

Article

The Complex Variable Solution for the Displacement and Stress Caused by Shallow Rectangular Tunnel Grouting Pressure Considering the Gravity Effect

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Abstract

This study presents an analytical elastic solution using the complex variable approach to determine the soil disturbance surrounding a shallow rectangular tunnel, taking into account the effect of gravity and considering the influence of grouting pressure applied at the tunnel boundary. A theoretical calculation model is first established by applying stress boundary conditions along the tunnel boundary. To handle the geometric complexity of the rectangular cavity, a conformal mapping function suitable for non-circular opening is adopted. This function maps the original half-infinite region with a rectangular cavity onto an annulus in the transformed plane, thereby converting the irregular cavity problem into a more tractable configuration. The boundary conditions at the ground surface and the tunnel interface are then addressed using Fourier series expansion. By solving these series-expanded equations, all unknown coefficients of the analytical functions are determined, leading to an analytical elastic solution for soil disturbance. Numerical verification validates the accuracy of the proposed approach. Moreover, parametric studies are conducted to evaluate the effects of soil mechanical properties and tunnel geometry on the soil disturbance. The findings offer a valuable reference for future investigations into construction-related issues associated with rectangular tunnels.

Keywords: shallow rectangular tunnel; complex variable solution; conformal mapping; grouting pressure; gravity effect



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1. Introduction

With the acceleration of urbanization, the aboveground space in cities is gradually becoming saturated. To improve the efficiency of underground space usage, shallow rectangular tunnels have been commonly adopted for infrastructure such as subway stations and utility tunnels. However, owing to their unfavorable cross-sectional geometry and shallow depth below ground, these tunnels are prone to pronounced stress concentrations and considerable deformations. Furthermore, during the construction of rectangular tunnels, grouting operations can disturb the surrounding soil, thereby affecting tunnel stability and safety. Consequently, it is necessary to predict the displacement and stress fields caused by shallow rectangular tunnel grouting pressure.

In practical engineering, the prediction of soil deformation primarily involves the following methods: analytical, numerical analysis, and empirical methods. The empirical

method is based on field-measured ground settlement data, from which theoretical empirical formulas are developed to evaluate the surface settlement caused by tunnel excavation. Among empirical methods, Peck [1] proposed that the ground surface settlement caused by tunneling typically follows a Gaussian distribution. Subsequently, many researchers have modified Peck's formula by incorporating field measurements and experimental data [2,3]. However, the empirical method is directly fitted to field observations, which rely on empirical correlations and experience, lacking a rigorous analytical basis. In numerical analysis, for complex soil properties and geometric conditions, numerical methods can account for the combined influence of factors such as ground conditions and excavation cross-sectional shapes, thereby offering effective support for approximately predicting ground movements [4,5]. Nevertheless, numerical simulation also has notable drawbacks: under complex stress states, its accuracy is constrained by the limitations of the constitutive models employed, and it involves high computational costs and long processing times associated with modeling and computation. Typical analytical methods mainly include the stochastic medium theory, the complex variable theory, the virtual image method, and the stress function method. The stochastic medium theory is originally introduced by Litwiniszyn [6] to explore surface displacements in rock strata resulting from coal mining operations. Sagaseta [7] employed the virtual image method to calculate surface settlement caused by tunneling in a semi-infinite domain. Verruijt and Booker [8], building upon Sagaseta's work, derived theoretical formulas for calculating the horizontal and vertical displacement fields resulting from circular tunnel excavation. With respect to the stress function approach, Bobet [9] and later Chou and Bobet [10] further developed the analytical framework to yield an elastic solution for ground surface deformations caused by tunnel excavation. The stress function approach is limited to deep tunnel mechanical frameworks, whereas the stochastic media method only yields the soil displacement field. By comparison, the complex variable technique [11–17] is applicable to shallow and deep tunnels and can concurrently derive the soil disturbance in the surrounding ground. Moreover, it resolves the singularity assumption issue associated with the virtual image approach. Muskhelishvili [18] introduced the complex variable approach as a standard methodology for solving cavity problems within elastic domains. The key step in this approach is to identify the appropriate conformal mapping function that maps any complicated region in the physical domain into a regular counterpart in the mapped plane.

For shallow circular tunnels, Verruijt [19,20] developed a conformal mapping equation that converts a half-infinite region with a circular cavity into an annulus in mapped plane and presented an elastic solution to the issue of surface movements. Wang et al. [21] characterized tunneling-induced soil movements in clay and suggested that applying non-uniform displacement boundary conditions along the tunnel perimeter yields more accurate predictions of ground settlements during excavation. Fu et al. [22] combined the complex variable approach with the Schwartz alternating technique to derive closed-form analytical expressions for the stresses and displacements surrounding two parallel tunnels. Fang et al. [23] put forward an analytical solution that incorporates the seepage effect of underwater circular tunnels and examined the influence of seepage on effective stress distribution around shallow submerged tunnels. Zhang et al. [24] took into account soil–lining interaction and developed a complex variable solution that captures the responses of both the lining and soil to tunnel excavation, while also considering the effect of fluctuating groundwater levels. For non-circular tunnels, Zeng et al. [25] first presented a new conformal mapping function capable of transforming a half-infinite region with a non-circular cavity onto an annulus in the transformed plane. On this basis, Tong et al. [26] improved the solution strategy for the mapping function coefficients by introducing the least-squares method. In addition, researchers such as Fang et al. [27,28] and Ye and Ai [29]

have employed the generalized complex variable theory to solve the disturbance issues of non-circular tunnels.

However, most of the aforementioned studies have not considered the effect of soil gravity. For shallow tunnels, the influence of gravitational loading is particularly significant; neglecting it in the analysis would lead to deficiencies in the underlying theoretical framework. During the construction of shallow tunnels, the removal of soil creates a weight differential that gives rise to a buoyancy effect. To resolve this problem, Strack and Verruijt [30] derived analytical function expressions accounting for gravity and proposed a complex variable solution that incorporates the buoyancy effect. Subsequently, many researchers have studied the buoyancy effect induced by the self-weight of the excavated soil. Strack and Verruijt [31] employed a simple elastic model of homogeneous isotropic soil to examine the effect of the buoyancy effect on surface settlement curves. Their results indicate that considering the buoyancy effect leads to a reduction in both the width of the settlement trough and the magnitude of ground surface settlement. Building on this work, Jain and Kumar [32,33] presented analytical solutions for stresses and displacements at any arbitrary location within the problem domain under the buoyancy effect, based on the method of a single circular tunnel with non-uniform deformation. The above-mentioned studies on the buoyancy effect have mainly focused on shallow circular tunnels; however, research on the buoyancy effect induced by the self-weight of excavated soil for shallow non-circular tunnels remains relatively limited.

This study proposes an analytical approach to determine the soil disturbance surrounding a shallow rectangular tunnel under grouting pressure, with soil gravity taken into consideration. A theoretical model is first constructed according to the boundary condition prescribed along the tunnel boundary. Through the non-circular conformal mapping function, the half-infinite region with a rectangular hole is transformed into an annulus in the mapped plane, and the complex variable method is then applied to derive analytic function expressions that incorporate the gravitational effect. To obtain the pending coefficients of analytic functions efficiently, Fourier series expansion is employed to represent the stress condition along the surface and the tunnel boundary as Fourier series. By coupling the governing equations for surface and tunnel boundary conditions, all pending coefficients are solved, leading to an elastic solution for the displacements and stresses. The feasibility of the presented method is confirmed through comparison of the theoretical results with finite element simulations. Additionally, parametric studies are performed to examine the effect of soil mechanical properties and tunnel geometry on the variation in the soil displacement and stress.

2. Problem Statement and Basic Theory

2.1. Problem Statement

This study focuses on the soil disturbance induced by grouting pressure during shallow rectangular tunnel construction. To analytically evaluate this problem, a simplified mechanical model is established based on the theory of elasticity. The following idealizations are adopted: (i) the soil is treated as an elastic, isotropic, and homogeneous medium; (ii) the analysis is performed under plane strain conditions, ignoring three-dimensional effects; and (iii) the surface condition is assumed to be stress-free. As illustrated in Figure 1, the grouting process is represented by a uniform pressure applied along the tunnel boundary. The soil properties include Young's modulus E , Poisson's ratio μ , and effective unit weight γ , and the tunnel geometry parameters are defined by its height $2h$, width $2b$, and burial depth D .

