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Pre-Metric Spaces Along with Different Types of Triangle Inequalities

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Abstract: The T_1-spaces induced by the pre-metric spaces along with many forms of triangle inequalities are investigated in this paper. The limits in pre-metric spaces are also studied to demonstrate the consistency of limit concept in the induced topologies.

Keywords: quasi-metric space; Hausdorff space; T_1-space; triangle inequalities

MSC: 54E35; 54E55

1. Introduction

Let X be a nonempty universal set, and let $d : X \times X \to \mathbb{R}_+$ be a nonnegative real-valued function defined on the product set $X \times X$. We say that (X, d) is a metric space if and only if the following conditions are satisfied:

- for any $x, y \in M$, $d(x, y) = 0$ implies $x = y$;
- (self-distance condition) for any $x \in M$, $d(x, x) = 0$;
- (symmetric condition) for any $x, y \in M$, $d(x, y) = d(y, x)$;
- (triangle inequality) for any $x, y, z \in M$, $d(x, z) \leq d(x, y) + d(y, z)$.

In the literature, different kinds of spaces are considered by weakening the above conditions. Wilson [1] says that (X, d) is a quasi-metric space when the symmetric condition is not satisfied; that is, the following conditions are satisfied:

- for any $x, y \in M$, $d(x, y) = 0$ if and only if $x = y$;
- for any $x, y, z \in M$, $d(x, z) \leq d(x, y) + d(y, z)$.

After that, many authors (referring to [2–15] and the references therein) also defined the quasi-metric space as follows:

- for any $x, y \in M$, $d(x, y) = 0 = d(y, x)$ if and only if $x = y$;
- for any $x, y, z \in M$, $d(x, z) \leq d(x, y) + d(y, z)$.

However, these two definitions are not equivalent. The reason is that $d(x, y) = 0$ does not necessarily imply $d(y, x) = 0$, since the symmetric condition is not satisfied. It is clear to see that, in the Wilson's sense, we also have $d(y, x) = 0$ if and only if $y = x$.

Wilson [16] also says that (X, d) is a semi-metric space when the triangle inequality is not satisfied; that is, the following conditions are satisfied:

- for any $x, y \in M$, $d(x, y) = 0$ if and only if $x = y$;
- for any $x, y \in M$, $d(x, y) = d(y, x)$.

On the other hand, Matthews [11] says that (X, d) is a partial metric space if and only if the following conditions are satisfied:

- for any $x, y \in M$, $x = y$ if and only if $d(x, x) = d(x, y) = d(y, y)$;
- for any $x, y \in M$, $d(x, x) \leq d(x, y)$;
- for any $x, y \in M$, $d(x, y) = d(y, x)$.
- for any $x, y, z \in M$, $d(x, z) \leq d(x, y) + d(y, z) - d(y, y)$.

The partial metric space does not assume the self-distance condition $d(x, x) = 0$.

In this paper, we shall consider a so-called pre-metric space in which we just assume that $d(x, y) = 0$ implies $x = y$ for any $x, y \in X$. In other words, the pre-metric space does not assume the self-distance condition and symmetric condition. Since the triangle inequality plays a very important role, without considering the symmetric condition, the triangle inequality can be considered in four forms, which was not discussed in the literature. Based on the four different kinds of triangle inequalities, we can induce the T_1-space space from the pre-metric space under some suitable conditions.

This paper is organized as follows. In Section 2 , we propose the so-called pre-metric space in which four forms of triangle inequalities are considered and studied. Many basic properties are also obtained for further investigation. In Section 3, we induce the T_1-space from a given pre-metric space under some suitable assumptions. In Section 4, the limits in pre-metric space are also studied. We present the consistency of limit concepts in the pre-metric space and the induced topologies.

2. Definitions and Properties

In this section, we shall introduce the concept of pre-metric space, and the four concepts of triangle inequalities. We also derive some interesting properties that will be used in the further study. Without considering the symmetric condition, we first introduce four types of triangle inequality as follows.

Definition 1. *Let X be a nonempty universal set, and let d be a mapping defined on $X \times X$ into $\mathbb{R}_+$.*

- *We say that d satisfies the* $\bowtie$-triangle inequality *if and only if the following inequality is satisfied:*

$$d(x, y) + d(y, z) \geq d(x, z) \text{ for all } x, y, z \in X.$$

- *We say that d satisfies the* $\triangleright$-triangle inequality *if and only if the following inequality is satisfied:*

$$d(x, y) + d(z, y) \geq d(x, z) \text{ for all } x, y, z \in X.$$

- *We say that d satisfies the* $\triangleleft$-triangle inequality *if and only if the following inequality is satisfied:*

$$d(y, x) + d(y, z) \geq d(x, z) \text{ for all } x, y, z \in X.$$

- *We say that d satisfies the* $\diamond$-triangle inequality *if and only if the following inequality is satisfied:*

$$d(y, x) + d(z, y) \geq d(x, z) \text{ for all } x, y, z \in X.$$

It is obvious that if d satisfies the symmetric condition, then the concepts of $\bowtie$-triangle inequality, $\triangleright$-triangle inequality, $\triangleleft$-triangle inequality and $\diamond$-triangle inequality are all equivalent.

