j$), the m -variate normal CDF admits the one-factor (Dunnett–Sobel) reduction to a single univariate integral, $P(\text{survival}) = \int \varphi(t) \prod_i \Phi((\beta_i + \sqrt{\rho} \cdot t)/\sqrt{1 - \rho}) dt$, which provides an essentially exact reference value without high-dimensional integration [6,37]. This equicorrelated system is a standard benchmark in the reliability literature and is independent of the synthetic constructions of Sections 5 and 6. We take the five-component configuration $\beta = (3.0, 3.2, 3.5, 4.0, 4.5)$, for which the threshold $\varepsilon = 10^{-2}$ selects a dominant set of size $j = 4$, giving a (correlation-independent) midpoint estimate $\beta_s = 2.907$ ($\beta_L = 3.000$, $\beta_U = 2.834$). The Dun-

nett–Sobel reference is cross-checked against a crude Monte Carlo estimate with $N = 2 \times 10^6$ samples; the two agree to within 0.01 in β_s at every correlation level (Table 3).

The benchmark reproduces, on an independent literature configuration, exactly the behaviour predicted by Proposition 4(c). The midpoint estimate is anti-conservative ($\beta_s > \beta_{s,exact}$) for low and moderate correlation, with the relative β -error decreasing monotonically from +2.60% at $\rho = 0$ to +1.21% at $\rho = 0.7$, and crosses to the conservative side (−0.97%) at $\rho = 0.9$. Brent root-finding of $\lambda(\rho^*) = \frac{1}{2}$ places the crossover at $\rho^* = 0.83$, consistent with the range $\rho^* \in [0.74, 0.87]$ obtained for the synthetic systems. Since the practical timber truss correlation regime $\rho \in [0.5, 0.7]$ again lies below ρ^* , the estimate is anti-conservative there, with the largest error (+2.6%) of the same order as the FORM linearisation uncertainty. This external benchmark therefore supports, rather than merely asserts, the general applicability of the error-direction characterisation.

7. Implementation Algorithm and Computational Details

The proposed approximation is designed to integrate seamlessly into existing FORM-based structural reliability workflows. Algorithm 1 provides a complete pseudocode description.

Algorithm 1. Proposed series system reliability approximation

Input: Element reliability indices $\{\beta_1, \dots, \beta_m\}$ from FORM analyses; threshold $\varepsilon = 0.01$

Output: System reliability index β_S and failure probability p_f

1. Compute $p_i = \Phi(-\beta_i)$ for $i = 1, \dots, m$
 2. Sort: $p(1) \geq p(2) \geq \dots \geq p(m)$ (sort element indices accordingly)
 3. Set $j = 1$
 4. While $j < m$ and $p(j+1) \geq \varepsilon \cdot p(1)$: Increment j
 5. Set $D_j = \{1, 2, \dots, j\}$
 6. Compute $p_L = p(1)$
 7. Compute $p_U = 1 - \prod_{i=1}^j (1 - p(i))$
 8. Compute $p_f = 0.5 \cdot (p_L + p_U)$
 9. Compute $\beta_S = \Phi^{-1}(1 - p_f)$
 10. Return $\beta_S, p_f, j, p_L, p_U$
-

The algorithm requires as input only the vector of element reliability indices, which are a standard output of any FORM-based reliability analysis. The computations in steps 1–9 involve only elementary arithmetic and the standard normal CDF and its inverse, both available as built-in functions in standard mathematical, statistical, and spreadsheet software. No specialised reliability software is required for the system-level computation.

For the timber truss application described in Section 8, the FORM analyses require OpenSees 2.2.2 [30] with a probabilistic analysis module and take approximately 30 s per element per truss configuration on a standard workstation. The subsequent system reliability approximation (Algorithm 1) adds less than 1 millisecond per configuration. For the full robustness assessment workflow covering 6 truss configurations $\times$ 37 elements $\times$ 5 damage scenarios = 1110 FORM analyses, the total computation time is approximately 10 h for FORM and negligible for the system approximation.

8. Application to Timber Truss Robustness Assessment

8.1. Truss Configurations and Probabilistic Model

The approximation is applied to the six timber truss configurations (RN1–RN6) used in a companion robustness assessment study [4,39]. The configurations differ in web topology, span (15–20 m), material grade, and connection system (Figure 4). All trusses are

designed in accordance with EN 1995-1-1 [38] using the probabilistic material parameters for GL24h glulam established experimentally in Reference [31].

