

Article

Semi-Analytical Solution for the Steady-State Temperature Field of a Shallow-Buried Twin-Pipe Ground Freezing System

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Abstract

Accurate prediction of the temperature field around shallow-buried freezing pipes is important for the design and assessment of artificial ground freezing systems in shallow tunnel and underground structure construction. This study develops a semi-analytical series solution for the steady-state temperature field around a twin-pipe freezing unit in a semi-infinite domain under Dirichlet boundary conditions. The model assumes steady-state heat conduction in homogeneous and isotropic soil with prescribed pipe wall and ground surface temperatures; transient phase change, groundwater seepage, and thermo-hydro-mechanical coupling are not considered. Conformal mappings are used to transform the original semi-infinite and eccentric domains into bounded circular domains. The temperature field is decomposed into two subproblems by superposition, and the corresponding general solutions are expressed using Fourier series. The unknown coefficients are determined through boundary discretization and the solution of a finite linear algebraic system. The proposed method is compared with ANSYS Fluent results in the transformed domain at six selected points, yielding a pointwise NRMSE of 0.758% at a truncation order of 20. Parametric analyses further illustrate the qualitative sensitivity of the calculated temperature field to different ground surface temperatures, pipe radii, burial depths, and pipe spacings. The proposed method provides a preliminary semi-analytical framework for rapid quasi-steady temperature field estimation and qualitative parameter sensitivity analysis of shallow-buried twin-pipe artificial ground freezing systems.

Keywords: shallow-buried freezing pipes; frozen curtain; temperature field; conformal mapping; semi-analytical solution



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1. Introduction

Artificial ground freezing (AGF) is a temporary ground improvement technique in which heat is extracted from the ground through embedded freezing pipes to increase soil strength and form a low-permeability frozen barrier. Owing to its adaptability to weak and water-rich strata, AGF has been widely applied in shaft sinking, metro cross-passages, tunnel construction, foundation pit reinforcement, and groundwater control [1–3]. Recent engineering applications have further demonstrated its effectiveness in twin-tunnel construction [4], river-crossing tunnel projects [5], metro station construction [6], shallow-buried tunnel engineering [7], and metro cross-passage reinforcement projects [8,9]. Experimental investigations have also examined the thermal and deformation responses of soil–structure interfaces during freezing [10].

The performance and safety of an AGF system depend strongly on the development of the temperature field around the freezing pipes. Accurate temperature field prediction is therefore essential for estimating the formation and spatial distribution of frozen soil and for selecting appropriate pipe arrangements and operating conditions. In a freezing pipe array, the thermal interaction between adjacent pipes governs the overlap and connection of the corresponding frozen regions. Previous analytical, experimental, and numerical studies have investigated single-row, double-row, and multiple-pipe freezing configurations and have demonstrated that pipe spacing and pipe arrangement significantly affect the development of the frozen region. Therefore, two adjacent parallel freezing pipes can be treated as a representative local unit for examining the effects of pipe geometry and external thermal boundaries on the temperature field.

The freezing process is inherently transient and may involve water–ice phase change, groundwater seepage, moisture migration, frost heave, and coupled thermo-hydro-mechanical processes. Numerical methods are consequently widely used to predict frozen wall development under complex engineering conditions. Existing studies have established coupled heat-and-mass-transfer models under high seepage velocities [11,12], considered uncertainties in pipe inclination and soil thermal conductivity [13], and developed thermo-hydro-mechanical models for soil freezing and frost-heave deformation [14]. Other investigations have addressed the spatial variability of hydrothermal properties [15], high-velocity groundwater flow [16], and interaction effects among multiple freezing pipes [17]. Recent field monitoring and three-dimensional numerical studies have further improved the representation of frozen curtain development in metro cross-passages [8,9], while experimental research has shown that soil and structural interfaces may substantially affect temperature evolution and deformation during freezing [10]. Although these methods can describe the freezing process in detail, they generally require extensive material parameters, geometric modeling, mesh generation, and repeated numerical computation when boundary conditions or pipe configurations change.

During the later stage of active freezing and the maintenance stage, the growth rate of the frozen region becomes relatively slow, and the temperature field may be approximated as quasi-steady over a certain period. Under such conditions, analytical and semi-analytical methods remain useful for rapidly evaluating temperature distributions and identifying the effects of geometric and boundary parameters. Compared with purely numerical approaches, these methods can provide more explicit relationships between the governing parameters and the resulting temperature field and can reduce repeated modeling work during parametric analysis.

Analytical solutions for artificial ground freezing temperature fields have been developed using Green's functions, virtual heat sources, potential function superposition, separation of variables, and complex variable methods. Hu et al. [18] derived an analytical solution for the steady-state temperature field of a circular frozen wall formed by a single freezing pipe. Fang and Hu [19] established a generalized analytical solution for a single-row pipe freezing system, whereas Hu et al. [20] extended the method to double-row pipe freezing configurations. An analytical solution based on singular perturbation theory has also been developed for the two-phase Stefan problem [21]. More recently, adjusted analytical solutions have been proposed for the preliminary thermal design of AGF systems [22], while complex variable and potential superposition methods have been applied to annular freezing pipe layouts [23]. Analytical solutions have also been developed for large-scale freezing projects after excavation [24] and for annular double-loop freezing pipes around adjacent urban tunnels [25]. These studies demonstrate the efficiency and physical interpretability of analytical methods in ground freezing analysis.

Rapid temperature field prediction methods based on physics-informed neural networks have also emerged. Li et al. developed a physics-informed neural network model for the steady-state temperature field around a single freezing pipe [26]. Liu et al. subsequently proposed a hard-constrained framework for multi-pipe heat transfer problems, in which Dirichlet boundary conditions were explicitly embedded into the network architecture [27]. These recent developments highlight the increasing need for efficient temperature field prediction methods. However, neural network-based approaches still require network construction, iterative training, and parameter tuning, whereas an explicit semi-analytical solution remains valuable for directly revealing the relationships among pipe geometry, boundary conditions, and temperature distribution.

To clarify the novelty of the present study, a concise comparison between representative existing studies and the present work is provided in Table 1.

Table 1. Concise comparison between existing studies and the present study.

Study Group	Representative Studies	Domain and Surface Boundary	Pipe Layout and Eccentricity	Method and Validation	Main Limitation/Difference from the Present Study
Row pipe freezing solutions	Fang and Hu [19]; Hu et al. [20]	Simplified freezing domains; ground surface boundary not emphasized	Single-row or double-row pipes; eccentric mapped boundaries are not the focus	Analytical solutions; compared with theoretical, numerical, or engineering references	Useful for pipe freezing systems, but not focused on shallow-buried semi-infinite domains with explicit ground surface effects
Circular or annular freezing solutions	Hu et al. [18]; Hong et al. [23]; Zhou et al. [25]	Circular, cylindrical, or annular domains; surface boundary usually simplified	Circular or annular pipe layouts; regular geometry	Analytical or semi-analytical methods; numerical or case-based comparison	Mainly applicable to regular circular or annular configurations rather than shallow-buried parallel twin pipes
Post-excavation freezing solutions	Zhang et al. [24]	Large-scale post-excavation freezing domain; engineering boundary conditions	Project-specific pipe layout; eccentric mapped boundary is not considered	Analytical solution and optimization; engineering or numerical comparison	Focuses on post-excavation freezing rather than pre-excavation shallow-buried twin-pipe freezing
Mathematical and conformal mapping methods	Lv and Li [28]; Fan et al. [29]; Schiefermayr and Sète [30]; Ivanshin and Shirokova [31]; Hakula et al. [32]	Doubly or multiply connected mathematical domains; not AGF-specific	General boundary configurations; no specific freezing pipe layout	Analytical, boundary discretization, or conformal mapping methods; mathematical or numerical examples	Provides a mathematical basis, but lacks AGF-specific shallow-buried twin-pipe boundary treatment

