

Article

Numerical Investigation of Mixed Mode I-III Fracture Behavior in Sandstone Under Static and Dynamic Loading

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Abstract

Mixed mode I–III fracture commonly occurs in rock masses under complex three-dimensional stress states, yet the combined effects of loading conditions and fracture mode remain unclear. In this study, a heterogeneous sandstone disc model is developed in ABAQUS by coupling the Drucker–Prager elastoplastic model, an equivalent-strain damage model, and cohesive elements. The model is validated against static Brazilian splitting tests and dynamic SHPB tests. ENDB specimens are then employed to investigate mode I, mode III, and mixed mode I–III fracture under static and dynamic loading with offset angles ranging from 0° to 62.5°. As the offset angle increases, crack propagation evolves from straight tensile extension to deflected, twisted, and fragmented patterns dominated by anti-plane shear. Dynamic loading intensifies crack segmentation, localized damage, and transient instability while increasing the peak load, fracture toughness, and fracture energy. In contrast, the effective fracture toughness and fracture energy decrease monotonically with increasing offset angle under both loading conditions. A linear trend consistent with $R^2 > 0.99$ is observed between the mode I–III mixing coefficient and the effective fracture toughness based on single-realization simulations, the effective fracture toughness under dynamic loading is approximately 2.22 times that under static loading based on direct comparison of calculated values. These findings improve the understanding of loading-dependent mixed mode I–III fracture in sandstone and provide guidance for rock mass stability assessment.



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Keywords: mixed mode I–III fracture; loading condition; cohesive element; fracture toughness; fracture energy

1. Introduction

Rock fracture is a progressive process involving microcrack initiation, propagation, coalescence, and eventual macroscopic failure [1,2]. Due to the ubiquitous presence of inherent structural discontinuities, including joints, fissures, and mineral interfaces, crack tips in natural rock masses are commonly subject to complex three-dimensional stress states [3,4]. Crack propagation generally occurs in a mixed mode rather than a pure fracture

mode. Therefore, understanding the mixed mode fracture behavior of rock is essential to reveal the failure mechanisms of rock masses and improve the stability assessment of underground engineering.

In engineering practice, crack tips in rocks are typically subject to combined tensile and shear stresses, leading to mixed mode I-II fracture [5]. Currently, numerous scholars conduct systematic research on the behaviors, characteristics, and crack propagation mechanisms of mixed mode I-II rock fracture. Wu et al. systematically analyze the instability characteristics of mixed mode I-II fracture under various crack inclination angles, revealing that the crack initiation process is in good agreement with the MTS criterion [6]. Wang et al. employ the discrete element method to investigate the effects of specimen size and shape on mixed mode I-II fracture behavior. They find that specimen size significantly influences mechanical parameters such as peak load and fracture toughness, whereas specimen geometry affects not only the fracture toughness but also the crack propagation modes and the predictive accuracy of theoretical criteria [7]. Cao et al. utilize SHPB tests to study the fracture behavior of sandstone under dynamic mixed mode I-II loading conditions, discovering that both the peak load and fracture toughness increase with an increasing strain rate, and that different loading angles exert distinct influences on the dynamic mode I and mode II stress intensity factors [8]. Li et al. adopt a discrete element numerical approach to investigate the dynamic mixed mode I-II fracture behavior of fractured rock, indicating that crack failure is primarily governed by micro-tension and that mixed loading conditions affect the crack propagation paths [9]. Additionally, Li et al. examine the dynamic mixed mode I-II crack propagation behavior and fracture surface morphology of rock under different loading rates and axial pre-stresses, elucidating the variation laws of effective fracture toughness [10].

However, rock fracture behavior exhibits not only two-dimensional fracture characteristics but also mixed mode I-III fracture patterns with three-dimensional features [11]. This fracture mode simultaneously contains opening-mode cracks and out-of-plane shear tearing-mode cracks. Aliha et al. employ cracked disc specimens to study the fracture toughness of rock under mixed mode I-III loading conditions, finding that as the loading mode shifts from mode I to mode III, the failure load changes significantly, whereas the overall fracture toughness shows a decreasing trend, which demonstrates that the mode III effect reduces the fracture resistance of rocks [12]. Yang et al. investigate the mixed mode I-III fracture behavior of diorite under high-temperature conditions, and their results indicate that temperature significantly influences the fracture toughness of diorite under various fracture modes [13]. Yan et al. conduct dynamic mode III rock fracture tests using axially notched flat Brazilian disc specimens, discovering that the shear failure mechanism plays a dominant role in the mode III fracture process, while the fracture toughness exhibits a distinct variation law under high loading rates [14]. Qin et al. perform a comparative study on the dynamic and static mode III fracture of sandstone, and the results reveal that a pronounced loading rate effect exists in both mode III and mode I fractures [15]. Fahem et al. utilize a combined tension–torsion loading apparatus to investigate the dynamic response of materials under mixed mode I-III fracture conditions, finding that the fracture initiation toughness increases with an increasing loading rate [16].

In summary, while research on the mixed mode I-II fracture behavior of rocks is relatively mature, studies on the mixed mode I-III fracture behavior predominantly focus on single loading and environmental conditions. A systematic investigation into the behaviors and characteristics of mixed mode I-III fracture under various loading conditions remains lacking. Therefore, this paper constructs a sandstone model incorporating cohesive elements and a random Voronoi structure based on the ABAQUS. It verifies the feasibility of the numerical model through Brazilian splitting and SHPB tests. Furthermore, this study

conducts numerical experiments on the mixed mode I-III fracture of sandstone under both static and dynamic loading conditions. Finally, it investigates the fracture characteristics, as well as the variation laws of fracture toughness and fracture energy in the mixed mode I-III fracture of sandstone, thereby providing a reference for improving the mixed mode rock fracture theory.

2. Numerical Methods

2.1. The Constitutive Equation in Simulation

The deformation, failure, and crack propagation behavior of rock under static and dynamic loading conditions are highly complex. These processes typically exhibit multi-stage characteristics, including pre-peak, peak, and post-peak responses. An appropriate constitutive model is essential for accurately describing the mechanical behavior of rock during loading.

In this study, a coupled constitutive framework is adopted. The Drucker–Prager elastoplastic model is used to describe the pre-peak load-bearing capacity and pressure-dependent shear yielding of the rock. Post-peak stiffness degradation is captured using an equivalent-strain-based damage model. Crack initiation, propagation, and coalescence are simulated by cohesive elements embedded along potential crack paths. The corresponding constitutive formulations are presented as follows.

2.1.1. Drucker–Prager Elastoplastic Model

Under small strain conditions, the total strain of the material can be decomposed into the sum of elastic strain and plastic strain, namely:

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p \quad (1)$$

where ε_{ij} denotes the total strain tensor, ε_{ij}^e is the elastic strain tensor, and ε_{ij}^p is the plastic strain tensor. During the elastic stage, the sandstone matrix is assumed to obey the generalized Hooke's law, and the effective stress can be expressed as:

$$\tilde{\sigma}_{ij} = C_{ijkl}^e (\varepsilon_{kl} - \varepsilon_{kl}^p) \quad (2)$$

where $\tilde{\sigma}_{ij}$ denotes the effective stress tensor, and C_{ijkl}^e represents the elastic stiffness tensor. Considering the pronounced pressure-sensitive behavior of sandstone under combined compression–shear loading, the Drucker–Prager yield criterion is adopted to describe its yielding behavior [17]:

$$F = \sqrt{J_2} + \alpha I_1 - k = 0 \quad (3)$$

where I_1 is the first invariant of the stress tensor, J_2 is the second invariant of the deviatoric stress tensor, α is the friction parameter, and k is the cohesion term. To establish the correspondence between the Drucker–Prager criterion and the strength parameters of the Mohr–Coulomb criterion, the parameters can be expressed as:

$$\alpha = \frac{2 \sin \phi}{\sqrt{3}(3 - \sin \phi)}, \quad k = \frac{6c \cos \phi}{\sqrt{3}(3 - \sin \phi)} \quad (4)$$

where ϕ is the internal friction angle, and c is the cohesion. Substituting Equation (4) into Equation (3) yields the Drucker–Prager yield function for the sandstone matrix:

$$F = \sqrt{J_2} + \frac{2 \sin \phi}{\sqrt{3}(3 - \sin \phi)} I_1 - \frac{6c \cos \phi}{\sqrt{3}(3 - \sin \phi)} = 0 \quad (5)$$

The increment of plastic strain is determined using a non-associated flow rule:

$$d\epsilon_{ij}^p = d\lambda \frac{\partial G}{\partial \sigma_{ij}} \quad (6)$$

where $d\lambda$ is the plastic multiplier, and G is the plastic potential function. In this study, the plastic potential function is defined as:

$$G = \sqrt{J_2} + \alpha_g I_1 \quad (7)$$

$$\alpha_g = \frac{2 \sin \psi}{\sqrt{3}(3 - \sin \psi)} \quad (8)$$

where ψ is the dilation angle. The above constitutive functions enable an accurate representation of the pre-peak load-bearing behavior, plastic yielding, and volumetric dilation of sandstone under combined compressive and shear loading conditions.

