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A Temperature-Dependent Discrete Contact Model for Linear Viscoelastic Bodies

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Abstract

A temperature-dependent discrete contact model for linear viscoelastic bodies is developed to predict normal contact forces at different prescribed temperatures. The formulation combines the Lee–Radok contact theory, the equivalent contact concept, the time–temperature superposition principle, and a generalized Maxwell representation. A global fitting procedure is employed to represent the equivalent contact relaxation modulus over specified time and temperature ranges, allowing for the fitted parameters to be reused without separate identification at each temperature. For two viscoelastic bodies at different temperatures, a best fit equivalent temperature is introduced to approximate their combined temperature-dependent relaxation response. Based on the resulting equivalent relaxation modulus, a discrete contact force expression is obtained through numerical evaluation of the hereditary convolution. The framework covers contact between dissimilar or identical viscoelastic bodies and between elastic and viscoelastic bodies. Numerical comparisons are conducted using finite element simulations of PTFE-based material pairs and steel–viscoelastic pairs over 0–50 °C. The maximum fitting error of the equivalent contact relaxation modulus is 2.62%, and the maximum error associated with the equivalent temperature approximation is 10.6% over the investigated domain. The calculated force histories show good agreement with the finite element results during loading. These results demonstrate that the proposed framework provides a unified representation of temperature-dependent viscoelastic contact while avoiding repeated fitting of equivalent contact parameters over the prescribed temperature range.

Keywords: linear viscoelasticity; time–temperature superposition; discrete contact model; finite element validation

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1. Introduction

Viscoelastic materials are widely used in engineering applications involving repeated contact and impact, such as polymer bearings, sealing components, biomedical devices, damping systems, and particulate materials [1–3]. Compared with purely elastic materials, viscoelastic materials exhibit pronounced time-dependent mechanical behaviors, including stress relaxation, creep, and hysteresis dissipation. These characteristics significantly affect the evolution of contact forces and energy dissipation during dynamic interactions [4–6]. Therefore, accurate contact models for viscoelastic bodies are essential

for predicting the dynamic response, wear behavior, and service reliability of engineering systems.

Recent work also shows why constitutive viscoelasticity must be retained when actual components and process equipment are analyzed. Bulk relaxation was measured to control the evolving normal and tangential forces between hydrogel particles [7]; viscoelastic contact laws have been embedded in discrete element simulations of an agitated drum [8]; and numerical–experimental studies of polyamide journal bearings have demonstrated temperature- and load-dependent hysteretic losses [9]. Frequency–temperature constitutive models developed for tire compounds likewise show that a single room-temperature stiffness is inadequate for contact-related dynamic predictions [10], while a contact rheological DEM law has recently been used for polymer powder laser sintering [11]. These examples span particulate processes, bearings, tires, and soft-material contacts, and they motivate a reduced contact law that retains material memory and temperature dependence without the cost of resolving every real structure by a full continuum model. For elastic materials, Hertz theory provides the fundamental solution for normal contact between two smooth bodies and forms the basis of many contact force models used in multibody dynamics. However, the classical Hertz model neglects energy dissipation during contact processes. To address this limitation, several dissipative contact force models have been developed, including the Kelvin–Voigt model, the Hunt–Crossley model, the Lankarani–Nikravesh model, and the Flores model [12–14]. These models extend Hertzian contact by incorporating damping terms associated with impact energy loss. Although they have been successfully applied to elastic or weakly dissipative systems, they cannot accurately describe the intrinsic time-dependent constitutive behavior of viscoelastic materials.

The contact mechanics of linear viscoelastic materials have been extensively investigated over the past decades. Lee and Radok employed the correspondence principle of linear viscoelasticity to derive the contact solution for a rigid indenter pressing into a viscoelastic half-space [15]. Graham independently obtained an equivalent formulation through a variable transformation technique [16]. However, these solutions are valid only when the contact area increases monotonically during loading. Ting subsequently extended the theory to unloading processes, while Greenwood further simplified the solution procedure through superposition principles [17,18]. Nevertheless, the resulting formulations remain mathematically complicated and require detailed knowledge of loading histories. Their direct implementation in multibody dynamic simulations involving repeated loading–unloading cycles is therefore challenging. Consequently, simplified Lee–Radok-based approaches are still widely adopted in engineering applications.

To facilitate engineering applications, several simplified viscoelastic contact models have been proposed. Zheng [19] introduced a semi-empirical formulation based on finite element results. Jian [20] derived an SLS-based normal contact law for spheres made of the same viscoelastic material. Olsson [21] further established a contact model for spheres with different viscoelastic properties by introducing an equivalent relaxation modulus and an incremental formulation, enabling practical applications in discrete element simulations. Ding [22] subsequently formulated an explicit SLS-based law for viscoelastic bodies with different elastic and Poisson properties and validated the compression and restitution responses using finite element and published experimental data. More recently, Ding et al. [23] used a generalized Maxwell model and a Taylor approximation to obtain an explicit force law for a rigid sphere contacting a viscoelastic half-space. The 2024 comparative review by Ding et al. [5] shows that the accuracy and cost of these reduced laws depend strongly on the assumed rheology and on how the dissipative term is approximated. However, Jian’s SLS formulation has a restricted material scope, and neither the Olsson model nor the recent Ding formulations includes an explicit temperature state. If

the isothermal-equivalent modulus procedure of Olsson is applied directly to temperature-dependent materials, a new equivalent relaxation spectrum must be computed and fitted for each selected temperature pair.

Temperature strongly changes the relaxation spectrum of viscoelastic materials. Under time–temperature superposition, an increase in temperature shifts relaxation toward shorter times, whereas cooling shifts it toward longer times [10,24,25]. Continuum thermoviscoelastic contact analyses can account for this effect, surface roughness, and lubrication, but their field solution is generally too expensive for repeated contacts in multi-body or large-particle simulation [4,9]. Conversely, representative reduced normal force laws remain isothermal [20,22,23], and the recent comparative review of viscoelastic particle models does not identify a reduced formulation that accepts two different material temperatures [5]. Time–temperature superposition provides temperature-shifted material functions, but it does not by itself provide the equivalent contact relaxation modulus of two dissimilar materials, particularly when the two bodies have different uniform temperatures. The resulting gap is a discrete, temperature-aware contact law that can reuse reference temperature material spectra over a prescribed thermal domain without reconstructing and re-fitting the equivalent contact properties at every operating condition.

Table 1 distinguishes the established building blocks from the contribution of the present study. The Lee–Radok correspondence solution, Olsson’s equivalent contact transformation, time–temperature superposition and the WLF shift factor, the generalized Maxwell representation, and numerical evaluation of hereditary convolution are established methods and are not claimed here as individually novel. Their direct combination, however, still leaves an identification problem: the equivalent relaxation spectrum of dissimilar contacting materials is not known a priori over temperature, and no single physical temperature represents two bodies held at different temperatures.

Table 1. Comparison of representative linear viscoelastic normal contact formulations.

Model	Material Scope	Temperature Treatment and Refitting	Unloading and Detachment	Computational Character	Validation and Applicable Regime
Lee–Radok/Graham [15,16]	Rigid sphere viscoelastic half-space; equivalent two-body mapping under Hertz assumptions	Isothermal relaxation function; temperature-specific input may be supplied, but no cross-temperature mapping	Not supported; contact area must increase monotonically	Hereditary convolution; moderate per time step	Classical analytical solution; linear viscoelasticity, small strain, smooth axisymmetric contact, loading only; no stated temperature range
Jian et al. [20]	Two identical viscoelastic spheres; SLS rheology	No explicit temperature state; parameters must be changed when material properties change	Restitution is approximated; no rigorous Ting-type contact area history	Explicit law; low	Numerical and experimental comparisons; quasi-static small deformation and SLS/weak-to-moderate viscoelasticity; no stated temperature range
Olsson and Jelagin [21]	Different or identical viscoelastic spheres; equivalent generalized Maxwell spectrum	Isothermal; if extended directly, the equivalent spectrum is recomputed and fitted for each	Lee–Radok loading law is reused; no rigorous unloading or detachment treatment	Inverse Laplace transform and fit offline; incremental convolution online; moderate	Numerical demonstrations for DEM; linear viscoelasticity, small deformation, smooth

temperature-depend- ent material pair					spheres; no stated temperature range
Ding et al. [22]	Viscoelastic spheres with differ- ent elastic and Poisson properties; SLS rheology	No explicit tempera- ture state; tempera- ture-dependent con- stitutive parameters would need separate identification	Compression and restitution ap- proximated; no Ting-type con- tact-edge history	Explicit time-de- pendent law; low	Finite element and published experi- mental compari- sons; quasi-static small deformation and SLS rheology; no stated tempera- ture range
Ding et al. [23]	Rigid sphere visco- elastic half-space; generalized Max- well rheology	No explicit tempera- ture state; relaxation spectrum and scale factor are identified at the test condition	Compression and restitution ap- proximated; no contact-edge his- tory solution	Taylor-based ex- plicit law; low to moderate	Relaxation experi- ment and finite ele- ment validation; smooth small-strain contact, evaluated at 20 °C
Present model	Different or identi- cal viscoelastic spheres and elastic- viscoelastic pairs; generalized Max- well rheology	TTS plus shift fac- tors; one global of- fline fit over the pre- scribed domain; best fit equivalent tem- perature for unequal body temperatures; no refit inside the do- main	No rigorous un- loading or de- tachment treat- ment; the Lee– Radok loading relation is reused	Full-history trape- zoidal convolution in the reported im- plementation; per- step work grows with stored his- tory.	Finite element nu- merical verification for PTFE/PTFE- composite and steel pairs over 0–50 °C; linear viscoelastic- ity, smooth spheres, small deformation, constant body tem- peratures, and load- ing-dominated his- tories.
Ting/Greenwood [17,18]	Axisymmetric in- denter–viscoelastic half-space; equiva- lent two-body mapping under Hertz assumptions	Isothermal relaxation functions; no cross- temperature map- ping	Treats unloading through non- monotonic con- tact edge history; detachment re- quires history-de- pendent solution	History-depend- ent integral equa- tions with un- known contact- edge times; rela- tively high com- plexity	Analytical/semi-an- alytical; linear visco- elasticity, smooth axisymmetric con- tact; no stated tem- perature range

Abbreviations: VE: viscoelastic; SLS: standard linear solid; TTS: time–temperature superposition; GA: genetic algorithm; N: number of Maxwell branches.