Example 1. *We define a function $d : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $d(x, y) = \max\{x, y\}$. Then $d(x, x) = x$ for any $x \geq 0$, which also says that $d(x, x)$ is not always zero. It is not hard to check*

$$\max\{x, y\} + \max\{y, z\} \geq \max\{x, z\},$$

which also says that

$$d(x, y) + d(y, z) \geq d(x, z).$$

This shows that d satisfies the $\bowtie$-triangle inequality. Since d also satisfies the symmetric condition, it means that all the four forms of triangle inequalities are equivalent. However, since $d(x,x) > 0$ for $x > 0$, it says that $(\mathbb{R}_+, d)$ is still not a metric space.

Example 2. *We define a function $d : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by*

$$d(x,y) = \begin{cases} x & \text{if } x \geq y \\ 2y - x & \text{if } x < y. \end{cases}$$

Then $d(x,x) = x$ for any $x \geq 0$, which also says that $d(x,x)$ is not always zero. For $x > y$, we see that $d(x,y) = x$ and $d(y,x) = 2x - y$, which says that $d(x,y) \neq d(y,x)$ in general; that is, the symmetric condition is not satisfied. It is not hard to check

$$d(x,y) + d(y,z) \geq d(x,z).$$

This shows that d also satisfies the $\bowtie$-triangle inequality.

Examples 1 and 2 say that $d(x,x) \neq 0$ for $x \neq 0$. Therefore, we propose the following definition.

Definition 2. *Let X be a nonempty universal set, and let d be a mapping defined on $X \times X$ into $\mathbb{R}_+$. We say that (X,d) is a* pre-metric space *if and only if $d(x,y) = 0$ implies $x = y$ for any $x, y \in X$.*

We see that (X,d) is a *quasi-metric space* if and only if (X,d) is a pre-metric space satisfying the $\bowtie$-triangle inequality and $d(x,x) = 0$ for all $x \in X$.

Example 3. *Examples 1 and 2 are pre-metric spaces, since it is not hard to check that $d(x,y) = 0$ implies $x = y = 0$ based on the nonnegativity.*

Remark 1. *Let (X,d) be a pre-metric space. Then $d(x,y) = 0$ implies $x = y$, which also implies $d(x,y) = 0 = d(x,x) = d(y,x)$ without needing the symmetric condition. We remark that this symmetric situation can only happen when $d(x,y) = 0$ or $d(y,x) = 0$. Therefore, if $d(x,y) > 0$ then we cannot have $d(x,y) = d(y,x)$ in general. On the other hand, we also see that $d(x,y) = 0$ or $d(y,x) = 0$ implies $d(x,x) = 0$. However, this situation does not say $d(x,x) = 0$ for all $x \in X$. We can just say that $d(x,x) = 0$ when $d(x,y) = 0$ for some $x, y \in X$. In other words, we can just say that $d(x,x) = 0$ for some $x \in X$. This situation can also be realized from Example 2.*

Proposition 1. *Let X be a nonempty universal set, and let d be a mapping defined on $X \times X$ into $\mathbb{R}_+$. Suppose that the following conditions are satisfied:*

- *$d(x,x) = 0$ for all $x \in X$;*
- *d satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality or the $\diamond$-triangle inequality.*

Then d satisfies the symmetric condition.

Proof. Suppose that d satisfies the $\triangleright$-triangle inequality. Then, given any $x, y \in X$, we have

$$d(x,y) \leq d(x,x) + d(y,x) = d(y,x).$$

By interchanging the roles of x and y, we can also obtain $d(y,x) \leq d(x,y)$. This shows that $d(x,y) = d(y,x)$. The other cases of satisfying the $\triangleleft$-triangle inequality and the $\diamond$-triangle inequality can be similarly obtained. This completes the proof. □

Remark 2. *Suppose that $d(x,x) = 0$ for all $x \in X$, and that d satisfies the $\circ$-triangle inequality for some $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then, using Proposition 1, we see that all the four forms of triangle inequalities are equivalent.*

3. T_1-Space

We want to show that the pre-metric space along with the different kinds of triangle inequalities can induce the T_1-Space based on the concepts of open balls defined below.

Definition 3. *Let (X, d) be a pre-metric space. Given $r > 0$, the* open balls *centered at x are denoted and defined by*

$$B^{\triangleleft}(x;r) = \{y \in X : d(x,y) < r\}$$

and

$$B^{\triangleright}(x;r) = \{y \in X : d(y,x) < r\}.$$

Let $\mathcal{B}^{\triangleleft}$ denote the family of all open balls $B^{\triangleleft}(x;r)$, and let $\mathcal{B}^{\triangleright}$ denote the family of all open balls $B^{\triangleright}(x;r)$.

In the sequel, we also assume that the open balls $B^{\triangleleft}(x;r)$ and $B^{\triangleright}(x;r)$ are nonempty for each $x \in X$ and $r > 0$. In other words, given any $x \in X$ and $r > 0$, we assume that there exist y_1 and y_2 such that $d(x, y_1) < r$ and $d(x, y_2) < r$, respectively. It is also clear that if d satisfies the symmetric condition, then

$$B^{\triangleleft}(x;r) = B^{\triangleright}(x;r).$$

In this case, we simply write $B(x;r)$ to denote the open balls centered at x, and write $\mathcal{B}$ to denote the family of all open balls $B(x;r)$.

