

Article

Configuration Assignments to $K = 1/2$ Three-Quasiparticle Bands

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Abstract

In the present work, configuration assignments to $K = 1/2$ bandheads of given three-quasiparticle (3qp) quadruplets experimentally observed and yet to be observed in odd-A nuclei lying in the rare-earth mass region are carried out based on an improved version of an earlier empirical formulation. The important contributions of the irrotational component, residual n - p interactions, and rotor-particle coupling terms are explored and highlighted. The magnitude of the RPC term is found to be comparatively smaller (≈ 5 – 10 keV) than the irrotational and residual interactions and can affect the 3qp bandheads with small energy separation. We confirm the tentative configuration assignments of three $K = 1/2$ bandheads having configurations $1/2[411]_{\pi} \otimes 7/2[523]_{\pi} \otimes 5/2[523]_{\nu}$ and $1/2[411]_{\pi} \otimes 9/2[514]_{\pi} \otimes 7/2[514]_{\nu}$ in ^{175}Yb and $9/2[514]_{\pi} \otimes 7/2[514]_{\nu} \otimes 1/2[521]_{\nu}$ in ^{177}Ta nuclides. The average deviation between experimental data and present model calculations for ^{175}Yb and ^{177}Ta is 37.9 keV and 163.2 keV, as compared to the earlier model estimates, which yielded a deviation of 249.7 keV and 222.3 keV, respectively. We also resolved some issues pertaining to ambiguous configuration assignments of 3qp quadruplets observed in ^{177}Lu , ^{173}Ta , and ^{179}Ta nuclides. Encouraged by the good overall agreement among experimental and calculated bandhead energies, we have extended these calculations to predict the locations of 18 $K = 1/2$ bandheads, which are members of 3qp quadruplets of $^{171,175,177}\text{Lu}$, $^{173,179,181}\text{Ta}$, and $^{179,183}\text{Re}$ nuclides but have not been observed experimentally and will be useful for predicting bandheads associated with various 3qp states.

Keywords: three-quasiparticle quadruplets; bandhead energies; residual interactions; configuration assignments; $K = 1/2$ bands; Gallagher–Moszkowski splitting; Newby shifts

1. Introduction

In odd-A nuclei, the interplay between collective rotation and intrinsic quasiparticle excitations determines the spectroscopic properties and hence plays a crucial role in understanding nuclear structure. Approximately 1 MeV of energy is required to generate the first $8\hbar$ of collective angular momentum [1]. The lowest excitation, i.e., one-quasiparticle (1qp) states, arise from the unpaired nucleon coupled to a rotating even-even core, while the first-seniority excitations, namely the three-quasiparticle (3qp) states, involve coupling of this unpaired nucleon with a broken Cooper pair. Bandhead energies of these configurations serve as fingerprints for identifying specific quasiparticle alignments, mixing among various states, and also provide essential input for theoretical models such as the particle plus rotor model, projected shell model, and cranked shell model to explain the properties associated with rotational bands. Specifically, the lowest three-quasiparticle (3qp) excitations come into existence by breaking of one proton/neutron pair provided the



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excitation energy is more than that of the average pairing gap, which is about 1–1.5 MeV for rare-earth nuclei [2]. In principle, two classifications of 3qp states exist: one with all three nucleons of the same kind (*nnn/ppp*) and the other having a combination of two kinds of nucleons (*npp/pnn*). Further, the coupling of the projections of intrinsic angular momenta of individual particles (labeled as k_1 , k_2 and k_3) on the symmetry axis leads to the formation of a 3qp quadruplet with resultant $K = |k_1 \pm k_2 \pm k_3|$, where K is the projection of total angular momentum on the symmetry axis [3]. Although eight sign combinations can formally be generated by independently assigning positive and negative signs to the three intrinsic angular-momentum projections, only four distinct values of K are physically independent, therefore reducing the eight formal combinations to the four unique values, $|k_1 + k_2 + k_3|$, $|k_1 + k_2 - k_3|$, $|k_1 - k_2 + k_3|$, and $|-k_1 + k_2 + k_3|$, which are degenerate in the absence of residual neutron–proton (*n-p*) interaction.

Its main components of residual interaction are the Gallagher–Moszkowski (GM) splitting [4] and the Newby shift (NS) [5] energies. The GM splitting originates from the parallel and antiparallel coupling of the proton and neutron angular momenta, while the NS introduces an additional energy displacement between the even- and odd-spin members of $K = 0$ bands. In the absence of these effects, all four members of a given 3qp quadruplet would energetically degenerate. However, this degeneracy is removed by the perturbation introduced by residual *n-p* interactions, and the members of a given quadruplet become split.

In the present study, we have calculated the bandhead energies of $K = 1/2$ states pertaining to 3qp quadruplets having Nilsson’s configurations $5/2[523]_v \otimes 1/2[411]_\pi \otimes 7/2[523]_\pi$ and $1/2[411]_\pi \otimes 9/2[514]_\pi \otimes 7/2[514]_v$ in ^{175}Yb , and $9/2[514]_\pi \otimes 7/2[514]_v \otimes 1/2[521]_v$ in ^{177}Ta nuclides, and obtained reasonable agreement with the experimental data. The results of the present calculations are in good agreement with experimental data, which shows the vital contributions of ignored terms (irrotational, rotor–particle, and particle–particle couplings) in earlier model calculations [3]. Additionally, we have tried to resolve ambiguous configuration assignments of 3qp quadruplets observed in ^{177}Lu , ^{173}Ta , and ^{179}Ta nuclides, which could not be confirmed from the available experimental data. We found that the irrotational contribution and rotor–particle coupling terms play a key role, which were neglected in the earlier models and contribute to the bandhead energies of $K = 1/2$ bands only. We have also extended these calculations for the prediction of 18 $K = 1/2$ bandheads, which are yet to be observed experimentally. The identification and understanding of such states contribute significantly to improving the understanding of quasiparticle coupling schemes, pairing correlations, and residual interactions in the nuclear many-body system. The structure of this paper is as follows: Section 2 provides a brief overview of the model’s methodology. Section 3 presents a detailed discussion of the results obtained for various nuclides and, finally, in Section 4, we summarize the main conclusions of this study.

2. Theoretical Framework

The detailed formulation of the present theoretical framework, which is an extended and improved version of the earlier empirical formulation [3] for the calculation of bandhead energies of a particular 3qp quadruplet, is based on the following assumptions: (i) the unperturbed excitation energy of a three-quasiparticle (3qp) configuration is obtained as the sum of the corresponding one-quasiparticle (1qp) excitation energies of the valence nucleons in neighboring odd-A nuclei; (ii) the effective moment of inertia of a 3qp configuration can be estimated either from the experimentally observed rotational levels of the same band using an evaluated set of nuclear data [6], when available, or from the average of moments of the inertia of the neighboring odd-A nuclei and the even–even core; and (iii) the non-diagonal contributions arising from the rotor–particle coupling (RPC) and

particle–particle coupling (PPC) interactions are assumed to be negligible, and therefore only their diagonal matrix elements are included in the bandhead energy calculations. In view of the above-said assumptions, the total excitation energy of a particular member of a given 3qp quadruplet is given by

$$E(K) = E_{\text{pairing}} + E_{qp} + E_{\text{rot}} + E_{\text{irrot}} + E_{\text{res}} + E_{\text{rpc}} + E_{\text{ppc}} \quad (1)$$

The proton and neutron pairing energies are calculated using the four-point mass difference formulae [2], which provide an estimate of the energy required to break a nucleon pair. The explicit forms of these formulae are given below:

$$\Delta_p = \frac{1}{4} \{B(N, Z - 2) - 3B(N, Z - 1) + 3B(N, Z) - B(N, Z + 1)\} \quad (2)$$

$$\Delta_n = \frac{1}{4} \{B(Z, N - 2) - 3B(Z, N - 1) + 3B(Z, N) - B(Z, N + 1)\} \quad (3)$$