Table 1. *Cont.*

Study Group	Representative Studies	Domain and Surface Boundary	Pipe Layout and Eccentricity	Method and Validation	Main Limitation/Difference from the Present Study
Rapid prediction methods	Li et al. [26]; Liu et al. [27]	AGF heat transfer domains; boundary conditions embedded in the model	Single-pipe or multi-pipe configurations; geometry is handled numerically	Physics-informed neural networks; compared with numerical or reference data	Requires network construction, training, and parameter tuning; does not provide an explicit semi-analytical solution
Present study	This work	Semi-infinite shallow-buried domain; prescribed Dirichlet ground surface temperature as an equivalent boundary	Two parallel freezing pipes; eccentric mapped pipe boundary treated explicitly	Semi-analytical series solution using conformal mapping, superposition, Fourier series, and boundary discretization; transformed domain pointwise comparison with ANSYS Fluent	Provides a preliminary semi-analytical framework for shallow-buried twin-pipe freezing; full physical domain validation, mesh/boundary independence, full-field error assessment, and quantitative frozen region indicators require further study

As shown in Table 1, existing studies have provided valuable solutions for pipe freezing systems, circular or annular freezing configurations, post-excavation freezing problems, general heat conduction domains, and rapid prediction methods. However, the steady-state temperature field around two shallow-buried parallel freezing pipes in a semi-infinite domain with a prescribed ground surface temperature has not been sufficiently addressed. In particular, the eccentric boundary generated after conformal mapping requires additional treatment. In a shallow-buried freezing system, the ground surface acts as an important thermal boundary and may result in a vertically asymmetric temperature field and frozen region. Therefore, a semi-analytical method that explicitly accounts for the ground surface boundary and permits rapid variation in pipe radius, burial depth, and pipe spacing is required.

The main novelty of this study lies in establishing an explicit semi-analytical framework for a shallow-buried twin-pipe freezing unit in a semi-infinite domain, where the ground surface boundary and the eccentric mapped pipe boundary are treated simultaneously. This distinguishes the present solution from existing analytical solutions for regular pipe arrangements or annular domains and provides a direct basis for rapid preliminary evaluation of geometric and boundary condition effects.

In this study, a semi-analytical series solution is developed for the steady-state temperature field around two shallow-buried parallel freezing pipes in a semi-infinite domain

under Dirichlet boundary conditions. The twin-pipe configuration is treated as a representative freezing unit in shallow tunnel and underground structure construction. Multiple conformal mappings are used to transform the original semi-infinite and eccentric domains into bounded circular domains. The temperature field is decomposed into two subproblems using the superposition method, and the corresponding general solutions are expressed using Fourier series. A boundary discretization method is subsequently employed to determine the unknown coefficients by converting the continuous boundary conditions into a finite system of linear algebraic equations. The proposed solution is compared with ANSYS Fluent results in the transformed domain and is then applied to investigate the qualitative effects of ground surface temperature, pipe radius, burial depth, and pipe spacing under prescribed quasi-steady assumptions.

2. Materials and Methods

2.1. Mathematical Model

The configuration shown in Figure 1 represents a basic twin-pipe freezing unit that can form part of a larger frozen curtain system in shallow tunnel or underground structure construction. To simplify the analytical derivation of the temperature field around this twin-pipe unit, the original physical domain is first transformed by conformal mapping. As shown in Figure 1, two freezing pipes with the same radius are arranged parallel to each other in a semi-infinite soil domain. The ground surface is taken as the x -axis of the original Cartesian coordinate system, with the positive y -axis directed downward. The centers of the two freezing pipes are located at the same burial depth. In the original physical plane, r denotes the radius of the freezing pipe, h denotes the burial depth from the ground surface to the pipe center, and d denotes the center-to-center distance between the two freezing pipes.

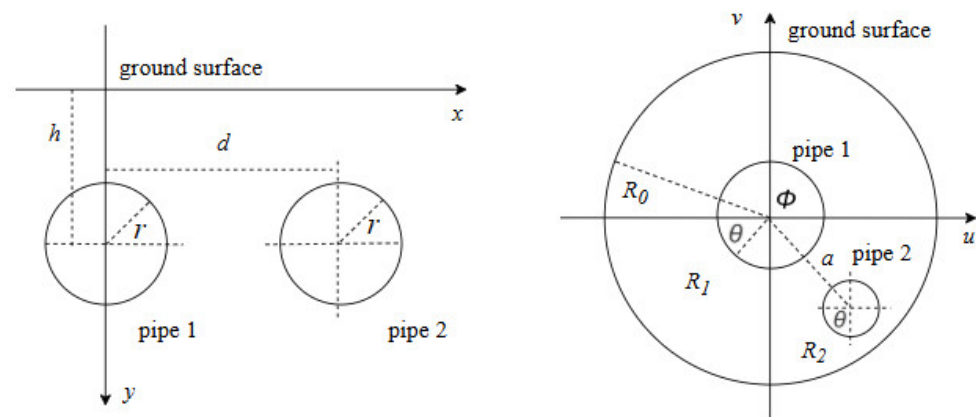


Figure 1. Geometric configurations in the physical plane and the w_1 -plane before and after the first conformal mapping.

The complex coordinate in the original physical plane is defined as

$$z = x + iy \quad (1)$$

where z denotes the complex variable in the original physical plane, x and y are the Cartesian coordinates, and i is the imaginary unit.

For compactness, the following auxiliary parameter is introduced:

$$\eta = \sqrt{h^2 - r^2} \quad (2)$$

To transform the semi-infinite physical plane into a bounded circular domain, the first conformal mapping is introduced as

$$w_1 = Re^{i\psi} = u + iv = \frac{(x + iy) - i\eta}{(x + iy) + i\eta} \quad (3)$$

where w_1 denotes the complex variable after the first conformal mapping; R and ψ are the dimensionless radial and angular coordinates of an arbitrary point in the w_1 -plane, respectively; and u, v are the corresponding Cartesian coordinate components in the mapped plane.

Since w_1 is defined as the ratio of two complex quantities with the same length dimension, the mapped coordinates u and v , together with the geometric parameters R_0 , R_1 , R_2 , and a in the w_1 -plane, are dimensionless.

This mapping converts the original semi-infinite region into a finite circular region in the w_1 -plane. The ground surface boundary in the original physical plane is mapped onto a unit circle in the w_1 -plane, whose radius is given by $R_0 = 1$.

Here, R_0 denotes the dimensionless radius of the mapped ground surface boundary. Meanwhile, the boundary of pipe 1 is transformed into a circle concentric with the mapped ground surface circle. The corresponding radius is

$$R_1 = \frac{r}{h + \eta} \quad (4)$$

where R_1 denotes the dimensionless radius of the mapped circle corresponding to pipe 1. Unlike pipe 1, pipe 2 is transformed into an eccentric circle in the mapped plane. Its dimensionless radius is denoted by R_2 , which can be expressed as

$$R_2 = \frac{2r\eta}{d^2 + 2\eta(h + \eta)} \quad (5)$$

The dimensionless distance from the center of the mapped circle corresponding to pipe 2 to the origin of the w_1 -plane is denoted by a , and is given by

$$a = \frac{d\sqrt{d^2 + 4\eta^2}}{d^2 + 2\eta(h + \eta)} \quad (6)$$

The angular position of the center of the mapped circle of pipe 2 is expressed as

$$\Phi = \arctan\left(\frac{2\eta}{d}\right) \quad (7)$$

where Φ denotes the angular position of the center of the mapped circle corresponding to pipe 2 with respect to the center of the mapped ground surface circle. Therefore, after the first conformal mapping, the ground surface is represented by a unit circle with radius $R_0 = 1$, pipe 1 is represented by a concentric circle with dimensionless radius R_1 , and pipe 2 is represented by an eccentric circle characterized by the dimensionless radius R_2 , center distance a , and angular position Φ . This transformation establishes the geometric basis for the subsequent formulation of the dimensionless boundary value problem. In Figure 1, θ denotes the parametric angle of an arbitrary point on the mapped pipe boundary, and its discrete form is denoted by θ_n in the subsequent boundary discretization.

