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Some New Results Involving the Generalized Bose–Einstein and Fermi–Dirac Functions

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Abstract: In this paper, we obtain a new series representation for the generalized Bose–Einstein and Fermi–Dirac functions by using fractional Weyl transform. To achieve this purpose, we obtain an analytic continuation for these functions by generalizing the domain of Riemann zeta functions from $(0 < \Re(s) < 1)$ to $(0 < \Re(s) < \mu)$. This leads to fresh insights for a new generalization of the Riemann zeta function. The results are validated by obtaining the classical series representation of the polylogarithm and Hurwitz–Lerch zeta functions as special cases. Fractional derivatives and the relationship of the generalized Bose–Einstein and Fermi–Dirac functions with Apostol–Euler–Nörlund polynomials are established to prove new identities.

Keywords: Fermi–Dirac function; Bose–Einstein function; Weyl transform; series representation

1. Introduction

The importance of the Fermi–Dirac and Bose–Einstein functions emerges from their fundamental presence in quantum physics and related sciences. Unlike the classical mechanics of particles, where the Maxwell distribution is used to study the velocity of classical gas molecules, the quantum gas is analyzed by using the Fermi–Dirac and Bose–Einstein functions. The distinct particles obey Fermi–Dirac statistics, while the indistinct particles follow Bose–Einstein statistics. All particles have a spin in relation to the usual theory. Fermions have half-integer spin and bosons have integer spin. The Fermi–Dirac and Bose–Einstein distribution functions are used to analyze them in the language of mathematics and physics. Indistinguishable particles that are not categorized through either of the aforementioned types are called anyons. The extensions of the Bose–Einstein and Fermi–Dirac functions interpolate between the two. Therefore, Chaudhry et al. [1] proposed that the extensions of the Bose–Einstein and Fermi–Dirac functions may help to describe anyons. In this paper, we generalize the results of Chaudhry and Qadir [2] by proving a general representation theorem to establish a new series representation of the generalized Bose–Einstein and Fermi–Dirac functions. However, we also discuss the fractional derivative, and the relationship of the generalized Bose–Einstein and Fermi–Dirac functions with Apostol–Euler–Nörlund polynomials. Before we provide our research results, it is necessary to enlist all the basic definitions and preliminaries that are required to present and understand this work.

2. Materials and Methods

2.1. Generalized Bose–Einstein and Fermi–Dirac Functions

During the course of our investigation, we consider the subsequent usual notations:

$$\mathbb{N} := \{1, 2, 3 \ldots.\},\ \mathbb{N} \cup \{0\} = \mathbb{N}_0; \mathbb{Z}^- = \{-1, -2, -3 \ldots\}.$$

In addition, $\mathbb{Z}$ is the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive numbers, and $\mathbb{C}$ is the set of complex numbers, $s = \sigma + i\tau$. Gamma function $\Gamma(s)$ as a generalization of factorials is also used here as a basic special function. For a detailed study of gamma and related functions, we refer the interested reader to [3,4].

More recently, Bayad and Chikhi [5] introduced and studied the generalized Fermi–Dirac functions given by ([5], (p. 12), Equation (45))

$$\begin{gathered}\Theta_\nu\ (s, \mu; x)\ :=\ \frac{\Gamma(\mu)}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-\nu(x+t)}}{(e^{x+t}+1)^\mu}\, dt \\ (\Re(x) \geq 0,\ \Re(\nu) > -\Re(\mu) \wedge \Re(s) > \Re(\mu) > 0 \text{ when } e^{-x} \neq -1)\end{gathered} \tag{1}$$

and their series representation is given by ([5], (p. 12), Equation (46))

$$\Theta_\nu\ (s, \mu; x) := \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(\mu+n) e^{-(\nu+\mu+n)x}}{n!(\nu+\mu+n)^s}. \tag{2}$$

For $\mu = 1$, in Equation (1) the extended Fermi–Dirac functions ([6], (p. 113), Equation (3.14)) are given here by

$$\begin{gathered}\Theta_\nu(s; x) := \Theta_\nu\ (s, 1; x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-\nu(t+x)}}{e^{t+x}+1}\, dt \\ (\Re(x) \geq 0,\ \Re(\nu) > -1),\end{gathered} \tag{3}$$

and for $\nu = 0$ and $\mu = 1$ in Equation (1), the original Fermi–Dirac function is given by ([6], (p. 109), Equation (1.12))

$$\mathcal{F}_{s-1}(x)\ := \Theta_0\ (s, 1; x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{t+x}+1} dt\ (\Re(x) \geq 0; \Re(s) > 0). \tag{4}$$