The binding energies used in proton and neutron pairing are taken from recent atomic mass evaluations [7]. The unperturbed excitation energy of a three-quasiparticle (3qp) quadruplet is obtained by summing the excitation energies of the corresponding one-quasiparticle (1qp) states of the three valence nucleons observed in neighboring odd-A nuclei (i.e., $E_{qp} = E_{qp}^{(1)} + E_{qp}^{(2)} + E_{qp}^{(3)}$). The E_{rot} is the rotational energy and can be obtained using the following expression: $E_{\text{rot}} = \hbar^2/2\mathfrak{I}(I(I + 1) - K^2)$. The irrotational correction originates from the recoil term, i.e., $H_{\text{rec}} = \hbar^2/2\mathfrak{I}(j^2 - j_z^2)$ represents the contribution of the intrinsic quasiparticle angular momentum to the rotational energy. Its explicit expression is given below.

$$\langle IMK'\alpha' | H_{\text{irrot}} | IMK\alpha \rangle = \frac{\hbar^2}{2\mathfrak{I}} \left[\begin{aligned} &\left(\sum_{j_1} |C_{k_1}^{j_1}|^2 j_1(j_1 + 1) - k_1^2 \right) + \\ &\left(\sum_{j_2} |C_{k_2}^{j_2}|^2 j_2(j_2 + 1) - k_2^2 \right) + \\ &\left(\sum_{j_3} |C_{k_3}^{j_3}|^2 j_3(j_3 + 1) - k_3^2 \right) \end{aligned} \right] \delta_{K'K} \delta_{\alpha'\alpha} \quad (4)$$

where j_1, j_2, j_3 and k_1, k_2, k_3 are the intrinsic angular momenta of three valence nucleons and their corresponding projections on the symmetric axis, respectively, and $|C_{k_1}^{j_1}|^2, |C_{k_2}^{j_2}|^2$, and $|C_{k_3}^{j_3}|^2$ are Nilsson's amplitudes [8]. The three summations represent the contributions of the individual quasiparticles to the total irrotational correction, weighted by their respective Nilsson amplitudes, while the Kronecker delta ensures that the correction is diagonal in the intrinsic basis. The contribution of residual n - p interaction energy (E_{res}) responsible for the breaking of degeneracy among different members of a given 3qp quadruplet is given as

$$\begin{aligned} E_{\text{res}} = & \sigma_{(1,2)} \left\{ \left[\frac{1}{2} - \delta_{\Sigma(1,2),0} \right] E_{(1,2)}^{\text{split}} - \delta_{K(1,2),0} E_{(1,2)}^N \Pi_{(1,2)} \right\} \\ & + \sigma_{(2,3)} \left\{ \left[\frac{1}{2} - \delta_{\Sigma(2,3),0} \right] E_{(2,3)}^{\text{split}} - \delta_{K(2,3),0} E_{(2,3)}^N \Pi_{(2,3)} \right\} \\ & + \sigma_{(1,3)} \left\{ \left[\frac{1}{2} - \delta_{\Sigma(1,3),0} \right] E_{(1,3)}^{\text{split}} - \delta_{K(1,3),0} E_{(1,3)}^N \Pi_{(1,3)} \right\} \end{aligned} \quad (5)$$

The term $\delta_{\Sigma(x,y),0} = 1$, if the intrinsic spins of coupled particles are anti-parallel and $\delta_{\Sigma(x,y),0} = 0$, if the intrinsic spins of coupled particles are parallel; $\sigma_{(x,y)} = 1$ for like particles and $\sigma_{(x,y)} = -1$ for unlike particles. The term $\Pi_{(x,y)} = +1$ or -1 for positive or negative parity of (x, y) combination. The $E_{(i,j)}^{\text{split}}$ ($i, j = 1, 2, 3; i \neq j$) is the GM splitting energy [4] and

$E_{(i,j)}^N$ is the Newby shift energy [5]. The diagonal contribution of rotor–particle coupling (E_{rpc}) and particle–particle coupling (E_{ppc}) terms to the bandhead energies of a given 3qp quadruplet is calculated as

$$E_{rpc} = \delta_{K, \frac{1}{2}} \frac{\hbar^2}{2\mathcal{I}} (-1)^{I+\frac{1}{2}} \left(I + \frac{1}{2} \right) \left\{ (\delta_{\sigma_{+++}} + \delta_{\sigma_{++-}}) \left(\langle k_1 \rho_1 | j_{1+} | -k_1 \rho_1 \rangle \delta_{k_1, \frac{1}{2}} \right) + \right. \\ \left. (\delta_{\sigma_{-++}} + \delta_{\sigma_{+-+}}) \left(\langle k_2 \rho_2 | j_{2+} | -k_2 \rho_2 \rangle \delta_{k_2, \frac{1}{2}} \right) + (\delta_{\sigma_{-+-}} + \delta_{\sigma_{-++}}) \left(\langle k_3 \rho_3 | j_{3+} | -k_3 \rho_3 \rangle \delta_{k_3, \frac{1}{2}} \right) \right\} \quad (6)$$

$$E_{ppc} = \delta_{K, \frac{1}{2}} \frac{\hbar^2}{2\mathcal{I}} \left\{ (\delta_{\sigma_{+++}} + \delta_{\sigma_{++-}}) \left(\langle k_1 \rho_1 | j_{1+} | -k_1 \rho_1 \rangle \langle k_2 \rho_2 | j_{2+} | -k_2 \rho_2 \rangle \delta_{k_1, \frac{1}{2}} \delta_{k_2, \frac{1}{2}} \delta_{k_3, \frac{1}{2}} \right) + \right. \\ \left. (\delta_{\sigma_{-++}} + \delta_{\sigma_{+-+}}) \left(\langle k_1 \rho_1 | j_{1+} | -k_1 \rho_1 \rangle \langle k_3 \rho_3 | j_{3+} | -k_3 \rho_3 \rangle \delta_{k_1, \frac{1}{2}} \delta_{k_3, \frac{1}{2}} \delta_{k_2, \frac{1}{2}} \right) + \right. \\ \left. (\delta_{\sigma_{-+-}} + \delta_{\sigma_{-++}}) \left(\langle k_2 \rho_2 | j_{2+} | -k_2 \rho_2 \rangle \langle k_3 \rho_3 | j_{3+} | -k_3 \rho_3 \rangle \delta_{k_2, \frac{1}{2}} \delta_{k_3, \frac{1}{2}} \delta_{k_1, \frac{1}{2}} \right) \right\} \quad (7)$$

The symbols σ_{+++} , σ_{++-} , σ_{+-+} , and σ_{-++} represent the four distinct couplings, i.e., $K = |k_1 + k_2 + k_3|$, $|k_1 + k_2 - k_3|$, $|k_1 - k_2 + k_3|$, and $|-k_1 + k_2 + k_3|$; here, k_1 , k_2 , k_3 are the projections of intrinsic angular momenta of individual nucleons on the nuclear symmetric axis. The single-particle wavefunctions and decoupling parameters used in the present calculations are obtained using Nilsson’s Model [8] with potential strength parameters and equilibrium deformation parameters (quadrupole (ε_2) and hexadecapolar deformation parameters (ε_4)) from Jain et al. [9] and Moller et al. [10], respectively.

3. Results and Discussion

The input parameters used in present modeling of the bandhead energy of 3qp quadruplets, i.e., the proton pairing gaps (Δ_p) and neutron pairing gaps (Δ_n), inertia parameters ($\hbar^2/2\mathcal{I}$), nuclear deformation (ε_2 , ε_4), and interaction matrix elements (J_+), are listed in Table 1, and the procedures to obtain these are given in Section 2. The values of quasiparticle energies ($E_{qp}^{(1)}$, $E_{qp}^{(2)}$, $E_{qp}^{(3)}$) of odd nucleons involved in the formation of the given 3qp configuration are derived from the systematics of 1qp states observed in neighboring odd-A nuclei presented in Figure 1a–d. Special care is exercised to extract these one-quasiparticle energies from the evaluated set of nuclear data [6] and may be useful for experimentalists to identify the location of unknown one-quasiparticle states. The Newby shift $E_{(i,j)}^N$ and the GM splitting energies $E_{(i,j)}^{split}$ ($i, j = 1, 2, 3; i \neq j$) among the triplet ($\uparrow\uparrow$ or $\downarrow\downarrow$) and the singlet ($\downarrow\uparrow$ or $\uparrow\downarrow$) members of a two-quasiparticle (2qp) configuration were extracted from the experimental data of neighboring even-even/odd-odd nuclides, if available; else, these values were taken from earlier works [11,12] and are listed in Table 2.