The conformal mappings used in this study are valid under the following geometric conditions. In the present formulation, the pipe centers must be located below the ground surface with $h > r$, and the two freezing pipes must not overlap, i.e., $d > 2r$. In addition, the mapping parameters should remain finite so that no singularity is introduced

in the transformed domains. In the calculation procedure, physical points are first transformed from the original z -plane to the w_1 -plane using Equation (3). For points associated with subproblem 2, the rotated coordinates are then transformed to the w_2 -plane using Equations (14)–(17). The obtained mapped coordinates are substituted into the truncated Fourier series solutions and the final temperature is recovered through the dimensionless temperature definition in Equation (8).

2.2. General Solution

After the first conformal mapping, the original semi-infinite physical domain is transformed into a bounded circular region in the w_1 -plane. Since the steady-state heat conduction problem considered in this study contains no internal heat source, the temperature field satisfies the Laplace equation. To simplify the subsequent derivation, the temperature is expressed in dimensionless form as

$$T = \frac{t - t_0}{t_1 - t_0} \quad (8)$$

where T is the dimensionless temperature, t is the physical temperature, t_0 is the ground surface temperature, and t_1 is the wall temperature of pipe 1. According to this definition, $T = 0$ corresponds to the mapped ground surface boundary, and $T = 1$ corresponds to the mapped boundary of pipe 1.

In the w_1 -plane, the dimensionless Laplace equation in the polar coordinate system can be written as

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial T}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 T}{\partial \psi^2} = 0 \quad (9)$$

where R and ψ denote the dimensionless radial coordinate and angular coordinate of a point in the w_1 -plane, respectively. The corresponding dimensionless boundary conditions are given by

$$\begin{cases} T|_{\Gamma_0} = 0 \\ T|_{\Gamma_1} = 1 \\ T|_{\Gamma_2} = \frac{t_2 - t_0}{t_1 - t_0} \end{cases} \quad (10)$$

where Γ_0 , Γ_1 , and Γ_2 denote the mapped boundaries of the ground surface, pipe 1, and pipe 2, respectively, and t_2 is the wall temperature of pipe 2. If the two pipes have the same wall temperature, then $t_2 = t_1$, and the dimensionless temperature on the boundary of pipe 2 is also equal to 1.

It should be noted that the above formulation allows different pipe wall temperatures for the two freezing pipes. However, the transformed domain numerical comparison and parametric analysis in the present study focus on the common engineering case with equal pipe wall temperatures, i.e., $t_1 = t_2$.

Because the governing equation is linear, the total dimensionless temperature field can be decomposed into two temperature components by the superposition method:

$$T = T'_1 + T'_2 \quad (11)$$

where T'_1 and T'_2 are the temperature components associated with pipe 1 and pipe 2, respectively. Each subproblem satisfies the Laplace equation in the mapped domain.

For subproblem 1, the governing equation and boundary condition are

$$\begin{cases} \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial T'_1}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 T'_1}{\partial \psi^2} = 0 \\ T'_1|_{R_0} = 0 \end{cases} \quad (12)$$

For subproblem 2, the governing equation and boundary condition are similarly written as:

$$\begin{cases} \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial T'_2}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 T'_2}{\partial \psi^2} = 0 \\ T'_2|_{R_0} = 0 \end{cases} \quad (13)$$

In the w_1 -plane, pipe 1 is represented by a circle concentric with the mapped ground surface boundary, whereas pipe 2 is represented by an eccentric circle. Therefore, it is difficult to directly solve subproblem 2 using the separation-of-variables and Fourier series methods. To overcome this problem, the second conformal mapping w_2 is introduced to transform the eccentric circular boundary of pipe 2 into a concentric circular boundary, as shown in Figure 2.

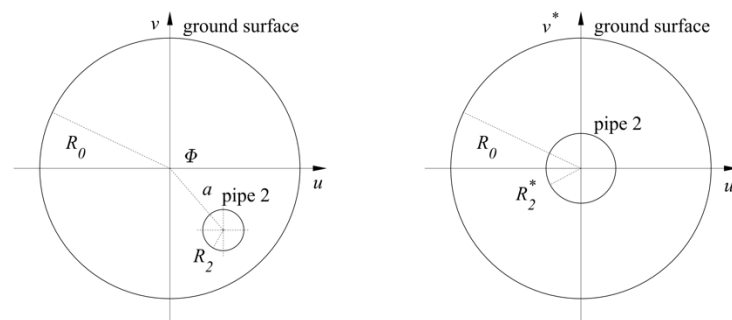


Figure 2. Geometric configurations in the w_1 - and w_2 -planes before and after the second conformal mapping.

Before the second conformal mapping, the coordinate system is rotated by the angle Φ . The Cartesian coordinates in the rotated coordinate system are given by

$$\begin{cases} u' = u \cos \Phi + v \sin \Phi \\ v' = v \cos \Phi - u \sin \Phi \end{cases} \quad (14)$$

where u' and v' denote the Cartesian coordinates in the rotated coordinate system.

The second conformal mapping is then defined as

$$w_2 = R^* e^{i\psi^*} = u^* + iv^* = \frac{u' + iv' - \lambda}{1 - \lambda(u' + iv')} \quad (15)$$

$$\lambda = \frac{1 + a_1 a_2 - \sqrt{(1 - a_1^2)(1 - a_2^2)}}{a_1 + a_2} \quad (16)$$

$$\begin{cases} a_1 = \frac{a - R_2}{R_0} \\ a_2 = \frac{a + R_2}{R_0} \end{cases} \quad (17)$$

where u^* and v^* are the Cartesian coordinates in the w_2 -plane; R^* and ψ^* are the corresponding dimensionless radial and angular coordinates, respectively; λ is the mapping parameter used to transform the eccentric boundary of pipe 2 into a concentric circular boundary; and a_1 and a_2 are the dimensionless u' -coordinates of the two intersection points between the mapped boundary of pipe 2 and the u' -axis after coordinate rotation.

After the second conformal mapping, the governing equation and boundary condition for subproblem 2 can be rewritten in the w_2 -plane as

$$\begin{cases} \frac{1}{R^*} \frac{\partial}{\partial R^*} \left(R^* \frac{\partial T'_2}{\partial R^*} \right) + \frac{1}{(R^*)^2} \frac{\partial^2 T'_2}{\partial (\psi^*)^2} = 0 \\ T'_2|_{R^*=R_0^*} = 0 \end{cases} \quad (18)$$

where R_0^* denotes the transformed outer boundary in the w_2 -plane after the second conformal mapping. It should be distinguished from $R_0 = 1$, which denotes the mapped ground surface boundary in the w_1 -plane.

Using the method of separation of variables, the general solutions of the two subproblems can be expressed as Fourier series. For subproblem 1, the general solution is

$$T_1' = a_0^1 \ln(R) + \sum_{k=1}^{\infty} A_k^1 (R^k - R^{-k}) \cdot \sin(k\psi) + \sum_{k=1}^{\infty} B_k^1 (R^k - R^{-k}) \cdot \cos(k\psi) \quad (19)$$

For subproblem 2, after the second conformal mapping, the general solution is

$$T_2' = a_0^2 \ln(R^*) + \sum_{k=1}^{\infty} A_k^2 ((R^*)^k - (R^*)^{-k}) \cdot \sin(k\psi^*) + \sum_{k=1}^{\infty} B_k^2 ((R^*)^k - (R^*)^{-k}) \cdot \cos(k\psi^*) \quad (20)$$

where a_0^1 and a_0^2 are the logarithmic term coefficients; A_k^1 , B_k^1 , A_k^2 , and B_k^2 are the Fourier coefficients to be determined. k is the Fourier series index; R and ψ are the polar coordinates in the w_1 -plane; and R^* and ψ^* are the corresponding polar coordinates in the w_2 -plane. These coefficients cannot be determined independently because the two pipe boundaries act simultaneously in the same mapped domain. Therefore, the boundary conditions on both pipe boundaries need to be imposed simultaneously.

For practical calculations, the infinite Fourier series are truncated at order m , namely $k = 1, 2, \dots, m$, where m is the truncation order and also controls the number of boundary discretization points. To determine the unknown coefficients, the boundary discretization method is adopted. Each pipe boundary is uniformly divided into $2m + 1$ discrete points.