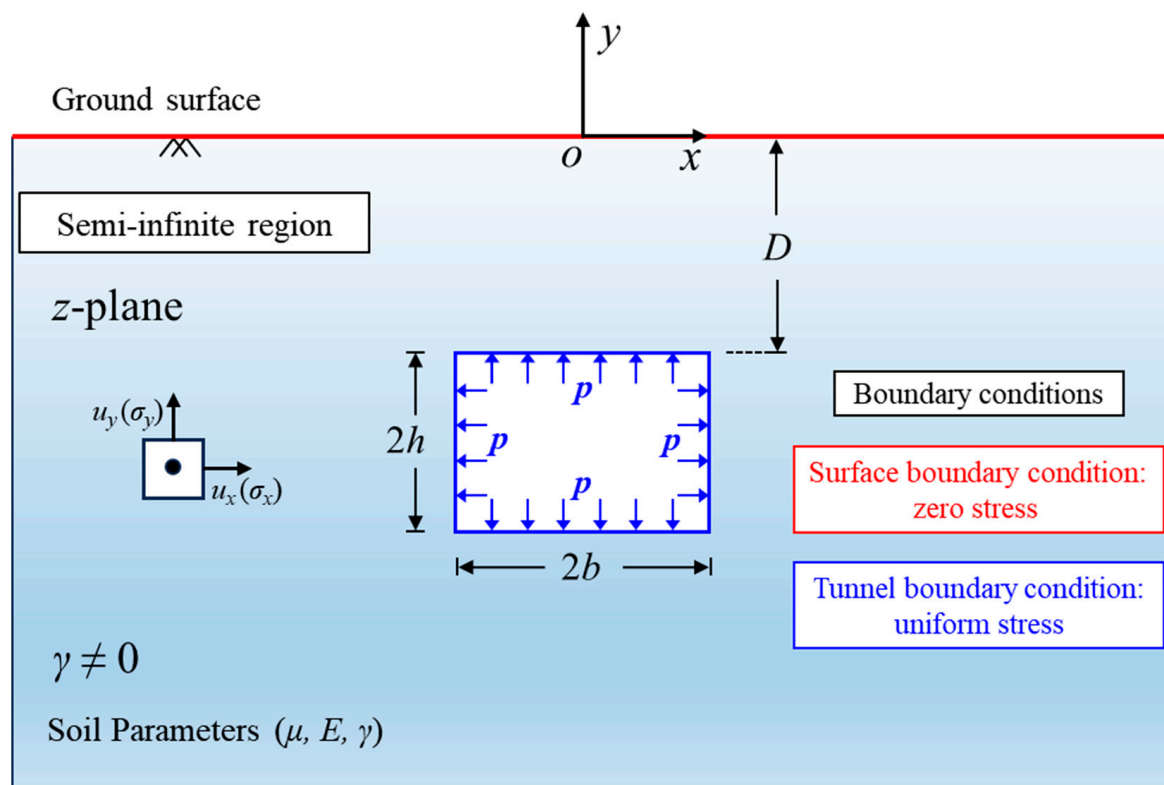


Figure 1. Shallow buried rectangular tunnel grouting model diagram.

2.2. Conformal Mapping

Conformal mapping is a mathematical technique that transforms complex geometries into simpler geometries in the mapped plane, and it is particularly valuable for addressing elastic half-plane problems with intricate boundary conditions. Following the mapping function developed by Zeng et al. [25], the elastic half-plane containing a rectangular cavity in the physical plane is mapped onto an annulus in the mapped plane. The corresponding mapping equation is given as follows:

$$z = \omega(\zeta) = ic \frac{1 + \zeta}{1 - \zeta} + i \sum_{k=1}^n \chi_k (\zeta^k - \zeta^{-k}) \quad (1)$$

where $z = x + iy = re^{i\theta}$ denotes an arbitrary point in the physical plane, with (x, y) and (r, θ) being the Cartesian and polar coordinate variables in that plane, respectively; $\zeta = \xi + i\eta = \rho e^{i\vartheta}$ denotes an arbitrary point in the image plane, with (ξ, η) and (ρ, ϑ) being the Cartesian and polar coordinate variables in the image plane. The quantities c and χ_k are the undetermined coefficients of the mapping function, and the procedure for obtaining their values will be elaborated in the subsequent sections.

An accurate conformal mapping function is a critical prerequisite for solving the associated boundary value problem. Figure 2 displays the process of conformal transformation between the physical and image planes: the elastic half-plane with a rectangular cavity is transformed onto an annular region in the transformed plane, with d representing the depth from surface to the tunnel center. Under this mapping, the characteristic points P_0, P_1, P_2 , and P_3 in the physical plane correspond to the points P'_0, P'_1, P'_2 , and P'_3 in the transformed plane, respectively. The surface is transformed to the outer circumference of the annulus, whereas the tunnel periphery is mapped to its inner boundary. On the outer circumference of the annulus, $\rho = 1$ and $\zeta = e^{i\theta}$; on the inner boundary, $\rho = I$ and $\zeta = Ie^{i\theta}$, where I denotes the radius of the annulus's inner boundary. To determine the explicit

form of this mapping function, the undetermined coefficients must be obtained, and the procedure for solving them will be detailed in the following sections:

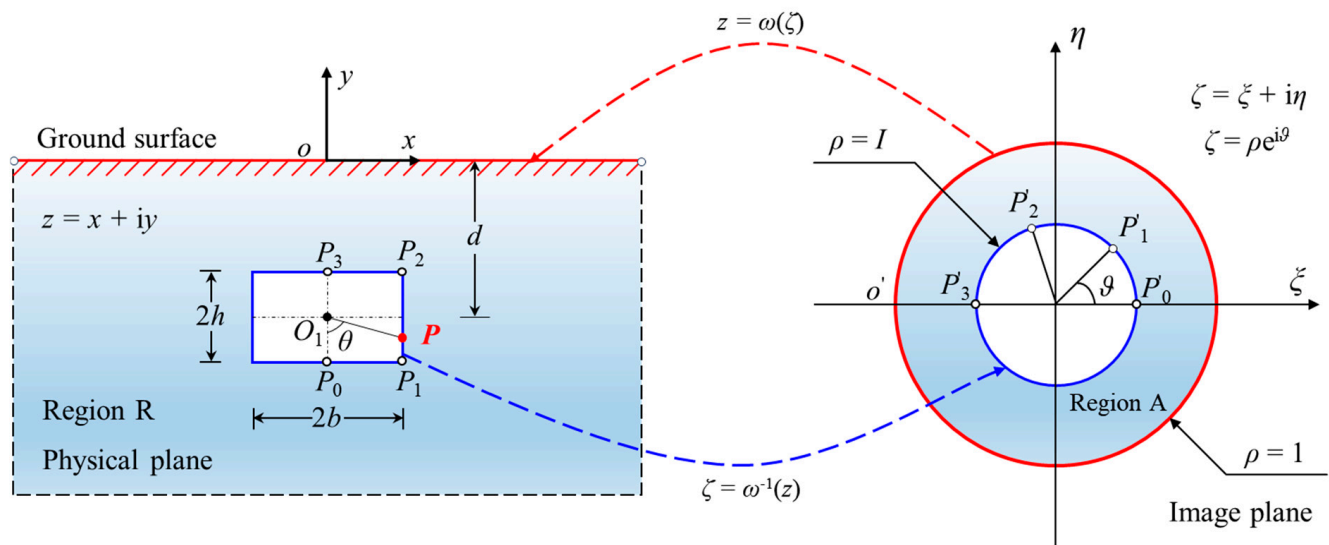


Figure 2. Conformal transformation process of rectangular hole.

- (i) In the physical plane, $m + 1$ sampling points are selected counterclockwise along the rectangular cavity boundary with P_0 serving as the starting point. Simultaneously, in the image plane, $m + 1$ sampling points are selected counterclockwise along the annulus's inner boundary, with P'_0 taken as the origin point. A strict one-to-one correspondence between the sampling points in the two planes is then established according to the sequential order in which they are traversed.
- (ii) On the basis of this correspondence, substituting the coordinates of each sampling point from both planes into Equation (1) yields a system of $m + 1$ equations involving $k + 1$ undetermined coefficients:

$$i \begin{bmatrix} \frac{1+\zeta_0}{1-\zeta_0} & \zeta_0 - \zeta_0^{-1} & \zeta_0^2 - \zeta_0^{-2} & \cdots & \zeta_0^k - \zeta_0^{-k} \\ \frac{1+\zeta_1}{1-\zeta_1} & \zeta_1 - \zeta_1^{-1} & \zeta_1^2 - \zeta_1^{-2} & \cdots & \zeta_1^k - \zeta_1^{-k} \\ \frac{1+\zeta_2}{1-\zeta_2} & \zeta_2 - \zeta_2^{-1} & \zeta_2^2 - \zeta_2^{-2} & \cdots & \zeta_2^k - \zeta_2^{-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1+\zeta_m}{1-\zeta_m} & \zeta_m - \zeta_m^{-1} & \zeta_m^2 - \zeta_m^{-2} & \cdots & \zeta_m^k - \zeta_m^{-k} \end{bmatrix} \begin{bmatrix} c \\ \chi_1 \\ \chi_2 \\ \vdots \\ \chi_k \end{bmatrix} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} \quad (2)$$

Equation (1) can be expressed in matrix form as follows:

$$BX = C \quad (3)$$

where B denotes the coefficient matrix determined by the angular coordinates of the sampling points; $X = [c \ \chi_1 \ \chi_2 \ \dots \ \chi_k]^T$ is the vector containing the unknown coefficients for the conformal mapping function; and $C = [z_1 \ z_2 \ \dots \ z_m]^T$ is the vector containing the coordinates of the sampling points along the tunnel boundary.

- (iii) The least squares method is employed to solve the above system of equations. By setting $X = (B^T B)^{-1} B^T C$, the coefficients after the first iteration are obtained. Subsequently, the coefficient vector X and the interior boundary sampling points ζ_m of the annular domain in the mapped plane are substituted into the mapping relationship (Equation (3)) to compute the vector of mapped coordinates $C^{(1)}$ corresponding to the second iterative cycle. This vector is then corrected using the actual perimeter of the

tunnel boundary, and the aforementioned procedure is carried out cyclically until the specified tolerance criterion is met:

$$\text{Norm}(X^{(nm)} - X^{(nm-1)}) < 10^{-10} \quad (4)$$

Subsequently, based on the relationship between the prescribed radius and the morphological variation in the mapped curve, the true value of the radius of the annulus's inner boundary can be back-calculated using the bisection method, thereby fully determining all coefficients of the mapping function for the rectangular cavity.

3. Analytical Functions and Boundary Conditions

3.1. Expression of the Analytical Functions

When employing the complex variable approach for solving plane elasticity problems, the core steps can be summarized as follows: (1) mapping the complex geometry onto a simpler geometry in the mapped plane through conformal mapping; (2) determining the analytic functions through the prescribed boundary conditions.

