

Article

On Solutions to the Set-Theoretical Yang-Baxter Equation in Wajsberg-Algebras

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Abstract: In this work, we introduce Wajsberg algebras which are equivalent structures to MV-algebras in their implicational version, and then we define new notions and give new solutions to the set-theoretical Yang-Baxter equation by using Wajsberg algebras.

Keywords: Yang-Baxter equation; Wajsberg-algebra

1. Introduction

The Yang-Baxter equation which was initially used in theoretical physics [1] and statical mechanics [2–4] has gradually attracted the attention of researchers from various areas of science. In particular, this equation is considered in areas such as link invariant, C^* -algebras, conformal field theory, quantum computing, quantum groups, quantum mechanics, knot theory, integrable systems, non-commutative geometry, etc. (see, for example, [5–11]).

The problem to find and study (set-theoretical) solutions of the Yang-Baxter equation has attracted many authors.

The Yang-Baxter equation involves a linear operator $R : V \otimes V \longrightarrow V \otimes V$, where V is a vector space and has the form

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \text{ in } \text{End}(V \otimes V \otimes V)$$

where $1 \leq n, m \leq 3$ and R^{nm} means R acting in the n -th and m -th components. In the last years, many set-theoretical solutions of this equation have given rise to the connection with various mathematical structures, such as quantum binomial algebras [12,13], semigroups of I-type and Bieberbach groups [14,15], bijective 1-cocycles [16], semisimple minimal triangular Hopf algebras [17], dynamical systems [18], and geometric crystals [19].

Since Wajsberg has shown that ∞ -valued Lukasiewicz logics were complete with respect to the axioms postulated by Lukasiewicz, these logics were shown to play an important role in the study of quantum physics. We then wish to investigate the Yang-Baxter equation rather in relation with Wajsberg-algebras than quantum physics.

The set-theoretical solutions to the Yang-Baxter equation using MV-algebras were given by [11].

In this paper, we give some solutions to the set-theoretical Yang-Baxter equation in Wajsberg algebras.

2. Preliminaries

In this section, we present some definitions and properties of Wajsberg-algebras.

Definition 1. [20] A Wajsberg algebra (briefly, a W -algebra) is a structure $\langle A, \longrightarrow, \neg, 1 \rangle$ satisfying the following equations, where A is a nonempty set, \neg is a unary operation on A , \longrightarrow is a binary operation on A , and 1 is a distinguished element of A :

- (W1) $1 \rightarrow a = a$
- (W2) $(a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c)) = 1$
- (W3) $(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a$
- (W4) $(\neg a \rightarrow \neg b) \rightarrow (b \rightarrow a) = 1.$

Moreover, bounded commutative BCI/BCK-algebras (BCK-algebras are special cases of BCI-algebras, for example, see [21]), in their implicational notation, are known as Wajsberg algebras (see also [22]).

Lemma 1. [20] *Let $\langle A, \rightarrow, 1 \rangle$ be a system satisfying (W1), (W2) and (W3). Then the following properties hold for every a, b and c in A :*

- (W5) $a \rightarrow a = 1$
- (W6) *If $a \rightarrow b = b \rightarrow a = 1$, then $a = b$.*
- (W7) $a \rightarrow 1 = 1$
- (W8) $a \rightarrow (b \rightarrow a) = 1$
- (W9) *If $a \rightarrow b = b \rightarrow c = 1$, then $a \rightarrow c = 1$.*
- (W10) $(a \rightarrow b) \rightarrow ((c \rightarrow a) \rightarrow (c \rightarrow b)) = 1$
- (W11) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c).$

Lemma 2. [20] *The following equations hold in every W-algebra:*

- (i) $\neg 1 \rightarrow a = 1,$
- (ii) $\neg a = a \rightarrow \neg 1,$
- (iii) $\neg \neg a = a,$
- (iv) $a \rightarrow b = \neg b \rightarrow \neg a.$

3. Solutions to the Yang-Baxter Equation in W-Algebras

In this section, we provide solutions to the set-theoretical Yang-Baxter equation in W-algebras. Let V be a vector space over a field F . We denote by $\tau : V \otimes V \rightarrow V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$ and by $I : V \rightarrow V$ the identity map over the space V ; for a F -linear map $R : V \otimes V \rightarrow V \otimes V$, let $R^{12} = R \otimes I, R^{23} = I \otimes R$, and $R^{13} = (I \otimes \tau)(R \otimes I)(\tau \otimes I)$.

Definition 2. [10] *A Yang-Baxter operator is an invertible F -linear map $R : V \otimes V \rightarrow V \otimes V$, and it satisfies the braid condition (also called the Yang-Baxter equation):*

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \tag{1}$$

If R satisfies Equation (1), then both $R \circ \tau$ and $\tau \circ R$ satisfy the quantum Yang-Baxter equation:

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \tag{QYBE}$$

The following definition enables us to constitute a relationship between the set-theoretical Yang-Baxter equation and W-algebras.

Definition 3. [10] *Let X be a set and $S : X^2 \rightarrow X^2, S(p_1, p_2) = (p'_1, p'_2)$ be a map. The map S is a solution to the set-theoretical Yang-Baxter equation if it satisfies the following identity:*

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}, \tag{2}$$

where

$$S^{12} : X^3 \rightarrow X^3, \quad S^{12}(p_1, p_2, p_3) = (p'_1, p'_2, p_3),$$

$$S^{23} : X^3 \longrightarrow X^3, \quad S^{23}(p_1, p_2, p_3) = (p_1, p'_2, p'_3),$$

$$S^{13} : X^3 \longrightarrow X^3, \quad S^{13}(p_1, p_2, p_3) = (p'_1, p_2, p'_3).$$

Now, we provide solutions to the set-theoretical Yang-Baxter equation by using W-algebras.