Proposition 2. *Let (X, d) be a pre-metric space.*

(i) Given any $x \in X$, we have the following properties.

- *Suppose that $d(x,x) = 0$. Then $x \in B^{\triangleleft}(x;r) \in \mathcal{B}^{\triangleleft}$ and $x \in B^{\triangleright}(x;r) \in \mathcal{B}^{\triangleright}$ for all $r > 0$.*
- *Suppose that $x \in B^{\triangleleft}(x;r)$ for all $r > 0$, or that $x \in B^{\triangleright}(x;r)$ for all $r > 0$. Then $d(x,x) = 0$.*

(ii) If $x \neq y$, then there exist $r_1 > 0$ and $r_2 > 0$ such that $y \notin B^{\triangleleft}(x;r_1)$ and $y \notin B^{\triangleright}(x;r_2)$.

(iii) For each $x \in X$, we have the following properties.

- *Given any $B^{\triangleleft}(x;r) \in \mathcal{B}^{\triangleleft}$, there exists $n \in \mathbb{N}$ such that $B^{\triangleleft}(x;\frac{1}{n}) \subseteq B^{\triangleleft}(x;r)$.*
- *Given any $B^{\triangleright}(x;r) \in \mathcal{B}^{\triangleright}$, there exists $n \in \mathbb{N}$ such that $B^{\triangleright}(x;\frac{1}{n}) \subseteq B^{\triangleright}(x;r)$.*

Proof. The first statement of part (i) is obvious. To prove the second statement of part (i), we take a sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers such that it is decreasing to zero. Then we have $d(x,x) < r_n$ for all n, which implies $d(x,x) = 0$ by taking $n \to \infty$. To prove part (ii), since $x \neq y$, it follows that $d(x,y) > 0$ and $d(y,x) > 0$ by the definition of pre-metric space. Using the denseness of $\mathbb{R}$, there exists $r_1 > 0$ such that $0 < r_1 < d(x,y)$, which also says that $y \notin B^{\triangleleft}(x;r_1)$. We also have $y \notin B^{\triangleright}(x;r_2)$ for some $r_2 > 0$ satisfying $0 < r_2 < d(y,x)$. Part (iii) follows from the existence of a positive integer n with $1/n < r$. This completes the proof. □

Proposition 3. *Let (X, d) be a pre-metric space. Then we have the following inclusions.*

(i) Suppose that d satisfies the $\bowtie$-triangle inequality.

- *Given any $y \in B^{\triangleleft}(x;r)$, there exists $\bar{r} > 0$ such that $B^{\triangleleft}(y;\bar{r}) \subseteq B^{\triangleleft}(x;r)$.*
- *Given any $y \in B^{\triangleright}(x;r)$, there exists $\bar{r} > 0$ such that $B^{\triangleright}(y;\bar{r}) \subseteq B^{\triangleright}(x;r)$.*

(ii) Suppose that d satisfies the $\triangleright$-triangle inequality. Given any $y \in B^{\triangleleft}(x;r)$, there exists $\bar{r} > 0$ such that $B^{\triangleright}(y;\bar{r}) \subseteq B^{\triangleleft}(x;r)$ and $B^{\triangleright}(y;\bar{r}) \subseteq B^{\triangleright}(x;r)$.

(iii) Suppose that d satisfies the $\triangleleft$-triangle inequality. Given any $y \in B^{\triangleright}(x;r)$, there exists $\bar{r} > 0$ such that $B^{\triangleleft}(y;\bar{r}) \subseteq B^{\triangleright}(x;r)$ and $B^{\triangleleft}(y;\bar{r}) \subseteq B^{\triangleleft}(x;r)$.

(iv) Suppose that d satisfies the $\diamond$-triangle inequality.

- *Given any $y \in B^{\triangleleft}(x;r)$, there exists $\bar{r} > 0$ such that $B^{\triangleleft}(y;\bar{r}) \subseteq B^{\triangleright}(x;r)$.*
- *Given any $y \in B^{\triangleright}(x;r)$, there exists $\bar{r} > 0$ such that $B^{\triangleright}(y;\bar{r}) \subseteq B^{\triangleleft}(x;r)$.*

(v) *Suppose that d satisfies the $\triangleright$-triangle inequality and the $\triangleleft$-triangle inequality.*

- *Given any $y \in B^{\triangleleft}(x;r)$, there exists $\bar{r} > 0$ such that $B^{\triangleleft}(y;\bar{r}) \subseteq B^{\triangleleft}(x;r)$.*
- *Given any $y \in B^{\triangleright}(x;r)$, there exists $\bar{r} > 0$ such that $B^{\triangleright}(y;\bar{r}) \subseteq B^{\triangleright}(x;r)$.*

Proof. To prove part (i), for $y \in B^{\triangleleft}(x;r)$ and $z \in B^{\triangleleft}(y;\bar{r})$, let $\bar{r} \leq r - d(x,y)$. Using the $\bowtie$-triangle inequality, we have

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \bar{r} \leq d(x,y) + r - d(x,y) = r,$$

which says that $z \in B^{\triangleleft}(x;r)$. Therefore we obtain the inclusion $B^{\triangleleft}(y;\bar{r}) \subseteq B^{\triangleleft}(x;r)$. For $y \in B^{\triangleright}(x;r)$ and $z \in B^{\triangleright}(y;\bar{r})$, let $\bar{r} \leq r - d(y,x)$. Then we can similarly obtain the inclusion $B^{\triangleright}(y;\bar{r}) \subseteq B^{\triangleright}(x;r)$.