In order to assess the effect of an irrotational term, RPC-couplings and PPC-couplings in the bandhead energy calculations and hence configuration assignment of the $K = 1/2$ (experimentally observed or yet to be observed) 3qp bands, we extracted 223 three-quasiparticle quadruplets lying in $153 \leq A \leq 187$, $63 \leq Z \leq 78$, $88 \leq N \leq 112$ mass region, and identified seventy eight 80 $K = 1/2$ bands from the available experimental data [13,14]. The experimental data available as of June 2026 were consulted in the present work. Out of the 80 three-quasiparticle $K = 1/2$ bands, there are 48 three-quasiparticle quadruplets having one of the quadruplet members as a $K = 1/2$ bandhead, and only three are experimentally observed. Similarly, there are 16 three-quasiparticle quadruplets having two bandheads (out of four) as $K = 1/2$ members, and none of them is experimentally observed.

Table 1. Input parameters used in the present calculations.

Nuclide	Nilsson's Conf. *	Δ_p/Δ_n (keV) [2]	$\frac{\hbar^2}{2\mathcal{S}} (\hbar^2\text{keV}^{-1})$ [6]	Deformation		Matrix Elements (J_+) [8]
				ε_2 [10]	ε_4 [10]	
^{175}Yb	A $\otimes$ G $\otimes$ H	693.97	13.28	0.26	0.07	$\langle \frac{1}{2}[411] J_+ \frac{1}{2}[411] \rangle : -0.685$
	G $\otimes$ I $\otimes$ B		14.54			
^{171}Lu	J $\otimes$ C $\otimes$ D	728.10	13.05	0.26	0.05	$\langle \frac{1}{2}[521] J_+ \frac{1}{2}[521] \rangle : -0.867$
^{175}Lu	J $\otimes$ B $\otimes$ D	571.10	18.65	0.26	0.07	$\langle \frac{1}{2}[521] J_+ \frac{1}{2}[521] \rangle : -0.848$
^{177}Lu	J $\otimes$ B $\otimes$ E	458.22	11.47	0.25	0.08	$\langle \frac{1}{2}[510] J_+ \frac{1}{2}[510] \rangle : -0.844$
	J $\otimes$ B $\otimes$ D		19.88			$\langle \frac{1}{2}[521] J_+ \frac{1}{2}[521] \rangle : -0.845$
^{173}Ta	J $\otimes$ C $\otimes$ D	793.58	8.51	0.24	0.04	$\langle \frac{1}{2}[521] J_+ \frac{1}{2}[521] \rangle : -0.894$
^{177}Ta	I $\otimes$ B $\otimes$ D	702.78	6.14	0.24	0.06	$\langle \frac{1}{2}[521] J_+ \frac{1}{2}[521] \rangle : -0.875$
^{179}Ta	I $\otimes$ D $\otimes$ F	478.33	12.11	0.23	0.07	$\langle \frac{1}{2}[521] J_+ \frac{1}{2}[521] \rangle : -0.868$
^{181}Ta	I $\otimes$ E $\otimes$ F	610.89	9.59	0.23	0.08	$\langle \frac{1}{2}[510] J_+ \frac{1}{2}[510] \rangle : -0.849$
^{179}Re	I $\otimes$ F $\otimes$ D	806.23	9.13	0.21	0.06	$\langle \frac{1}{2}[521] J_+ \frac{1}{2}[521] \rangle : -0.889$
^{183}Re	I $\otimes$ F $\otimes$ E	850.45	10.48	0.21	0.07	$\langle \frac{1}{2}[510] J_+ \frac{1}{2}[510] \rangle : -0.860$

* A: 5/2[523]_v, B: 7/2[514]_v, C: 7/2[633]_v, D: 1/2[521]_v, E: 1/2[510]_v, F: 9/2[624]_v, G: 1/2[411]_π, H: 7/2[523]_π, I: 9/2[514]_π, J: 7/2[404]_π.

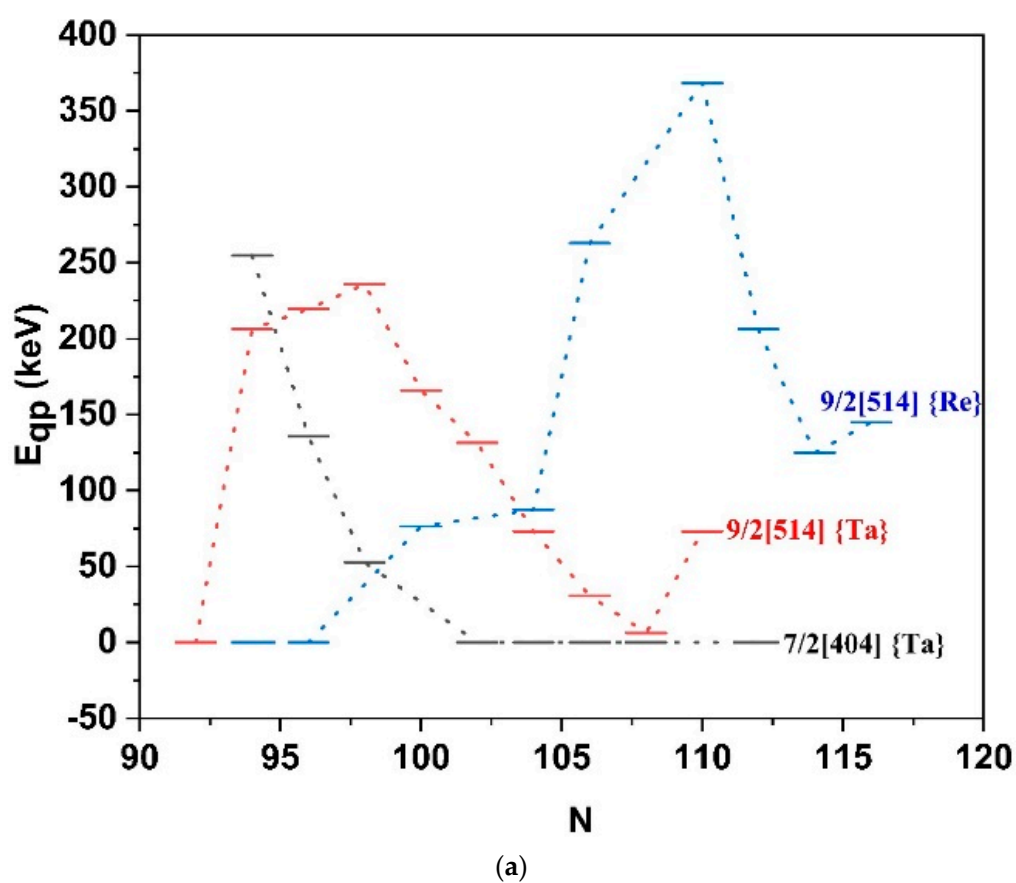


Figure 1. Cont.