Theorem 1. Let $\langle A, \longrightarrow, \neg, 1 \rangle$ be a W-algebra. Then $S(a, b) = (\neg a \longrightarrow b, \neg 1)$ is a solution to the set-theoretical Yang-Baxter equation.

Proof. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(a, b, c) = (\neg a \longrightarrow b, \neg 1, c), \quad S^{23}(a, b, c) = (a, \neg b \longrightarrow c, \neg 1).$$

For all $(a, b, c) \in A^3$, we get

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(a, b, c) &= (S^{12} \circ S^{23})(S^{12}(a, b, c)) \\ &= (S^{12} \circ S^{23})(\neg a \longrightarrow b, \neg 1, c) \\ &= S^{12}(S^{23}(\neg a \longrightarrow b, \neg 1, c)) \\ &= S^{12}(\neg a \longrightarrow b, \neg \neg 1 \longrightarrow c, \neg 1) \\ &= S^{12}(\neg a \longrightarrow b, 1 \longrightarrow c, \neg 1) && \text{(Lemma 2 (iii))} \\ &= S^{12}(\neg a \longrightarrow b, c, \neg 1) && \text{(W1)} \\ &= (\neg(\neg a \longrightarrow b) \longrightarrow c, \neg 1, \neg 1) \\ &= (\neg c \longrightarrow (\neg b \longrightarrow a), \neg 1, \neg 1) && \text{(Lemma 2 (iii)-(iv))} \\ &= (\neg b \longrightarrow (\neg c \longrightarrow a), \neg 1, \neg 1) && \text{(W11)} \\ &= (\neg b \longrightarrow (\neg a \longrightarrow c), \neg 1, \neg 1) && \text{(Lemma 2 (iii)-(iv))} \\ &= (\neg a \longrightarrow (\neg b \longrightarrow c), \neg 1, \neg 1) && \text{(W11)} \\ &= (\neg a \longrightarrow (\neg b \longrightarrow c), \neg \neg 1 \longrightarrow \neg 1, \neg 1) && \text{((W1) and Lemma 2 (iii))} \\ &= S^{23}(\neg a \longrightarrow (\neg b \longrightarrow c), \neg 1, \neg 1) \\ &= S^{23}(S^{12}(a, \neg b \longrightarrow c, \neg 1)) \\ &= (S^{23} \circ S^{12})(a, \neg b \longrightarrow c, \neg 1) \\ &= (S^{23} \circ S^{12})(S^{23}(a, b, c)) \\ &= (S^{23} \circ S^{12} \circ S^{23})(a, b, c) \end{aligned}$$

Then, $S(a, b) = (\neg a \longrightarrow b, \neg 1)$ is a solution to the set-theoretical Yang-Baxter equation in the W-algebra A . \square

Lemma 3. Let $\langle A, \longrightarrow, \neg, 1 \rangle$ be a W-algebra. Then $S(a, b) = (\neg b, \neg a)$ is a solution to the set-theoretical Yang-Baxter equation.

Theorem 2. Let $\langle A, \longrightarrow, \neg, 1 \rangle$ be a W-algebra. Then $S(a, b) = ((a \longrightarrow b) \longrightarrow b, b)$ is a solution to the set-theoretical Yang-Baxter equation.

Proof. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(a, b, c) = ((a \longrightarrow b) \longrightarrow b, b, c), \quad S^{23}(a, b, c) = (a, (b \longrightarrow c) \longrightarrow c, c).$$

For all $(a, b, c) \in A^3$, we have

$$\begin{aligned}
 (S^{12} \circ S^{23} \circ S^{12})(a, b, c) &= S^{12}(S^{23}(S^{12}(a, b, c))) \\
 &= S^{12}(S^{23}((a \rightarrow b) \rightarrow b, b, c)) \\
 &= S^{12}((a \rightarrow b) \rightarrow b, (b \rightarrow c) \rightarrow c, c) \\
 &= (((a \rightarrow b) \rightarrow b) \rightarrow \\
 &\quad ((b \rightarrow c) \rightarrow c)) \rightarrow \\
 &\quad ((b \rightarrow c) \rightarrow c), \\
 &\quad (b \rightarrow c) \rightarrow c, c) \\
 &= (((a \rightarrow b) \rightarrow b) \rightarrow \\
 &\quad ((c \rightarrow b) \rightarrow b)) \rightarrow \\
 &\quad ((c \rightarrow b) \rightarrow b), \\
 &\quad (b \rightarrow c) \rightarrow c, c) \tag{W3} \\
 &= (((c \rightarrow b) \rightarrow ((a \rightarrow b) \\
 &\quad \rightarrow b) \rightarrow b)) \rightarrow ((c \rightarrow \\
 &\quad b) \rightarrow b), (b \rightarrow c) \rightarrow c, c) \tag{W11} \\
 &= (((c \rightarrow b) \rightarrow ((b \rightarrow \\
 &\quad (a \rightarrow b)) \rightarrow (a \rightarrow b))) \\
 &\quad \rightarrow ((c \rightarrow b) \rightarrow b), \\
 &\quad (b \rightarrow c) \rightarrow c, c) \tag{W3} \\
 &= (((c \rightarrow b) \rightarrow (a \rightarrow b)) \\
 &\quad \rightarrow ((c \rightarrow b) \rightarrow b), \\
 &\quad (b \rightarrow c) \rightarrow c, c) \tag{(W8) and (W1)} \\
 &= ((a \rightarrow ((b \rightarrow c) \rightarrow c)) \\
 &\quad \rightarrow ((b \rightarrow c) \rightarrow c), \\
 &\quad (b \rightarrow c) \rightarrow c, c) \tag{(W11) and (W3)}
 \end{aligned}$$

