

Article

# Separability of Topological Groups: A Survey with Open Problems

Arkady G. Leiderman <sup>1,\*</sup> and Sidney A. Morris <sup>2,3,†</sup>

<sup>1</sup> Department of Mathematics, Ben-Gurion University of the Negev, P.O. Box 653, Beer Sheva 84105, Israel

<sup>2</sup> Centre for Informatics and Applied Optimization (CIAO), Federation University Australia, P.O. Box 663, Ballarat 3353, Australia; morris.sidney@gmail.com

<sup>3</sup> Department of Mathematics and Statistics, La Trobe University, Melbourne 3086, Australia

\* Correspondence: arkady@math.bgu.ac.il; Tel.: +972-8-647-7811

† These authors contributed equally to this work.

Received: 25 Nvember 2018; Accepted: 25 December 2018; Published: 29 December 2018



**Abstract:** Separability is one of the basic topological properties. Most classical topological groups and Banach spaces are separable; as examples we mention compact metric groups, matrix groups, connected (finite-dimensional) Lie groups; and the Banach spaces  $C(K)$  for metrizable compact spaces  $K$ ; and  $\ell_p$ , for  $p \geq 1$ . This survey focuses on the wealth of results that have appeared in recent years about separable topological groups. In this paper, the property of separability of topological groups is examined in the context of taking subgroups, finite or infinite products, and quotient homomorphisms. The open problem of Banach and Mazur, known as the Separable Quotient Problem for Banach spaces, asks whether every Banach space has a quotient space which is a separable Banach space. This paper records substantial results on the analogous problem for topological groups. Twenty open problems are included in the survey.

**Keywords:** separable topological group; subgroup; product; isomorphic embedding; quotient group; free topological group

## 1. Introduction

All topological spaces and topological groups are assumed to be Hausdorff and all topological spaces are assumed to be infinite unless explicitly stated otherwise.

The fundamental topological operations which produce new topological groups from given ones are:

- (1) taking subgroups;
- (2) taking finite or infinite products;
- (3) open continuous homomorphic images = quotient images;
- (4) (topological group) isomorphic embeddings.

A topological space which has a dense countable subspace is called *separable*.

The main aim of this survey paper is to present systematically the results concerning the behavior of separability of topological groups with respect to the topological operations listed above and make clear which problems are open. Much of the material is from the recent publications [1–8].

Informally speaking, this survey contributes to the manifestation of the phenomenon that the structure of topological groups is much more sensitive to the presence of countable topological properties than is the structure of general topological spaces.

Section 2 sketches the relevant background results about separability of general topological spaces. Section 3 is devoted to the closed subgroups of separable topological groups and isomorphic

embeddings into separable topological groups. Section 4 is devoted to the products of separable topological groups and the products of separable topological vector spaces. Section 5 is devoted to the separable quotient problem for general topological groups. Section 6 is devoted to the metrizable and separable quotients of free topological groups. Section 7 deals with the question of when an abstract group can be equipped with a separable topological group topology.

At the end of most sections we pose open problems, 20 problems in all. Throughout the paper anything labeled as a Problem is an open problem. We also have many Questions, for which an answer is provided.

The reader is advised to consult the monographs of Engelking [9] and Arhangel'skii and Tkachenko [10] for any notions which are not explicitly defined in our paper.

## 2. Separability of Topological Spaces

The *weight*  $w(X)$  of a topological space  $X$  is defined as the smallest cardinal number  $|\mathcal{B}|$ , where  $\mathcal{B}$  is a base of the topology on  $X$ . The *density character*  $d(X)$  of a topological space  $X$  is  $\min\{|A| : A \text{ is dense in } X\}$ . Recall that if  $d(X) \leq \aleph_0$ , then we say that the space  $X$  is *separable*. We denote by  $\mathfrak{c}$  the cardinality of continuum.

A topological space  $X$  is said to be *hereditarily separable* if  $X$  and every subspace of  $X$  is separable. A topological space is said to be *second countable* if its topology has a countable base. A topological space  $X$  is said to have a *countable network* if there exists a countable family  $\mathcal{B}$  of (not necessarily open) subsets such that each open set of  $X$  is a union of members of  $\mathcal{B}$ .

- Any space with a countable network is hereditarily separable;
- a metrizable space is separable if and only if it is second countable;
- any continuous image of a separable space is separable;
- countable networks are preserved by continuous images.

### 2.1. Weight of Separable Topological Spaces

**Theorem 1.** (De Groot, ([11], Theorem 3.3)) *If  $X$  is a separable regular space, then  $w(X) \leq \mathfrak{c}$ . More generally, every regular space  $X$  satisfies  $w(X) \leq 2^{d(X)}$ , and then  $|X| \leq 2^{w(X)} \leq 2^{2^{d(X)}}$ .*

Compact dyadic spaces are defined to be continuous images of generalized Cantor cubes  $\{0, 1\}^\kappa$ , where  $\kappa$  is an arbitrary cardinal number. It is well-known ([12], Theorem 10.40) that every compact group is dyadic.

**Proposition 1.** (Engelking, ([13], Theorem 10)) *Let  $\kappa$  be an infinite cardinal. A compact dyadic space  $K$  with  $w(K) \leq 2^\kappa$  satisfies  $d(K) \leq \kappa$ . In particular, if  $w(K) \leq \mathfrak{c}$ , then  $K$  is separable.*

### 2.2. Products of (Hereditarily) Separable Topological Spaces

**Theorem 2.** (Hewitt–Marczewski–Pondiczery, [9]) *Let  $\{X_i : i \in I\}$  be a family of topological spaces and  $X = \prod_{i \in I} X_i$ , where  $|I| \leq 2^\kappa$  for some cardinal number  $\kappa \geq \omega$ . If  $d(X_i) \leq \kappa$  for each  $i \in I$ , then  $d(X) \leq \kappa$ . In particular, the product of no more than  $\mathfrak{c}$  separable spaces is separable.*

**Remark 1.** The Sorgenfrey line  $\mathbb{S}$  is a hereditarily separable space whose square  $\mathbb{S} \times \mathbb{S}$  has the uncountable discrete subspace  $\{(x, -x) : x \in \mathbb{S}\}$  as a subspace and so  $\mathbb{S} \times \mathbb{S}$  is not hereditarily separable.

**Proposition 2** ([1]). *Let  $X$  be a hereditarily separable space and  $Y$  a space with a countable network. Then the product  $X \times Y$  is also hereditarily separable.*

### 2.3. Closed Embeddings into Separable Topological Spaces

Any compact space of weight not greater than  $\mathfrak{c}$  homeomorphically embeds into the separable compact cube  $[0, 1]^{\mathfrak{c}}$ .

A Tychonoff space  $X$  is called *pseudocompact* if every continuous real-valued function defined on  $X$  is bounded. Similarly to Theorem 11 one can prove:

**Proposition 3.** *Every Tychonoff space of weight not greater than  $\mathfrak{c}$  is homeomorphic to the closed subspace of a separable pseudocompact space.*

### 2.4. Separable Quotient Spaces of Topological Spaces

Assume  $\varphi: X \rightarrow Y$  is a mapping such that (1)  $\varphi$  is surjective, (2)  $\varphi$  is continuous, and (3) for  $U \subseteq Y$ ,  $\varphi^{-1}(U)$  is open in  $X$  implies that  $U$  is open in  $Y$ . In this case, the mapping  $\varphi$  is called a *quotient mapping*.

Every closed mapping and every open mapping is a quotient mapping.