Accordingly, the principal methodological novelty comprises two linked offline identification steps: (i) a global fit of one generalized Maxwell representation of the equivalent contact relaxation modulus over the prescribed time-common-temperature domain, replacing a sequence of independent temperature-by-temperature fits; and (ii) a best fit equivalent temperature approximation for two dissimilar viscoelastic bodies held at different uniform temperatures. The reported discrete implementation evaluates the hereditary term by full-history trapezoidal convolution. It is therefore an implementation of the fitted constitutive functions, but an $O(N)$ recursive update and a quantified online speedup are not claimed in this study. For identical material pairs and elastic–viscoelastic pairs, analytical simplifications remove part or all of the equivalent spectrum fitting procedure, as detailed in Section 2.4.2.

The framework therefore covers different and identical linear viscoelastic spheres as well as elastic–viscoelastic pairs. Its loading response is compared with finite element reference calculations for dissimilar PTFE-based pairs and steel–viscoelastic pairs over 0–50

°C. These comparisons test the implementation and internal consistency of the reduced formulation; because the analytical and finite element models share constitutive assumptions, they are termed numerical verification rather than independent physical validation. The intended regime is smooth frictionless spherical contact, linear viscoelasticity, small deformation, spatially uniform and constant body temperatures during each contact event, and histories dominated by loading. The model does not rigorously resolve unloading contact edge history, restitution, energy dissipation, or detachment.

2. Materials and Methods

2.1. Contact Model for Elastic Materials

In multibody dynamics, solving for contact forces from the relative positions and velocities of components within joints is fundamental to modeling. Such contact force models are typically derived from the contact between two spheres in a general sense.

The mathematical problem considered here consists of two smooth spheres with radii R_1 and R_2 whose centers remain collinear with the common contact normal. If $d(t)$ is the center distance, the non-negative relative penetration is $\delta(t) = \max[0, R_1 + R_2 - d(t)]$, and the normal force acts along the line of centers. The effective radius satisfies $1/R^* = 1/R_1 + 1/R_2$. Each body is treated as homogeneous and isotropic, with a spatially uniform prescribed temperature T_i during an individual contact event. Tangential traction, adhesion, plasticity, surface roughness, and thermal expansion are excluded. The Hertz half-space approximation additionally requires the contact radius to remain small compared with both sphere radii and the strains to remain within the linear viscoelastic range.

Based on Hertz contact theory, the Hertz contact force model is one of the most well-known contact models and can be expressed as

$$f(t) = K\delta^n(t), \quad (1)$$

where t denotes the time variable; $f(t)$ represents the elastic contact force; K is the contact stiffness parameter [26]; $\delta(t)$ is the relative penetration depth between the two elastic spheres; n is the nonlinear power exponent, which is a material and geometric characteristic parameter; according to Hertz contact theory, n equals 3/2.

However, the Hertz contact force model does not account for energy dissipation during contact between two spheres. For more accurate analysis, many scholars have proposed contact force models that consider energy dissipation. Representative and widely used ones include the Kelvin–Voigt contact force model [27], the Hunt–Crossley contact force model [28], the Lankarani–Nikravesh contact force model [13], and the Flores contact force model [14]. Recently, Rodrigues Da Silva [26] comprehensively reviewed the contact force models considering energy dissipation and compared various models.

It is observed that all contact force models considering energy dissipation conform to the following form:

$$f(t) = K\delta^n(t) + \chi\delta^n(t)\dot{\delta}(t), \quad (2)$$

where χ is the hysteresis damping factor, which differs among various contact models; specific expressions can be found in reference [26]; $\dot{\delta}(t)$ is the relative penetration velocity. Equation (2) clearly presents the elastic term and the viscous term. Therefore, in deriving the temperature-dependent viscoelastic contact model in this paper, we also strive to keep the formulation as close as possible to the form shown in Equation (2).

2.2. Contact Model for Linear Viscoelastic Materials

Unlike elastic materials, linear viscoelastic materials do not respond instantaneously to excitation; rather, the response is time-dependent, which greatly increases the complexity of the model. Currently, the more established viscoelastic contact models are all derived from a rigid sphere indenting a viscoelastic half-space; according to Hertz contact theory, such models are equivalent to those derived from the contact between two spheres.

Lee and Radok [15] used the linear viscoelastic correspondence principle to derive the contact model for a rigid sphere indenting a viscoelastic half-space; Graham [16] obtained the same model through an ingenious substitution, as follows:

$$F(t) = \frac{4}{3R} \int_0^t E(t-\tau) \frac{d}{d\tau} a^3(\tau) d\tau, \quad t < t_m, \quad (3)$$

where $F(t)$ is the viscoelastic contact force; R is the radius of the rigid sphere; $E(t)$ is the relaxation modulus of the viscoelastic half-space; $a(t)$ is the instantaneous radius of the contact area; t_m is the end time of the loading process, i.e., the instant when $a(t)$ reaches its maximum. Owing to certain assumptions made in the derivation, Equation (3) is only applicable to the loading process, i.e., the process during which the instantaneous contact radius continuously increases.

According to linear viscoelastic theory, the relaxation modulus $E(t)$ of a viscoelastic material is usually described by the generalized Maxwell model, as shown in Figure 1.

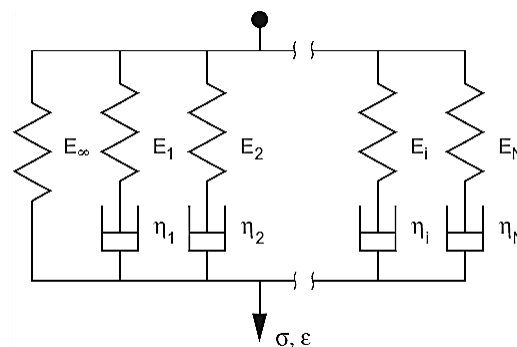


Figure 1. Schematic diagram of the generalized Maxwell model.

The generalized Maxwell model can be expressed by the Prony series, and its form is as follows:

$$E(t) = E_\infty + \sum_{j=1}^N E_j \exp(-t/\tau_j) = E_0 - \sum_{j=1}^N E_j [1 - \exp(-t/\tau_j)], \quad (4)$$

where E_∞ is the long-term equilibrium relaxation modulus; E_j is the stiffness coefficient of the j -th element of the generalized Maxwell model (the j -th Prony coefficient); N is the number of elements in the generalized Maxwell model (the number of exponential terms in the Prony series); E_0 is the glassy modulus; τ_j is the relaxation time, $\tau_j = \eta_j / E_j$; η_j is the damping coefficient of the j -th element of the generalized Maxwell model. The above parameters are usually obtained by fitting Equation (4) to experimental data.

Theoretically, when N is sufficiently large, the generalized Maxwell model can accurately describe the linear viscoelastic behavior of viscoelastic materials; when $N = 1$,

the generalized Maxwell model degenerates into the widely used standard linear solid (SLS) model.

During the loading process, the relative penetration depth $\delta(t)$ and the instantaneous contact radius $a(t)$ always satisfy the following relationship, which can be obtained from Hertz theory:

$$a^2(t) = R\delta(t), \quad (5)$$

Therefore, Equation (3) can be rewritten in the following form:

$$F(t) = \frac{4\sqrt{R}}{3} \int_0^t E(t-\tau) \frac{d}{d\tau} \delta^{\frac{3}{2}}(\tau) d\tau \quad t < t_m, \quad (6)$$

Ting [17] derived the contact force model for the unloading process, i.e., the process in which $\delta(t)$ and $a(t)$ decrease, as follows:

$$F(t) = \frac{4}{3R} \int_0^{t_1} E(t-\tau) \frac{d}{d\tau} a^3(\tau) d\tau \quad t > t_m, \quad (7)$$

where $t_1 < t_m$ and satisfies $a(t_1) = a(t) < a(t_m)$; during unloading, the relationships between $\delta(t)$ and $a(t)$ no longer satisfy Equation (5), and both t_1 and t_m are unknown quantities to be solved. The solution process for these unknown relationships and quantities was greatly simplified by Greenwood [18] through a clever transformation pair and the superposition principle.

However, although the solution procedure has been greatly simplified, solving Equation (7) remains quite complicated, and it also requires knowledge of the variation in contact force during the unloading process. In general, however, contact force models are usually embedded in the modeling and solution of dynamical systems, where the typical usage scenario is to determine the contact force from the known relative penetration depth and relative penetration velocity. Moreover, the contact force includes inertial forces and is deeply coupled with the form of relative penetration motion, which introduces further difficulties in solving Equation (7) and may even make it unsolvable. In addition, vibration, as a core issue in dynamics, introduces an enormous number of complex loading–unloading cycles that are difficult to estimate. The models of Ting and Greenwood are applicable only to a single cycle. Although Graham [29] derived solutions for multiple cycles, the complexity increases even further, making them difficult to apply in practical dynamic studies.