and

$$\begin{aligned}
 (S^{23} \circ S^{12} \circ S^{23})(a, b, c) &= S^{23}(S^{12}(S^{23}(a, b, c))) \\
 &= S^{23}(S^{12}(a, (b \rightarrow c) \rightarrow c, c)) \\
 &= S^{23}((a \rightarrow ((b \rightarrow c) \rightarrow c)) \\
 &\quad \rightarrow ((b \rightarrow c) \rightarrow c), \\
 &\quad (b \rightarrow c) \rightarrow c, c) \\
 &= ((a \rightarrow ((b \rightarrow c) \rightarrow c)) \rightarrow \\
 &\quad ((b \rightarrow c) \rightarrow c), (((b \rightarrow \\
 &\quad c) \rightarrow c) \rightarrow c) \rightarrow c, c) \\
 &= ((a \rightarrow ((b \rightarrow c) \rightarrow c)) \\
 &\quad \rightarrow ((b \rightarrow c) \rightarrow c), \\
 &\quad (b \rightarrow c) \rightarrow c, c) \tag{(W3), (W8) and (W1), respectively).}
 \end{aligned}$$

Then, $S(a, b) = ((a \rightarrow b) \rightarrow b, b)$ is a solution to the set-theoretical Yang-Baxter equation in the W-algebra A . \square

Corollary 1. Let $\langle A, \rightarrow, \neg, 1 \rangle$ be a W-algebra. Then $S(a, b) = ((b \rightarrow a) \rightarrow a, b)$, $S(a, b) = ((b \rightarrow a) \rightarrow a, a)$ and $S(a, b) = ((a \rightarrow b) \rightarrow b, a)$ are solutions to the set-theoretical Yang-Baxter equation.

Proof. The proof is completed from (W3) and Theorem 2. \square

Proposition 1. [23] Let A be a W -algebra. The binary relation \leq defined on A as follows

$$a \leq b \text{ if and only if } a \longrightarrow b = 1$$

is a partial order on A .

Proposition 2. [23] Let A be a W -algebra and \leq be a partial order on A . If the join and the meet operators are defined by

$$p \vee q = (p \longrightarrow q) \longrightarrow q$$

and

$$p \wedge q = \neg(p \longrightarrow \neg(p \longrightarrow q)),$$

then this partial order determines a lattice on A .

Proposition 3. [23] Any W -algebra satisfies the following implications and equations

- (1) $\neg(a \vee b) = \neg a \wedge \neg b$
- (2) $\neg(a \wedge b) = \neg a \vee \neg b$
- (3) $(a \vee b) \longrightarrow c = (a \longrightarrow c) \wedge (b \longrightarrow c)$
- (4) $a \longrightarrow (b \vee c) = (a \longrightarrow b) \vee (a \longrightarrow c)$
- (5) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$
- (6) $a \longrightarrow (b \wedge c) = (a \longrightarrow b) \wedge (a \longrightarrow c)$
- (7) $(a \longrightarrow b) \vee (b \longrightarrow a) = 1$

Lemma 4. Any W -algebra A is a Boolean algebra if and only if $\neg a \longrightarrow a = a$ for all $a \in A$.

Proof. (\Rightarrow) Assume that a W -algebra A is a Boolean algebra. By using properties of Boolean algebras and proposition 2, we obtain

$$1 = \neg a \vee a = (\neg a \longrightarrow a) \longrightarrow a \tag{*1}$$

for all $a \in A$. Since we already have that $a = 1 \longrightarrow a$ for all $a \in A$ from (W1), we get

$$\begin{aligned} a &= 1 \longrightarrow a && \text{(W1)} \\ &= ((\neg a \longrightarrow a) \longrightarrow a) \longrightarrow a && \text{(*1)} \\ &= (a \longrightarrow (\neg a \longrightarrow a)) \longrightarrow (\neg a \longrightarrow a) && \text{(W3)} \\ &= 1 \longrightarrow (\neg a \longrightarrow a) && \text{(W8)} \\ &= (\neg a \longrightarrow a). && \text{(W1)} \end{aligned}$$

Consequently, $\neg a \longrightarrow a = a$ for all $a \in A$.

(\Leftarrow) Suppose that A is a W -algebra and let $\neg a \longrightarrow a = a$ for all $a \in A$. Since

$$\begin{aligned} \neg a \vee a &= ((\neg a \longrightarrow a) \longrightarrow a) && \text{(Proposition 2)} \\ &= a \longrightarrow a && \text{(Hypothesis)} \\ &= 1 && \text{(W5)} \end{aligned}$$

and

$$\begin{aligned} \neg a \wedge a &= \neg(\neg a \longrightarrow \neg(\neg a \longrightarrow a)) && \text{(Proposition 2)} \\ &= \neg(\neg a \longrightarrow \neg a) && \text{(Hypothesis)} \\ &= \neg 1 && \text{(W5)} \\ &= 0, \end{aligned}$$

we have that the negation \neg is a complementation, and the least element of A is 0 defined as $\neg 1 = 0$ and 1 is the greatest element in A . Thus, A is a bounded lattice with a complementation.

