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Time-Varying LQR Control with Matrix Smoothing for Continuous-Time Switched Systems Under Time-Dependent Switching

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Abstract

This paper addresses the analysis and control of continuous-time linear switched systems within the linear time-varying (LTV) framework. A correspondence between switched and LTV system representations is first established, enabling the use of LTV tools for stability and controllability analysis under arbitrary switching sequences. Two state-feedback synthesis approaches are considered. The first is a switched LQR design formulated via Linear Matrix Inequalities using Multiple Lyapunov Functions. The second is a continuous-time LTV-LQR obtained by solving the differential Riccati equation. To mitigate the discontinuities associated with hard switching, a matrix smoothing strategy based on the partition of unity is introduced. This approach generates continuous approximations of the plant matrices utilized exclusively for control synthesis in the time-varying Riccati framework, while the actual plant retains its hard-switched dynamics. Numerical simulations for representative switched systems illustrate that the proposed smoothed LTV formulation leads to improved closed-loop behavior compared to the switched LQR baseline, including reduced control discontinuities and lower values of the quadratic performance index for the considered scenarios. The results indicate that the method is suitable for applications where abrupt control variations are undesirable.

Keywords: continuous-time; LQR control; matrix smoothing; switched system; time-dependent switching; time-varying system; differential Riccati equation

1. Introduction

Switched linear systems represent a distinct and challenging class of dynamical models, defined by a finite collection of continuous-time linear subsystems and a supervisory logic governing transitions between them [1]. In engineering practice, this structure naturally describes physical plants operating across multiple configurations, such as power electronic converters or advanced robotic systems [2–4], as well as systems employing multiple controllers that switch to ensure desired performance under varying conditions [5,6].

From a control-theoretic perspective, the analysis and control of such systems is nontrivial, as the overall system trajectory depends strongly on the switching signal. Consequently, classical analysis and design techniques developed for time-invariant systems may become inadequate. Due to the explicitly time-dependent structure, switched systems can be interpreted as a particular class of time-varying systems.



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The analysis of stability and controllability in switched systems introduces additional challenges arising from structural transitions rather than the individual subsystem dynamics. In particular, asymptotic stability of all subsystems does not guarantee stability under arbitrary switching signals, while appropriately designed switching can stabilize systems composed entirely of unstable modes [7].

Classical analytical approaches rely on Common Lyapunov Functions (CLFs), Multiple Lyapunov Functions (MLFs), or dwell-time constraints [8–13]. These methods often result in conservative designs or computationally demanding Linear Matrix Inequality (LMI) problems [14]. Similarly, assessing controllability under switching cannot rely on standard time-invariant rank conditions, requiring a more general dynamical formulation [11,15]. These limitations complicate optimal controller synthesis and motivate alternative modeling approaches.

The synthesis of optimal Linear Quadratic Regulator (LQR) controllers for switched systems has led to multiple design methodologies. A dominant class of approaches employs LMI formulations based on Multiple Lyapunov Functions to ensure stability and impose bounds on the quadratic cost functional [11,16]. Within this framework, techniques such as de Oliveira’s extended variable transformation combined with the Schur complement are used to reduce conservatism [17–19].

Alternatively, optimization-based approaches, including dynamic programming and model predictive control (MPC), aim at determining optimal switching sequences [20]. However, these methods either preserve a piecewise-constant control structure or involve significant computational complexity. This motivates continuous formulations in which LQR synthesis is performed directly using the continuous-time differential Riccati equation (DRE) [21,22].

Formulating the continuous LTV feedback synthesis via matrix smoothing for an executed hard-switched system links this approach to Linear Parameter-Varying (LPV) [23,24] and gain-scheduling [25–27] paradigms. Standard LPV and gain-scheduling methods utilize smooth interpolation parameters to transition between local linear modes. However, they typically treat scheduling variables as exogenous signals, introducing conservatism and limiting their capacity to optimally exploit explicit, time-triggered switching sequences.

In this paper, a continuous time-varying LQR strategy is developed and compared with a Switched LMI-LQR approach. To mitigate discontinuities inherent to hard-switching, a matrix smoothing procedure satisfying the partition of unity is incorporated exclusively for control synthesis purposes, while the actual system operates under hard-switched plant dynamics. By applying sigmoidal smoothing in this manner, the feedback gains are computed from continuous matrix trajectories, allowing the differential Riccati equation to be solved without processing step discontinuities. Numerical simulations conducted on unstable systems indicate that the proposed continuous LTV formulation reduces the LQR cost functional, shortens the settling time, and produces a smooth control input.

The paper is organized as follows. In Section 2, the switched state-space model is introduced, and the equivalence between the switched system and the continuous LTV formulation is established. Section 3 reviews stability using the Differential Lyapunov Equation and controllability using the continuous Gramian. Section 4 develops both the Switched LMI-LQR and continuous LTV-LQR synthesis methods. Section 5 presents a numerical comparison of both approaches. Finally, Section 6 summarizes the results and outlines future directions.

The following notation is used throughout the paper: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^{n \times m}$ the set of real matrices ($\mathbb{R}^n = \mathbb{R}^{n \times 1}$), $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers, and $\mathbb{I}_n$ the identity matrix. For a symmetric matrix $P \in \mathbb{R}^{n \times n}$, $P \succ 0$ ($P \succeq 0$) de-

notes positive definiteness (semidefiniteness), while $P \prec 0$ ($P \preccurlyeq 0$) denotes negative definiteness (semidefiniteness).

2. Preliminaries

2.1. Continuous-Time Switched Systems Under Time-Dependent Switching

Consider a continuous-time linear switched system described by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the input vector. The matrices $A_k \in \mathbb{R}^{n \times n}$, $B_k \in \mathbb{R}^{n \times m}$, for $k \in \{1, 2, \dots, N\}$, define a finite set of linear subsystems. The switching signal $\sigma(t) \in \{1, 2, \dots, N\}$ is assumed to be piecewise constant, with a sequence of switching instants $t_i \geq 0$ and $t_{i+1} > t_i$. On each interval $t \in [t_i, t_{i+1})$, the active subsystem is given by $\sigma(t) = k$ for some $k \in \{1, 2, \dots, N\}$.

The state trajectory is assumed to be continuous at switching instants, i.e.,

$$x(t_i^-) = x(t_i^+) \quad \forall i \in \mathbb{Z}_{\geq 0}, \quad (2)$$

which implies that switching affects only the system dynamics and not the state value.

Due to the presence of multiple switching instants, the state trajectory cannot be expressed using a single subsystem flow and instead results from a recursive composition of solutions over successive intervals. Expanding this construction explicitly leads to a structured expression involving accumulated state-transition mappings and input contributions across all preceding switching periods. The corresponding representation is formalized in the following result.

Theorem 1. *The solution to Equation (1) for known initial condition $x(t_0) \in \mathbb{R}^n$, given input $u(t) \in \mathbb{R}^m$ and known switching instants t_i , $i \in \mathbb{Z}_{\geq 0}$, is given by*

$$x(t) = \Phi(t, t_i) \left[\prod_{j=0}^{i-1} \Phi(t_{i-j}, t_{i-j-1}) x(t_0) + \sum_{j=1}^i \left(\Psi_{i-j-1} \Theta(t_j, t_{j-1}) \right) \right] + \Theta(t, t_i), \quad (3)$$

where

$$\Phi(t_i, t_j) = e^{A_{\sigma(t_j)}(t_i - t_j)}, \quad \Theta(t_i, t_j) = \int_{t_j}^{t_i} e^{A_{\sigma(t_j)}(t_i - \tau)} B_{\sigma(t_j)} u(\tau) d\tau, \quad (4)$$

$$\Psi_k = \prod_{l=i-k}^i \Phi(t_{2i-l-k}, t_{2i-l-k-1}), \quad \Psi_{-1} = \mathbb{I}_n.$$

Proof. It is well-known [28] that the solution to Equation (1) in the interval $t \in [t_i, t_{i+1})$ is given by

$$x(t) = e^{A_{\sigma(t_i)}(t - t_i)} x(t_i) + \int_{t_i}^t e^{A_{\sigma(t_i)}(t - \tau)} B_{\sigma(t_i)} u(\tau) d\tau, \quad (5)$$

where $x(t_i) \in \mathbb{R}^n$ is the initial condition at the switching instant t_i , $i \in \mathbb{Z}_{\geq 0}$. Substituting $t = t_{i+1}$ to Equation (5), we can compute the initial conditions $x(t_i)$ using the recursive formula

$$x(t_{i+1}) = e^{A_{\sigma(t_i)}(t_{i+1} - t_i)} x(t_i) + \int_{t_i}^{t_{i+1}} e^{A_{\sigma(t_i)}(t_{i+1} - \tau)} B_{\sigma(t_i)} u(\tau) d\tau, \quad (6)$$

Substituting (6) for $i = 0, 1, 2, \dots$ into (5) and introducing (4), we obtain Formula (3). $\square$

2.2. Equivalent LTV Representation of a Switched System

The switched system (1) can be interpreted as a special case of a linear time-varying (LTV) system with piecewise-constant system matrices. In particular, the dynamics can be rewritten in the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad (7)$$

where the matrices $A(t) = A_{\sigma(t)}$ and $B(t) = B_{\sigma(t)}$ can be expressed as functions of subsystem matrices $A_k, B_k, k \in \{1, \dots, N\}$ using Lagrange interpolation [29,30]

$$A(t) = \sum_{k=1}^N \left(A_k \prod_{\substack{m=1 \\ m \neq k}}^N \frac{\sigma(t) - m}{k - m} \right), \quad B(t) = \sum_{k=1}^N \left(B_k \prod_{\substack{m=1 \\ m \neq k}}^N \frac{\sigma(t) - m}{k - m} \right). \quad (8)$$

It is well-known [28] that the solution of (7) for a given initial condition $x(t_0) \in \mathbb{R}^n$ is given by

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau)d\tau, \quad (9)$$

where

$$\Phi(t, t_0) = \mathbb{I}_n + \int_{t_0}^t A(\tau)d\tau + \int_{t_0}^t A(\tau) \int_{t_0}^{\tau} A(\tau_1)d\tau_1d\tau + \dots \quad (10)$$

The following standard properties of the state transition matrix will be used in the subsequent analysis.

Lemma 1 ([28]). *The transition matrix $\Phi(t, t_0)$ defined by (10) satisfies the following conditions:*

- (1) $\Phi(t_0, t_0) = \mathbb{I}_n$;
- (2) $\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$ for any t, t_1 and t_0 ;
- (3) $\Phi(t, t_0) = e^{A(t-t_0)}$ if the elements of the matrix A are constant.

To relate the switched-system representation derived in Theorem 1 with the LTV formulation, we now compare the corresponding state trajectories under piecewise-constant switching. The following result establishes the equivalence between these representations.