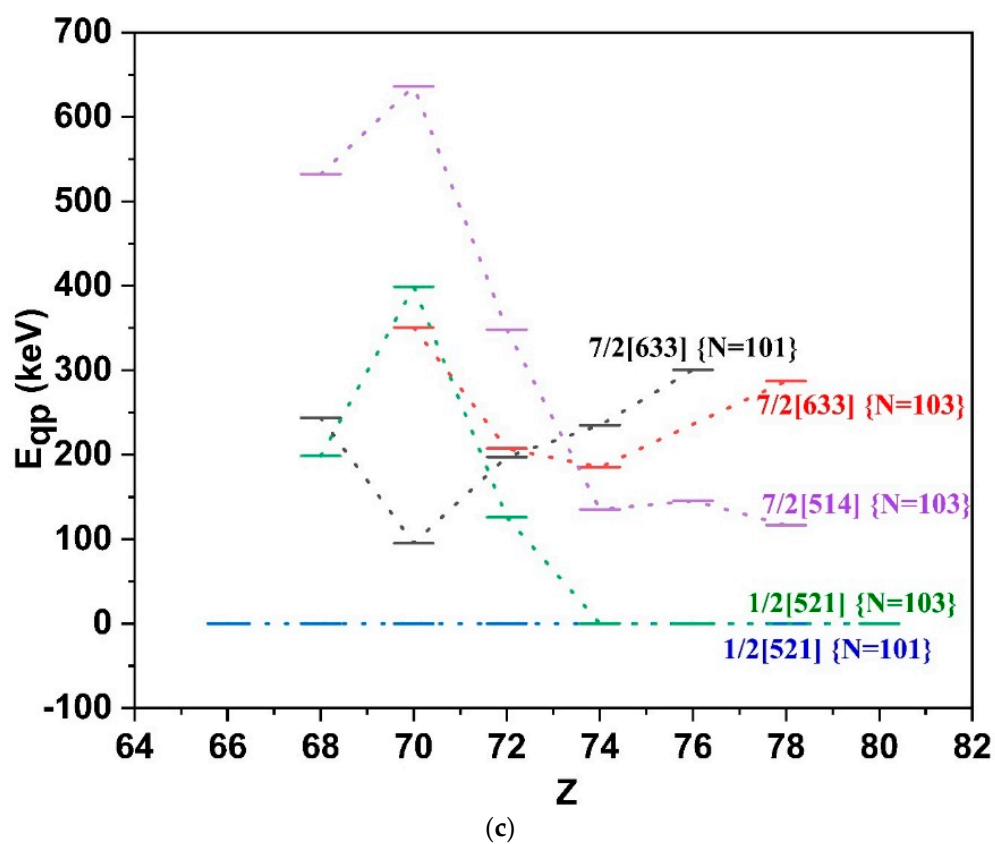
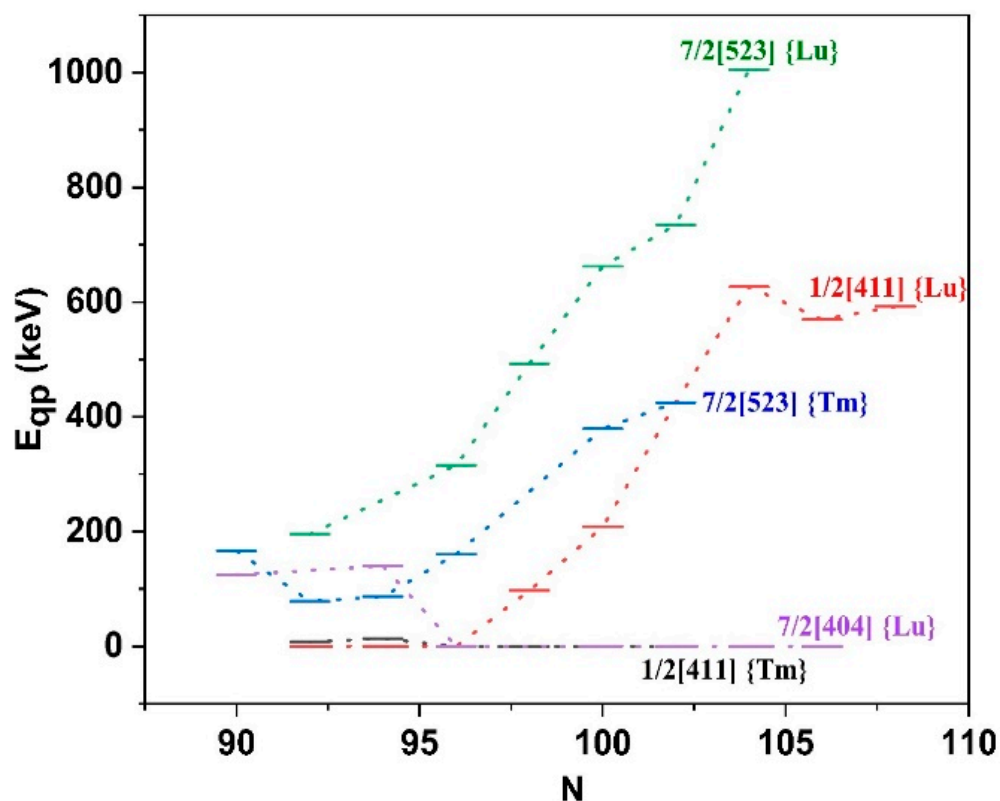


Figure 1. Cont.

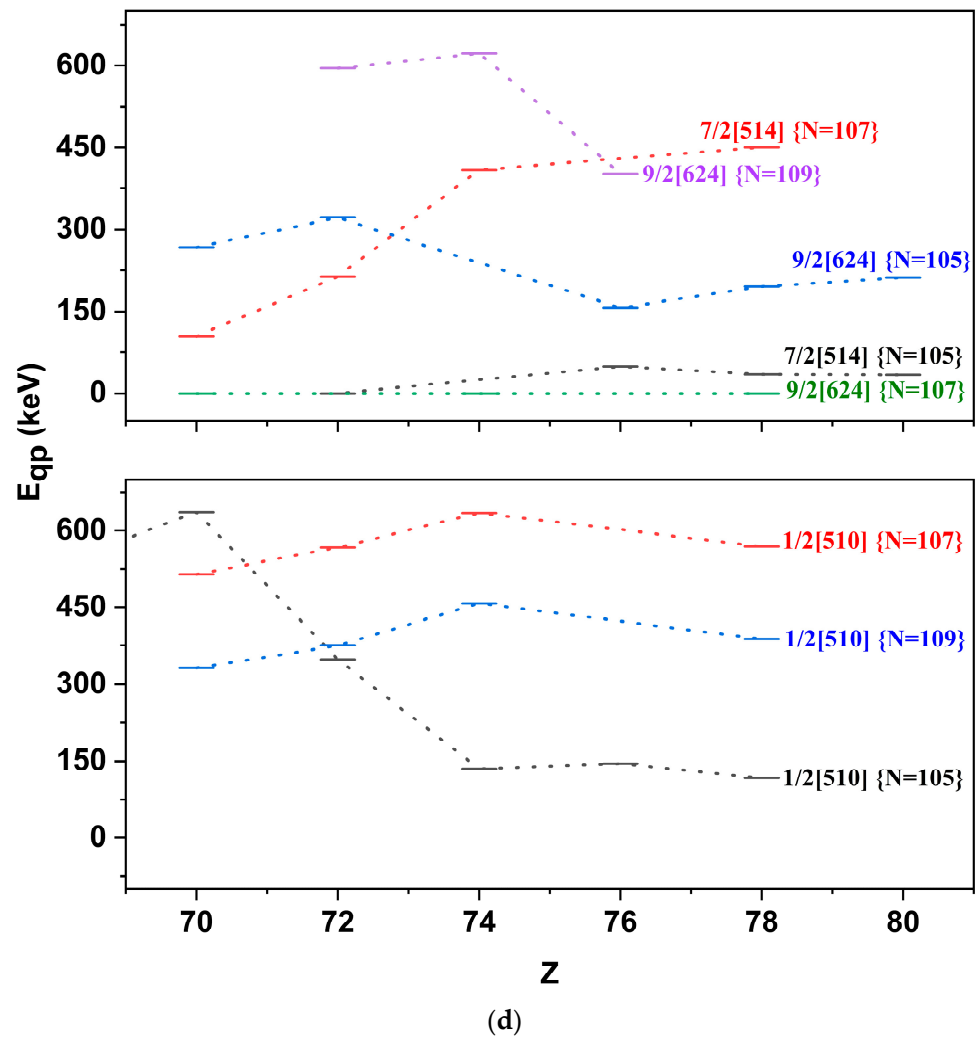


Figure 1. (a) Systematic trends of 1-quasiproton energies of Ta and Re isotopes [6]. (b) Same as (a), but for Lu and Tm isotopes [6]. (c) Systematic trends of 1-quasineutron energies in N = 101, 103 isotones [6]. (d): Same as (c) but for N = 105, 107, 109 isotones [6].

Table 2. Compilation of GM splitting and Newby shift energies for proton–neutron, neutron–neutron, and proton–proton 2qp configurations involved in present model calculations.