2.1.2. Equivalent-Strain-Based Damage Model

To capture the post-peak softening behavior and brittle failure of sandstone, a damage variable is further introduced to characterize the degradation of material stiffness. The constitutive stress relationship for the damaged matrix is expressed as [18]:

$$\sigma_{ij} = (1 - D) \tilde{\sigma}_{ij} = (1 - D) C_{ijkl}^e (\epsilon_{kl} - \epsilon_{kl}^p) \quad (9)$$

Subsequently, a piecewise damage evolution law is adopted to describe the progression from damage initiation to complete material failure [19]:

$$D = \begin{cases} 0, & \bar{\epsilon} \leq \epsilon_0 \\ \frac{\bar{\epsilon} - \epsilon_0}{\epsilon_f - \epsilon_0}, & \epsilon_0 < \bar{\epsilon} < \epsilon_f \\ 1, & \bar{\epsilon} \geq \epsilon_f \end{cases} \quad (10)$$

where $\bar{\epsilon}$ is the equivalent strain, ϵ_0 is the damage initiation strain, and ϵ_f is the complete damage strain. When $\bar{\epsilon} \leq \epsilon_0$, the material remains in the elastic state. When $\bar{\epsilon} > \epsilon_f$, the material is considered to have undergone complete failure. To reduce the mesh-size dependency associated with the damage softening process, the complete damage strain is determined based on the fracture energy and the characteristic element length, namely [20]:

$$\epsilon_f = \epsilon_0 + \frac{2G_f}{E\epsilon_0 l_{ch}} \quad (11)$$

where G_f is the fracture energy, E is the elastic modulus, and l_{ch} is the characteristic element length. This formulation establishes a direct relationship between the post-peak softening behavior and the material's energy dissipation capacity, thereby providing a clear energy-based interpretation of the damage evolution process.

Since the subsequent analysis focuses on the mixed mode fracture behavior of sandstone, damage evolution cannot be adequately characterized using the normal strain alone. Therefore, the normal opening, tangential sliding, and torsional shear components are incorporated into the definition of the equivalent strain as follows [21]:

$$\bar{\epsilon} = \omega_I \langle \epsilon_n \rangle + \omega_{II} |\gamma_s| + \omega_{III} |\gamma_t| \quad (12)$$

where ϵ_n is the normal strain, γ_s is the tangential shear strain, and γ_t is the torsional shear strain. $\langle \rangle$ denote the Macaulay bracket, which retain only the opening displacement contribution, i.e., $\langle \epsilon_n \rangle = (\epsilon_n + |\epsilon_n|)/2$. ω_I , ω_{II} , and ω_{III} are the weighting coefficients

corresponding to mode I, mode II, and mode III fracture, respectively, and they satisfy the following constraint:

$$\omega_I + \omega_{II} + \omega_{III} = 1 \quad (13)$$

In this study, the values of the three weighting coefficients are selected as 0.45, 0.1 and 0.45, respectively. A small mode II component is retained to account for local in-plane shear deformation caused by crack deviation and heterogeneous grain-scale interactions.

2.1.3. Cohesive Element

Cohesive elements are a class of interface elements used to represent interfacial damage and crack propagation within materials. By inserting zero-thickness cohesive elements between bulk elements, potential fracture interfaces are introduced, enabling the simulation of the entire crack evolution process—from initiation and propagation to complete coalescence—within a finite element framework. Before damage initiation, the cohesive interface follows a traction–separation relationship [22]:

$$\mathbf{t} = \mathbf{k}\boldsymbol{\delta} \quad (14)$$

where $\mathbf{t} = [t_n, t_s, t_t]^T$ is the traction vector, $\boldsymbol{\delta} = [\delta_n, \delta_s, \delta_t]^T$ is the interface separation displacement vector, and $\mathbf{k}$ is the initial interface stiffness matrix. Crack initiation is governed by the maximum nominal stress criterion:

$$\max\left(\frac{\langle t_n \rangle}{t_n^0}, \frac{\langle t_s \rangle}{t_s^0}, \frac{\langle t_t \rangle}{t_t^0}\right) = 1 \quad (15)$$

where t_n^0 , t_s^0 , and t_t^0 are the interfacial strength thresholds in the normal, shear, and torsional directions, respectively. Once the interface enters the damage evolution stage, the cohesive traction can be expressed as [23]:

$$\mathbf{t} = (1 - D)\mathbf{k}\boldsymbol{\delta} \quad (16)$$

2.2. Verification

The proposed constitutive model is implemented in ABAQUS 2024 through the user-defined material subroutines UMAT/VUMAT. To validate the numerical model, the simulation results are compared with those obtained from laboratory quasi-static Brazilian splitting tests and dynamic SHPB tests, thereby evaluating the model's capability to reproduce the experimental responses. The Brazilian splitting test and SHPB test were selected for model validation because they represent two typical loading regimes of rock fracture, namely quasi-static tensile failure and dynamic impact failure. Validation against Brazilian splitting tests verifies the capability of the proposed model to reproduce tensile crack initiation and propagation under static loading, whereas SHPB validation evaluates its ability to capture dynamic fracture characteristics.

The sandstone specimens were collected from the roof strata of a coal mine. After coring, the rock blocks were machined into standard disc specimens with a diameter of 50 mm and a thickness of 25 mm. Static Brazilian splitting tests were conducted using a TAW-2000 (Chaoyang Test Instrument Co., Ltd., Changchun, China) microcomputer-controlled electro-hydraulic servo rock testing system in Figure 1 at a constant displacement rate of 0.002 mm/s. Dynamic impact tests were carried out using an SHPB system, as illustrated in Figure 2, with an impact gas pressure of 0.4 MPa.



Figure 1. Brazilian splitting test apparatus and schematic diagram.

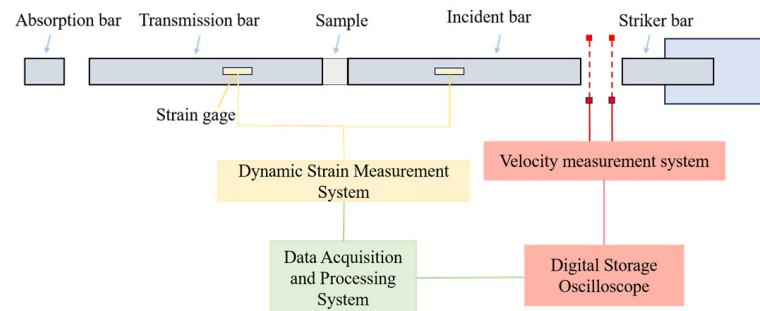


Figure 2. Schematic diagram of split-Hopkinson pressure bar (SHPB) system.

The material parameters used in the numerical simulations are listed in Table 1. To improve computational efficiency, the dynamic loading model is simplified by retaining only four components: the striker bar, incident bar, specimen, and transmission bar. A Voronoi tessellation algorithm is introduced to represent the heterogeneous internal structure of the rock specimen. The number of Voronoi cells used in the simulations was set at 100. The comparison between the crack propagation patterns obtained from the experiments and the numerical simulations is shown in Figure 3. The developed numerical model shows close agreement with the experimental results in terms of crack initiation, propagation paths, and the final failure patterns associated with crack coalescence and through-going fracture. The comparison between the load–displacement curves obtained from the experiments and the numerical simulations is shown in Figure 4. For the static Brazilian splitting test, the simulated peak load is 6.53 kN, while the experimental value is 5.99 kN, resulting in a relative error of 9.02%. For the dynamic SHPB test, the simulated peak load is 23.07 kN, compared with the experimental value of 21.39 kN, corresponding to a relative error of 7.85%. In addition, the root mean square error (RMSE) values of the static loading and dynamic loading curves before the peak are 0.43 and 1.27, respectively. The overall trends and stage-dependent characteristics of the curves agree well, indicating that the numerical model established based on the constitutive equations presented in Section 2.1 accurately reproduces the mechanical behavior of sandstone under both static and dynamic loading conditions. Furthermore, the model effectively captures the macroscopic fracture and instability processes of the specimen during loading, providing a reliable basis for subsequent numerical investigations of mixed mode I–III fracture behavior in sandstone.