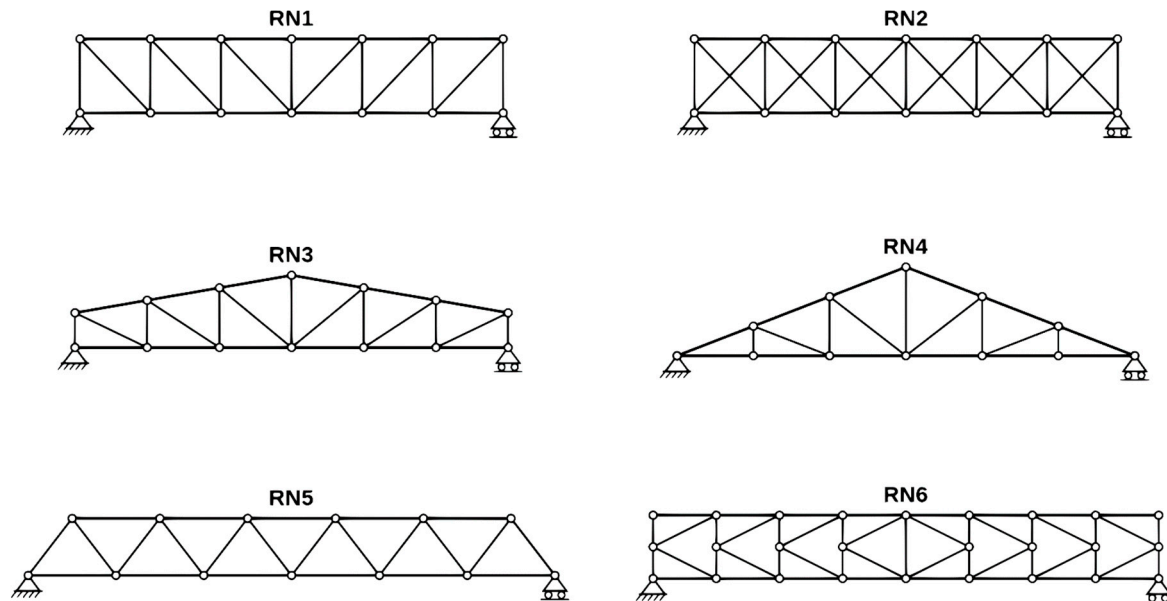


Figure 4. Timber truss configurations used in the robustness assessment application.

The probabilistic model follows JCSS [40] with modifications based on experimental data from Reference [31]. Material strength properties are modelled as lognormal variables. The inter-element strength correlation coefficient is taken as $\rho = 0.7$ for elements within the same truss (same material batch), consistent with the experimental evidence in Reference [36]. Since $\rho = 0.7$ lies below the crossover ρ^* identified in Proposition 4(c) and Section 6 ($\rho^* \in [0.74, 0.87]$ for all test systems considered), the approximation is expected to be anti-conservative for these timber truss configurations, with the magnitude of the bias bounded by Equation (10). Connection stiffness variability is modelled using the coefficient of variation reported in companion testing programmes [4].

8.2. Results and Robustness Classification

Table 4 summarises the system reliability indices for the intact state β_s and the minimum and maximum F_{rob} and I_{rob} values for each truss configuration across all considered single-element damage scenarios. The values reproduce those reported in the doctoral thesis [31]. The robustness factor $F_{rob} = -\log_{10}(p_{sf,dmg}/p_{f,norm})$ is computed as the negative base-10 logarithm of the ratio between the damaged-state system failure probability and a normalised target failure probability, following the formulation in the doctoral thesis [31]. The normalised failure probability is $p_{f,norm} = 8.0 \times 10^{-5}$, corresponding to a target reliability index $\beta_{s,norm} = 3.775$. Classification ranges per [31] are: $F_{rob} \in [-4.097, -3.097]$ no robustness; $F_{rob} \in (-3.097, -2.097]$ low; $F_{rob} \in (-2.097, -1.097]$ medium; $F_{rob} \in (-1.097, 0]$ high; $F_{rob} > 0$ ideal robustness. Table 4 represents: β_s intact is the intact-state system reliability index (always non-negative, ranging from 3.95 to 6.05 across the six configurations). $F_{rob,min}$ and $F_{rob,max}$ are the minimum and maximum robustness factors across all single-element damage scenarios, $F_{rob} = -\log_{10}(p_{sf,dmg}/p_{f,norm})$. $I_{rob,min}$ and $I_{rob,max}$ are the robustness indices, $I_{rob} = \max(\beta_{s,dmg}/\beta_{s,norm}, 0)$. Classifications per [31] are: R0 = no robustness ($F_{rob,min} \leq -3.097$), R1 = low, R2 = medium, R3 = high, R4 = ideal. It is important to note: F_{rob} is a normalised logarithmic robustness factor with classification range $[-4.10, +\infty)$, distinct from the system reliability index β_s , which is reported in the second column. The value $F_{rob,min} = -4.10$ corresponds to complete system collapse after damage ($p_{sf,dmg} \rightarrow 1$), is the lower bound of the

