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Minimum-Phase Preserving Balanced Truncation with Data-Driven Order Scoring

Thang Ngoc Pham ¹, Hoa Thi Phuong Nguyen ^{1,*}, Hong-Son Vu ¹, Khanh Tuan Do ¹ and Huy-Du Dao ^{2,*}

¹ Faculty of Electronics and Electrical Engineering, Hung Yen University of Technology and Education, Hung Yen 17000, Vietnam; phamngocthangutehy@gmail.com (T.N.P.); hongson.ute@gmail.com (H.-S.V.); kxanhdotuan@gmail.com (K.T.D.)

² Faculty of Electronics, Thai Nguyen University of Technology, Thai Nguyen 251750, Vietnam

* Correspondence: nguyenphuonghoa@utehy.edu.vn (H.T.P.N.); daohuydu@tnut.edu.vn (H.-D.D.)

Abstract

This paper investigates the problem of model-order reduction for linear systems arising from minimum-phase circuits and filters, where stability and frequency characteristics must be preserved. The MPPBT framework uses a Riccati–Lyapunov Gramian pair and constructs a balancing transformation with the sequence of MPPBT singular values, from which a relative error bound in the H_∞ norm and a scoring function, S_β are derived to select the model order. We apply the algorithm to a fourth-order Butterworth low-pass filter with a full-order state dimension, $n = 4$, and reduced-order models with $r = 1, 2, 3$ are examined. The results show that the model with $r = 3$ yields an H_∞ error of approximately 6.3×10^{-3} and an H_2 error of approximately 2.2×10^{-3} . The model with $r = 1$ gives an H_∞ error of approximately 1.46 and an H_2 error of approximately 4.8×10^{-1} . The model with $r = 2$ attains a composite score of $S_\beta \approx 0.16$, preserves stability and the minimum-phase property, and is regarded as a balanced choice between accuracy and complexity. A further comparison on an RLC ladder circuit of order $n = 15$ shows that at $r = 3$, MPPBT achieves the lowest H_∞ error among BT, PRBT, and MPPBT, while retaining an H_2 error close to that of BT.

Keywords: balanced truncation; Butterworth filter; minimum phase; model-order reduction; multi-criteria scoring

1. Introduction

The model-order-reduction (MOR) problem for linear time-invariant (LTI) dynamical systems continues to play a foundational role in numerical simulation, optimization, identification, and control design when the state dimension increases due to spatial discretization or due to interconnection of multiple subsystems. Recent surveys indicate a general trend in MOR from the objective of good approximation toward structured approximation, meaning that the reduced model is required to preserve physical-mathematical invariants (e.g., stability, passivity, and energy dissipation) in order to avoid qualitative mismatch in closed-loop operation or in extrapolation beyond the training-data regime [1,2].

Among structural frameworks, port-Hamiltonian (pH) representations and dissipative Hamiltonian variants are often used to encode energy constraints and passivity directly. Studies on identification and data fitting in pH form from frequency responses and

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emphasize that even at the stage of constructing the full-order model, passivity preservation and order minimization constitute necessary conditions to ensure usability in inversion or system reversal [3]. On that basis, surveys on structure-preserving MOR for interconnected pH systems, including nonlinear cases, have systematized projection and interpolation strategies that maintain energy constraints and have made explicit the difficulties that arise when additional frequency-domain conditions or parameter constraints are introduced [4]. Recent data-driven approaches combining the Loewner framework for pH systems and network systems enable constructing reduced models directly from data, retaining pH properties to a certain extent when the projection operators and interpolation constraints are designed appropriately [5]. Studies on structure-preserving discretization and MOR for boundary-controlled pH systems have shown that passivity preservation must be ensured throughout the chain: discretization $\rightarrow$ frequency-domain interpolation $\rightarrow$ physically meaningful projection construction [6]. At a more general level, monograph chapters on reduction for descriptor pH-DAE systems have clarified the role of structured interpolation and the handling of algebraic constraints to avoid spurious dynamics [7]. To reduce optimization costs in structured MOR algorithms, adaptive sampling strategies for pH systems demonstrate clear effectiveness when the number of subproblem solves is controlled by the fitting error [8].

In many circuit–mechanical–electrical applications, model data may appear in descriptor form and require a reducer that is not only passive but also structure-compatible with algebraic constraints. In that direction, optimization-based MOR for port-Hamiltonian descriptor systems systematizes a transfer-function fitting procedure under pH constraints imposed on the reduced model, limiting the risk of passivity loss induced by purely projection-based reduction [9]. Within the same viewpoint, structured optimization-based MOR (SOBMOR) provides an approach that fits the transfer function while locking pH/structured properties, which is suitable when preserving system properties is prioritized over optimizing a single error norm [10]. From a broader perspective, dissipativity-preserving methods based on moment matching extend the scope of property preservation to general dissipative systems, in which stability and dissipation inequalities are maintained regardless of interpolation-point placement [11]. Beyond moment matching, a spectral-factorization-based direction for passivity preservation has shown that passive reduced models can be designed without explicit pH constraints by exploiting the positive-real structure of the transfer function [12]. For symmetric second-order systems, positive-real balancing enables order reduction while retaining the physical structure of the mechanical system (mass–damping–stiffness), which avoids non-physical modes and supports post-reduction physical interpretation [13].

When the objective includes preserving subtler frequency-domain properties, especially those related to system zeros, existing results remain scattered across problem classes. For pH systems, interval-limited balanced truncation has been proposed to reduce order over a frequency band or time window of interest while maintaining structure [14]. The guarantees primarily focus on passivity and stability rather than directly targeting minimum-phase preservation. On the quantitative side, error-bounding results and controller reduction for pH systems based on balancing show how to exploit degrees of freedom in weighting to support controller synthesis under error constraints [15]. Studies on a posteriori error bounds for pH systems continue to improve the ability to assess reduced-model reliability through auxiliary problems for the error system [16]. For positive-real systems, mixed-Gramian balanced truncation (MGBT) with error bounds reduces computational load relative to classical PRBT and provides controlled error assessments [17]. Mixed positive-bounded balanced truncation targets simultaneous preservation of positive realness and bounded realness, which is suitable when a system is passive and subject to gain constraints [18]. Phase-angle balanced truncation within conic positive real

frameworks incorporate transfer-function phase information into the reduction criterion, improving phase matching relative to purely energy-based criteria [19]. From the dissipation-inequality viewpoint, extended balancing for LTI systems provides a unified LMI characterization and suggests a priori error bounds and links to structure-preserving reduction for the pH subclass [20]. These works mainly address passivity, dissipativity, sector conditions, and phase matching in an approximate sense. Minimum-phase preservation, which is directly tied to zero structure and stable invertibility, is rarely treated as a central constraint of balanced-truncation algorithms.

From a practical perspective, preserving the minimum-phase property during model reduction is important for filter design, signal processing, and control applications. A minimum-phase system admits a stable and causal inverse; hence, a reduced model retaining this property can be used reliably in inverse filtering, equalization, signal reconstruction, and controller synthesis without introducing unstable inverse dynamics [21,22]. Moreover, among causal systems having the same magnitude response, a minimum-phase realization has the smallest group delay and the most front-loaded impulse-response energy, which supports reliable phase behavior and favorable transient characteristics after reduction [21].

Beyond property constraints, a practical challenge is the selection of the order r in balanced truncation when the system is large and/or only a finite operating domain is of interest. Frequency-limited balancing techniques have been developed with low-rank approximations to address large-scale frequency-limited Gramian problems [23]. Time-limited balancing has formed a separate methodological branch, in which Gramian computation and error bounds are adapted to a time window of interest [24]. Recent results on numerical computation and output bounds for discrete-time systems increase robustness when applied in simulation and digital control [25]. Applications in data assimilation show that time-limited balanced truncation can serve as an effective preprocessing tool to reduce inference costs when accuracy is required only over an observation interval [26]. For second-order systems, frequency–time-limited balancing has been extended in a structure-preserving direction to avoid breaking the physical description of the system [27]. Recent short contributions on low-rank approximations for frequency-limited Gramians of discrete-time systems further strengthen the computational basis for variants of balancing with a domain of interest [28]. Even when Hankel/Gramian-type error bounds are available, the practice of choosing r often relies on heuristic thresholds, particularly when trade-offs are required among norm errors, transient responses, and phase deviations in the frequency domain.

A related line focuses on frequency-domain approximation criteria and weighted H_2 norm settings. Work on frequency-weighted H_2 -optimal MOR based on oblique projection shows that priority can be concentrated on important frequency bands through weighting, rather than relying only on full-band controllability–observability balancing [29]. In a time-window-compatible direction, modeling in a time-limited H_2 setting under relative error has been proposed to reduce bias when the signal amplitude varies strongly over time [30]. The H_2 relative error setting in Systems & Control Letters provides an additional view on how to pose approximation problems so that the acceptable error level is reflected on the system’s scale [31]. Unstable stochastic systems impose additional constraints on Gramians and indicate the need to screen or preprocess system properties before applying standard balancing techniques [32]. These results suggest that a controlled reduction workflow must manage baseline properties (stability, minimality, and structure) and integrate multi-domain evaluation criteria (time and frequency) to select r consistently.

For order selection, automated schemes are beginning to appear, yet they often optimize a single metric. Energy-based order selection and scale homogenization in

parameter–state spaces through empirical cross Gramians and symmetrizers have been proposed as a mechanism for automatic thresholding in identification and parameterization [33]. In non-intrusive ROM contexts, a Pareto criterion between error and model complexity is used to select models on the Pareto frontier, emphasizing the multi-objective nature of the problem [34]. Existing approaches typically emphasize structure (pH and dissipativity) without directly constraining minimum-phase properties, or they emphasize a single evaluation domain (e.g., H_2/H_∞ or time-limited only) without providing a quantitative scoring mechanism that consistently aggregates time-response errors, magnitude–phase deviations over frequency, and characteristic system indices.

From these observations, this study develops a theoretical framework and mathematical arguments for Gramian-based balanced truncation under system-property preservation constraints. At a minimum, the framework preserves stability and is further extended to incorporate minimum-phase preservation. The resulting method is referred to as the MPPBT algorithm. In addition, a quantitative scoring procedure is developed to determine the optimal reduced order r .

The proposed method follows a controlled reduction workflow. First, the original system is screened to verify the baseline properties of stability, minimality, and minimum-phase behavior. Next, controllability and observability Gramians are used to quantify state importance. The system is then transformed into balanced coordinates, in which the states are ordered according to their contributions, allowing for low-importance states to be truncated to construct a reduced-order model.

Theoretical analysis establishes conditions under which the reduced model preserves stability. Moreover, when the MPPBT singular values associated with the truncated states are sufficiently small, the reduced model can also retain the minimum-phase property.

To avoid heuristic order selection, the study introduces a multi-domain evaluation framework consisting of norm errors, time-domain response indices, frequency-domain magnitude and phase deviations, and characteristic system indices. These criteria are normalized to a common scale and combined with the model-order measure to form a trade-off score. The resulting aggregate criterion provides a Pareto-based interpretation of the optimal order-selection problem. The effectiveness of MPPBT in preserving stability and minimum-phase properties is demonstrated using a fourth-order Butterworth low-pass filter derived from a one-plus-alpha filter structure, as reported in [35]. The framework is further validated on a 15th-order passive RLC ladder circuit, where MPPBT is compared against classical BT and PRBT.

2. Mathematical Framework of Minimum Phase Preserving Balanced Truncation (MPPBT)

Definition 1. Given matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{m \times n}$, $\mathbf{D} \in \mathbb{R}^{m \times m}$, consider the (square) LTI system with $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t), \mathbf{y}(t) \in \mathbb{R}^m$: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$, and the transfer function $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$. The system $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is called MPPBT-admissible if: (i) $\mathbf{A}$ is Hurwitz: $\sigma(\mathbf{A}) \subset \{z \in \mathbb{C} : \Re z < 0\}$; (ii) $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{C}, \mathbf{A})$ is observable; (iii) $\mathbf{D}$ is nonsingular (equivalently $\mathbf{R}_0 := \mathbf{D}\mathbf{D}^T > 0$); (iv) $\mathbf{G}(s)$ is minimum phase: it has no RHP zeros, i.e., there is no z with $\Re z \geq 0$ such that $\text{rank } \mathbf{G}(z) < n$ rank $\mathbf{G}$ (for square full-normal-rank $\mathbf{G}$, equivalently $\det \mathbf{G}(z) \neq 0$ for $\Re z \geq 0$) [12,36,37].