According to the investigation of Strack and Verruijt [30], with the effect of soil self-weight taken into account, the general expressions for the analytic functions can be written as

$$\varphi(z) = -\frac{F_x + iF_y}{2\pi(1 + \kappa)} [\kappa \ln(z - \bar{z}_a) + \ln(z - z_a)] + \varphi_0(z) \quad (5)$$

$$\psi(z) = \frac{F_x - iF_y}{2\pi(1 + \kappa)} [\ln(z - \bar{z}_a) + \kappa \ln(z - z_a)] + \psi_0(z) \quad (6)$$

where z_a is an arbitrary point within the cavity; $\kappa = 3 - 4\mu$ is a parameter related to Poisson's ratio; and w_t is defined as the weight ratio of the lining to the excavated soil, which directly governs the magnitude of the buoyancy effect and quantitatively characterizes the buoyancy induced by soil removal. Correspondingly, the resultant force components are taken as $F_x = 0$ and $F_y = (1 - w_t)\gamma S$, with S being the cross-sectional area of the tunnel. $\varphi_0(z)$ and $\psi_0(z)$ are single-valued analytic functions.

By employing the conformal transformation $z = \omega(\zeta)$ to map the above fundamental relations from the physical plane to the mapped plane, the analytic functions in the image plane, $\varphi(z)$ and $\psi(z)$ are obtained as follows:

$$\varphi(z) = \varphi[\omega(\zeta)] = \varphi(\zeta) \quad (7)$$

$$\psi(z) = \psi[\omega(\zeta)] = \psi(\zeta) \quad (8)$$

From Equations (7) and (8), the explicit expression of analytical function can be obtained:

$$\varphi(\zeta) = -\frac{F_x + iF_y}{2\pi(1 + \kappa)} [\kappa \ln(z - \bar{z}_a) + \ln(z - z_a)] + \varphi_0(\zeta) \quad (9)$$

$$\psi(\zeta) = \frac{F_x - iF_y}{2\pi(1 + \kappa)} [\kappa \ln(z - z_a) + \ln(z - \bar{z}_a)] + \psi_0(\zeta) \quad (10)$$

where $\varphi_0(z)$ and $\psi_0(z)$ can be expanded into Laurent series in the image plane:

$$\varphi_0(\zeta) = \sum_{k=-\infty}^{+\infty} e_k \zeta^k \quad (11)$$

$$\psi_0(\zeta) = \sum_{k=-\infty}^{+\infty} f_k \zeta^k \quad (12)$$

where e_k and f_k are the unknown coefficients of the analytic functions.

3.2. Stress Boundary Condition Along the Ground Surface

This study addresses the problem of soil disturbance caused by shallow rectangular tunnel grouting pressure, where the boundary conditions of primary concern include a traction-free ground surface and a uniform pressure boundary. Based on complex variable theory, the general form of the stress boundary condition can be expressed as follows:

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = X_c + iY_c \quad (13)$$

In accordance with the assumptions adopted in this paper, the surface is set to be a traction-free boundary ($X_c = Y_c = 0$), and the corresponding boundary condition equation can be expressed as

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = 0 \quad (14)$$

Substituting Equations (9) and (10) into Equation (14) yields

$$\varphi_0(z) + z\overline{\varphi'_0(z)} + \overline{\psi_0(z)} = \frac{F_x - iF_y}{2\pi(1 + \kappa)} \left[\frac{\kappa z}{\bar{z} - z_a} + \frac{z}{\bar{z} - \bar{z}_a} \right] \quad (15)$$

To simplify Equation (15), the following definitions are introduced:

$$Q(z) = \frac{F_x - iF_y}{2\pi(1 + \kappa)} \left[\frac{\kappa z}{\bar{z} - z_a} + \frac{z}{\bar{z} - \bar{z}_a} \right] \quad (16)$$

Applying the chain rule yields the following relation:

$$\varphi'_0(z) = \frac{d\varphi_0(z)}{d\zeta} \frac{d\zeta}{dz} = \frac{\varphi'_0(\zeta)}{\omega'(\zeta)} \quad (17a)$$

$$\psi'_0(z) = \frac{d\psi_0(z)}{d\zeta} \frac{d\zeta}{dz} = \frac{\psi'_0(\zeta)}{\omega'(\zeta)} \quad (17b)$$

$$\varphi''_0(z) = \frac{d\varphi'_0(z)}{d\zeta} \frac{d\zeta}{dz} = \frac{\varphi''_0(\zeta)\omega'(\zeta) - \varphi'_0(\zeta)\omega''(\zeta)}{[\omega'(\zeta)]^3} \quad (17c)$$

Similarly, by transforming the fundamental relations from the physical plane to the image plane via the conformal mapping and substituting Equations (17a)–(17c) into Equation (15), the traction-free boundary condition equation on the ground surface expressed in terms of ζ is obtained as follows:

$$\varphi_0(\zeta) + \frac{\omega(\zeta)\overline{\varphi'_0(\zeta)}}{\omega'(\zeta)} + \overline{\psi_0(\zeta)} = Q(\zeta) \quad (18)$$

For the surface boundary, $\zeta = e^{i\theta} = \sigma$, from which it follows that

$$\varphi_0(\sigma) + \frac{\omega(\sigma)\overline{\varphi'_0(\sigma)}}{\omega'(\sigma)} + \overline{\psi_0(\sigma)} = Q(\sigma) \quad (19)$$

3.3. Stress Boundary Condition Along the Tunnel Boundary

For the tunnel boundary, the stress condition is also given by Equation (13), where the right-hand side terms are evaluated as follows:

$$X_c + iY_c = i \int (g_x + ig_y) ds = -pz \quad (20)$$

where g_x and g_y are the horizontal and vertical components of the surface traction applied on the tunnel boundary, respectively, and s denotes the arc length measured along the tunnel boundary.

Substituting Equation (20) into Equation (13) yields the governing equation for the stress boundary condition at the tunnel as follows:

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = -pz \quad (21)$$

Substituting Equations (9) and (10) into Equation (21) yields

$$\begin{aligned} \varphi_0(z) + z\overline{\varphi'_0(z)} + \overline{\psi_0(z)} = & \frac{F_x + iF_y}{2\pi(1+\kappa)} [\kappa \ln(z - \bar{z}_a) + \ln(z - z_a)] \\ & - \frac{F_x + iF_y}{2\pi(1+\kappa)} [\kappa \ln(\bar{z} - \bar{z}_a) + \ln(\bar{z} - z_a)] + \frac{F_x - iF_y}{2\pi(1+\kappa)} \left[\frac{\kappa z}{\bar{z} - z_a} + \frac{z}{\bar{z} - \bar{z}_a} \right] - pz \end{aligned} \quad (22)$$

To simplify Equation (22), the following definitions are introduced:

$$\begin{aligned} T(z) = & \frac{F_x + iF_y}{2\pi(1+\kappa)} [\kappa \ln(z - \bar{z}_a) + \ln(z - z_a)] - \frac{F_x + iF_y}{2\pi(1+\kappa)} [\kappa \ln(\bar{z} - \bar{z}_a) + \ln(\bar{z} - z_a)] \\ & + \frac{F_x - iF_y}{2\pi(1+\kappa)} \left[\frac{\kappa z}{\bar{z} - z_a} + \frac{z}{\bar{z} - \bar{z}_a} \right] - pz \end{aligned} \quad (23)$$

Substituting Equations (1) and (17) into Equation (22) yields the governing equation for the stress boundary condition of the tunnel in the image plane:

$$\varphi_0(\zeta) + \frac{\omega(\zeta)\overline{\varphi'_0(\zeta)}}{\omega'(\zeta)} + \overline{\psi_0(\zeta)} = T(\zeta) \quad (24)$$

On the tunnel boundary, $\zeta = e^{i\theta} = I\sigma$, from which it follows that

$$\varphi_0(I\sigma) + \frac{\omega(I\sigma)\overline{\varphi'_0(I\sigma)}}{\omega'(I\sigma)} + \overline{\psi_0(I\sigma)} = T(I\sigma) \quad (25)$$

4. The Derivation of the Analytical Solution

4.1. Solving Equations for the Coefficients of Analytical Function

The governing equations for the traction-free boundary condition on the ground surface (Equation (19)) and the stress boundary condition on the tunnel boundary (Equation (25)) together form a system from which the explicit expressions of the analytic functions can be determined. To obtain the unknown coefficients in these analytic functions, a series expansion method is employed herein. However, certain complex terms in the equations cannot be directly expanded into a standard series form; therefore, these terms are transformed into Fourier series by means of the Fourier transform. The resulting expansions of the complex terms appearing in Equations (19) and (25) are presented as follows:

$$\frac{\omega(\sigma)}{\omega'(\sigma)} = \frac{ic \frac{1+\sigma}{1-\sigma} + i \sum_{k=1}^n \chi_k(\sigma^k - \sigma^{-k})}{\frac{-i2c}{(1-\sigma^{-1})^2} - \sum_{k=1}^n k\chi_k(\sigma^{1-k} + \sigma^{1+k})} \quad (26a)$$

$$Q(\sigma) = \frac{F_x - iF_y}{2\pi(1+\kappa)} \left[ic \frac{1+\sigma}{1-\sigma} + i \sum_{k=1}^n \chi_k(\sigma^k - \sigma^{-k}) \right] \left\{ \begin{aligned} & \frac{\sigma - 1}{-2c\sigma i - i(\sigma - 1) \sum_{k=1}^n \chi_k(\sigma^{-k} - \sigma^k)} \\ & + \frac{\sigma - 1}{-2ci - i(\sigma - 1) \sum_{k=1}^n \chi_k(\sigma^{-k} - \sigma^k)} \end{aligned} \right\} \quad (26b)$$

$$\frac{\omega(I\sigma)}{\omega'(I\sigma)} = \frac{ic \frac{1+I\sigma}{1-I\sigma} + i \sum_{k=1}^n \chi_k (I^k \sigma^k - I^{-k} \sigma^{-k})}{\frac{-i2c}{(1-I^{-1}\sigma^{-1})^2} - \sum_{k=1}^n k \chi_k (I^{k-1} \sigma^{1-k} + I^{-k-1} \sigma^{k+1})} \quad (26c)$$

$$T(I\sigma) = \frac{F_x + iF_y}{2\pi(1+\kappa)} [\kappa \ln(z - \bar{z}_a) + \ln(z - z_a)] - \frac{F_x + iF_y}{2\pi(1+\kappa)} [\kappa \ln(\bar{z} - \bar{z}_a) + \ln(\bar{z} - z_a)] \\ + \frac{F_x - iF_y}{2\pi(1+\kappa)} \left[\frac{\kappa z}{\bar{z} - z_a} + \frac{z}{\bar{z} - \bar{z}_a} \right] - p\omega(I\sigma) \quad (26d)$$