Papangelo and Ciavarella [30], Jian et al. [20], and Olsson and Jelagin [21] show that Lee–Radok-based engineering approximations can remain useful in selected displacement-controlled or loading-dominated situations. This evidence does not remove the underlying limitation: after the contact area reaches its maximum, the monotonic contact area condition is violated, and the Ting/Greenwood contact edge history is not recovered. The present work therefore uses Equation (6) as an approximation outside monotonic loading and enforces a non-adhesive contact condition by setting $F_n(t) = \max[0, F(t)]$. Results during unloading are shown only to expose the discrepancy and are not used to claim restitution, hysteretic energy, detachment, or repeated cycle accuracy.

2.3. Time–Temperature Superposition Principle

Temperature has a stronger effect on the relaxation response of viscoelastic polymers than on the elastic modulus of many metals. Under the time–temperature superposition (TTS) principle, increasing temperature shifts a relaxation process toward shorter reduced times, whereas cooling shifts it toward longer reduced times. Equation (8) expresses this horizontal shift for a thermorheologically simple material.

$$E(t, T) = E\left(\frac{t}{\phi_T}, T_0\right), \quad (8)$$

where T is the temperature variable; $E(t, T)$ is the relaxation modulus of the viscoelastic material, where the independent variable now includes temperature T because the temperature effect is considered; ϕ_T is the temperature shift factor; and T_0 is the reference temperature for the relaxation modulus of the viscoelastic material.

There are many models for describing the temperature shift factor. In this study, the universal form of the WLF model is adopted, as shown in the following equation [31]:

$$\log_{10}(\phi_T) = -\frac{8.86(T - T_s)}{101.6 + (T - T_s)}, \quad (9)$$

where T_s is an adjustable temperature parameter.

According to the study by Schwarzl et al. [32], Equation (9) is applicable within the temperature range of $T_g \sim T_g + 100^\circ\text{C}$, where T_g is the glass transition temperature of the viscoelastic material, and $T_s = T_g + 50$.

The universal WLF form is used here as a proof-of-concept constitutive assumption; it is not asserted to be material-specific for the selected PTFE grades. Thermorheological simplicity can fail when different relaxation mechanisms have different temperature dependences [10,24]. Accordingly, 0–50 °C is the numerical calibration domain investigated in this paper, not an experimentally demonstrated service range for either material.

If material-specific DMA or multi-temperature relaxation data are available, the shift factor in Equation (8) should be identified directly and the global fit repeated using those data. Arrhenius [33], Bueche [34], or another shift law may replace Equation (9) without changing the equivalent contact construction. When a material spectrum is tabulated at a reference temperature T_0 different from the WLF anchor temperature, the shift factor used in Equation (8) must be normalized so that at $T_0 = 1$. No transient heat transfer or thermo-mechanical coupling is solved in the present study.

2.4. Temperature-Dependent Discrete Contact Model for Linear Viscoelastic Bodies

2.4.1. Equivalent Contact Relaxation Modulus for Two Different Materials

Analogous to Hertz theory, Olsson [21] considered that the contact between two linear viscoelastic spheres made of different materials can be equivalently treated as a rigid sphere with an equivalent radius indenting into an infinite elastic half-space with an equivalent relaxation modulus. In this way, Equation (3) becomes applicable. The equivalent radius of the rigid sphere is given as follows:

$$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}, \quad (10)$$

where R^* is the equivalent radius of the rigid sphere; R_1 and R_2 are the radii of the original two viscoelastic spheres in contact, respectively.

When temperature is taken into account, the relaxation moduli of the two spheres can be expressed by the generalized Maxwell model as shown in Equation (4). For clarity, the expressions for the relaxation moduli of the two spheres are repeated here:

$$\begin{aligned} E_i(t, T) &= E_i \left(\frac{t}{\phi_{iT}}, T_{i0} \right) \\ &= E_{i\infty} + \sum_{j=1}^N E_{ij} \exp \left[-t / (\phi_{iT} \tau_{ij}) \right] \\ &= E_{i0} - \sum_{j=1}^N E_{ij} \left\{ 1 - \exp \left[-t / (\phi_{iT} \tau_{ij}) \right] \right\} \end{aligned} \quad (11)$$

since temperature is taken into account, the independent variable of the relaxation modulus includes the temperature variable T ; the subscripts $i = 1, 2$ are used to distinguish the two contacting spheres, sphere 1 and sphere 2.

The equivalent relaxation modulus of the viscoelastic half-space can be expressed as follows:

$$\frac{1}{s\tilde{\Psi}(s, T_e)} = \frac{1 - (s\tilde{\mu}_1)^2}{s\tilde{E}_1(s, T_1)} + \frac{1 - (s\tilde{\mu}_2)^2}{s\tilde{E}_2(s, T_2)}, \quad (12)$$

where $\tilde{y}(s)$ denotes the Laplace transform of the function $y(t)$ in the time domain, and s is the transform variable; $\Psi(t, T)$ is the relaxation modulus of the viscoelastic half-space, defined as the equivalent contact relaxation modulus; μ_i is the viscoelastic Poisson's ratio; T_1 and T_2 are the temperatures of the two contacting bodies; T_e is the equivalent temperature. T_e denotes a best fit equivalent temperature introduced to approximate the temperature-dependent equivalent contact relaxation modulus. It should be noted that T_e does not represent a rigorous thermodynamic temperature, but rather an optimization variable determined by minimizing the discrepancy between the target equivalent relaxation modulus and its generalized Maxwell approximation over the prescribed time–temperature range.

Theoretically, the equivalent contact relaxation modulus $\Psi(t, T_e)$ can be obtained by performing the inverse Laplace transform on $\tilde{\Psi}(s, T_e)$. However, calculations show that even when $N = 1$ in $E_i(t, T)$, the inverse Laplace transform of $\tilde{\Psi}(s, T_e)$ does not yield a practically useful concise analytical solution, let alone when $N > 1$. Therefore, the inverse Laplace transform must be carried out numerically.

In order to obtain an analytical form of $\Psi(t, T_e)$, it is expressed using the generalized Maxwell model as follows:

$$\begin{aligned} \Psi(t, T_e) &= \Psi_{\infty} + \sum_{j=1}^N \Psi_j \exp \left[-t / (\phi_{zT_e} \tau_j) \right] \\ &= \Psi_0 - \sum_{j=1}^N \Psi_j \left\{ 1 - \exp \left[-t / (\phi_{zT_e} \tau_j) \right] \right\} \end{aligned} \quad (13)$$

where

$$\log_{10}(\phi_{zT_e}) = -\frac{C_{z1}(T_e - T_z)}{C_{z2} + (T_e - T_z)}, \quad (14)$$

In Equations (13) and (14), Ψ_{∞} , Ψ_0 , Ψ_j , τ_j , C_{z1} , C_{z2} , T_z and T_e are all unknowns; by performing some fitting of the results of the inverse Laplace transforms of Equations (12)–(14), the above unknown parameters can be obtained, as follows:

(1) Calculation of the glassy (instantaneous) modulus Ψ_0 and the long-term equilibrium modulus Ψ_∞

According to elastic contact theory, the glassy (instantaneous) modulus Ψ_0 and the long-term equilibrium modulus Ψ_∞ can be calculated as follows:

$$\Psi_0 = \frac{1}{\frac{1-\nu_1(0,T_{10})^2}{E_1(0,T_{10})} + \frac{1-\nu_2(0,T_{20})^2}{E_2(0,T_{20})}} = \Psi_\infty + \sum_{j=1}^N \Psi_j, \quad (15)$$

(2) Calculation of the relaxation time τ_j

When $N = 1$, since the SLS model cannot accurately describe the viscoelastic material properties, the relaxation time τ_j is still treated as an optimization variable to improve the accuracy of the SLS as much as possible; when $N \geq 2$, in order for the generalized Maxwell model to cover the entire time range of the data, the relaxation times τ_j are chosen as follows [35]:

$$\tau_j = 10^{j - \text{ceil}(\frac{N}{2})}, \quad (16)$$

where ceil means the ceiling of $N/2$.

(3) Establishment of the fitting model

First, the fitting model assumes that the temperatures of the two viscoelastic contact bodies are equal, i.e., $T_1 = T_2 = T_e = T$; the time is discretized into k points over the time range $t = [0, t_{\max}]$ of interest; the temperature is discretized into m points over the temperature range $T = [T_{\min}, T_{\max}]$ of interest; the fitting is achieved by minimizing the following residual:

$$R_p = \sum_{n=1}^m \sum_{p=1}^k \left[\frac{\hat{\Psi}(t_p, T_n) - \left(\Psi_\infty + \sum_{j=1}^N \Psi_j \exp(-t_p / (\phi_{zT_n} \tau_j)) \right)}{\Psi_0} \right]^2, \quad (17)$$

where t_p denotes each discrete time point, $p = 1 \sim k$; T_n denotes each discrete temperature point, $n = 1 \sim m$; and $\hat{\Psi}(t_p, T_n)$ denotes the result of the inverse Laplace transform evaluated at $t = t_p$ and $T = T_n$.

When $N \geq 2$, there are $N + 3$ optimization variables in total; when $N = 1$, since the relaxation time is also an optimization variable, there are $N + 4 = 5$ optimization variables. Since the optimization objective function is too complex, a genetic algorithm is adopted for optimization.

The above optimization problem needs to satisfy two constraints:

Constraint 1: Since each stiffness coefficient in the generalized Maxwell model must be greater than 0, the optimization variables must be greater than 0; i.e., $\Psi_j > 0, j = 1 \sim N$.

Constraint 2: Since the glassy modulus Ψ_∞ needs to be greater than 0, $\Psi_0 - \sum_{j=1}^N \Psi_j$ also needs to be greater than 0; i.e., $\Psi_\infty = \Psi_0 - \sum_{j=1}^N \Psi_j > 0$.