Definition 2. Define two linear operators $\mathcal{L}_{\mathbf{A}}(\mathbf{X}) := \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^T$, $\mathcal{R}_{\mathbf{A}}(\mathbf{X}) := \mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A}$. The MPPBT controllability Gramian is the (unique) SPD solution $\mathbf{P} = \mathbf{P}^T$ of the Lyapunov equation in (1). From $\mathbf{P}$, define $\mathfrak{B}_w := \mathbf{P}\mathbf{C}^T + \mathbf{B}\mathbf{D}^T$ and $\mathbf{R}_0 := \mathbf{D}\mathbf{D}^T$; define $\mathcal{M}(\mathbf{Q}) := (\mathbf{C} - \mathfrak{B}_w^T \mathbf{Q})^T \mathbf{R}_0^{-1} (\mathbf{C} - \mathfrak{B}_w^T \mathbf{Q})$. The MPPBT observability Gramian is defined as the minimal (stabilizing) SPD solution $\mathbf{Q} = \mathbf{Q}^T = \mathbf{Q}_w^T$ of the Riccati equation in (2): [12,36,37].

$$\mathcal{L}_A(\mathbf{P}) + \mathbf{B}\mathbf{B}^\top = \mathbf{0} \Leftrightarrow \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top + \mathbf{B}\mathbf{B}^\top = \mathbf{0} \quad (1)$$

$$\mathcal{R}_A(\mathbf{Q}) + \mathcal{M}(\mathbf{Q}) = \mathbf{0} \Leftrightarrow \mathbf{A}^\top \mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathcal{M}(\mathbf{Q}) = \mathbf{0} \quad (2)$$

The existence and uniqueness of these Gramians, together with the square-root factorizations and singular value decomposition used in Theorem 1, are established in Appendix A (Lemmas A1–A3).

Theorem 1. Define $\mathbf{T} := \mathbf{R}\mathbf{V}\boldsymbol{\Sigma}^{-1/2}$, $\mathbf{T}^{-1} := \boldsymbol{\Sigma}^{-1/2}\mathbf{U}^\top\mathbf{L}^\top$. For $\tilde{\mathbf{A}} := \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$, $\tilde{\mathbf{B}} := \mathbf{T}^{-1}\mathbf{B}$, $\tilde{\mathbf{C}} := \mathbf{C}\mathbf{T}$, $\tilde{\mathbf{D}} := \mathbf{D}$, the transformed Gramians satisfy $\tilde{\mathbf{P}} := \mathbf{T}^{-1}\mathbf{P}\mathbf{T}^{-\top} = \boldsymbol{\Sigma}$ and $\tilde{\mathbf{Q}} := \mathbf{T}^\top\mathbf{Q}\mathbf{T} = \boldsymbol{\Sigma}$ [37,38].

Proof of Theorem 1. $\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$ since $\mathbf{T}^{-1}\mathbf{T} = \boldsymbol{\Sigma}^{-1/2}\mathbf{U}^\top(\mathbf{L}^\top\mathbf{R})\mathbf{V}\boldsymbol{\Sigma}^{-1/2} = \mathbf{I}$. Next, $\tilde{\mathbf{P}} = (\mathbf{T}^{-1}\mathbf{R})(\mathbf{T}^{-1}\mathbf{R})^\top$ with $\mathbf{T}^{-1}\mathbf{R} = \boldsymbol{\Sigma}^{-1/2}\mathbf{U}^\top\mathbf{H} = \boldsymbol{\Sigma}^{1/2}\mathbf{V}^\top$, hence $\tilde{\mathbf{P}} = \boldsymbol{\Sigma}$. Similarly, $\tilde{\mathbf{Q}} = (\mathbf{L}^\top\mathbf{T})^\top(\mathbf{L}^\top\mathbf{T})$ with $\mathbf{L}^\top\mathbf{T} = \mathbf{H}\mathbf{V}\boldsymbol{\Sigma}^{-1/2} = \mathbf{U}\boldsymbol{\Sigma}^{1/2}$, hence $\tilde{\mathbf{Q}} = \boldsymbol{\Sigma}$. $\square$

Lemma 1. In balanced coordinates, introduce the state partition and induced matrix blocks as follows (3) [37]:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \tilde{\mathbf{C}} = [\mathbf{C}_1 \quad \mathbf{C}_2] \quad (3)$$

Moreover, partition $\boldsymbol{\Sigma} = \text{diag}(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2)$, where $\boldsymbol{\Sigma}_1 = \text{diag}(\sigma_1, \dots, \sigma_r)$, $\boldsymbol{\Sigma}_2 = \text{diag}(\sigma_{r+1}, \dots, \sigma_n)$. The order- r model in balanced coordinates is $\mathbf{G}_r(s) = \mathbf{C}_1(s\mathbf{I}_r - \mathbf{A}_{11})^{-1}\mathbf{B}_1 + \mathbf{D}$. Define the projection matrices $\mathbf{V}_r := \mathbf{T} \begin{bmatrix} \mathbf{I}_r \\ \mathbf{0} \end{bmatrix}$, $\mathbf{W}_r := \mathbf{T}^{-\top} \begin{bmatrix} \mathbf{I}_r \\ \mathbf{0} \end{bmatrix}$; then the reduced realization is (4) [37]:

$$\mathbf{A}_r := \mathbf{W}_r^\top \mathbf{A} \mathbf{V}_r, \mathbf{B}_r := \mathbf{W}_r^\top \mathbf{B}, \mathbf{C}_r := \mathbf{C} \mathbf{V}_r, \mathbf{D}_r := \mathbf{D} \quad (4)$$

is an order- r realization of $\mathbf{G}_r$. Moreover, the sub-Gramians of the retained part satisfy $\mathbf{P}_{11} = \boldsymbol{\Sigma}_1, \mathbf{Q}_{11} = \boldsymbol{\Sigma}_1$.

Proof of Lemma 1. The definition of $\mathbf{V}_r, \mathbf{W}_r$ implies that $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$ realizes $\mathbf{G}_r$ and that the retained sub-Gramians satisfy $\mathbf{P}_{11} = \boldsymbol{\Sigma}_1, \mathbf{Q}_{11} = \boldsymbol{\Sigma}_1$ by construction of balanced coordinates. $\square$

Theorem 2. For an MPPBT-admissible system and the partition in (3), the matrix $\mathbf{A}_{11}$ is Hurwitz; hence $\mathbf{G}_r$ is stable [37].

Proof of Theorem 2. In balanced coordinates, the controllability Gramian equation reads $\tilde{\mathbf{A}}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\tilde{\mathbf{A}}^\top + \tilde{\mathbf{B}}\tilde{\mathbf{B}}^\top = \mathbf{0}$. Taking the $(1,1)$ block gives $\mathbf{A}_{11}\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_1\mathbf{A}_{11}^\top + \mathbf{B}_1\mathbf{B}_1^\top = \mathbf{0}$. Let $\lambda \in \sigma(\mathbf{A}_{11})$ and $\mathbf{w} \neq \mathbf{0}$ be a left eigenvector, $\mathbf{w}^\top \mathbf{A}_{11} = \lambda \mathbf{w}^\top$. Premultiplying/postmultiplying by $\mathbf{w}^\top, \mathbf{w}$ yields $2\Re(\lambda) \mathbf{w}^\top \boldsymbol{\Sigma}_1 \mathbf{w} + \|\mathbf{B}_1^\top \mathbf{w}\|_2^2 = 0$. If $\Re(\lambda) \geq 0$, then $\mathbf{B}_1^\top \mathbf{w} = \mathbf{0}$, contradicting controllability of $(\mathbf{A}_{11}, \mathbf{B}_1)$ (Hautus). Thus $\Re(\lambda) < 0$ for all λ , so $\mathbf{A}_{11}$ is Hurwitz. $\square$

Theorem 3. Assume $\mathbf{G}(s)$ is minimum phase. If there exists r such that $\|\mathbf{E}_r\|_{H_\infty} < 1$, where $\mathbf{E}_r(s) := \mathbf{G}(s)^{-1}(\mathbf{G}(s) - \mathbf{G}_r(s))$, then $\mathbf{G}_r(s)$ is also minimum phase [36].

Proof of Theorem 3. For $\Re(s) \geq 0$, $\mathbf{G}_r(s) = \mathbf{G}(s)(\mathbf{I} - \mathbf{E}_r(s))$. If $\|\mathbf{E}_r\|_{H_\infty} < 1$, then $\mathbf{I} - \mathbf{E}_r(s)$ is invertible for all $\Re(s) \geq 0$ (Neumann series), and since $\mathbf{G}(s)$ is invertible there, $\det(\mathbf{G}_r(s)) \neq 0$ on $\Re(s) \geq 0$. $\square$

Theorem 4. For the relative error operator $\mathbf{E}_r(s) := \mathbf{G}(s)^{-1}(\mathbf{G}(s) - \mathbf{G}_r(s))$, define the scalar function and accumulated factor as (5). Assume $\sigma_i < 1$ for all i , then $\|\mathbf{E}_r\|_{H_\infty} \leq \Psi_r$ [36].

$$\begin{aligned}\varphi(\sigma) &:= \frac{1+\sigma}{1-\sigma} - 1 = \frac{2\sigma}{1-\sigma} \\ \Psi_r &:= \prod_{i=r+1}^n (1 + \varphi(\sigma_i)) - 1 = \prod_{i=r+1}^n \frac{1+\sigma_i}{1-\sigma_i} - 1\end{aligned}\quad (5)$$

The following theorem bounds the H_∞ -norm bound of the relative error. Assume $\sigma_i < 1$ for all i , then $\|\mathbf{E}_r\|_{H_\infty} \leq \Psi_r$.

Proof of Theorem 4. In the BST relative-error construction, truncating each balanced component $i > r$ yields a stable perturbation factor $I + \Delta_i$, satisfying $\|\Delta_i\|_{\mathcal{H}_\infty} \leq \varphi(\sigma_i)$.

Therefore, the accumulated relative error resulting from truncating the states $i = r + 1, \dots, n$ is bounded by the product of the individual perturbation factors. By the submultiplicativity of the H_∞ norm and the triangle inequality, we obtain

$$\left\| \prod_{i=r+1}^n (I + \Delta_i) - I \right\|_{\mathcal{H}_\infty} \leq \prod_{i=r+1}^n (1 + \|\Delta_i\|_{\mathcal{H}_\infty}) - 1 \leq \prod_{i=r+1}^n (1 + \varphi(\sigma_i)) - 1. \quad (6)$$

Since $1 + \varphi(\sigma_i) = (1 + \sigma_i)/(1 - \sigma_i)$, the right-hand side of (6) is equal to Ψ_r . Hence, $\|\mathbf{E}_r\|_{\mathcal{H}_\infty} \leq \Psi_r$, which proves the result. $\square$