By employing the Fourier series expansion technique, the terms in Equations (26a)–(26d) are expanded into Fourier series (the complete expansion of Equation (26d) is omitted here due to its excessive length):

$$\frac{\omega(\sigma)}{\omega'(\sigma)} = \sum_{k=-\infty}^{\infty} F_k \sigma^k \quad (27a)$$

$$Q(\sigma) = \sum_{k=-\infty}^{+\infty} L_k \sigma^k \quad (27b)$$

$$\frac{\omega(I\sigma)}{\omega'(I\sigma)} = \sum_{k=-\infty}^{\infty} Y_k \sigma^k \quad (27c)$$

$$T(I\sigma) = \sum_{k=-\infty}^{+\infty} J_k \sigma^k \quad (27d)$$

Substituting Equations (11), (12), and (27a)–(27d) into the stress boundary conditions expressed by Equations (19) and (25) yields the equation system used to solve for the coefficients in the analytic functions, as follows:

$$\sum_{k=-\infty}^{+\infty} \left[e_k + \sum_{v=-\infty}^{\infty} (v F_{k+v-1} \bar{e}_v) + \bar{f}_{-k} \right] \sigma^k = \sum_{k=-\infty}^{+\infty} L_k \sigma^k \quad (28a)$$

$$\sum_{k=-\infty}^{+\infty} \left[I^k e_k + \sum_{v=-\infty}^{+\infty} (v I^{k-1} Y_{k+v-1} \bar{e}_v) + \alpha^{-k} \bar{f}_{-k} \right] \sigma^k = \sum_{k=-\infty}^{+\infty} J_k \sigma^k \quad (28b)$$

Combining and simplifying the expressions in Equation (28a-b) yields

$$\bar{f}_{-k} = L_k - e_k - \sum_{v=-\infty}^{\infty} (v F_{k+v-1} \bar{e}_v) \quad (29a)$$

$$I^k e_k + \sum_{v=-\infty}^{+\infty} (v I^{k-1} Y_{k+v-1} \bar{e}_v) + I^{-k} \bar{f}_{-k} = J_k \quad (29b)$$

Substituting Equation (29a) into Equation (29b) yields the equation for determining e_k :

$$e_k (I^k - I^{-k}) - \sum_{v=-\infty}^{+\infty} v (Y_{k+v-1} I^v - F_{k+v-1} I^{-k}) \bar{e}_v = J_k - L_k I^{-k} \quad (30)$$

By jointly solving Equations (29a) and (30), all the undetermined coefficients associated with the analytic functions can be derived, thereby fully determining their explicit expressions.

4.2. Solution of the Stress and Displacement

According to the complex variable theory of plane elasticity, the soil disturbance can be derived from the analytic functions, and the corresponding equations are as follows:

$$\sigma_x^* + \sigma_y^* = 4\text{Re} \left[\frac{\varphi'(\zeta)}{\omega'(\zeta)} \right] \quad (31a)$$

$$\sigma_y^* - \sigma_x^* + 2i\tau_{xy}^* = 2 \left[\frac{\overline{\omega(\zeta)} \varphi''(\zeta) \omega'(\zeta) - \omega''(\zeta) \varphi'(\zeta)}{\omega'^3(\zeta)} + \frac{\psi'(\zeta)}{\omega'(\zeta)} \right] \quad (31b)$$

$$2G(u_x + iu_y) = \kappa \varphi(\zeta) - \frac{\omega(\zeta)}{\omega'(\zeta)} \overline{\varphi'(\zeta)} - \overline{\psi(\zeta)} \quad (31c)$$

In Equations (31a)–(31c), σ_x^* and σ_y^* denote the normal stress components in the horizontal and vertical directions, respectively; τ_{xy}^* is the shear stress component in the x - y plane; $G = E/(2 + 2\mu)$ stands for the shear modulus of the soil; and u_x and u_y are the horizontal and vertical displacements, respectively. The stresses and displacements in the soil surrounding the tunnel can be obtained from this equation.

In addition, prior to tunnel excavation, an initial gravitational stress field exists in the ground, and its stress components are expressed as

$$\begin{cases} \sigma_x^0 = K_0 \gamma y \\ \sigma_y^0 = \gamma y \\ \tau_{xy}^0 = 0 \end{cases} \quad (32)$$

where $K_0 = \mu/(1 - \mu)$ is the coefficient of lateral earth pressure. By superposing the stress field yielded by the analytic functions onto the original gravitational stress field given by Equation (32), the total stresses surrounding the tunnel are obtained as follows:

$$\sigma_x = \sigma_x^* + \sigma_x^0 \quad (33a)$$

$$\sigma_y = \sigma_y^* + \sigma_y^0 \quad (33b)$$

$$\tau_{xy} = \tau_{xy}^* + \tau_{xy}^0 \quad (33c)$$

5. Verification of the Analytical Solution

To validate the proposed theoretical solution, a finite element model simulating the grouting process of a shallow rectangular tunnel was developed using Abaqus 2021, with all computational parameters listed in Table 1. Figure 3 illustrates the meshing scheme and the boundary conditions applied at different analysis stages: a locally refined mesh is adopted around the rectangular cavity, while a coarser mesh is used in the far-field region, resulting in a total of 14,850 elements.

In the numerical simulation, the buoyancy effect induced by the self-weight of the excavated soil (Stage I) and the uniform grouting pressure applied on the tunnel boundary (Stage II) were solved independently, and their combined effect was subsequently determined by linear superposition. The boundary conditions and material settings for Stage I are as follows: the bottom of the model is fully fixed, and the lateral sides are horizontally restrained. Following the approach of Yang et al. [34], the buoyancy effect due to soil removal is reproduced by assigning an extremely small unit weight (0.01 N/m^3) to the surrounding soil, setting the unit weight of the excavated soil to $(1 - w_t)\gamma$, and increasing the elastic modulus of the excavated soil to 100 times that of the surrounding soil. In Stage II, the soil is regarded as an elastic material without considering gravity, and the same stress boundary condition as that adopted in the theoretical model is imposed on the tunnel periphery. After superposition of the results from the two stages, a comparison with the theoretical solution is presented in Figure 4. The theoretical predictions (the following parameters are used for derivation: the number of sampling points $m = 1000$, the truncation number $k = 140$, and the series $n = 15$) agree closely with the finite element results, and the error between the analytical results and the FEM results is approximately 1.03%, which confirms the accuracy of the present solution.

Table 1. Parameters in numerical simulation.

Parameter	Value	Parameter	Value
μ	0.2	D	3 m
h	0.5 m	b	0.75 m
w_t	0.35	γ	18 kN/m ³
E	10 MPa		