S.No.	Nilsson Configuration $\Omega[Nn_z\Lambda]\Sigma$	K^π		GM Splitting (Newby Shift) Energies (keV) [11,12]
		Triplet State (K^T)	Singlet State (K^S)	
1.	9/2[514] $\pi\uparrow\otimes$ 9/2[624] $\nu\uparrow$	9 [−]	0 [−]	142(−2)
2.	1/2[411] $\pi\downarrow\otimes$ 7/2[514] $\nu\downarrow$	4 [−]	3 [−]	120
3.	9/2[514] $\pi\uparrow\otimes$ 7/2[514] $\nu\downarrow$	8 ⁺	1 ⁺	100
4.	7/2[404] $\pi\downarrow\otimes$ 7/2[514] $\nu\downarrow$	7 [−]	0 [−]	255(−69)
5.	9/2[514] $\pi\uparrow\otimes$ 1/2[510] $\nu\uparrow$	5 ⁺	4 ⁺	143
6.	7/2[404] $\pi\downarrow\otimes$ 1/2[510] $\nu\uparrow$	4 [−]	3 [−]	101
7.	9/2[514] $\pi\uparrow\otimes$ 1/2[521] $\nu\downarrow$	5 ⁺	4 ⁺	171
8.	7/2[404] $\pi\downarrow\otimes$ 1/2[521] $\nu\downarrow$	4 [−]	3 [−]	77
9.	7/2[404] $\pi\downarrow\otimes$ 7/2[633] $\nu\uparrow$	7 ⁺	0 ⁺	126(−40)

Table 2. Cont.

S.No.	Nilsson Configuration $\Omega[Nn_z\Lambda]\Sigma$	K^π		GM Splitting (Newby Shift) Energies (keV) [11,12]
		Triplet State (K^T)	Singlet State (K^S)	
10.	$9/2[514]_{\pi}\uparrow\otimes 1/2[411]_{\pi}\downarrow$	5^-	4^-	400
11.	$9/2[624]_{\nu}\uparrow\otimes 1/2[521]_{\nu}\downarrow$	5^-	4^-	255
12.	$7/2[514]_{\nu}\downarrow\otimes 1/2[521]_{\nu}\downarrow$	4^+	3^+	295
13.	$7/2[633]_{\nu}\uparrow\otimes 1/2[521]_{\nu}\downarrow$	4^-	3^-	428
14.	$9/2[624]_{\nu}\uparrow\otimes 1/2[521]_{\nu}\downarrow$	5^-	4^-	321
15.	$9/2[624]_{\nu}\uparrow\otimes 1/2[510]_{\nu}\downarrow$	5^-	4^-	300
16.	$7/2[514]_{\nu}\downarrow\otimes 1/2[510]_{\nu}\downarrow$	4^+	3^+	376
17.	$5/2[523]_{\nu}\downarrow\otimes 1/2[411]_{\pi}\downarrow$	3^-	2^-	53
18.	$1/2[411]_{\pi}\downarrow\otimes 7/2[523]_{\pi}\uparrow$	4^-	3^-	345
19.	$5/2[523]_{\nu}\downarrow\otimes 7/2[523]_{\pi}\uparrow$	6^-	1^-	146
20.	$9/2[514]_{\pi}\uparrow\otimes 7/2[514]_{\nu}\downarrow$	8^+	1^+	135
21.	$7/2[514]_{\nu}\downarrow\otimes 1/2[521]_{\nu}\downarrow$	4^+	3^+	376
22.	$9/2[514]_{\pi}\uparrow\otimes 1/2[521]_{\nu}\downarrow$	5^+	4^+	152
23.	$5/2[402]_{\pi}\uparrow\otimes 5/2[512]_{\nu}\uparrow$	5^-	0^-	135(−25)
24.	$5/2[512]_{\nu}\uparrow\otimes 7/2[633]_{\nu}\uparrow$	6^-	1^-	386
25.	$5/2[402]_{\pi}\uparrow\otimes 7/2[633]_{\nu}\uparrow$	6^-	1^-	126
26.	$7/2[404]_{\pi}\downarrow\otimes 9/2[624]_{\nu}\uparrow$	8^+	1^+	129

3.1. Configuration Assignment to Experimentally Observed $K = 1/2$ Bands

Out of the 80 $K = 1/2$ bands collected by us, only three $K = 1/2$ members having Nilsson configurations $1/2[411]_{\pi}\otimes 7/2[523]_{\pi}\otimes 5/2[523]_{\nu}$ and $1/2[411]_{\pi}\otimes 9/2[514]_{\pi}\otimes 7/2[514]_{\nu}$ in ^{175}Yb [15] and $9/2[514]_{\pi}\otimes 7/2[514]_{\nu}\otimes 1/2[521]_{\nu}$ in ^{177}Ta [16,17] are experimentally observed. Therefore, the bandhead energies of these $K = 1/2$ members can be compared with the present model calculations. The limited number of experimentally established $K = 1/2$ three-quasiparticle bandheads presently available precludes a statistically extensive validation of the model. Nevertheless, all experimentally known cases satisfying the selection criteria were included in the present study.

In Figures 2 and 3, the results of present model calculations are compared with experimental data of three quadruplets based on the above-said configurations observed in ^{175}Yb [15] and ^{177}Ta [16,17]. Notably, the high- K members of above-said 3qp quadruplets are not observed due to extremely high hindrance factors offered to β -decay from ^{175}Tm ground state $1/2^+[411]_{\pi}$ to either $K^\pi = 13/2^+$ or $K^\pi = 17/2^+$ high- K bandheads in ^{175}Yb [15] and to EC-decay from ^{177}W excited state $5/2^- [512]_{\nu}$ to $K^\pi = 15/2^-$ bandhead in ^{177}Ta [16]. However, the $K^\pi = 17/2^-$ bandhead observed in the ^{177}Ta nuclide is populated tentatively in a high-spin reaction [17], which is suitable for populating high-spin states and accessing high- K multi-quasiparticle configurations, but such reactions have relatively weak cross-section to populate low-lying states.

From Figures 2 and 3, it is notable that the revised model calculations are in better agreement with experimental data, which indicates the importance of the previously neglected contributions [3]. The average deviation between experimental data and present model calculations for ^{175}Yb and ^{177}Ta is 37.9 keV and 163.2 keV, as compared to the earlier model estimates, which give an rms deviation of 249.7 keV and 222.3 keV, respectively.

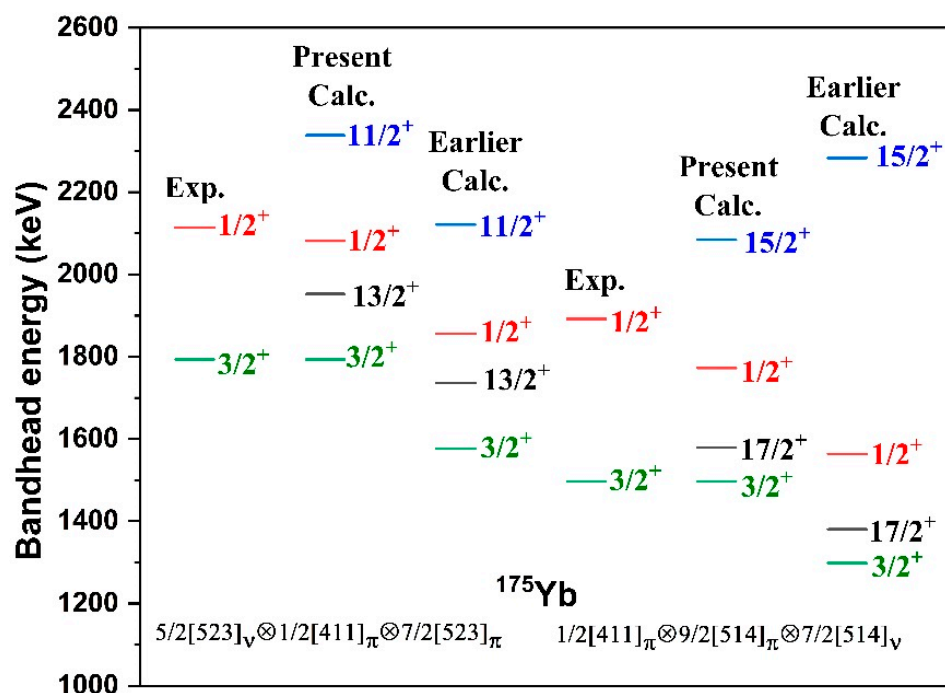


Figure 2. The results of present model calculations compared with experimental data and earlier model calculations for 3qp quadruplets observed in the ^{175}Yb nuclide [15].