Table 1. Material parameters in numerical simulation.

Parameter	Value	Parameter	Value
Density ρ (kg/m ³)	2500	Elastic modulus E (GPa)	20
Poisson's ratio ν	0.24	Internal friction angle ϕ	30°
Cohesion c (MPa)	10	Damage initiation strain ε_0	0.0015
Dilation angle ψ	12°	Complete damage strain ε_f	0.0025
Characteristic element length l_{ch} (m)	0.005	Fracture energy G_f (N/m)	50 (Static) 75 (Dynamic)
Cohesive stiffness (N/m ³)	5×10^{14}	Cohesive strengths $t_n^0/t_s^0/t_t^0$ (MPa)	5/1/1

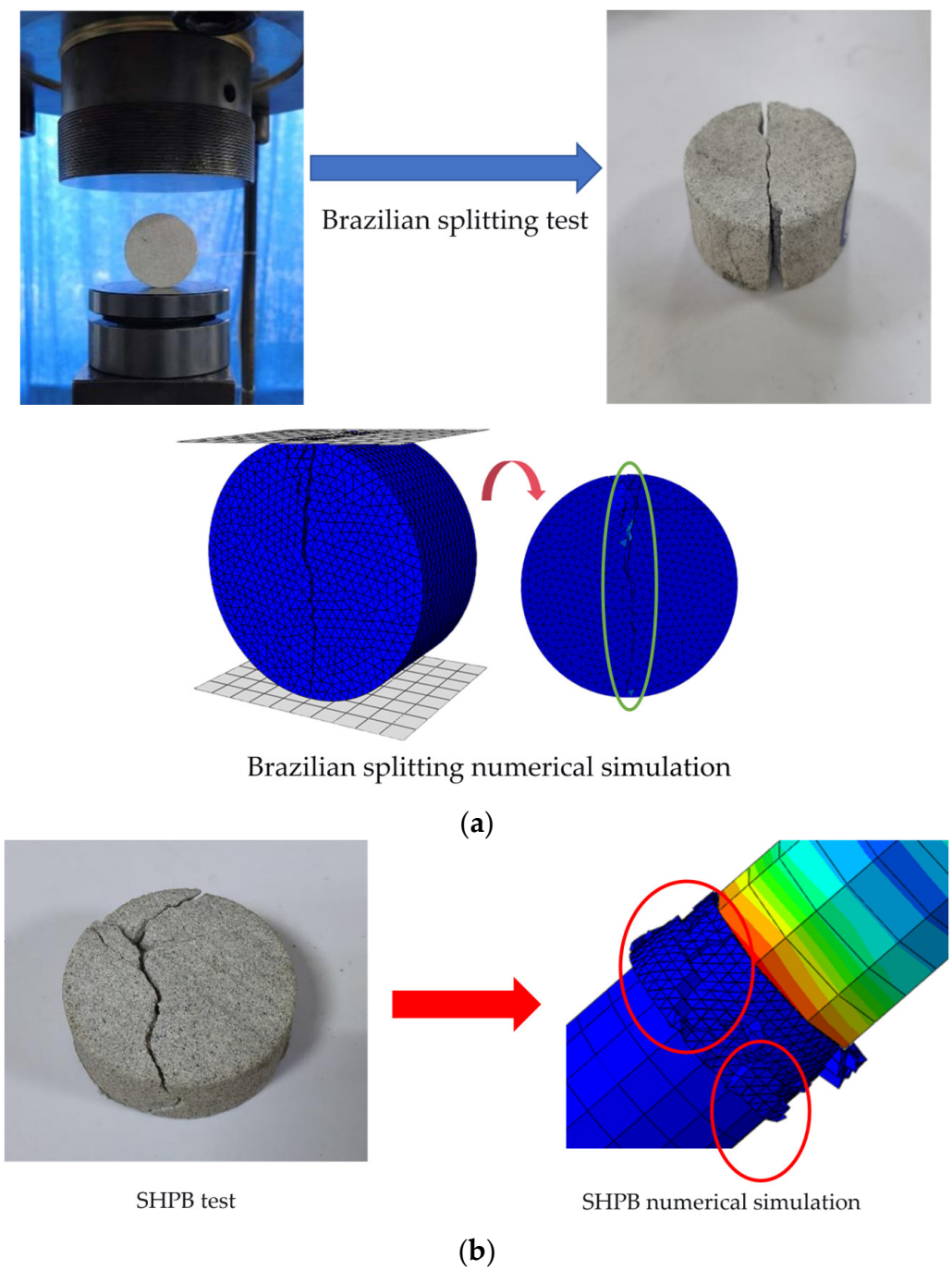


Figure 3. Crack propagation patterns in specimens from laboratory tests and numerical simulations. (a) Static loading. (b) Dynamic loading.

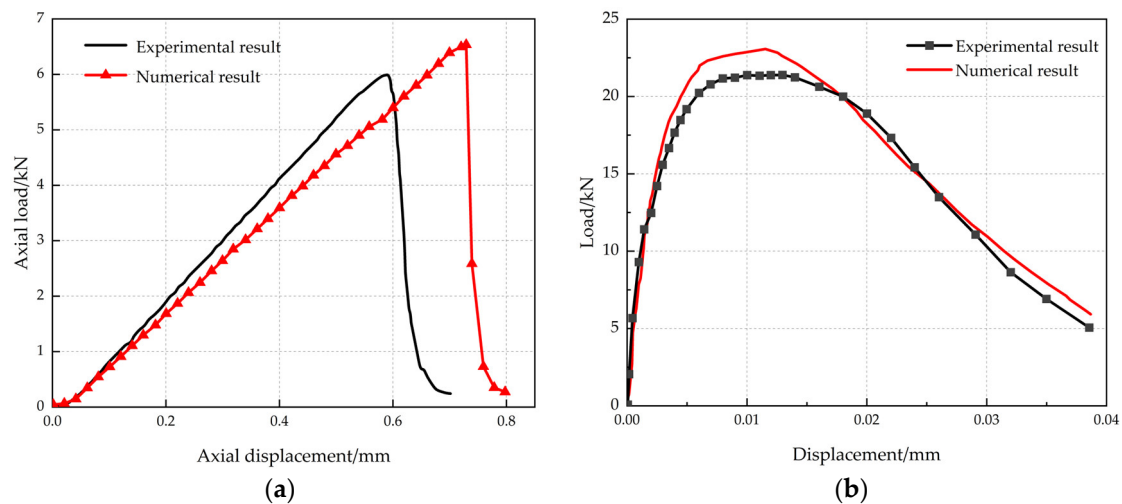


Figure 4. Load–displacement curves from laboratory experiments and numerical simulations. (a) Static loading. (b) Dynamic loading.

2.3. Mode I, Mode III and Mixed Mode I–III Loading

2.3.1. Numerical Model

Edge-notched disc bend (ENDB) specimens are employed to investigate the fracture response of sandstone under static and dynamic loading conditions for mode I, mode III, and mixed mode I–III fractures. The geometric configuration of the specimen under static loading is shown in Figure 5a. A pre-crack passing through the center and extending along the radial direction is introduced on the end face of a circular disc with a radius of 25 mm and a thickness of 25 mm. The pre-crack has a depth of 10 mm and an opening width of 0.5 mm. A loading bar is placed on the upper surface of the specimen. Different loading conditions are achieved by varying the offset angle α between the loading bar and the pre-crack. Two support bars are arranged in parallel on the lower surface, with a spacing of $2S = 47.5$ mm. The bars are modeled as rigid bodies to minimize the influence of their deformation on the results. The mesh element type was set as C3D8R and mesh element size was set at 5 mm. Static loading analyses are performed using the ABAQUS/Standard implicit solver. A displacement-controlled loading rate is applied to the upper loading bar, consistent with the experimental conditions. Dynamic loading analyses are conducted using the ABAQUS/Explicit solver. The specimen geometry remains identical to that described above, and the loading configuration is established based on the SHPB impact test, the same 0.4 MPa impact pressure is applied to the striker bar, as illustrated in Figure 5b.