Theorem 2. *For the continuous-time linear switched system (1) with matrices $A(t) = A_{\sigma(t)}$, $B(t) = B_{\sigma(t)}$ defined by (8), the solutions (3) and (9) are equivalent.*

Proof. Note that $A(t) = A_{\sigma(t_i)}$ and $B(t) = B_{\sigma(t_i)}$ within each interval $t \in [t_i, t_{i+1})$. Therefore, Formulas (4) and (10) describe the same matrix $\Phi(t_i, t_j)$. Consequently, using the properties of this matrix given in Lemma 1 for Equation (3), we obtain

$$\begin{aligned}
x(t) &= \Phi(t, t_i) \Phi(t_i, t_{i-1}) \cdots \Phi(t_1, t_0) x(t_0) \\
&\quad + \Phi(t, t_i) \Phi(t_i, t_{i-1}) \cdots \Phi(t_2, t_1) \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau) d\tau \\
&\quad + \Phi(t, t_i) \Phi(t_i, t_{i-1}) \cdots \Phi(t_3, t_2) \int_{t_1}^{t_2} \Phi(t_2, \tau) B(\tau) u(\tau) d\tau \\
&\quad + \cdots + \Phi(t, t_i) \int_{t_{i-1}}^{t_i} \Phi(t_i, \tau) B(\tau) u(\tau) d\tau + \int_{t_i}^t \Phi(t, \tau) B(\tau) u(\tau) d\tau \\
&= \Phi(t, t_0) x(t_0) + \int_{t_0}^{t_1} \Phi(t, \tau) B(\tau) u(\tau) d\tau \\
&\quad + \int_{t_1}^{t_2} \Phi(t, \tau) B(\tau) u(\tau) d\tau + \cdots + \int_{t_{i-1}}^{t_i} \Phi(t, \tau) B(\tau) u(\tau) d\tau \\
&\quad + \int_{t_i}^t \Phi(t, \tau) B(\tau) u(\tau) d\tau = \Phi(t, t_0) x(t_0) + \int_{t_0}^t \Phi(t, \tau) B(\tau) u(\tau) d\tau,
\end{aligned} \tag{11}$$

which leads directly to Formula (9). This demonstrates that the solutions to Equations (1) and (7) under (8) are equivalent. $\square$

Having established the equivalent LTV representation of the switched system, we now analyze its structural properties relevant for control design. In particular, conditions for stability and controllability are examined prior to controller synthesis.

3. Stability and Controllability of Continuous-Time Switched Systems

Based on the LTV representation introduced in Section 2, we analyze the stability and controllability properties of the system. These properties are examined as prerequisites for the subsequent controller synthesis.

3.1. Stability Analysis

Since the stability properties of system (1) are determined by the unforced dynamics, we consider the case $u(t) = 0$. Using the LTV representation introduced in Section 2, the system reduces to

$$\dot{x}(t) = A(t)x(t). \tag{12}$$

In the context of LTV systems, exponential stability is considered, as it implies Uniform Asymptotic Stability (UAS) and provides a convergence rate independent of the initial time t_0 .

Definition 1. The continuous-time switched system (1) is called exponentially stable if there exist constants $M \geq 1$ and $\alpha > 0$ such that the solution $x(t)$ satisfies

$$\|x(t)\| \leq M e^{-\alpha(t-t_0)} \|x(t_0)\| \tag{13}$$

for all $t \geq t_0$ and any initial conditions $x(t_0) \in \mathbb{R}^n$.

It is well-known [31] that the system (7) (or equivalently (1), by Theorem 2) is exponentially stable if and only if there exists a positive definite Lyapunov function $V(t, x)$ such that its time derivative along the trajectories of the system is negative definite, i.e.,

$$0 < c_1 \|x(t)\|^2 \leq V(t, x) = x^T(t)P(t)x(t) \leq c_2 \|x(t)\|^2 \quad (14)$$

and

$$\frac{dV(t, x)}{dt} = -x^T(t)Q(t)x(t) \leq -c_3 \|x\|^2 < 0, \quad (15)$$

where $P(t)$ and $Q(t)$ are positive definite symmetric matrices with the constraints

$$c_1 \mathbb{I}_n \preceq P(t) \preceq c_2 \mathbb{I}_n, \quad c_3 \mathbb{I}_n \preceq Q(t) \preceq c_4 \mathbb{I}_n \quad (16)$$

for all $t \geq t_0$. The coefficients $c_k > 0$ are given by

$$\begin{aligned} c_1 &= \inf_{t \geq 0} \lambda_{\min}(P(t)), \quad c_2 = \sup_{t \geq 0} \lambda_{\max}(P(t)), \\ c_3 &= \inf_{t \geq 0} \lambda_{\min}(Q(t)), \quad c_4 = \sup_{t \geq 0} \lambda_{\max}(Q(t)), \end{aligned} \quad (17)$$

where $\lambda_{\max}$ and $\lambda_{\min}$ denote the maximum and minimum eigenvalue of the matrix, respectively.

For a given positive-definite symmetric matrix $Q(t)$, the matrix $P(t)$ can be obtained as the solution of the differential Lyapunov equation (DLE) of the form [32]

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) - Q(t), \quad t \geq t_0. \quad (18)$$

For numerical purposes, this equation is often solved using backward integration by introducing a reverse time variable $\tau = t_f - t$, which implies $\dot{P}(t) = -\dot{P}(\tau)$. Under this transformation, Equation (18) can be rewritten as

$$\dot{P}(\tau) = P(\tau)A(\tau) + A^T(\tau)P(\tau) + Q(\tau). \quad (19)$$

For systems with unstable dynamics, backward integration with a constant weighting matrix Q may lead to numerical growth of $P(t)$ along the integration horizon. To mitigate this effect, a time-varying matrix $Q(t)$ can be defined as

$$Q(t) = Q(t_0) - \left(A^T(t)P(t_0) + P(t_0)A(t) \right), \quad (20)$$

where $P(t_0) \succ c_1 \mathbb{I}_n$ and $Q(t_0) \succ c_3 \mathbb{I}_n$. Therefore, from (19) and (20), we have

$$\dot{P}(\tau) = [P(\tau) - P(t_0)]A(\tau) + A^T(\tau)[P(\tau) - P(t_0)] + Q(t_0). \quad (21)$$

This time-varying formulation alters the driving term of the differential equation. By embedding the initial states $P(t_0)$ and $Q(t_0)$ into $Q(t)$, it counteracts the immediate uncontrolled growth triggered by a constant weight matrix, thereby improving the numerical conditioning during the backward integration process.

Introducing the tracking error matrix $\tilde{P}(t) = P(t) - P(t_0)$, one can reformulate Equation (21) into the form

$$\dot{\tilde{P}}(\tau) = \tilde{P}A(\tau) + A^T(\tau)\tilde{P} + Q(t_0). \quad (22)$$

At any given instant $t \in \mathbb{R}$, the system operates in a specific mode k such that $A(t) = A_k$. Evaluating (20) at this instance yields

$$Q(t) = Q(t_0) - (A_k^T P(t_0) + P(t_0) A_k) \quad (23)$$

To prove the asymptotic stability of the system, let us make the following assumptions.

- (1) Suppose that there exist constant matrices $P(t_0) \succ 0$ and $Q(t_0) \succ 0$ that satisfy the finite set of LMIs for all constituent subsystems, i.e.,

$$A_k^T P(t_0) + P(t_0) A_k \prec Q(t_0) \quad \forall k \in \{1, 2, \dots, N\}. \quad (24)$$

- (2) The set of subsystems can be partitioned into stable modes S_s and unstable modes S_u such that $\{A_1, \dots, A_N\} = S_s \cup S_u$. We define the uniform stability margin factor $\beta > 0$ as the largest scalar such that there exists a constant $M_s \geq 1$ satisfying

$$\|e^{A_j t}\| \leq M_s e^{-\beta t} \quad \forall t \geq 0, \quad \forall j \in S_s, \quad (25)$$

and the maximum instability growth factor $\gamma > 0$ as the smallest scalar such that there exists a constant $M_u \geq 1$ satisfying

$$\|e^{A_l t}\| \leq M_u e^{\gamma t} \quad \forall t \geq 0, \quad \forall l \in S_u, \quad (26)$$

where $\|\cdot\|$ denotes the induced matrix 2-norm. The switching signal satisfies an activation time ratio such that the total active time of unstable modes T_u relative to stable modes T_s satisfies

$$\frac{T_u}{T_s} < \frac{\beta}{\gamma}. \quad (27)$$

Theorem 3. *The continuous-time switched system (1) is exponentially stable if conditions (24) and (27) are satisfied. Under these conditions, the matrices $P(t)$ and $Q(t)$ remain strictly positive definite and uniformly bounded from above for all $t \in [t_0, t_f]$.*

Proof. First, we start with the verification of the positive definiteness and boundedness of $Q(t)$. From (24), we have the upper bound $A_k^T P(t_0) + P(t_0) A_k \prec Q(t_0)$ for all $k \in \{1, 2, \dots, N\}$. Applying this bound directly to (23) yields

$$Q(t) \succ Q(t_0) - Q(t_0) = 0 \quad \forall t \in [t_0, t_f]. \quad (28)$$

Thus, $Q(t) \succ 0$ holds uniformly. Since the matrix set $\{A_1, \dots, A_N\}$ is finite and constant, $c_3 \mathbb{I}_n \preccurlyeq Q(t) \preccurlyeq c_4 \mathbb{I}_n$ is satisfied for $0 < c_3 \leq c_4 < \infty$ as defined in (16).

Now, let us verify the positive definiteness and boundedness of $P(t)$. Equation (22) is integrated backward over $\tau \in [0, t_f - t_0]$, starting from $\tilde{P}(0) = P(t_f) - P(t_0) \neq 0$. For any piecewise interval $\tau \in [\tau_k, \tau_{k+1})$, where $A(\tau) = A_k$, the induced 2-norm of the backward trajectory satisfies

$$\|\tilde{P}(\tau)\| \leq \|e^{A_k(\tau-\tau_k)}\|^2 \|\tilde{P}(\tau_k)\| + \int_{\tau_k}^{\tau} \|e^{A_k(\tau-s)}\|^2 \|Q(t_0)\| ds. \quad (29)$$

Applying the exponential bounds from (25) and (26) to (29) and evaluating the definite integrals yields

- (1) for $A_k \in S_s$: $\|\tilde{P}(\tau)\| \leq M_s^2 e^{-2\beta(\tau-\tau_k)} \|\tilde{P}(\tau_k)\| + M_s^2 \frac{\|Q(t_0)\|}{2\beta} (1 - e^{-2\beta(\tau-\tau_k)});$
- (2) for $A_k \in S_u$: $\|\tilde{P}(\tau)\| \leq M_u^2 e^{2\gamma(\tau-\tau_k)} \|\tilde{P}(\tau_k)\| + M_u^2 \frac{\|Q(t_0)\|}{2\gamma} (e^{2\gamma(\tau-\tau_k)} - 1).$

Using $(1 - e^{-2\beta\Delta\tau}) < 1$ and $(e^{2\gamma\Delta\tau} - 1) < e^{2\gamma\Delta\tau}$, recursive compounding via mathematical induction backward across all N_s switching intervals results in the global majorant

$$\|\tilde{P}(\tau)\| \leq \mu^{N_s} e^{2\gamma T_u(0,\tau) - 2\beta T_s(0,\tau)} \|\tilde{P}(0)\| + \|Q(t_0)\| \sum_{k=0}^{N_s-1} \mu^{N_s-k} \Delta\tau_k e^{2\gamma T_u(\tau_k,\tau) - 2\beta T_s(\tau_k,\tau)}, \quad (30)$$

where $\mu = \max(M_s^2, M_u^2) \geq 1$ and $\Delta\tau_k = \tau_{k+1} - \tau_k$. Although the activation time ratio in (27) is defined in forward time to guarantee the stability of the state dynamics, mapping this sequence to the backward integration horizon ensures that the cumulative exponential argument satisfies $2\gamma T_u(a, b) - 2\beta T_s(a, b) < 0$ for any interval $[a, b]$. Since the switching signal over the finite horizon $[t_0, t_f]$ guarantees a finite number of switches N_s , the amplification factor μ^{N_s} remains strictly bounded. This guarantees $\|\tilde{P}(\tau)\| \leq \tilde{c}_2 < \infty$, satisfying $P(t) \preccurlyeq c_2 \mathbb{I}_n$ in (16) for all $t \in [t_0, t_f]$.