Similarly, the generalized Bose–Einstein functions $\Psi_\nu(s,\ \alpha\,; x)$, which are defined by ([5], (p. 13), Equation (51)), are as follows

$$\begin{gathered}\Psi_\nu(s, \mu; x)\ :=\ \frac{\Gamma(\mu)}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-\nu t}}{(e^{t+x}-1)^\mu}\, dt \\ (\Re(x) \geq 0, \Re(\nu) > -\Re(\mu) \wedge\ \Re(s) > \Re(\mu) > 0 \text{ when } e^{-x} = 1\ \wedge\ \Re(s) > 0\ ),\end{gathered} \tag{5}$$

and their series representation is given by ([5], (p. 13), Equation (52))

$$\Psi_\nu\ (s, \mu; x) := \sum_{n=1}^{\infty} \frac{\Gamma(\mu+n) e^{-(\nu+\mu+n)x}}{n!(\nu+\mu+n)^s}. \tag{6}$$

For $\mu = 1$, the extended Bose–Einstein functions ([6], (p. 115), Equation (4.4)) are given here by

$$\Psi_\nu(s; x)\ := \Psi_\nu(s, 1; x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-\nu t}}{e^{t+x}-1}\, dt\ (\Re(x) \geq 0,\ \Re(\nu) > -1), \tag{7}$$

and the original Bose–Einstein function is given by ([6], (p. 109), Equation (1.13)).

$$\mathcal{B}_{s-1}(x)\ := \Psi_0\ (s, 1; x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{t+x}-1}\, dt\ (\Re(x) \geq 0; \Re(s) > 1). \tag{8}$$

For further study of the Fermi–Dirac and Bose–Einstein functions, we refer the interested reader to [7–9]. The reduction and duality theorems for these functions are given by ([5], (p. 12–13))

$$\Theta_{\nu-M}(s;\ \mu+M;x) = \sum_{m=0}^{M} R_1(M\ ,\ m,-\nu)\Theta_\nu(s-m,\mu;x), \tag{9}$$

$$\Theta_\nu(s-M;\mu;x) = \sum_{m=0}^{M}\ (-1)^{M-m}R(M\ ,\ m,-\nu)\Theta_{\nu-m}(s\ ,\ \mu+m;x), \tag{10}$$

$$\Psi_{\nu-M}(s;\ \mu+M;x) = \sum_{m=0}^{M} R_1(M\ ,\ m,-\nu)\Psi_\nu(s-m,\ \mu;x), \tag{11}$$

$$\Psi_\nu(s-M;\ \mu;x) = \sum_{m=0}^{M}(-1)^{M-m}R(M\ ,\ m,-\nu)\Psi_{\nu-m}(s\ ,\ \mu+m;x), \tag{12}$$

respectively, where $R_1(M\ ,\ m,-\nu)$ and $R(M\ ,\ m,-\nu)$ are the polynomials having explicit representations in terms of Stirling numbers. For examples and details see Carlitz [10,11]. More recently, Tassaddiq [12,13] considered the λ-generalized extended Fermi–Dirac functions and λ-generalized extended Bose–Einstein functions as a transformed form of Srivastava's λ-generalized Hurwitz–Lerch zeta functions ([14], (p. 1487), Equation (1.14)). In this research, we generalize the results of Chaudhry and Qadir [2]. To achieve this goal, it is important to briefly highlight their relationship with the zeta functions. It should be noted that for $x = 0$, the Bose–Einstein and Fermi–Dirac functions are related to the Riemann zeta functions respectively.

$$\zeta(s) := \mathcal{B}_{s-1}(0); \Re(s) > 1 \tag{13}$$

$$\zeta(s)\left(1-2^{1-s}\right) := \mathcal{F}_{s-1}(0); \Re(s) > 0. \tag{14}$$

The polylogarithm function is an important function in the study of theory of polymers that was introduced and investigated by Truesdell [15]

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}\ (s \in \mathbb{C},\ |z|< 1; \Re(s) >1,\ |z| = 1). \tag{15}$$

It generalizes the Riemann zeta function, as we have

$$\mathrm{Li}_s(1) = \phi(1,s) = \zeta(s)\ (\Re(s) > 1\ ), \tag{16}$$

and it can also be represented as an integral

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t - z}\,dt\ (s \in \mathbb{C} \text{ when } |z|< 1; \Re(s) >1 \text{ and when } |z| = 1). \tag{17}$$

In our present analysis, we are especially interested in Lindelöf's representation of these functions given by ([15], (p. 149), Equation (13)),

$$\begin{gathered}\mathrm{Li}_s(z) = \Gamma(1-s)(\log z)^{s-1} + \textstyle\sum_{n=0}^{\infty} \zeta(s-n)\frac{(\log z)^n}{n!} \\ (\left|\log z\right| < 2\pi, s \neq 1,2,3,\ldots, \nu \neq 0,-1,-2,\ldots,).\end{gathered} \tag{18}$$