To prove part (ii), for $y \in B^{\triangleleft}(x;r)$ and $z \in B^{\triangleright}(y;\bar{r})$, let $\bar{r} \leq r - d(x,y)$. Using the $\triangleright$-triangle inequality, we have

$$d(x,z) \leq d(x,y) + d(z,y) < d(x,y) + \bar{r} \leq d(x,y) + r - d(x,y) = r,$$

which says that $z \in B^{\triangleleft}(x;r)$. Therefore we obtain the inclusion $B^{\triangleright}(y;\bar{r}) \subseteq B^{\triangleleft}(x;r)$. We can similarly obtain the inclusion $B^{\triangleright}(y;\bar{r}) \subseteq B^{\triangleright}(x;r)$.

Parts (iii) and (iv) can be similarly obtained. To prove the first statement of part (v), using part (ii), we can take $\bar{r}^* > 0$ such that $B^{\triangleright}(y;\bar{r}^*) \subseteq B^{\triangleleft}(x;r)$. Using part (iii), we can also take $\bar{r} > 0$ such that $B^{\triangleleft}(y;\bar{r}) \subseteq B^{\triangleright}(y;\bar{r}^*)$. This shows that $B^{\triangleleft}(y;\bar{r}) \subseteq B^{\triangleleft}(x;r)$. The second statement of part (v) can be similarly obtained. This completes the proof. $\square$

Proposition 4. *Let (X,d) be a pre-metric space. Then we have the following inclusions.*

(i) *Suppose that d satisfies the $\bowtie$-triangle inequality.*

- *If $x \in B^{\triangleleft}(x_1,r_1) \cap B^{\triangleleft}(x_2,r_2)$, then there exists $r_3 > 0$ such that*
$$B^{\triangleleft}(x,r_3) \subseteq B^{\triangleleft}(x_1,r_1) \cap B^{\triangleleft}(x_2,r_2).$$
- *If $x \in B^{\triangleright}(x_1,r_1) \cap B^{\triangleright}(x_2,r_2)$, then there exists $r_3 > 0$ such that*
$$B^{\triangleright}(x,r_3) \subseteq B^{\triangleright}(x_1,r_1) \cap B^{\triangleright}(x_2,r_2).$$

(ii) *Suppose that d satisfies the $\triangleright$-triangle inequality. If $x \in B^{\triangleleft}(x_1,r_1) \cap B^{\triangleleft}(x_2,r_2)$, then there exists $r_3 > 0$ such that*
$$B^{\triangleright}(x,r_3) \subseteq B^{\triangleleft}(x_1,r_1) \cap B^{\triangleleft}(x_2,r_2) \text{ and } B^{\triangleright}(x,r_3) \subseteq B^{\triangleright}(x_1,r_1) \cap B^{\triangleright}(x_2,r_2).$$

(iii) *Suppose that d satisfies the $\triangleleft$-triangle inequality. If $x \in B^{\triangleright}(x_1,r_1) \cap B^{\triangleright}(x_2,r_2)$, then there exists $r_3 > 0$ such that*
$$B^{\triangleleft}(x,r_3) \subseteq B^{\triangleright}(x_1,r_1) \cap B^{\triangleright}(x_2,r_2) \text{ and } B^{\triangleleft}(x,r_3) \subseteq B^{\triangleleft}(x_1,r_1) \cap B^{\triangleleft}(x_2,r_2).$$

(iv) *Suppose that d satisfies the $\diamond$-triangle inequality.*

- *If $x \in B^{\triangleleft}(x_1,r_1) \cap B^{\triangleleft}(x_2,r_2)$, then there exists $r_3 > 0$ such that*
$$B^{\triangleleft}(x,r_3) \subseteq B^{\triangleright}(x_1,r_1) \cap B^{\triangleright}(x_2,r_2).$$
- *If $x \in B^{\triangleright}(x_1,r_1) \cap B^{\triangleright}(x_2,r_2)$, then there exists $r_3 > 0$ such that*
$$B^{\triangleright}(x,r_3) \subseteq B^{\triangleleft}(x_1,r_1) \cap B^{\triangleleft}(x_2,r_2).$$

(v) *Suppose that d satisfies the* $\triangleright$*-triangle inequality and the* $\triangleleft$*-triangle inequality. We have the following inclusions.*

- *If* $x \in B^{\triangleleft}(x_1, r_1) \cap B^{\triangleleft}(x_2, r_2)$, *then there exists* $r_4 > 0$ *such that*

$$B^{\triangleleft}(x, r_4) \subseteq B^{\triangleleft}(x_1, r_1) \cap B^{\triangleleft}(x_2, r_2).$$