The majority of topological properties are not preserved by quotient mappings. For instance, a quotient space of a metric space need not be a Hausdorff space, and a quotient space of a separable metric space need not have a countable base.

A surjective continuous mapping  $\varphi: X \rightarrow Y$  is said to be *R-quotient* [14] if for every real-valued function  $f$  on  $Y$ , the composition  $f \circ \varphi$  is continuous if and only if  $f$  is continuous. Clearly, every quotient mapping is *R-quotient*, but the converse is false.

Let  $\varphi: X \rightarrow Y$  be a surjective continuous mapping, where the space  $Y$  is Tychonoff. Then  $Y$  admits the finest topology, say,  $\sigma$  such that the mapping  $\varphi: X \rightarrow (Y, \sigma)$  is *R-quotient*. The topology  $\sigma$  of  $Y$  is initial with respect to the family of real-valued functions  $f$  on  $Y$  such that the composition  $f \circ \varphi$  is continuous. It is easy to see that the space  $(Y, \sigma)$  is also Tychonoff and that  $\sigma$  is finer than the original topology of  $Y$ . We say that  $\sigma$  is the *R-quotient topology* on  $Y$  (with respect to  $\varphi$ ). Notice that the mapping  $\varphi: X \rightarrow (Y, \sigma)$  remains continuous.

**Proposition 4** ([2]). *Every continuous mapping of a pseudocompact space onto a first countable Tychonoff space is R-quotient. Therefore for every pseudocompact space there exists an R-quotient mapping onto infinite subset of the closed unit interval  $[0, 1]$ . Every locally compact space also admits an R-quotient mapping onto an infinite subset of the closed unit interval  $[0, 1]$ .*

Recall that the class of Lindelöf  $\Sigma$ -spaces is the smallest class of topological spaces which contains all compact and all separable metrizable topological spaces, and is closed with respect to countable products, closed subspaces, and continuous images (see [10], Section 5.3).

**Proposition 5** ([2]). *For every Lindelöf  $\Sigma$ -space (in particular,  $\sigma$ -compact space)  $X$  there exists an R-quotient mapping onto an infinite space with a countable network.*

### 2.5. Open Problems

**Problem 1** ([2]). *Does there exist an R-quotient mapping from each Tychonoff space onto an infinite subspace of  $[0, 1]$ ?*

Furthermore, a more particular question below is open:

**Problem 2** ([2]). *Does there exist an R-quotient mapping from each Lindelöf space onto an infinite separable Tychonoff space?*

### 3. Subgroups of Separable Topological Groups

#### 3.1. Topological Groups with a Dense Compactly Generated Subgroup

A topological group  $G$  is said to be *compactly generated* if it has a compact subspace  $K$  such that the smallest subgroup of  $G$  which contains  $K$  is  $G$  itself. The topological group  $G$  is said to be *finitely generated modulo open sets* if for every open set  $U \subseteq G$ , there exists a finite set  $F \subset G$  such that the smallest subgroup of  $G$  which contains  $F \cup U$  is  $G$  itself.

Since every metrizable compact space is separable, it is easy to see that every metrizable topological group which has a dense compactly generated subgroup is separable. The next theorem says under what conditions the converse is also true.

**Theorem 3** ([15]). *A metrizable topological group  $G$  has a dense compactly generated subgroup if and only if it is separable and finitely generated modulo open sets.*

**Corollary 1.** *Let  $G$  be a metrizable connected topological group. Then  $G$  is separable if and only if it has a dense compactly generated subgroup.*

**Corollary 2.** *Let  $G$  be an additive topological group of a metrizable topological vector space. Then  $G$  is separable if and only if it has a dense compactly generated subgroup.*

#### 3.2. Characterization of Subgroups of Separable Topological Groups

A topological group is said to be  $\omega$ -*narrow* ([10], Section 3.4) if it can be covered by countably many translations of every neighborhood of the identity element. It is known that every separable topological group is  $\omega$ -narrow (see [10], Corollary 3.4.8). The class of  $\omega$ -narrow groups is productive and hereditary with respect to taking arbitrary subgroups ([10], Section 3.4), so  $\omega$ -narrow groups need not be separable. In fact, an  $\omega$ -narrow group  $G$  can have uncountable cellularity, i.e., there is an uncountable family of disjoint non-empty open subsets in  $G$  (see [10], Example 5.4.13).

The following theorem characterizes the class of  $\omega$ -narrow topological groups.

**Theorem 4.** (Guran, [16]) *A topological group  $G$  is  $\omega$ -narrow if and only if it is topologically isomorphic to a subgroup of a product of second countable topological groups.*

A topological group which has a local base at the identity element consisting of open subgroups is called *protodiscrete*. A complete protodiscrete group is said to be *prodiscrete*. Protodiscrete topological groups are exactly the totally disconnected pro-Lie groups ([17], Proposition 3.30).

**Theorem 5** ([4]). *A (protodiscrete abelian) topological group  $H$  is topologically isomorphic to a subgroup of a separable (prodiscrete abelian) topological group if and only if  $H$  is  $\omega$ -narrow and satisfies  $w(H) \leq \mathfrak{c}$ .*

It is natural to compare the restrictions on a given topological group  $G$  imposed by the existence of either a topological embedding of  $G$  into a separable regular space or a topological isomorphism of  $G$  onto a subgroup of a separable topological group.

Let us note that the first of the two classes of topological groups is strictly wider than the second one. In order to show this, consider an arbitrary discrete group  $G$  satisfying  $\omega < |G| \leq \mathfrak{c}$ . Then  $G$  embeds as a *closed subspace* into the separable space  $\mathbb{N}^{\mathfrak{c}}$  [9], where  $\mathbb{N}$  is the set of positive integers endowed with the discrete topology. However,  $G$  does not admit a topological isomorphism onto a subgroup of a separable topological group. Indeed, every subgroup of a separable topological group is  $\omega$ -narrow by Theorem 5. Since the discrete group  $G$  is uncountable, it fails to be  $\omega$ -narrow.

The above observation makes it natural to restrict our attention to  $\omega$ -narrow topological groups when considering embeddings into separable topological groups. It turns out that in the class of

$\omega$ -narrow topological groups, the difference between the two types of embeddings disappears, even if we require an embedding to be closed.

In the next result, which complements Theorem 5, we identify a large class of topological groups with the class of closed subgroups of separable path-connected, locally path-connected topological groups.

**Theorem 6** ([4]). *The following are equivalent for an arbitrary  $\omega$ -narrow topological group  $G$ :*

- (a)  $G$  is homeomorphic to a subspace of a separable regular space;
- (b)  $G$  is topologically isomorphic to a subgroup of a separable topological group;
- (c)  $G$  is topologically isomorphic to a closed subgroup of a separable path-connected, locally path-connected topological group.

Next, we consider the following question: Let  $G$  be a separable topological group. Under what conditions is every closed subgroup of  $G$  separable?

Historically, the first non-trivial result is due to Itzkowitz [18].

**Theorem 7.** *Let  $G$  be a separable compact topological group. Then every closed subgroup of  $G$  is separable.*

Please note that  $\sigma$ -compactness of  $G$  is not a sufficient condition.

**Example 1.** Let  $X$  be any separable compact space which contains a closed non-separable subspace  $Y$ . The free abelian topological group  $A(Y)$  naturally embeds into  $A(X)$  as a closed subgroup. Then  $A(X)$  is a separable  $\sigma$ -compact group, while  $A(Y)$  is not separable—otherwise  $Y$  would be separable (see [19], Lemma 3.1).