For the mapped boundary of pipe 1, the coordinates of the n -th discrete point in the w_1 -plane are expressed as

$$\begin{cases} u_n = R_1 \cdot \cos(\theta_n) \\ v_n = R_1 \cdot \sin(\theta_n) \end{cases} \quad (21)$$

For the mapped boundary of pipe 2, the coordinates of the discrete points in the w_1 -plane are expressed as

$$\begin{cases} u_n = R_2 \cdot \cos(\theta_n) + a \cdot \cos(\Phi) \\ v_n = R_2 \cdot \sin(\theta_n) + a \cdot \sin(\Phi) \end{cases} \quad (22)$$

where $\theta_n = \frac{2\pi \cdot n}{2m+1}$, $n = 0, 1, 2, \dots, 2m$

Here, u_n and v_n are the Cartesian coordinates of the n -th discrete point in the w_1 -plane; θ_n is the parametric angle used for boundary discretization; R_1 and R_2 are the dimensionless radii of the mapped circles corresponding to pipe 1 and pipe 2, respectively; a is the dimensionless distance from the center of the mapped circle of pipe 2 to the origin of the w_1 -plane; and Φ is the angular position of the center of the mapped circle corresponding to pipe 2. The parameter θ_n should be distinguished from ψ_n and ψ_n^* , which denote the angular coordinates of the discrete points in the w_1 -plane and w_2 -plane, respectively.

After the discrete boundary points are obtained, their corresponding polar coordinates in the w_1 -plane and w_2 -plane can be determined and substituted into the general solutions of T_1' and T_2' . For the discrete points on the boundary of pipe 1, the total dimensionless temperature satisfies

$$T_1' + T_2' = 1 \quad (23)$$

that is,

$$\begin{aligned} 1 = & a_0^1 \ln(R_n) + a_0^2 \ln(R_n^*) + \sum_{k=1}^m A_k^1 (R_n^k - R_n^{-k}) \cdot \sin(k\psi_n) + \sum_{k=1}^m B_k^1 (R_n^k - R_n^{-k}) \cdot \cos(k\psi_n) \\ & + \sum_{k=1}^m A_k^2 (R_n^{*k} - R_n^{*-k}) \cdot \sin(k\psi_n^*) + \sum_{k=1}^m B_k^2 (R_n^{*k} - R_n^{*-k}) \cdot \cos(k\psi_n^*) \end{aligned} \quad (24)$$

For the discrete points on the boundary of pipe 2, the total dimensionless temperature satisfies

$$T'_1 + T'_2 = \frac{t_2 - t_0}{t_1 - t_0} \quad (25)$$

that is,

$$\begin{aligned} \frac{t_2 - t_0}{t_1 - t_0} = & a_0^1 \ln(R_n) + a_0^2 \ln(R_n^*) + \sum_{k=1}^m A_k^1 (R_n^k - R_n^{-k}) \cdot \sin(k\psi_n) + \sum_{k=1}^m B_k^1 (R_n^k - R_n^{-k}) \cdot \cos(k\psi_n) \\ & + \sum_{k=1}^m A_k^2 (R_n^{*k} - R_n^{*-k}) \cdot \sin(k\psi_n^*) + \sum_{k=1}^m B_k^2 (R_n^{*k} - R_n^{*-k}) \cdot \cos(k\psi_n^*) \end{aligned} \quad (26)$$

where R_n and ψ_n are the radial and angular coordinates of the n -th discrete point in the w_1 -plane, and R_n^* and ψ_n^* are the corresponding coordinates after the second conformal mapping.

The two sets of discrete boundary equations form a finite linear algebraic system. Since each pipe boundary is uniformly discretized into $2m + 1$ points, the two pipe boundaries provide $4m + 2$ equations in total. These equations correspond to the $4m + 2$ unknown coefficients in the truncated Fourier series solution. The resulting linear system can be written in compact matrix form as

$$\mathbf{M}\mathbf{c} = \mathbf{b} \quad (27)$$

where $\mathbf{M}$ is the coefficient matrix assembled from the logarithmic, sine, and cosine terms evaluated at all boundary discretization points, $\mathbf{c}$ is the vector of unknown coefficients, and $\mathbf{b}$ is determined by the prescribed dimensionless temperatures on the pipe boundaries. The coefficient vector is expressed as

$$\mathbf{c} = [a_0^1, a_0^2, A_1^1, B_1^1, \dots, A_m^1, B_m^1, A_1^2, B_1^2, \dots, A_m^2, B_m^2]^T \quad (28)$$

If the discrete equations on pipe 1 are arranged first and those on pipe 2 are arranged second, the right-hand-side vector can be written as

$$\mathbf{b} = \left[\underbrace{1, \dots, 1}_{2m+1}, \underbrace{\frac{t_2 - t_0}{t_1 - t_0}, \dots, \frac{t_2 - t_0}{t_1 - t_0}}_{2m+1} \right]^T \quad (29)$$

For the common case considered in the transformed domain numerical comparison and parametric analysis, the two freezing pipes have the same wall temperature, i.e., $t_1 = t_2$. In this case, all entries of $\mathbf{b}$ are equal to 1.

Each row of $\mathbf{M}$ is obtained by substituting one boundary discretization point into the truncated forms of Equations (19) and (20). By solving this linear system, the unknown coefficients can be obtained. Substituting these coefficients back into the truncated Fourier series expressions yields the semi-analytical solution of the steady-state temperature field.

Increasing the truncation order m generally improves the approximation accuracy of the boundary conditions, but it also increases the size of the coefficient matrix and may affect numerical conditioning. Therefore, m should be selected by balancing solution accuracy and numerical stability. In the present study, the convergence analysis in Section 3 is used to determine an appropriate truncation order. The overall computational procedure of the proposed semi-analytical method is summarized in Algorithm 1.

Algorithm 1. Computational procedure of the proposed semi-analytical method

Input: pipe radius r , burial depth h , pipe spacing d , ground surface temperature t_0 , pipe wall temperatures t_1 and t_2 , and truncation order m .

Step 1: Calculate the auxiliary parameter η and the mapping parameters R_1 , R_2 , a , and Φ using Equations (2)–(7).

Step 2: Generate $2m + 1$ uniformly distributed boundary points on each mapped pipe boundary according to Equations (21) and (22).

Step 3: For each boundary point, calculate its polar coordinates in the w_1 -plane. For points associated with pipe 2, perform the coordinate rotation and the second conformal mapping to obtain the corresponding coordinates in the w_2 -plane.

Step 4: Substitute the mapped boundary coordinates into the truncated Fourier series expressions of T'_1 and T'_2 .

Step 5: Assemble the coefficient matrix $\mathbf{M}$ and the right-hand-side vector $\mathbf{b}$ according to the prescribed pipe wall temperatures.

Step 6: Solve the linear algebraic system $\mathbf{M}\mathbf{c} = \mathbf{b}$ to obtain the unknown Fourier coefficients.

Step 7: Substitute the obtained coefficients into the truncated series to reconstruct the dimensionless temperature field.

Step 8: Convert the dimensionless temperature T to the physical temperature t using Equation (8).

Step 9: Extract and plot the calculated 0°C isotherm for frozen region interpretation.

3. Transformed Domain Numerical Comparison and Convergence Analysis

A transformed domain numerical comparison was conducted using ANSYS Fluent to examine the implementation consistency of the proposed semi-analytical formulation and its convergence with respect to the truncation order m . It should be noted that the numerical model was constructed in the first mapped domain rather than in the original physical domain. Therefore, the comparison presented in this section should be interpreted as a transformed domain pointwise comparison, not as a fully independent physical domain validation of the complete conformal mapping procedure.

The radius of each pipe was set to 0.10 m, the pipe-center burial depth was 0.75 m, and the center-to-center pipe spacing was 0.50 m. The wall temperature of both pipes was prescribed as -7°C , and the ground surface temperature was prescribed as 12°C .