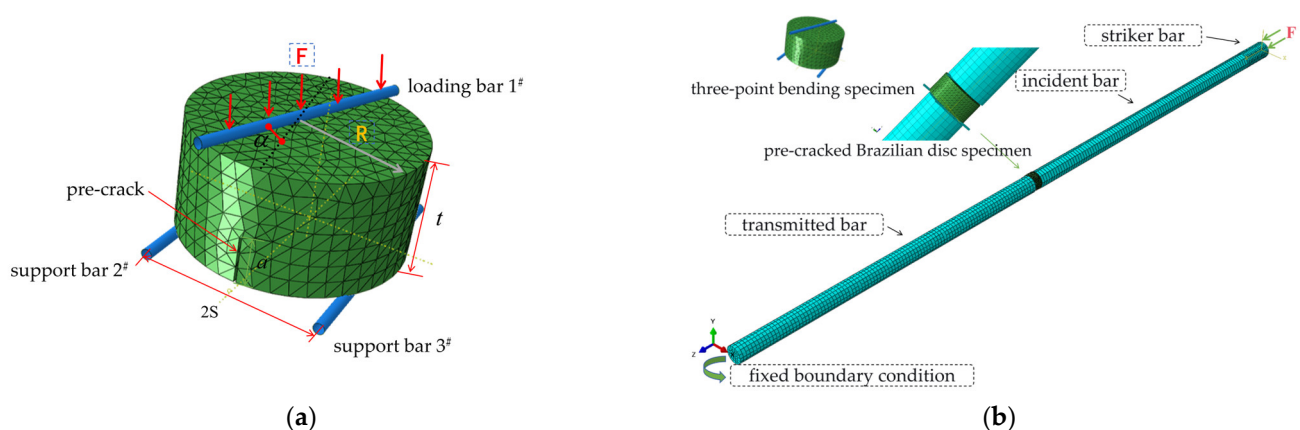


Figure 5. Numerical simulation models. (a) Static loading method. (b) Dynamic loading method.

2.3.2. Fracture Toughness

In fracture mechanics, the stress intensity factor K is a physical parameter used to characterize the intensity of the stress field in the vicinity of a crack tip. The stress intensity factors for mode I and mode III fractures are calculated as follows [24,25]:

$$K_I = \frac{6PS\sqrt{\pi a}}{Rt^2} Y_I(a/t, S/R, \alpha) \quad (17)$$

$$K_{III} = \frac{6PS\sqrt{\pi a}}{Rt^2} Y_{III}(a/t, S/R, \alpha) \quad (18)$$

where K_I and K_{III} are the mode I and mode III stress intensity factors, respectively; P is the applied load; S is the half span of the single-side support; R is the specimen radius; t is the specimen thickness; and a is the pre-crack depth. Y_I and Y_{III} are the dimensionless geometry factors, which account for the effects of the crack length ratio a/t , the span ratio S/R , and the offset angle α on the stress field correction. The relationships between Y_I , Y_{III} and α are shown in Figure 6 [12]. In this study, α is set to 0° , 10° , 30° , 50° , and 62.5° . When $\alpha = 0^\circ$, the crack tip is dominated by tensile stress and the fracture can be regarded as pure mode I. As α increases, the shear stress component at the crack tip gradually increases, and the specimen transitions from pure mode I to mixed mode I–III fracture. When $\alpha = 62.5^\circ$, the crack tip is primarily subjected to shear stress parallel to the crack front, which can be considered as pure mode III fracture. It should be noted that for dynamic loading cases, the stress intensity factors are calculated using the peak load after dynamic equilibrium is achieved. The influence of stress wave oscillation and inertial effects is minimized by adopting the peak transmitted force corresponding to the critical fracture stage.

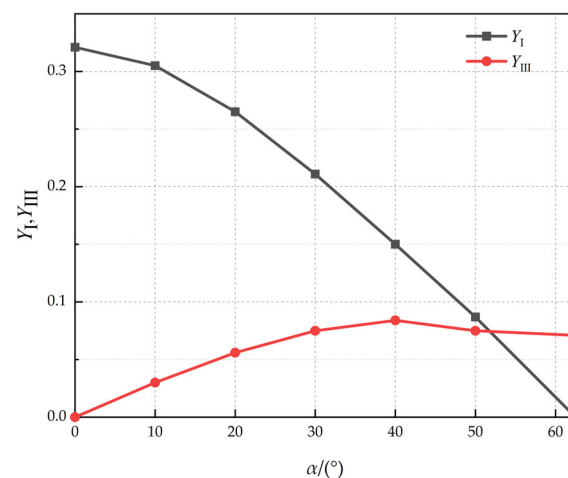


Figure 6. The relationship between geometry factors and offset angle.

3. Results

3.1. Crack Propagation Analysis

As α increases, the fracture mode of the specimen evolves from mode I dominated to mode III dominated, accompanied by corresponding changes in crack propagation behavior. Figures 7 and 8 illustrate the crack propagation patterns of specimens with different α under static loading conditions. The final failure morphologies differ significantly among the specimens. When $\alpha = 0^\circ$, the crack primarily propagates upward along the extension line of the pre-crack, and the overall fracture pattern is approximately symmetric in the left–right direction. In the side view, the crack surface is relatively smooth and regular, indicating that crack propagation is predominantly governed by mode I tensile opening. When $\alpha = 10^\circ$, the main crack still initiates from the tip of the pre-crack and propagates upward; however,

the crack path exhibits a slight deflection, indicating that the shear stress component begins to influence crack propagation while tensile stress remains dominant under mixed mode loading conditions. When α increases to 30° , crack deflection becomes more pronounced. In the front view, the main crack no longer propagates along the extension line of the pre-crack, whereas in the side view, the crack surface exhibits torsional characteristics, indicating the occurrence of mixed mode I–III fracture. When $\alpha = 50^\circ$, crack propagation becomes more complex. In addition to the significant deflection of the main crack path, localized damage zones also appear in the intact region away from the crack. The side view shows more pronounced curvature of the crack surface, and the fracture pattern evolves from a single dominant crack to the combined development of the main crack and multiple localized secondary cracks. When α reaches 62.5° , the spatial characteristics of crack propagation become most pronounced. In the front view, the crack propagation paths become widely dispersed, accompanied by more severe localized damage, while in the side view, the crack surface exhibits a stronger twisting tendency. At this stage, the shear stress component dominates the mixed mode fracture behavior, and the crack surfaces predominantly slide in the direction of shear stress.