To satisfy the lower-bound constraint in (16), we verify that $P(t) \succ 0$ holds for all $t \in [t_0, t_f]$. The trajectory being the solution of Equation (18) can be analytically evaluated in forward time from the given initial condition $P(t_0) \succ 0$ as

$$P(t) = \Phi^T(t_0, t) P(t_0) \Phi(t_0, t) + \int_{t_0}^t \Phi^T(s, t) Q(s) \Phi(s, t) ds, \quad (31)$$

where $\Phi(s, t)$ is the state transition matrix associated with $A(t)$ in forward time. Since the given initial condition satisfies $P(t_0) \succ 0$, and the transition matrix is nonsingular, the unforced component remains strictly positive definite ($\Phi^T P(t_0) \Phi \succ 0$) across the horizon. Furthermore, as already proven, the time-varying matrix satisfies $Q(s) \succ 0$ for all $s \in [t_0, t]$, which guarantees that the forced integral response component in (31) is also strictly positive definite for all $t > t_0$. As $P(t)$ is formed by the sum of two strictly positive definite matrix trajectories, it follows directly that $P(t) \succ 0$ holds for all $t \in [t_0, t_f]$. This establishes the existence of the strictly positive uniform lower bound $c_1 \mathbb{I}_n \preccurlyeq P(t)$ required by (16) for $c_1 = \inf_{t \geq 0} \lambda_{\min}(P(t)) > 0$. $\square$

3.2. Controllability Analysis

Before proceeding with controller synthesis, it is necessary to verify the controllability of the system.

Definition 2. The continuous-time switched system (1) is called globally controllable on the interval $[t_0, t_f]$ if, for any initial state $x(t_0) = x_0 \in \mathbb{R}^n$ and any target state $x_f \in \mathbb{R}^n$, there exists a control input $u(t)$ defined on $t \in [t_0, t_f]$ that steers the system state from x_0 to $x(t_f) = x_f$.

Before proceeding with the synthesis of the optimal LQR controller, it is essential to formally verify the open-loop controllability of the system. For LTV systems, the classical Kalman rank criterion is insufficient [33]. A necessary and sufficient condition for the global controllability of system (1) over the time interval $t \in [t_0, t_f]$ is the strict positive definiteness of the controllability Gramian, $W_c(t_0, t_f) \succ 0$, defined as [33]

$$W_c(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_f, \tau) B(\tau) B^T(\tau) \Phi^T(t_f, \tau) d\tau, \quad (32)$$

where $\Phi(t, \tau)$ denotes the state transition matrix satisfying the conditions of Lemma 1.

For numerical evaluation, the Gramian can be obtained as the solution of the differential equation [33]:

$$\dot{W}_c(t) = A(t)W_c(t) + W_c(t)A^T(t) + B(t)B^T(t), \quad W_c(t_0) = 0. \quad (33)$$

In summary, evaluating the solution of (33) at the final time t_f yields the controllability Gramian $W_c(t_0, t_f)$. If this matrix is strictly positive definite, the open-loop LTV system is guaranteed to be globally controllable, thereby ensuring that a well-defined optimal control law can be synthesized via the LQR framework.

4. Synthesis of LQR Control for Continuous-Time Switched Systems

In this section, the conventional switched LQR framework, well established in the literature [1,11], is analyzed alongside a time-varying LQR formulation incorporating a matrix smoothing strategy. The comparison focuses on the resulting control signal characteristics and closed-loop behavior under switching conditions.

The Linear Quadratic Regulator problem is defined through the quadratic performance index

$$J = \int_{t_0}^{\infty} \left(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right) dt, \quad (34)$$

where $Q(t) = Q^T(t) \succcurlyeq 0$ and $R(t) = R^T(t) \succ 0$ represent the state and control weight matrices, respectively. For a continuous-time linear system, the optimal control law that minimizes (34) is given by the state-feedback relation

$$u(t) = -K(t)x(t), \quad K(t) = R^{-1}(t)B^T(t)P(t), \quad (35)$$

where $P(t)$ is obtained by solving the corresponding Riccati equation.

Under switching or time-varying dynamics, maintaining continuity of the feedback law becomes nontrivial due to discontinuities in the system matrices. Therefore, two synthesis approaches are considered. First, a switched LQR design based on LMIs and Multiple Lyapunov Functions is presented. Subsequently, a time-varying LQR formulation employing matrix smoothing is introduced to mitigate discontinuities in the control law.

4.1. Switched LQR Design Based on LMIs and Multiple Lyapunov Functions

The switched LQR design is constructed by assigning a constant state-feedback gain K_k to each subsystem $\dot{x}(t) = A_k x(t) + B_k u(t)$ for $k \in \{1, \dots, N\}$, together with a quadratic Lyapunov function $V_k(x) = x(t)^T P_k x(t)$, where $P_k = P_k^T \succ 0$ is a strictly positive-definite matrix.

For each subsystem, the standard LQR condition can be expressed as a Riccati-type inequality

$$A_k^T P_k + P_k A_k - P_k B_k R^{-1} B_k^T P_k + Q \prec 0. \quad (36)$$

The primary numerical difficulty in applying convex solvers directly arises from the nonlinear quadratic term $-P_k B_k R^{-1} B_k^T P_k$. To obtain convex conditions for the synthesis of the pairs (P_k, K_k) , the problem is reformulated within an LMI framework using the extended de Oliveira variable transformation [18,19] together with the standard Schur complement LQR parameterization [17].

In this design, the state weighting matrix is factorized as $Q = (Q^{1/2})^T Q^{1/2}$, while the control weighting matrix is selected as $R = \mathbb{I}_m$. By introducing the change of variables $X_k = P_k^{-1} \succ 0$ and $L_k = K_k G$, where G is a slack variable shared across all subsystems,

the synthesis conditions can be expressed as a system of Linear Matrix Inequalities (LMIs) through the application of the Schur complement

$$\begin{bmatrix} A_k G + G^T A_k^T + B_k L_k + L_k^T B_k^T & (X_k - G + A_k G + B_k L_k)^T & L_k^T & G^T Q^{1/2} \\ X_k - G + A_k G + B_k L_k & -X_k & 0 & 0 \\ L_k & 0 & -\mathbb{I}_m & 0 \\ Q^{1/2} G & 0 & 0 & -\mathbb{I}_n \end{bmatrix} \prec 0, \quad (37)$$

$$k \in \{1, 2, \dots, N\}.$$

The condition for global stability under non-uniform switching intervals is guaranteed via algebraic boundary constraints of the Multiple Lyapunov Functions (MLFs). In particular, a scalar coefficient $\mu_{\text{worst}} > 1$ is introduced to bound the admissible increase of the Lyapunov function at switching instants.

To reduce conservatism in the MLF constraints, the coefficient μ_{worst} is determined numerically for a given non-uniform switching schedule using a line-search bisection algorithm. A bounded search interval $[\mu_{\min}, \mu_{\max}]$, with $\mu_{\min} \rightarrow 1^+$, is initialized, and the feasibility of the LMI system is evaluated iteratively.

If the system is feasible for a candidate value, the upper bound is updated; otherwise, the lower bound is increased. The procedure terminates when the interval width satisfies a predefined numerical tolerance. The resulting μ_{worst} corresponds to the smallest value for which the LMI conditions remain feasible, providing a non-conservative bound on the admissible increase of the Lyapunov function. This constraint limits the effect of switching over the shortest operating interval $\Delta T_{\min} = \min_k \{\Delta T_k\}$.

Consequently, the MLF boundary constraints implemented in the dual space ($X_k = P_k^{-1}$) are formulated as

$$X_k \preceq \mu_{\text{worst}} X_{k+1}, \quad k \in \{1, \dots, N-1\}, \quad X_N \preceq \mu_{\text{worst}} X_1. \quad (38)$$

Due to the inverse mapping between the Lyapunov matrix and its dual representation, the inequalities in (38) correspond to the energy jump conditions $P_{k+1} \preceq \mu_{\text{worst}} P_k$, which bound the variation of the Lyapunov function at switching instants.

The LMI system is solved numerically using the YALMIP parser with the SeDuMi solver in the MATLAB environment [34]. The obtained matrices are used to recover the feedback gains via the relationships $P_k = X_k^{-1}$ and $K_k = L_k G^{-1}$. These gains define the switched control law $u(t) = -K(t)x(t)$, where the feedback gain matrix takes the value K_k when the k -th subsystem is active.

4.2. Time-Varying LQR Design with Matrix Smoothing

The first stage of the dynamic time-varying LQR controller synthesis involves transforming the hard-switching schedule of the stationary subsystems into a continuous representation. To eliminate discontinuities and reduce abrupt parameter variations, the partition of unity principle is applied. This approach ensures that the continuous time-varying plant matrices $A(t)$ and $B(t)$ are constructed as a weighted combination of the subsystem matrices, governed by time-varying coefficients

$$A_{\text{smooth}}(t) = \sum_{k=1}^N y_k(t) A_k, \quad B_{\text{smooth}}(t) = \sum_{k=1}^N y_k(t) B_k, \quad (39)$$

where N denotes the total number of subsystems, and the normalized weighting functions $y_k(t)$ satisfy the partition of unity identity at any given time instant:

$$\sum_{k=1}^N y_k(t) = 1 \quad \forall t \in \mathbb{R}. \quad (40)$$

To implement the partition of unity for an arbitrary number of subsystems N switching periodically with a total cycle period T , a smooth windowing scheme is introduced. Let $\{t_1, t_2, \dots, t_N\}$ denote the sequence of switching instants within one period, where t_k corresponds to the activation time of the k -th subsystem, and $t_{N+1} = t_1 + T$. The duration of the k -th operating mode is given by $\Delta T_k = t_{k+1} - t_k$.

The smoothing mechanism is based on unnormalized windowing functions $w_k(t)$, constructed using generalized logistic (sigmoidal) functions. To ensure continuity across the periodic boundary T under non-uniform and asymmetric switching intervals, a local time variable is defined using a wrapped periodic distance function relative to each switching instant

$$d_k(t) = \text{mod}(t_{\text{mod}} - t_k + \delta_k, T) - \delta_k, \quad k \in \{1, \dots, N\}, \quad (41)$$

where $t_{\text{mod}} = \text{mod}(t, T)$, and $\delta_k \in (0, T)$ denotes a dynamic phase shift associated with the k -th transition point.

Unlike a fixed half-period shift, each δ_k is dynamically selected to place the modulo discontinuity in a region where the corresponding weighting functions remain inactive, thereby preserving continuity of the constructed profiles:

$$\delta_k = \text{mod}(t_k + T - t_{\text{mid},j}, T), \quad (42)$$

where $t_{\text{mid},j}$ represents the geometric midpoint of a complementary inactive subsystem interval, whose index $j \in \{1, 2, \dots, N\}$ is chosen such that $j \neq k$ and $j \neq k - 1$ to guarantee that the discontinuity falls outside the active transition zone.

The transition dynamics at the boundaries of successive subsystems are described by sigmoidal activation functions $s_k(t)$, defined by

$$s_k(t) = \frac{1}{1 + e^{-\alpha_{\text{shared}} d_k(t)}}, \quad k \in \{1, \dots, N\}. \quad (43)$$

To ensure consistent transition characteristics across non-uniform operating intervals, a common steepness parameter α_{shared} is used for all switching transitions. This parameter is defined based on the shortest operating interval as

$$\alpha_{\text{shared}} = \frac{\gamma}{\Delta T_{\min} \cdot \kappa}, \quad (44)$$

where $\Delta T_{\min} = \min_k \{\Delta T_k\}$ determines the minimum subsystem duration, $\kappa \in (0, 1)$ is a constant scaling factor defining the relative transition width, and $\gamma = 10$ is an invariant empirical constant ensuring asymptotic saturation. Utilizing a shared slope based on $\Delta T_{\min}$ prevents overlapping tails from non-adjacent transitions and ensures that the rise and fall times are identical across all operating modes.