It remains to see that A is distributive. By Proposition 3 (5), we already have that $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all $a, b, c \in A$. Besides, we obtain that

$$\begin{aligned}
 (a \vee b) \wedge c &= \neg((a \vee b) \longrightarrow \neg((a \vee b) \longrightarrow c)) && \text{(Proposition 2)} \\
 &= \neg((\neg c \longrightarrow \neg(a \vee b)) \longrightarrow \neg(a \vee b)) && \text{(Lemma 2 (iii)–(iv))} \\
 &= \neg((\neg(a \vee b) \longrightarrow \neg c) \longrightarrow \neg c) && \text{(W3)} \\
 &= \neg((c \longrightarrow (a \vee b)) \longrightarrow \neg c) && \text{(Lemma 2 (iv))} \\
 &= \neg(((c \longrightarrow a) \vee (c \longrightarrow b)) \longrightarrow \neg c) && \text{(Proposition 3 (4))} \\
 &= \neg(((c \longrightarrow a) \longrightarrow \neg c) \wedge ((c \longrightarrow b) \longrightarrow \neg c)) && \text{(Proposition 3 (3))} \\
 &= \neg((c \longrightarrow a) \longrightarrow \neg c) \vee \neg((c \longrightarrow b) \longrightarrow \neg c) && \text{(Proposition 3 (2))} \\
 &= \neg((\neg a \longrightarrow \neg c) \longrightarrow \neg c) \vee \neg((\neg b \longrightarrow \neg c) \longrightarrow \neg c) && \text{(Lemma 2 (iii)–(iv))} \\
 &= \neg(\neg a \vee \neg c) \vee \neg(\neg b \vee \neg c) && \text{(Proposition 2)} \\
 &= (\neg \neg a \wedge \neg \neg c) \vee (\neg \neg b \wedge \neg \neg c) && \text{(Proposition 3 (1))} \\
 &= (a \wedge c) \vee (b \wedge c). && \text{(Lemma 2 (iii))}
 \end{aligned}$$

Hence, A is a bounded distributive lattice with a complementation, that is, a W -algebra A is a Boolean algebra. \square

The proof of the following theorem is given by F.F. Nichita.

Theorem 3. [9] *Let $(A, \vee, \wedge, 0, 1, \neg)$ be a Boolean algebra. Then $S(a, b) = (a \vee b, a \wedge b)$ is a solution to the set-theoretical Yang-Baxter equation.*

Proof. Let $(A, \vee, \wedge, 0, 1, \neg)$ be a Boolean algebra. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(a, b, c) = (a \vee b, a \wedge b, c), \quad S^{23}(a, b, c) = (a, b \vee c, b \wedge c).$$

For all $(a, b, c) \in A^3$, we obtain

$$\begin{aligned}
 (S^{12} \circ S^{23} \circ S^{12})(a, b, c) &= S^{12}(S^{23}(S^{12}(a, b, c))) \\
 &= S^{12}(S^{23}(a \vee b, a \wedge b, c)) \\
 &= S^{12}(a \vee b, (a \wedge b) \vee c, (a \wedge b) \wedge c) \\
 &= ((a \vee b) \vee ((a \wedge b) \vee c), (a \vee b) \wedge ((a \wedge b) \vee c), (a \wedge b) \wedge c) \\
 &= (((a \vee b) \vee (a \wedge b)) \vee c, (a \vee b) \wedge ((a \vee c) \wedge (b \vee c)), (a \wedge b) \wedge c) \\
 &= ((a \vee (b \vee (a \wedge b))) \vee c, ((a \vee b) \wedge (a \vee c)) \wedge (b \vee c), a \wedge (b \wedge c)) \\
 &= ((a \vee b) \vee c, (a \vee (b \wedge c)) \wedge (b \vee c), a \wedge (b \wedge c)) \\
 &= (a \vee (b \vee c), (a \wedge (b \vee c)) \vee ((b \wedge c) \wedge (b \vee c)), a \wedge (b \wedge c)) \\
 &= (a \vee (b \vee c), (a \wedge (b \vee c)) \vee (b \wedge c), a \wedge (b \wedge c))
 \end{aligned}$$

and

$$\begin{aligned}
 (S^{23} \circ S^{12} \circ S^{23})(a, b, c) &= S^{23}(S^{12}(S^{23}(a, b, c))) \\
 &= S^{23}(S^{12}(a, b \vee c, b \wedge c)) \\
 &= S^{23}(a \vee (b \vee c), a \wedge (b \vee c), b \wedge c) \\
 &= (a \vee (b \vee c), (a \wedge (b \vee c)) \vee (b \wedge c), (a \wedge (b \vee c)) \wedge (b \wedge c)) \\
 &= (a \vee (b \vee c), (a \wedge (b \vee c)) \vee (b \wedge c), a \wedge ((b \vee c) \wedge (b \wedge c))) \\
 &= (a \vee (b \vee c), (a \wedge (b \vee c)) \vee (b \wedge c), a \wedge (b \wedge c)).
 \end{aligned}$$

Then, $S(a, b) = (a \vee b, a \wedge b)$ is a solution to the set-theoretical Yang-Baxter equation. \square

Theorem 4. *Let $(A, \longrightarrow, \neg, 1)$ be a W -algebra. If the identities*

$$\neg((\neg a \rightarrow b) \rightarrow \neg c) = (a \rightarrow \neg c) \rightarrow \neg(b \rightarrow \neg c) \tag{3}$$

or

$$(a \rightarrow \neg b) \rightarrow c = \neg((\neg a \rightarrow c) \rightarrow \neg(\neg b \rightarrow c)) \tag{4}$$

are satisfied for all a, b and c in A , then

$$S(a, b) = ((a \rightarrow b) \rightarrow b, \neg(a \rightarrow \neg(a \rightarrow b)))$$

and

$$S(a, b) = (\neg a \rightarrow b, \neg(a \rightarrow \neg b))$$

are solutions to the set-theoretical Yang-Baxter equation in the W -algebra A .