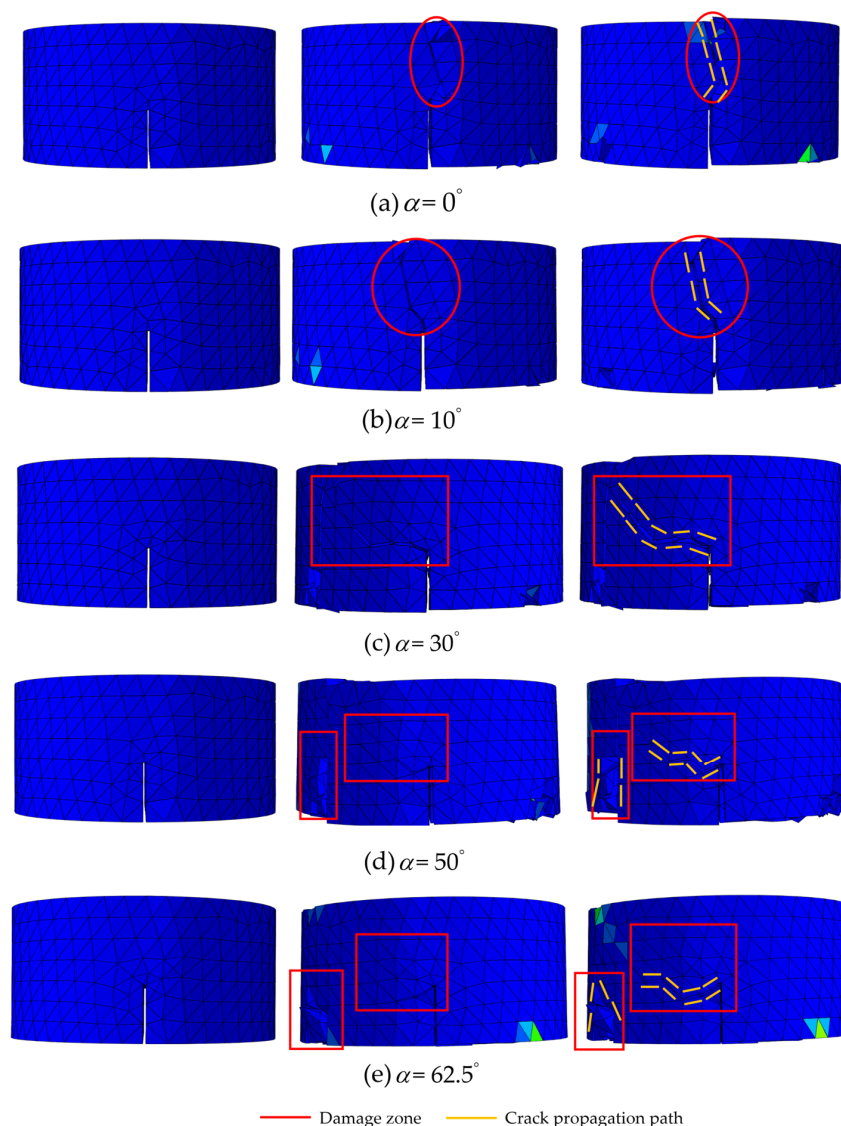


Figure 7. Front view of crack propagation of specimens under static loading with different offset angles.

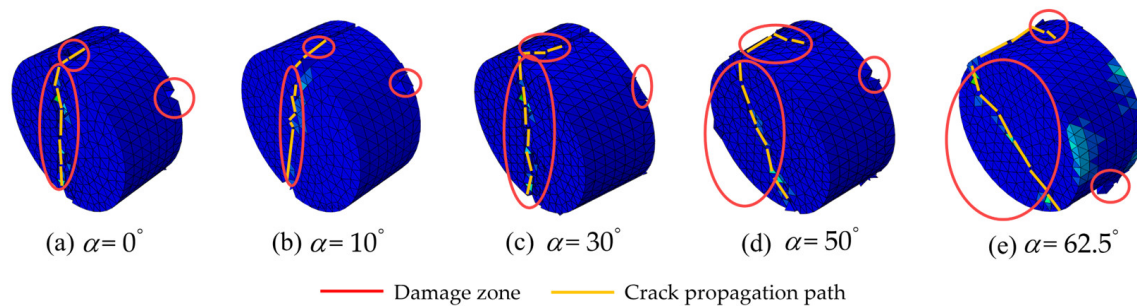


Figure 8. Side view of crack propagation of specimens under static loading with different offset angles.

Figure 9 illustrates the crack propagation patterns of specimens with different α under dynamic loading conditions. Compared with static loading, the crack propagation paths under dynamic loading are less continuous. Localized damage and crack-tip deterioration around the pre-crack are more pronounced, exhibiting stronger characteristics of transient instability. When $\alpha = 0^\circ$, the main crack still initiates from the tip of the pre-crack and generally propagates along its extension direction. The fracture mode remains mode I dominated. However, compared with static loading, localized damage appears in the surrounding region during crack propagation, indicating that the higher stress level near the crack tip under high strain-rate loading promotes rapid crack initiation and facilitates faster crack coalescence. When $\alpha = 10^\circ$, the main crack still propagates generally upward. However, the crack path exhibits noticeable deflection, and the continuity between adjacent crack segments is reduced. When $\alpha = 30^\circ$, the main crack exhibits a zigzag propagation pattern, and distinct crack deflection zones are observed in localized regions. This indicates the occurrence of mixed mode I–III fracture, in which crack propagation is jointly governed by tensile opening and anti-plane tearing. When α increases to 50° , the crack morphology becomes more complex. In addition to the further deterioration of the regularity of the main crack path, the extent of localized damage within the specimen also increases. Crack propagation is characterized by the coexistence of segmented growth and localized instability. This indicates that, under dynamic loading, as α continues to increase, the shear stress component becomes increasingly dominant, whereas the influence of the pre-crack on crack propagation gradually weakens. When α reaches 62.5° , the spatial irregularity of crack propagation becomes most pronounced. The fracture pattern is mainly characterized by the combined development of localized failure zones and segmented cracks. The crack propagation paths become highly dispersed, and the generated cracks tend to distribute over a larger region and form multiple fracture segments, leading to a relatively extended failure process.

By comparing the crack propagation behavior under static and dynamic loading conditions, it is observed that the mixed mode I–III fracture response of the specimens is strongly influenced by the offset angle α . Under both loading conditions, the main crack initiates near the tip of the pre-crack, and as α increases, the crack propagation path evolves from an approximately straight trajectory at small angles to a deflected propagation pattern. At larger offset angles, the cracks exhibit pronounced spatial twisting and localized damage. These observations indicate that the offset angle is a critical factor governing the mixed stress state at the crack tip and the crack propagation direction under both static and dynamic loading conditions.

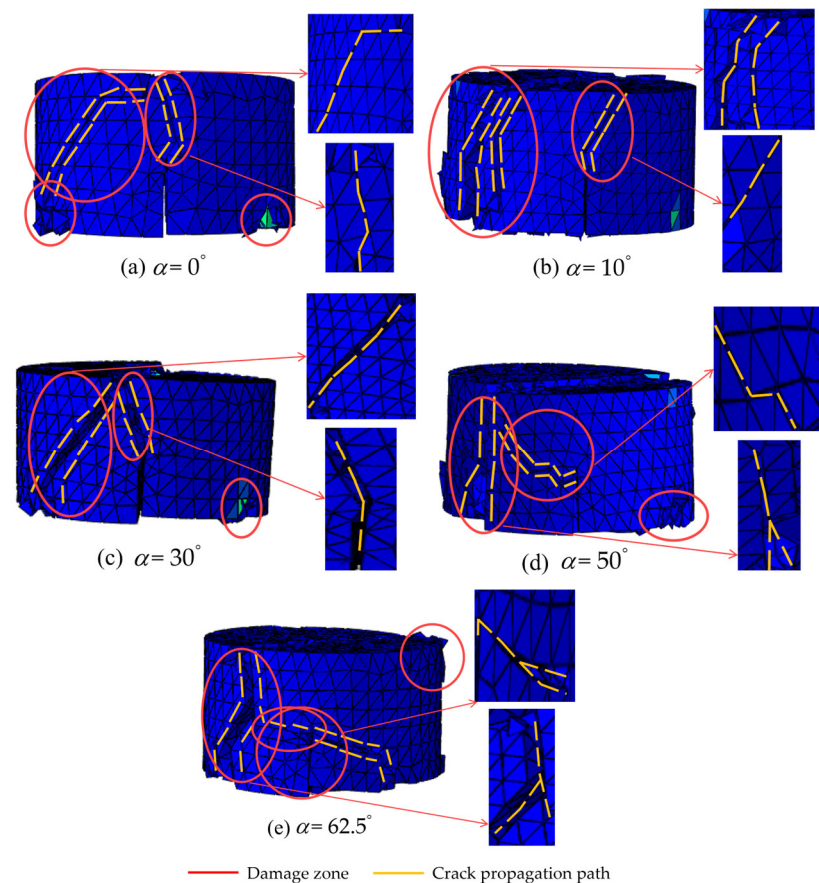


Figure 9. Crack propagation of specimens under dynamic loading with different offset angles and detail enlargement.

The primary differences between the two loading conditions are reflected in the stability, continuity, and extent of localized damage during crack propagation. Under static loading, crack growth is relatively slow, the main crack path is clearer, and the fracture surface is more regular, making the gradual transition from straight propagation to deflected growth easier to observe. Under dynamic loading, crack propagation occurs at a higher rate, and the combined effects of stress waves and high strain rates make the crack more susceptible to sudden instability. Consequently, the continuity of the crack path is reduced, and localized damage becomes more pronounced.