Corollary 1. Given $\varepsilon > 0$, if there exists r such that $\Psi_r \leq \varepsilon$, then $\|\mathbf{E}_r\|_{H_\infty} \leq \varepsilon$. For each $r \in \mathcal{R}$, consider $\mathbf{G}(s)$ and $\mathbf{G}_r(s)$. Define norm errors as (7); for step-response errors, define the step error signal as (8); and for $t \in [0, T]$, define the time-domain criteria by (9) [1,2,36]:

$$\phi_1(r) := \|\mathbf{G} - \mathbf{G}_r\|_{H_\infty}, \phi_2(r) := \|\mathbf{G} - \mathbf{G}_r\|_{H_2} \quad (7)$$

$$e_{st}^{(r)}(t) := y_{st}(t) - y_{st,r}(t) \quad (8)$$

$$\phi_3(r) := \int_0^T |e_{st}^{(r)}(t)|^2 dt \text{ (ISE)},$$

$$\phi_4(r) := \int_0^T |e_{st}^{(r)}(t)| dt \text{ (IAE)},$$

$$\phi_5(r) := \int_0^T t |e_{st}^{(r)}(t)|^2 dt \text{ (ITSE)},$$

$$\phi_6(r) := \int_0^T t |e_{st}^{(r)}(t)| dt \text{ (ITAE)},$$

$$\phi_7(r) := \frac{1}{T} \int_0^T |e_{st}^{(r)}(t)|^2 dt \text{ (MSE)}, \quad (9)$$

$$\phi_8(r) := \sqrt{\phi_7(r)} \text{ (RMSE)},$$

$$\phi_9(r) := \max_{t \in [0, T]} |e_{st}^{(r)}(t)|,$$

$$\phi_{10}(r) := \min_{t \in [0, T]} e_{st}^{(r)}(t),$$

$$\phi_{11}(r) := \frac{1}{T} \int_0^T e_{st}^{(r)}(t) dt$$

For impulse-response errors, define the impulse error signal as (10) [1,2]:

$$e_{imp}^{(r)}(t) := y_{imp}(t) - y_{imp,r}(t) \quad (10)$$

Define $\phi_{12}(r), \dots, \phi_{20}(r)$ analogously by replacing $e_{st}^{(r)}$ with $e_{imp}^{(r)}$ in (9). For frequency-domain errors (for a chosen SISO channel or applied entrywise), use the magnitude/phase decomposition in (11) [2,29]:

$$G(j\omega) = M(\omega)e^{j\theta(\omega)}, G_r(j\omega) = M_r(\omega)e^{j\theta_r(\omega)} \quad (11)$$

with $\Delta M_{dB}^{(r)}(\omega) := 20\log_{10} M(\omega) - 20\log_{10} M_r(\omega)$ and $\Delta\theta^{(r)}(\omega) := \theta(\omega) - \theta_r(\omega)$ (unwrap on $\Omega \subset \mathbb{R}_+$), we define (12) [2,29]:

$$\begin{aligned} \phi_{21}(r) &:= \min_{\omega \in \Omega} |\Delta M_{dB}^{(r)}(\omega)|, \\ \phi_{22}(r) &:= \frac{1}{|\Omega|} \int_{\Omega} |\Delta M_{dB}^{(r)}(\omega)| d\omega, \\ \phi_{23}(r) &:= \max_{\omega \in \Omega} |\Delta M_{dB}^{(r)}(\omega)|, \\ \phi_{24}(r) &:= \min_{\omega \in \Omega} |\Delta\theta^{(r)}(\omega)|, \\ \phi_{25}(r) &:= \frac{1}{|\Omega|} \int_{\Omega} |\Delta\theta^{(r)}(\omega)| d\omega, \\ \phi_{26}(r) &:= \max_{\omega \in \Omega} |\Delta\theta^{(r)}(\omega)| \end{aligned} \quad (12)$$

For reduced-model properties, we have (13) [2]:

$$\phi_{27}(r) := \text{dcgain}(\Sigma_r), \phi_{28}(r) := PM(r) \quad (13)$$

which are the DC gain and phase margin of the order- r model, respectively. Collect them into the vector $\boldsymbol{\phi}(r) := (\phi_1(r), \dots, \phi_{N_c}(r))^T \in \mathbb{R}^{N_c}$.

Lemma 2. Define $\phi_j^{\min} := \min_{r \in \mathcal{R}} \phi_j(r)$, $\phi_j^{\max} := \max_{r \in \mathcal{R}} \phi_j(r)$, and pick a small $\varepsilon > 0$. The min–max normalized criterion is defined by (14). Then $0 \leq \bar{\phi}_j(r) \leq 1$ for all $r \in \mathcal{R}$ and all j .

$$\bar{\phi}_j(r) := \frac{\phi_j(r) - \phi_j^{\min}}{\max(\phi_j^{\max} - \phi_j^{\min}, \varepsilon)} \quad (14)$$

Then for all $r \in \mathcal{R}$ and all j , $0 \leq \bar{\phi}_j(r) \leq 1$.

Proof of Lemma 2. By definition, $\phi_j^{\min} \leq \phi_j(r) \leq \phi_j^{\max} \Rightarrow 0 \leq \phi_j(r) - \phi_j^{\min} \leq \phi_j^{\max} - \phi_j^{\min}$. Also $\max(\phi_j^{\max} - \phi_j^{\min}, \varepsilon) \geq \phi_j^{\max} - \phi_j^{\min} \geq 0$ and $\max(\cdot) \geq \varepsilon > 0$, hence $0 \leq \bar{\phi}_j(r) \leq \frac{\phi_j^{\max} - \phi_j^{\min}}{\max(\phi_j^{\max} - \phi_j^{\min}, \varepsilon)} \leq 1$. $\square$

Definition 3. Choose an index subset $\mathcal{J}_{sel} := \{j_1, \dots, j_{N_{sel}}\} \subset \{1, \dots, N_c\}$. Choose weights $\mathbf{w}_{err} := (w_1, \dots, w_{N_{sel}})^T$, with $w_\ell \geq 0$ and $\sum_{\ell=1}^{N_{sel}} w_\ell = 1$. The composite error index is defined by (15). From Lemma 2 and $\sum_\ell w_\ell = 1$, it follows that $0 \leq \mathcal{J}_{err}(r) \leq 1$, where we define $r_{\min} := \min_{r \in \mathcal{R}} r$, $r_{\max} := \max_{r \in \mathcal{R}} r$. The order-complexity index is defined by (16). Then $0 \leq \mathcal{J}_{ord}(r) \leq 1$ and $\mathcal{J}_{ord}(r)$ is nondecreasing in r . With $\beta \in [0, 1]$, the MPPBT composite score is defined by (17). Hence $0 \leq \mathcal{S}_\beta(r) \leq 1$. Cases: $\beta = 0$ (error only), $\beta = 1$ (order only), $0 < \beta < 1$ (trade-off).

$$\mathcal{J}_{err}(r) := \sum_{\ell=1}^{N_{sel}} w_\ell \bar{\phi}_{j_\ell}(r), r \in \mathcal{R} \quad (15)$$

$$\mathcal{J}_{ord}(r) := \frac{r - r_{\min}}{\max(r_{\max} - r_{\min}, \varepsilon)}, r \in \mathcal{R} \quad (16)$$

$$\mathcal{S}_\beta(r) := (1 - \beta) \mathcal{J}_{err}(r) + \beta \mathcal{J}_{ord}(r), r \in \mathcal{R} \quad (17)$$

Lemma 3. For fixed r , $\mathcal{S}_\beta(r)$ is affine in β on $[0, 1]$, as in (18). If $\mathcal{J}_{err}(r_1) \leq \mathcal{J}_{err}(r_2)$ and $r_1 \leq r_2$, then $\mathcal{S}_\beta(r_1) \leq \mathcal{S}_\beta(r_2)$ for all $\beta \in [0, 1]$.

$$\mathcal{S}_\beta(r) = \mathcal{J}_{err}(r) + \beta(\mathcal{J}_{ord}(r) - \mathcal{J}_{err}(r)) \quad (18)$$

Proof of Lemma 3. The equation in (18) is a rearrangement of (17). If $r_1 \leq r_2$, then by (16) $\mathcal{J}_{ord}(r_1) \leq \mathcal{J}_{ord}(r_2)$. Combining with $\mathcal{J}_{err}(r_1) \leq \mathcal{J}_{err}(r_2)$ and multiplying each inequality by nonnegative coefficients $(1 - \beta)$, β , then summing yields the claim. $\square$

Theorem 5. For each $\beta \in [0, 1]$, the minimization over $\mathcal{R}$ attains at least one minimizer (19):

$$r_\beta^* \in \arg \min_{r \in \mathcal{R}} \mathcal{S}_\beta(r) \quad (19)$$

Proof of Theorem 5. Since $\mathcal{R}$ is finite, the set $\{\mathcal{S}_\beta(r) : r \in \mathcal{R}\}$ is a finite nonempty subset of $\mathbb{R}$. Therefore, it has a minimum, which proves (19). $\square$

Corollary 2. Define the bi-objective mapping $\mathcal{F} : \mathcal{R} \rightarrow [0, 1]^2$ by $\mathcal{F}(r) := (\mathcal{J}_{err}(r), \mathcal{J}_{ord}(r))^\top$. For $\beta \in (0, 1)$, every minimizer r_β^* of (19) is Pareto-nondominated for the bi-objective problem $\min(\mathcal{J}_{err}(r), \mathcal{J}_{ord}(r)), r \in \mathcal{R}$, i.e., there is no $r \in \mathcal{R}$ such that $\mathcal{J}_{err}(r) \leq \mathcal{J}_{err}(r_\beta^*)$, $\mathcal{J}_{ord}(r) \leq \mathcal{J}_{ord}(r_\beta^*)$, with at least one strict inequality. Conversely, every supported Pareto-nondominated point on the convex hull of $\{\mathcal{F}(r) : r \in \mathcal{R}\}$ is a minimizer of $\mathcal{S}_\beta$ for some $\beta \in (0, 1)$.

Proof of Corollary 2. Let $\beta \in (0, 1)$, and suppose r_β^* is optimal for $\mathcal{S}_\beta$. If there existed r with $\mathcal{J}_{err}(r) \leq \mathcal{J}_{err}(r_\beta^*)$ and $\mathcal{J}_{ord}(r) \leq \mathcal{J}_{ord}(r_\beta^*)$ and at least one strict inequality, then by (17) with positive weights $(1 - \beta) > 0$, $\beta > 0$, $\mathcal{S}_\beta(r) < \mathcal{S}_\beta(r_\beta^*)$, contradicting optimality. Hence r_β^* is nondominated. For the converse, a supported nondominated point admits a supporting hyperplane with some weight vector $\lambda \succ 0$; setting $\beta = \lambda_2 / (\lambda_1 + \lambda_2) \in (0, 1)$ yields a weighted-sum scalarization equivalent to minimizing $\mathcal{S}_\beta$, hence the point is optimal for some β . $\square$

Algorithm 1 summarizes the Minimum-Phase Preserving Balanced Truncation (MPPBT) workflow: it computes MPPBT-type Gramian factors via Lyapunov–Riccati solvers, constructs the MPPBT balancing transformation from the SVD of $L^T R$, and generates reduced models G_r for $r = 1, \dots, n - 1$. Each candidate order is then evaluated using a unified multi-domain metric suite and selected by a min–max normalized composite score that trades accuracy against model complexity while enforcing stability and minimum-phase feasibility when required.

Algorithm 1. Pseudocode of the Minimum-Phase Preserving Balanced Truncation (MPPBT) algorithm

Input: (num, den), opts=(solver, ϵ), grids (t, w), selected criteria J_{sel} , weights (w_err, β)

1 sys ← ss(tf(num, den)); (A, B, C, D) ← (sys.A, sys.B, sys.C, sys.D); if cols(B) < rows(C) then (A, B, C, D) ← (A^T, C^T, B^T, D^T)

2 R ← lyap_factor(A, B); Bw ← B · D^T + R(R^TC^T); D ← (rank(D) low or ||D|| small)? ϵI : R-from-qr(D^T)

3 $\tilde{A}$ ← A - Bw((D^TD) \ C); $\tilde{B}$ ← Bw/D; $\tilde{C}$ ← D \ C; L ← riccati_factor($\tilde{A}$, $\tilde{B}$, $\tilde{C}$, opts.solver)

4 [U, Σ , V] ← svd(L^TR); T ← R V $\Sigma^{\{-1/2\}}$; Tinv ← $\Sigma^{\{1/2\}} \setminus (U^T L^T)$

5 Precompute full responses of G(A, B, C, D): step/impulse on t, bode on w

6 For $r = n - 1 \dots 1$: Vr ← T(:, 1:r); Wr ← (Tinv(1:r, :))^T; sysr ← ss(Wr^TAVr, Wr^TB, CVr, D); compute $\phi(r)$ (all criteria)

7 Normalize: $\phi_j(r) = (\phi_j(r) - \min_r \phi_j) / (\max(\max_r \phi_j - \min_r \phi_j, \epsilon))$

```

8   $S_{\beta}(r) = (1-\beta) \cdot \Sigma_{\{j \in J_{sel}\}} w_{err}(j) \phi_j(r) + \beta \cdot (r - r_{min}) / \max(r_{max} - r_{min}, \epsilon)$ 
9   $r^* \leftarrow \operatorname{argmin}_r S_{\beta}(r); \quad r^*_{SM} \leftarrow \operatorname{argmin}_{\{r: \text{stableAminphase}\}} S_{\beta}(r) \text{ (if any)}$ 
Output:  $HSV = \operatorname{diag}(\Sigma), \varphi(r), S_{\beta}(r), r^*, r^*_{SM}$ 

```

3. Results and Discussion

Implement the MPPBT algorithm (Algorithm 1) in MATLAB R2022a, then apply model-order reduction to the 4th-order Butterworth low-pass filter model described in [35] to obtain lower-order models. Tables 1–7 report the quantitative results, and Figures 1–19 illustrate the qualitative comparisons.

Table 1. Norm errors of MPPBT.

Quantity	r = 3	r = 2	r = 1
$\ G - G_r\ _{\infty}$	6.322192×10^{-3}	8.459487×10^{-2}	1.458838
$\ G - G_r\ _2$	2.228581×10^{-3}	4.218250×10^{-2}	4.832579×10^{-1}

Table 2. Time-domain step error metrics.