Let us recall that a topological group  $G$  is called *feathered* if it contains a non-empty compact subset with a countable neighborhood base in  $G$ . Equivalently,  $G$  is feathered if it contains a compact subgroup  $K$  such that the quotient space  $G/K$  is metrizable (see [10], Section 4.3). All metrizable groups and all locally compact groups are feathered. Notice also that the class of feathered groups is closed under taking countable products (see [10], Proposition 4.3.13).

**Theorem 8** ([4]). *Let a feathered topological group  $G$  be isomorphic to a subgroup of a separable topological group. Then  $G$  is separable.*

Since the class of feathered topological groups includes both locally compact and metrizable groups, Theorem 8 provides a generalization of the results of Comfort and Itzkowitz [19] for locally compact groups and also well-known results for metrizable groups [20,21].

**Corollary 3.** *If a locally compact topological group  $G$  is isomorphic to a subgroup of a separable topological group, then  $G$  is separable.*

**Corollary 4.** *If a metrizable group  $G$  is isomorphic to a subgroup of a separable topological group, then  $G$  is separable.*

Recall that a non-empty class  $\Omega$  of topological groups is said to be a *variety* [22–27] if it is closed under the operations of taking subgroups, quotient groups, (arbitrary) cartesian products and isomorphic images. Let  $\mathcal{C}$  be a class of topological groups and let  $\mathfrak{V}(\mathcal{C})$  be the intersection of all varieties containing  $\mathcal{C}$ . Then  $\mathfrak{V}(\mathcal{C})$  is said to be the *variety generated by  $\mathcal{C}$* . With the help of results from [26] Corollary 4 can be extended as follows.

**Corollary 5.** *If  $\mathcal{C}$  is any class of separable abelian topological groups, then every metrizable group in  $\mathfrak{W}(\mathcal{C})$  is separable.*

It is clear that Corollary 5 would be false if the metrizability condition is deleted or replaced by feathered or even compact.

**Problem 3.** *Does Corollary 5 remain valid if we drop the assumption that all groups in the class  $\mathcal{C}$  are abelian?*

**Remark 2** ([4]). (1) A discrete (hence locally compact and metrizable) topological group  $G$  homeomorphic to a closed subspace of a separable Tychonoff space is not necessarily separable. Indeed, it suffices to consider the Niemytzki plane which contains a discrete copy of the real numbers, the  $X$ -axis. Therefore Theorem 8 and Corollaries 3 and 4 would not be valid if the group  $G$  were assumed to be a subspace of a separable Hausdorff (or even Tychonoff) space rather than a subgroup of a separable topological group.

(2) The separable connected pro-Lie group  $G = \mathbb{R}^{\mathfrak{c}}$  contains a closed non-separable subgroup. To see this, we consider the closed subgroup  $\mathbb{Z}^{\mathfrak{c}}$  of  $G$ . By a theorem of Uspenskij [28], the group  $\mathbb{Z}^{\mathfrak{c}}$  contains a subgroup  $H$  of uncountable cellularity. The closure of  $H$  in  $G$ , say,  $K$  is a closed non-separable subgroup of  $G$ . Note that the group  $K$  cannot be almost connected. (See the next subsection for discussion of almost connected groups.)

(3) A natural question is whether a connected metrizable group must be separable if it is a subspace of a separable Hausdorff (or regular) space. Again the answer is 'No'. Indeed, consider an arbitrary connected metrizable group  $G$  of weight  $\mathfrak{c}$ . For example, one can take  $G = C(X)$ , the Banach space of continuous real-valued functions on a compact space  $X$  satisfying  $w(X) = \mathfrak{c}$ , endowed with the sup-norm topology. Since  $w(G) = \mathfrak{c}$ , the space  $G$  is homeomorphic to a subspace of the Tychonoff cube  $I^{\mathfrak{c}}$ , where  $I = [0, 1]$  is the closed unit interval. Thus  $G$  embeds as a subspace in a separable regular space, but both the density and weight of  $G$  are equal to  $\mathfrak{c}$ .

### 3.3. Separability of Pro-Lie Groups

Early this century Hofmann and Morris, [17,29], introduced the class of *pro-Lie groups*, which consists of projective limits of finite-dimensional Lie groups and proved that it contains all compact groups, all locally compact abelian groups, and all connected locally compact groups and is closed under the formation of products and closed subgroups. They defined a topological group  $G$  to be *almost connected* if the quotient group of  $G$  by the connected component of its identity is compact [17]. Of course all compact groups, all connected topological groups and all finite or infinite products of a set of topological groups, each factor of which is either a connected topological group or a compact group, are almost connected.

Below we consider topological groups which are *homeomorphic* to a subspace of a separable Hausdorff space.

**Theorem 9** ([4]). *Let  $G$  be an almost connected pro-Lie group. If  $G$  is homeomorphic to a subspace of a separable Hausdorff space, then  $G$  is separable.*

This result can be strengthened as follows.

**Theorem 10** ([4]). *Let  $G$  be an  $\omega$ -narrow topological group which contains a closed subgroup  $N$  such that  $N$  is an almost connected pro-Lie group and the quotient space  $G/N$  is locally compact. If  $G$  is homeomorphic to a subspace of a separable Hausdorff space, then  $G$  is separable.*

Unlike the case of almost connected pro-Lie groups, closed subgroups of separable prodiscrete abelian groups can fail to be separable.

**Proposition 6** ([4]). *Closed subgroups of separable prodiscrete abelian groups need not be separable.*

### 3.4. Closed Topologically Isomorphic Embeddings into Separable Topological Groups

Theorems 11 and 12 given below show that there is a wealth of separable pseudocompact topological (abelian) groups with closed non-separable subgroups.

**Theorem 11** ([4]). *Every precompact topological group of weight  $\leq \mathfrak{c}$  is topologically isomorphic to a closed subgroup of a separable, connected, pseudocompact group  $H$  of weight  $\leq \mathfrak{c}$ .*

Here is the abelian version.

**Theorem 12** ([4]). *Every precompact abelian group of weight  $\leq \mathfrak{c}$  is topologically isomorphic to a closed subgroup of a separable, connected, pseudocompact abelian group  $H$  of weight  $\leq \mathfrak{c}$ .*

**Theorem 13** ([4]). *Under the Continuum Hypothesis CH, there exists a separable countably compact abelian topological group  $G$  which contains a closed non-separable subgroup.*

**Problem 4.** *Does there exist in ZFC a separable countably compact topological group which contains a non-separable closed subgroup?*

## 4. Products of Separable Topological Groups/Locally Convex Spaces

### 4.1. Strongly Separable Topological Groups

Let us say that a topological group  $G$  is *strongly separable* (briefly, *S-separable*) if for any topological group  $H$  such that every closed subgroup of  $H$  is separable, the product  $G \times H$  has each of its closed subgroups separable. What are the *S-separable* groups?

One of the main statements is the following result which can be reformulated by saying that every separable compact group is *S-separable*.

**Theorem 14** ([1]). *Let  $G$  be a separable compact group and  $H$  be a topological group in which all closed subgroups are separable. Then all closed subgroups of the product  $G \times H$  are separable.*

It is not clear to what extent one can generalize Theorem 14 by weakening the compactness assumption on  $G$ . However by Theorem 17 some additional conditions on the groups  $G$  and/or  $H$  have to be imposed.