Under static loading, crack propagation exhibits a distinct staged and progressive evolution, whereas under dynamic loading, fracture is dominated by instantaneous crack penetration and a more complex failure morphology. For the same α , specimens under dynamic loading exhibit crack deflection, crack segmentation, and localized fragmentation earlier than those under static loading, indicating that the anti-plane tearing effect has a more pronounced influence on crack propagation under high strain-rate conditions.

Overall, the comparison of crack propagation under static and dynamic loading reveals the differences in the fracture response of sandstone at different loading rates. The static loading results provide insight into the progressive crack growth and stability evolution of rock masses under long-term loading, whereas the dynamic loading results are more applicable to understanding the rapid instability and failure behavior of rock masses subject to impact or dynamic disturbances.

3.2. Load–Displacement Curves Analysis

Figure 10 presents the load–displacement curves of the specimens during loading, providing further insight into the mechanical behavior of mixed mode I–III fracture. Under static loading, the load–displacement curves for different α exhibit a continuous increase before reaching the peak load, followed by a rapid post-peak drop, indicating a typical brittle fracture response. As α increases, both the peak load and the corresponding peak displacement show an overall increasing trend. This indicates that, as the fracture mode evolves from pure mode I to mixed mode I–III and eventually to pure mode III, a greater external load is required to induce failure, implying an increase in the resistance to crack propagation. Compared with the static loading results, the specimens under dynamic loading exhibit higher peak loads but smaller corresponding peak displacements for all offset angles. This behavior indicates a pronounced strain-rate strengthening effect in sandstone under impact loading, enabling the specimens to sustain higher loads within a much smaller displacement range. Meanwhile, the post-peak portions of the dynamic load–displacement curves exhibit a more gradual decline than those under static loading, with this trend being particularly evident for $\alpha = 50^\circ$ and 62.5° . The dynamic loading condition exhibits a more distributed and progressive failure behavior; this phenomenon results from the enhanced development of distributed microcracks, crack branching, and segmented fracture propagation under high strain-rate loading, which increases the energy dissipation capacity during failure.

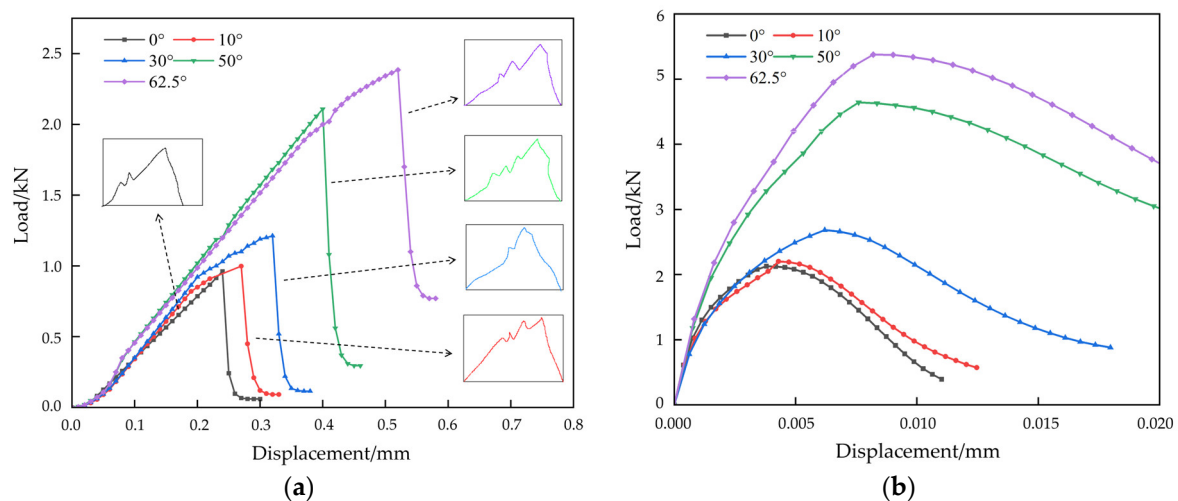


Figure 10. Load–displacement curves of each specimen. (a) Static loading. (b) Dynamic loading.

Overall, the influence of the offset angle on the mixed mode I–III fracture response exhibits a consistent trend under both static and dynamic loading conditions. However, dynamic loading further amplifies the differences in the post-peak mechanical response among specimens with different α . As α increases, the proportion of the shear-stress component at the tip of the pre-crack gradually increases within the mixed mode stress field. Consequently, both the peak load and the corresponding peak displacement increase, indicating that a greater contribution from the shear stress component in mixed mode fracture requires higher load-bearing capacity and greater energy dissipation before specimen failure.

3.3. Fracture Toughness Analysis

Fracture toughness is a measure of a material's resistance to unstable crack propagation and is defined as the critical value of the stress intensity factor at the peak load. In this study, the effective fracture toughness, K_{eff} , is adopted to characterize the resistance of the specimen to unstable crack propagation under mixed mode loading. It is calculated as follows:

$$K_{\text{eff}} = \sqrt{K_{\text{If}}^2 + \frac{K_{\text{IIIIf}}^2}{1-\nu}} \quad (19)$$

where K_{If} and K_{IIIIf} are the critical mode I and mode III stress intensity factors, respectively, and ν is Poisson's ratio. The calculated fracture toughness values for all specimens are summarized in Table 2.

Table 2. The calculated results of fracture toughness for each specimen.

Loading Condition	$\alpha/(^{\circ})$	Peak Load $P_c/(\text{kN})$	$K_{\text{If}}/(\text{MPa} \cdot \text{m}^{0.5})$	$K_{\text{IIIIf}}/(\text{MPa} \cdot \text{m}^{0.5})$	$K_{\text{eff}}/(\text{MPa} \cdot \text{m}^{0.5})$
Static	0	0.962	0.494	0.000	0.494
	10	0.989	0.483	0.047	0.486
	30	1.212	0.409	0.145	0.442
	50	2.109	0.293	0.253	0.413
	62.5	2.385	0.000	0.271	0.311
Dynamic	0	2.128	1.092	0.000	1.092
	10	2.201	1.074	0.106	1.080
	30	2.683	0.905	0.322	0.978
	50	4.644	0.646	0.577	0.925
	62.5	5.386	0.000	0.612	0.702

Figure 11 illustrates the variation trends of mode I and mode III fracture toughness with different α . As α increases, K_{If} decreases monotonically under both static and dynamic loading conditions, while K_{IIIIf} increases monotonically. The effective fracture toughness K_{eff} also shows a decreasing trend with increasing α , indicating that as the sandstone disc transitions from mode I-dominated fracture to mode III-dominated fracture, its resistance to unstable crack propagation under mixed mode conditions gradually decreases. At the same α , the fracture toughness under dynamic loading is generally higher than that under static loading, reflecting a dynamic strengthening effect induced by increased loading rate. This indicates that, without changing the mixed mode fracture mechanism, a higher loading rate requires greater energy to initiate crack propagation and to trigger unstable crack growth at the tip of the pre-existing crack during dynamic loading.

To quantitatively characterize the respective contributions of mode I and mode III in the mixed fracture regime, a mode I–III mixing coefficient M_{13}^e is introduced. This coefficient varies continuously from 1 (pure mode I) to 0 (pure mode III), and is calculated as follows [24]:

$$M_{13}^e = \frac{2}{\pi} \tan^{-1} \left(\frac{K_{\text{I}}}{K_{\text{III}}} \right) \quad (20)$$

The relationship between the mixing coefficient M_{13}^e and the effective fracture toughness K_{eff} is shown in Figure 12. Single-realization simulation results show that under static loading, a linear positive correlation is observed between M_{13}^e and K_{eff} , with a fitted slope of 0.181 and an R^2 value of 0.992. Under dynamic loading, a similar linear positive correlation is obtained, with a higher slope of 0.391 and an R^2 value of 0.995. The increase in the slope of the fitted curves from 0.181 to 0.391 indicates that, as M_{13}^e increases and the proportion of mode I fracture becomes more dominant, K_{eff} exhibits a more pronounced

increase under dynamic loading. This suggests that the influence of loading rate on the mixed mode fracture behavior is more significant under dynamic conditions.