The Hurwitz–Lerch zeta function ([16], (p. 27)), as a generalization of the polylogarithm, is given by

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}\ (a\in\mathbb{C}\backslash\mathbb{Z}^-;s\in\mathbb{C}\text{ when }|z|<1;\ \Re(s)>1\text{ when }|z|=1). \tag{19}$$

It has a meromorphic extension to the whole complex s-plane, while it has a simple singularity at $s = 1$ of residue 1. It is also represented by ([16], (p. 27), Equation (1.6))

$$\Phi(z,s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-at}}{1-ze^{-t}}\,dt\ (|z|<1\Rightarrow\Re(s)>0;\Re(a)>0;z=1\Rightarrow\Re(s)>1). \tag{20}$$

Apart from other applications, the Hurwitz–Lerch zeta function is the most general function in the original zeta family. For example, different values of the involved parameters in (19–20) yield the following relationships with the polylogarithm, Hurwitz, and Riemann zeta functions, respectively:

$$\mathrm{Li}_s(z):=\sum_{n=1}^{\infty}\frac{z^n}{n^s}=z\Phi(s,z,1), \tag{21}$$

$$\zeta(s,a):=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}=\Phi(s,1,a), \tag{22}$$

$$\zeta(s):=\sum_{n=1}^{\infty}\frac{1}{n^s}=\Phi(s,1,1)=\zeta(s,1). \tag{23}$$

For our purposes, it is important to note that the Hurwitz–Lerch zeta function has a series representation ([16], (pp. 28–29))

$$\begin{gathered}\Phi(z,s,\nu)=\frac{\Gamma(1-s)}{z^{\nu}}\left(\log\frac{1}{z}\right)^{s-1}+z^{-\nu}\sum_{n=0}^{\infty}\zeta(s-n,\nu)\frac{(\log z)^n}{n!}\\ \left(\left|\log z\right|<2\pi\,,s\neq 1,2,3,\ldots,\nu\neq 0,-1,-2,\ldots,\right)\end{gathered} \tag{24}$$

that generalizes Lindelöf's representation (18).

Further to all of the above discussion, Chaudhry et al. [17] defined a new generalization of the Riemann zeta function in the critical strip by

$$\Xi_a(s;x):=\frac{1}{\Gamma(s)}\int_x^{\infty}(t-x)^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)e^{-at}\,dt\ (0<\Re(s)<1:x\geq 0;a\geq 0). \tag{25}$$

The Riemann hypothesis is a well-known unsolved problem in analytic number theory [18]. It states that "all the non-trivial zeros of the zeta function exist on the line $s = 1/2$". These zeros seem to be complex conjugates and are hence symmetrical on this line. The Riemann zeta function in the critical strip is defined and studied in [18]

$$\zeta(s):=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)dt\ (0<\Re(s)<1), \tag{26}$$

which can be obtained as a special case of Equation (25) by putting $x = a = 0$.

2.2. A Class $\Re_{\infty}(A,P,\delta)$ of Functions and the Representation Theorem

More recently Chaudhry and Qadir [19] discussed some important classes of functions. The statements of this section are taken from [19–21].

We first give a brief introduction to the function spaces $H(\xi;\eta)$ and $H(\infty;\eta)$. The elements of $H(\xi;\eta)$ are particular functions $f\in C^{\infty}(0,\infty)$ that satisfy the following conditions

1. $\int_0^T f(t)dt$ is well defined for $T \in [0, \infty)$;
2. $f(t) = O(t^{-\eta})$ $(t \to 0^+)$;
3. $f(t) = O(t^{-\xi})$ $(t \to \infty)$.

Furthermore, if $f(t) = O(t^{-\xi})$ $(t \to \infty; \xi \in \mathbb{R}_0^+)$, then $f(t) \in H(\infty;\eta)$. We can note that $H(\infty;\eta) \subset H(\xi;\eta)$ $(\forall \xi \in \mathbb{R}_0^+)$.