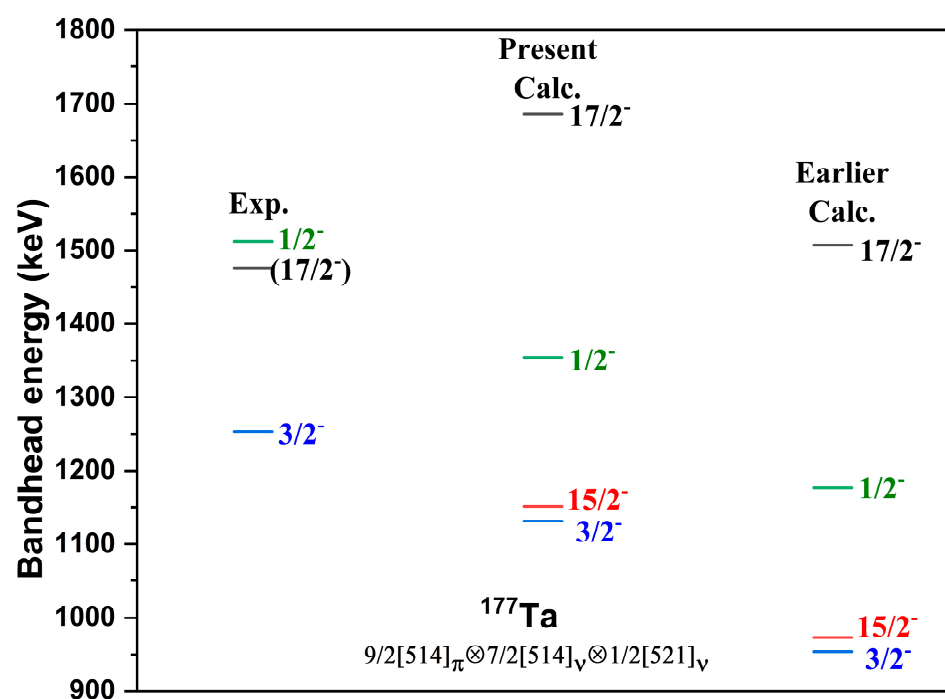


Figure 3. Same as Figure 2 but for the ^{177}Ta nuclide [16,17].

To illustrate the relative importance of the correction terms incorporated in the present semi-empirical model, the contribution of irrotational correction, residual neutron–proton interaction, and rotor–particle coupling (RPC) term for all experimentally established members of the GM quadruplets observed in ^{175}Yb and ^{177}Ta are summarized in Table 3. In addition, the calculated bandhead energies obtained in the present work are compared with both the available experimental data and the calculated energies using an earlier semi-empirical model.

Table 3. Contributions of the irrotational correction, residual neutron–proton interaction, and rotor–particle coupling (RPC) terms for all experimentally established members of the Gallagher–Moszkowski quadruplets, together with a comparison of the present calculations, earlier calculations, and experimental bandhead energies (BE). Σ_1 , Σ_2 , and Σ_3 represent the intrinsic spins of valence particles forming the given 3qp state. Double arrow is spin or unlike particles and single arrow denotes the spin or like particles.

Nuclide	Nilsson Config.	K^π ($\Sigma_1\Sigma_2\Sigma_3$)	Irrotational Correction (keV)	Residual Interaction (keV)	RPC (keV)	BE (expt.) (keV)	BE Present Cal. (keV)	BE Earlier Cal. (keV)
^{175}Yb	$5/2[523]_\nu$, $\otimes 1/2[411]_\pi$	$1/2^+$ ($\downarrow\downarrow\downarrow$)	216.86	73.3	9.06	2114.1	2083.37	1857.44
	$\otimes 7/2[523]_\pi$	$3/2^+$ ($\uparrow\downarrow\downarrow$)	216.86	−219.3	-	1793.4	1794.99	1578.13
	$1/2[411]_\pi$	$1/2^+$ ($\downarrow\downarrow\downarrow$)	198.97	90.0	9.92	1891.8	1772.68	1563.79
	$\otimes 9/2[514]_\pi$, $\otimes 7/2[514]_\nu$	$3/2^+$ ($\downarrow\downarrow\uparrow$)	198.7	−190.0	-	1497.3	1497.8	1298.33
^{177}Ta	$9/2[514]_\pi$	$1/2^-$ ($\downarrow\downarrow\downarrow$)	177.72	44.50	5.37	1512.5	1354.59	1171.50
	$\otimes 7/2[514]_\nu$	$3/2^-$ ($\uparrow\uparrow\downarrow$)	177.72	−162.0	-	1253.3	1131.23	955.13
	$\otimes 1/2[521]_\nu$	$(17/2^-)$ ($\uparrow\downarrow\downarrow$)	177.72	314.0	-	1475.9	1684.4	1506.5

It is evident that the irrotational correction provides the largest contribution to the calculated bandhead energies. For all configurations considered in the present work, the magnitude lies between approximately 178 and 217 keV and remains the same for all members belonging to a given three-quasiparticle configuration. This behavior is expected because the irrotational correction originates from the recoil term of the particle–rotor Hamiltonian and depends only on the intrinsic quasiparticle configuration through the Nilsson wave functions. Consequently, it shifts the energies of all members of a given GM quadruplet by nearly the same amount without altering their relative ordering. The residual neutron–proton interaction exhibits a different behavior. Unlike the irrotational correction, its magnitude and sign depend strongly on the intrinsic spin coupling of the valence particles. Positive values are obtained for the $K = 1/2$ members, whereas negative values are found for the corresponding $K = 3/2$ members of the same GM quadruplet. This behavior reflects the effect of different Gallagher–Moszkowski coupling combinations and plays a decisive role in determining the relative spacing and ordering of the bandheads within a quadruplet.

The magnitude of the RPC term is comparatively small, with values of about 5–10 keV, and contributes only to the $K = 1/2$ bandheads satisfying the condition $K = k_1 + k_2 + k_3 = 1/2$. Although its magnitude is much smaller than that of the irrotational correction or the residual neutron–proton interaction, it should not be regarded as negligible, particularly for the cases where the energy separation between bandheads is small. For example, for the configuration $9/2[514]_\pi \otimes 7/2[514]_\nu \otimes 9/2[624]_\nu$ in ^{177}Lu [18], the experimentally observed bandheads correspond to $K^\pi = 7/2^+$ (1336.5 keV), $25/2^+$ (1325.0 keV), and $11/2^+$ (1230.4 keV), while a $K^\pi = (7/2^+)$ state was tentatively assigned at 1241.5 keV. The $(7/2^+, 25/2^+)$ and $(11/2^+, 7/2^+)$ bandheads are separated by only about 10–12 keV, indicating a nearly degenerate level structure. Under such circumstances, even the relatively small RPC correction term becomes important for reproducing the correct relative ordering of the bandheads. The PPC correction term is non-zero only for the special three-quasiparticle configurations in which the intrinsic angular-momentum projections (k_1, k_2, k_3) satisfy $k_1 = k_2 = k_3 = 1/2$, resulting in $|K| = 1/2$. None of the configurations considered in the present work satisfy this condition; therefore, the PPC contribution vanishes for all cases investigated here. Nevertheless, this term was included in the formalism to provide a complete description of the interactions among the valence particles whenever such configurations occur.

A comparison of the present calculations with the earlier semi-empirical results demonstrates the importance of incorporating these additional correction terms. For example, in ^{177}Ta , the calculated energy of the experimentally observed $K^\pi = 3/2^-$ bandhead is improved from 955.13 keV in the earlier calculation to 1131.23 keV in the present work, compared with the experimental value of 1253.3 keV. Similar improvements are observed for the remaining experimentally known members of the GM quadruplets. The only notable exception is the tentative $K^\pi = (17/2^-)$ bandhead in ^{177}Ta , for which the earlier semi-empirical calculation happens to be closer to the experimental value than the present calculation. However, it should be noted that this bandhead has only been assigned tentatively in the experimental level scheme. Excluding this tentative assignment, the present calculations show consistently improved agreement with the experimental data.