Quantity (Unit)	r = 3	r = 2	r = 1
ISE	2.915615×10^{-5}	2.892838×10^{-2}	1.663140
IAE	1.044059×10^{-2}	4.290313×10^{-1}	3.351246
ITSE	2.579913×10^{-4}	2.197318×10^{-1}	1.295462×10^1
ITAE	8.607820×10^{-2}	3.047574×10^0	2.247815×10^1
MSE	2.921667×10^{-6}	2.893326×10^{-3}	1.664653×10^{-1}
RMSE	1.709289×10^{-3}	5.378965×10^{-2}	4.080016×10^{-1}
$e_{\max}$	2.774808×10^{-4}	9.171338×10^{-3}	2.741545×10^{-1}
$e_{\min}$	-4.235058×10^{-3}	-8.223176×10^{-2}	-7.970201×10^{-1}
e_{mean}	-9.635793×10^{-4}	-4.075042×10^{-2}	-2.011260×10^{-1}

Note: Boldface indicates the performance metrics used to calculate the proposed composite score for data-driven order selection.

Table 3. Time-domain impulse error metrics.

Quantity (Unit)	r = 3	r = 2	r = 1
ISE	4.270527×10^{-6}	1.778347×10^{-3}	2.046788×10^{-1}
IAE	5.202806×10^{-3}	1.044938×10^{-1}	1.345368
ITSE	3.160098×10^{-5}	7.202528×10^{-3}	8.643756×10^{-1}
ITAE	3.578148×10^{-2}	4.447436×10^{-1}	6.351753
MSE	4.268459×10^{-7}	1.776574×10^{-4}	2.045124×10^{-2}
RMSE	6.533344×10^{-4}	1.332882×10^{-2}	1.430078×10^{-1}
$e_{\max}$	3.343979×10^{-4}	1.277047×10^{-2}	2.574344×10^{-1}
$e_{\min}$	-1.097076×10^{-3}	-2.590301×10^{-2}	-1.770087×10^{-1}
e_{mean}	-4.234182×10^{-4}	-8.215721×10^{-3}	-7.966997×10^{-2}

Note: Boldface indicates the performance metrics used to calculate the proposed composite score for data-driven order selection.

Table 4. Frequency-domain magnitude and phase errors.

Quantity (Unit)	r = 3	r = 2	r = 1
$\Delta \text{Mag}_{\min}$ (dB)	7.889897×10^{-8}	5.231418×10^{-6}	8.983673×10^{-5}
$\Delta \text{Mag}_{\text{mean}}$ (dB)	1.256822×10^{-2}	2.253959×10^{-1}	2.793135×10^0
$\Delta \text{Mag}_{\max}$ (dB)	5.455007×10^{-2}	7.050728×10^{-1}	7.792448×10^0
$\Delta \text{Phase}_{\min}$ (deg)	3.792099×10^{-6}	7.836970×10^{-5}	7.557749×10^{-6}
$\Delta \text{Phase}_{\text{mean}}$ (deg)	5.584468×10^{-2}	9.067782×10^{-1}	1.285721×10^1
$\Delta \text{Phase}_{\max}$ (deg)	3.113833×10^{-1}	4.210708×10^0	4.681356×10^1

Note: Boldface indicates the performance metrics used to calculate the proposed composite score for data-driven order selection.

Table 5. Time-domain step response of reduced-order models.

Parameter	r = 3	r = 2	r = 1
Rise time (s)	2.140426×10^0	2.777723×10^0	1.671521×10^1
Settling time (s)	3.213342×10^0	4.941336×10^0	2.975641×10^1
Overshoot (%)	8.334950×10^{-1}	0	0
Peak value	1.014140×10^0	1.084011×10^0	2.456650×10^0
Peak time (s)	4.758380×10^0	1.334777×10^1	5.570795×10^1

Table 6. DC gain and phase margin of reduced-order models.

Quantity	r = 3	r = 2	r = 1
DC gain	1.005757	1.084030	2.458273
Phase margin (deg)	170.3888	154.0017	114.1226

Table 7. The H_∞ and H_2 norm errors of BT, PRBT, and MPPBT for $r = 3$.

Quantity	BT	PRBT	MPPBT
$\ G - G_r\ _\infty$	0.0468	0.0632	0.0413
$\ G - G_r\ _2$	0.0430	0.0501	0.0435

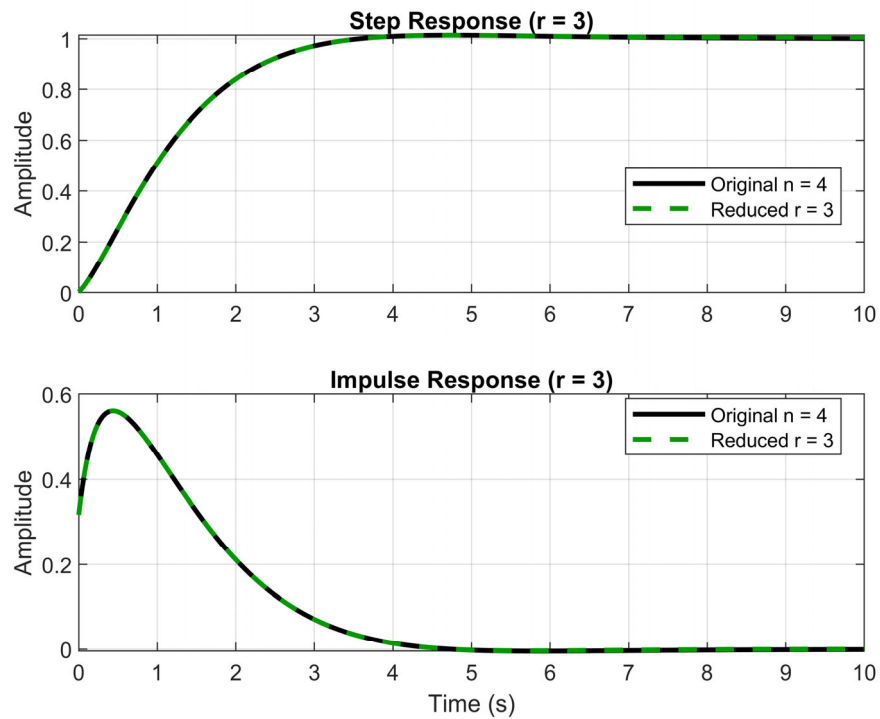


Figure 1. Step and impulse response comparison for reduced order $r = 3$.

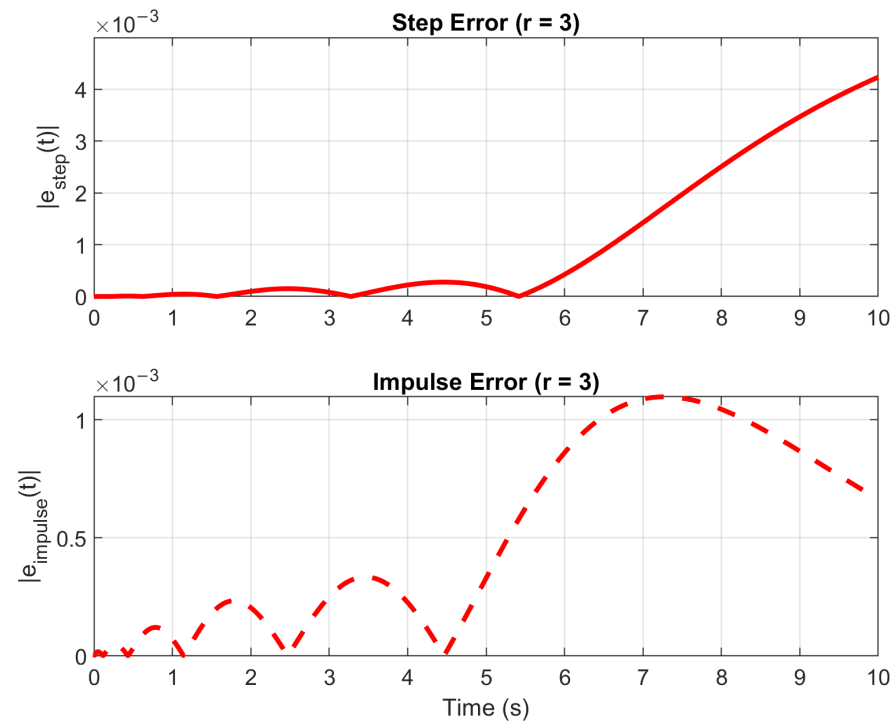


Figure 2. Step and impulse response errors for reduced order $r = 3$.

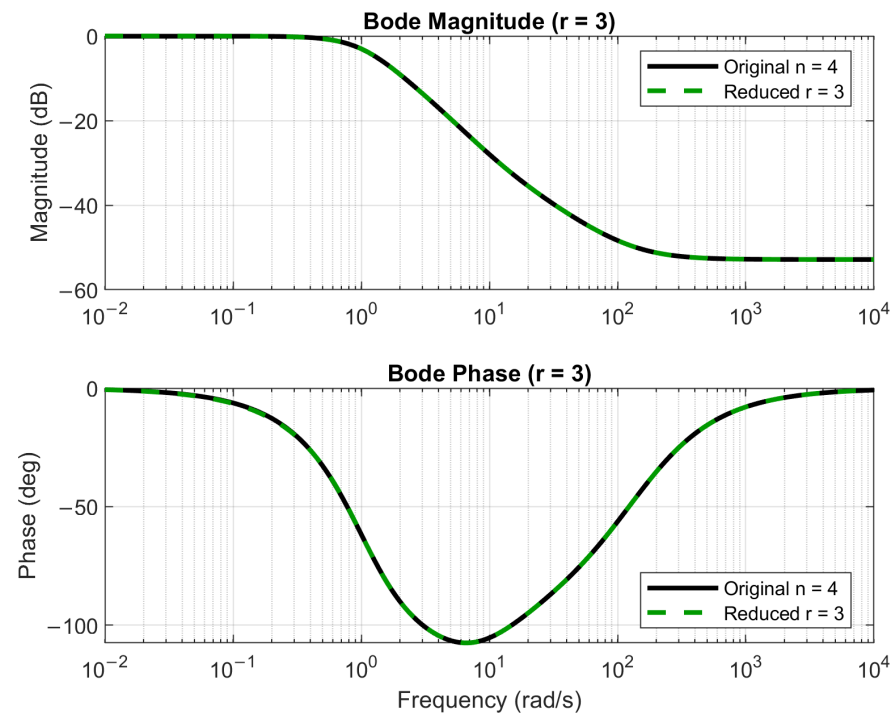


Figure 3. Bode magnitude and phase response comparison for reduced order $r = 3$.

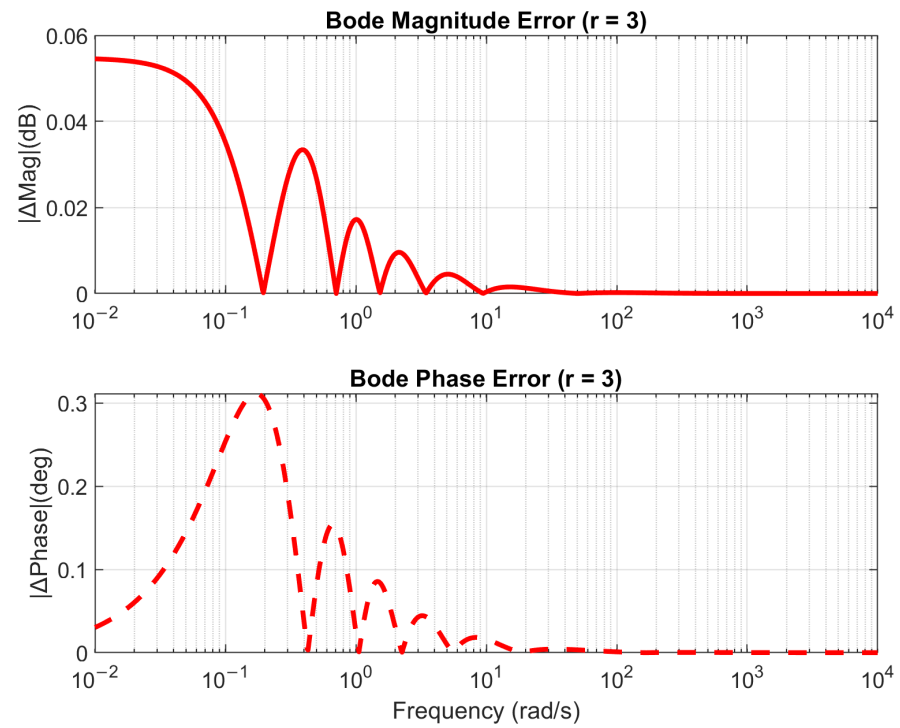


Figure 4. Bode magnitude and phase errors for reduced order $r = 3$.