- *If* $x \in B^{\triangleright}(x_1, r_1) \cap B^{\triangleright}(x_2, r_2)$, *then there exists* $r_4 > 0$ *such that*

$$B^{\triangleright}(x, r_4) \subseteq B^{\triangleright}(x_1, r_1) \cap B^{\triangleright}(x_2, r_2).$$

Proof. To prove part (i), for $x \in B^{\triangleleft}(x_1, r_1)$ and $x \in B^{\triangleleft}(x_2, r_2)$, using part (i) of Proposition 3, there exist $\bar{r}_1$ and $\bar{r}_2$ such that

$$B^{\triangleleft}(x, \bar{r}_1) \subseteq B^{\triangleleft}(x_1, r_1) \text{ and } B^{\triangleleft}(x, \bar{r}_2) \subseteq B^{\triangleleft}(x_2, r_2).$$

We take $r_3 = \min\{\bar{r}_1, \bar{r}_2\}$. Then

$$B^{\triangleleft}(x, r_3) \subseteq B^{\triangleleft}(x, \bar{r}_1) \cap B^{\triangleleft}(x, \bar{r}_2) \subseteq B^{\triangleleft}(x_1, r_1) \cap B^{\triangleleft}(x_2, r_2).$$

Therefore we obtain the first inclusion. The second inclusion can be similarly obtained.

To prove part (ii), for $x \in B^{\triangleleft}(x_1, r_1)$ and $x \in B^{\triangleleft}(x_2, r_2)$, using part (ii) of Proposition 3, there exist $\bar{r}_1$ and $\bar{r}_2$ such that

$$B^{\triangleright}(x, \bar{r}_1) \subseteq B^{\triangleleft}(x_1, r_1) \text{ and } B^{\triangleright}(x, \bar{r}_1) \subseteq B^{\triangleright}(x_1, r_1).$$

and

$$B^{\triangleright}(x, \bar{r}_2) \subseteq B^{\triangleleft}(x_2, r_2) \text{ and } B^{\triangleright}(x, \bar{r}_2) \subseteq B^{\triangleright}(x_2, r_2).$$

Let $r_3 = \min\{\bar{r}_1, \bar{r}_2\}$. Then

$$B^{\triangleright}(x, r_3) \subseteq B^{\triangleright}(x, \bar{r}_1) \cap B^{\triangleright}(x, \bar{r}_2) \subseteq B^{\triangleleft}(x_1, r_1) \cap B^{\triangleleft}(x_2, r_2)$$

and

$$B^{\triangleright}(x, r_3) \subseteq B^{\triangleright}(x, \bar{r}_1) \cap B^{\triangleright}(x, \bar{r}_2) \subseteq B^{\triangleright}(x_1, r_1) \cap B^{\triangleright}(x_2, r_2).$$

Therefore we obtain the desired inclusions.

Parts (iii) and (iv) can be similarly obtained. To prove the first statement of part (v), using part (iii) of Proposition 3 and part (ii) of this proposition, we can find $r_4 > 0$ such that

$$B^{\triangleleft}(x, r_4) \subseteq B^{\triangleright}(x, r_3) \subseteq B^{\triangleleft}(x_1, r_1) \cap B^{\triangleleft}(x_2, r_2).$$

The second statement can be similarly obtained. This completes the proof. □

Proposition 5. *Let* (X, d) *be a pre-metric space. Suppose that* $x \neq y$. *Then we have the following properties.*

(i) *Suppose that d satisfies the* $\bowtie$*-triangle inequality or the* $\diamond$*-triangle inequality. Then* $B^{\triangleleft}(x; r) \cap B^{\triangleright}(y; r) = \emptyset$ *and* $B^{\triangleright}(x; r) \cap B^{\triangleleft}(y; r) = \emptyset$ *for some* $r > 0$.

(ii) *Suppose that d satisfies the* $\triangleright$*-triangle inequality. Then* $B^{\triangleleft}(x; r) \cap B^{\triangleleft}(y; r) = \emptyset$ *for some* $r > 0$.

(iii) *Suppose that d satisfies the* $\triangleleft$*-triangle inequality. Then* $B^{\triangleright}(x; r) \cap B^{\triangleright}(y; r) = \emptyset$ *for some* $r > 0$.

Proof. Since $x \neq y$, it says that $d(x, y) > 0$ and $d(y, x) > 0$. We consider the following cases.

- Suppose that d satisfies the $\triangleright$-triangle inequality. Let $r \leq d(x, y)/2$. We are going to prove $B^{\triangleleft}(x; r) \cap B^{\triangleleft}(y; r) = \emptyset$ by contradiction. Suppose that $z \in B^{\triangleleft}(x; r) \cap B^{\triangleleft}(y; r)$. Since d satisfies the $\triangleright$-triangle inequality, it follows that

$$d(x, y) \leq d(x, z) + d(y, z) < r + r = 2r \leq d(x, y),$$

which is a contradiction. Suppose that d satisfies the $\triangleleft$-triangle inequality. Then we can similarly obtain the desired result.