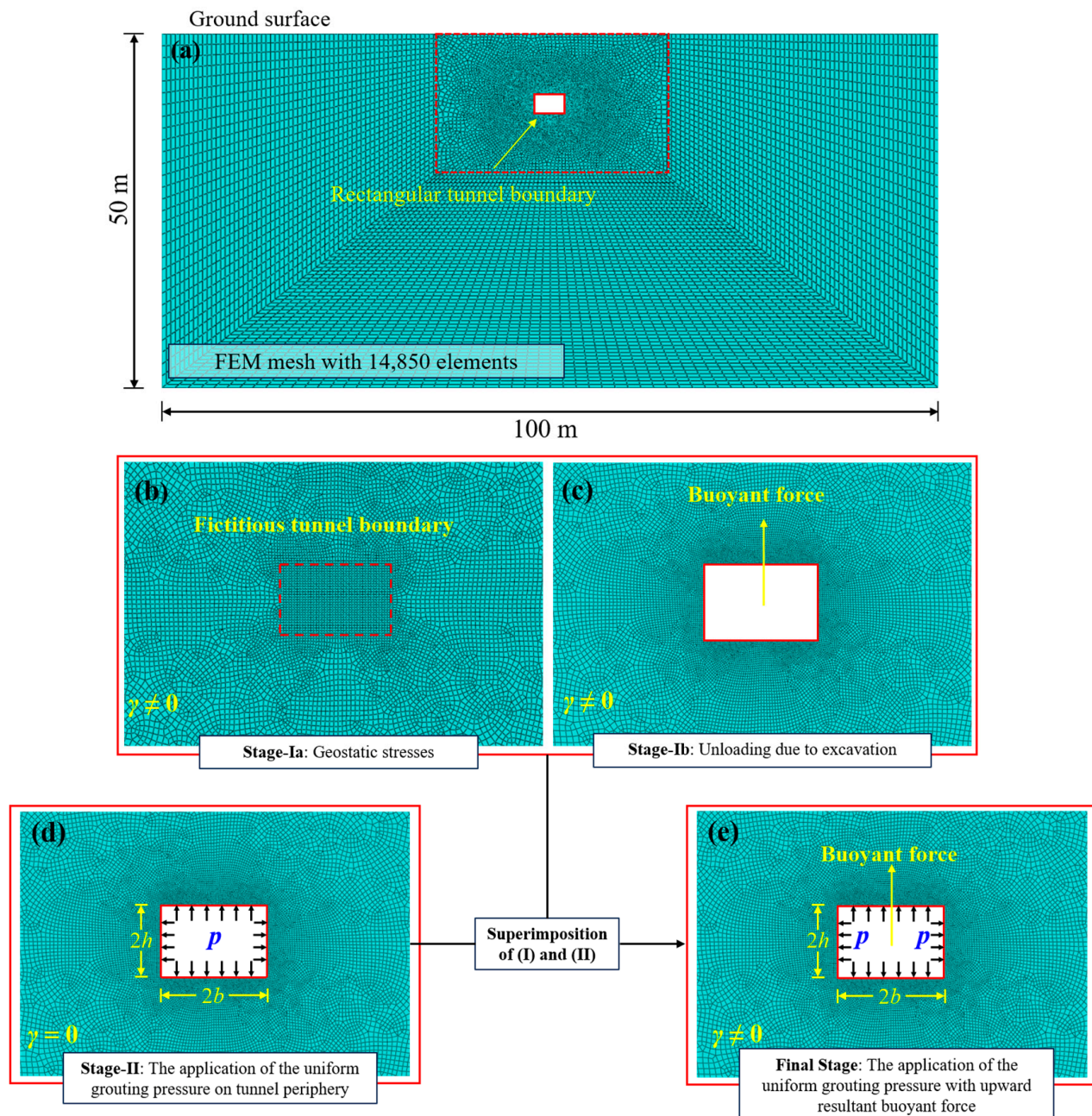


Figure 3. The diagram of infinite element mesh and the boundary conditions applied at different stages: (a) Mesh configuration and geometric model with applied boundary conditions; (b,c) Simulation stages of the buoyancy effect; (d) Application of uniform grouting pressure along the tunnel periphery within a weightless medium; (e) Combined effects of buoyancy and uniform grouting pressure on the tunnel boundary.

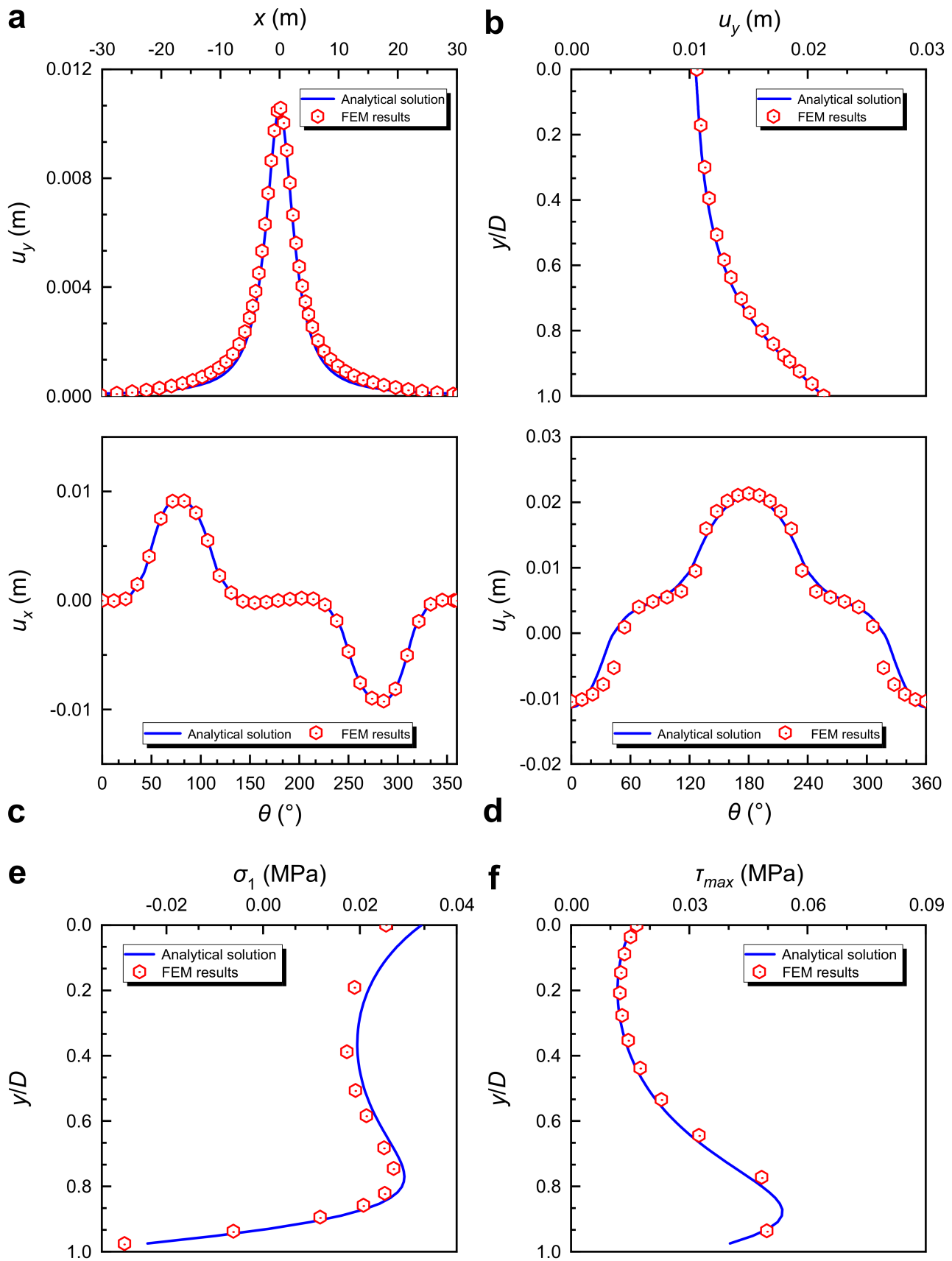


Figure 4. Comparison between the analytical solution and finite element method results: (a) surface vertical displacement; (b) vertical displacement from ground to tunnel crown; (c) horizontal displacement around the tunnel; (d) vertical displacement around the tunnel; (e) first principal stress from ground to tunnel crown; (f) maximum shear stress from ground to tunnel crown.

6. Parametric Analyses

To investigate the effect of grouting pressure on the soil disturbance, a systematic parametric analysis was carried out in this section, focusing on key parameters including burial depth, aspect ratio, the buoyancy factor, and Poisson's ratio. The soil and tunnel parameters adopted in the analysis are listed in Table 2. The baseline case is defined as follows: Poisson's ratio $\mu = 0.3$, burial depth $D = 5$ m, a rectangular tunnel of height $2h = 3$ m and width $2b = 2$ m (aspect ratio $h/b = 1.5$, and the cross-sectional area of the rectangular tunnel is always fixed in the parameter analysis of the tunnel aspect ratio), buoyancy factor $w_t = 0.65$, and grouting pressure $p = 0.2$ MPa. Other parametric cases are generated by varying only one parameter from this baseline configuration.

Table 2. All the parameters in parametric analyses.

Parameter	Value
M	0.1, 0.25, 0.35, 0.5
b/a	1/2, 2/3, 3/2, 2
D/m	3, 5, 7, 9
w_t	0, 0.35, 0.65, 1

6.1. Influence of Poisson's Ratio

Figure 5a displays the ground surface vertical displacement against the horizontal distance x from the tunnel centerline across a range of Poisson's ratios. For all Poisson's ratios considered, the surface vertical displacement profiles exhibit a similar trend. A larger Poisson's ratio results in a smaller maximum displacement. Figure 5b illustrates the distribution of vertical displacement from the tunnel crown (corresponds to the upper part of the tunnel) to the ground surface with the normalized distance y/D under various μ . With an increase in μ , the vertical displacement near the surface diminishes, whereas the displacement near the crown remains virtually unaffected. Figure 5c,d present the variation in the displacement in the vertical and horizontal directions at the horizontal centerline of the tunnel with the normalized distance $2x/a$ at varying μ . With a rising μ , the vertical displacement observed along this path progressively diminishes; the horizontal displacement near the opening increases, whereas far away from the opening, it decreases. Figure 5e,f further display the distributions of the first principal stress (σ_1) and the maximum shear stress (τ_{max}) along the path of crown-to-surface with y/D under various μ . The first principal stress exhibits an overall S-shaped trend with increasing y/D ; at a given depth, it increases slightly with Poisson's ratio, whereas the maximum shear stress remains essentially unaffected by changes in μ .