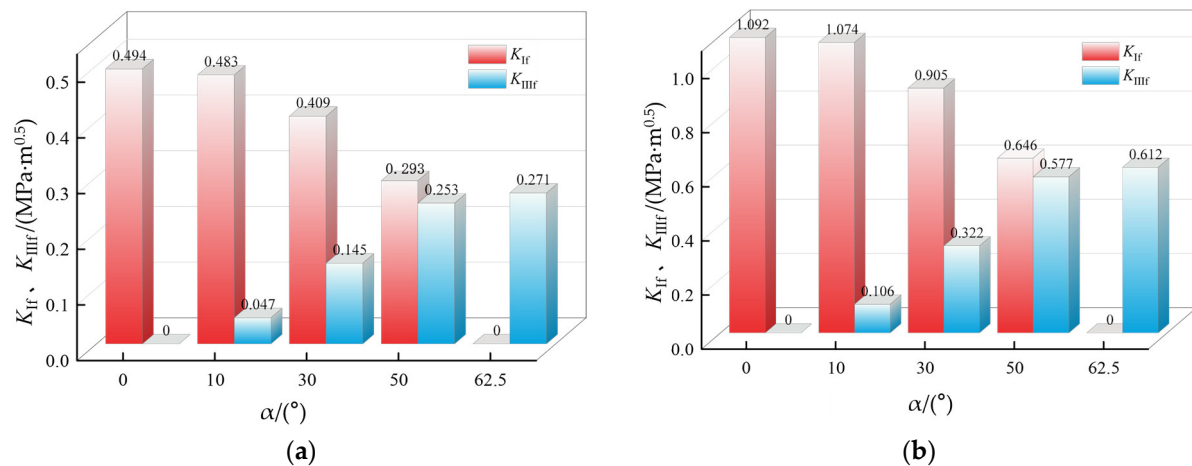


Figure 11. The relationship between mode I and mode III fracture toughness with different offset angles for each specimen. (a) Static loading. (b) Dynamic loading.

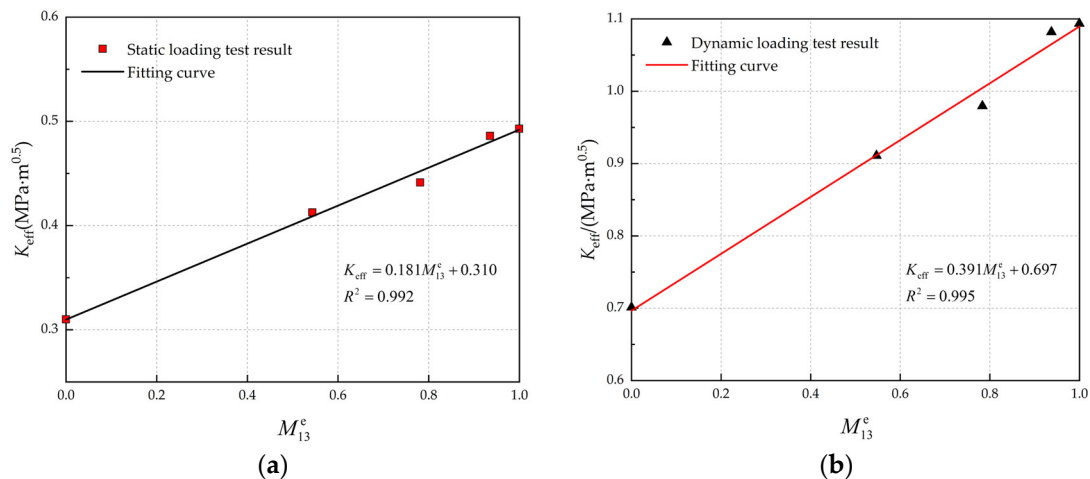


Figure 12. The relationship between mixing coefficient and effective fracture toughness. (a) Static loading. (b) Dynamic loading.

By comparing the two fitted curves, let K_{eff}^s denote the effective fracture toughness under static loading and K_{eff}^d denote the effective fracture toughness under dynamic loading. A proportional relationship exists between the dynamic and static curves, and by eliminating M_{13}^e from the governing equations, the following relationship can be obtained:

$$K_{eff}^d \approx 2.16K_{eff}^s + 0.027 \quad (21)$$

This indicates that, under the same loading conditions and specimen parameters, the effective fracture toughness under dynamic loading can be approximated as 2.16 times that under static loading. Meanwhile, the difference between the two is expressed as:

$$\Delta K_{eff} = K_{eff}^d - K_{eff}^s = 0.21M_{13}^e + 0.387 \quad (22)$$

That is, the enhancement magnitude of dynamic loading relative to static loading increases with M_{13}^e , indicating that in mode I-dominated mixed fracture conditions, an increase in loading rate improves the load-bearing capacity of the specimen. Based on the combined analysis of both static and dynamic curves, the following conclusions can be drawn: (1) Mixed mode effect: Under both static and dynamic loading conditions, K_{eff} increases with increasing M_{13}^e , indicating that a higher proportion of mode I fracture in the mixed mode regime leads to an increase in effective fracture toughness. Conversely, as the proportion of mode III fracture increases, K_{eff} gradually decreases. (2) Loading rate effect: Under dynamic loading, the overall values of the $K_{\text{eff}}-M_{13}^e$ relationship are higher than those under static loading, the directly calculated dynamic effective fracture toughness was approximately 2.22 times the static value, and the slope of the fitted curve is also larger. This indicates that dynamic loading not only increases the magnitude of K_{eff} , but also enhances its sensitivity to changes in the mixed fracture mode. (3) Proportional relationship: Under the present loading conditions, a proportional relationship exists between the dynamic and static curves, satisfying $K_{\text{eff}}^d \approx 2.16K_{\text{eff}}^s$. This clearly establishes a quantitative control law of the mixed fracture mode on the effective fracture resistance in sandstone mixed mode I–III fracture, and further reveals the significant strengthening effect induced by dynamic loading as well as the enhanced sensitivity of fracture resistance to the degree of mixed mode.

3.4. Fracture Energy Analysis

Fracture toughness reflects the resistance of a material to unstable crack propagation at the onset of failure, whereas fracture energy characterizes the energy dissipated during crack propagation. The parameters of the cohesive elements directly govern the evolution of interfacial damage and failure behavior in the numerical model. Therefore, establishing the relationships among these three quantities is essential for further investigating the characteristics of mixed mode I–III fracture in sandstone.

In classical fracture mechanics, for mode I fracture, the fracture energy can be expressed as:

$$G_I = \frac{K_I^2}{E'} \quad (23)$$

For the problem of plane strain, $E' = E / (1 - \nu^2) = 21.23 \text{ GPa}$.

For mode III fracture:

$$G_{III} = \frac{K_{III}^2}{2\mu} \quad (24)$$

where $\mu = E / 2(1 + \nu) = 8.06 \text{ GPa}$.

For mixed mode I–III fracture:

$$G_{I-III} = G_I + G_{III} = \frac{1 - \nu^2}{E} K_{\text{eff}}^2 \quad (25)$$

According to the traction–separation relationship of the bilinear cohesive softening law, the elastic stage is defined by $\tau = K_p \delta$, and damage initiates when $\delta_0 = \tau_{\text{max}} / K_p$. Let δ_f denote the relative interface displacement at complete failure under linear softening. The fracture energy, G_c , is defined as the area under the τ – δ curve, namely:

$$G_c = \int_0^{\delta_f} \tau(\delta) d\delta = \frac{1}{2} \tau_{\text{max}} \delta_f \quad (26)$$

When fracture energy is calculated from the cohesive element properties, the critical fracture energy under mixed mode fracture is characterized using the BK (Benzeggagh–Kenane) mixed mode fracture criterion. The corresponding equivalent fracture energy is expressed as follows:

$$G_{BK} = G_{Ic} + (G_{IIIc} - G_{Ic})\beta^\eta \quad (27)$$

where β denotes the energy partition ratio, defined as $\beta = G_{III}/(G_I + G_{III})$, and η is the mixed mode exponent used to describe the influence of variations in the shear-energy contribution on the fracture-energy threshold, in this study, $\eta = 0.74$.