The unnormalized windowing function $w_k(t)$ for each subsystem is constructed by combining the corresponding activation and deactivation profiles

$$w_k(t) = s_k(t) \cdot (1 - s_{k+1}(t)) \quad (45)$$

with the boundary condition $s_{N+1}(t) = s_1(t)$ due to the periodic nature of the switching sequence.

Finally, to ensure compliance with the partition of unity and to avoid numerical inconsistencies due to overlapping transitions, the normalized weighting functions $y_k(t)$ are obtained using

$$y_k(t) = \frac{w_k(t)}{\sum_{j=1}^N w_j(t)}, \quad k \in \{1, \dots, N\}. \quad (46)$$

The comparison between the hard-switched profile and its continuously transformed counterpart obtained via the matrix smoothing procedure is illustrated in Figure 1.

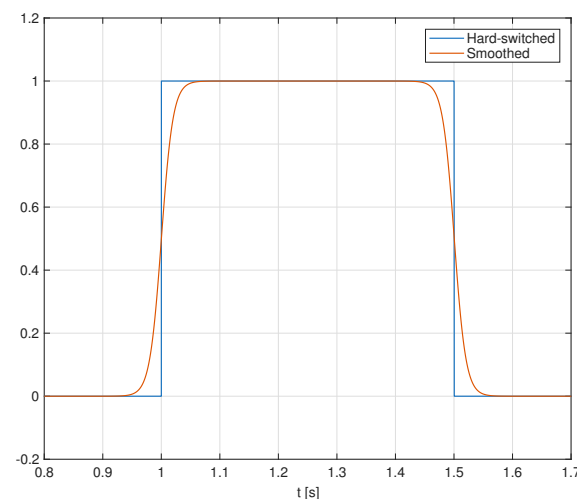


Figure 1. Comparison of hard-switched and smoothed subsystem transition profiles.

The matrices $A_{\text{smooth}}(t)$ and $B_{\text{smooth}}(t)$ are subsequently embedded into the standard linear time-varying differential Riccati equation [21,22], formulated as

$$\begin{aligned} \dot{P}(t) = & -P(t)A_{\text{smooth}}(t) - A_{\text{smooth}}^T(t)P(t) \\ & + P(t)B_{\text{smooth}}(t)B_{\text{smooth}}^T(t)P(t) - Q_{\text{smooth}}(t), \end{aligned} \quad (47)$$

where the inverse control weighting matrix is assumed to be constant and identity-valued, i.e., $R^{-1}(t) = \mathbb{I}_m$. The time-varying state weighting matrix $Q_{\text{smooth}}(t)$ is constructed analogously to (20), taking the form

$$Q_{\text{smooth}}(t) = Q(t_0) - \left(A_{\text{smooth}}^T(t)P(t_0) + P(t_0)A_{\text{smooth}}(t) \right). \quad (48)$$

To solve Equation (47) numerically, a virtual backward time variable $\tau = t_f - t$ is introduced, which reverses the sign of the time derivative, analogously to the procedure used for solving the differential Lyapunov equation discussed in Section 3.1. The backward integration is initialized with constant matrices $P(t_0) \succ 0$ and $Q(t_0) \succ 0$. Due to the partition of unity property where $\sum y_k(t) = 1$ and $y_k(t) \geq 0$, multiplying the constituent LMI constraints (24) by $y_k(t)$ and summing over $k \in \{1, \dots, N\}$ yields

$$A_{\text{smooth}}^T(t)P(t_0) + P(t_0)A_{\text{smooth}}(t) \prec Q(t_0) \quad \forall t \in [t_0, t_f]. \quad (49)$$

Substituting (49) into (48) ensures $Q_{\text{smooth}}(t) \succ 0$ holds continuously over the entire time horizon.

By introducing the tracking error matrix $\tilde{P}(t) = P(t) - P(t_0)$ and mapping (47) to the backward time variable τ , the nonlinear error dynamics are formulated as

$$\dot{\tilde{P}}(\tau) = \tilde{P}(\tau)A_{\text{smooth}}(\tau) + A_{\text{smooth}}^T(\tau)\tilde{P}(\tau) - P(\tau)B_{\text{smooth}}(\tau)B_{\text{smooth}}^T(\tau)P(\tau) + Q(t_0). \quad (50)$$

Since $P(\tau)B_{\text{smooth}}(\tau)B_{\text{smooth}}^T(\tau)P(\tau) \succcurlyeq 0$, the corresponding negative quadratic term in (50) is uniformly negative semidefinite. Furthermore, because $A_{\text{smooth}}(\tau)$ is confined within the convex polytope generated by the constituent modes, its instantaneous growth and decay rates are strictly bounded by the uniform factors β and γ defined in (25) and (26). Consequently, the nonlinear error trajectory in (50) is strictly upper-bounded by the global exponential boundary established in Theorem 3. This guarantees that the matrix smoothing process preserves both the uniform upper-boundedness and the strict positive definiteness ($P(t) \succ 0$) of the solution over the horizon, preventing numerical chattering.

Once the time-varying solution matrix $P(t)$ is computed over the entire time horizon, the corresponding feedback gain matrix $K(t)$ is defined by

$$K(t) = B_{\text{smooth}}^T(t)P(t). \quad (51)$$

To extend the controller to an infinite time horizon, a single period of the steady-state gain profile $K(t)$, obtained after the decay of transient effects, is periodically repeated to define the final control law, denoted as $K_{\infty}(t) = K_{\infty}(t + T)$ for all $t \geq 0$.

4.3. Comparative Analysis of Controller Synthesis Approaches

The first limitation of the Multiple Lyapunov Function (MLF) approach within the LMI formulation (37), incorporating the extended de Oliveira variable transformation and the Schur complement LQR parameterization, is related to the use of a single time-invariant slack variable G shared across all subsystems. For switching systems with significantly different control input matrices B_k , this constraint restricts the feasible solution space and may lead to numerical infeasibility of the LMI conditions.

Furthermore, the piecewise-constant gain architecture K_k inherent to the standard MLF framework does not account for the future evolution of the plant dynamics. As a result, abrupt variations in the control signal may occur at discrete switching instants.

In contrast, the backward-in-time integration of the nonlinear differential Riccati equation in the LTV formulation introduces a form of anticipatory behavior. The time-varying trajectory $P(t)$ acts as a predictor of the system evolution, allowing the control law to adjust the actuator inputs smoothly in advance of the switching instants.

Finally, a difference arises in the treatment of the switching schedule and the optimization horizon. While the MLF-based LMI synthesis incorporates the switching schedule through static energy decay relations, such as the inter-mode inequalities $X_{k+1} \preccurlyeq \mu X_k$ evaluated over an infinite horizon, the LQR cost functional is effectively constrained by time-invariant bounds.

In contrast, the continuous-time LTV-LQR formulation directly accounts for the evolution of the system over the actual finite operating intervals of the switching sequence. This formulation incorporates the time-varying plant dynamics and the smoothed matrix transitions, reducing the conservatism associated with stationary multi-model assumptions.

5. Numerical Analysis of the Proposed Approach

This section delivers a comparative numerical analysis of the investigated control frameworks using three systematically structured case studies designed to evaluate closed-loop robustness under challenging dynamic configurations.

First, the baseline scenario considers three unstable subsystems characterized exclusively by purely real eigenvalues, establishing the fundamental operational limits of the controllers.

Second, the structural complexity is elevated by introducing unstable complex-conjugate eigenvalues into one of the operating modes, which consequently induces oscillatory behavior in the system.

Third, the configuration with the oscillatory mode is evaluated under an accelerated periodic switching schedule to investigate the performance of the frameworks during transitions with short dwell times.

5.1. Example 1

This example focuses on the numerical verification of the nominal switched system, analyzing the performance indicators under the non-oscillatory subsystem configurations.

5.1.1. Considered Numerical Model

Consider the continuous-time linear switched system (1) consisting of $N = 3$ subsystems with the matrices $A_k, B_k, k \in \{1, 2, 3\}$ given by

$$\begin{aligned} A_1 &= \begin{bmatrix} 1.50 & 0.80 & -0.25 & 0.15 \\ 0.35 & 1.20 & 0.40 & -0.10 \\ -0.15 & 0.20 & -1.50 & 0.50 \\ 0.05 & -0.10 & 0.30 & -3.00 \end{bmatrix}, & B_1 &= \begin{bmatrix} 1.20 & 0.15 \\ 0.10 & 1.50 \\ 0.05 & 0.10 \\ 0.15 & 0.05 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.90 & 0.25 & 0.60 & -0.40 \\ 0.15 & -0.80 & 0.10 & 0.30 \\ 0.45 & -0.20 & 1.10 & 0.15 \\ -0.10 & 0.15 & 0.20 & -2.50 \end{bmatrix}, & B_2 &= \begin{bmatrix} 1.40 & 0.10 \\ 0.05 & 0.15 \\ 0.15 & 1.10 \\ 0.05 & 0.05 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} -4.00 & 0.15 & 0.35 & 0.10 \\ 0.50 & 1.30 & 0.60 & -0.20 \\ 0.15 & 0.25 & 0.80 & 0.40 \\ -0.20 & 0.10 & 0.35 & 0.50 \end{bmatrix}, & B_3 &= \begin{bmatrix} 0.05 & 0.05 \\ 1.10 & 0.20 \\ 0.15 & 1.30 \\ 0.25 & 0.90 \end{bmatrix}. \end{aligned} \quad (52)$$

The switching function is assumed to be periodic, $\sigma(t) = \sigma(t + T)$ with the fundamental period $T = 5$ s. Within the first period $t \in [0, 5)$, the switching schedule is defined as

$$\sigma(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1, \\ 2 & \text{for } 1 \leq t < 3, \\ 3 & \text{for } 3 \leq t < 5. \end{cases} \quad (53)$$

5.1.2. Stability Test

To verify the open-loop stability of the autonomous continuous-time switched system ($u(t) = 0$), the modified differential Lyapunov Equation (21) is solved using backward numerical integration. The time-varying state matrix $A(t)$ is constructed from the subsystem matrices defined in (52) according to the switching schedule $\sigma(t)$ given by (53). The integration is initialized with the matrices $P(0) = 0.25 \cdot \mathbb{I}_4$, $Q(0) = 5 \cdot \mathbb{I}_4$ satisfying (24). According to Theorem 3, this guarantees the positive definiteness and uniform boundedness of the weighting matrix $Q(t)$, shifting the verification of stability entirely to the spectrum of the matrix $P(t)$. The integration is performed over the time interval $t \in [0, 50]$ s. The obtained spectral bounds are given by

$$-3.13 \cdot 10^{42} \leq \lambda(P(t)) \leq 9.30 \cdot 10^{54}, \quad 4.03 \leq \lambda(Q(t)) \leq 7.02. \quad (54)$$

While the eigenvalues of the dynamically scaled weighting matrix $Q(t)$ remain strictly positive and well-bounded, the spectrum of the Lyapunov matrix $P(t)$ collapses into the negative domain. This violation of the lower spectral bound ($c_1 < 0$) unmasks the severe underlying instability of the open-loop configuration. This behavior is further illustrated by the divergence of the state trajectories, as shown in Figure 2.