A numerical model of the first mapped domain, i.e., the w_1 -plane, was constructed in ANSYS DesignModeler and solved with ANSYS Fluent (Ansys, Inc., Canonsburg, PA, USA; <https://www.ansys.com/products/fluids/ansys-fluent>, accessed on 11 June 2026) to provide reference results. In this study, ANSYS Fluent was not used for flow simulation but as an independent finite volume solver for the steady-state heat conduction equation. Fluent was selected because it can solve the energy equation in complex domains with curved boundaries, supports unstructured meshes and local refinement, and allows convenient extraction of temperature values at specified points. Therefore, the Fluent results were used as numerical reference data for checking the transformed domain solution and the convergence behavior of the truncated Fourier series formulation. As shown in Figure 3, the mapped ground surface boundary and each pipe boundary were discretized into 100 segments. An unstructured triangular mesh was generated within the computational domain, with local refinement applied near the pipe boundaries to resolve the relatively large temperature gradients.

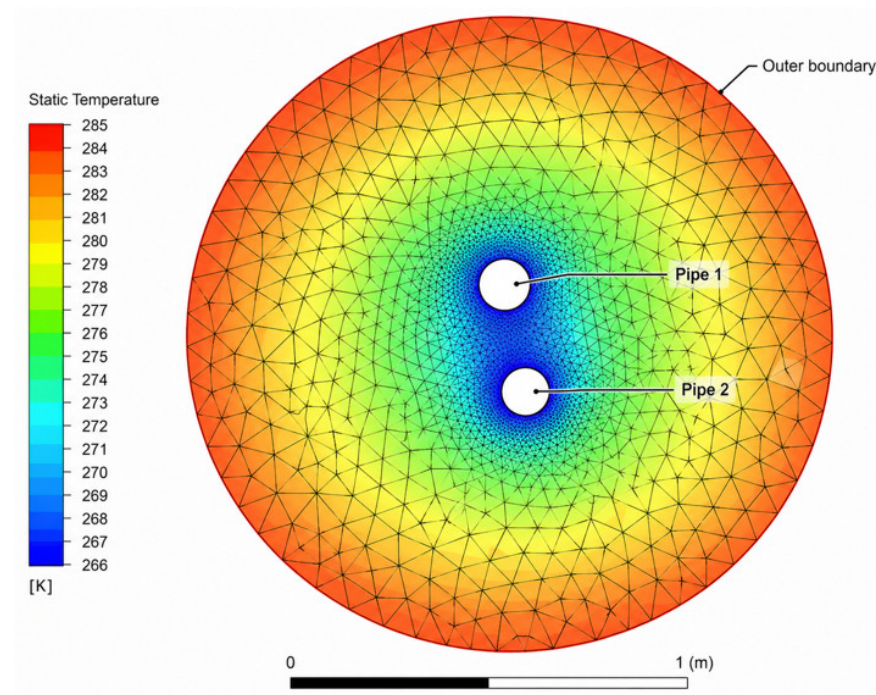


Figure 3. Transformed domain numerical model in the first mapped domain: steady-state temperature contours and locally refined computational mesh near the mapped freezing pipe boundaries.

The steady-state energy equation was solved for a homogeneous and isotropic soil domain with constant thermal conductivity and no internal heat generation. Since only Dirichlet temperature boundary conditions were imposed, the normalized temperature distribution is independent of the absolute value of thermal conductivity.

Local mesh refinement was applied around the two pipe boundaries, where relatively large temperature gradients occur. The residual convergence criterion for the energy equation was set to 10^{-6} , and the monitored temperatures at the selected test points were checked to ensure that the transformed domain numerical solution reached a stable steady state. However, because the present comparison was mainly intended to examine the transformed domain implementation and series convergence, a systematic mesh independence study was not included in this section. Therefore, the numerical results should not be interpreted as a mesh-independent physical domain benchmark.

Six test points were uniformly selected along the horizontal centerline connecting the two pipe centers in the original Cartesian coordinate system. Their x -coordinates ranged from 0.20 to 0.30 m at intervals of 0.02 m, while the y -coordinate was fixed at 0.75 m. The test points in the original physical plane were transformed into the corresponding locations in the w_1 -plane using the first conformal mapping before the numerical temperatures were extracted. The temperatures obtained from ANSYS Fluent in kelvin were converted to degrees Celsius before comparison with the semi-analytical results. The temperature results obtained from the two methods at the six test points were then compared. The truncation order was varied from $m = 5$ to $m = 20$ in increments of 1, and the resulting temperatures are compared in Figure 4.

As shown in Figure 4, the semi-analytical results exhibit noticeable fluctuations at low truncation orders but rapidly converge toward the numerical results as m increases. These fluctuations are mainly caused by the truncation of the Fourier series and the limited number of boundary discretization points at low truncation orders. When m is small, the truncated series cannot fully represent the boundary temperature variation and the relatively large temperature gradients near the pipe boundaries, resulting in pointwise fluctuations. As m increases, more Fourier terms and boundary points are included, the

boundary conditions are better approximated, and the fluctuations are rapidly reduced. When $m \geq 8$, the calculated temperatures remain nearly unchanged and agree closely with the numerical results at all six test points. These results indicate that the truncated semi-analytical solution achieves good pointwise agreement with the transformed domain numerical reference results under the considered comparison condition.

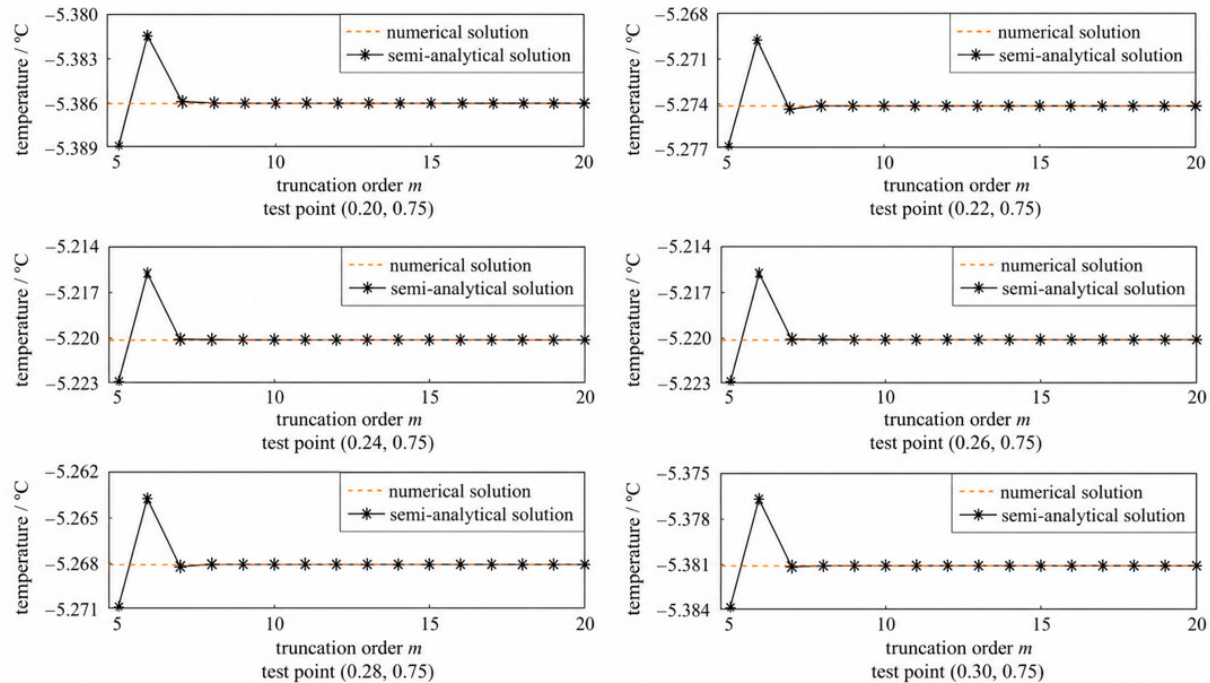


Figure 4. Comparison between the semi-analytical and numerical temperatures at six test points for different truncation orders ($r = 0.10$ m, $h = 0.75$ m, $d = 0.50$ m, $t_1 = t_2 = -7$ °C, and $t_0 = 12$ °C).