F_{rob} classification scale defined in [31], and is mathematically equivalent to $-\log_{10}(1/p_{f,\text{norm}})$ with $p_{f,\text{norm}} = 8.0 \times 10^{-5}$; it is not an anomalous negative reliability index.

Table 4. Robustness assessment results for six timber truss configurations [31].

Config.	β_s Intact	$F_{\text{rob,min}}$	$F_{\text{rob,max}}$	$I_{\text{rob,min}}$	$I_{\text{rob,max}}$	Class
RN1	4.07	−4.10	−2.58	0.00	0.50	R0
RN2	6.05	−2.74	+0.06	0.52	1.04	R1
RN3	3.95	−4.10	−2.76	0.00	0.45	R0
RN4	4.20	−4.10	−1.23	0.00	0.79	R0
RN5	4.39	−4.10	+0.98	0.00	1.14	R0
RN6	4.27	−4.10	+2.19	0.00	1.29	R0

The formal robustness class is governed by the worst-case robustness factor $F_{\text{rob,min}}$ (class R0 when $F_{\text{rob,min}} \leq -3.097$). On this basis, every configuration is classified R0 except RN2, whose worst-case factor $F_{\text{rob,min}} = -2.74$ places it in class R1: in five of the six topologies, at least one element is collapse-critical, so that its removal drives the damaged-state system failure probability towards unity ($F_{\text{rob,min}} = -4.10$). A clear progression is nevertheless visible in the best-case robustness factor $F_{\text{rob,max}}$ and in the maximum robustness index $I_{\text{rob,max}}$, which rise systematically from RN1 ($F_{\text{rob,max}} = -2.58$, $I_{\text{rob,max}} = 0.50$) to RN6 ($F_{\text{rob,max}} = +2.19$, $I_{\text{rob,max}} = 1.29$) as web redundancy increases. This is physically consistent: X-web and K-web topologies provide alternative load paths for the majority of single-element damage scenarios, whereas N-web trusses have virtually no redundancy. The more redundant topologies therefore tolerate most element removals far better, even though each still contains at least one element whose loss is critical and so retains the R0 worst-case class. The dominant element count $j = 3\text{--}5$ confirms that the threshold criterion $\varepsilon = 10^{-2}$ correctly identifies 3–5 critical elements across all configurations, validating the computational efficiency claim.

The approximation error relative to the Ditlevsen narrow bounds (computed separately for validation) is within 1.5% of the system reliability index for all six intact-state configurations and within 3% for the damaged states. No configuration is reclassified relative to the Ditlevsen-based result, confirming that the approximation error does not affect the robustness class assignment for these trusses. This timber truss application should be read as illustrative rather than as an independent experimental validation of the method. It reproduces the configurations, the probabilistic model, and the reference robustness results of the companion doctoral study [31], and its purpose is to demonstrate how the approximation is embedded in a complete robustness-classification workflow. The inter-element correlation ($\rho = 0.7$), the connection models, and the single-element damage scenarios are adopted from [4,31] and inherit the modelling assumptions and idealisations made there; they are not re-derived or re-measured here. Consequently, the close agreement with the Ditlevsen bounds reported above demonstrates internal consistency of the system reliability step, not the physical accuracy of the underlying truss reliabilities, which would require dedicated structural testing beyond the scope of this paper.