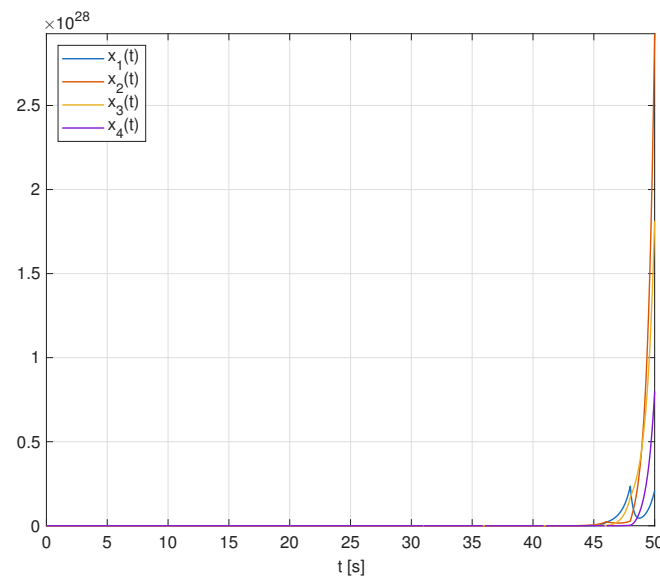


Figure 2. Open-loop state trajectories for the initial condition $x_0 = [10; 10; 10; 10]^T$.

5.1.3. Controllability Test

To numerically verify the global controllability condition over the time interval $t \in [0, 50]$ s, the differential matrix Equation (33) is integrated forward in time starting from the zero initial condition $W_c(0) = 0$. The eigenvalues of the controllability Gramian $W_c(0, 50)$ satisfy

$$4.29 \cdot 10^{-4} \leq \lambda(W_c(0, 50)) \leq 3.32 \cdot 10^{54}. \quad (55)$$

Since $\lambda_{\min}(W_c(0, 50)) = 4.29 \cdot 10^{-4} > 0$, the controllability Gramian is strictly positive definite, i.e., $W_c(0, 50) \succ 0$. Therefore, the switched system is globally controllable over the considered time interval.

This result indicates that, despite the instability of the open-loop dynamics, the control input has sufficient capability to steer the state trajectories over the entire state space, satisfying the conditions required for subsequent LQR controller synthesis.

5.1.4. Switched LMI-LQR Synthesis

To stabilize the unstable open-loop configuration, the Linear Matrix Inequalities (LMIs) defined in (37) and (38) are solved using the YALMIP parser with the SeDuMi solver [34]. The weighting matrix is selected as $Q = 5 \cdot \mathbb{I}_4$. The corresponding state-feedback gains K_k are computed as

$$K_1 = \begin{bmatrix} 4.47 & 0.44 & -0.36 & -0.47 \\ 0.48 & 2.43 & 3.20 & -5.32 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 3.46 & 0.40 & 0.65 & -1.53 \\ 0.09 & -0.19 & 4.50 & -3.67 \end{bmatrix}, \quad (56)$$

$$K_3 = \begin{bmatrix} 1.07 & 3.57 & 4.98 & -9.65 \\ 0.12 & -0.24 & 2.64 & -0.73 \end{bmatrix}.$$

The dynamic behavior of the synthesized switched LQR framework was validated through a closed-loop numerical simulation under the identical hard-switching schedule $\sigma(t)$. The effectiveness of the synthesized control law is illustrated by the control input vector $u(t)$ and the corresponding state trajectories, shown in Figure 3 for the initial condition $x_0 = [10; 10; 10; 10]^T$. The piecewise-constant nature of the feedback gain $K(t)$ leads to visible discontinuities in the control signals at the switching instants.

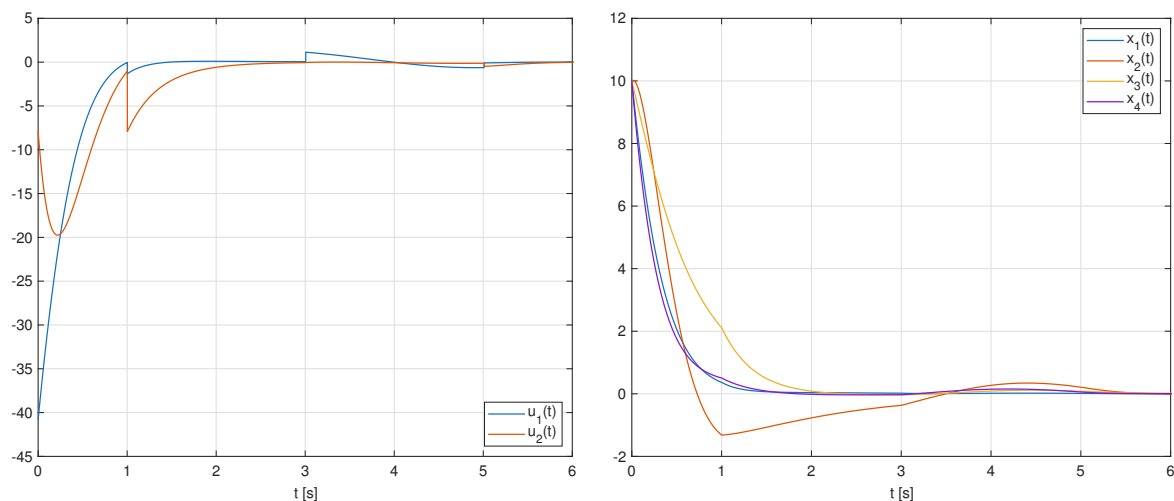


Figure 3. Closed-loop response for the initial condition $x_0 = [10; 10; 10; 10]^T$ using the Switched LMI-LQR approach: control signals (**left**) and state variable trajectories (**right**) governed by the feedback gain matrix defined in (56).

As a result, the control input exhibits abrupt variations corresponding to mode transitions. The state trajectories, however, remain bounded and converge towards the desired equilibrium, indicating that the Switched LMI-LQR controller stabilizes the system under the given switching schedule.

5.1.5. Time-Varying LQR with Matrix Smoothing Synthesis

To eliminate discontinuities in the control input, the time-varying LQR controller is synthesized by directly solving the continuous-time differential Riccati equation. In this framework, it is important to note that the underlying subsystem switching schedule remains governed by the hard-switching sequence defined by the periodic function $\sigma(t)$. However, the plant matrices $A(t)$ and $B(t)$ are continuously smoothed using the sigmoidal windowing functions satisfying the partition of unity. The feedback gain matrix $K(t)$ is then computed by integrating the Riccati Equation (47) with (48) backward in time, using the same initial conditions as specified in Section 5.1.2, i.e., $P(0) = 0.25 \cdot \mathbb{I}_4$ and $Q(0) = 5 \cdot \mathbb{I}_4$.

The continuous matrix transition behavior is determined by the configuration of the windowing algorithm parameters. To preserve symmetry between the activation and deactivation slopes under non-uniform intervals, the transition width scaling factor is set to a constant value of $\kappa = 0.2$. This configuration ensures that the transition duration occupies 20% of the shortest operating window ($\Delta T_{\min} = 1$ s), resulting in $\Delta t = 0.2$ s. The corresponding shared steepness parameter is given by $\alpha_{\text{shared}} = 50$.

The resulting time-varying feedback gain matrix $K(t)$ is illustrated in Figure 4.

The control inputs and state trajectories corresponding to this approach are shown in Figure 5. Compared to the hard-switching LQR, the control signal does not exhibit abrupt variations at switching instants. As a result, smoother actuator behavior is observed. The state trajectories remain bounded and converge to the equilibrium, indicating stabilization of the closed-loop system under the considered switching conditions.

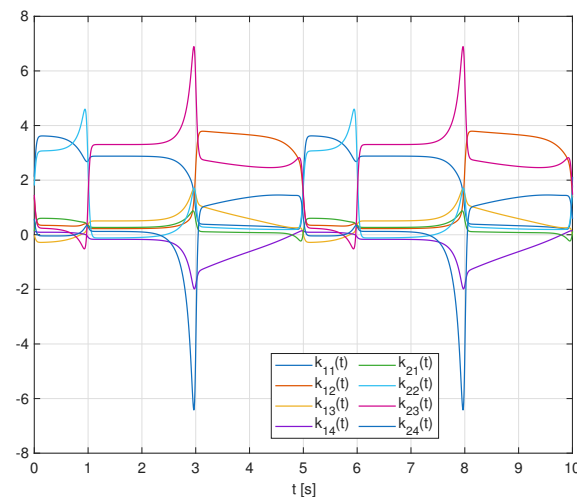


Figure 4. Time plot of the continuous state-feedback gain matrix $K(t)$ generated via Riccati equation with the matrix smoothing.

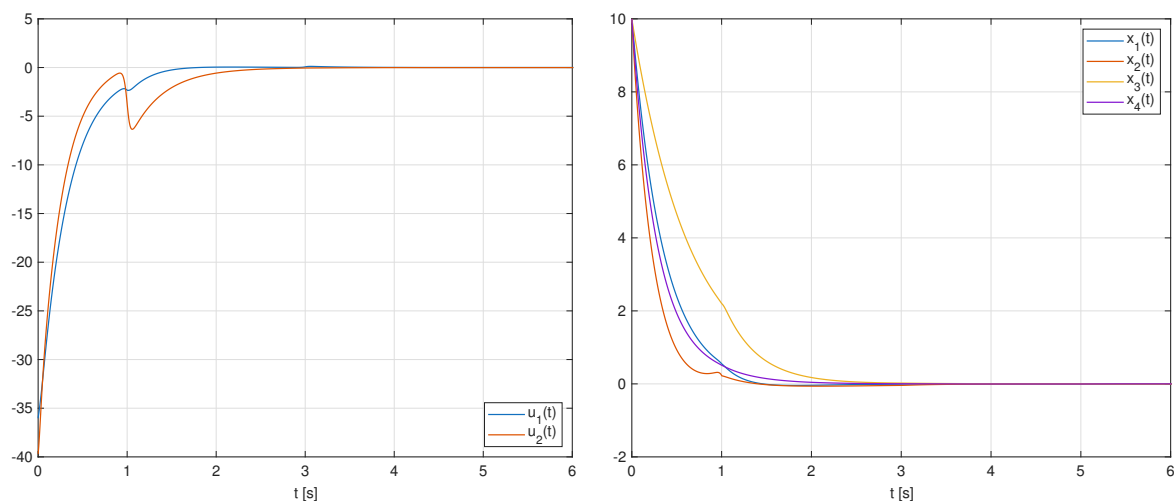


Figure 5. Closed-loop response for the initial condition $x_0 = [10; 10; 10; 10]^T$ using the time-varying LQR approach: control signals (**left**) and state variable trajectories (**right**) governed by the feedback gain matrix depicted in Figure 4.

5.1.6. Comparison of Different LQR Approaches

To quantitatively assess and compare the performance of the developed control strategies, a comparative analysis is conducted across five distinct LQR configurations. The evaluation considers the transition from the stationary switching paradigm to various formulations of the continuous-time and non-smooth time-varying LQR frameworks, with particular attention to the effects of matrix smoothing and dynamic weight scaling.