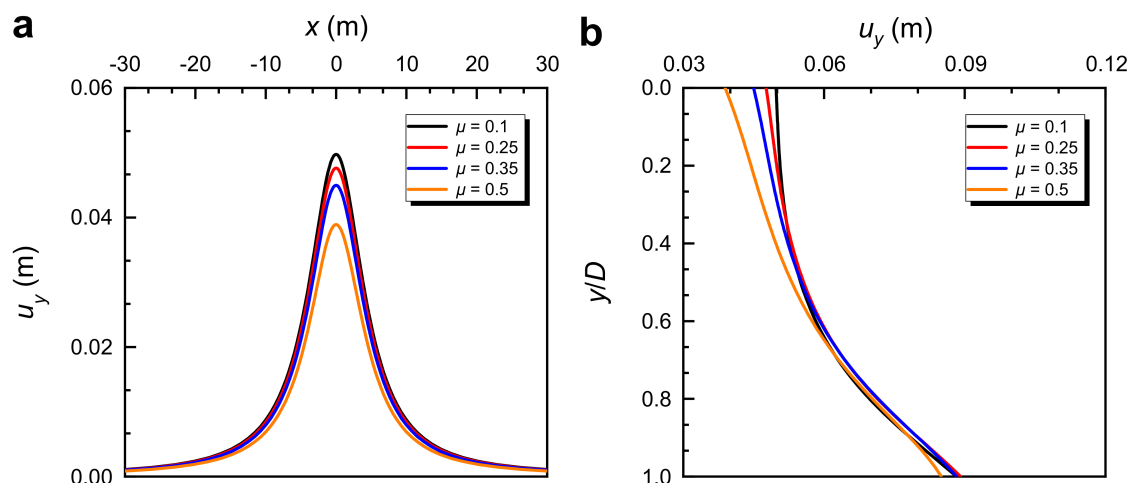


Figure 5. Cont.

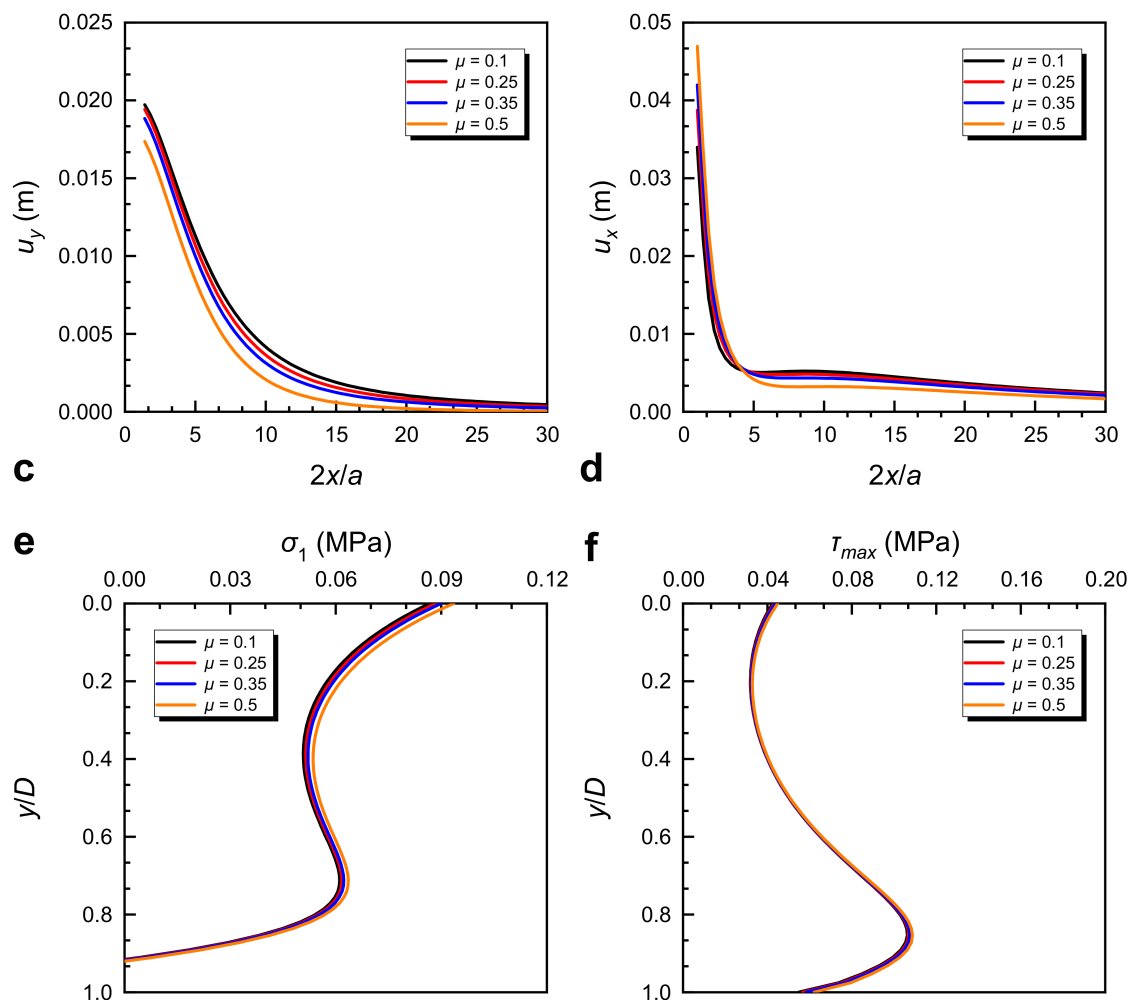


Figure 5. The effect of Poisson's ratio on soil displacement and stress: (a) surface vertical displacement; (b) vertical displacement from ground to tunnel crown; (c) horizontal displacement along the horizontal centerline of the tunnel; (d) vertical displacement along the horizontal centerline of the tunnel; (e) first principal stress from ground to tunnel crown; (f) maximum shear stress from ground to tunnel crown.

6.2. Influence of Aspect Ratio

Figure 6a presents the ground surface vertical displacement as a function of x for different h/b . In the region near the centerline, the surface vertical displacement decreases as the aspect ratio increases, whereas farther away from the centerline, the trend reverses, with the displacement increasing at larger aspect ratios. Figure 6b displays the variation in the vertical displacement from the tunnel crown to the surface with y/D for various h/b . Within the range of low aspect ratios, the vertical displacement changes more pronouncedly along this path, as evidenced by the considerably larger differences between displacement curves corresponding to different low aspect ratios compared with those at higher aspect ratios. The maximum vertical displacement increases gradually with a decreasing aspect ratio. Figure 6c,d show the variation in the displacement in the vertical and horizontal directions at the horizontal centerline of the tunnel against the $2x/a$ at varying h/b . With an increasing h/b , the vertical displacement profile becomes increasingly gentle, and the peak vertical displacement near the opening decreases accordingly. For the horizontal displacement, the distribution patterns across all cases remain similar, while the displacement magnitudes increase with a larger aspect ratio. Figure 6e,f depict the variation in σ_1 and τ_{max} along the path of crown-to-surface with y/D for different aspect ratios. At low aspect ratio, the σ_1

curve exhibits an overall S-shaped trend; with a continuous increase in the aspect ratio, this curve gradually transitions to a pattern of first increasing and then decreasing. A reduction in the aspect ratio causes an increase in both σ_1 and τ_{max} near the ground surface, while their values near the tunnel crown decrease.

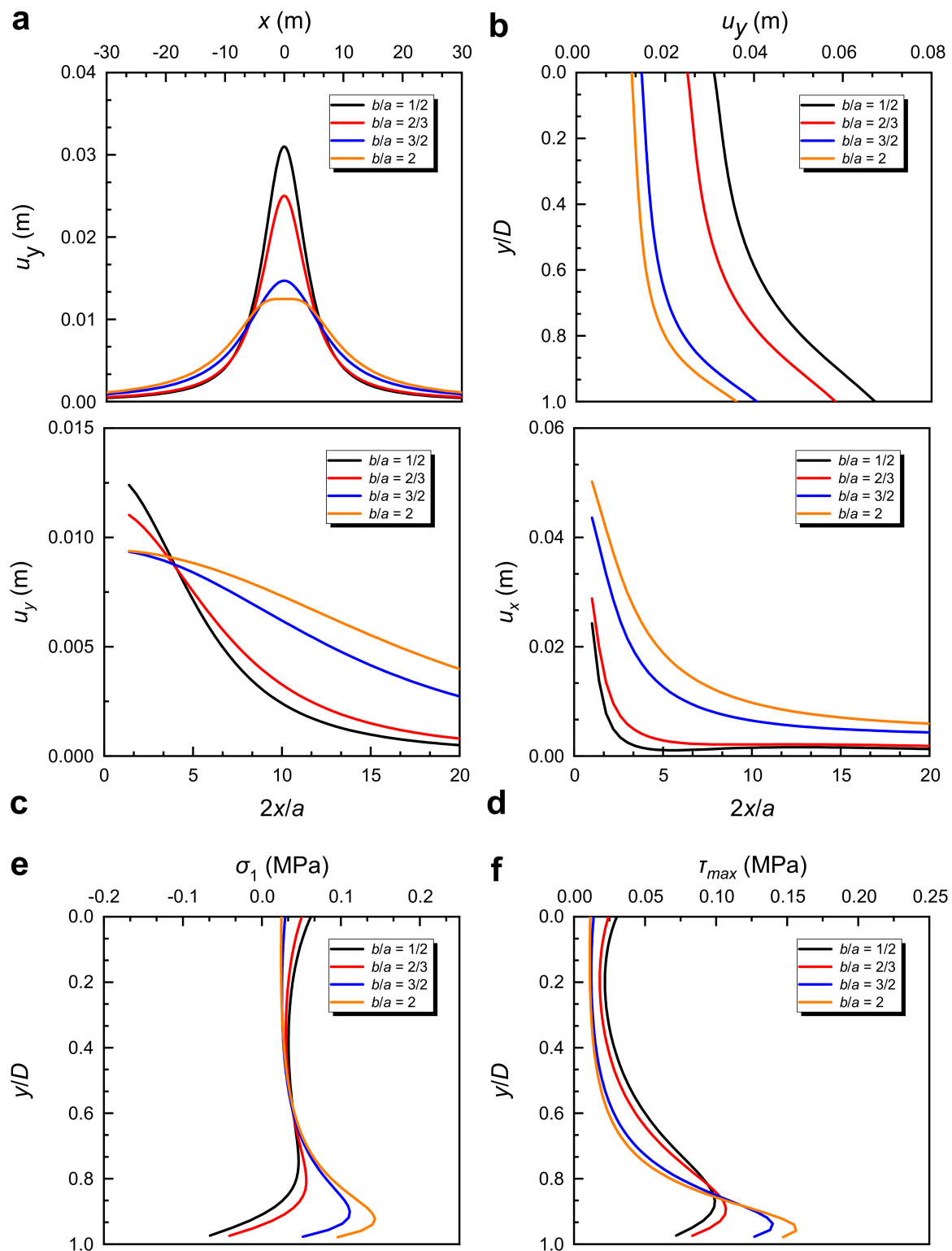


Figure 6. The effect of aspect ratio on soil displacement and stress: (a) surface vertical displacement; (b) vertical displacement from ground to tunnel crown; (c) horizontal displacement along the horizontal centerline of the tunnel; (d) vertical displacement along the horizontal centerline of the tunnel; (e) first principal stress from ground to tunnel crown; (f) maximum shear stress from ground to tunnel crown.