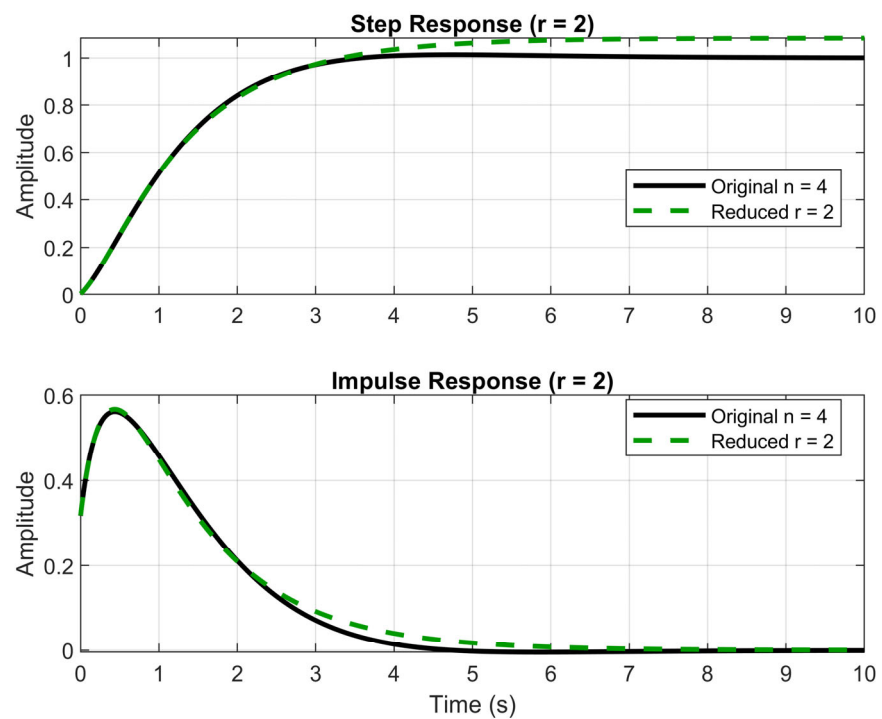


Figure 5. Step and impulse response comparison for reduced order $r = 2$.

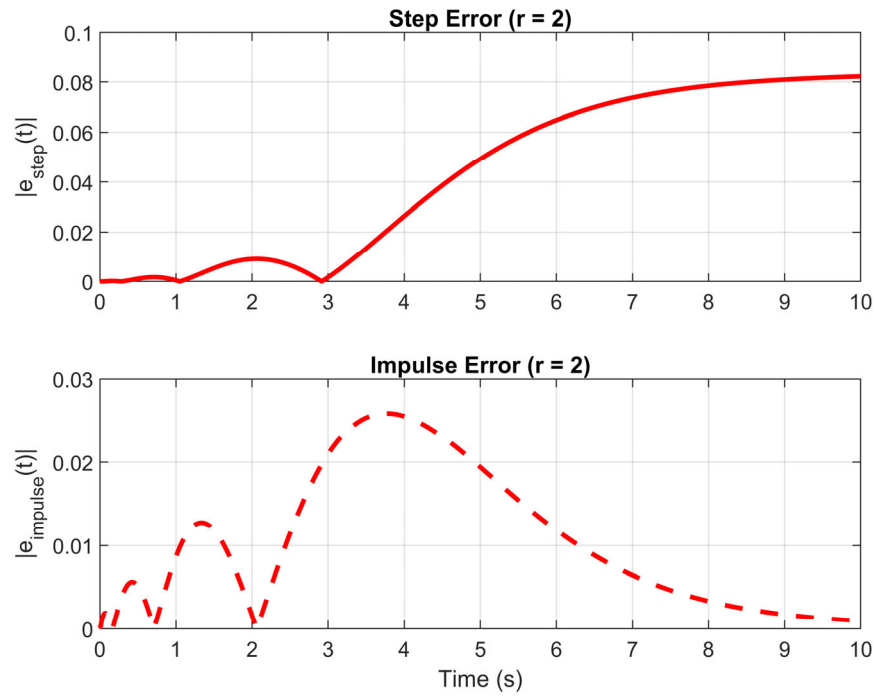


Figure 6. Step and impulse response errors for reduced order $r = 2$.

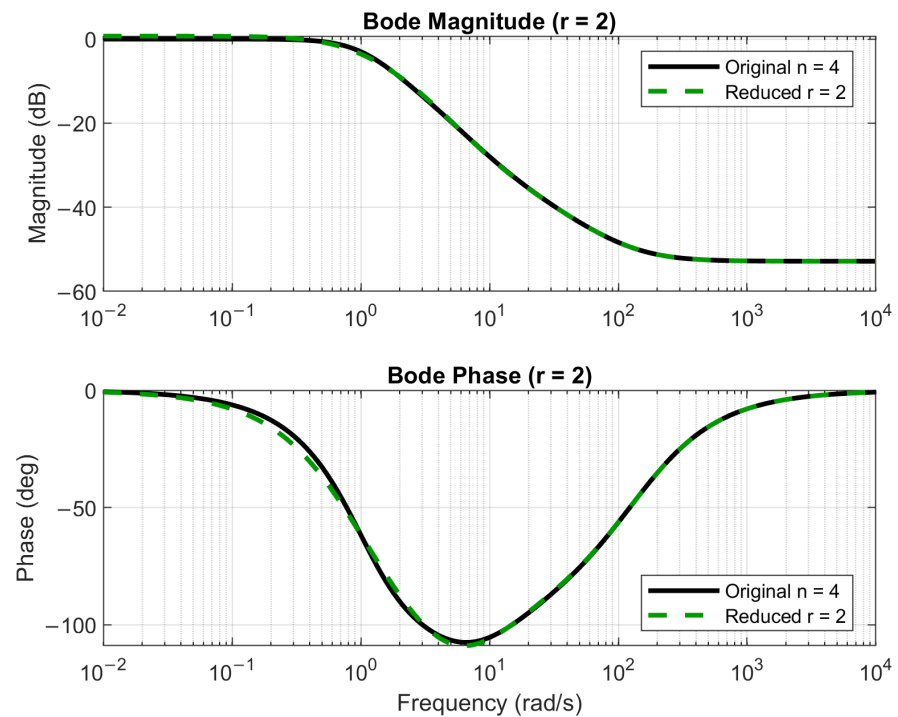


Figure 7. Bode magnitude and phase response comparison for reduced order $r = 2$.

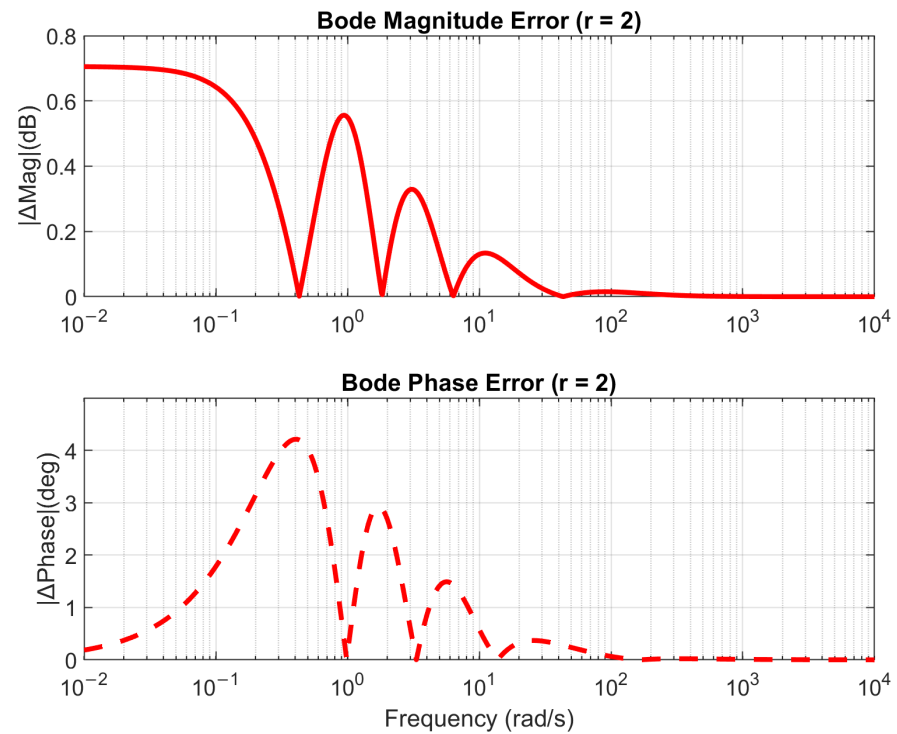


Figure 8. Bode magnitude and phase errors for reduced order $r = 2$.

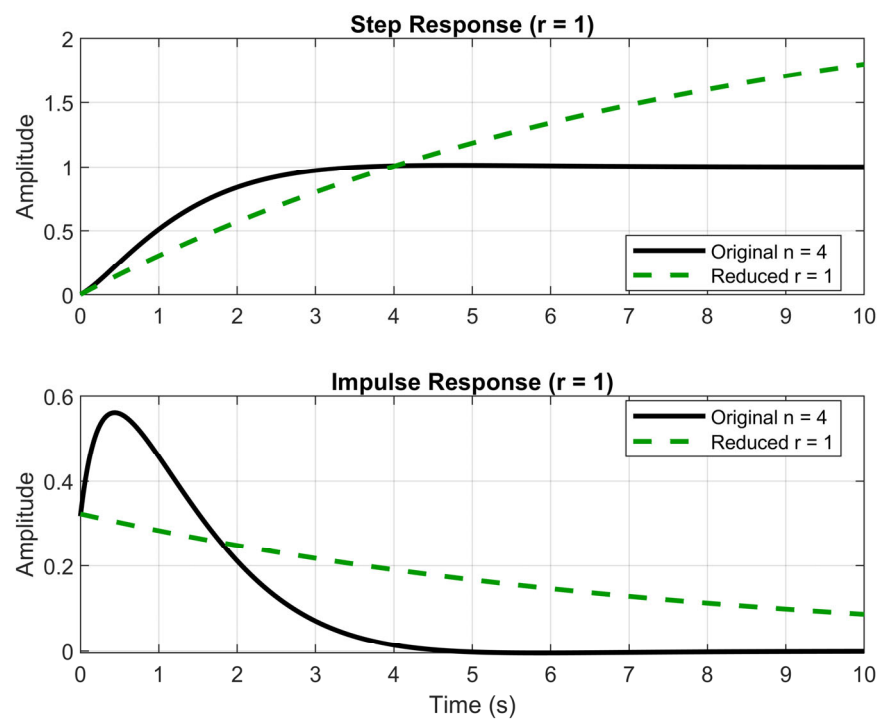


Figure 9. Step and impulse response comparison for reduced order $r = 1$.

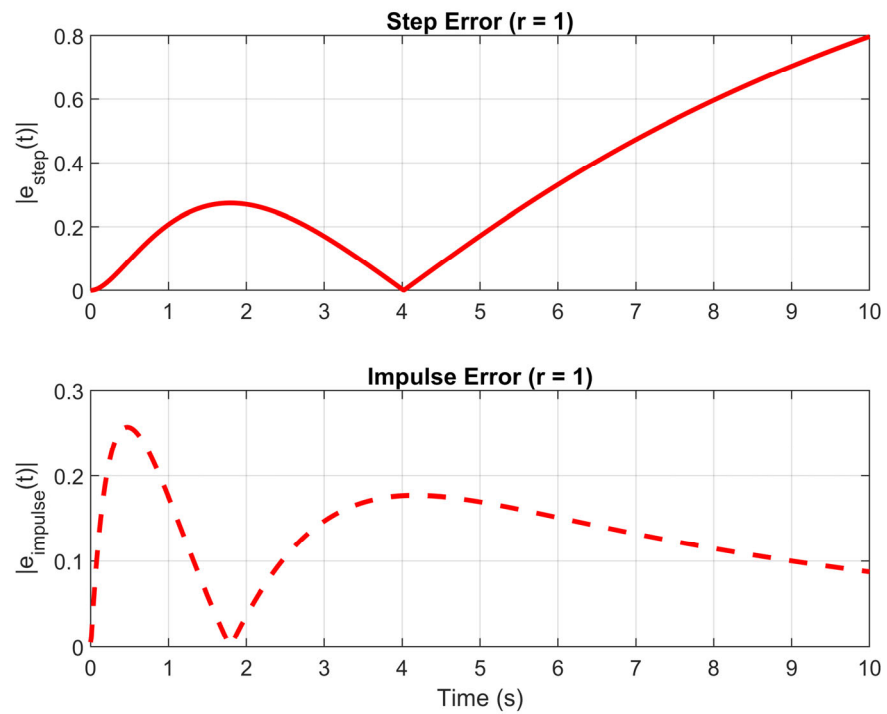


Figure 10. Step and impulse response errors for reduced order $r = 1$.