(4) Solution of the equivalent temperature

Since it is assumed that the two contacting viscoelastic bodies have the same temperature during the fitting process for solving the unknown parameters, when the temperatures of the two viscoelastic bodies are different, in order to use the fitted Equation (13), it

is necessary to calculate the equivalent contact temperature T_e of the two viscoelastic bodies, as follows:

$$T_e = aT_1 + bT_2, \quad (18)$$

where T_1 and T_2 are the temperatures of the two viscoelastic materials; a and b are the temperature weighting coefficients for the two materials. To solve for a and b , the following operation is required: within the temperature range $T = [T_{min}, T_{max}]$ of interest, discretize the temperature into s points; a and b should minimize the following residual:

$$R_t = \sum_{p=1}^k \sum_{q=1}^s \sum_{r=1}^s \left[\frac{\hat{\Psi}(t_p, T_{1q}, T_{2r}) - \left(\Psi_\infty + \sum_{j=1}^N \Psi_j \exp(-t_p / (\phi_{zT_e} \tau_j)) \right)}{\Psi_0} \right]^2, \quad (19)$$

where T_{1q} and T_{2r} are the temperature discrete points of the two viscoelastic bodies, respectively; $\hat{\Psi}(t_p, T_n)$ is the result of the inverse Laplace transform evaluated at $t = t_p$, $T_1 = T_{1q}$, and $T_2 = T_{1r}$; this optimization problem has only two optimization variables, but the objective function is still relatively complex, so a genetic algorithm is still used for solving; the above problem needs to satisfy two constraints:

Constraint 1: According to the definition of weighting coefficients, the sum of a and b should be 1; i.e., $a + b = 1$; Constraint 2: $a, b \in [0, 1]$.

Once the time and temperature ranges of interest are known, before performing the contact calculation for viscoelastic bodies of different materials, it is only necessary to carry out steps (1)–(4) above once; when the time or temperature changes, the equivalent contact relaxation modulus given by Equation (13) can be directly used for calculation without the need for repeated fitting.

Admittedly, when considering the case where the two viscoelastic bodies are at different temperatures, an approximate method is adopted in solving for the temperature-dependent equivalent contact relaxation modulus. This approximation reduces the dimensionality of the offline equivalent temperature identification. Over the sampled temperature domain of 0–50 °C, the maximum relative discrepancy was 10.6%, and the discrepancy generally decreased as the temperature difference between the two bodies decreased. Its suitability therefore depends on the accuracy required for a particular application; no general computational speed or accuracy claim is made here.

2.4.2. Equivalent Contact Relaxation Modulus for Other Contact Forms

Here, “other forms of contact” refer to contact between two viscoelastic bodies of the same material and contact between an elastic body and a viscoelastic body. These two forms can simplify, to some extent, the solution of the temperature-dependent equivalent contact relaxation modulus.

(1) Temperature-dependent equivalent contact relaxation modulus for two viscoelastic bodies of the same material

When the two viscoelastic bodies are composed of the same material, the index i in Equation (11) takes only the value 1, and the viscoelastic Poisson’s ratio of the material is assumed constant. When the two viscoelastic bodies are at the same temperature, at any temperature T the following conclusion can be drawn:

$$\Psi(t, T) = \frac{E_1(t, T)}{2(1 - \mu_1^2)}, \quad (20)$$

Since the material is represented by the generalized Maxwell model, the following conclusion can be obtained by applying the Laplace transform to Equation (20):

$$\tilde{E}_1(s, T) = \phi_{1T} \tilde{E}_1(\phi_{1T}s, T_{10}), \quad (21)$$

Therefore, the equivalent contact relaxation modulus also conforms to the form shown in Equation (13), namely:

$$\tilde{\Psi}(s, T) = \phi_{1T} \tilde{\Psi}(\phi_{1T}s, T_{10}), \quad (22)$$

Equation (22) indicates that the equivalent contact relaxation modulus has the same reference temperature and temperature shift factor as the material's relaxation modulus; when the two viscoelastic bodies are at different temperatures, according to Equation (12), it can be obtained that

$$\begin{aligned} \frac{1}{s\tilde{\Psi}(s, T_e)} &= \frac{1-\mu_1^2}{s\tilde{E}_1(s, T_1)} + \frac{1-\mu_1^2}{s\tilde{E}_1(s, T_2)} \\ &= \frac{1-\mu_1^2}{\phi_{1T_1} s\tilde{E}_1(\phi_{1T_1}s, T_{10})} + \frac{1-\mu_1^2}{\phi_{1T_2} s\tilde{E}_1(\phi_{1T_2}s, T_{10})}, \end{aligned} \quad (23)$$

where T_1 and T_2 denote the different temperatures of the two materials.

Equation (23) still cannot be analytically inverse Laplace-transformed. Combining Equations (22) and (23), it can be seen that when the two viscoelastic bodies are of the same material, the combined contact relaxation modulus can be simplified directly from Equation (23) without performing the fitting procedure described in Section 2.4.1 (1)–(3), which greatly reduces the computational effort; however, it is still necessary to fit and solve for the equivalent temperature T_e according to the method described in Section 2.4.1 (4).

(2) Temperature-dependent equivalent contact relaxation modulus for an elastic body and a viscoelastic body

When the object numbered $i = 1$ is an elastic body and the object numbered $i = 2$ is a viscoelastic body, the contact between the two constitutes an elastic–viscoelastic body contact. It is assumed that the properties of the elastic material are not affected by temperature. At the reference temperature of the viscoelastic body, according to Equation (12), we obtain

$$\frac{1}{s\tilde{\Psi}(s, T_{20})} = \frac{1-\mu_1^2}{E_1} + \frac{1-\mu_2^2}{s\tilde{E}_2(s, T_{20})}, \quad (24)$$

When the viscoelastic body is at an arbitrary temperature T , since the properties of the elastic material do not vary with temperature, the equivalent temperature T_e is always equal to the viscoelastic body temperature $T_2 = T$; according to Equation (24), we can obtain

$$\frac{1}{s\tilde{\Psi}(s, T_e)} = \frac{1}{s\tilde{\Psi}(s, T)} = \frac{1-\mu_1^2}{E_1} + \frac{1-\mu_2^2}{\phi_{2T} s\tilde{E}_2(\phi_{2T}s, T_{20})}, \quad (25)$$

When a generalized Maxwell model represented by a Prony series with $N > 2$ is adopted, it is still difficult to obtain analytical inverse Laplace transform solutions for Equations (24) and (25); however, by comparing these two equations, it can be obtained that

$$\tilde{\Psi}(s, T_e) = \tilde{\Psi}(s, T) = \phi_{2T} \tilde{\Psi}(\phi_{2T}s, T_{20}), \quad (26)$$

Equations (24) and (25) indicate that, for elastic–viscoelastic contact with the viscoelastic material represented by a generalized Maxwell model with more than two Maxwell branches ($N > 2$), the equivalent contact relaxation modulus only needs to be fitted at the reference temperature of the viscoelastic material using the procedure proposed by Ols-son. Its temperature–time dependence can then be directly characterized by the tempera- ture shift factor of the viscoelastic material, without the need to determine an equivalent temperature. Consequently, the fitting procedure described in Section 2.4.1 is no longer required, further reducing the computational cost of the fitting process.

For elastic–viscoelastic contact when the viscoelastic material is described by the standard linear solid (SLS) model ($N = 1$), the above approach can be further simplified. Specifically, by substituting the elastic modulus E_1 of the elastic material and the SLS model $E_2(t, T) = E_{\infty 2} + E_2 \exp(-t / (\phi_{2T} \tau_2))$ of the viscoelastic material into Equation (25), an analytical inverse Laplace transform can be obtained as

$$\Psi(t, T_e) = \Psi_{\infty} + \Psi \exp(-t / (\phi_T \tau)), \quad (27)$$

where

$$\begin{aligned} \Psi_{\infty} &= \frac{-E_1 E_{\infty 2}}{\mu_1^2 E_{\infty 2} + \mu_2^2 E_1 - E_1 - E_{\infty 2}}, \\ \tau &= \frac{(E_2 \mu_1^2 + \mu_1^2 E_{\infty 2} + \mu_2^2 E_1 - E_2 - E_1 - E_{\infty 2}) \tau_2}{\mu_1^2 E_{\infty 2} + \mu_2^2 E_1 - E_1 - E_{\infty 2}}, \\ \phi_T &= \phi_{2T}, \\ \Psi &= \frac{E_1^2 E_2 (\mu_2^2 - 1)}{(\mu_1^2 E_{\infty 2} + \mu_2^2 E_1 - E_1 - E_{\infty 2}) * (-E_2 \mu_1^2 - \mu_1^2 E_{\infty 2} - \mu_2^2 E_1 + E_2 + E_1 + E_{\infty 2})}, \\ T_e &= T, \end{aligned}$$

In the case of elastic–viscoelastic contact where the viscoelastic body is represented by the standard linear solid (SLS) model ($N = 1$), no fitting procedure is required, nor is it necessary to solve for an equivalent temperature. However, since the SLS model may not always accurately capture the constitutive behavior of viscoelastic materials, Equation (27) is subject to certain inherent errors.

2.4.3. Temperature-Dependent Discrete Contact Force Model

The equivalent contact relaxation modulus derived in Sections 2.4.1 and 2.4.2 can be written in a unified form as follows:

$$\Psi(t, T_e) = \Psi_{\infty} + \hat{\Psi}(t, T_e), \quad (28)$$

Substituting Equations (28) and (5) into Equation (3), the temperature-dependent vis- coelastic contact force can be expressed in the following form:

$$\begin{aligned} F(t, T_e) &= \frac{4\sqrt{R^*}}{3} \Psi_{\infty} [\delta^{3/2}(t) - \delta^{3/2}(0)] \\ &\quad + \frac{4\sqrt{R^*}}{3} \int_0^t \hat{\Psi}(t - \tau, T_e) \frac{3}{2} \delta^{1/2}(\tau) \dot{\delta}(\tau) d\tau, \\ &= F_e(t, T_e) + F_v(t, T_e) \end{aligned} \quad (29)$$

where

$$F_e(t, T_e) = \frac{4\sqrt{R^*}}{3} \Psi_\infty [\delta^{3/2}(t) - \delta^{3/2}(0)], \quad (30)$$

$$F_v(t, T_e) = \frac{4\sqrt{R^*}}{3} \int_0^t \hat{\Psi}(t - \tau, T_e) \frac{3}{2} \delta^{1/2}(\tau) \dot{\delta}(\tau) d\tau, \quad (31)$$

Equation (30) corresponds to the equivalent elastic force component, while Equation (31) corresponds to the equivalent viscous force component.