6.3. Influence of Burial Depth

Figure 7a illustrates the ground surface vertical displacement against x under different D . With increasing burial depth, the peak vertical ground displacement diminishes progressively. Specifically, in the region near the centerline, the vertical displacement diminishes with increasing depth, whereas farther away from the centerline, the trend reverses and the displacement increase with depth. Figure 7b presents the variation in the vertical displacement from the tunnel crown to the ground surface with the normalized distance y/D . As D increases, the vertical displacement at all depths exhibits a consistent decreasing trend: shallower depths correspond to greater displacement. Figure 7c,d illustrate the variation in the displacement in the vertical and horizontal directions at the horizontal centerline of the tunnel with the $2x/a$ under different burial depths. For the vertical displacement, a rise in burial depth causes a diminish in displacement when $2x/a$ is within the range of 0 to 5, whereas the trend reverses for $2x/a > 5$, where the displacement increases with depth. For the horizontal displacement, a shallower burial depth results in larger displacements in the region away from the opening, and this effect is most pronounced at $D = 3$ m. Figure 7e,f further depict the variation in σ_1 and τ_{max} along the path of crown-to-surface with y/D under different D . At a relatively shallow depth (e.g., $D = 3$ m), the first principal stress decreases monotonically with depth, while the maximum shear stress exhibits an S-shaped trend. With an increasing D , both the curves of σ_1 and τ_{max} transition to a pattern of first increasing and then decreasing. Moreover, both σ_1 and τ_{max} at the ground surface increase consistently with greater D .

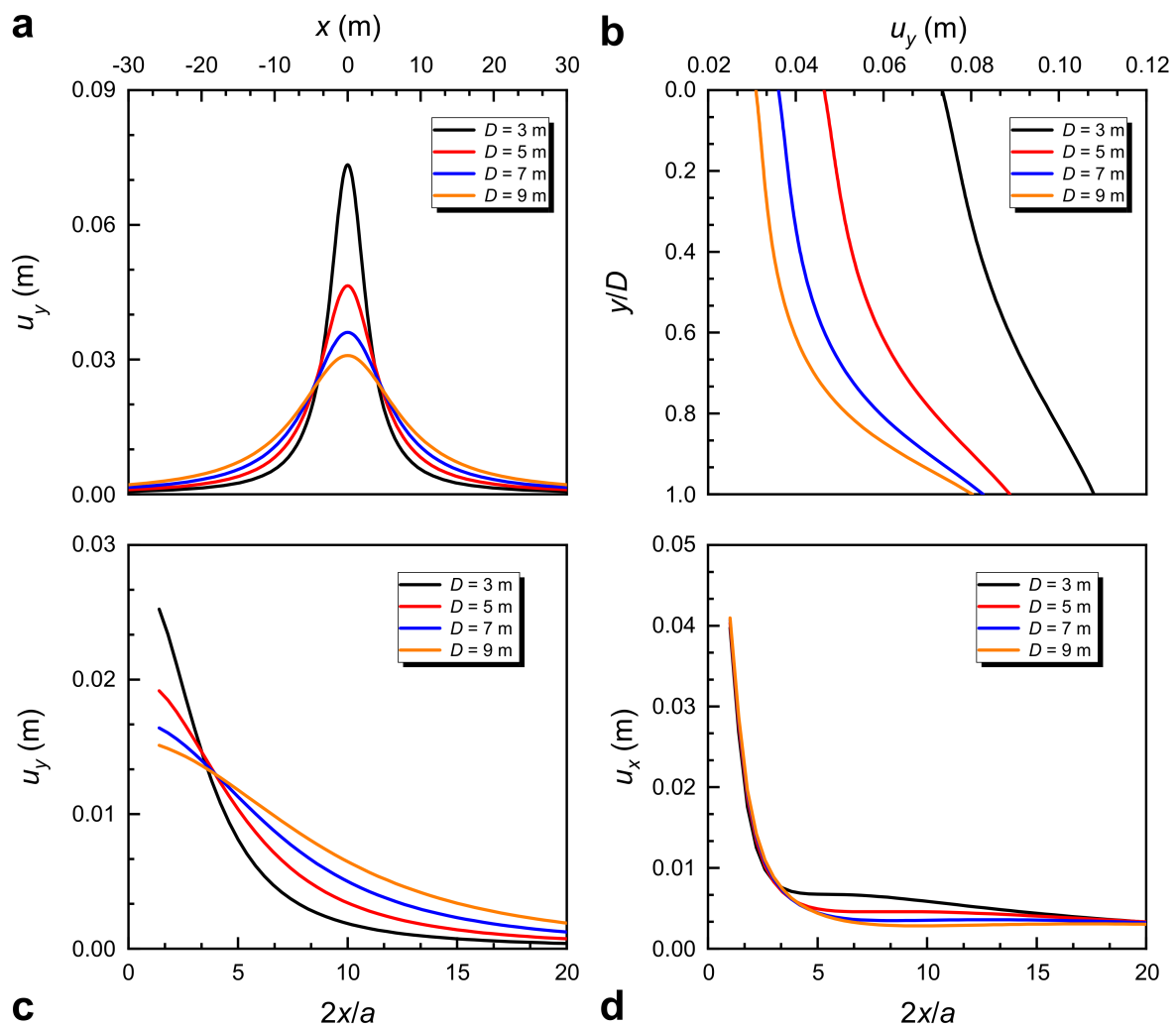


Figure 7. Cont.

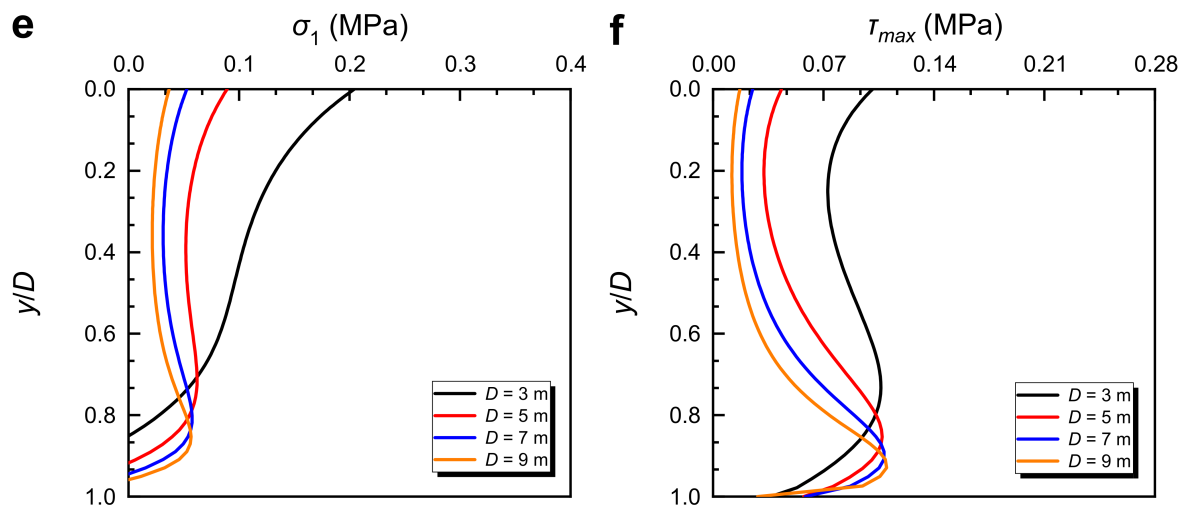


Figure 7. The effect of burial depth on soil displacement and stress: (a) surface vertical displacement; (b) vertical displacement from ground to tunnel crown; (c) horizontal displacement along the horizontal centerline of the tunnel; (d) vertical displacement along the horizontal centerline of the tunnel; (e) first principal stress from ground to tunnel crown; (f) maximum shear stress from ground to tunnel crown.

6.4. Influence of the Buoyancy Factor

Figure 8a,b present, respectively, the variation in ground surface vertical displacement and the variation in vertical displacement along the crown-to-surface path with the y/D under various values of w_t . The maximum surface vertical displacement decreases as w_t rises; a smaller w_t corresponds to a more pronounced buoyancy effect and the displacement on the top of tunnel boundary rises with decreasing w_t . Notably, w_t influences only the amplitude of the vertical displacement without altering its distribution pattern along the depth. Figure 8c,d present the variation in the displacement in the vertical and horizontal directions at the horizontal centerline of the tunnel with the $2x/a$ for various w_t . It is observed that a smaller w_t yields a pronounced increase in the maximum vertical displacement, together with an overall rise in the horizontal displacement along this path. Figure 8e,f show the distribution of σ_1 and τ_{max} along the path of crown-to-surface with y/D under various w_t . Both σ_1 and τ_{max} increase with rising w_t , with σ_1 being more sensitive to changes in the buoyancy factor.