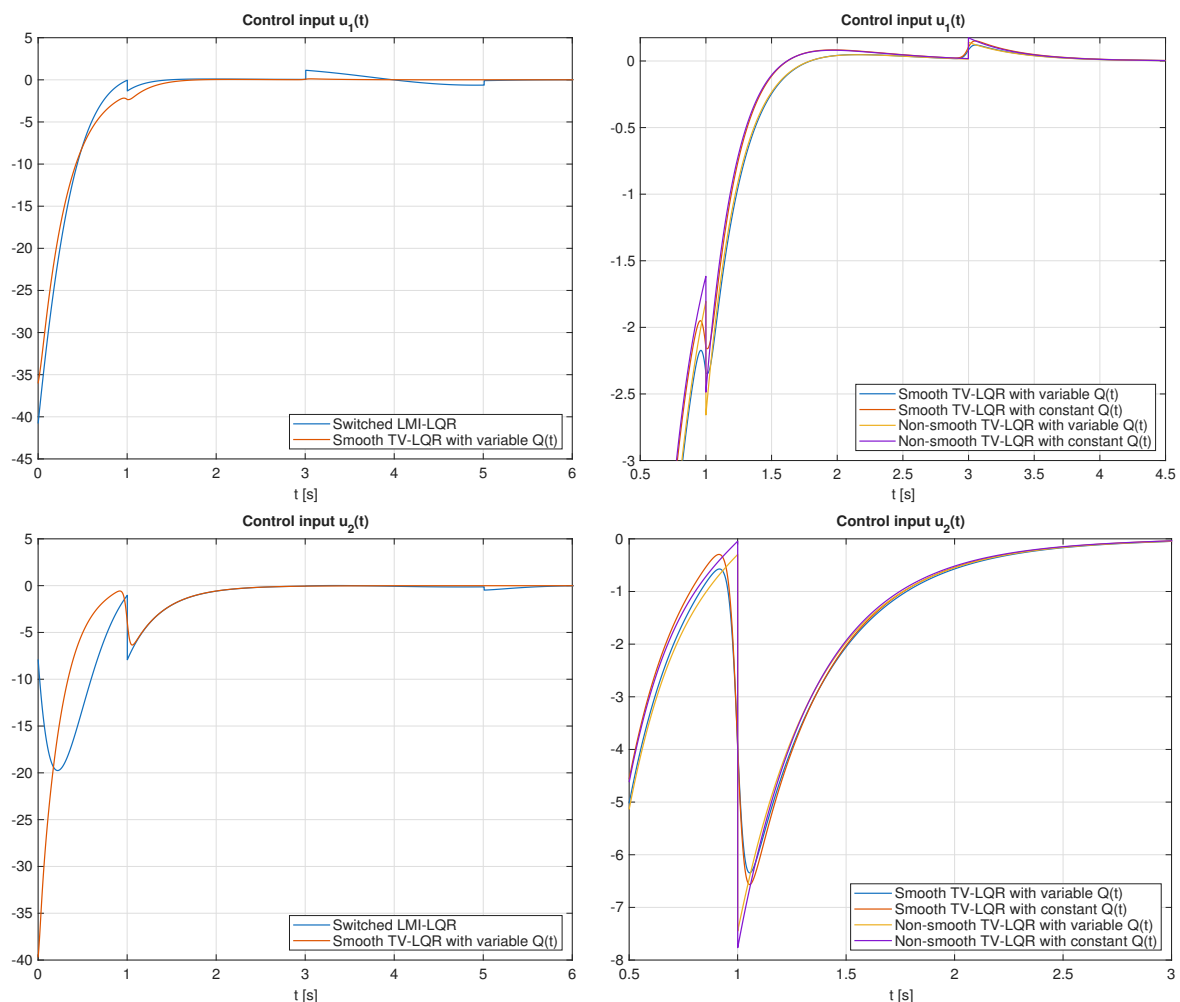
The performance assessment is based on two primary criteria: the cumulative LQR cost functional defined in (34), and the settling time defined as the time required for the state trajectory norm to enter and remain within a bounded region, i.e., $\max_i |x_i(t)| < 0.01$.

The quantitative results obtained from the closed-loop simulations for all investigated control variants are summarized in Table 1.

The data presented in Table 1 and the control input responses depicted in Figure 6 highlight several operational trade-offs among the investigated control strategies.

Table 1. Numerical performance indices and settling times for the investigated LQR variants.

Control Approach	Performance Index (34)	Settling Time to $\max_i x_i(t) < 0.01$
Switched LMI-LQR	921.00	6.17 s
Smooth time-varying LQR with variable $Q(t)$ defined by (48)	810.45	3.62 s
Smooth time-varying LQR with constant $Q(t) = 5 \cdot \mathbb{I}_4$	812.71	3.66 s
Non-smooth time-varying LQR with variable $Q(t)$ defined by (20)	814.15	3.69 s
Non-smooth time-varying LQR with constant $Q(t) = 5 \cdot \mathbb{I}_4$	816.09	3.71 s

**Figure 6.** Closed-loop control effort comparison: control input signals under Switched LMI-LQR and smooth time-varying LQR (left), and fragments of control inputs across all investigated time-varying LQR variants excluding the underperforming Switched LQR configuration (right).

First, the Switched LMI-LQR framework is characterized by a higher performance index and a longer settling time, indicating that the piecewise-constant gain structure introduces conservatism into the control design. In contrast, the time-varying LQR variants

exhibit comparable baseline performance, with noticeable improvements obtained through dynamic weight scaling.

Specifically, introducing a time-varying weighting matrix $Q(t)$ reduces the total LQR cost functional and shortens the closed-loop response.

Finally, while the matrix smoothing process does not significantly affect the absolute value of the LQR cost functional, the resulting sigmoidal transitions lead to shorter settling times compared to the non-smooth counterparts, while eliminating discontinuities in the control input at the switching instants.

5.2. Example 2

This example investigates the closed-loop response under heightened structural complexity, where unstable complex-conjugate eigenvalues are introduced into the second subsystem matrix.

5.2.1. System Configuration Under Oscillatory Behavior

To evaluate the impact of oscillatory modes on the system's structural properties, we introduce a modification to the previously defined linear continuous-time switched plant (52). Specifically, the state matrix of the second subsystem is replaced by

$$A_2 = \begin{bmatrix} 0.65 & 2.50 & 0.50 & -0.30 \\ -2.50 & 0.65 & 0.15 & 0.20 \\ 0.35 & -0.20 & -0.85 & 0.15 \\ -0.10 & 0.15 & 0.20 & -2.35 \end{bmatrix}, \quad (57)$$

while the remaining matrices A_k (for $k \in \{1, 3\}$) and B_k (for $k \in \{1, 2, 3\}$) from (52) remain strictly unchanged. The system operates under the identical periodic switching schedule $\sigma(t)$ specified in (53).

5.2.2. Stability and Controllability Test

The stability and controllability verification for this configuration is conducted similarly to the procedures detailed in Sections 5.1.2 and 5.1.3. The numerical integration is initialized with the matrices $P(0) = 0.25 \cdot \mathbb{I}_4$ and $Q(0) = 5 \cdot \mathbb{I}_4$, which satisfy the initial LMI condition (24). The simulation is performed over the identical time horizon $t \in [0, 50]$ s, and the resulting spectral bounds for the trajectories $P(t)$ and $Q(t)$ are obtained as

$$-1.38 \cdot 10^{33} \leq \lambda(P(t)) \leq 4.16 \cdot 10^{47}, \quad 4.03 \leq \lambda(Q(t)) \leq 7.02. \quad (58)$$

The invariance of the spectral limits for the matrix $Q(t)$ stems from the fact that its global extrema are dominated by the structure of the first and third subsystem, which remain unchanged. Consequently, the collapse of the matrix $P(t)$ into the negative domain confirms that the open-loop system is unstable, as illustrated by the state trajectories in Figure 7.

The eigenvalues of the controllability Gramian $W_c(0, 50)$ satisfy

$$4.25 \cdot 10^{-4} \leq \lambda(W_c(0, 50)) \leq 1.26 \cdot 10^{47}. \quad (59)$$

Since $\lambda_{\min}(W_c(0, 50)) = 4.25 \cdot 10^{-4} > 0$, the controllability Gramian is strictly positive definite, i.e., $W_c(0, 50) \succ 0$. Therefore, the switched system is globally controllable over the considered time interval.

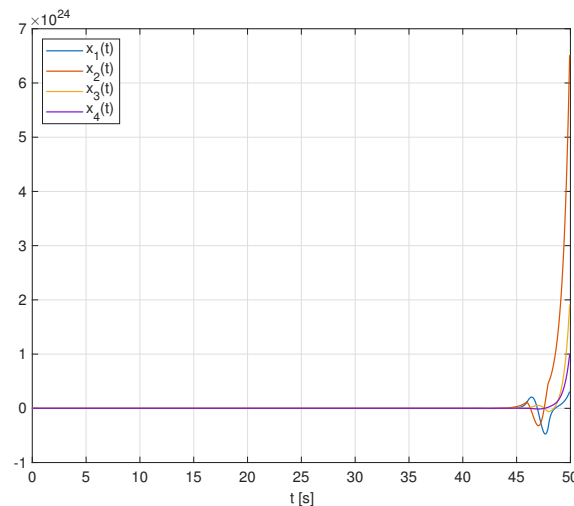


Figure 7. Open-loop state trajectories for the initial condition $x_0 = [10; 10; 10; 10]^T$ with the modified matrix A_2 introducing oscillatory mode.

5.2.3. LQR Controllers Synthesis

The Switched LMI-LQR is synthesized through the LMIs defined in (37) and (38). The weighting matrix is selected as previously, i.e., $Q = 5 \cdot \mathbb{I}_4$. The corresponding state-feedback gains K_k are computed as

$$K_1 = \begin{bmatrix} 2.58 & -2.27 & -5.54 & 9.85 \\ 1.56 & 2.28 & 3.29 & -6.15 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1.34 & -1.66 & -5.76 & 11.45 \\ 0.10 & -0.17 & 2.46 & -3.69 \end{bmatrix}, \quad (60)$$

$$K_3 = \begin{bmatrix} 1.76 & 5.71 & 8.89 & -17.29 \\ -0.07 & -0.81 & 1.56 & 1.20 \end{bmatrix}.$$

The effectiveness of the synthesized control law is illustrated by the control input vector $u(t)$ and the corresponding state trajectories, shown in Figure 8 for the initial condition $x_0 = [10; 10; 10; 10]^T$.

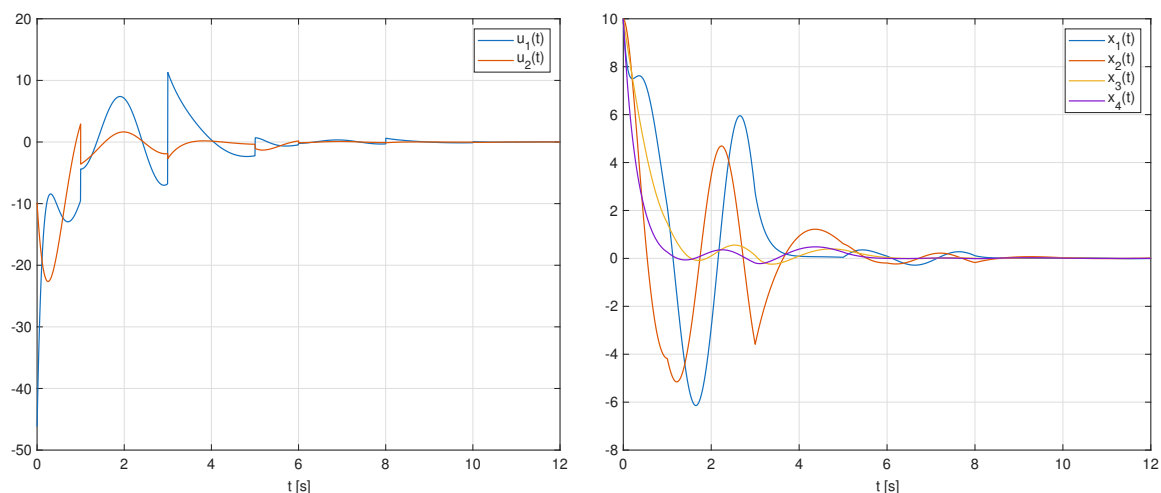


Figure 8. Closed-loop response for the initial condition $x_0 = [10; 10; 10; 10]^T$ using the Switched LMI-LQR approach: control signals (**left**) and state variable trajectories (**right**) governed by the feedback gain matrix defined in (60) for the plant with the modified matrix A_2 introducing oscillatory mode.

The time-varying LQR is synthesized by solving the differential Riccati Equation (47) with (48). The feedback gain matrix $K(t)$ is computed by using the same initial conditions as specified in Section 5.1.5, i.e., $P(0) = 0.25 \cdot \mathbb{I}_4$ and $Q(0) = 5 \cdot \mathbb{I}_4$. The smoothing algorithm

parameters are set to $\kappa = 0.2$ and $\alpha_{\text{shared}} = 50$, similarly as in Section 5.1.5. The resulting time-varying feedback gain matrix $K(t)$ is illustrated in Figure 9.