Clearly, we have

$$f(t) = e^{-bt} \in H(\infty, 0)\ (b > 0). \tag{27}$$

The Mellin transform of $f \in H(\xi;\eta)$ is defined by

$$f_M(s) = M[f(t);s] := \int_0^\infty f(t)t^{s-1}dt\ (s = \sigma + i\tau,\ \eta < \Re(s) < \xi). \tag{28}$$

The fractional Weyl transform of $f \in H(\xi;0)$ is defined by

$$\begin{gathered}\Omega(s;x) := W^{-s}[f(t)](x) := \tfrac{1}{\Gamma(s)}M[f(t+x);s] \\ = \tfrac{1}{\Gamma(s)}\int_0^\infty f(t+x)t^{s-1}dt = \tfrac{1}{\Gamma(s)}\int_x^\infty f(t)(t-x)^{s-1}dt;\ (s = \sigma + i\tau,\ 0 < \Re(s) < \xi\ ,\ x \geq 0).\end{gathered} \tag{29}$$

Considering $\Re(s) \leq 0$, we define the Weyl transform of $\omega \in H(\xi;0)$ as follows,

$$\Omega(s;x) := W^{-s}[f(t)](x) := (-1)^n \frac{d^n}{dx^n}(\Omega(n+s;x)),\ (0 \leq n + \Re(s) < \xi), \tag{30}$$

and

$$\Omega(0;x) := \omega(x). \tag{31}$$

We can rewrite Equation (30) alternately as

$$\begin{gathered}\Omega(-s;x) := W^s[\omega(t)](x) = (-1)^n \tfrac{d^n}{dx^n}\left(W^{-(n-s)}[\omega(t)](x)\right) \\ =: (-1)^n \tfrac{d^n}{dx^n}(\Omega(n-s;x))\ (0 \leq n - \Re(s) < \xi, \Re(s) > 0).\end{gathered} \tag{32}$$

For these formulae, $n \geq \Re(s)$ where n is the positive and smallest such integer. For $s = n$ in Equation (32), we get

$$\Omega(-n;x) := W^n[\omega(t)](x) := (-1)^n \frac{d^n}{dx^n}(\Omega(0;x)) = (-1)^n \frac{d^n}{dx^n}(\omega(x)). \tag{33}$$

Note that $\{W^s\}$ $(s \in \mathbb{C})$ satisfies

$$W^{-(\mu+s)}[\omega(t)](x) = W^{-\mu}[\Omega(s;t)](x) = \Omega(s+\mu;x) \tag{34}$$

the multiplicative group property. For further detailed study of Weyl and related integral transforms, we refer the interested reader to [22–24].

The space of analytic functions [20,21] as discussed by Hardy is reviewed here as follows: Let $0 < \delta < 1$ and $H(\delta) := \{s = \sigma + i\tau : \Re(s) \geq -\delta\}$ be the half space. Further, for an analytic function $\phi(s)$, $s \epsilon H(\delta)$, suppose that $0 < A < \pi$ and

$$\Re = \Re(A, P, \delta) := \left\{\phi(s) : \left|\phi(s)\right| \leq Ce^{P\sigma + A|\tau|}\right\} \tag{35}$$

is called the Hardy space of analytic functions that restricts the parameter A to lie in $(0, \pi)$. Consider a function $\phi \in \mathfrak{R}$ and define

$$\Phi(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin \pi s}\phi(-s)x^{-s}ds \ (0 < c < \delta), \tag{36}$$

such that the kernel is majorized by

$$e^{-(\pi-A)|\tau|}e^{-Pc}x^{-c}, \ (x > 0). \tag{37}$$

These are uniformly convergent in an interval of $0 < x \leq x_0 \leq X < \infty$. Therefore, the function $\Phi(x)$ is regular, and represented by the integral (36), for all positive x. We combine these classes to define a new class of functions for our purposes. Assume that $\omega(0) := \Omega(0;0)$ is well defined and $\omega(x) := \Omega(0;x)$ $(x \geq 0)$. Then, $\omega \in \mathfrak{R}_\infty(A, P, \delta)$ iff $\omega \in H(\delta;0)$ and

$$\left|\frac{\Omega(s;0)}{\Gamma(1-s)}\right| \leq Ce^{\sigma p+A|\tau|} \ (0 \leq \mathfrak{R}(s) < \delta). \tag{38}$$

Theorem 1. *Let* $\varphi \in H(\delta;0)$ *and* $\Phi(s;x)$ $(x > 0)$ *be its Weyl transform; then, the series representation is*

$$\Phi(s;x) = \sum_{n=0}^{\infty} \frac{\Phi(s-n;0)(-x)^n}{n!} \ (0 \leq \mathfrak{R}(s) < \delta, 0 < x < \infty). \tag{39}$$

Proof. Since $\varphi \in \mathfrak{R}_\infty(A, P, \delta)$, the inverse Mellin transform is

$$\begin{aligned}\Phi(0;x) &:= \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \varphi_M(s)x^{-s}ds \\ &= \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\Phi(s;0)x^{-s}ds = \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tfrac{\pi\Phi(s;0)}{\sin(\pi s)\Gamma(1-s)}x^{-s}ds \\ &\quad (0 \leq c < \delta \ , \ x > 0),\end{aligned} \tag{40}$$

well defined because the integrand is majorized by a constant multiple of $e^{-(\pi-A)|\tau|}e^{-Pc}x^{-c}$.