The normalized root mean square error (NRMSE) was used to quantify the difference between the semi-analytical and numerical solutions and was calculated as

$$\text{NRMSE} = \frac{\sqrt{\frac{1}{N} \sum_{j=1}^N (t_{a,j} - t_{n,j})^2}}{t_{\max} - t_{\min}} \times 100\% \quad (30)$$

where $N = 6$; $t_{a,j}$ and $t_{n,j}$ denote the semi-analytical and numerical temperatures at the j -th test point, respectively; and $t_{\max}$ and $t_{\min}$ are the maximum and minimum numerical temperatures among the six test points, respectively. It should be emphasized that the NRMSE defined in Equation (30) is a pointwise error indicator based only on the six selected test points. It does not represent a full-domain error measure of the reconstructed temperature field in the original physical domain. Therefore, the present comparison is used only to examine local agreement and convergence behavior under the selected transformed domain condition.

The pointwise comparison in Table 2 further shows close agreement between the semi-analytical and numerical solutions at $m = 20$. According to the convergence curves in Figure 4, the semi-analytical temperatures at the six selected test points almost coincide with the corresponding numerical results, with absolute deviations within approximately 0.002 °C. The NRMSE calculated from the six selected test points is 0.758%, indicating close pointwise agreement under the considered transformed domain comparison condition. However, this value should not be interpreted as a full-field accuracy indicator because the comparison does not include global temperature field errors or 0 °C isotherm position errors.

Table 2. Pointwise comparison between transformed domain numerical temperatures and rounded semi-analytical temperatures at selected test points when $m = 20$.

Test Point	(x)/m	(y)/m	Numerical Result/°C	Semi-Analytical Result/°C	Absolute Error/°C	Relative Error/%
P1	0.2	0.75	−5.38698	−5.387	<0.002	<0.04
P2	0.22	0.75	−5.27445	−5.274	<0.002	<0.04
P3	0.24	0.75	−5.21899	−5.219	<0.002	<0.04
P4	0.26	0.75	−5.21899	−5.219	<0.002	<0.04
P5	0.28	0.75	−5.27445	−5.274	<0.002	<0.04
P6	0.3	0.75	−5.38698	−5.387	<0.002	<0.04

4. Parametric Analysis and Discussion

4.1. Parametric Effects and Engineering Implications

For engineering interpretation purposes, the two-pipe configuration is treated as a basic freezing unit, and the calculated frozen bridge between the two pipes is used to describe the local connectivity of the predicted frozen region under the quasi-steady assumptions of the present model. In the following parametric analysis, the frozen region is defined as the area enclosed by the calculated 0 °C isotherm. Unless otherwise specified, the pipe radius, burial depth, pipe spacing, and pipe wall temperature were fixed at 0.10 m, 0.75 m, 0.50 m, and −7 °C, respectively. The prescribed ground surface temperatures from 10 °C to 35 °C are used here as equivalent Dirichlet boundary temperatures for sensitivity analysis. They are not intended to represent instantaneous measured surface temperatures at a specific project site. Instead, they represent different possible surface thermal states that may arise from seasonal environmental conditions, surface structures, or heat exchange between the ground surface and the surroundings. Similarly, the pipe wall temperature of −7 °C is prescribed as a representative low-temperature boundary for the simplified quasi-steady analysis. Therefore, the following results should be interpreted as sensitivity trends under equivalent boundary conditions rather than direct predictions for a specific field project. For the subsequent analyses, the ground surface temperature was fixed at 12 °C. In each case, only one parameter was varied while the others were held constant. In shallow artificial ground freezing projects, heat exchange through the ground surface may be affected by ambient conditions and nearby structures. The resulting ground surface temperature can therefore substantially influence the temperature field, frozen bridge geometry, and local connectivity of the calculated frozen region. It should be noted that the following parametric cases are semi-analytical sensitivity calculations under prescribed assumptions. They are not independent numerical validation cases. Therefore, the results are used to illustrate the qualitative influence of each parameter on the calculated 0 °C isotherm, rather than to prove the accuracy of the method over the entire parameter range.

To clarify the engineering interpretation of the calculated frozen region morphology, several design-oriented scalar indicators can be defined from the calculated 0 °C isotherm. The frozen bridge thickness at the midpoint, denoted as B_f , describes the local thermal connection between the two pipe-associated frozen zones. The area enclosed by the calculated 0 °C isotherm, denoted as A_f , represents the overall size of the predicted frozen region. The upper and lower extents of the calculated frozen region, denoted as H_u and H_l , describe the vertical development of the 0 °C isotherm toward the ground surface and deeper soil, respectively. The minimum distance between two pipe-associated frozen zones, denoted as $D_{\min}$, can be used to identify whether the two frozen zones are thermally connected. The critical pipe spacing, denoted as d_{cr} , corresponds to the spacing at which the calculated frozen region changes from connected to disconnected. In the present study,

these quantities are used as interpretive indicators for discussing the contour results, while their systematic numerical extraction is reserved for future quantitative post-processing.

The ground surface temperature was varied from 10 °C to 35 °C in increments of 5 °C, and the corresponding steady-state temperature contours were calculated using the semi-analytical solution. As shown in Figure 5, the calculated contours suggest that the temperature gradient near the ground surface is stronger than that in the deeper soil, leading to a vertically asymmetric calculated frozen region. As the prescribed ground surface temperature increases, the upper part of the calculated 0 °C isotherm tends to move downward, while the lower part changes less noticeably. This indicates a reduction in the apparent frozen bridge thickness at the midpoint under the present quasi-steady assumptions. This trend can be attributed to the stronger equivalent heat input from the warmer ground surface boundary, which suppresses the upward development of the calculated frozen region.

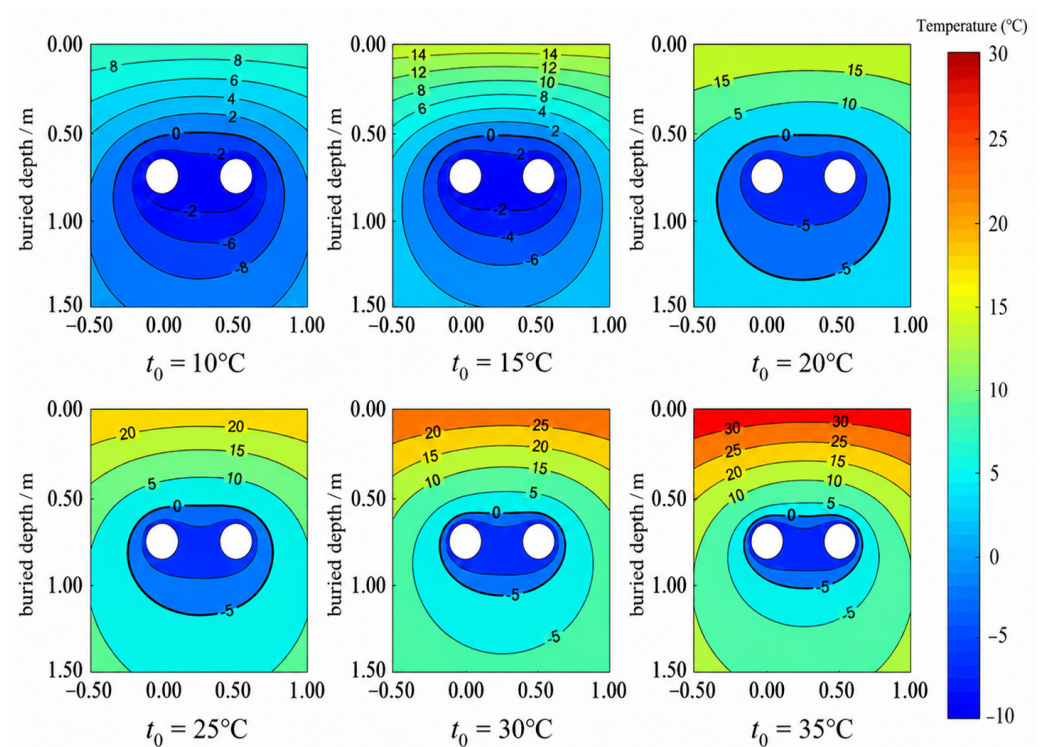


Figure 5. Steady-state temperature contours under different ground surface temperatures ($t_0 = 10\text{--}35\text{ }^{\circ}\text{C}$), with $r = 0.10\text{ m}$, $h = 0.75\text{ m}$, $d = 0.50\text{ m}$, and $t_1 = t_2 = -7\text{ }^{\circ}\text{C}$. The bold contour denotes the calculated $0\text{ }^{\circ}\text{C}$ isotherm, and all panels use identical contour intervals and coordinate units of meters.