9. Practical Guidelines for Implementation

Based on the formal properties and numerical results, the following guidelines are recommended for practitioners applying the proposed approximation:

Guideline 1—Threshold selection. The threshold $\varepsilon = 10^{-2}$ (Equation (8)) is recommended as a balance between accuracy and computational economy. For structures where classification is critical and adjacent classes are close, $\varepsilon = 10^{-3}$ can be used to include more

elements. For screening purposes where only the most critical element governs, $j = 1$ (lower bound only) provides a conservative quick estimate.

Guideline 2—Borderline cases. When the computed β_S falls within ± 0.2 of a robustness class boundary, the Ditlevsen narrow bounds should be computed to confirm the classification. The boundary ± 0.2 corresponds to a factor of approximately 1.5 in the failure probability scale ($\Delta\beta = 0.2 \rightarrow \Delta p/p \approx \Phi(\beta + 0.2)/\Phi(\beta) - 1 \approx 50\%$ for $\beta \approx 3.2$). This margin is larger than the observed approximation errors ($\leq 3\%$ in β) for the timber truss configurations studied.

Guideline 3—Correlation. The inter-element strength correlation coefficient ρ should be estimated from experimental data when available. In the absence of data, $\rho = 0.7$ is recommended for trusses fabricated from a single material batch, consistent with Reference [36] and the JCSS model [40]. This value lies below the crossover ρ^* identified in Proposition 4(c), so the approximation is expected to be anti-conservative for typical timber truss applications. The magnitude of the anti-conservative bias is bounded above by Equation (10) and quantified numerically in Sections 5 and 6, where the maximum observed β_{error} is 2.6%.

Guideline 4—Software integration. Algorithm 1 can be implemented in any scripting environment in fewer than 20 lines of code. A reference Python implementation is available from the author upon request. The only external dependency is the standard normal CDF (scipy.stats.norm.cdf in Python, NORMSDIST in Excel, pnorm in R).

Guideline 5—Reporting. When reporting system reliability results, the following quantities should be provided: the number of elements m , the number of dominant elements j , the threshold ε , the lower bound β_L , the upper bound β_U , the approximation β_S , and the expected error direction (anti-conservative for $\rho < \rho^*$, conservative for $\rho > \rho^*$, per Proposition 4(c)). This enables readers to independently verify the approximation and assess its adequacy for the specific application.

Engineering Scope and Safety Implications

The proposed midpoint approximation is anti-conservative in the practical inter-element correlation range $\rho \in [0.5, 0.7]$ characteristic of timber elements cut from a single material batch, as quantified by Proposition 4 and the extended numerical study of Section 6. The maximum relative β -error observed across all six numerical test problems and the equicorrelated literature benchmark (Sections 5 and 6) is +2.6%, corresponding to underestimation of the failure probability by approximately one order of magnitude near the target $\beta_{S, \text{norm}} = 3.775$. The acceptability of this direction depends on the engineering context.

The approximation is recommended for (i) preliminary screening and robustness classification of large portfolios of structures, where the goal is to identify configurations requiring further analysis rather than to certify a specific design; (ii) robustness-class assignment under EN 1991-1-7, where the classes themselves span more than one order of magnitude in p_{sf} and a +2.6% β -error rarely changes the assignment; (iii) design optimisation loops where the inner-loop reliability evaluation is repeated many times and full Ditlevsen evaluation is computationally prohibitive; and (iv) educational and teaching contexts.

The approximation is not recommended as the sole basis for (i) final certification of safety-critical structures (Ditlevsen narrow bounds or exact multivariate-normal CDF should be used instead); (ii) structures where partial safety factors do not already provide a margin above the FORM linearisation uncertainty (typically 2–9% in β_S); and (iii) systems where the practical correlation regime may exceed the crossover $\rho^* \approx 0.85$ (in which case the error direction reverses; see Proposition 4(c)).

When the approximation is used at $\varepsilon = 10^{-2}$, a working safety margin of +0.10 in β_S (equivalently, a factor 1.5–2.0 reduction in p_{sf}) above the regulatory target is recommended

to absorb the anti-conservative bias. This margin is well within the partial-safety-factor framework of EN 1990 [1] and does not impose additional design penalty in practice. For applications requiring a strict safe-side guarantee, the Sørensen complement-product upper bound p_u should be used instead of the midpoint, accepting an accuracy reduction by approximately a factor of two relative to the midpoint estimate.