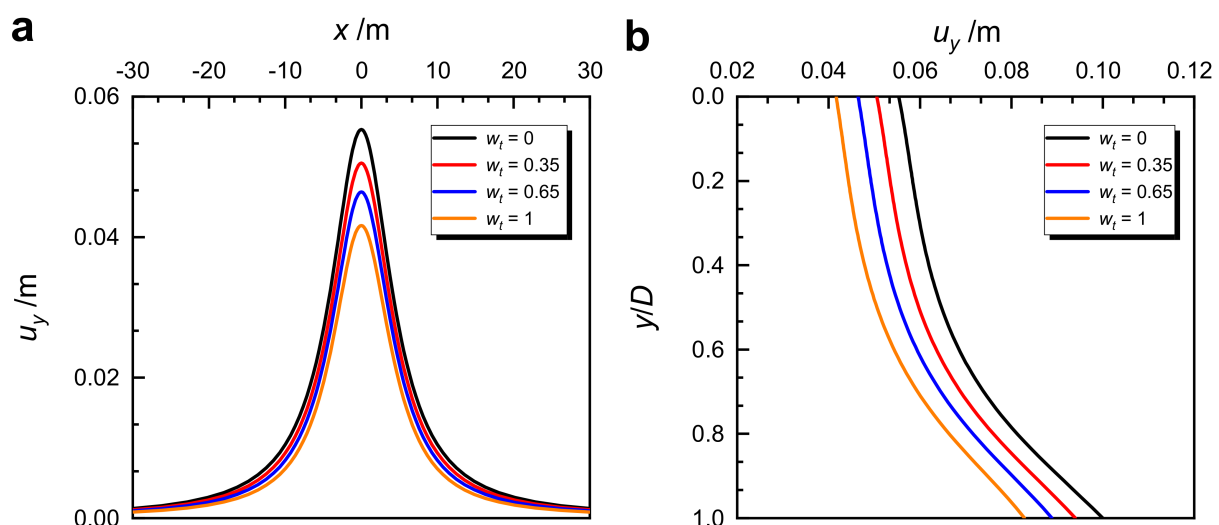


Figure 8. Cont.

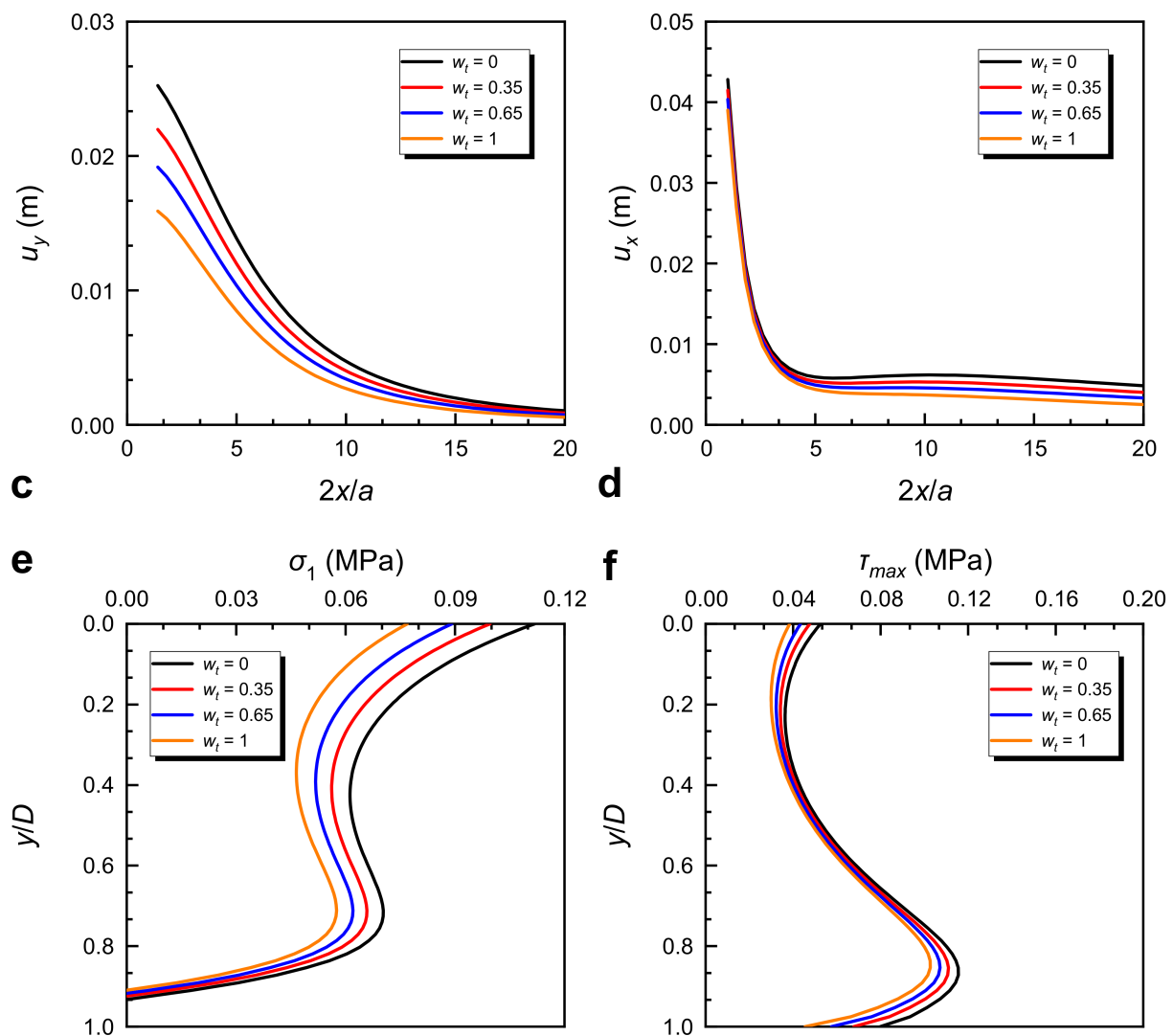


Figure 8. The effect of buoyancy factor on soil displacement and stress: (a) surface vertical displacement; (b) vertical displacement from ground to tunnel crown; (c) horizontal displacement along the horizontal centerline of the tunnel; (d) vertical displacement along the horizontal centerline of the tunnel; (e) first principal stress from ground to tunnel crown; (f) maximum shear stress from ground to tunnel crown.

7. Conclusions

This paper derives an analytical solution for soil displacement and stress induced by grouting pressure in a shallow rectangular tunnel by using the conformal mapping and complex variable theory. In the derivation, the buoyancy effect caused by the self-weight of the excavated soil is incorporated, and the solution procedure is simplified through Fourier series expansion. Comparisons with finite element results show good agreement, confirming the validity of the proposed analytical solution. Moreover, a parametric analysis is performed to examine the effect of key factors—including aspect ratio, Poisson's ratio, the buoyancy effect, and burial depth on the soil disturbance. The main conclusions are as follows:

- (1) Increasing μ suppresses the vertical displacement at the surface and the path along the crown to surface. Along the horizontal centerline, a larger Poisson's ratio leads to a smaller maximum vertical displacement, with a more pronounced increase in horizontal displacement near the opening and a decrease in the far field. In terms of stress, the first principal stress slightly increases with a higher Poisson's ratio, while the maximum shear stress remains nearly unaffected.

- (2) Near the tunnel centerline, a larger aspect ratio can effectively reduce the surface vertical displacement, but it tends to increase the surface displacement in the far-field region. Along the horizontal centerline, increasing the aspect ratio results in a gentler vertical displacement profile and a smaller maximum vertical displacement at the opening, while the horizontal displacement increases globally. The stress distribution pattern also changes with the aspect ratio—under low aspect ratios, the first principal stress exhibits an S-shaped profile with depth, whereas it transitions to an increasing-then-decreasing pattern at higher aspect ratios.
- (3) Increasing the burial depth generally reduces the displacement magnitudes around the tunnel, most notably diminishing the surface vertical displacements, while the horizontal displacement is remarkably insensitive to depth changes. Along the horizontal centerline, vertical displacement decreases with increasing depth within $2x/a < 5$, but the trend reverses in the region away from the opening. The burial depth also reshapes the stress distribution: at shallow depths, σ_1 decreases monotonically with depth and τ_{max} follows an S-shaped curve; with a larger burial depth, both stress quantities evolve into a pattern of first increasing and then decreasing.
- (4) Enhancing the buoyancy effect significantly increases the vertical displacements, yet it has virtually no impact on the horizontal displacements. For the path at the tunnel horizontal centerline, a smaller w_t yields a larger maximum vertical displacement and an overall increase in horizontal displacement. Conversely, weakening the buoyancy effect leads to an overall increase in both σ_1 and τ_{max} , with σ_1 being more sensitive to changes in the buoyancy factor.

The presented analytical method offers a reference for addressing similar problems. Moreover, the proposed analytical solution can provide a guide for designing and constructing rectangular tunnels. Nevertheless, the assumptions of elastic and isotropic soil behavior are idealized and differ from practical soil properties. Future research could consider treating the soil as an elastoplastic or anisotropic medium to establish corresponding theoretical models, thereby making the results more consistent with practical engineering conditions.

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