Proof. Substituting 1 instead of a and b , or $\neg 1$ instead of a and b in identities (3) and (4), respectively, we obtain $c = \neg c \rightarrow c$ or $\neg c = c \rightarrow \neg c$ by Lemma 2 (i) and (iii), and (W1). Then A is a Boolean algebra by Lemma 4. From the definitions of join and meet operators and Theorem 3,

$$\begin{aligned} S(a, b) &= ((a \rightarrow b) \rightarrow b, \neg(a \rightarrow \neg(a \rightarrow b))) \\ &= (a \vee b, a \wedge b) \end{aligned}$$

is a solution to the set-theoretical Yang-Baxter equation in the W -algebra A . Additionally, putting simultaneously $\neg b$ instead of b and b instead of c in identity (4), we get

$$(a \rightarrow \neg \neg b) \rightarrow b = \neg((\neg a \rightarrow b) \rightarrow \neg(\neg \neg b \rightarrow b)). \tag{*2}$$

Thus, we have

$$\begin{aligned} (a \rightarrow b) \rightarrow b &= (a \rightarrow \neg \neg b) \rightarrow b && \text{(Lemma 2 (iii))} \\ &= \neg((\neg a \rightarrow b) \rightarrow \neg(b \rightarrow b)) && \text{(*2)} \\ &= \neg((\neg a \rightarrow b) \rightarrow \neg 1) && \text{(W5)} \\ &= \neg \neg(\neg a \rightarrow b) && \text{(Lemma 2 (ii))} \\ &= \neg a \rightarrow b. && \text{(Lemma 2 (iii))} \end{aligned}$$

By applying Lemma 2 (iii)–(iv) to identity (3), we get

$$\neg(c \rightarrow \neg(\neg a \rightarrow b)) = (c \rightarrow \neg a) \rightarrow \neg(c \rightarrow \neg b). \tag{*3}$$

Therefore, we obtain

$$\begin{aligned} \neg(a \rightarrow \neg(a \rightarrow b)) &= \neg(a \rightarrow \neg(\neg \neg a \rightarrow b)) && \text{(Lemma 2 (iii))} \\ &= (a \rightarrow \neg \neg a) \rightarrow \neg(a \rightarrow \neg b) && \text{(*3)} \\ &= (a \rightarrow a) \rightarrow \neg(a \rightarrow \neg b) && \text{(Lemma 2 (iii))} \\ &= 1 \rightarrow \neg(a \rightarrow \neg b) && \text{(W5)} \\ &= \neg(a \rightarrow \neg b). && \text{(W1)} \end{aligned}$$

Hence, $S(a, b) = (\neg a \rightarrow b, \neg(a \rightarrow \neg b))$ is a solution to the set-theoretical Yang-Baxter equation in the W -algebra A . \square

Remark 1. Notice that the map $S(a, b) = (a \rightarrow b, a)$ is a solution to the set-theoretical Yang-Baxter equation in Boolean algebras (Theorem 5.1 in [24]) while it is not a solution in Wajsberg algebras. Indeed, in W -algebras, we have

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(a, b, c) &= S^{12}(S^{23}(S^{12}(a, b, c))) \\ &= S^{12}(S^{23}(a \rightarrow b, a, c)) \\ &= S^{12}(a \rightarrow b, a \rightarrow c, a) \\ &= ((a \rightarrow b) \rightarrow (a \rightarrow c), a \rightarrow b, a) \end{aligned}$$

and

$$\begin{aligned} (S^{23} \circ S^{12} \circ S^{23})(a, b, c) &= S^{23}(S^{12}(S^{12}(a, b, c))) \\ &= S^{23}(S^{12}(a, b \rightarrow c, b)) \\ &= S^{23}((a \rightarrow b) \rightarrow c, a, b) \\ &= (a \rightarrow (b \rightarrow c), a \rightarrow b, a). \end{aligned}$$

Then, we get $(S^{12} \circ S^{23} \circ S^{12})(a, b, c) \neq (S^{23} \circ S^{12} \circ S^{23})(a, b, c)$ for all a, b, c .

However, under the condition that $a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c)$ holds for any a, b, c , we have that $S(a, b) = (a \rightarrow b, a)$ is a solution to the set-theoretical Yang-Baxter equation in W -algebras.

Lemma 5. The following equations hold in every W -algebra:

- (i) $(a \rightarrow \neg b) \rightarrow (\neg a \rightarrow b) = \neg a \rightarrow b$ which is equivalent to $\neg(\neg a \rightarrow b) \rightarrow \neg(a \rightarrow \neg b) = \neg a \rightarrow b$
- (ii) $(a \rightarrow b) \rightarrow \neg((\neg a \rightarrow \neg b) \rightarrow \neg b) = a$
- (iii) $(a \rightarrow \neg b) \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c) = (a \rightarrow \neg c) \rightarrow \neg((\neg a \rightarrow c) \rightarrow \neg b)$

Proof.