3.2. Configuration Assignment to High-K States Having One of the Members as $K = 1/2$

3.2.1. Three-Quasiparticle State at 1479.7 keV in ^{173}Ta

The high-spin rotational structure was investigated by Carlson et al. [19]. Based on the coupling of specific proton orbitals ($7/2[404]_\pi$, $5/2[502]_\pi$) and neutron orbitals ($1/2[521]_\nu$, $5/2[512]_\nu$, $7/2[633]_\nu$) near the Fermi surface and comparisons of calculated $B(M1)/B(E2)$, $|g_k - g_r|$ values obtained using branching ratios with semi-classical formula, two probable three-quasiparticle configurations were suggested for state at 1479.7 keV. These three-quasiparticle configurations are $K^\pi = 15/2^- : 7/2[404]_\pi \otimes 7/2[633]_\nu \otimes 1/2[521]_\nu$ and $K^\pi = 17/2^- : 5/2[402]_\pi \otimes 5/2[512]_\nu \otimes 7/2[633]_\nu$; however, no conclusive distinction between the configurations was made. The involvement of the $i_{13/2}$ neutron orbital (i.e., $7/2[633]_\nu$) is supported by the observed experimental alignment at low frequencies. To resolve this ambiguous assignment, calculations were performed to estimate the bandhead energies for each member of the proposed 3qp configurations, and the calculated energies are listed in Table 4. Based on present calculations, we propose the following two situations for the 3qp state under discussion:

- If the observed state at 1479.7 keV is not the bandhead and the two initial levels were not observed experimentally, then $K^\pi = 13/2^- : 7/2[404]_\pi \otimes 7/2[633]_\nu \otimes 1/2[521]_\nu$ configuration (at 1470.8 keV) is an appropriate assignment to the state under discussion as compared to $K^\pi = 15/2^- : 7/2[404]_\pi \otimes 7/2[633]_\nu \otimes 1/2[521]_\nu$ (at 974.3 keV) and $K^\pi = 17/2^- : 5/2[402]_\pi \otimes 5/2[512]_\nu \otimes 7/2[633]_\nu$ (at 1324.5 keV), as listed in Table 4. Under this situation, we propose $7/2[404]_\pi \otimes 7/2[633]_\nu \otimes 1/2[521]_\nu$ configuration to two experimentally unobserved $K = 1/2$ members corresponding to coupling $1/2^- (\downarrow\downarrow\downarrow)$ and $1/2^- (\uparrow\uparrow\downarrow)$.
- If the observed state at 1479.7 keV is the bandhead, then $K^\pi = 17/2^- : 5/2[402]_\pi \otimes 5/2[512]_\nu \otimes 7/2[633]_\nu$ (at 1324.5 keV) could be the better candidate as compared to $K^\pi = 15/2^- : 7/2[404]_\pi \otimes 7/2[633]_\nu \otimes 1/2[521]_\nu$ (at 974.3 keV) for the 3qp state under discussion. More detailed experimental investigations are required to address the present issue pertaining to ambiguous configuration assignments.

3.2.2. Three-Quasiparticle State at 1627.6 keV in ^{179}Ta

The energy level at 1627.6 keV was first identified by Barneoud et al. [20], but with tentative spin and parity assignment; later on, Kondev et al. [21] refined its spin, parity, and configuration assignment. The possible spin assignments (i.e., $I = 19/2$, $21/2$, or $23/2$) for this state are based on the anisotropy coefficient of the depopulating 376.1 keV transition. However, the spin assignment $I = 23/2$ was ruled out based on the relatively low intensity of the 376.1 keV transition as compared to the 289.9 keV transition (which connects the $23/2^-$ and $21/2^-$ levels within the band built over the 320 ns isomeric state). Based on the above arguments and decay properties (branching ratios, g_k factors and aligned angular mo-

menta), Kondev et al. [21], suggested two possible candidate configurations i.e., $K^\pi = 19/2^+$: $9/2[514]_\pi \otimes 1/2[521]_\nu \otimes 9/2[624]_\nu$ or $K^\pi = 21/2^-$: $5/2[402]_\pi \otimes 9/2[624]_\nu \otimes 7/2[514]_\nu$ for the state at 1627.6 keV. To resolve this ambiguity, we carried out calculations to estimate the energy of the bandhead for each configuration. These calculations (shown in Table 4) reveal that the energy of the $K^\pi = 19/2^+$: $9/2[514]_\pi \otimes 1/2[521]_\nu \otimes 9/2[624]_\nu$ matches closely with the experimental data as compared to the $K^\pi = 21/2^-$: $5/2[402]_\pi \otimes 9/2[624]_\nu \otimes 7/2[514]_\nu$ configuration which has a larger energy discrepancy ≈ 433.4 keV. Therefore, based on the better match between the experimental and calculated energies, we propose the $K^\pi = 19/2^+$: $9/2[514]_\pi \otimes 1/2[521]_\nu \otimes 9/2[624]_\nu$ configuration as a more suitable candidate for the state observed at 1627.6 keV. As discussed for ^{173}Ta , there are two $K = 1/2$ members corresponding to the $9/2[514]_\pi \otimes 1/2[521]_\nu \otimes 9/2[624]_\nu$ quadruplet; hence, we propose the same configuration for these two experimentally unobserved $K = 1/2$ members corresponding to couplings $1/2^- (\uparrow\downarrow\downarrow)$ and $1/2^- (\downarrow\downarrow\uparrow)$.

Table 4. Calculated bandhead energies compared with experimental energies of 3qp quadruplets observed in $^{173,179}\text{Ta}$ [19–21] and ^{177}Lu [22,23] nuclides. The bold values represent the most suitable/probable configurations based on the present work.

Nuclide	Nilsson Config *	K^π	$\frac{\hbar^2}{2\mathcal{I}}$ ($\hbar^2/\text{keV}^{-1}$)	Irrotational Correction (keV)	Residual Interaction (keV)	RPC (keV)	BE (Expt.) (keV)	BE (Calc.) (keV)
^{173}Ta	A ⊗ D ⊗ E	17/2[−]	8.1	205.4	62.5		1479.7	1324.5
		3/2 [−]			−197.5			1007.9
		7/2 [−]			−163.5			1058.1
		7/2 [−]			348.5			1570.1
	B ⊗ E ⊗ F	15/2 [−]	8.5	151.6	−189.5			974.3
		13/2[−]			315.5			1470.8
		1/2 [−]			72.5	7.6		1184.4
		1/2 [−]			−278.5	7.6		833.4
	C ⊗ F ⊗ G	19/2⁺	12.1	224.2	−146.0		1627.6	1335.5
		1/2 ⁺			319.0			1691.5
		17/2 ⁺			4.0			1473.4
		1/2 ⁺			−173.0	10.5		1199.5
^{179}Ta	A ⊗ G ⊗ H	21/2 [−]	5.1	195.8	−137.5			1194.2
		7/2 [−]			−69.5			1226.0
		3/2 [−]			255.5			1541.1
		11/2 [−]			−48.5			1257.6
	C ⊗ H ⊗ I	17/2[−]	12.3	198.1	−209.5		1437.9	1253.5
		15/2 [−]			309.5			1760.3
		3/2 [−]			66.5			1443.7
		1/2 [−]			−166.50	10.3		1198.5
^{177}Lu	B ⊗ G ⊗ I	17/2 [−]	9.7	261.9	265.0			1646.3
		15/2 [−]			−136.0			1235.6
		1/2 [−]			−164.0	8.2		1147.9
		3/2 [−]			35.0			1348.5

* A: 5/2[402]_π, B: 7/2[404]_π, C: 9/2[514]_π, D: 5/2[512]_ν, E: 7/2[633]_ν, F: 1/2[521]_ν, G: 9/2[624]_ν, H: 7/2[514]_ν, I: 1/2[510]_ν.