The proof of Theorem 14 relies on the fact that for a compact factor  $G$  the projection  $G \times H$  onto  $H$  is a closed mapping. In the next proposition we present another situation when the projection  $G \times H \rightarrow H$  turns out to be a closed mapping.

**Proposition 7** ([1]). *Let  $G$  be a countably compact topological group and  $H$  a separable metrizable topological group. If all closed subgroups of  $G$  are separable, then all closed subgroups of the product  $G \times H$  are separable.*

Every countable group is *S-separable*. The theorem below unites both the compact and countable classes of groups.

**Theorem 15** ([1]). *A topological group  $G$  is *S-separable* provided it contains a separable compact subgroup  $K$  such that the quotient space  $G/K$  is countable.*

**Proposition 8** ([1]). *The class of *S-separable* groups is closed under the operations:*

- (1) *finite products;*
- (2) *taking closed subgroups;*

(3) taking continuous homomorphic images.

#### 4.2. Product of Two Separable Precompact/Pseudocompact Groups

**Theorem 16** ([1]). Assume that  $2^{\omega_1} = \mathfrak{c}$ . Then there exist pseudocompact abelian topological groups  $G$  and  $H$  such that all closed subgroups of  $G$  and  $H$  are separable, but the product  $G \times H$  contains a closed non-separable  $\sigma$ -compact subgroup.

Recently Zhiqiang Xiao, Sánchez and Tkachenko [6] presented the first example in ZFC of precompact (but not necessarily pseudocompact) abelian topological groups  $G$  and  $H$  with the similar properties.

**Theorem 17** ([6]). There exist precompact abelian topological groups  $G$  and  $H$  such that all closed subgroups of  $G$  and  $H$  are separable, but the product  $G \times H$  contains a closed non-separable subgroup.

Also the authors of [6] improved upon Theorem 16 by constructing, under the assumption of  $2^{\omega_1} = \mathfrak{c}$ , a pseudocompact abelian topological group  $G$  such that every closed subgroup of  $G$  is separable, but the square  $G \times G$  contains a closed non-separable  $\sigma$ -compact subgroup.

Finally, the following result is obtained under the assumption of Martin's Axiom and negation of the Continuum Hypothesis  $MA \& \neg CH$ .

**Theorem 18** ([6]). Assume  $MA \& \neg CH$ . Then there exist countably compact Boolean topological groups  $G$  and  $H$  such that all closed subgroups of  $G$  and  $H$  are separable, but the product  $G \times H$  contains a closed non-separable subgroup.

#### 4.3. Product of Two Separable Pseudocomplete Locally Convex Spaces

**Proposition 9** ([1]). Let  $K$  be a finite-dimensional Banach space and let  $L$  be a topological vector space in which all closed vector subspaces are separable. Then all closed vector subspaces of the product  $K \times L$  are separable.

**Remark 3.** It is not known whether Proposition 9 remains valid for an arbitrary separable Banach space  $K$ .

**Theorem 19** ([1]). Assume that  $2^{\omega_1} = \mathfrak{c}$ . Then there exist pseudocomplete locally convex spaces  $K$  and  $L$  such that all closed vector subspaces of  $K$  and  $L$  are separable, but the product  $K \times L$  contains a closed non-separable  $\sigma$ -compact vector subspace.

#### 4.4. Product of Continuum Many Separable Locally Convex Spaces

The classical Hewitt–Marczewski–Pondiczery Theorem 2 implies that the product of no more than  $\mathfrak{c}$  separable topological spaces is separable. Domański [30] gave an example of a non-separable complete locally convex space which can be embedded as a closed vector subspace of a product of  $\mathfrak{c}$  copies of the Banach space  $c_0$ . Later he extended this result in show that every product of  $\mathfrak{c}$  copies of any infinite-dimensional Banach space has non-separable closed vector subspaces [31]. In fact, Domański proved in [31] that if  $E_i$ ,  $i \in I$ , with  $\text{card}(I) = \mathfrak{c}$  are separable topological vector spaces whose completions are not  $q$ -minimal, then the product  $\prod_{i \in I} E_i$  has a non-separable closed vector subspace. (A topological vector space  $E$  is called  $q$ -minimal if it and all its quotient spaces are minimal, while  $E$  is called *minimal* if it does not admit a strictly weaker Hausdorff topological vector space topology).

Similarly to the variety of topological groups defined earlier, a non-empty class  $\Omega$  of locally convex spaces is said to be a *variety* [27,32–34] if it is closed under the operations of taking subspaces, quotient spaces, (arbitrary) cartesian products and isomorphic images. Let  $\mathcal{C}$  be a class of locally convex spaces, denote by  $\mathfrak{V}(\mathcal{C})$  the intersection of all varieties containing  $\mathcal{C}$ . Then  $\mathfrak{V}(\mathcal{C})$  is said to be the *variety generated by  $\mathcal{C}$* . We repeat that if  $\mathcal{C}$  consists of a single object  $E$ , then  $\mathfrak{V}(\mathcal{C})$  is written as  $\mathfrak{V}(E)$ .

**Theorem 20** ([3]). *Let  $I$  be an index set and  $E_i$  a locally convex space for each  $i \in I$ . If at least  $\mathfrak{c}$  of the  $E_i$  are not in  $\mathfrak{B}(\mathbb{R})$ , or equivalently do not have the weak topology, then the product  $\prod_{i \in I} E_i$  has a non-separable closed vector subspace.*

**Theorem 21** ([3]). *Let  $I$  be any index set and each  $E_i, i \in I$ , a separable locally convex space. If  $X$  is a vector subspace of  $\prod_{i \in I} E_i$  with  $Y$  a closed vector subspace of  $X$  such that either*

- (a)  *$Y$  is metrizable and  $X/Y$  is separable, or*
- (b)  *$Y$  is separable and  $X/Y$  is metrizable,*

*then  $X$  is separable.*

Let  $C_p(X, E)$  denote the space of all continuous  $E$ -valued functions on  $X$  endowed with the pointwise convergence topology, where  $E$  is a locally convex space. The space  $C_p(X, E)$  is a vector subspace of  $E^X$  endowed with the product topology.

**Corollary 6** ([3]). *Let  $X$  be a Tychonoff space. If  $E$  is a separable locally convex space, then every metrizable vector subspace of  $C_p(X, E)$  is separable.*

**Proposition 10** ([3]). *Let  $X$  be a Tychonoff space such that every closed vector subspace of  $C_p(X)$  is separable. Then every closed subset  $F$  of  $X$  is a  $G_\delta$ -set, that is  $F = \bigcap_{i=1}^{\infty} U_i$ , where each  $U_i$  is open in  $X$ .*

**Example 2** ([3]). Let  $\mathbb{M}$  denote the Michael line. Then  $C_p(\mathbb{M})$  is a separable locally convex space containing a non-separable closed vector subspace.