Familiar Cauchy's theorem for complex numbers is used to invert the Mellin transform in Equation (40). The integrand has singularities of order 1 at $s = -n$ $(n = 0, 1, 2, 3, \ldots)$ with residues $\frac{\Phi(-n;0)(-x)^n}{n!}$. Therefore,

$$\Phi(0;x) = \sum_{n=0}^{\infty} \frac{(-1)^n\Phi(-n;0)x^n}{n!} \ (0 < x < e^{-P}). \tag{41}$$

Because $\varphi \in H(\delta;0)$, the series (41) extends uniquely for the Weyl transform as

$$(s;x) = \sum_{n=0}^{\infty} \frac{\Phi(s-n;0)(-x)^n}{n!} \ (0 < x < e^{-P} , 0 \leq \mathfrak{R}(s) < \delta). \tag{42}$$

□

Theorem 2. *Let* $\varphi \in \mathfrak{R}_\infty(A, P, \delta)$ *and*

$$\psi(t) = \lambda t^{-\mu} + \varphi(t) \ (\mu > 0). \tag{43}$$

Then, the Weyl transform $\Psi(s;x)$ *has a closed form representation*

$$\Psi(s;x) \ = \lambda\frac{\Gamma(\mu-s)}{\mu}x^{s-\mu} + \sum_{n=0}^{\infty} \frac{\Phi(s-n;0)(-x)^n}{n!}(0 \leq \mathfrak{R}(s) < \min(\delta, \mu); 0 < x < e^{-P}). \tag{44}$$

Proof. An application of the linearity property of Weyl's transform to Equation (43) gives

$$\begin{aligned}\Psi(s;x) = W^{-s}[\psi(t)](x) \ &= \lambda W^{-s}[t^{-\mu}](x) + \Phi(s;x) \\ (0 \leq \Re(s) < \min(\delta,\mu); 0 < x < e^{-p}).\end{aligned} \tag{45}$$

However, (see [22], (p. 249)) we have

$$W^{-s}[t^{-\mu}](x) = \frac{\Gamma(\mu - s)}{\mu} x^{s-\mu};\ (0 < \Re(s) < \mu; 0 < x < \infty). \tag{46}$$

From Equations (38), (42), and (46) we arrive at Equation (44). □

Example 1. *Define*

$$\varphi(t) := \frac{1}{e^t - 1} - \frac{1}{t}\ (t > 0). \tag{47}$$

Note that $\varphi \in \Re_\infty(\frac{\pi}{2}, \ln(1/2\pi), \delta)$ *and*

$$\Phi(s,0) = \zeta(s)\ (0 < \Re(s) < 1). \tag{48}$$

Hence, we have an expansion

$$\Phi(0,x) = \frac{1}{e^x - 1} - \frac{1}{x} = \zeta(0) + \sum_{n=1}^{\infty} \frac{(-1)^n \zeta(-n) x^n}{n!}\ (0 < x < 2\pi), \tag{49}$$

which is the standard result. Using

$$\zeta(-n) = -\frac{B_n}{n+1}\ (n = 0,1,2,3,\ldots), \tag{50}$$

we can rewrite $\Phi(s;x)$ *in terms of Bernoulli numbers.*

3. Results

Application of the General Representation Theorem to the Generalized Bose–Einstein and Fermi–Dirac and Related Functions

In this section, we first evaluate the fractional Weyl transform for the function involved in the integrand of generalized Bose–Einstein functions and then analytically continued this function in the interval $(0 < \Re(s) < \mu)$, namely the generalized critical strip.

Remark 1. *To apply the general representation theorem, we first discussed analytic continuation of the Bose–Einstein function in the critical strip. The integral representation (5) of the generalized Bose–Einstein function* $\Psi_\nu\ (s, \mu; 0)$ *can be continued to the domain,* $0 < \Re(s) < \mu$, *where a particular case of this domain* $0 < \Re(s) < 1$ *is known as the critical strip for the zeta function. For* $\Re(s) > \mu$, *we may write in the usual sense as we write for the zeta function* ([18], (p. 37))

$$\begin{aligned}\Gamma(s)\Psi_\nu\ (s,\mu;0) = &\int_0^1 \left(\frac{e^{-\nu t}}{(e^t - 1)^\mu} - \frac{1}{t^\mu}\right) t^{s-1} dt + \frac{1}{s-\mu} \\ &+ \int_1^\infty \frac{e^{-\nu t}}{(e^t - 1)^\mu} t^{s-1} dt,\end{aligned} \tag{51}$$