(i)

$$\begin{aligned} (a \rightarrow \neg b) \rightarrow (\neg a \rightarrow b) &= 1 \rightarrow ((a \rightarrow \neg b) \rightarrow (\neg a \rightarrow b)) && (W1) \\ &= ((\neg a \rightarrow b) \rightarrow ((a \rightarrow \neg b) \rightarrow (\neg a \rightarrow b))) \rightarrow ((a \rightarrow \neg b) \rightarrow (\neg a \rightarrow b)) && (W8) \\ &= (((a \rightarrow \neg b) \rightarrow (\neg a \rightarrow b)) \rightarrow (\neg a \rightarrow b)) \rightarrow (\neg a \rightarrow b) && (W3) \\ &= ((a \rightarrow \neg b) \vee (\neg a \rightarrow b)) \rightarrow (\neg a \rightarrow b) && (Proposition 2) \\ &= 1 \rightarrow (\neg a \rightarrow b) && (Proposition 3 (7)) \\ &= \neg a \rightarrow b && (W1) \end{aligned}$$

is equivalent to

$$\neg(\neg a \rightarrow b) \rightarrow \neg(a \rightarrow \neg b) = (a \rightarrow \neg b) \rightarrow (\neg a \rightarrow b) = \neg a \rightarrow b$$

from Lemma 2 (iv).

(ii)

$$\begin{aligned}
 (a \longrightarrow b) \longrightarrow \neg((\neg a \longrightarrow \neg b) \longrightarrow \neg b) &= (a \longrightarrow b) \longrightarrow \\
 &\quad \neg(\neg a \vee \neg b) && \text{(Proposition 2)} \\
 &= (a \longrightarrow b) \longrightarrow (a \wedge b) && \text{(Proposition 3 (1))} \\
 &= ((a \longrightarrow b) \longrightarrow a) \wedge \\
 &\quad ((a \longrightarrow b) \longrightarrow b) && \text{(Proposition 3 (6))} \\
 &= ((a \longrightarrow b) \longrightarrow a) \wedge \\
 &\quad ((b \longrightarrow a) \longrightarrow a) && \text{(W3)} \\
 &= ((a \longrightarrow b) \vee \\
 &\quad (b \longrightarrow a)) \longrightarrow a && \text{(Proposition 3 (3))} \\
 &= 1 \longrightarrow a && \text{(Proposition 3 (7))} \\
 &= a && \text{(W1)}
 \end{aligned}$$

(iii) By Proposition 3 (7), we already have that $(a \longrightarrow \neg b) \vee (\neg b \longrightarrow a) = 1$ and $(a \longrightarrow \neg c) \vee (\neg c \longrightarrow a) = 1$. Thus, from definition of the join operator \vee , $a \longrightarrow \neg b = 1$ or $\neg a \longrightarrow b = \neg b \longrightarrow a = 1$ (Lemma 2 (iii)-(iv)) and $a \longrightarrow \neg c = 1$ or $\neg a \longrightarrow c = \neg c \longrightarrow a = 1$ (Lemma 2 (iii)-(iv)).

Case 1. Assume that $a \longrightarrow \neg b = 1$ and $a \longrightarrow \neg c = 1$. Then by using $a \longrightarrow \neg b = 1$, we get

$$(a \longrightarrow \neg b) \longrightarrow \neg((\neg a \longrightarrow b) \longrightarrow \neg c) = \neg((\neg a \longrightarrow b) \longrightarrow \neg c) \tag{W1}$$

and by using $a \longrightarrow \neg c = 1$ we have

$$(a \longrightarrow \neg c) \longrightarrow \neg((\neg a \longrightarrow c) \longrightarrow \neg b) = \neg((\neg a \longrightarrow c) \longrightarrow \neg b). \tag{W1}$$

Case 2. Assume that $a \longrightarrow \neg b = 1$ and $\neg a \longrightarrow c = 1$. Then by using $a \longrightarrow \neg b = 1$, we have

$$(a \longrightarrow \neg b) \longrightarrow \neg((\neg a \longrightarrow b) \longrightarrow \neg c) = \neg((\neg a \longrightarrow b) \longrightarrow \neg c) \tag{W1}$$

and by using $\neg a \longrightarrow c = 1$ we get

$$(a \longrightarrow \neg c) \longrightarrow \neg((\neg a \longrightarrow c) \longrightarrow \neg b) = (a \longrightarrow \neg c) \longrightarrow b. \tag{((W1) and Lemma 2 (iii))}$$

Case 3. Suppose that $\neg a \longrightarrow b = 1$ and $a \longrightarrow \neg c = 1$. Then by using $\neg a \longrightarrow b = 1$, we get

$$(a \longrightarrow \neg b) \longrightarrow \neg((\neg a \longrightarrow b) \longrightarrow \neg c) = (a \longrightarrow \neg b) \longrightarrow c \tag{((W1) and Lemma 2 (iii))}$$

and by using $a \longrightarrow \neg c = 1$ we have

$$(a \longrightarrow \neg c) \longrightarrow \neg((\neg a \longrightarrow c) \longrightarrow \neg b) = \neg((\neg a \longrightarrow c) \longrightarrow \neg b). \tag{W1}$$