- Suppose that d satisfies the $\bowtie$-triangle inequality. Let $r \leq d(x,y)/2$. For $z \in B^{\triangleleft}(x;r) \cap B^{\triangleright}(y;r)$, it follows that
$$d(x,y) \leq d(x,z) + d(z,y) < r + r = 2r \leq d(x,y),$$
which is a contradiction. On the other hand, let $r \leq d(y,x)/2$, for $z \in B^{\triangleright}(x;r) \cap B^{\triangleleft}(y;r)$, it follows that
$$d(y,x) \leq d(y,z) + d(z,x) < r + r = 2r \leq d(y,x),$$
which is a contradiction. Suppose that d satisfies the $\diamond$-triangle inequality. Then we can similarly obtain the desired result.

This completes the proof. $\square$

Theorem 1. *Let (X,d) be a pre-metric space. Define*

$$\tau^{\triangleleft} = \left\{ O^{\triangleleft} \subseteq X : x \in O^{\triangleleft} \text{ if and only if there exist } r > 0 \text{ such that } x \in B^{\triangleleft}(x;r) \subseteq O^{\triangleleft} \right\}. \tag{1}$$

and

$$\tau^{\triangleright} = \left\{ O^{\triangleright} \subseteq X : x \in O^{\triangleright} \text{ if and only if there exist } r > 0 \text{ such that } x \in B^{\triangleright}(x;r) \subseteq O^{\triangleright} \right\}. \tag{2}$$

Suppose that d satisfies the $\bowtie$-triangle inequality. Then we have the following results.

- *Assume additionally that $d(x,x) = 0$ for all $x \in X$, or that $x \in B^{\triangleleft}(x;r)$ for all $x \in X$ and $r > 0$. Then $(X,\tau^{\triangleleft})$ is a T_1-space such that $\mathcal{B}^{\triangleleft}$ is a base for the topology $\tau^{\triangleleft}$.*
- *Assume additionally that $d(x,x) = 0$ for all $x \in X$, or that $x \in B^{\triangleright}(x;r)$ for all $x \in X$ and $r > 0$. Then $(X,\tau^{\triangleright})$ is a T_1-space such that $\mathcal{B}^{\triangleright}$ is a base for the topology $\tau^{\triangleright}$.*

The T_1-spaces $(X,\tau^{\triangleleft})$ and $(X,\tau^{\triangleright})$ also satisfy the first axiom of countability. Moreover, $B^{\triangleleft}(x;r)$ is a $\tau^{\triangleleft}$-open set and $B^{\triangleright}(x;r)$ is a $\tau^{\triangleright}$-open set.

Proof. Using part (i) of Proposition 2 and part (i) of Proposition 4, we see that $\tau^{\triangleleft}$ is a topology such that $\mathcal{B}^{\triangleleft}$ is a base for $\tau^{\triangleleft}$. Part (ii) of Proposition 2 says that $(X,\tau^{\triangleleft})$ is a T_1-space. Part (iii) of Proposition 2 says that there exists a countable local base at each $x \in X$ for $\tau^{\triangleleft}$, which also says that $\tau^{\triangleleft}$ satisfies the first axiom of countability. Regarding $\tau^{\triangleright}$, we can similarly obtain the desired results. Finally, part (i) of Proposition 3 says that $B^{\triangleleft}(x;r)$ is a $\tau^{\triangleleft}$-open set and $B^{\triangleright}(x;r)$ is a $\tau^{\triangleright}$-open set. This completes the proof. $\square$

We remark that, in Theorem 1, although we assume $d(x,x) = 0$ for all $x \in X$, (X,d) is not necessarily a metric space, since the symmetric condition is still not satisfied. The following example provides this observation.

Example 4. *We define a function $d : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by*

$$d(x,y) = \begin{cases} x & \text{if } x > y \\ 0 & \text{if } x = y \\ 2y - x & \text{if } x < y. \end{cases}$$

Then $d(x,x) = 0$ for all $x \in X$. By referring to Example 2, we also see that the symmetric condition is not satisfied, and that d satisfies the $\bowtie$-triangle inequality. Using Theorem 1, we can induce two T_1-spaces $(\mathbb{R}_+,\tau^{\triangleleft})$ and $(\mathbb{R}_+,\tau^{\triangleright})$. Moreover, the spaces $(\mathbb{R}_+,\tau^{\triangleleft})$ and $(\mathbb{R}_+,\tau^{\triangleright})$ also satisfy the first axiom of countability.

4. Limits in Pre-Metric Space

Let (X,d) be a pre-metric space. Since the symmetric condition is not necessarily satisfied, the different concepts of limit are proposed below.

Definition 4. *Let (X,d) be a pre-metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X.*

- *We write $x_n \xrightarrow{d^{\triangleright}} x$ as $n \to \infty$ if and only if $d(x_n,x) \to 0$ as $n \to \infty$.*
- *We write $x_n \xrightarrow{d^{\triangleleft}} x$ as $n \to \infty$ if and only if $d(x,x_n) \to 0$ as $n \to \infty$.*
- *We write $x_n \xrightarrow{d} x$ as $n \to \infty$ if and only if*

$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0.$$

The uniqueness of limits will be discussed below.