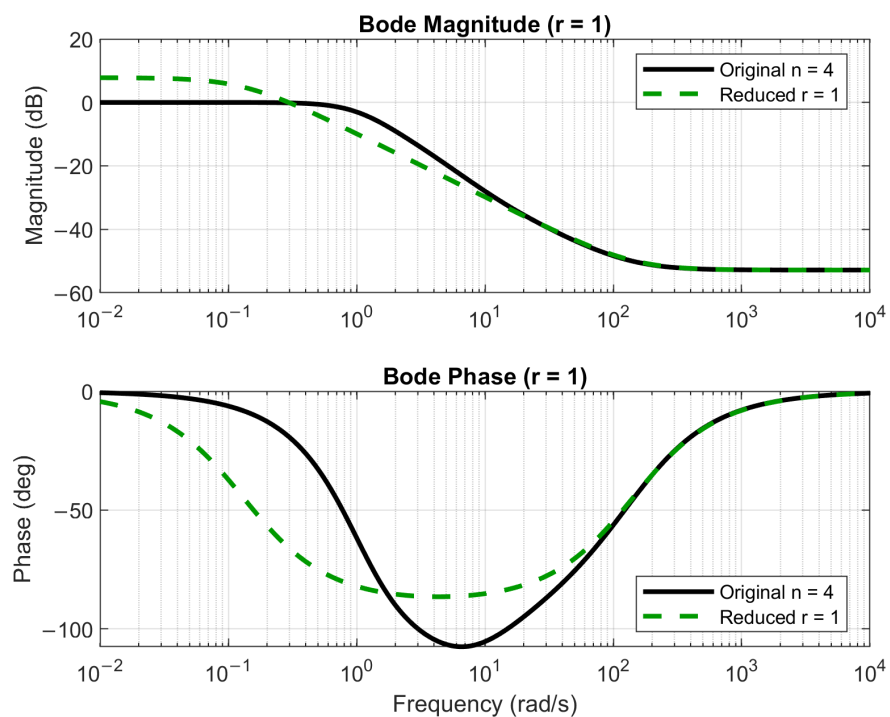


Figure 11. Bode magnitude and phase response comparison for reduced order $r = 1$.

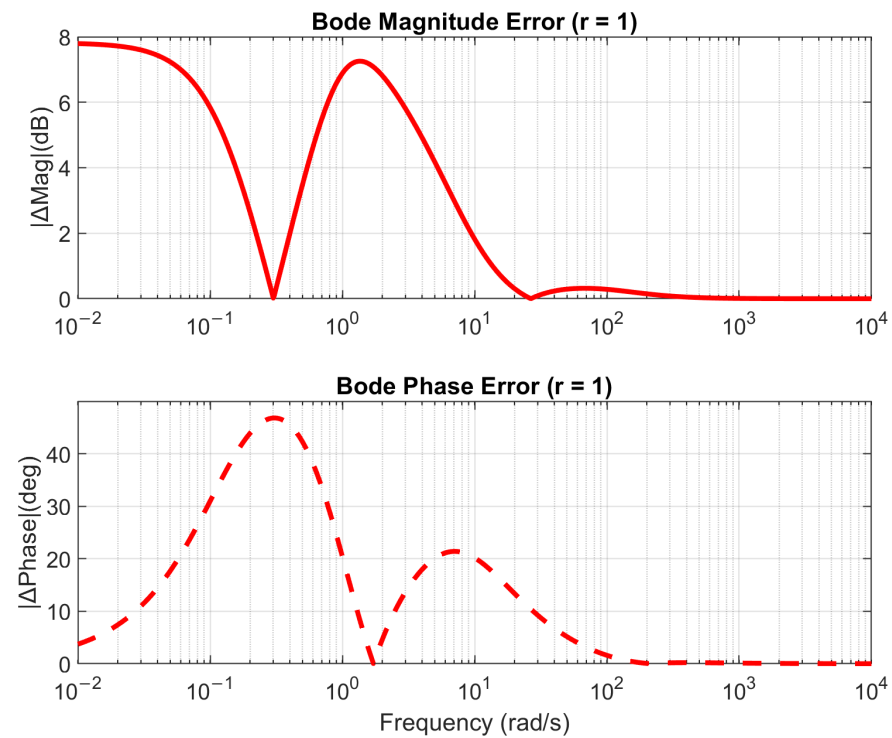


Figure 12. Bode magnitude and phase errors for reduced order $r = 1$.



Figure 13. Composite performance score for different reduced orders.

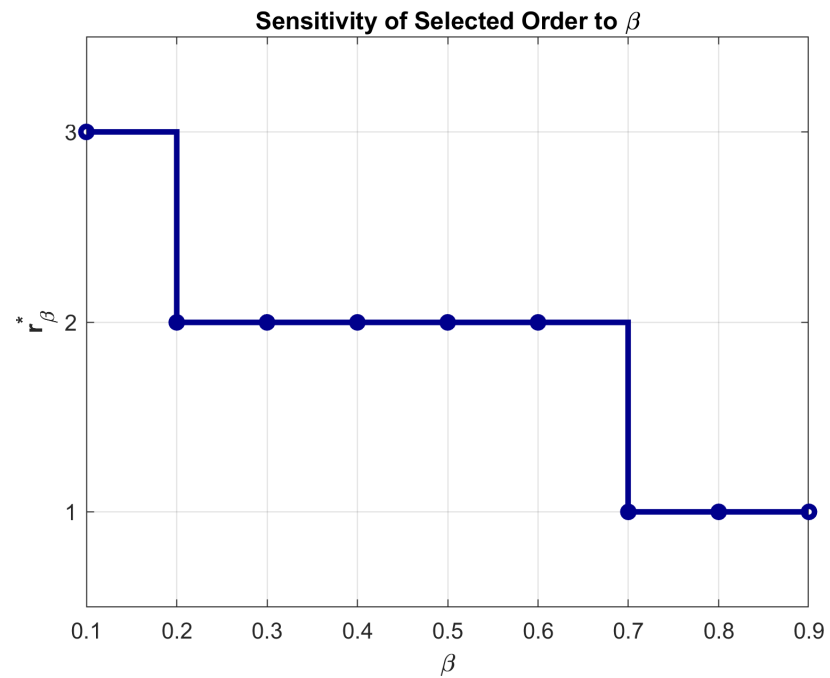


Figure 14. Sensitivity of the selected reduced order to the parameter β .

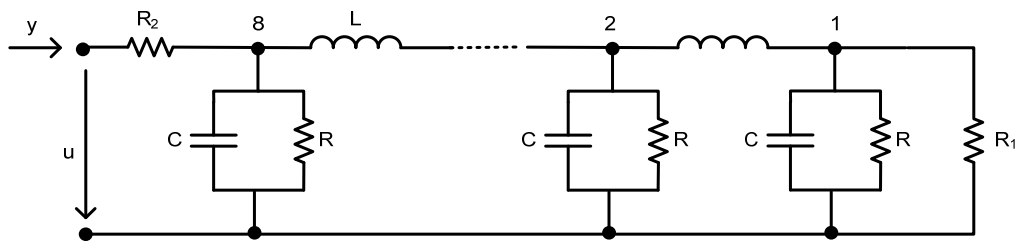


Figure 15. RLC ladder circuit.

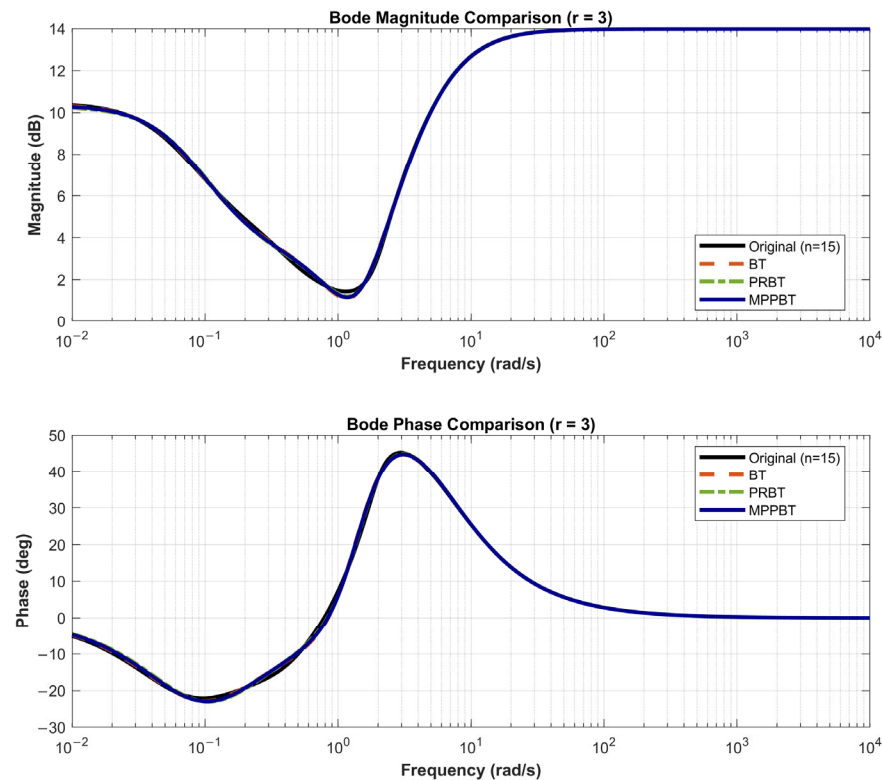


Figure 16. Bode-response comparison of the original 15th-order model and the 3rd-order models obtained using BT, PRBT, and MPPBT.

3.1. MPPBT Reduction Results

Table 1 provides the H_∞ and H_2 norm errors for the reduced-order models. The model with $r = 3$ has very small errors, with $H_\infty \approx 6.32 \times 10^{-3}$ and $H_2 \approx 2.23 \times 10^{-3}$, so it preserves almost full accuracy. The model with $r = 2$ increases the errors to the range of 10^{-2} – 10^{-1} , and the model with $r = 1$ gives errors of order 10^0 , which shows a clear degradation of approximation quality.

Table 2 reports the time-domain error indices of the step responses. The highlighted measures, namely ISE, IAE, ITSE, ITAE, and RMSE, quantify complementary aspects of the transient mismatch: cumulative squared error, cumulative absolute error, late-time squared error, late-time absolute error, and effective error magnitude, respectively. For $r = 3$, all five representative indices remain small, with $ISE \approx 2.92 \times 10^{-5}$, $IAE \approx 1.04 \times 10^{-2}$, $ITSE \approx 2.58 \times 10^{-4}$, $ITAE \approx 8.61 \times 10^{-2}$, and $RMSE \approx 1.71 \times 10^{-3}$. When the order is reduced to $r = 2$ and $r = 1$ these indices increase consistently, indicating progressively poorer reproduction of both the overall and late transient dynamics. The unhighlighted quantities are additionally reported to indicate the range and signed bias of the response error.

Table 3 provides an additional time-domain validation of the order selected. As expected, the third-order model yields the smallest impulse-response errors; however, this improvement is obtained at a higher model complexity. The selected second-order model maintains low impulse-response errors, with $ISE = 1.78 \times 10^{-3}$, $IAE = 1.04 \times 10^{-1}$, $ITSE = 7.20 \times 10^{-3}$, $ITAE = 4.45 \times 10^{-1}$, and $RMSE = 1.33 \times 10^{-2}$, while using one fewer state than the third-order model. In contrast, the first-order model exhibits substantially larger errors across all highlighted measures, indicating that the reduction to $r = 1$ causes an unacceptable loss of transient fidelity.