The effects of pipe radius, burial depth, and pipe spacing were subsequently investigated using the same one-factor-at-a-time approach.

The pipe radius was varied from 0.10 m to 0.20 m in increments of 0.02 m. As shown in Figure 6, the calculated contours indicate that increasing the pipe radius tends to enlarge the predicted frozen region. The expansion is more evident below the pipes than above them because the prescribed ground surface temperature constrains the upward development of the calculated $0\text{ }^{\circ}\text{C}$ isotherm. From the perspective of the scalar indicators defined above, a larger pipe radius would be expected to increase the area enclosed by the calculated $0\text{ }^{\circ}\text{C}$ isotherm and the lower extent of the predicted frozen region under the same boundary conditions.

The burial depth was varied from 0.75 m to 1.25 m in increments of 0.10 m. As shown in Figure 7, increasing the burial depth weakens the influence of the prescribed

ground surface temperature on the temperature field around the pipes. The calculated 0°C isotherm becomes more uniformly distributed around the two pipes, and the upper portion of the predicted frozen region becomes less compressed by the surface boundary. This suggests that the upper and lower extents of the calculated frozen region become more balanced as the pipe centers move farther away from the ground surface. However, this thermal trend should be interpreted together with construction constraints because a greater burial depth may increase drilling difficulty and construction cost.

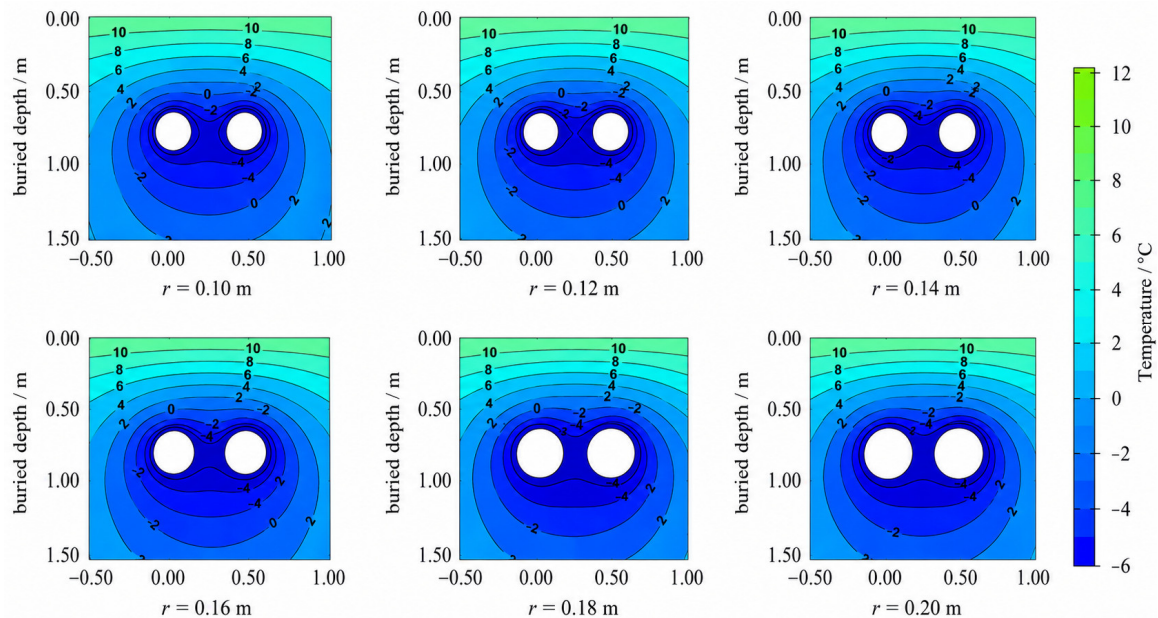


Figure 6. Steady-state temperature contours under different pipe radii ($r = 0.10\text{--}0.20\text{ m}$), with $h = 0.75\text{ m}$, $d = 0.50\text{ m}$, $t_1 = t_2 = -7^{\circ}\text{C}$, and $t_0 = 12^{\circ}\text{C}$. The bold contour denotes the calculated 0°C isotherm, and all panels use identical contour intervals and coordinate units of meters.

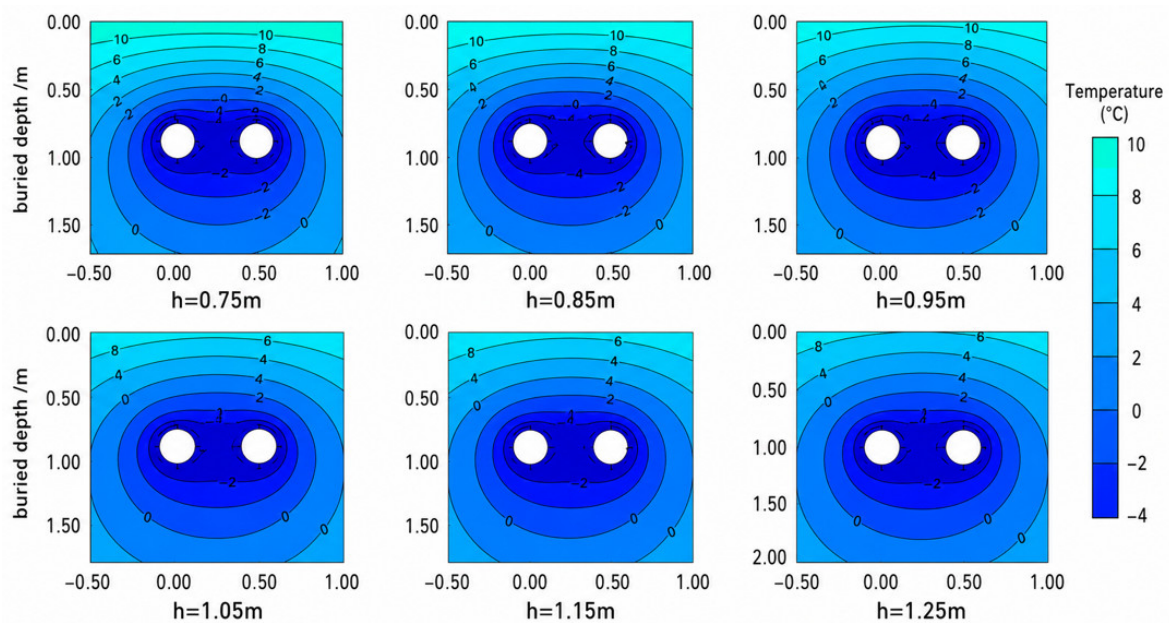


Figure 7. Steady-state temperature contours under different burial depths ($h = 0.75\text{--}1.25\text{ m}$), with $r = 0.10\text{ m}$, $d = 0.50\text{ m}$, $t_1 = t_2 = -7^{\circ}\text{C}$, and $t_0 = 12^{\circ}\text{C}$. The bold contour denotes the calculated 0°C isotherm, and all panels use identical contour intervals and coordinate units of meters.

The pipe spacing was varied from 0.5 m to 1.0 m in increments of 0.1 m. As shown in Figure 8, increasing the pipe spacing weakens the thermal interaction between the two freezing pipes. The calculated frozen bridge between the pipe-associated frozen zones becomes progressively thinner, and the overlap of the two local frozen regions decreases. Under the present contour-based interpretation, this trend suggests that a sufficiently large pipe spacing may lead to loss of local connectivity between the two calculated frozen zones. However, the exact critical spacing d_{cr} should be determined through quantitative extraction of the 0 °C isotherm in future post-processing rather than by visual inspection alone.