It should be noted that the fracture energy under dynamic loading is calculated using the same formulation as that under static loading. This is because crack propagation is analyzed in this study based on the energy-release-rate theory within the framework of linear elastic fracture mechanics. Within this theoretical framework, the relationship between fracture energy and the stress intensity factor remains unchanged regardless of the loading mode.

The fracture energy values calculated using the conventional fracture-mechanics approach and the cohesive element property-based approach are summarized in Table 3. For both methods, the discrepancies between the calculated fracture energies and the simulation-derived values of G are within $\pm 5\%$, indicating good comparability. The conventional fracture-mechanics approach characterizes the energy condition at the onset of critical crack propagation, whereas the cohesive element property-based approach represents the equivalent energy dissipation during the interface failure process from crack initiation to complete failure. Thus, the two methods characterize the energy features of the mixed mode fracture process from complementary perspectives.

Table 3. The calculated results of fracture energy for each specimen.

Loading Condition	$\alpha/(^{\circ})$	$G_I/(N/m)$	$G_{III}/(N/m)$	$G_{I-III}/(N/m)$	$G_{BK}/(N/m)$	$G/(N/m)$
Static	0	11.50	0.00	11.50	11.33	11.43
	10	10.99	0.14	11.13	11.10	11.12
	30	7.88	1.30	9.19	9.78	9.49
	50	4.05	3.97	8.04	7.43	7.72
	62.5	0.00	4.55	4.55	4.77	4.65
Dynamic	0	56.19	0.00	56.19	56.79	56.49
	10	54.35	0.70	55.05	55.00	55.03
	30	38.59	6.43	45.02	48.68	46.85
	50	19.66	20.64	40.31	36.36	38.34
	62.5	0.00	23.22	23.22	23.22	23.22

The variations in the simulation-derived fracture energy values for all specimens under static and dynamic loading conditions are shown in Figure 13. At all offset angles of α , the fracture energy under dynamic loading is substantially higher than that under static loading, indicating that crack instability and propagation in the disc specimens require a greater energy input under dynamic loading. As α increases, the fracture energy decreases monotonically under both static and dynamic loading conditions. An increase in α corresponds to a greater contribution of mode III fracture to the mixed mode response. Consequently, less energy is dissipated during mode III-dominated fracture, and cracks are more likely to undergo unstable propagation at a lower energy level.

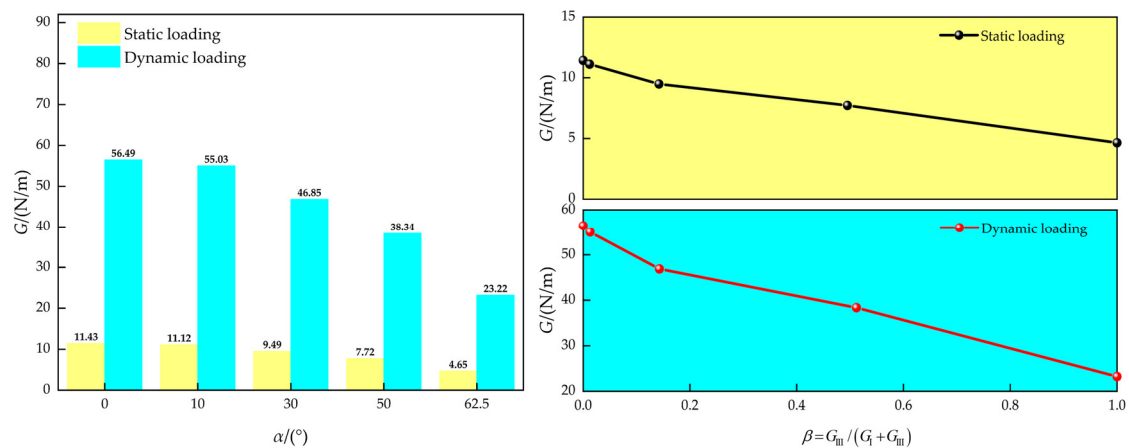


Figure 13. Fracture energy statistics of each specimen under static and dynamic loading conditions.

4. Discussion and Limitation

The present study developed a heterogeneous sandstone numerical model incorporating a Voronoi microstructure, a Drucker–Prager constitutive model, and cohesive elements to investigate the fracture behavior of ENDB specimens under static and dynamic mixed mode I–III loading. The numerical framework successfully reproduced the main fracture characteristics and provided a systematic analysis of crack evolution, fracture toughness, fracture energy, and their dependence on the mixed mode loading configuration. These results contribute to the understanding of fracture mechanisms in heterogeneous rock materials subjected to static and dynamic loading conditions.

Nevertheless, several limitations of the present study should be acknowledged. First, the fracture behaviors presented in this work are entirely based on numerical simulations. Although the proposed model was validated against Brazilian splitting and SHPB experiments, these validations mainly represent the tensile and dynamic mechanical responses of the sandstone. Direct experimental validation of the mixed mode I–III fracture behavior has not yet been performed. Additionally, comparisons between the simulation results and previous findings, such as those documented in Ref. [12], are not feasible due to discrepancies in rock type and specimen geometry. Therefore, the numerical results should be interpreted as a mechanism-oriented investigation rather than a fully experimentally validated prediction.

Second, the heterogeneous sandstone was represented by a single Voronoi realization for each loading condition. While the generated microstructure captures the essential heterogeneity of the rock, the statistical variability associated with different random realizations was not evaluated. Future studies will consider multiple independent Voronoi realizations to quantify the scatter of crack propagation characteristics, fracture toughness, and fracture energy, thereby improving the statistical reliability of the numerical predictions.

Third, only one dynamic loading condition was considered in the present work. Consequently, the influence of strain rate on crack initiation, propagation behavior, fracture toughness, and fracture energy under mixed mode I–III loading has not been systematically investigated. In addition, the fracture-energy formulation adopted in this study follows the quasi-static LEFM framework to provide a consistent basis for comparing different loading conditions after dynamic force equilibrium was achieved. Further refinement of dynamic fracture-energy formulations and other fracture indices under different loading rates is therefore warranted.

Finally, the crack evolution presented in this study was primarily evaluated through qualitative observations of crack initiation, propagation, and failure morphology. Quantitative descriptors, such as crack deflection angle, twist angle, crack surface characteristics,

and damaged volume, were not included. Future work will develop crack morphology characterization methods and combine them with dedicated mixed mode I–III experiments to establish a more comprehensive relationship between fracture geometry and macroscopic fracture resistance.

5. Conclusions

This study developed a heterogeneous sandstone numerical model to investigate mixed mode I–III fracture behavior under static and dynamic loading. The main conclusions are as follows:

- (1) A coupled numerical framework incorporating the Drucker–Prager elastoplastic model, equivalent-strain-based damage evolution, cohesive elements, and a Voronoi random structure was established in ABAQUS. Comparisons with static Brazilian splitting and dynamic SHPB tests demonstrated good agreement in crack initiation location, crack propagation path, final failure pattern, and load–displacement response, confirming that the model can reproduce the fracture and instability behavior of sandstone under different loading rates.
- (2) As the offset angle increased from 0° to 62.5° , the fracture mode gradually changed from mode I-dominated tensile opening to mode III-dominated anti-plane shear. Correspondingly, crack paths evolved from approximately straight propagation to deflected, twisted, and locally fragmented patterns. Compared with static loading, dynamic loading promoted earlier crack deflection, segmented propagation, localized damage, and sudden instability, indicating that high loading rates enhance the influence of anti-plane shear on mixed mode I–III fracture evolution.
- (3) Peak load increased with offset angle under both loading conditions, whereas the effective fracture toughness and fracture energy decreased monotonically as the mode III contribution increased. Dynamic loading significantly enhanced the fracture resistance and energy dissipation capacity of sandstone. Under the present conditions, the directly calculated dynamic effective fracture toughness was approximately 2.22 times the static value. In addition, the slope ratio of the linear fitting relationships between effective fracture toughness and mode I–III mixing coefficient was 2.16, indicating that dynamic loading enhances the sensitivity of fracture resistance to the variation in mixed mode composition. The mode I–III mixing coefficient exhibited a linear positive trend with effective fracture toughness, demonstrating that the mixed mode composition and loading rate jointly govern the fracture resistance of sandstone. In addition, it should be noted that above relationships are derived from single-realization simulations per condition and should be regarded as indicative trends rather than statistically validated scaling laws.

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