#### 4.5. Open Problems

**Problem 5** ([1]). *Find the frontiers of the class of  $S$ -separable topological groups:*

- (a) *Is every separable locally compact group  $S$ -separable?*
- (b) *Is the abelian topological group  $\mathbb{R}$  of all real numbers with the Euclidean topology  $S$ -separable? Does there exist a separable metrizable group which is not  $S$ -separable?*
- (c) *Is the free topological group on the closed unit interval  $[0, 1]$   $S$ -separable?*

The following problem arises in an attempt to generalize Proposition 7:

**Problem 6** ([1]). *Let  $G$  be a countably compact topological group such that all closed subgroups of  $G$  are separable, and  $H$  a topological group with a countable network. Are the closed subgroups of  $G \times H$  separable?*

Denote by  $\mathfrak{S}$  the smallest class of topological groups which is generated by all compact separable groups and all countable groups and is closed under the operations listed in (1)–(3) of Proposition 8. It is not difficult to verify that if  $G \in \mathfrak{S}$ , then  $G$  contains a compact separable subgroup  $K$  such that the quotient space  $G/K$  is countable. In the next problem we ask if this property characterizes the groups from  $\mathfrak{S}$ :

**Problem 7** ([1]). *Does a topological group  $G$  belong to the class  $\mathfrak{S}$  if and only if  $G$  contains a compact separable subgroup  $K$  such that the quotient space  $G/K$  is countable?*

**Problem 8** ([1]). *Are Theorems 16 and 19 valid in ZFC alone?*

**Problem 9** ([3]). *Characterize those Tychonoff spaces  $X$  such that all closed vector subspaces of  $C_p(X)$  are separable.*

## 5. The Separable Quotient Problem for General Topological Groups

Let us begin this section with a famous unsolved problem in Banach space theory. The Separable Quotient Problem for Banach Spaces has its roots in the 1930s and is due to Stefan Banach and Stanisław Mazur.

**Problem 10.** (*Separable Quotient Problem for Banach Spaces*) Does every infinite-dimensional Banach space have a quotient Banach space which is separable and infinite-dimensional?

In the literature many special cases of the Separable Quotient Problem for Banach Spaces have been proved, for instance:

- Every infinite-dimensional reflexive Banach space has a separable infinite-dimensional quotient Banach space (Pełczyński, 1964).
- Every Banach space  $C(K)$ , where  $K$  is a compact space, has a separable infinite-dimensional quotient Banach space (Rosenthal, 1969; Lacey, 1972).
- Every Banach dual of any infinite-dimensional Banach space,  $E^*$ , has a separable infinite-dimensional quotient Banach space (Argyros, Dodos, Kanellopoulos, 2008).

However, the general Problem 10 remains unsolved.

Turning to locally convex spaces one can state the analogous problem.

**Question 1.** (*Separable Quotient Problem for Locally Convex Spaces*) Does every infinite-dimensional locally convex space have a quotient locally convex space which is separable and infinite-dimensional?

- Every infinite-dimensional Fréchet space which is non-normable has the separable metrizable topological vector space  $\mathbb{R}^\omega$  as a quotient space (Eidelheit, 1936).

Please note that there are many other partial positive solutions in the literature to Problem 1 (see [35]). However, Kąkol, Saxon and Todd [36] answered Question 1 in the negative. Recall that a barrel in a topological vector space is a convex, balanced, absorbing and closed set. A Hausdorff topological vector space  $E$  is called barreled if every barrel in  $E$  is a neighborhood of the zero element.

**Theorem 22** ([36]). *There exists an infinite-dimensional barreled locally convex space without any quotient space which is an infinite-dimensional separable locally convex space.*

Now we formulate various natural versions of the Separable Quotient Problem(s) for Topological Groups. Unless explicitly stated otherwise the results presented in this section are from the paper [5].

**Problem 11.** (*Separable Quotient Problem for Topological Groups*) Does every non-totally disconnected topological group have a quotient group which is a non-trivial separable topological group?

**Problem 12.** (*Separable Infinite Quotient Problem for Topological Groups*) Does every non-totally disconnected topological group have a quotient group which is an infinite separable topological group?

**Problem 13.** (*Separable Metrizable Quotient Problem for Topological Groups*) Does every non-totally disconnected topological group have a quotient group which is a non-trivial separable metrizable topological group?

**Problem 14.** (*Separable Infinite Metrizable Quotient Problem for Topological Groups*) Does every non-totally disconnected topological group have a quotient group which is an infinite separable metrizable topological group?

It is natural to consider these questions for various prominent classes of topological groups such as Banach spaces, locally convex spaces, compact groups, locally compact groups, pro-Lie groups, pseudocompact groups, and precompact groups. The paper [7] provides an interesting solution for Banach spaces.

**Theorem 23** ([7]). *Let  $E$  be a locally convex space (over  $\mathbb{R}$  or  $\mathbb{C}$ ). If  $E$  has a subspace which is an infinite-dimensional Fréchet space, then  $E$  has the (infinite separable metrizable) tubby torus group  $\mathbb{T}^\omega$  as a quotient group.*

**Corollary 7** ([7]). *Every infinite-dimensional Fréchet space, and in particular every infinite-dimensional Banach space, has the (infinite separable metrizable) tubby torus group  $\mathbb{T}^\omega$  as a quotient group.*

We denote by  $\varphi$  the complete countable infinite-dimensional locally convex space which is the strong dual of the locally convex space  $\mathbb{R}^\omega$ .

**Remark 4** ([7]). There is no continuous surjective homomorphism of the separable locally convex space  $\varphi$  onto the tubby torus  $\mathbb{T}^\omega$ .

Corollary 7 suggests the following unsolved problem, a negative answer for which would immediately yield a negative answer to the Banach-Mazur Separable Quotient Problem for Banach Spaces, Problem 10.

**Problem 15.** *Does every infinite-dimensional Banach space have a quotient group which is homeomorphic to  $\mathbb{R}^\omega$ ?*

A topological group  $G$  is said to be a SIN-group if every neighborhood of the identity of  $G$  contains a neighborhood of the identity which is invariant under all inner automorphisms. Of course every abelian topological group and every compact group is a SIN-group.

As we noted in Section 2 the cardinality of any regular separable topological space is not greater than  $2^c$ . This makes the following statement interesting.

**Theorem 24** ([37]). *Let  $G$  be an abelian topological group or more generally a SIN-group. If  $G$  has a quotient group which is separable, then it also has a quotient group of cardinality not greater than  $c$ .*

We now consider several natural questions which are special cases of Problems 11, 12, 13, and 14.

**Question 2. (Separable Quotient Problem for Locally Compact Abelian Groups)** *Does every infinite locally compact abelian group have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable?*

The non-abelian version of Question 2 is:

**Question 3. (Separable Quotient Problem for Locally Compact Groups)** *Does every non-totally disconnected locally compact group have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable?*

As a special case of Question 3 we have:

**Question 4. (Separable Quotient Problem for Compact Groups)** *Does every infinite compact group have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable?*

### 5.1. Locally Compact Groups and Pro-Lie Groups

In this subsection we present a positive answer to each of Question 2 (i), (ii), (iii), and (iv) and Question 4 (i), (ii), (iii), and (iv), and a partial answer to Question 3. Satisfying results have been proved for pro-Lie groups. Stronger structural results have been obtained for compact abelian groups, connected compact groups, and totally disconnected compact groups.

**Theorem 25.** *Every non-separable compact abelian group  $G$  has a quotient group  $Q$  which is a countably infinite product of non-trivial compact finite-dimensional Lie groups. The quotient group,  $Q$ , is therefore an infinite separable metrizable group.*

**Theorem 26.** *Every non-separable connected compact group,  $G$ , has a quotient group,  $Q$ , which is a countably infinite product of non-trivial compact finite-dimensional Lie groups. The quotient group,  $Q$ , is therefore an infinite separable metrizable group.*

**Remark 5.** No discrete group has a quotient group which is a countably infinite product of non-trivial topological groups since every quotient of a discrete group is evidently discrete. In particular then, a locally compact abelian group need not have a quotient group which is a countably infinite product of non-trivial topological groups.