which is true by analytic continuation for $\Re(s) > 0$. *For these values* $0 < \Re(s) < \mu$, *we get*

$$\frac{1}{s-\mu} = \int_1^{\infty} \frac{t^{s-1}}{t^{\mu}} dt\,, \tag{52}$$

such that we can write

$$\Gamma(s)\Psi_{\nu}\,(s,\mu;0) = \int_0^{\infty}\left(\frac{e^{-\nu t}}{(e^t-1)^{\mu}} - \frac{1}{t^{\mu}}\right)t^{s-1}dt\ (0 < \Re(s)< \mu; \Re(\nu) >0). \tag{53}$$

Putting $\nu = 0$ *in Equation (53), we get the representation*

$$\Gamma(s)\Psi_0\,(s,\mu;0) = \int_0^{\infty}\left(\frac{1}{(e^t-1)^{\mu}} - \frac{1}{t^{\mu}}\right)t^{s-1}dt\ (0 < \Re(s) < \mu). \tag{54}$$

Putting $\nu = 0;\ \mu = 1$ *in Equation (53), the classical representation (26) for the Riemann zeta function is recovered.*

Remark 2. *The series representation (24) for the Hurwitz–Lerch function is proved in* ([16], (p. 28)) *by using the following steps.*

1. *Using the contour integral to state the involved function*
2. *Using the Cauchy residue theorem from complex analysis*
3. *Using the following identity known as Hurwitz formula* [16]

$$\zeta(s,\nu) = 2(2\pi)^{s-1}\Gamma(1-s)\sum_{n=1}^{\infty}\frac{\sin(2\pi n\nu + \frac{\pi s}{2})}{n^{1-s}}\ (\Re(s) < 0\,,\ 0 < \nu \le 1). \tag{55}$$

In this section, we have obtained a new series representation for the generalized Bose–Einstein and Fermi–Dirac functions. We have shown that the above stated results (18) and (24) for the polylogarithm and Hurwitz–Lerch functions are special cases by using the fractional Weyl transform.

Theorem 3. *Show that the generalized Fermi–Dirac functions have a series representation*

$$\Theta_{\nu}(s,\mu;x) := \Gamma(\mu)\sum_{M=0}^{\infty}\frac{\Theta_{\nu}\,(s-M,\mu;0)x^M}{M!} = \Gamma(\mu)\sum_{M=0}^{\infty}\frac{\Sigma_{m=0}^{M}\ (-1)^{M-m}R(M\,,\ m,-\nu)\Theta_{\nu-m}(s\,,\ \mu+m;0)}{M!}x^M \tag{56}$$
$$(0 \le \Re(s) < \mu;\ \nu \neq 0,-1,-2,\ldots).$$

Proof. The generalized Fermi–Dirac function (1) can be written as

$$\begin{aligned}\Theta_{\nu}(s,\mu;x) &= \frac{\Gamma(\mu)}{\Gamma(s)}\int_x^{\infty} e^{-\nu t}\frac{(t-x)^{s-1}}{(e^t+1)^{\mu}}dt\\ &= \frac{\Gamma(\mu)}{\Gamma(s)}\int_x^{\infty}\left[\frac{e^{-\nu t}}{(e^t+1)^{\mu}}\right](t-x)^{s-1}dt\\ &= \Gamma(\mu)W^{-s}\left[\frac{e^{-\nu t}}{(e^t+1)^{\mu}}\right](x) = \Gamma(\mu)\sum_{M=0}^{\infty}\frac{\Theta_{\nu}\,(s-M,\mu;0)}{M!}x^M\ ,\end{aligned} \tag{57}$$

which leads to the required result by using Equations (10) and (39). □

Corollary 1. *The Fermi–Dirac function has a representation* ([2], Equation (4.2)):

$$F_{s-1}(x) := \sum_{M=0}^{\infty}\frac{\left(1-2^{M-s+1}\right)\zeta(s-M)x^M}{M!}\ . \tag{58}$$

Proof. This result follows by putting $\mu = 1; \nu = 0$ in Equation (56) and using Equation (14). □

Theorem 4. *Show that the generalized Bose–Einstein functions have a series representation:*

$$\Psi_\nu\,(s,\mu;x) = \frac{\Gamma(\mu)\Gamma(\mu-s)}{\mu}x^{s-\mu} + \Gamma(\mu)\sum_{M=0}^{\infty}\frac{(-1)^M\Psi_\nu(s-M\,,\,\mu;0)}{M!}x^M$$
$$(0 \leq \Re(s) < \mu;\ \nu \neq 0,-1,-2,\ldots). \tag{59}$$