Proposition 6. *Let (X,d) be a pre-metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X.*

(i) Suppose that d satisfies the $\bowtie$-triangle inequality or $\diamond$-triangle inequality. If $x_n \xrightarrow{d^{\triangleleft}} x$ and $x_n \xrightarrow{d^{\triangleright}} y$, then $x = y$.

(ii) Suppose that d satisfies the $\triangleleft$-triangle inequality. If $x_n \xrightarrow{d^{\triangleright}} x$ and $x_n \xrightarrow{d^{\triangleright}} y$, then $x = y$. In other words, the $d^{\triangleright}$-limit is unique.

(iii) Suppose that d satisfies the $\triangleright$-triangle inequality. If $x_n \xrightarrow{d^{\triangleleft}} x$ and $x_n \xrightarrow{d^{\triangleleft}} y$, then $x = y$. In other words, the $d^{\triangleleft}$-limit is unique.

Proof. To prove part (i), we first assume that d satisfies the $\bowtie$-triangle inequality. Then

$$d(x,y) \leq d(x,x_n) + d(x_n,y) \to 0 + 0 = 0,$$

which says that $x = y$. Now suppose that d satisfies the $\diamond$-triangle inequality. Then

$$d(y,x) \leq d(x_n,y) + d(x,x_n) \to 0 + 0 = 0$$

which also says that $x = y$. The other cases can be similarly obtained. This completes the proof. □

Let (X,τ) be a topological space. The sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x \in X$ with respect to the topology is denoted by $x_n \xrightarrow{\tau} x$ as $n \to \infty$.

Proposition 7. *Let (X,d) be a pre-metric space. Suppose that d satisfies the $\bowtie$-triangle inequality or the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality. Assume that $d(x,x) = 0$ for all $x \in X$. Then the following statements hold true.*

(i) Let $\tau^{\triangleright}$ be the topology defined by (1) in Theorem 1, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Then $x_n \xrightarrow{\tau^{\triangleright}} x$ as $n \to \infty$ if and only if $x_n \xrightarrow{d^{\triangleright}} x$ as $n \to \infty$.

(ii) Let $\tau^{\triangleleft}$ be the topology defined by (2) in Theorem 1, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Then $x_n \xrightarrow{\tau^{\triangleleft}} x$ as $n \to \infty$ if and only if $x_n \xrightarrow{d^{\triangleleft}} x$ as $n \to \infty$.

Proof. Under the assumptions, Theorem 1 says that we can induce two topologies $\tau^{\triangleright}$ and $\tau^{\triangleleft}$. It suffices to prove part (i). Suppose that $x_n \xrightarrow{\tau^{\triangleright}} x$ as $n \to \infty$. Given any $\epsilon > 0$, there exits $n_\epsilon \in \mathbb{N}$ such that $x_n \in B^{\triangleright}(x;\epsilon)$ for all $n \geq n_\epsilon$, i.e., $d(x_n,x) < \epsilon$ for all $n \geq n_\epsilon$. This says that $d(x_n,x) \to 0$ as $n \to \infty$. Conversely, if $d(x_n,x) \to 0$ as $n \to \infty$, then, given any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $d(x_n,x) < \epsilon$ for all $n \geq n_\epsilon$, which says that $x_n \in B^{\triangleright}(x;\epsilon)$ for all $n \geq n_\epsilon$. This shows that $x_n \xrightarrow{\tau^{\triangleright}} x$ as $n \to \infty$, and the proof is complete. □

Let (X, d) be a pre-metric space. We consider the following sets

$$\bar{B}^{\triangleleft}(x;r) = \{y \in X : d(x,y) \leq r\} \text{ and } \bar{B}^{\triangleright}(x;r) = \{y \in X : d(y,x) \leq r\}.$$

If the symmetric condition is satisfied, then we simply write $\bar{B}(x;r)$. We are going to consider the closeness of $\bar{B}^{\triangleleft}(x;r)$ and $\bar{B}^{\triangleright}(x;r)$. Let us recall that, given a topological space (X, τ), we say that a subset F of X is τ-closed if and only if τ-$\mathrm{cl}(F) = F$, where τ-$\mathrm{cl}(F)$ denotes the τ-closure of F.

Proposition 8. *Let (X, d) be a pre-metric space. Suppose that d satisfies $d(x, x) = 0$ for all $x \in X$ and the $\bowtie$-triangle inequality. We have the following results.*

- *$\bar{B}^{\triangleleft}(x;r)$ is $\tau^{\triangleright}$-closed. In other words, we have $\tau^{\triangleright}$-$\mathrm{cl}(\bar{B}^{\triangleleft}(x;r)) = \bar{B}^{\triangleleft}(x;r)$.*
- *$\bar{B}^{\triangleright}(x;r)$ is $\tau^{\triangleleft}$-closed. In other words, we have $\tau^{\triangleleft}$-$\mathrm{cl}(\bar{B}^{\triangleright}(x;r)) = \bar{B}^{\triangleright}(x;r)$.*