**Theorem 27.** *Every non-separable connected locally compact abelian group,  $G$ , has a quotient group,  $Q$ , which is a countably infinite product of non-trivial compact finite-dimensional Lie groups. The quotient group,  $Q$ , is therefore an infinite separable metrizable group.*

**Theorem 28.** *Every infinite totally disconnected compact group  $G$  has a quotient group,  $Q$ , which is homeomorphic to a countably infinite product of finite discrete topological groups. The quotient group,  $Q$ , is thus homeomorphic to the Cantor space and therefore is an infinite separable metrizable group.*

Below a positive answer to Question 4 (i), (ii), (iii), and (iv) is presented.

**Theorem 29. (Separable Quotient Theorem for Compact Groups)** *Let  $G$  be an infinite compact group. Then  $G$  has a quotient group which is an infinite separable metrizable (compact) group.*

With the help of Theorem 29 a positive answer to Question 2 (i), (ii), (iii), and (iv) is given.

**Theorem 30. (Separable Quotient Theorem for Locally Compact Abelian Groups)** *Let  $G$  be an infinite locally compact abelian group. Then  $G$  has a quotient group which is an infinite separable metrizable group.*

Recall that a *proto-Lie group* is defined in ([17], Definition 3.25) to be a topological group  $G$  for which every neighborhood of the identity contains a closed normal subgroup  $N$  such that the quotient group  $G/N$  is a Lie group. If  $G$  is also a complete topological group, then it is said to be a *pro-Lie group*. If  $G$  is a proto-Lie group (respectively, pro-Lie group) with all the quotient Lie groups  $G/N$  discrete then  $G$  is said to be *protodiscrete* (respectively, *prodiscrete*). It is immediately clear that if  $G$  is a proto-Lie group which is not a Lie group, then it is not topologically simple.

**Theorem 31. (Separable Quotient Theorem for Proto-Lie Groups)** *Let  $G$  be an infinite proto-Lie group which is not protodiscrete; that is,  $G$  is not totally disconnected. Then  $G$  has a quotient group which is an infinite separable metrizable (Lie) group.*

**Theorem 32. (Separable Quotient Theorem for  $\sigma$ -compact Pro-Lie groups)** *Let  $G$  be an infinite  $\sigma$ -compact pro-Lie group. Then  $G$  has a quotient group which is an infinite separable metrizable group.*

Another significant generalization of Theorem 30 is Theorem 33.

**Theorem 33. (Separable Quotient Theorem for Abelian Pro-Lie groups)** *Let  $G$  be an infinite abelian pro-Lie group. Then  $G$  has a quotient group which is an infinite separable metrizable group.*

The next theorem, which generalizes Theorem 29, provides a partial but significant answer to Question 3.

**Theorem 34. (Separable Quotient Theorem for  $\sigma$ -compact Locally Compact Groups)** *Every infinite  $\sigma$ -compact locally compact group has a quotient group which is an infinite separable metrizable group.*

**Corollary 8. (Separable Quotient Theorem for Almost Connected Locally Compact Groups)** *Every infinite almost connected locally compact group has a quotient group which is an infinite separable metrizable group.*

### 5.2. $\sigma$ -Compact Groups, Lindelöf $\Sigma$ -Groups and Pseudocompact Groups

Recall that the class of Lindelöf  $\Sigma$ -groups contains all  $\sigma$ -compact and all separable metrizable topological groups, and is closed with respect to countable products, closed subgroups, and continuous homomorphic images (see [10], Section 5.3).

**Proposition 11. (Separable Quotient Theorem for Lindelöf  $\Sigma$ -groups)** *Let  $G$  be an infinite Lindelöf  $\Sigma$ -group. Then  $G$  has a quotient group which is infinite and separable. Indeed, the topology of  $G$  is initial with respect to the family of quotient homomorphisms onto infinite groups with a countable network.*

Since every  $\sigma$ -compact topological group is evidently a Lindelöf  $\Sigma$ -group, the next result is immediate from Proposition 11.

**Corollary 9. (Separable Quotient Theorem for  $\sigma$ -compact Groups)** *Let  $G$  be an infinite  $\sigma$ -compact topological group. Then  $G$  has a quotient group which is infinite and separable. Indeed, the topology of  $G$  is initial with respect to the family of quotient homomorphisms of the group onto infinite groups with a countable network.*

Regarding Question 14, one might reasonably ask: If the topological group  $G$  has a quotient group which is infinite and separable, does  $G$  necessarily have a quotient group which is infinite, separable and metrizable? This question is answered negatively in the next Proposition 12.

**Proposition 12.** *There exists a countably infinite precompact abelian group  $H$  such that every quotient group of  $H$  is either trivial or non-metrizable.*

We now consider pseudocompact groups.

**Theorem 35. (Separable Quotient Theorem for Pseudocompact Groups)** *The topology of every infinite pseudocompact topological group,  $G$ , is initial with respect to the family of quotient homomorphisms onto infinite compact metrizable groups. In particular,  $G$  has a quotient group which is infinite separable compact and metrizable.*

### 5.3. A Precompact Topological Group Which Does Not Admit Separable Quotient Group

In this section, we show that Theorem 35 cannot be extended to precompact topological groups, even in the weak form of the existence of nontrivial separable quotients.

**Theorem 36.** *There exists an uncountable dense subgroup  $G$  of the compact group  $\mathbb{T}^c$  satisfying  $\dim G = 0$  such that every countable subgroup of  $G$  is closed in  $G$  and every uncountable subgroup of  $G$  is dense in  $G$ . Hence every quotient group of  $G$  is either trivial or non-separable.*

In fact, every power of the group  $G$  in Theorem 36 does not have non-trivial separable quotients.

**Theorem 37.** *Let  $G \subset \mathbb{T}^c$  be the group constructed in Theorem 36 and let  $\tau \geq 1$  be a cardinal number. Then every quotient group of  $G^\tau$  is either trivial or non-separable.*

A topological group is said to be a  $G_\sigma$ -group if it has a dense subgroup  $H$  which is the union of a strictly increasing sequence of closed topological subgroups. Finally, we show that there exists a  $G_\sigma$ -group without a nontrivial separable quotient group.

**Theorem 38.** *For every cardinal  $\tau \geq c$ , there exists a precompact topological abelian group  $H$  satisfying  $\dim G = 0$  and with the following properties:*

- (a)  $w(H) = \tau$ ;
- (b)  $H = \bigcup_{n \in \omega} H_n$ , where  $H_0 \subset H_1 \subset H_2 \subset \dots$  are proper closed subgroups of  $H$ ;
- (c) every quotient group of  $H$  is either trivial or non-separable.

The class of  $\mathbb{R}$ -factorizable groups (see [10], Chapter 8) contains all pseudocompact groups as well as  $\sigma$ -compact groups. We note that by Theorem 8.1.9 of [10], a locally compact group is  $\mathbb{R}$ -factorizable if and only if it is  $\sigma$ -compact. Since the group  $G$  in Theorem 36 is precompact, it is  $\mathbb{R}$ -factorizable according to ([10], Corollary 8.1.17).

**Corollary 10.** *There exist infinite  $\mathbb{R}$ -factorizable groups without non-trivial separable or metrizable quotients.*

#### 5.4. Open Problems

We note that Question 3 formulated earlier in Section 5 has not been fully answered, so we state it now as an unsolved problem.