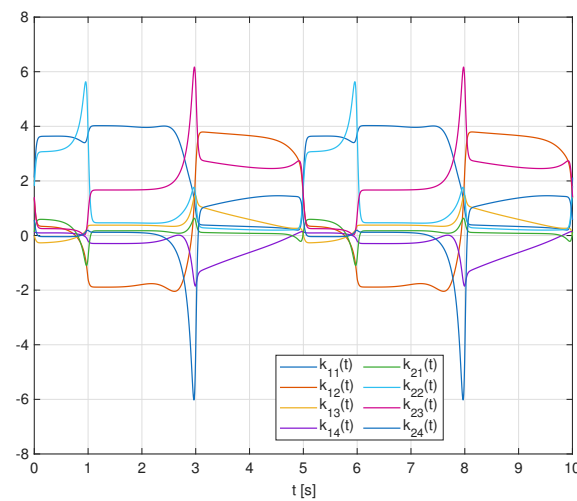


Figure 9. Time plot of the continuous state-feedback gain matrix $K(t)$ generated via Riccati equation with the matrix smoothing for the plant with the modified matrix A_2 introducing oscillatory mode.

The control inputs and state trajectories corresponding to this approach are shown in Figure 10.

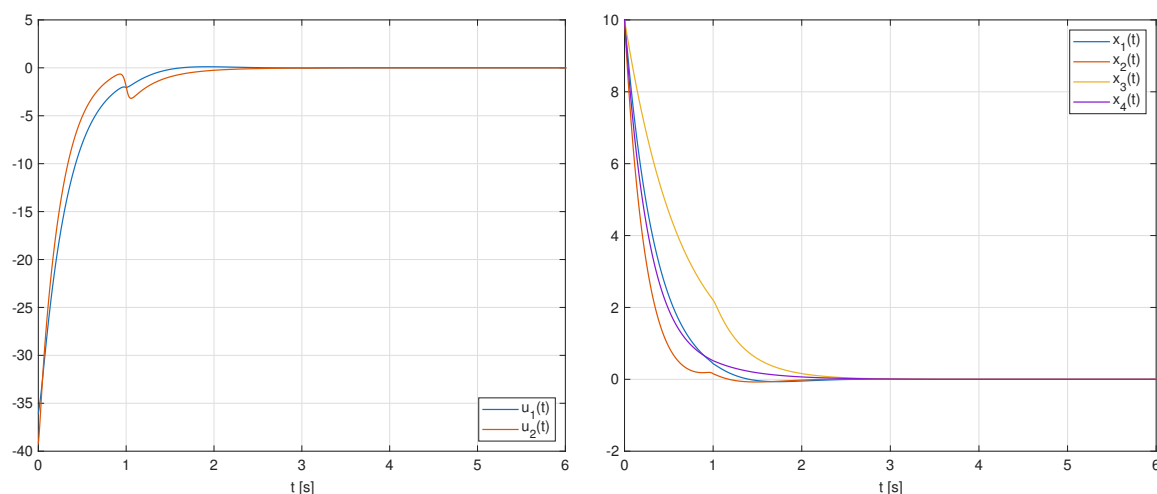


Figure 10. Closed-loop response for the initial condition $x_0 = [10; 10; 10; 10]^T$ using the time-varying LQR approach: control signals (**left**) and state variable trajectories (**right**) governed by the feedback gain matrix depicted in Figure 9 for the plant with the modified matrix A_2 introducing oscillatory mode.

5.2.4. Comparison of Different LQR Approaches for the Oscillatory Plant

A comparative analysis is conducted across five distinct LQR configurations, similarly to Section 5.1.6. The quantitative results obtained for the modified plant are summarized in Table 2.

Under the oscillatory mode (A_2) configuration, the performance differences between the investigated frameworks become more pronounced. The Switched LMI-LQR controller exhibits a substantial performance penalty: the settling time increases more than twofold, and the LQR performance index grows by a factor of over 1.5 (see Figure 8).

Table 2. Numerical performance indices and settling times for the investigated LQR variants with the modified plant introducing oscillatory mode.

Control Approach	Performance Index (34)	Settling Time to $\max_i x_i(t) < 0.01$
Switched LMI-LQR	1470.20	12.83 s
Smooth time-varying LQR with variable $Q(t)$ defined by (48)	803.08	2.98 s
Smooth time-varying LQR with constant $Q(t) = 5 \cdot \mathbb{I}_4$	805.32	3.02 s
Non-smooth time-varying LQR with variable $Q(t)$ defined by (20)	805.97	3.03 s
Non-smooth time-varying LQR with constant $Q(t) = 5 \cdot \mathbb{I}_4$	807.36	3.05 s

Conversely, the time-varying LQR variants retain superior closed-loop dynamics. The matrix smoothing process provides benefits by accelerating convergence. Furthermore, introducing the dynamic time-varying weighting matrix $Q(t)$ yields a shorter settling time and a lower LQR cost functional value compared to the time-invariant $Q(t)$ configuration, as observed in Figure 10.

Notably, the time-varying LQR frameworks exhibit even better transient response characteristics under the oscillatory mode configuration than in the non-oscillatory scenario with purely real eigenvalues.

5.3. Example 3

This scenario evaluates the dynamics of the investigated frameworks by combining the oscillatory plant configuration with significantly shortened subsystem dwell times.

5.3.1. System Configuration Under Oscillatory Behavior and Reduced Switching Intervals

In this configuration, the physical structure of the continuous-time linear switched system is retained directly from the previous oscillatory case study. Specifically, the plant dynamics are governed by the state matrix A_2 defined in (57), while the remaining constituent matrices A_k (for $k \in \{1, 3\}$) and B_k (for $k \in \{1, 2, 3\}$) from (52) remain unchanged.

The switching function is assumed to be periodic, $\sigma(t) = \sigma(t + T)$, with a shortened fundamental period $T = 0.5$ s. Within the first period $t \in [0, 0.5)$ s, the accelerated switching schedule is defined as

$$\sigma(t) = \begin{cases} 1 & \text{for } 0 \leq t < 0.1, \\ 2 & \text{for } 0.1 \leq t < 0.3, \\ 3 & \text{for } 0.3 \leq t < 0.5. \end{cases} \quad (61)$$

5.3.2. Stability and Controllability Test

The stability and controllability verification for this configuration is conducted similarly to the procedures detailed in Sections 5.1.2 and 5.1.3. The numerical integration is initialized with the matrices $P(0) = 0.25 \cdot \mathbb{I}_4$ and $Q(0) = 5 \cdot \mathbb{I}_4$, which satisfy the initial LMI condition (24). The simulation is performed over the time horizon $t \in [0, 50]$ s, and the resulting spectral bounds for the trajectories $P(t)$ and $Q(t)$ are obtained as

$$-103.60 \leq \lambda(P(t)) \leq 4.12 \cdot 10^{13}, \quad 4.03 \leq \lambda(Q(t)) \leq 7.02. \quad (62)$$

The collapse of the matrix $P(t)$ into the negative domain confirms that the open-loop system is unstable, as illustrated by the state trajectories in Figure 11.

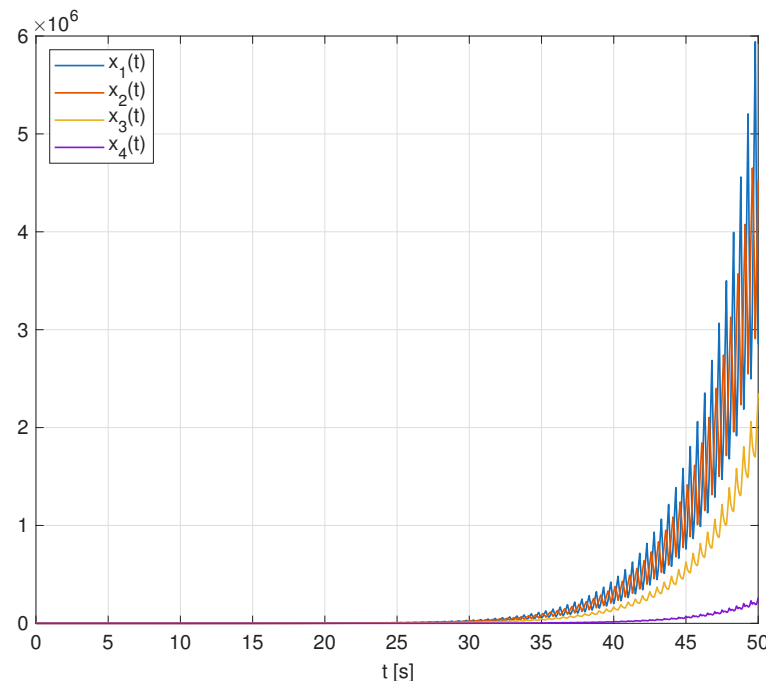


Figure 11. Open-loop state trajectories for the initial condition $x_0 = [10; 10; 10; 10]^T$ with the oscillatory mode and reduced switching intervals.

The eigenvalues of the controllability Gramian $W_c(0, 50)$ satisfy

$$0.09 \leq \lambda(W_c(0, 50)) \leq 6.02 \cdot 10^{12}. \quad (63)$$

Since $\lambda_{\min}(W_c(0, 50)) = 0.09 > 0$, the controllability Gramian is strictly positive definite, i.e., $W_c(0, 50) \succ 0$. Therefore, the switched system is globally controllable over the considered time interval.

5.3.3. LQR Controllers Synthesis

The Switched LMI-LQR is synthesized through the LMIs defined in (37) and (38). The weighting matrix is selected as previously, i.e., $Q = 5 \cdot \mathbb{I}_4$. Due to the robust nature of the Switched LMI-LQR framework, which guarantees closed-loop stability under arbitrary switching schedules, the synthesis algorithm remains fundamentally independent of the operating interval durations. Consequently, the state-feedback gains K_k computed previously in (60) are directly applied to this scenario, where they remain strictly identical:

$$K_1 = \begin{bmatrix} 2.58 & -2.27 & -5.54 & 9.85 \\ 1.56 & 2.28 & 3.29 & -6.15 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1.34 & -1.66 & -5.76 & 11.45 \\ 0.10 & -0.17 & 2.46 & -3.69 \end{bmatrix}, \quad (64)$$

$$K_3 = \begin{bmatrix} 1.76 & 5.71 & 8.89 & -17.29 \\ -0.07 & -0.81 & 1.56 & 1.20 \end{bmatrix}.$$

The effectiveness of the synthesized control law is illustrated by the control input vector $u(t)$ and the corresponding state trajectories, shown in Figure 12 for the initial condition $x_0 = [10; 10; 10; 10]^T$.