Proof. Under the assumptions, Theorem 1 says that we can induce two topologies $\tau^{\triangleright}$ and $\tau^{\triangleleft}$ satisfying the first axiom of countability. To prove the first statement, for $y \in \tau^{\triangleright}$-$\mathrm{cl}(\bar{B}^{\triangleleft}(x;r))$, since $(X, \tau^{\triangleright})$ satisfies the first axiom of countability, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ in $\bar{B}^{\triangleleft}(x;r)$ such that $y_n \xrightarrow{\tau^{\triangleright}} y$ as $n \to \infty$. We also have $d(x, y_n) \leq r$ for all n. By part (i) of Proposition 7, we have $d(y_n, y) \to 0$ as $n \to \infty$ for all. The $\bowtie$-triangle inequality says that

$$d(x,y) \leq d(x,y_n) + d(y_n,y) \leq r + d(y_n,y) \to r \text{ as } n \to \infty,$$

which shows $y \in \bar{B}^{\triangleleft}(x;r)$. Therefore we obtain $\tau^{\triangleright}$-$\mathrm{cl}(\bar{B}^{\triangleleft}(x;r)) = \bar{B}^{\triangleleft}(x;r)$. The second statement can be similarly obtained. This completes the proof. □

Proposition 9. *Let (X, d) be a pre-metric space. Suppose that the following conditions are satisfied.*

- *d satisfies the $\triangleright$-triangle inequality and the $\triangleleft$-triangle inequality simultaneously.*
- *$d(x, x) = 0$ for all $x \in X$.*

Then d satisfies the symmetric condition; that is, (X, d) is a metric space.

Proof. Using part (i) of Proposition 2 and part (v) of Proposition 4, we see that $\tau^{\triangleleft}$ defined by (1) in Theorem 1 is a topology such that $\mathcal{B}^{\triangleleft}$ is a base for $\tau^{\triangleleft}$. Part (iii) of Proposition 2 says that there exists a countable local base at each $x \in X$ for $\tau^{\triangleleft}$, which also says that $\tau^{\triangleleft}$ satisfies the first axiom of countability. For $y \in \tau^{\triangleleft}$-$\mathrm{cl}(\bar{B}^{\triangleleft}(x;r))$, the first axiom of countability says that there exists a sequence $\{y_n\}_{n=1}^{\infty}$ in $\bar{B}^{\triangleleft}(x;r)$ such that $y_n \xrightarrow{\tau^{\triangleleft}} y$ as $n \to \infty$. We also have $d(x, y_n) \leq r$ for all n. By part (ii) of Proposition 7, we have $d(y, y_n) \to 0$ as $n \to \infty$. The $\triangleright$-triangle inequality says that

$$d(x,y) \leq d(x,y_n) + d(y,y_n) \leq r + d(y,y_n) \to r \text{ as } n \to \infty,$$

which shows $y \in \bar{B}^{\triangleleft}(x;r)$, i.e., $\tau^{\triangleleft}$-$\mathrm{cl}(\bar{B}^{\triangleleft}(x;r)) = \bar{B}^{\triangleleft}(x;r)$. On the other hand, we also have

$$d(y,x) \leq d(y,y_n) + d(x,y_n) \leq d(y,y_n) + r \to r \text{ as } n \to \infty,$$

which shows $y \in \bar{B}^{\triangleright}(x;r)$. Therefore we obtain the inclusion $\tau^{\triangleleft}$-$\mathrm{cl}(\bar{B}^{\triangleleft}(x;r)) \subseteq \bar{B}^{\triangleright}(x;r)$, which also says that $\bar{B}^{\triangleleft}(x;r) \subseteq \bar{B}^{\triangleright}(x;r)$. Now, given any $x, y \in X$ with $y \neq x$, we have $d(x, y) > 0$. Let $r = d(x, y)$. Then $y \in \bar{B}^{\triangleleft}(x;r)$. This also says that $y \in \bar{B}^{\triangleright}(x;r)$, i.e.,

$$d(y,x) \leq r = d(x,y).$$

By interchanging the roles of x and y, we can similarly obtain $d(x, y) \leq d(y, x)$. This completes the proof. □

Remark 3. *Let (X, d) be a pre-metric space. Suppose that the conditions presented in Proposition 9 are satisfied. Then (X, d) turns into a metric space. It is well-known that the metric space (X, d) can induce a Hausdorff topological space. More precisely, using the notations in this paper, we see that $\tau^{\triangleleft} = \tau^{\triangleright}$ that is simply written as τ. In other words, (X, τ) is a Hausdorff space such that $\mathcal{B}$ is a base for the topology τ, where $\mathcal{B} = \mathcal{B}^{\triangleleft} = \mathcal{B}^{\triangleright}$. The Hausdorff space (X, τ) also satisfies the first axiom of countability. Moreover, $B(x; r)$ is a τ-open set and $\bar{B}(x; r)$ is a τ-closed set, where $B(x; r) = B^{\triangleleft}(x; r) = B^{\triangleright}(x; r)$ and $\bar{B}(x; r) = \bar{B}^{\triangleleft}(x; r) = \bar{B}^{\triangleright}(x; r)$.*

In a future study, we shall avoid to consider the conditions presented in Proposition 9. Otherwise, the study will become trivial, based on the results of conventional metric space.

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