**Problem 16. (Separable Quotient Problem for Locally Compact Groups)** *Does every infinite non-totally disconnected locally compact group have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable?*

Recall that an abelian topological group  $G$  is called a *reflexive topological group* if the natural map of  $G$  into its second dual group is a topological group isomorphism. The Pontryagin van-Kampen Theorem [38] says that every locally compact abelian group is reflexive. It is also known that every complete metrizable locally convex space, in particular every Banach space, is a reflexive topological group ([39], Proposition 15.2). Therefore the following unsolved problem arises naturally.

**Problem 17. (Separable Quotient Problem for Reflexive Topological Groups)** *Does every infinite reflexive abelian topological group,  $G$ , have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable?*

**Problem 18.** *Does there exist a precompact abelian group  $G$  as in Theorem 36 which has one of the following additional properties:*

- (a)  $G$  is connected;
- (b)  $G$  is Baire;
- (c)  $G$  is reflexive?

## 6. Quotient Groups of Free Topological Groups

Let, as usual,  $F(X)$  and  $A(X)$  denote the free topological group and the free abelian topological group of a Tychonoff space  $X$ , respectively.  $A(X)$  is a natural quotient group of  $F(X)$ , and for every  $X$  there is a quotient mapping from  $A(X)$  onto the group of integers  $\mathbb{Z}$  (see [10], Chapter 7).

### 6.1. Free Topological Groups Which Admit Second Countable Quotient Groups

A space  $X$  is called  $\omega$ -bounded if the closure of every countable subset of  $X$  is compact. Clearly, every compact space is  $\omega$ -bounded, while every  $\omega$ -bounded space is countably compact.

**Proposition 13** ([2]). *Let  $X$  be a non-scattered Tychonoff space. If  $X$  has one of the following properties (a) or (b), then both  $A(X)$  and  $F(X)$  admit an open continuous homomorphism onto the circle group  $\mathbb{T}$ :*

- (a)  $X$  is normal and countably compact;
- (b)  $X$  is  $\omega$ -bounded.

**Proposition 14** ([2]). *Let  $X$  be a scattered  $\omega$ -bounded Tychonoff space. Then every quotient group of  $F(X)$  and  $A(X)$  is either discrete and finitely generated (hence, countable) or non-metrizable.*

**Theorem 39** ([2]). *Let  $X$  be an  $\omega$ -bounded Tychonoff space. Then the following conditions are equivalent:*

- (a) Every metrizable quotient group of  $F(X)$  is discrete and finitely generated.
- (b) Every metrizable quotient group of  $F(X)$  is finitely generated.
- (c) Every metrizable quotient group of  $F(X)$  is countable.
- (d)  $X$  is scattered.

**Corollary 11** ([2]). *Let  $X$  be either the compact space of ordinals  $[0, \alpha]$  with the order topology or the one-point compactification of an arbitrary discrete space. Then every metrizable quotient group of  $F(X)$  or  $A(X)$  is discrete and finitely generated.*

It turns out that free topological groups on non-pseudocompact zero-dimensional spaces do have non-trivial metrizable quotient groups:

**Proposition 15** ([2]). *Let  $X$  be a non-pseudocompact zero-dimensional space. Then the groups  $F(X)$  and  $A(X)$  admit an open continuous homomorphism onto the (countably infinite separable metrizable) discrete group  $A(\mathbb{Z})$ .*

### 6.2. Free Topological Groups Which Admit Quotient Groups with a Countable Network

**Proposition 16** ([2]). *Let  $X$  be a locally compact or pseudocompact space. Then the groups  $A(X)$  and  $F(X)$  admit an open continuous homomorphism onto  $A(S)$ , where  $S$  is an infinite compact subspace of the closed unit interval  $[0, 1]$ , hence  $A(S)$  has a countable network.*

**Proposition 17** ([2]). *Let  $X$  be a Lindelöf  $\Sigma$ -space (in particular,  $\sigma$ -compact space). Then the groups  $A(X)$  and  $F(X)$  admit an open continuous homomorphism onto  $A(Y)$ , where  $Y$  has a countable network, hence  $A(Y)$  also has a countable network.*

### 6.3. Free Topological Groups Which Admit Separable Quotient Groups

**Theorem 40** ([2]). *Let  $X$  be a Tychonoff space satisfying the following conditions:*

- (1) the closure of every countable subset of  $X$  is countable and compact;
- (2) every countable compact subset of  $X$  is a retract of  $X$ .

*Then every separable quotient group of  $F(X)$  is countable.*

**Corollary 12** ([2]). *Let  $X$  be either the space of ordinals  $[0, \alpha)$  with the order topology or the one-point compactification of an arbitrary discrete space. Then every separable quotient group of  $F(X)$  or  $A(X)$  is countable.*

#### 6.4. Open Problems

Similarly to Problems 1 and 2 we can ask the following related questions.

**Problem 19** ([2]). *Does there exist an open continuous homomorphism of  $A(X)$  onto  $A(S)$ , where  $X$  is an arbitrary Tychonoff space and  $S$  is an infinite subspace of the closed unit interval  $[0, 1]$ ?*

Furthermore, a more particular question below is open:

**Problem 20** ([2]). *Does there exist an open continuous homomorphism of  $A(X)$  onto  $A(Y)$ , where  $X$  is an arbitrary Lindelöf space and  $Y$  is an infinite space which has a countable network?*

### 7. Separable Group Topologies for Abelian Groups

Which abelian groups  $G$  admit a separable Hausdorff group topology? To answer the question, the author of [8] considers three different cases:

Case 1. There is an element  $x \in G$  of infinite order;

Case 2.  $G$  is a bounded torsion group;

Case 3.  $G$  is an unbounded torsion group.

**Theorem 41** ([8]). *Let  $G$  be an abelian group with  $|G| \leq 2^c$ . Then  $G$  admits a separable, precompact, Hausdorff group topology.*

**Remark 6.** The “abelian” condition in Theorem 41 cannot be deleted as Shelah [40] proved that there exist non-abelian groups which admit no non-discrete Hausdorff topological group topology. Morris and Obraztsov ([41], Theorem L) produce an uncountable number of countably infinite groups each of which admits no non-discrete Hausdorff topological group topology; more precisely, they identify a continuum of pairwise non-isomorphic infinite groups,  $\{G_i : i \in I\}$ , of exponent  $p^2$ , for any sufficiently large prime  $p$ , where each proper subgroup of  $G_i$  is cyclic and each  $G_i$  does not admit any non-discrete Hausdorff topological group topology. It is also proved in [41] that there exist non-discrete Hausdorff topological groups  $G$  of each cardinality  $\aleph_n$  with no proper subgroup of the same cardinality as  $G$ . Such groups obviously have no quotient group of smaller cardinality than that of  $G$ .

**Remark 7.** Yves Cornulier noticed that every abelian group  $G$  of cardinality  $|G| \leq 2^c$  embeds as a subgroup of  $(\mathbb{Q} \times \mathbb{Q}/\mathbb{Z})^c$ , which is a separable Hausdorff topological group. Nevertheless, this remark does not prove Theorem 41, because even a closed subgroup of a separable group, with the induced topology, need not to be separable.

Yves Cornulier also kindly informed us that in sharp contrast to the abelian case, for every uncountable cardinal  $\tau$ , there exists a 2-step nilpotent group of cardinality  $\tau$  that has no Hausdorff separable group topology. So any reasonable potential generalization of Theorem 41 fails.