Proof. First, we note that the integral representation (5) can be rewritten as

$$\Psi_\nu\,(s,\mu;x) := \frac{\Gamma(\mu)}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-\nu t}}{(e^{t+x}-1)^\mu}\,dt = \frac{\Gamma(\mu)}{\Gamma(s)}\int_x^\infty e^{-\nu t}\frac{(t-x)^{s-1}}{(e^t-1)^\mu}dt, \tag{60}$$

which can be rearranged as follows

$$= \frac{\Gamma(\mu)}{\Gamma(s)}\int_x^\infty \left[\frac{e^{-\nu t}}{(e^t-1)^\mu} - \frac{1}{t^\mu} + \frac{1}{t^\mu}\right](t-x)^{s-1}dt\,. \tag{61}$$

Next, by making use of the definition of Weyl transform, we get

$$\Psi_\nu\,(s,\mu;x) = \Gamma(\mu)W^{-s}\left[\frac{e^{-\nu t}}{(e^t-1)^\mu} - \frac{1}{t^\mu}\right](x) + \Gamma(\mu)W^{-s}\left[\frac{1}{t^\mu}\right](x). \tag{62}$$

However, an application of the Weyl transform (46) along with an application of the general representation theorem (39) on the left hand side of the above Equation (62) leads to the required series representation. □

Corollary 2. *The Bose–Einstein function has a representation* ([2], Equation (4.7)):

$$\mathcal{B}_{s-1}(x) = \Gamma(1-s)x^{s-1} + \sum_{M=0}^{\infty}\frac{(-1)^n\zeta\,(s-M)x^M}{M!}. \tag{63}$$

Proof. The result follows by putting $\mu = 1; \nu = 0$ in Equation (59) and using Equation (13). □

Remark 3. *Putting* $\mu = 1; x = \log\frac{1}{z} \Rightarrow z = e^{-x}; -x = \log z$, *replacing* ν *by* $\nu - 1$ *in (59), and using the relation, ([6], Equation (4.5))* $\Psi_\nu(s;\ x) = e^{-(\nu+1)x}\Phi(e^{-x}\ ,\ s,\ \nu\ +\ 1)$, *we obtain*

$$z^\nu\Phi(z,s,\nu) = \Gamma(1-s)(\log\tfrac{1}{z})^{s-1} + \sum_{M=0}^{\infty}\zeta(s-M,\nu)\frac{(\log z)^M}{M!}\left(|\log z| < 2\pi\,, s \neq 1,2,3,\ldots,\nu \neq 0,-1,-2,\ldots\right), \tag{64}$$

which is exactly Equation (24). Further, by putting $\nu = 0$, *we deduce Lindelöf's representation (18) for the polylogarithm function.*

Remark 4. *The use of fractional derivatives and fractional integrals has become vital to solve many physical problems that were unsolvable otherwise, see for example* [25,26]. *For our interest, the Riemann–Liouville fractional derivative is defined by* ([22], (p. 70)) *and* [23].

$$\mathfrak{D}_z^\mu\{f(z)\} := \begin{cases} \frac{1}{\Gamma(-\mu)}\int_0^z (z-t)^{\mu-1}\ f(t)dt\ \Re(\mu) > 0 \\ \frac{d^m}{dz^m}\left\{\ \mathfrak{D}_z^{\mu-m}\{f(z)\}\right\}(m-1 \leqq \Re(\mu) < m(m \in \mathbb{N})). \end{cases} \tag{65}$$

It is important to notice from integral representations (1) and (5) that the functions $\Theta_\nu\,(s,\mu;x)$ and $\Psi_\nu\,(s,\mu;x)$ are in effect a Riemann–Liouville fractional derivative of the Fermi–Dirac and Bose–Einstein functions respectively given by

$$\Theta_\nu\,(s,\mu;x) = \frac{1}{\Gamma(\mu)}\,\mathfrak{D}_x^{\mu-1}\left\{e^{-x(\mu-1)}\Theta_\nu\,(s;x)\right\}; \Re(\mu) > 0, \tag{66}$$

$$\Theta_\nu\,(s,\mu;x) = \frac{1}{\Gamma(\mu)}\,\mathfrak{D}_x^{\mu-1}\left\{e^{-x(\mu-1)}\Theta_\nu\,(s;x)\right\}; \Re(\mu) > 0. \tag{67}$$