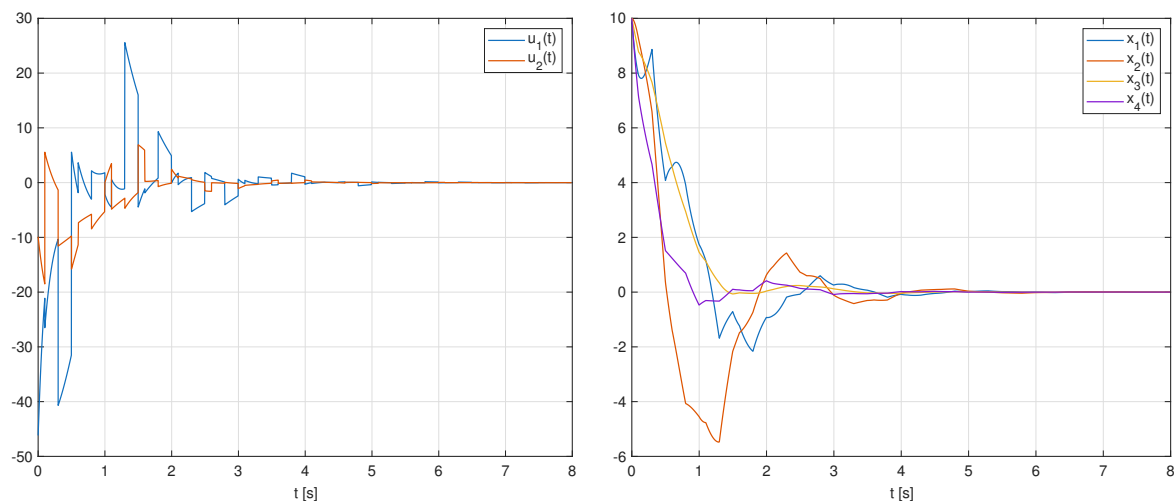


Figure 12. Closed-loop response for the initial condition $x_0 = [10; 10; 10; 10]^T$ using the Switched LMI-LQR approach: control signals (**left**) and state variable trajectories (**right**) governed by the feedback gain matrix defined in (64) for the plant with the oscillatory mode and reduced switching intervals.

The time-varying LQR is synthesized by solving the differential Riccati Equation (47). The feedback gain matrix $K(t)$ is computed by using the same initial conditions as specified in Sections 5.1.5 and 5.2.3, i.e., $P(0) = 0.25 \cdot \mathbb{I}_4$ and $Q(0) = 5 \cdot \mathbb{I}_4$. The smoothing algorithm parameters are set to $\kappa = 0.2$ and $\alpha_{\text{shared}} = 500$, similarly as in Sections 5.1.5 and 5.2.3. The resulting time-varying feedback gain matrix $K(t)$ is illustrated in Figure 13.

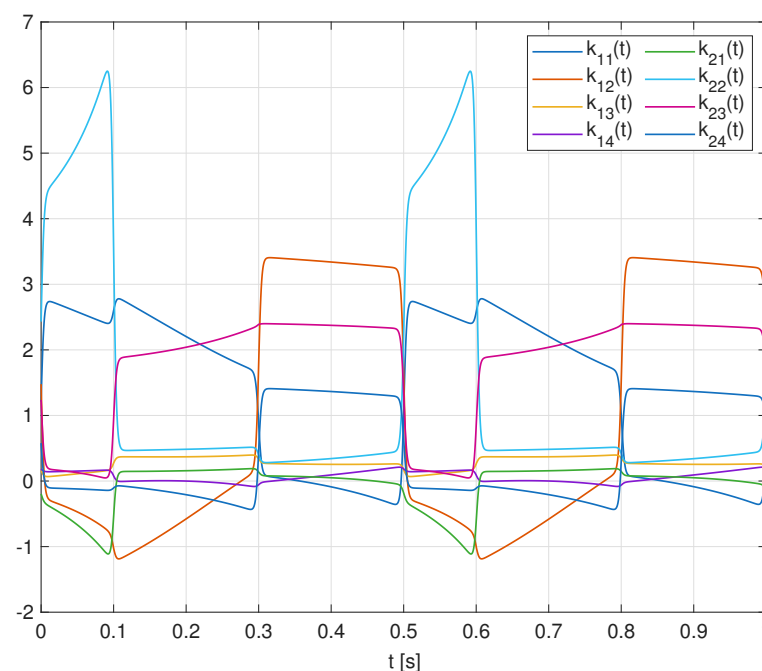


Figure 13. Time plot of the continuous state-feedback gain matrix $K(t)$ generated via Riccati equation with the matrix smoothing for the plant with the oscillatory mode and reduced switching intervals.

The control inputs and state trajectories corresponding to this approach are shown in Figure 14.

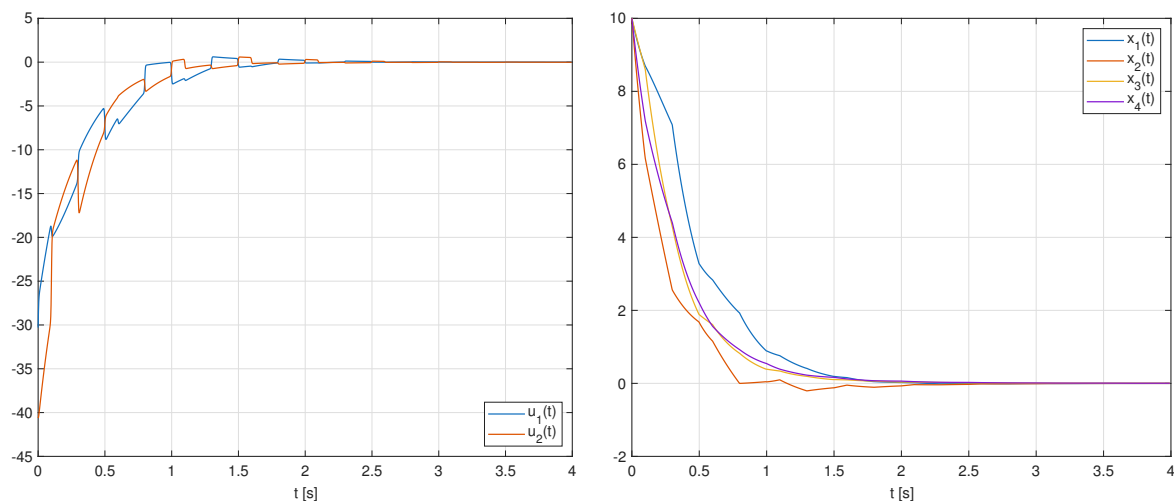


Figure 14. Closed-loop response for the initial condition $x_0 = [10; 10; 10; 10]^T$ using the time-varying LQR approach: control signals (**left**) and state variable trajectories (**right**) governed by the feedback gain matrix depicted in Figure 13 for the plant with the oscillatory mode and reduced switching intervals.

5.3.4. Comparison of Different LQR Approaches for the Oscillatory Plant with Reduced Switching Intervals

A comparative analysis is conducted across five distinct LQR configurations, similarly to Sections 5.1.6 and 5.2.4. The quantitative results obtained for the modified plant are summarized in Table 3.

Table 3. Numerical performance indices and settling times for the investigated LQR variants with the oscillatory mode and reduced switching intervals.

Control Approach	Performance Index (34)	Settling Time to $\max_i x_i(t) < 0.01$
Switched LMI-LQR	1314.26	6.87 s
Smooth time-varying LQR with variable $Q(t)$ defined by (48)	727.44	3.04 s
Smooth time-varying LQR with constant $Q(t) = 5 \cdot \mathbb{I}_4$	730.79	3.08 s
Non-smooth time-varying LQR with variable $Q(t)$ defined by (20)	728.04	3.05 s
Non-smooth time-varying LQR with constant $Q(t) = 5 \cdot \mathbb{I}_4$	730.89	3.08 s

Under the accelerated switching schedule with the oscillatory subsystem configuration, several key control characteristics are observed.

First, the time-varying LQR frameworks consistently outperform the Switched LMI-LQR controller, retaining superior trajectory tracking and closed-loop performance.

Notably, all investigated control variants exhibit improved performance indices under this higher switching frequency compared to the slower switching scenarios.

Within the time-varying LQR formulations, the benefits of the matrix smoothing process become less pronounced, providing only a marginal advantage over its non-smooth counterpart. This behavior indicates that the benefits of sigmoidal matrix transitions are

strictly bound to slower switching schedules with larger dwell times, whereas rapid cycles force the smoothing windows to collapse toward step-like discontinuities.

Conversely, embedding the time-varying weighting matrix $Q(t)$ continues to yield a performance advantage, resulting in improved transient dynamics and a lower cost functional value than the time-invariant $Q(t)$ configuration.

6. Concluding Remarks

This paper presents a framework for the control of continuous-time linear switched systems formulated within the linear time-varying (LTV) system setting. The theoretical foundation of this work establishes, in Theorem 2, that the state trajectories generated by a hard-switched system can be represented within the LTV framework. This equivalence enables the application of LTV system theory tools to the modeling, stability analysis, and control design of switched systems within a unified continuous-time formulation. Furthermore, based on this LTV representation, Theorem 3 provides the exponential stability conditions for the switched system under a dynamic continuous-time weighting matrix $Q(t)$.

Building on this foundation, the standard requirements of stability and controllability are evaluated. The Differential Lyapunov Equation (DLE) and the controllability Gramian are analyzed under arbitrary switching schedules. Two approaches to optimal state-feedback synthesis are considered: a Switched LMI-LQR framework based on the extended de Oliveira variable transformation and the Schur complement, and a continuous time-varying LQR formulation obtained by solving the differential Riccati equation (DRE) combined with a partition of unity-based matrix smoothing procedure applied to the plant dynamics.

The proposed methods are evaluated through three numerical case studies. The results indicate that the continuous LTV-LQR formulation consistently provides improved closed-loop performance compared to the Switched LMI-LQR approach. Under the nominal configuration, the smoothing-based design reduces the LQR cost functional and accelerates state convergence while producing continuous control inputs. When unstable complex-conjugate eigenvalues are introduced to induce oscillatory behavior, the Switched LMI-LQR framework exhibits a severe performance penalty. Conversely, the continuous-time LTV framework retains superior dynamics. Furthermore, under accelerated switching cycles, all variants show decreased cost functionals, while the benefits of the matrix smoothing process diminish as sigmoidal windows collapse toward step-like transitions. Notably, embedding the dynamic time-varying weighting matrix $Q(t)$ consistently delivers a distinct performance advantage over the time-invariant $Q(t)$ baseline across all evaluated scenarios.

The evaluated methods differ in the allocation and scale of their computational requirements. The Switched LMI-LQR framework concentrates its complexity offline due to the iterative bisection search and semidefinite programming (SDP) optimization. Its real-time implementation requires only conditional switching between fixed matrix gains, which induces minimal processor overhead. Conversely, the continuous-time LTV-LQR design requires the backward numerical integration of the differential Riccati equation. Because the switching schedule is known a priori, this integration is executed entirely offline. The real-time execution of the LTV-LQR requires continuous memory lookups and online interpolation of the gain trajectories $K(t)$. This makes it slightly more computationally demanding than the discrete switching mechanism. However, it completely avoids runtime matrix differential calculations and remains efficient for microprocessor implementation.

From a practical perspective, the proposed LTV reformulation offers distinct advantages over conventional switched-system analysis, particularly when the switching schedule is known in advance and smooth control actions are strictly required. Under these

conditions, the continuous feedback gain trajectories generated by the proposed framework effectively eliminate sharp control discontinuities, which is highly beneficial for reducing actuator wear, mitigating mechanical stress, and satisfying physical implementation constraints. Conversely, a primary limitation of the current methodology is its reliance on a priori knowledge of the switching schedule, which is indispensable for the offline backward integration of the differential Riccati equation. Consequently, the present formulation is most suitable for systems operating under deterministic, repetitive, or preplanned switching sequences, whereas its applicability to arbitrary, state-dependent, or stochastic switching scenarios remains limited within the current framework.

Finally, several directions for future work can be identified. First, extending the framework to include time-varying control weighting matrices $R(t)$ may enable adaptive shaping of the control effort during transition intervals. Second, alternative smoothing schemes, such as spline-based or trigonometric transitions, alongside a systematic parametric analysis of the slope steepness, could be explored to further optimize the numerical conditioning and continuity of the system matrices. Third, to mitigate the limitation of a perfectly known schedule, future extensions could incorporate adaptive formulations or robust control strategies capable of handling uncertain switching instants. Finally, extending the proposed approach to fractional-order switched systems constitutes a promising direction for modeling and control of systems with memory effects.

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