The elastic term $F_e(t, T_e)$ is straightforward to compute, whereas the viscous term $F_v(t, T_e)$ is difficult to obtain analytically and generally requires numerical evaluation. Since the contact force model is typically embedded in a dynamic analysis, if the time step of the dynamic analysis is constant and denoted as Δt , then $F_v(t, T_e)$ can be solved using a composite quadrature formula. The discrete time points coincide exactly with the time instants of the dynamic computation, at which the relative penetration depth δ and the relative velocity $\dot{\delta}$ of the two bodies are known. In this study, the composite trapezoidal rule is employed for numerical integration [36]. At the w time step, i.e., at $t = w\Delta t$, the equivalent viscous force term $F_v(w\Delta t, T_e)$ can be expressed as follows:

$$\begin{aligned} F_v(w\Delta t, T_e) &= \Delta t \sqrt{R^*} \left[\begin{aligned} &\hat{\Psi}(w\Delta t, T_e) \delta^{1/2}(0) \dot{\delta}(0) \\ &+ 2 \sum_{v=1}^{w-1} \hat{\Psi}(w\Delta t - v\Delta t, T_e) \delta^{1/2}(v\Delta t) \dot{\delta}(v\Delta t) \\ &+ \hat{\Psi}(0, T_e) \delta^{1/2}(w\Delta t) \dot{\delta}(w\Delta t) \end{aligned} \right] \\ &= 2\Delta t \sqrt{R^*} \left[\sum_{v=0}^w \hat{\Psi}(w\Delta t - v\Delta t, T_e) \delta^{1/2}(v\Delta t) \dot{\delta}(v\Delta t) \right] \\ &\quad - \Delta t \sqrt{R^*} \left[\begin{aligned} &\hat{\Psi}(w\Delta t, T_e) \delta^{1/2}(0) \dot{\delta}(0) \\ &+ \hat{\Psi}(0, T_e) \delta^{1/2}(w\Delta t) \dot{\delta}(w\Delta t) \end{aligned} \right] \end{aligned} \quad (32)$$

Equation (32) can be simplified as

$$\begin{aligned} F_v(w\Delta t, T_e) &= 2\Delta t \sqrt{R^*} \left[\hat{\Psi}_v(w\Delta t, T_e) * H_v(w\Delta t) \right]_{1:w+1} \\ &\quad - \Delta t \sqrt{R^*} \left[\begin{aligned} &\hat{\Psi}(w\Delta t, T_e) \delta^{1/2}(0) \dot{\delta}(0) \\ &+ \hat{\Psi}(0, T_e) \delta^{1/2}(w\Delta t) \dot{\delta}(w\Delta t) \end{aligned} \right] \end{aligned} \quad (33)$$

where $\hat{\Psi}_v(w\Delta t, T_e) = [0, \hat{\Psi}(\Delta t, T_e), \hat{\Psi}(2\Delta t, T_e), \dots, \hat{\Psi}(w\Delta t, T_e)]$ is the vector storing the history of $\hat{\Psi}(t, T_e)$; $H_v(w\Delta t) = [\delta^{1/2}(0) \dot{\delta}(0), \delta^{1/2}(\Delta t) \dot{\delta}(\Delta t), \delta^{1/2}(2\Delta t) \dot{\delta}(2\Delta t), \dots, \delta^{1/2}(w\Delta t) \dot{\delta}(w\Delta t)]$ is the vector storing the history of $\delta^{1/2}(t) \dot{\delta}(t)$; * denotes discrete convolution; and the notation $1:w+1$ indicates that only the first $w+1$ terms of the discrete convolution result are taken.

The reported implementation stores the complete penetration/velocity history and evaluates the discrete convolution directly. With M elapsed time steps, this requires $O(M)$ work at the current step, $O(M^2)$ work over the complete simulation, and $O(M)$ stored history per contact. Thus, the computational advantage established here is the removal of repeated temperature-by-temperature equivalent spectrum fitting inside the calibrated domain; no measured online speedup relative to inverse Laplace or finite element calculations is claimed. A Maxwell internal variable recurrence could reduce the online update to $O(N)$ work and memory per contact, but that recurrence is not implemented or benchmarked in the present study.

Therefore, through the simplification given by Equation (33), the temperature-dependent viscoelastic contact force can be reduced to the following form:

$$F(w\Delta t, T_e) = K \left[\delta^{3/2}(w\Delta t) - \delta^{3/2}(0) \right] + \chi \left[\hat{\Psi}_v(w\Delta t, T_e) * H_v(w\Delta t) \right]_{1:w+1} + C(w\Delta t, T_e)' \quad (34)$$

$$\text{where} \quad K = \frac{4\sqrt{R^*}}{3} \Psi_\infty \quad ; \quad \chi = 2\Delta t \sqrt{R^*} \quad ;$$

$$C(w\Delta t, T_e) = -\Delta t \sqrt{R^*} \left[\hat{\Psi}(w\Delta t, T_e) \delta^{1/2}(0) \dot{\delta}(0) - \hat{\Psi}(0, T_e) \delta^{1/2}(w\Delta t) \dot{\delta}(w\Delta t) \right]$$

Equations (28)–(34) provide a compact temperature-dependent loading force formulation with an explicit record of the penetration history.

3. Results and Discussion

The proposed formulation was compared with finite element reference solutions. Since the analytical and finite element calculations were based on the same material relaxation data and temperature shift relation, this comparison represents numerical verification of the model implementation rather than independent experimental validation. Figure 2 summarizes the model geometry, prescribed penetration history program, and finite element mesh.

The finite element calculations were performed using Abaqus/Standard 2021, while the inverse Laplace calculation, parameter fitting, equivalent temperature optimization, and discrete convolution were implemented in MATLAB R2022b. A consistent mm–MPa–s–N unit system was adopted. As shown in Figure 2, the radii of the two contacting bodies were $R = 5\text{mm}$ and $R = 10\text{mm}$. Owing to the two orthogonal symmetry planes, only one quarter of each hemisphere was modelled. The bodies were discretized using 260,824 and 465,876 four-node linear tetrahedral elements (C3D4), respectively, giving 726,700 solid elements in total. The mesh was refined around the contact region and gradually coarsened away from it.

The remote surface of Sphere 2 was fully constrained, whereas the remote surface of Sphere 1 was kinematically coupled to a reference point through which the prescribed normal motion was applied. Symmetry conditions were imposed on the $x-z$ and $y-z$ cut planes. The contact interface was defined using surface-to-surface contact with hard normal behavior and a frictionless tangential response. Geometric nonlinearity was disabled because the maximum penetration was only 0.1mm , corresponding to $\delta_{\max}/R_1 = 0.02$.

Initially, the two bodies were just in contact without preload, so that $\delta(0) = 0$. The prescribed history consisted of loading at 0.5mm/s from 0 to 0.2 s, holding at $\delta_{\max} = 0.10\text{mm}$ from 0.2 to 0.4 s, unloading at 0.5mm/s from 0.4 to 0.6 s, followed by a second loading–unloading cycle over 0.6 to 1.0 s. Therefore, the right-hand initial penetration rate was $\dot{\delta}(0^+) = 0.5\text{mm/s}$ rather than zero. Each body was assigned a spatially uniform and constant temperature during an individual simulation; transient heat transfer and temperature gradients were not considered.