**Author Contributions:** This survey is the joint effort of the authors, who contributed equally.

**Funding:** This research received no external funding.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Leiderman, A.G.; Tkachenko, M.G. Products of topological groups in which all closed subgroups are separable. *Topol. Appl.* **2018**, *241*, 89–101. [[CrossRef](#)]
2. Leiderman, A.G.; Tkachenko, M.G. Quotients of free topological groups. Unpublished word, 2018.
3. Kąkol, J.; Leiderman, A.G.; Morris, S.A. Nonseparable closed vector subspaces of separable topological vector spaces. *Monatsh. Math.* **2017**, *182*, 39–47; Erratum in **2017**, *182*, 49–50. [[CrossRef](#)]
4. Leiderman, A.G.; Morris, S.A.; Tkachenko, M.G. Density character of subgroups of topological groups. *Trans. Am. Math. Soc.* **2017**, *369*, 5645–5664. [[CrossRef](#)]
5. Leiderman, A.G.; Morris, S.A.; Tkachenko, M.G. The separable quotient problem for topological groups. *arXiv* **2017**, arXiv:1707.09546.
6. Xiao, Z.; Sánchez, I.; Tkachenko, M. Topological groups whose closed subgroups are separable and the product operation. *Topol. Appl.* **2019**, to appear.
7. Gabrielyan, S.S.; Morris, S.A. A topological group observation on the Banach-Mazur separable quotient problem. *Topol. Appl.* **2019**, to appear.
8. López, L.F.M. Continuous isomorphisms onto separable groups. *Appl. Gen. Topol.* **2012**, *13*, 135–150.
9. Engelking, R. *General Topology*; Heldermann Verlag: Berlin, Germany, 1989.
10. Arhangel'skii, A.V.; Tkachenko, M.G. *Topological Groups and Related Structures*; Atlantis Series in Mathematics, Volume I; Atlantis Press and World Scientific: Paris, France; Amsterdam, The Netherlands, 2008.
11. Hodel, R.E. Cardinal Functions I, Chapter 1. In *Handbook of Set-Theoretic Topology*; Kunen, K., Vaughan, J.E., Eds.; North Holland: Amsterdam, The Netherlands; New York, NY, USA; Oxford, UK, 1984.
12. Hofmann, K.H.; Morris, S.A. *The Structure of Compact Groups*, 3rd ed.; Walter de Gruyter GmbH: Berlin, Germany; Boston, MA, USA, 2013.
13. Engelking, R. Cartesian products and dyadic spaces. *Fundam. Math.* **1965**, *57*, 287–304. [[CrossRef](#)]
14. Karnik, S.M.; Willard, S. Natural covers and  $r$ -quotient mappings. *Can. Math. Bull.* **1982**, *25*, 456–462. [[CrossRef](#)]
15. Fujita, H.; Shakhmatov, D. Topological groups with dense compactly generated subgroups. *Appl. Gen. Topol.* **2002**, *3*, 85–89. [[CrossRef](#)]
16. Guran, I.I. On topological groups close to being Lindelöf. *Sov. Math. Dokl.* **1981**, *23*, 173–175.
17. Hofmann, K.H.; Morris, S.A. *The Lie Theory of Connected Pro-Lie Groups*; European Mathematical Society: Zurich, Switzerland, 2007.
18. Itzkowitz, G.L. On the density character of compact topological groups. *Fundam. Math.* **1972**, *75*, 201–203. [[CrossRef](#)]
19. Comfort, W.W.; Itzkowitz, G.L. Density character in topological groups. *Math. Ann.* **1977**, *226*, 223–227. [[CrossRef](#)]
20. Vidossich, G. Characterization of separability for LF-spaces. *Annales de L'institut Fourier* **1968**, *18*, 87–90. [[CrossRef](#)]
21. Lohman, R.H.; Stiles, W.J. On separability in linear topological spaces. *Proc. Am. Math. Soc.* **1974**, *42*, 236–237. [[CrossRef](#)]
22. Morris, S.A. Varieties of topological groups. *Bull. Aust. Math. Soc.* **1969**, *1*, 145–160. [[CrossRef](#)]
23. Morris, S.A. Varieties of topological groups II. *Bull. Aust. Math. Soc.* **1970**, *2*, 1–13. [[CrossRef](#)]
24. Morris, S.A. Varieties of topological groups III. *Bull. Aust. Math. Soc.* **1970**, *2*, 165–178. [[CrossRef](#)]
25. Morris, S.A. Varieties of topological groups a survey. *Colloq. Math.* **1982**, *2*, 147–165. [[CrossRef](#)]
26. Brooks, M.S.; Morris, S.A.; Saxon, S.A. Generating varieties of topological groups. *Proc. Edinb. Math. Soc.* **1973**, *18*, 191–197. [[CrossRef](#)]
27. Kopperman, R.; Mislove, M.W.; Morris, S.A.; Nickolas, P.; Pestov, V.; Svetlichny, S. Limit laws for wide varieties of topological groups. *Houst. J. Math.* **1996**, *22*, 307–328.
28. Uspenskij, V.V. On the Suslin number of subgroups of products of countable groups. *Acta Universitatis Carolinae—Mathematica et Physica* **1995**, *36*, 85–87.
29. Hofmann, K.H.; Morris, S.A. Pro-Lie groups: A survey with open problems. *Axioms* **2015**, *4*, 294–312. [[CrossRef](#)]
30. Domański, P. On the separable topological vector spaces. *Funct. Approx. Comment. Math.* **1984**, *14*, 117–122.

31. Domański, P. Nonseparable closed subspaces in separable products of topological vector spaces, and  $q$ -minimality. *Arch. Math.* **1983**, *41*, 270–275. [[CrossRef](#)]
32. Diestel, J.; Morris, S.A.; Saxon, S.A. Varieties of locally convex topological vector spaces. *Bull. Am. Math. Soc.* **1971**, *77*, 799–803. [[CrossRef](#)]
33. Diestel, J.; Morris, S.A.; Saxon, S.A. Varieties of linear topological spaces. *Trans. Am. Math. Soc.* **1972**, *172*, 207–230. [[CrossRef](#)]
34. Morris, S.A.; Diestel, J. Remarks on varieties of locally convex linear topological spaces. *J. Lond. Math. Soc.* **1974**, *8*, 271–278. [[CrossRef](#)]
35. Kąkol, J.; Saxon, S.A. Separable quotients in  $C_c(X)$ ,  $C_p(X)$ , and their duals. *Proc. Am. Math. Soc.* **2017**, *145*, 3829–3841. [[CrossRef](#)]
36. Kąkol, J.; Saxon, S.A.; Todd, A. Barrelled spaces with(out) separable quotients. *Bull. Aust. Math. Soc.* **2014**, *90*, 295–303. [[CrossRef](#)]
37. Hofmann, K.H.; Morris, S.A.; Nickolas, P.; Pestov, V. Small large subgroups of a topological group. *Note Math.* **1997**, *4*, 161–165.
38. Morris, S.A. *Pontryagin Duality and the Structure of Locally Compact Abelian Groups*; Cambridge University Press: Cambridge, UK, 1977; ISBN 0521215439.
39. Banaszczyk, W. *Additive Subgroups of Topological Vector Spaces*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1977; ISBN 0387539174.
40. Shelah, S. On a problem of Kurosh, Jonsson groups, and applications. In *Word Problems Vol II*; Amsterdam, The Netherlands, 1980; pp. 373–394.
41. Morris, S.A.; Obraztsov, V.N. Non-discrete topological groups with many discrete subgroups. *Topol. Appl.* **1998**, *84*, 105–120. [[CrossRef](#)]



© 2018 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).