3.2.3. Three-Quasiparticle State at 1437.9 keV in ^{177}Lu

The multi-quasiparticle rotational structure of ^{177}Lu was studied by T.R. McGoram [22] using γ -ray spectroscopy and incomplete fusion reactions, followed by Dracoulis et al. [23] using high-spin reactions. Dracoulis et al. [23] identified two isomeric states based on $K^\pi = 23/2^-$ ($\tau = 160$ days) and $K^\pi = 25/2^+$ ($\tau = 90$ ns); however, the isomeric state $K^\pi = 17/2^-$ ($\tau = 14$ ns) at energy 1437.9 keV could not be populated due to the inability of heavy ion reactions to populate a low-spin state. Based on the coupling of specific proton orbitals and neutron orbitals near the Fermi surface, possible candidates for $K^\pi = 17/2^-$ isomer are $7/2[404]_\pi \otimes 9/2[624]_v \otimes 1/2[510]_v$, $1/2[411]_\pi \otimes 7/2[514]_v \otimes 9/2[624]_v$, $9/2[514]_\pi \otimes 7/2[514]_v \otimes 1/2[510]_v$, and $7/2[404]_\pi \otimes 9/2[514]_\pi \otimes 1/2[411]_\pi$. On the basis of the agreement of experimental g_k -values (0.74 (5)) with Nilsson model prediction ($g_k = 0.69$), McGoram [22] suggested that the isomeric state at 1437.9 keV can be assigned as $K^\pi = 17/2^-$: $9/2[514]_\pi \otimes 7/2[514]_v \otimes 1/2[510]_v$ or $K^\pi = 17/2^-$: $7/2[404]_\pi \otimes 9/2[624]_v \otimes 1/2[510]_v$ configuration. To resolve this issue of configuration assignment, we calculated the bandhead energies of both the suggested configurations and present the results in Table 4. From the comparison of experimental and calculated bandhead energies for $K^\pi = 17/2^-$: $9/2[514]_\pi \otimes 7/2[514]_v \otimes 1/2[510]_v$ and $K^\pi = 17/2^-$: $7/2[404]_\pi \otimes 9/2[624]_v \otimes 1/2[510]_v$ configurations, we suggest that our calculations favor the former configuration as a more suitable candidate for the state at 1437.9 keV and propose the same configuration for experimentally unobserved $K = 1/2^-$ member of this quadruplet having intrinsic coupling $1/2^- (\downarrow\downarrow\uparrow)$.

3.3. Prediction of Bandhead Energies of $K = 1/2$ Bands

The present version of the model is more complete and realistic compared to its earlier version, as it includes irrotational terms and rotor–particle coupling terms, which were not considered in the earlier model and contribute significantly towards bandhead energies of $K = 1/2$ bands of given 3qp quadruplets. Motivated by the overall good agreement among experimental and calculated results, we predicted the bandhead energies of 18 $K = 1/2$, yet-to-be-observed states (listed in Table 5), which might be useful to experimentalists and future theoretical investigations.

Table 5. Calculated values of the irrotational term, residual n - p interactions, RPC terms, and the predicted bandhead energies (BE) of $K = 1/2$ states. Σ_1, Σ_2 , and Σ_3 represent the intrinsic spins of valence particles. Calculated bandhead energies indicated with an asterisk (*) correspond to the experimentally observed states, as shown in Figures 2 and 3.

Nuclide	Nilsson Config.	K^π ($\Sigma_1\Sigma_2\Sigma_3$)	Irrotational Correction (keV)	Residual Interaction (keV)	RPC (keV)	BE Present Cal. (keV)	BE Earlier Cal. (keV)
^{175}Yb	A $\otimes$ G $\otimes$ H	$1/2^+$ ($\downarrow\downarrow\downarrow$)	216.86	73.3	9.06	2083.37 *	1857.44
	G $\otimes$ I $\otimes$ B	$1/2^+$ ($\downarrow\downarrow\downarrow$)	198.97	90.0	9.92	1772.68 *	1563.79
^{171}Lu	J $\otimes$ C $\otimes$ D	$1/2^-$ ($\downarrow\downarrow\downarrow$)	163.05	72.5	11.31	1246.87	1072.51
		$1/2^-$ ($\uparrow\uparrow\downarrow$)		−278.5		895.87	721.51
^{175}Lu	J $\otimes$ B $\otimes$ D	$1/2^+$ ($\downarrow\uparrow\downarrow$)	196.15	−30.0	15.81	1559.9	1347.94
		$1/2^+$ ($\uparrow\downarrow\downarrow$)		423.0		2012.9	1800.94
^{177}Lu	J $\otimes$ B $\otimes$ E	$1/2^+$ ($\downarrow\uparrow\uparrow$)	179.81	435.0	9.68	1640.56	1451.07
		$1/2^+$ ($\uparrow\downarrow\uparrow$)		−42.0		1163.56	974.07
	J $\otimes$ B $\otimes$ D	$1/2^+$ ($\downarrow\uparrow\downarrow$)	213.98	10.5	16.78	1583.70	1352.94
		$1/2^+$ ($\uparrow\downarrow\downarrow$)		382.5		1955.70	1724.94

Table 5. Cont.

Nuclide	Nilsson Config.	K^π ($\Sigma_1 \Sigma_2 \Sigma_3$)	Irrotational Correction (keV)	Residual Interaction (keV)	RPC (keV)	BE Present Cal. (keV)	BE Earlier Cal. (keV)
^{173}Ta	$J \otimes C \otimes D$	$1/2^- (\downarrow\downarrow\downarrow)$	151.63	72.5	7.61	1184.37	1025.13
		$1/2^- (\uparrow\uparrow\downarrow)$		−278.5		833.37	674.13
^{177}Ta	$I \otimes B \otimes D$	$1/2^- (\downarrow\downarrow\downarrow)$	177.72	44.50	5.37	1354.59 *	1171.50
^{179}Ta	$I \otimes D \otimes F$	$1/2^+ (\uparrow\uparrow\downarrow)$	224.22	319.0	10.51	1702.05	1467.31
		$1/2^+ (\downarrow\downarrow\uparrow)$		−173.0		1210.05	975.32
^{181}Ta	$I \otimes E \otimes F$	$1/2^+ (\uparrow\uparrow\downarrow)$	241.96	−180.5	8.15	1190.32	940.21
		$1/2^+ (\downarrow\downarrow\uparrow)$		326.5		1697.32	1447.21
^{179}Re	$I \otimes F \otimes D$	$1/2^+ (\uparrow\uparrow\downarrow)$	210.25	286.0	8.11	1635.62	1417.26
		$1/2^+ (\downarrow\downarrow\uparrow)$		−140.0		1209.62	991.26
^{183}Re	$I \otimes F \otimes E$	$1/2^+ (\uparrow\uparrow\downarrow)$	239.92	−148.5	9.01	1882.44	1633.51
		$1/2^+ (\downarrow\downarrow\uparrow)$		294.5		2325.44	2076.51

* A: $5/2[523]_\nu$, B: $7/2[514]_\nu$, C: $7/2[633]_\nu$, D: $1/2[521]_\nu$, E: $1/2[510]_\nu$, F: $9/2[624]_\nu$, G: $1/2[411]_\pi$, H: $7/2[523]_\pi$, I: $9/2[514]_\pi$, J: $7/2[404]_\pi$.

4. Conclusions

The configuration assignments to $K = 1/2$ bandhead(s) of the given three-quasiparticle quadruplets experimentally observed and yet to be observed in odd-A nuclei lying in the rare-earth mass region are presented. The results of $K = 1/2$ bandheads pertaining to $1/2[411]_\pi \otimes 7/2[523]_\pi \otimes 5/2[523]_\nu$ and $1/2[411]_\pi \otimes 9/2[514]_\pi \otimes 7/2[514]_\nu$ configurations in ^{175}Yb and $9/2[514]_\pi \otimes 7/2[514]_\nu \otimes 1/2[521]_\nu$ configuration in ^{177}Ta nuclide are in better agreement with the experimental data as compared to the earlier calculations. The average deviation between experimental data and present model calculations for ^{175}Yb and ^{177}Ta is 37.9 keV and 163.2 keV, as compared to the earlier model estimates, which yielded a deviation of 249.7 keV and 222.3 keV, respectively. We also tried to resolve ambiguous configuration assignments of three quadruplets observed in $^{173,179}\text{Ta}$, ^{177}Lu , and also confirmed configuration assignments of 3 tentative $K = 1/2$ bands observed in ^{175}Yb , ^{177}Ta . Motivated by these results, we have also predicted the bandhead energy of 18 $K = 1/2$ bands, which may be useful for future experiments and theoretical calculations. The present calculations highlighted the contributions that arise from the irrotational motion of valence nucleons, residual interactions, and rotor–particle interactions, and suggest that these terms significantly contribute to determining the magnitude of the bandhead energies. Although the present semi-empirical model provides a reasonable description of the available experimental bandhead energies, it does not explicitly account for the changes in equilibrium deformation and pairing correlations associated with multi-quasiparticle excitations. These effects may influence the moment of inertia and the predicted bandhead energies and therefore represent a limitation of the present work. Their incorporation within a more microscopic description constitutes an important direction for future development of the model. We believe that present model calculations will support experimental efforts in identifying currently unobserved $K = 1/2$ members of various three-quasiparticle quadruplets. Additionally, these calculations will be helpful in the configuration assignments for the member states when more than one competing/mixed configuration is suggested, and experimental information for distinction is not adequate.

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