Remark 5. *The Apostol–Euler–Nörlund polynomials* $E_n^{(\mu)}(x;\lambda)$ [27,28] *are defined by the generating function*

$$\left(\frac{2}{\lambda e^t+1}\right)^{\mu} e^{vt} = \sum_{n=0}^{\infty} E_n^{(\mu)}(v;\lambda)\frac{t^n}{n!}; |t| < \left|\log(-\lambda)\right|; \lambda \neq -1 \tag{68}$$

and Bernoulli–Nörlund polynomials [27,28] *are defined by*

$$\left(\frac{t}{e^t-1}\right)^{\mu} e^{vt} = \sum_{n=0}^{\infty} B_n^{(\mu)}(v)\frac{t^n}{n!}; |t| < |2\pi|. \tag{69}$$

It is important to further mention that the relation of the generalized Fermi–Dirac and Bose–Einstein functions with Apostol—-Euler–Nörlund [27,28] polynomials can be established in view of integral representations (1) and (5), respectively, as follows.

Consider ([5], Equation (47))

$$\Theta_v\,(s,\mu;x) = e^{-x(v+\mu)}\zeta(s,\mu;\nu+\mu,-e^{-x}). \tag{70}$$

Now, replace ν by $\nu-\mu$ and $s=-m$ in Equation (70); we get

$$\Theta_{v-\mu}\,(-m,\mu;x) = e^{-xv}\zeta(-m,\mu;\nu,-e^{-x}). \tag{71}$$

Next, by using $\lambda = e^{-x}; \alpha = \mu$ in ([5], Equation (27)), we get

$$\Theta_{v-\mu}\,(-m,\mu;x) = e^{-xv}\Gamma(\mu)2^{-\mu}E_m^{(\mu)}(\nu;e^{-x}). \tag{72}$$

Similarly, by considering (5], Equation (53)) and replacing ν by $\nu-\mu$, $s=-m$, we get

$$\Psi_{v-\mu}\,(-m,\mu;x) = e^{-xv}\zeta(-m,\mu\,;\nu,e^{-x}). \tag{73}$$

Next, by using $\lambda = e^{-x}$ in ([5], Equation (27)) and using the result in the above Equation (73), we get

$$\Psi_{v-\mu}\,(-m,\mu;x) = e^{-xv}\Gamma(\mu)2^{-\mu}E_m^{(\mu)}(\nu;-e^{-x}). \tag{74}$$

For x = 0, in Equations (72) and (74), we get

$$\Theta_{v-\mu}\,(-m,\mu;0) = \Gamma(\mu)2^{-\mu}E_m^{(\mu)}(\nu;1)\Psi_{v-\mu}\,(-m,\mu;0) = \Gamma(\mu)2^{-\mu}E_m^{(\mu)}(\nu;-1),$$

which can be used in Equations (56) and (59) to obtain the representation in terms of special cases of Apostol–Euler–Nörlund polynomials $E_m^{(\mu)}(v;\mp 1)$. Considering the further restrictions $v=\mu=1$, we can get these relations in terms of commonly used Bernoulli and Euler numbers.

Remark 6. *One can note that Equation (59) may be stated alternately in terms of Stirling numbers by using Equation (13) as follows*

$$\Psi_{\nu}(s,\mu;x) = \frac{\Gamma(\mu)\Gamma(\mu-s)}{\mu}x^{s-\mu} + \Gamma(\mu)\sum_{M=0}^{\infty}\frac{(-1)^M\sum_{m=0}^{M}(-1)^{M-m}R(M,m,-\nu)\Psi_{\nu-m}(s,\mu+m;0)}{M!}x^M \tag{75}$$
$$(0 \leq \Re(s) < \mu;\ \nu \neq 0,-1,-2,\ldots).$$

4. Concluding Remarks

One important aspect in relation to the analysis of special functions is to study their representations. These special functions can be studied in different regions by using their series, asymptotic, and integral representations. This fact is also important when writing simpler mathematical proofs of known results. Here, we have provided a new series representation of the generalized Bose–Einstein and Fermi–Dirac functions by using a general representation theorem. To accomplish this work, we discussed an analytic continuation for these functions by generalizing the Riemann zeta function from $(0 < \Re(s) < 1)$ to $(0 < \Re(s) < \mu)$. This gives new insights for a possible generalization of the Rieman zeta function

$$\zeta^*_{\mu}(s) := \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\left(\frac{1}{(e^t-1)^{\mu}} - \frac{1}{t^{\mu}}\right)dt\ (0 < \Re(s) < \mu)$$

and will be discussed in more detail in our future research. Our results were validated by obtaining known series representations for the polylogarithm and the Hurwitz–Lerch zeta functions as special cases. A comparison of the known proof of their series representation was given with this new proof. It is hoped that the general representation theorem can also be applied to analyze other special functions.

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