Case 4. Suppose that $\neg a \longrightarrow b = 1$ and $\neg a \longrightarrow c = 1$. Then by using $\neg a \longrightarrow b = 1$, we get

$$(a \longrightarrow \neg b) \longrightarrow \neg((\neg a \longrightarrow b) \longrightarrow \neg c) = (a \longrightarrow \neg b) \longrightarrow c \tag{((W1) and Lemma 2 (iii))}$$

and by using $\neg a \longrightarrow c = 1$ we have

$$(a \longrightarrow \neg c) \longrightarrow \neg((\neg a \longrightarrow c) \longrightarrow \neg b) = (a \longrightarrow \neg c) \longrightarrow b. \tag{((W1) and Lemma 2 (iii))}$$

Hence, by Case 1, 2, 3 and 4,

$$(a \rightarrow \neg b) \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c) = (a \rightarrow \neg c) \rightarrow \neg((\neg a \rightarrow c) \rightarrow \neg b)$$

holds in every W -algebra. \square

Proposition 4. *The following identity*

$$(a \rightarrow \neg b) \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c) = \neg((\neg a \rightarrow b) \rightarrow \neg((a \rightarrow \neg b) \rightarrow c))$$

holds for every W -algebra.

Proof. By Proposition 3 (7), we already have that $(a \rightarrow \neg b) \vee (\neg b \rightarrow a) = 1$, and from definition of the join operator \vee , $a \rightarrow \neg b = 1$ or $\neg a \rightarrow b = \neg b \rightarrow a = 1$ (Lemma 2 (iii)–(iv)).

Assume that $a \rightarrow \neg b = 1$. Then, we get

$$\begin{aligned} (a \rightarrow \neg b) \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c) &= 1 \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c) && \text{(Hypothesis)} \\ &= \neg((\neg a \rightarrow b) \rightarrow \neg(1 \rightarrow c)) && \text{(W1)} \\ &= \neg((\neg a \rightarrow b) \rightarrow \neg((a \rightarrow \neg b) \rightarrow c)) && \text{(Hypothesis)} \end{aligned}$$

for all a, b, c in A .

Suppose that $\neg a \rightarrow b = 1$. Then, we obtain

$$\begin{aligned} (a \rightarrow \neg b) \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c) &= (a \rightarrow \neg b) \rightarrow \neg(1 \rightarrow \neg c) && \text{(Hypothesis)} \\ &= \neg(1 \rightarrow \neg((a \rightarrow \neg b) \rightarrow c)) && \text{((W1) and Lemma 2 (iii))} \\ &= \neg((\neg a \rightarrow b) \rightarrow \neg((a \rightarrow \neg b) \rightarrow c)) && \text{(Hypothesis)} \end{aligned}$$

for all a, b, c in A . \square

Corollary 2. *Let $(A, \rightarrow, \neg, 1)$ be a W -algebra. Then $S(a, b) = (\neg a \rightarrow b, \neg(a \rightarrow \neg b))$ is a solution to the set-theoretical Yang-Baxter equation in the W -algebra A .*

Proof. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(a, b, c) = (\neg a \rightarrow b, \neg(a \rightarrow \neg b), c), \quad S^{23}(a, b, c) = (a, \neg b \rightarrow c, \neg(b \rightarrow \neg c)).$$

For all $(a, b, c) \in A^3$, we get

$$\begin{aligned}
 (S^{12} \circ S^{23} \circ S^{12})(a, b, c) &= S^{12}(S^{23}(S^{12}(a, b, c))) \\
 &= S^{12}(S^{23}(\neg a \rightarrow b, \neg(a \rightarrow \neg b), z)) \\
 &= S^{12}(\neg a \rightarrow b, \neg\neg(a \rightarrow \neg b) \rightarrow c, \neg(\neg(a \rightarrow \neg b) \rightarrow \neg c)) \\
 &= (\neg(\neg a \rightarrow b) \rightarrow (\neg\neg(a \rightarrow \neg b) \rightarrow c), \\
 &\quad \neg((\neg a \rightarrow b) \rightarrow \neg(\neg\neg(a \rightarrow \neg b) \rightarrow c)), \\
 &\quad \neg(\neg(a \rightarrow \neg b) \rightarrow \neg c)) \\
 &= (\neg(\neg a \rightarrow b) \rightarrow ((a \rightarrow \neg b) \rightarrow c), \\
 &\quad \neg((\neg a \rightarrow b) \rightarrow \neg((a \rightarrow \neg b) \rightarrow c)), \\
 &\quad \neg(\neg(a \rightarrow \neg b) \rightarrow \neg c)) && \text{(Lemma 2 (iii))} \\
 &= ((a \rightarrow \neg b) \rightarrow (\neg(\neg a \rightarrow b) \rightarrow c), \\
 &\quad \neg((\neg a \rightarrow b) \rightarrow \neg((a \rightarrow \neg b) \rightarrow c)), \\
 &\quad \neg(\neg(a \rightarrow \neg b) \rightarrow \neg c)) && \text{(W11)} \\
 &= ((a \rightarrow \neg b) \rightarrow (\neg c \rightarrow (\neg a \rightarrow b)), \\
 &\quad \neg((\neg a \rightarrow b) \rightarrow \neg((a \rightarrow \neg b) \rightarrow c)), \\
 &\quad \neg(c \rightarrow (a \rightarrow \neg b))) && \text{(Lemma 2 (iii)-(iv))} \\
 &= (\neg c \rightarrow ((a \rightarrow \neg b) \rightarrow (\neg a \rightarrow b)), \\
 &\quad \neg((\neg a \rightarrow b) \rightarrow \neg((a \rightarrow \neg b) \rightarrow c)), \\
 &\quad \neg(c \rightarrow (a \rightarrow \neg b))) && \text{(W11)} \\
 &= (\neg c \rightarrow (\neg a \rightarrow b), \\
 &\quad \neg((\neg a \rightarrow b) \rightarrow \neg((a \rightarrow \neg b) \rightarrow c)), \\
 &\quad \neg(c \rightarrow (a \rightarrow \neg b))) && \text{(Lemma 5 (i))} \\
 &= (\neg a \rightarrow (\neg c \rightarrow b), \\
 &\quad \neg((\neg a \rightarrow b) \rightarrow \neg((a \rightarrow \neg b) \rightarrow c)), \\
 &\quad \neg(c \rightarrow (a \rightarrow \neg b))) && \text{(W11)} \\
 &= (\neg a \rightarrow (\neg b \rightarrow c), \\
 &\quad \neg((\neg a \rightarrow b) \rightarrow \neg((a \rightarrow \neg b) \rightarrow c)), \\
 &\quad \neg(c \rightarrow (a \rightarrow \neg b))) && \text{(Lemma 2 (iii)-(iv))} \\
 &= (\neg a \rightarrow (\neg b \rightarrow c), \\
 &\quad (a \rightarrow \neg b) \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c), \\
 &\quad \neg(c \rightarrow (a \rightarrow \neg b))) && \text{(Proposition 4)}
 \end{aligned}$$