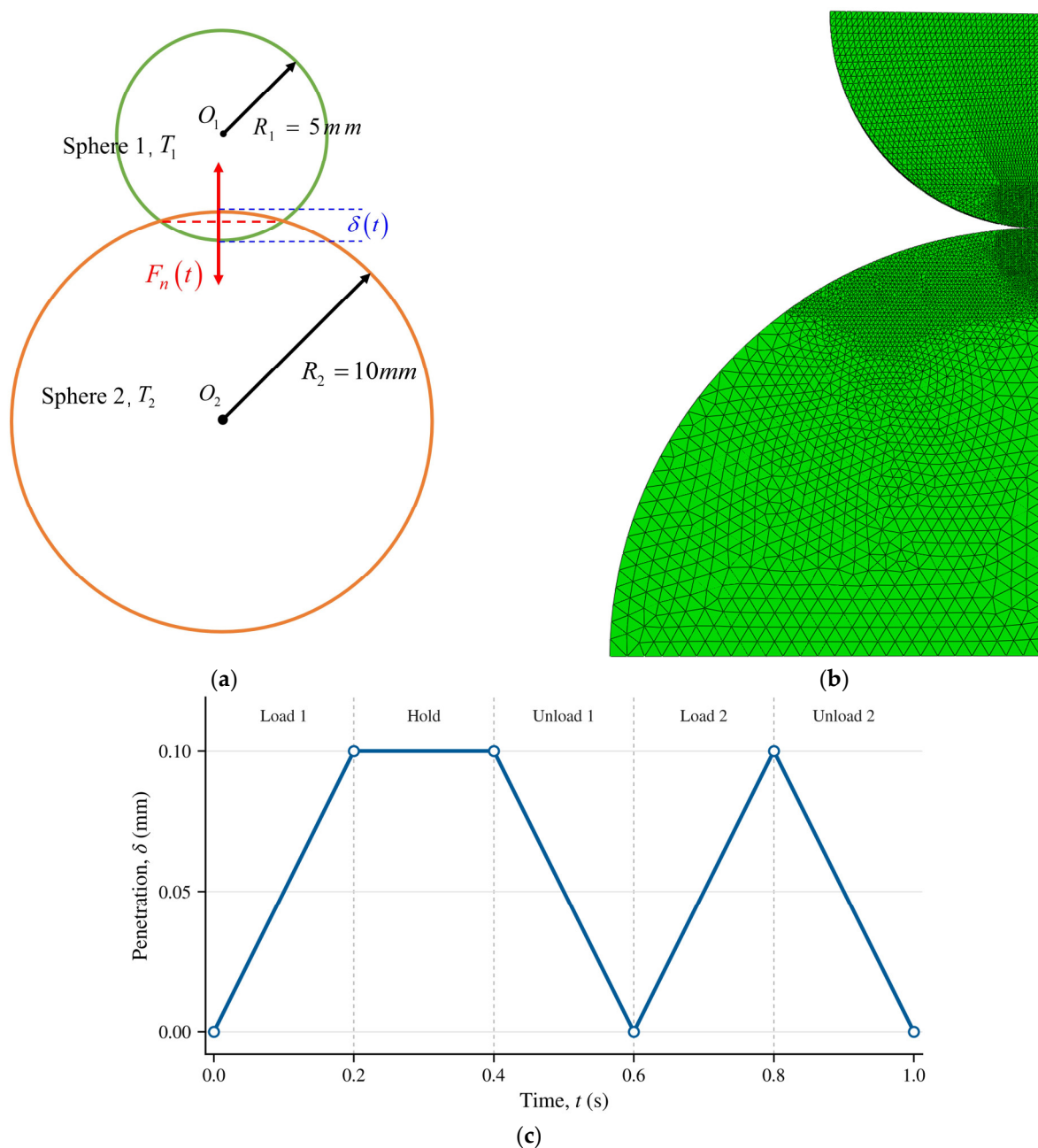


Figure 2. Model definition and numerical verification setup: (a) Two-sphere normal contact geometry; (b) locally refined quarter finite element mesh; (c) prescribed penetration history.

Cases 1~3 represent contact between different viscoelastic materials at the same temperature. Cases 4~6 correspond to contact between different viscoelastic materials at different temperatures. Cases 7~9 represent contact between an elastic material and a viscoelastic material at different temperatures. Since the mechanical properties of the elastic material are assumed to be temperature-independent, the temperature of the elastic body was kept constant throughout the analysis. In addition, contact between two identical viscoelastic materials can be regarded as a special case of contact between two different viscoelastic materials. To avoid unnecessary repetition, the corresponding validation results are not presented in this paper.

No new material characterization experiment was conducted in this study. The relaxation properties of unfilled PTFE and PTFE CG 32-3 were taken from the macro-indentation experiments reported in Ref. [37], in which the specimens were supplied by Ceproinv S.A. (Focșani, Romania). Their instantaneous moduli were 233.1 and 543.5 MPa,

respectively, and the instantaneous Poisson ratio was 0.46 for both materials. A density of 2100 kg/m^3 was used in the finite element models. The steel body was treated as a linear-elastic reference material with $E = 210 \text{ GPa}$ and $\nu = 0.25$.

Temperature dependence was described using the WLF relation with a reference temperature of 5°C , $C_1 = 8.86$, and $C_2 = 101.6^\circ\text{C}$, consistent with the Abaqus input files. For the equivalent contact relaxation modulus, eight Maxwell branches with relaxation times between 10^{-3} and 10^4 s were used. The contact force convolution was evaluated using a time step of 0.001 s. The Abaqus analyses employed an implicit dynamic procedure with automatic time incrementation and a suggested initial increment of 0.02 s. The contact force obtained from the quarter model was multiplied by four to recover the force of the complete configuration. Because the Lee–Radok formulation assumes a non-decreasing contact area, the loading and holding stages provide the principal basis for quantitative verification. The unloading results are retained to illustrate the history-dependent response, but they are not intended as rigorous validation of detachment or restitution.

3.1. Numerical Verification of the Equivalent Contact Relaxation Modulus and Equivalent Temperature

For elastic–viscoelastic contact, the equivalent contact relaxation modulus can be directly determined using the simplified approach proposed by Olsson [21]. Therefore, this section focuses on the additional numerical approximations introduced for contact between two different viscoelastic materials.

At the same temperature, the equivalent contact relaxation modulus obtained by discrete inverse Laplace transformation was compared with the globally fitted generalized Maxwell representation in Equation (13), as shown in Figure 3.

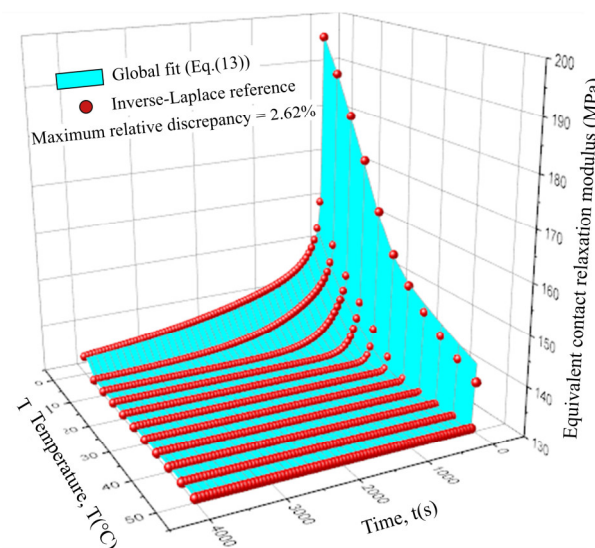


Figure 3. Comparison between the globally fitted equivalent contact relaxation modulus and the discrete inverse Laplace reference values under equal temperature conditions.

The fitted spectrum agrees closely with the discrete inverse Laplace reference values over the sampled time–temperature domain. The reported maximum relative discrepancy is 2.62%. This value represents the numerical fitting error introduced by the generalized Maxwell approximation rather than an experimental validation error.

For bodies at different temperatures, the accuracy of the equivalent temperature approximation was evaluated by comparing its predicted equivalent contact relaxation modulus with the corresponding discrete inverse Laplace results. Material 1 denotes PTFE CG 32-3, whereas Material 2 denotes unfilled PTFE. Because numerous temperature

combinations are possible, Figure 4 presents representative results with the temperature of Material 1 fixed at 0 and 30 °C.

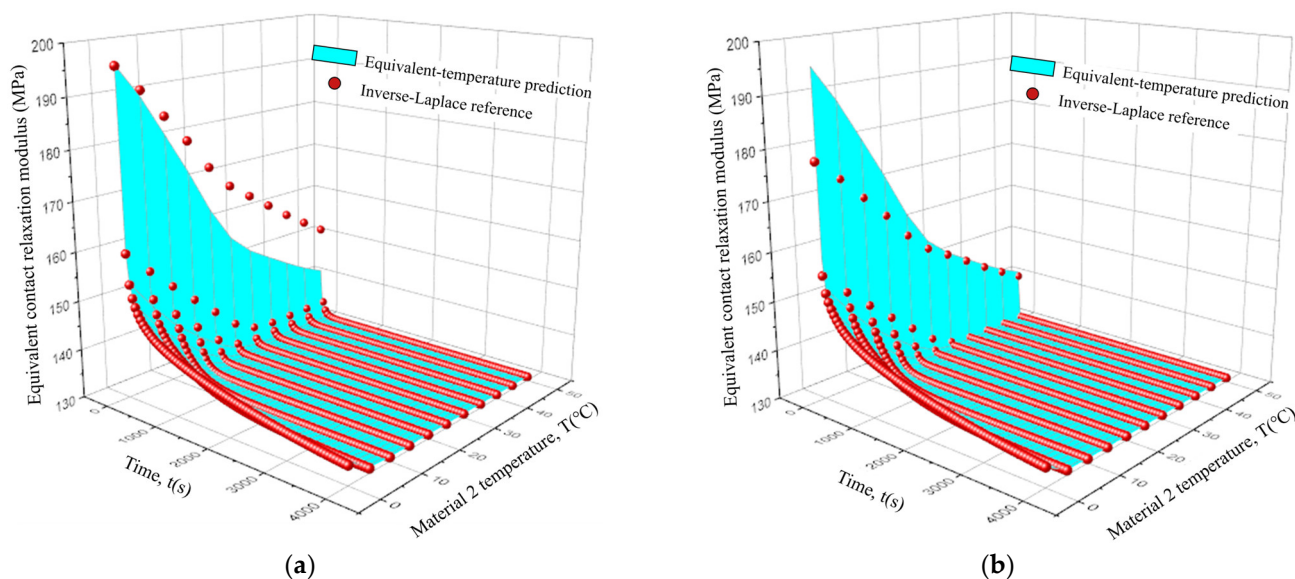


Figure 4. Numerical verification of the equivalent temperature approximation for two different viscoelastic materials maintained at different constant temperatures: (a) Material 1 at 0 °C; (b) Material 1 at 30 °C.

Over the sampled domain, the maximum discrepancy introduced by replacing the two actual material temperatures with a single equivalent temperature is 10.6%. The discrepancy generally decreases as the temperature difference between the two materials decreases. Therefore, the equivalent temperature approximation provides a practical representation of unequal temperature contact within the investigated time and temperature ranges, although its suitability depends on the accuracy required for a particular application.

3.2. Contact Between Different Viscoelastic Materials at the Same Temperature

The contact force histories calculated using the proposed model were compared with the finite element results for Cases 1–3 in Table 2. These cases correspond to contact between PTFE CG 32-3 and unfilled PTFE at common temperatures of 15, 30, and 45 °C, respectively. The results are shown in Figure 5.

Table 2. Contact configurations and temperature conditions.