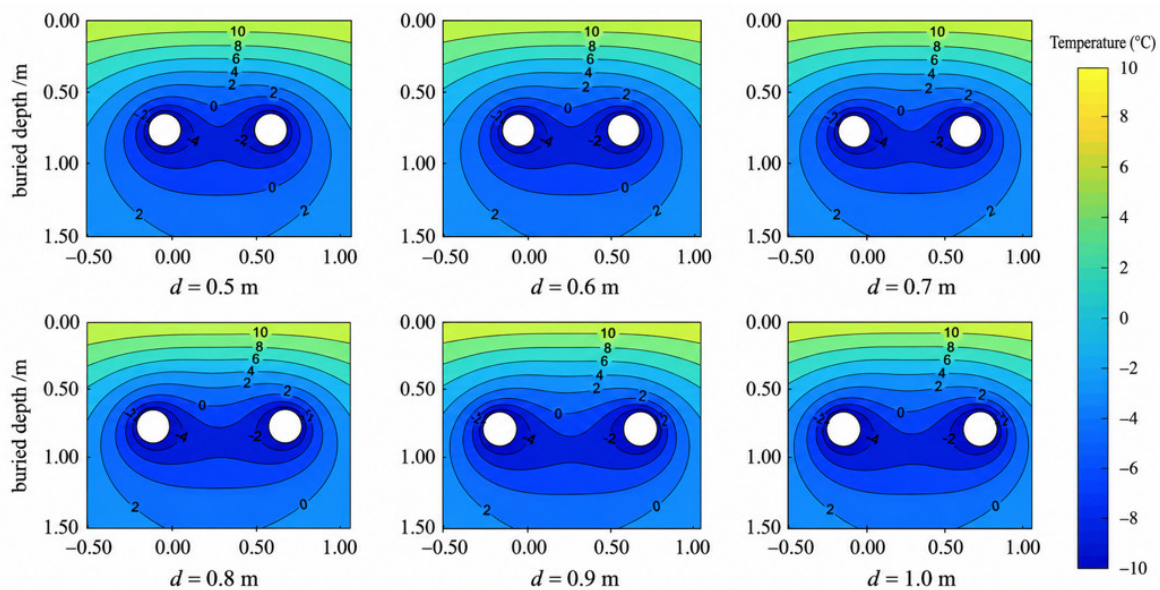


Figure 8. Steady-state temperature contours under different pipe spacings ($d = 0.5\text{--}1.0\text{ m}$), with $r = 0.10\text{ m}$, $h = 0.75\text{ m}$, $t_1 = t_2 = -7\text{ °C}$, and $t_0 = 12\text{ °C}$. The bold contour denotes the calculated 0 °C isotherm, and all panels use identical contour intervals and coordinate units of meters.

Taken together, these results illustrate the qualitative sensitivity of the calculated 0 °C isotherm to ground surface temperature, pipe radius, burial depth, and pipe spacing under the prescribed quasi-steady assumptions. They provide a preliminary basis for understanding the influence of geometric and boundary parameters, but further quantitative extraction of frozen region indicators is required before the method can be used for direct design assessment.

4.2. Research Limitations

The present model assumes steady-state heat conduction in homogeneous and isotropic soil with prescribed pipe wall and ground surface temperatures. Transient freezing, phase change effects, groundwater seepage, moisture migration, frost heave, and thermo-hydro-mechanical coupling are not explicitly considered. Therefore, the proposed solution is mainly applicable to quasi-steady conditions during the later stage of active freezing and the maintenance stage. Under these assumptions, the calculated 0 °C isotherm should be interpreted as a quasi-steady thermal indicator of the predicted frozen region geometry, rather than a complete representation of actual frozen curtain formation.

Another limitation of the present numerical comparison is that a systematic mesh independence and computational boundary independence analysis was not performed. The local mesh refinement and residual tolerance used in the transformed domain Fluent model define the numerical setup, but they do not replace a formal mesh sensitivity study. A more rigorous validation framework should include at least three systematically refined

meshes and enlarged lateral and lower boundaries in the original physical domain to evaluate the influence of mesh density and computational domain size on the temperature field near the freezing pipes and on the calculated 0 °C isotherm.

In addition, the current study considers only two parallel freezing pipes with equal radii and burial depths, and the proposed solution is only compared with transformed domain numerical results rather than experimental or field measurements. Although the transformed domain numerical comparison is useful for checking the mathematical implementation of the proposed semi-analytical solution, experimental or field data would provide a stronger basis for assessing its engineering applicability. Future work should extend the method to more complex geological conditions and pipe arrangements and further verify its applicability using experimental or field data.

The prescribed ground surface temperature used in this study can be regarded as an equivalent boundary temperature for preliminary quasi-steady analysis. This Dirichlet boundary condition is suitable when the surface thermal state can be approximated by a representative temperature over the considered period. In practical shallow AGF projects, however, the ground surface may also be affected by convective heat transfer, radiation, seasonal temperature variation, and surface structures. Under such conditions, Robin or Neumann boundary conditions may provide a more realistic description. The variation in the prescribed ground surface temperature from 10 °C to 35 °C in the present parametric analysis can be regarded as a preliminary sensitivity analysis of the equivalent surface thermal state. Under a Robin-type boundary condition, changes in the convective heat transfer coefficient and ambient temperature would affect the equivalent thermal influence of the ground surface and may alter the upper extent of the calculated 0 °C isotherm. Therefore, a rigorous Robin-type sensitivity analysis and the extension of the present semi-analytical solution to mixed or flux-type surface boundary conditions will be investigated in future work.

In addition, the present study does not provide full-domain error metrics between the semi-analytical solution and an independent physical domain numerical solution. For a more complete validation, future work should calculate the mean absolute error, maximum absolute temperature error, relative L_2 error, area difference enclosed by the calculated 0 °C isotherm, and mean and maximum positional errors of the 0 °C contour. These indicators would provide a more comprehensive assessment of both the temperature field and the predicted frozen region boundary.

Finally, the present parametric analysis mainly identifies the qualitative effects of ground surface temperature, pipe radius, burial depth, and pipe spacing on the calculated 0 °C isotherm. Design-oriented scalar indicators, such as frozen bridge thickness, 0 °C isotherm area, upper and lower frozen region extents, minimum isotherm distance, and critical pipe spacing for disconnection, will be extracted and analyzed quantitatively in future work.

5. Conclusions

This study developed a semi-analytical series solution for the steady-state temperature field around a shallow-buried twin-pipe freezing unit in a semi-infinite domain. The method combines conformal mapping, temperature field superposition, Fourier series expansion, and boundary discretization. The proposed formulation provides an explicit theoretical framework for estimating the quasi-steady temperature field under prescribed pipe wall and ground surface temperatures.

A transformed domain numerical comparison was conducted using ANSYS Fluent. The semi-analytical solution showed rapid convergence with increasing truncation order, and the pointwise NRMSE at six selected test points was 0.758% when $m = 20$. This

result indicates good local agreement under the selected transformed domain comparison condition. However, it should not be interpreted as a full physical domain validation or a full-field error assessment.

The parametric analysis qualitatively illustrated the effects of equivalent ground surface temperature, pipe radius, burial depth, and pipe spacing on the calculated 0 °C isotherm. Increasing the equivalent ground surface temperature compressed the upper part of the predicted frozen region, whereas increasing the pipe radius enlarged the calculated frozen region. Increasing burial depth weakened the influence of the ground surface boundary, and increasing pipe spacing reduced the local thermal connection between the two pipe-associated frozen zones. These results provide preliminary insight into the sensitivity of the quasi-steady temperature field, while independent physical domain validation, mesh and boundary independence analyses, full-field error evaluation, and quantitative extraction of frozen region indicators are still required for future engineering design applications.

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Abbreviations

The following abbreviations are used in this manuscript:

AGF	Artificial ground freezing
NRMSE	Normalized root mean square error

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