Table 4 summarizes the magnitude and phase errors in the frequency domain. As expected, $r = 3$ yields the smallest deviations. When $r = 2$ retains the main frequency-

response characteristics, with mean magnitude and phase errors of 2.25×10^{-1} dB and 9.07×10^{-1} deg, respectively, its maximum errors are limited to 7.05×10^{-1} dB and 4.21 deg. In contrast, $r = 1$ produces much larger magnitude and phase deviations.

Table 5 shows the step-response parameters of the reduced-order models. The model with $r = 3$ has a rise time of ≈ 2.14 s, a settling time of ≈ 3.21 s, an overshoot of $\approx 0.83\%$ and a peak value of ≈ 1.014 , so it is very close to the desired dynamics. In contrast, the model with $r = 2$ exhibits a slower response, reaches its peak later, and has a peak magnitude of approximately 1.084. Moreover, the first-order model ($r = 1$) has a settling time of approximately 29.76 s and a peak value of approximately 2.4567, thereby exhibiting substantially different response characteristics.

Table 6 shows the DC gain and phase margin of the reduced-order models. As the order decreases, the DC-gain deviation increases and the phase margin decreases. The second-order model retains a DC gain of 1.084 and a phase margin of 154.0 deg, indicating satisfactory low-frequency fidelity together with a substantial stability margin. Although the third-order model provides values closer to the full-order reference, the first-order model exhibits a noticeably larger DC-gain deviation and a reduced-phase margin. All reduced models remain stable and minimum phase.

Figure 1 illustrates the step and impulse responses of the original system of order $n = 4$ compared with the reduced model with $r = 3$. The two curves almost coincide over the entire interval $0 \div 10$ s, and the steady-state amplitude of the step response is approximately 1. The peak of the impulse response is about 0.56 at $t \approx 0.7$ s for both models.

Figure 2 shows the absolute errors of the step and impulse responses for $r = 3$. The amplitude $|e_{\text{step}}(t)|$ remains below about 4×10^{-3} over the interval $0 \div 10$ s. The amplitude $|e_{\text{impulse}}(t)|$ reaches a maximum of about 1.1×10^{-3} , which indicates that the time-domain deviation is very small.

Figure 3 shows the Bode magnitude and phase characteristics of the original system and the reduced model with $r = 3$. The magnitude stays close to 0 dB at low frequencies and decreases to about -60 dB at high frequencies with the two curves almost overlapping. The phase varies from 0° down to approximately -100° in the range of 10^{-2} – 10^4 rad/s without any clearly visible discrepancy.

Figure 4 describes the Bode magnitude and phase errors for $r = 3$. The magnitude error $|\Delta \text{Mag}|$ reaches a maximum of about 5.5×10^{-2} dB in the mid-frequency region. The phase error $|\Delta \text{Phase}|$ reaches a maximum of about 0.31° , concentrated around the phase crossover region, so its impact on controller design can be neglected.

Figure 5 shows the step and impulse responses when the reduced model with $r = 2$ is used. The step response of the reduced model reaches a peak of about 1.08 at $t \approx 13$ s, which produces a steady-state-level deviation compared with the original value of approximately 1. The impulse response still follows the general shape, but the tail decays more slowly, and the amplitude for $t > 3$ s is larger than that of the full-order system.

Figure 6 presents the absolute step and impulse response errors for the model with $r = 2$. The amplitude $|e_{\text{step}}(t)|$ increases and reaches about 8×10^{-2} at $t = 10$ s. The amplitude $|e_{\text{impulse}}(t)|$ attains a maximum of about 2.6×10^{-2} , so the time-domain error is larger by about one to two orders of magnitude compared with the case of $r = 3$.

Figure 7 shows the Bode characteristics of the original system and the model with $r = 2$. The magnitudes of the two models are almost identical over the entire frequency range, and the low-frequency value remains close to 0 dB. The phase exhibits a small deviation in the mid-frequency region around 0.1–1 rad/s but gradually converges at very low and very high frequencies.

Figure 8 depicts the Bode magnitude and phase errors corresponding to $r = 2$. The maximum magnitude error is approximately 0.7 dB, and the maximum phase error is

approximately 4.3° around a frequency of about 0.5 rad/s. The errors decrease rapidly toward nearly 0 when the frequency approaches the two ends of the range of 10^{-2} to 10^4 rad/s, so the reduced model still preserves the global magnitude-phase characteristics in a satisfactory manner.

Figure 9 shows the step and impulse responses of the original system with the order $n = 4$ compared with the reduced model with $r = 1$. The step response of the model with $r = 1$ increases slowly, does not reach steady state in $0 \div 10$ s, and has an amplitude at $t = 10$ s of approximately 1.8 while the original system settles around 1. The impulse response of the model with $r = 1$ decreases monotonically from about 0.33 to 0.1 over $0 \div 10$ s and misses the peak of about 0.56 at $t \approx 0.7$ s of the original system.

Figure 10 describes the absolute step and impulse response errors for the model with $r = 1$. The value $|e_{\text{step}}(t)|$ reaches about 0.28 at $t \approx 2$ s, drops close to 0 around $t \approx 4$ s, then increases gradually to about 0.8 at $t = 10$ s. The value $|e_{\text{impulse}}(t)|$ attains a maximum near 0.27 in the initial phase and remains above 0.1 over most of the observation interval, which reflects a rather large time-domain deviation.

Figure 11 depicts the Bode magnitude and phase characteristics of the original system and the model with $r = 1$. At low frequencies, the reduced model gives a magnitude of about $+7 \div 8$ dB, while the original system is close to 0 dB, and the two curves only merge around -60 dB in the high-frequency region. The phase of the model with $r = 1$ deviates significantly in the range of $10^{-2} \div 10^1$ rad/s with differences of several tens of degrees before converging back near 0° at very high frequencies.

Figure 12 shows the Bode magnitude and phase errors for $r = 1$. The magnitude error $|\Delta \text{Mag}|$ reaches a maximum of approximately 7.8 dB around frequencies of order $10^{-2} \div 10^0$ rad/s and only decreases toward 0 above 10^2 rad/s. The phase error $|\Delta \text{Phase}|$ reaches a maximum value of about 47° near 0.5 rad/s, which shows that the first-order model severely distorts the phase characteristic.

3.2. Order Selection and Sensitivity Analysis

The evaluation results for each criterion show that the model with $r = 3$ attains the lowest error in most indices. For example, the H_∞ norm at $r = 3$ is 6.322192×10^{-3} , whereas at $r = 1$ it increases to about 1.458838. The integral squared error of the step response ISE_{step} at $r = 3$ is 2.915615×10^{-5} , while at $r = 1$ it rises to 1.663140. The magnitude and phase boundary errors in the frequency domain indicate a significant degradation for the model with $r = 1$; for instance, the maximum magnitude error $\text{MagErr}_{\text{max}}$ at $r = 1$ reaches 7.792448 dB, whereas at $r = 3$ it is only about 5.455007×10^{-2} dB.

The metrics related to the impulse response exhibit the same trend. The integral squared error ISE_{imp} at $r = 3$ is 4.270527×10^{-6} , while at $r = 1$ it is 2.046788×10^{-1} . The RMS error of the impulse response at $r = 3$ is 6.533344×10^{-4} and increases to 1.430078×10^{-1} at $r = 1$. The mean phase error $\text{PhErr}_{\text{mean}}$ at $r = 1$ reaches 1.285721×10^1 deg, which is much larger than the value at $r = 3$, equal to 5.584468×10^{-2} deg.

Model $r = 3$ gives the best values for each individual criterion. The composite score shows that $r = 2$ achieves 1.556392×10^{-1} , which is smaller than $r = 3$ ($\approx 2.00 \times 10^{-1}$) and significantly lower than $r = 1$ ($\approx 8.00 \times 10^{-1}$). Figure 13 illustrates this trend, in which the CompositeScore curve decreases from $r = 1$ to $r = 2$ and then increases slightly at $r = 3$. The final result indicates that $r = 2$ is more suitable because it achieves a small error with a lower model order and still satisfies stability and minimum-phase properties.

To examine the robustness of the proposed order-selection procedure, a sensitivity analysis was conducted with respect to the trade-off parameter β . As illustrated in Figure

14, the selected reduced order varies in a consistent and interpretable manner as β changes. For a β ranging from 0 to 0.1, the method selects $r^* = 3$, since the error-related criteria are given dominant importance. When β ranges from 0.2 to 0.6, the selected order remains unchanged at $r^* = 2$, demonstrating that the recommended second-order model is robust over a relatively broad range of balanced weighting values. For larger values of β , namely $\beta \geq 0.7$, the procedure selects $r^* = 1$, reflecting the stronger emphasis on achieving a more compact reduced model. Therefore, the choice of $\beta = 0.2$ adopted in this study lies within the stable interval in which $r = 2$ is consistently selected. This result confirms that the proposed method provides a meaningful trade-off between approximation accuracy and model compactness.

3.3. Comparison with BT and PRBT

Following verification of the proposed MPPBT method, this subsection compares its model-reduction performance with that of conventional balanced truncation (BT) and positive-real balanced truncation (PRBT). The comparison is conducted using the passive RLC ladder circuit benchmark described in [39] and illustrated in Figure 15. The circuit comprises eight interconnected nodes and has a total order of $n = 15$, with the capacitor voltages and inductor currents selected as the state variables. For a fair comparison, BT, PRBT, and MPPBT are applied to the same full-order model at the same prescribed reduced order.

Having identified $r = 3$ through the proposed composite-score criterion, we next compare the resulting MPPBT model with 3rd-order models obtained by BT and PRBT.

Table 7 reports the H_∞ and H_2 approximation errors of BT, PRBT, and MPPBT for the RLC ladder circuit of order $n = 15$ at the selected reduced order $r = 3$. MPPBT achieves the smallest H_∞ error, equal to 4.13×10^{-2} , compared with 4.68×10^{-2} for BT and 6.32×10^{-2} for PRBT. Regarding the H_2 criterion, BT attains the lowest error of 4.30×10^{-2} , while MPPBT produces a very close value of 4.35×10^{-2} ; PRBT has a larger H_2 error of 5.01×10^{-2} . Therefore, MPPBT provides the best worst-case approximation performance in the H_∞ sense and remains nearly as accurate as BT in the H_2 sense. These results indicate that MPPBT achieves a favorable balance between approximation accuracy and its intended system-property-preservation capability.

Figure 16 compares the Bode magnitude and phase responses. All three reduced-order models reproduce the overall frequency-domain characteristics of the full-order system well. In particular, MPPBT closely follows the original response in both magnitude and phase, including the magnitude minimum near 1–2 rad/s and the phase peak around a few rad/s. Only minor deviations are observed in the low-frequency and transition regions, confirming the good approximation performance of MPPBT at $r = 3$.

Figure 17 shows the Bode magnitude and phase errors of the reduced-order models relative to the original system. The errors are concentrated mainly in the low-to-intermediate frequency range. For the magnitude response, MPPBT and PRBT exhibit similar error levels, while BT shows a slightly larger negative peak near 1 rad/s. For the phase response, MPPBT generally provides smaller peak deviations than BT, particularly around 1–2 rad/s, whereas PRBT shows somewhat larger deviations at very low frequencies. Overall, the Bode-error plots confirm that MPPBT achieves accurate magnitude and phase matching across the frequency range.

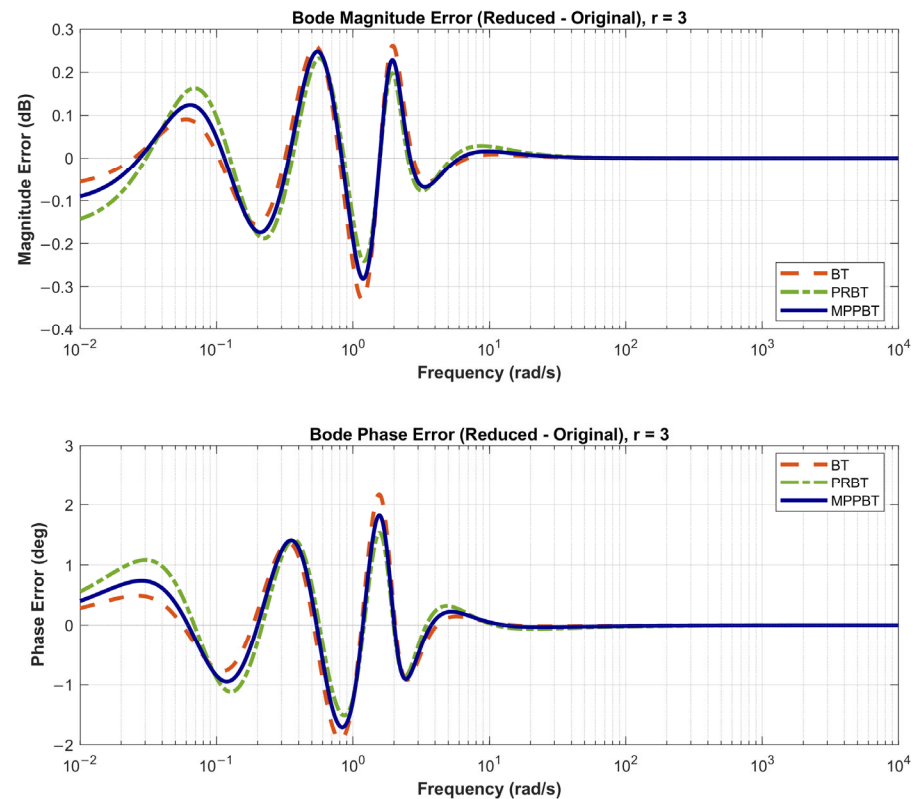


Figure 17. Bode magnitude and phase errors for the 3rd-order models obtained using BT, PRBT, and MPPBT.