and we have

$$\begin{aligned}
 (S^{23} \circ S^{12} \circ S^{23})(a, b, c) &= S^{23}(S^{12}(S^{23}(a, b, c))) \\
 &= S^{23}(S^{12}(a, \neg b \rightarrow c, \neg(b \rightarrow \neg c))) \\
 &= S^{23}(\neg a \rightarrow (\neg b \rightarrow c), \neg(a \rightarrow \neg(\neg b \rightarrow c)), \neg(b \rightarrow \neg c)) \\
 &= (\neg a \rightarrow (\neg b \rightarrow c), \\
 &\quad \neg\neg(a \rightarrow \neg(\neg b \rightarrow c)) \rightarrow \neg(b \rightarrow \neg c), \\
 &\quad \neg(\neg(a \rightarrow \neg(\neg b \rightarrow c)) \rightarrow \neg\neg(b \rightarrow \neg c))) \\
 &= (\neg a \rightarrow (\neg b \rightarrow c), \\
 &\quad (b \rightarrow \neg c) \rightarrow \neg(a \rightarrow \neg(\neg b \rightarrow c)), \\
 &\quad \neg(\neg(a \rightarrow \neg(\neg b \rightarrow c)) \rightarrow \\
 &\quad (b \rightarrow \neg c))) \tag{Lemma 2 (iv)} \\
 &= (\neg a \rightarrow (\neg b \rightarrow c), \\
 &\quad (b \rightarrow \neg c) \rightarrow \neg((\neg b \rightarrow c) \rightarrow \neg a), \\
 &\quad \neg(\neg(a \rightarrow \neg(\neg b \rightarrow c)) \rightarrow \\
 &\quad (b \rightarrow \neg c))) \tag{Lemma 2 (iii)-(iv)} \\
 &= (\neg a \rightarrow (\neg b \rightarrow c), \\
 &\quad (b \rightarrow \neg a) \rightarrow \neg((\neg b \rightarrow a) \rightarrow \neg c), \\
 &\quad \neg(\neg(a \rightarrow \neg(\neg b \rightarrow c)) \rightarrow \\
 &\quad (b \rightarrow \neg c))) \tag{Lemma 5 (iii)} \\
 &= (\neg a \rightarrow (\neg b \rightarrow c), \\
 &\quad (a \rightarrow \neg b) \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c), \\
 &\quad \neg(\neg(b \rightarrow \neg c) \rightarrow (a \rightarrow \\
 &\quad \neg(\neg b \rightarrow c)))) \tag{Lemma 2 (iii)-(iv)} \\
 &= (\neg a \rightarrow (\neg b \rightarrow c), \\
 &\quad (a \rightarrow \neg b) \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c), \\
 &\quad \neg(a \rightarrow (\neg(b \rightarrow \neg c) \rightarrow \\
 &\quad \neg(\neg b \rightarrow c)))) \tag{W11)} \\
 &= (\neg a \rightarrow (\neg b \rightarrow c), \\
 &\quad (a \rightarrow \neg b) \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c), \\
 &\quad \neg(a \rightarrow (b \rightarrow \neg c))) \tag{Lemma 5 (i)} \\
 &= (\neg a \rightarrow (\neg b \rightarrow c), \\
 &\quad (a \rightarrow \neg b) \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c), \\
 &\quad \neg(a \rightarrow (c \rightarrow \neg b))) \tag{Lemma 2 (iii)-(iv)} \\
 &= (\neg a \rightarrow (\neg b \rightarrow c), \\
 &\quad (a \rightarrow \neg b) \rightarrow \neg((\neg a \rightarrow b) \rightarrow \neg c), \\
 &\quad \neg(c \rightarrow (a \rightarrow \neg b))). \tag{W11)}
 \end{aligned}$$

Then, $S(a, b) = (\neg a \rightarrow b, \neg(a \rightarrow \neg b))$ is a solution to the set-theoretical Yang-Baxter equation in the W-algebra A . \square

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