Serial No.	Material		Temperature (°C)	
	Sphere 1	Sphere 2	Sphere 1	Sphere 2
1	PTFE CG 32-3	PTFE	15	15
2	PTFE CG 32-3	PTFE	30	30
3	PTFE CG 32-3	PTFE	45	45
4	PTFE CG 32-3	PTFE	0	15
5	PTFE CG 32-3	PTFE	0	30
6	PTFE CG 32-3	PTFE	0	45
7	Steel	PTFE CG 32-3	0	15
8	Steel	PTFE CG 32-3	0	30
9	Steel	PTFE CG 32-3	0	45

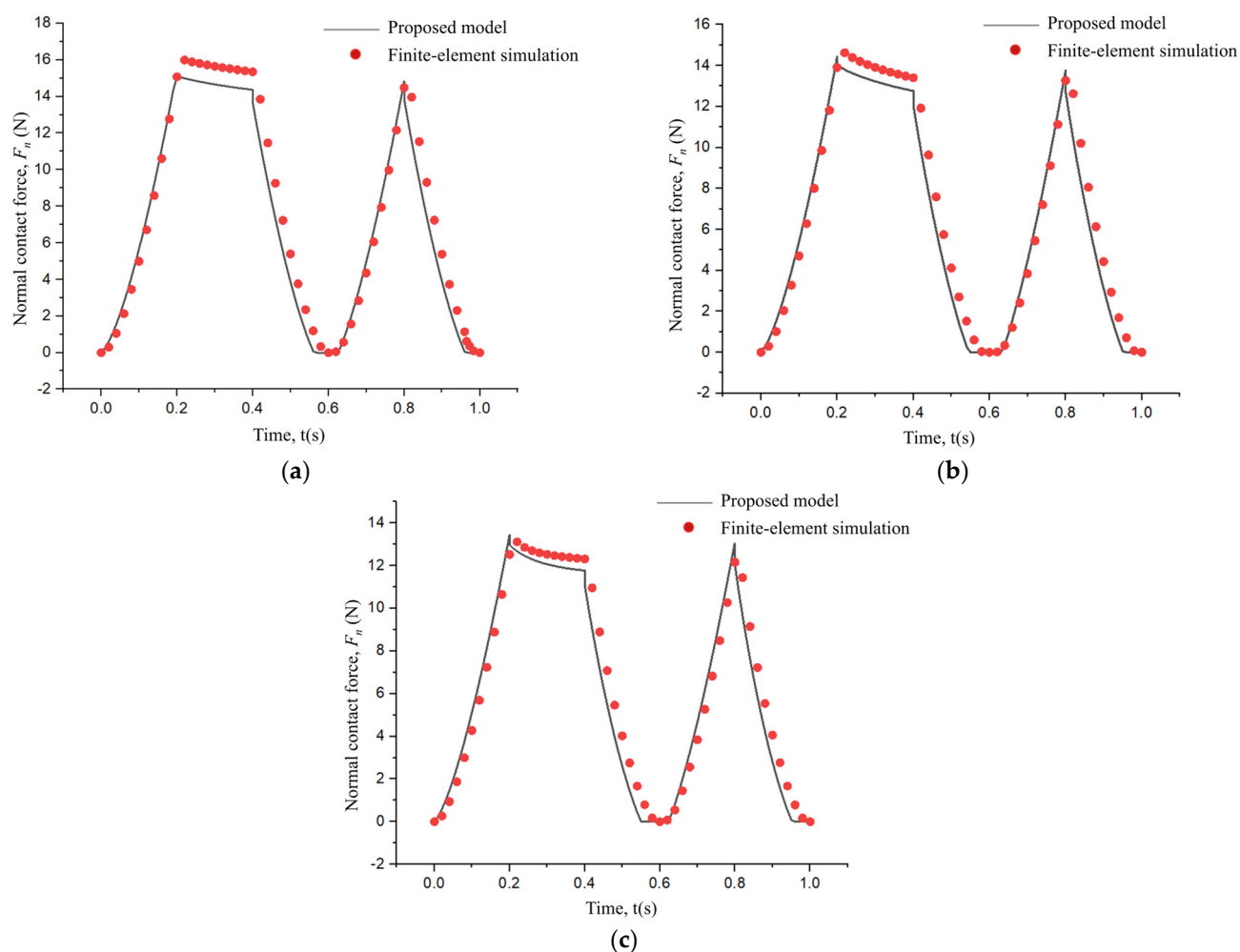


Figure 5. Comparison of contact forces for contact between different viscoelastic materials at the same temperature: (a) PTFE CG 32–3 and PTFE at 15 °C; (b) PTFE CG 32–3 and PTFE at 30 °C; (c) PTFE CG 32–3 and PTFE at 45 °C.

During both loading stages, the proposed model reproduces the overall magnitude and evolution of the finite element contact force. The contact force decreases as the common temperature increases, which is consistent with the shift toward faster material relaxation predicted by the WLF relation. The holding stage also demonstrates the time-dependent response of the contacting materials, although some differences remain near the end of loading and during the holding period. Since the prescribed penetration varies with time, Figure 5 presents force–time histories rather than direct force–displacement curves.

More evident differences occur during unloading because the Lee–Radok formulation assumes a monotonically increasing contact area. Consequently, the unloading curves are retained to illustrate the predicted history-dependent response but are not used to establish the accuracy of detachment or restitution.

3.3. Contact Between Different Viscoelastic Materials at Different Temperatures

The proposed model and the finite element method were used to calculate the contact force histories for Cases 4–6 in Table 2. In these cases, PTFE CG 32–3 was maintained at 0 °C, whereas the temperature of unfilled PTFE was set to 15, 30, and 45 °C. The corresponding comparisons are shown in Figure 6.

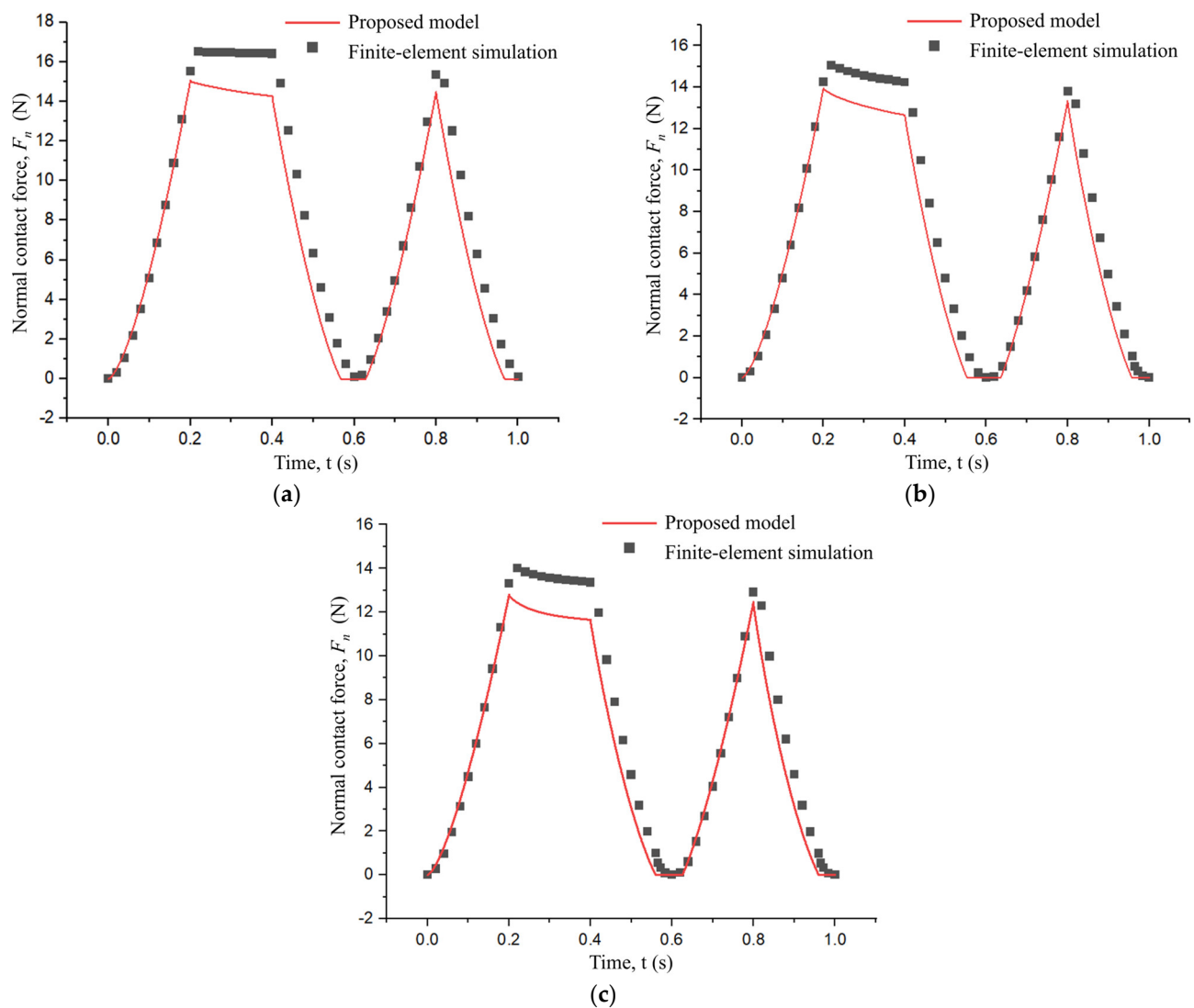


Figure 6. Comparison of contact forces for contact between different viscoelastic materials at different temperatures: (a) PTFE CG 32–3 at 0 °C and PTFE at 15 °C; (b) PTFE CG 32–3 at 0 °C and PTFE at 30 °C; (c) PTFE CG 32–3 at 0 °C and PTFE at 45 °C.