10. Conclusions

This paper has presented a formal analysis and numerical comparison of approximation methods for series structural system reliability, with application to timber truss robustness assessment. The principal conclusions are:

- (1) Existing methods—simple bounds, Sørensen’s complement-product reformulation, and Ditlevsen’s narrow bounds—are formally presented with complete derivations. Ditlevsen’s narrow bounds are more accurate but require $O(m^2)$ bivariate integrations; simple bounds require only component failure probabilities but are often too wide.
- (2) The proposed arithmetic-mean approximation satisfies four formal properties: consistency with bounds (Proposition 1), monotonicity in component failure probabilities (Proposition 2), convergence to the exact result for $j = m$ under independence (Proposition 3), and a characterisation of the error direction as a function of inter-element correlation via Slepian’s inequality [35] and the intermediate value theorem (Proposition 4). The latter result identifies a unique crossover $\rho^* \in (0, 1)$ below which the approximation is anti-conservative (β_s overestimates $\beta_{s,exact}$ by up to 2.6% in the test problems considered) and above which it is conservative. The constituent tools are classical—Slepian’s inequality [35] and the Boolean and Ditlevsen bounds [6,7]—and the contribution here is not a new inequality but their combination into an explicit, correlation-dependent error-direction criterion for the midpoint estimator, which we have not found stated in this form in the series system reliability literature surveyed.
- (3) Numerical validation for a ten-element series system shows errors of 0.5–2.1% in the system reliability index for $j = 10$, always in the unsafe direction for $\rho \leq 0.8$. The error grows as the dominant set is truncated below the elements selected by the threshold $\varepsilon = 10^{-2}$ (Table 1); because that threshold automatically retains every element contributing appreciably to the upper bound, the operative error in the recommended configuration stays within the few-percent band reported in Sections 5 and 6 and remains acceptable for robustness classification. It must be emphasised, however, that this validation—although expanded relative to the previous version through five additional synthetic systems and one literature benchmark (Section 6)—covers only a finite set of system sizes, correlation topologies, and reliability-index distributions and does not span all configurations and correlation structures encountered in engineering practice; results outside the examined range should be confirmed against Ditlevsen narrow bounds or exact integration.
- (4) Application to six timber truss configurations confirms that the approximation produces robustness class assignments identical to those obtained with Ditlevsen’s narrow bounds for all configurations studied. The dominant element count $j = 3\text{--}5$ is consistently identified by the threshold criterion, validating the computational efficiency of the approach.
- (5) The method requires only standard FORM outputs (element reliability indices) and $O(j)$ arithmetic operations, making it accessible without specialised reliability software. This enables its use in iterative design optimisation and large-scale robustness assessment workflows.

11. Future Research Directions

Several extensions of the proposed methodology are identified as priorities for future work:

(1) Parallel and mixed systems. The current approximation applies to series systems. Many structures contain both series and parallel subsystems: for instance, a timber truss with redundant web members may have parallel subsystems at the member level but a series system at the connection level. Extension of the formal properties to general series–parallel systems, using the sequential compounding framework of Kang and Song [11] as a starting point, is a natural next step.

(2) Time-variant reliability. The present analysis treats material properties as time-invariant random variables. Timber structures are subject to moisture-dependent strength degradation and creep, which introduce time-variant element failure probabilities. Extending the series system approximation to time-variant settings, following the framework of Köhler and Sørensen [18], would enable life-cycle robustness assessment.

(3) Non-Gaussian and non-linear limit states. The FORM-based approach introduces errors for highly non-linear limit-state functions or non-Gaussian basic variables. Second-order reliability methods (SORM) improve the component reliability estimates but complicate the correlation matrix computation. Integration of SORM-based component reliabilities with the proposed system approximation would extend its applicability to connection-critical failure modes.

(4) Experimental validation. The inter-element correlation coefficient $\rho = 0.7$ used in this study is based on a limited experimental dataset [31,36]. A systematic experimental programme measuring the correlation of material strength properties within timber batches from different species, drying methods, and grading systems would reduce uncertainty in the position of the practical correlation regime relative to the crossover ρ^* identified by Proposition 4(c).

(5) Integration with digital design workflows. The proliferation of BIM-based structural design tools creates an opportunity to implement the proposed approximation as a post-processing step in structural analysis software, enabling routine system reliability checks during the design phase. A plug-in implementation for OpenSees [30] or commercial software environments is planned as a near-term development. For a comprehensive overview of system reliability theories and their integration with structural optimisation, the reader is referred to Song et al. [41].

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