The time-domain responses in Figure 18 support the frequency-domain results. The BT, PRBT, and MPPBT reduced-order models closely overlap the original step and impulse responses, accurately reproducing the initial transient, the undershoot in the step response, and the subsequent recovery. Only negligible differences are visible among the reduced models over the displayed time interval. Therefore, at the selected reduced order $r = 3$, MPPBT achieves time-domain accuracy comparable to that of BT and PRBT while preserving the minimum-phase property.

Figure 19 shows the step- and impulse-response errors of the third-order reduced models with respect to the original RLC ladder system. The step-response errors of all methods remain small, with peak deviations below approximately 2.5×10^{-2} ; MPPBT generally lies between BT and PRBT over the displayed interval. For the impulse response, all errors decay rapidly and remain close to zero after the initial transient. MPPBT exhibits error behavior comparable to BT and PRBT, confirming that it accurately captures the dominant time-domain dynamics at $r = 3$.

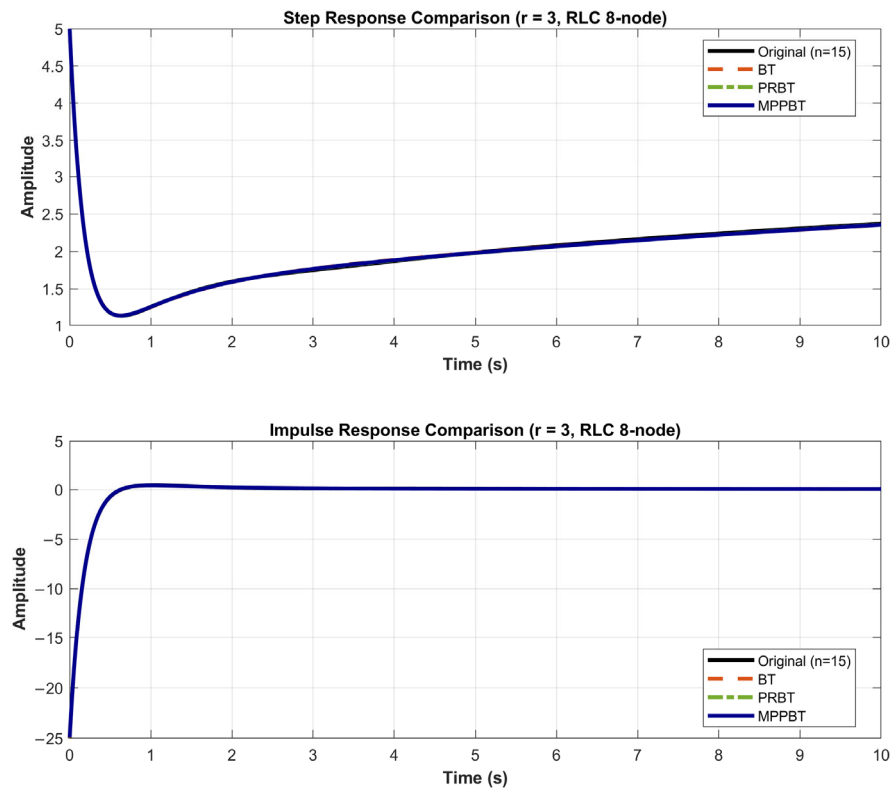


Figure 18. Time-domain response comparison of the original model and the 3rd-order models obtained using BT, PRBT, and MPPBT.

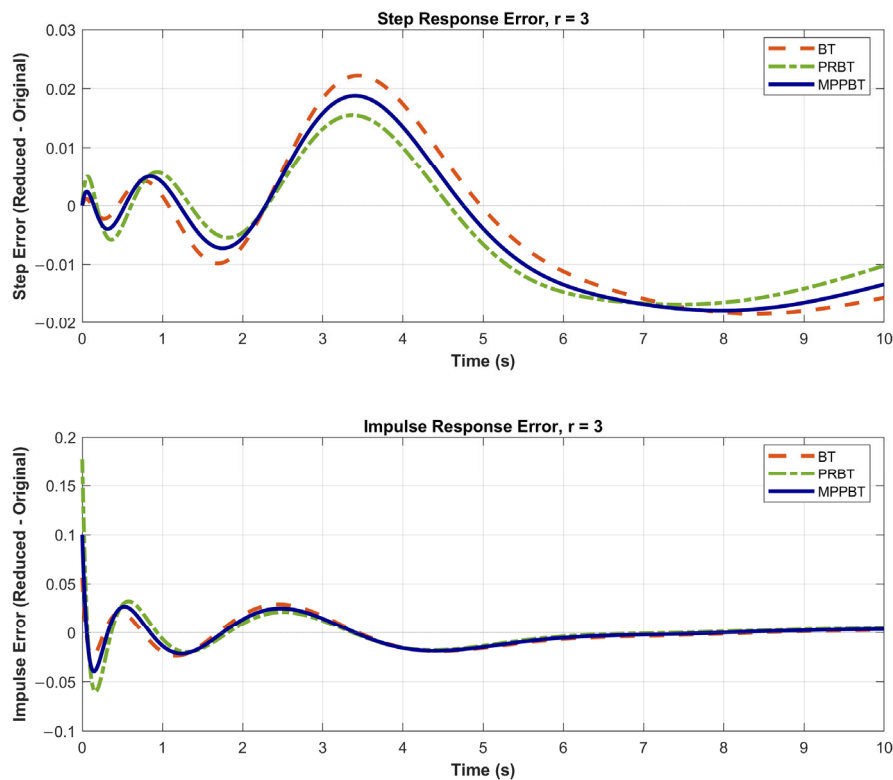


Figure 19. Step- and impulse-response errors of the third-order BT, PRBT, and MPPBT models relative to the original model.

Finally, the present example is of modest order and is intended primarily to validate the approximation and minimum-phase-preservation properties of MPPBT. The computational cost of Gramian-based reduction schemes involving Lyapunov and Riccati equations, including the relative cost implications of solving one Lyapunov equation and one Riccati equation, has been analyzed in detail in [40]. For large-scale implementations, low-rank matrix-equation solvers and Krylov-subspace techniques may be used to reduce the memory and computational burden; a detailed scalability study of MPPBT is left for future work.

4. Conclusions

This study presents a model-order-reduction framework for linear systems with minimum-phase behavior with emphasis on a Riccati–Lyapunov-type pair of Gramians and the MPPBT balancing transformation. The theoretical part establishes MPPBT-admissible conditions that guarantee the existence of a unique pair of positive definite Gramians P and Q and sets up a balancing transformation so that the two Gramians coincide in the new coordinates and are associated with a sequence of singular values, σ_i , that quantify controllability–observability energy. On this basis a relative error bound in the H_∞ norm is expressed through the function Ψ_r , which provides a direct link between the magnitude of σ_i and the quality of the reduced model in the minimum-phase setting.

The numerical assessment applies this framework to a fourth-order Butterworth low-pass filter with $n = 4$ and three reduced models with $r = 1, 2, 3$ to examine in detail frequency-domain and time-domain errors and amplitude–phase indices. The third-order model yields $H_\infty \approx 0.0063$ and $H_2 \approx 0.0022$, whereas the first-order model gives $H_\infty \approx 1.46$ and $H_2 \approx 0.483$, leading to pronounced deviations in the step response, impulse response, and frequency response. A metric set of multi-criteria together with a min–max normalization mechanism and the score function S_β under uniform weights and a fixed β shows that the second-order model attains $S_\beta \approx 0.16$, which is lower than the third-order model with $S_\beta = 0.20$ and much lower than the first-order model with $S_\beta = 0.80$, while preserving stability and the minimum-phase property of the original system. To further assess its accuracy relative to existing methods, MPPBT was compared with classical balanced truncation (BT) and positive-real balanced truncation (PRBT) on a 15th-order passive RLC ladder circuit at a common reduced order of $r = 3$. MPPBT achieved the smallest $\mathcal{H}_\infty$ error, 0.0413, compared with 0.0468 for BT and 0.0632 for PRBT, while its $\mathcal{H}_2$ error of 0.0435 remained close to that of BT, 0.0430. The Bode, step, and impulse responses further confirmed that MPPBT accurately reproduces the dominant frequency- and time-domain behavior of the full-order circuit. In addition, the sensitivity analysis of the trade-off parameter β showed that the score-based order-selection procedure changes in an interpretable manner as the relative importance of approximation accuracy and model compactness varies. In particular, the nominal choice of β lies in a stable interval in which the same reduced order is selected, supporting the robustness of the proposed scoring procedure.

These results open up the possibility of applying the MPPBT framework to multi-input multi-output systems in power electronic engineering and high-speed interconnect circuits where the minimum-phase structure plays a decisive role for control quality and design robustness. One direction focuses on refining the criterion vector and the weights in S_β for RLC ladder circuits, high-order electromechanical systems, and industrial control systems with many state variables so that the scoring matches requirements on amplitude, phase, energy, and transient time more closely. One further direction emphasizes extending the MPPBT theoretical foundation to classes of stochastic bilinear or quadratic–bilinear systems combined with data-driven balancing and machine learning techniques

in order to build a unified reduction scheme for large-scale minimum-phase systems in diverse engineering settings.

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Abbreviations

The following abbreviations are used in this manuscript:

ISE	Integral of Squared Error
IAE	Integral of Absolute Error
ITSE	Integral of Time-weighted Squared Error
ITAE	Integral of Time-weighted Absolute Error
MSE	Mean Squared Error
RMSE	Root Mean Squared Error
SISO	Single Input Single Output

Appendix A

Lemma A1. For an MPPBT-admissible system, Equation (1) has a unique solution, $P = P^T > 0$ and $P = \int_0^{+\infty} e^{At} B B^T e^{A^T t} dt$ [37].

Proof of Lemma A1. A Hurwitz $\Rightarrow \exists c, \alpha > 0: \|e^{At}\| \leq ce^{-\alpha t}$, so the integral converges and $P = P^T \geq 0$. Controllability of (A, B) implies $P > 0$. Differentiating $e^{At} P e^{A^T t}$ and integrating over $[0, \infty)$ yields (1). Injectivity of $\mathcal{L}_A$ on symmetric matrices ensures uniqueness. $\square$

Lemma A2. For an MPPBT-admissible system, Equation (2) admits a stabilizing solution, $Q = Q^T > 0$, such that the closed-loop matrix $A_Q := A - \mathfrak{B}_w R_0^{-1} (C - \mathfrak{B}_w^T Q)$ is Hurwitz [12,36].

Proof of Lemma A2. Equation (2) is a continuous-time “Schur/relative-error” Riccati equation associated with spectral-factor/relative-error constructions; under the standing assumptions (stability, D nonsingular, and minimum phase), a stabilizing solution exists and yields A_Q Hurwitz. $\square$

Lemma A3. Assume $P, Q > 0$. Take factors $P = R R^T$ and $Q = L L^T$, with invertible $R, L \in \mathbb{R}^{n \times n}$, and define $H := L^T R$. Then there exists an SVD $H = U \Sigma V^T$, with U, V orthonormal and $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$, $\sigma_1 \geq \dots \geq \sigma_n > 0$. The σ_i are the MPPBT singular values [37,38].

Proof of Lemma A3. $P, Q > 0 \Rightarrow R, L$ invertible $\Rightarrow H$ invertible $\Rightarrow \sigma_i > 0$. Standard SVD applies. $\square$

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