The proposed model captures the principal loading trend and the reduction in contact force associated with increasing PTFE temperature. This indicates that the combined time–temperature fitting procedure and equivalent temperature approximation can represent the main thermorheological effect in contact between bodies at different temperatures. Nevertheless, the model generally underestimates the finite element force near the end of loading and during the holding stage.

The observed differences can be partly attributed to replacing the two actual temperatures with a single fitted equivalent temperature. Additional differences may arise from the conversion of the uniaxial relaxation data into the shear and bulk Prony representations used in Abaqus and from the respective temporal discretizations. The same Lee–Radok unloading limitation identified in Section 3.2 also applies to these unequal temperature cases. Therefore, the present results primarily support loading-dominated application of the model within the investigated temperature range.

3.4. Contact Between an Elastic Material and a Viscoelastic Material at Different Temperatures

Cases 7~9 consider contact between an elastic steel body at 0 °C and PTFE CG 32-3 at 15, 30, and 45 °C. The contact force histories obtained using the proposed model and the finite element method are compared in Figure 7.

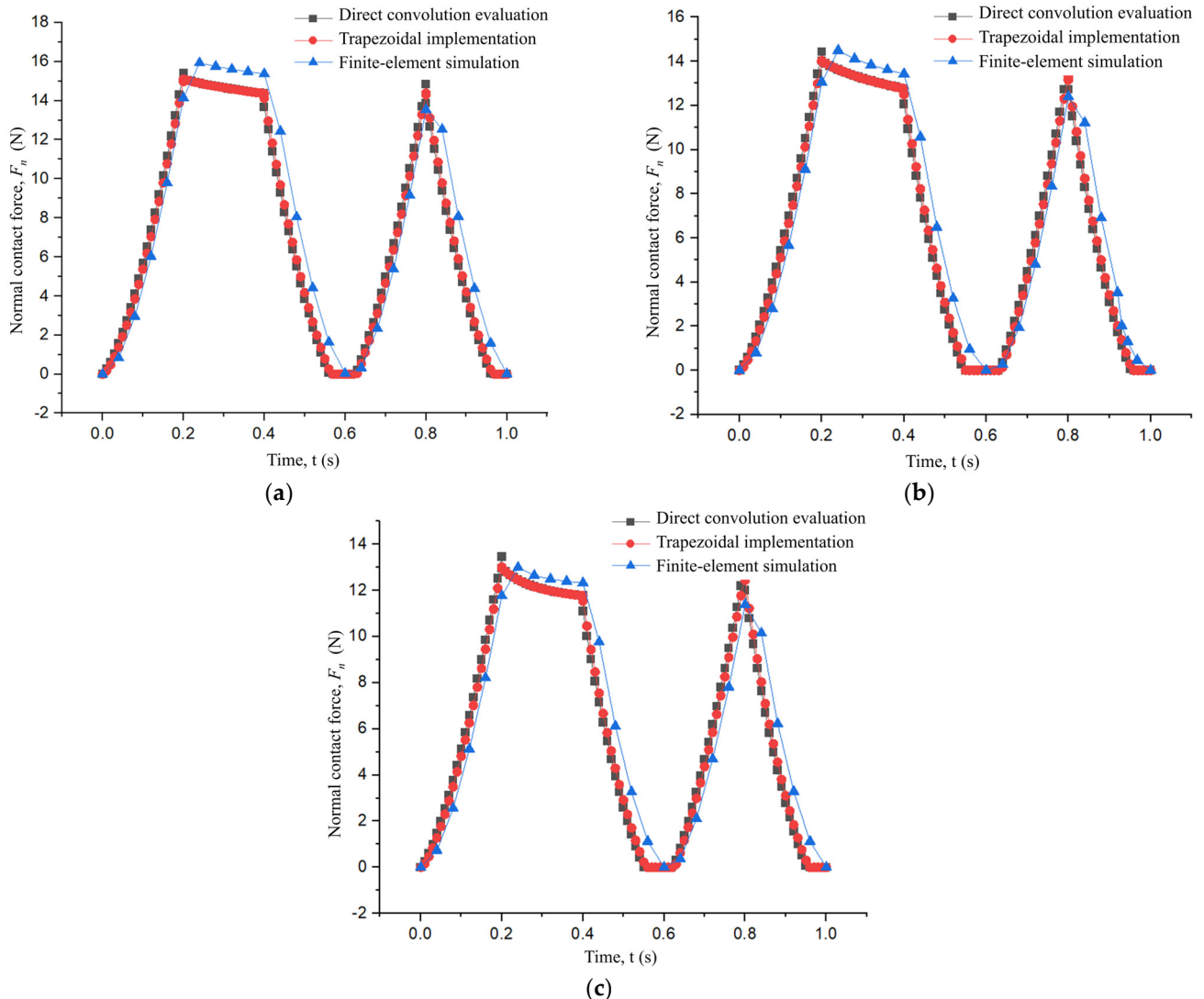


Figure 7. Comparison of contact forces for contact between an elastic material and a viscoelastic material at different temperatures: (a) Steel at 0 °C and PTFE CG 32-3 at 15 °C; (b) Steel at 0 °C and PTFE CG 32-3 at 30 °C; (c) Steel at 0 °C and PTFE CG 32-3 at 45 °C.

The results obtained from the composite-trapezoidal form of Equation (32) and the discrete convolution form of Equation (33) are nearly coincident, confirming the numerical consistency of the two implementations. Both reproduce the principal loading trend of the finite element reference results over the investigated temperature range.

For elastic–viscoelastic contact, only the viscoelastic material is temperature shifted. Therefore, the equivalent contact relaxation modulus can be fitted at the reference temperature and subsequently shifted using the WLF factor of the viscoelastic material, without determining a best fit equivalent temperature. This simplification reduces the offline identification required for unequal temperature contact between two viscoelastic materials.

Deviations remain during unloading because the simplified formulation does not account for the non-monotonic contact edge history. Moreover, no wall clock benchmark was performed; therefore, the present results demonstrate reduced offline identification

and loading force agreement rather than a quantified computational speedup or rigorous repeated contact capability.

3.5. Comparison with Existing Models and Limitations

The proposed formulation extends the equivalent contact construction of Olsson and Jelagin [21] by fitting a temperature-shifted equivalent relaxation spectrum over a prescribed time–temperature domain. Compared with the SLS-based explicit models of Jian et al. [20] and Ding et al. [22], the generalized Maxwell representation used here can describe a broader distribution of relaxation times. Unlike the generalized Maxwell approximation of Ding et al. [23], the present formulation explicitly introduces temperature through time–temperature superposition and, for two viscoelastic bodies at different temperatures, a best fit equivalent temperature approximation. A direct quantitative ranking of these models is not attempted because their materials, geometries, loading histories, and validation data differ.

The systematic unloading differences observed in Figures 5–7 are consistent with the theoretical distinction between the monotonically increasing contact assumed by Lee–Radok theory [15] and the non-monotonic contact edge histories treated by Ting and Greenwood [17,18]. Accordingly, the present formulation is intended primarily for loading-dominated contact between smooth, frictionless spheres under linear viscoelastic, small deformation conditions. It does not rigorously predict detachment, restitution, or post-separation recovery.

The present study provides internal numerical verification rather than independent physical validation. Its principal limitations are the absence of contact force experiments and a Ting/Greenwood benchmark, the lack of a formal mesh convergence series, the use of universal WLF parameters without material-specific multi-temperature characterization, and the relatively coarse time sampling used in the equivalent spectrum fitting. Within these limitations, the results demonstrate the ability of the proposed framework to represent the loading response and temperature-dependent trend of the selected contact systems under spatially uniform and constant temperatures.

4. Conclusions

A temperature-dependent discrete contact model for linear viscoelastic bodies has been developed in this study. The main conclusions are summarized as follows:

(1) The established Lee–Radok correspondence solution, equivalent contact concept, time–temperature superposition, generalized Maxwell representation, and numerical convolution were combined into a unified formulation. The principal contribution is the global fitting of a single temperature-shifted equivalent relaxation spectrum over a prescribed time–temperature domain, together with a best fit equivalent temperature approximation for two different viscoelastic bodies at unequal temperatures. This procedure avoids repeated identification of the equivalent contact spectrum at each operating temperature within the calibrated domain.

(2) Simplified formulations were obtained for special material combinations. For identical viscoelastic materials, the equivalent contact spectrum can be determined without performing the complete global fitting procedure, although an equivalent temperature may still be required when the two bodies are at different temperatures. For elastic–viscoelastic contact, the temperature dependence is governed directly by the shift factor of the viscoelastic material, and no equivalent temperature optimization is required. When the viscoelastic material is represented by an SLS model, a closed-form equivalent relaxation modulus can be obtained. The resulting hereditary contact law was implemented using full-history discrete convolution; however, no wall clock computational speedup is claimed.

(3) The globally fitted eight-branch equivalent relaxation spectrum was compared with discrete inverse Laplace reference values, giving a maximum relative discrepancy of 2.62% over the sampled domain. For two different viscoelastic materials at unequal temperatures, the maximum discrepancy introduced by the equivalent temperature approximation was 10.6%, and the discrepancy generally decreased as the temperature difference between the bodies decreased. Comparisons for the available viscoelastic–viscoelastic cases showed that the formulation reproduced the principal loading force response and the temperature-dependent trend. These comparisons constitute internal numerical verification rather than independent physical validation.

(4) Systematic differences were observed during unloading because the Lee–Radok formulation does not account for a decreasing contact area and the associated contact-edge history. The demonstrated applicability of the model is therefore restricted to loading-dominated, small deformation, frictionless spherical contact under spatially uniform and constant body temperatures. Rigorous treatment of unloading and detachment, material-specific multi-temperature characterization, mesh and time step convergence studies, and independent experimental validation remain subjects for further investigation.

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