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Article

Riemann–Liouville Operator in Weighted L_p Spaces via the Jacobi Series Expansion

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Abstract: In this paper, we use the orthogonal system of the Jacobi polynomials as a tool to study the Riemann–Liouville fractional integral and derivative operators on a compact of the real axis. This approach has some advantages and allows us to complete the previously known results of the fractional calculus theory by means of reformulating them in a new quality. The proved theorem on the fractional integral operator action is formulated in terms of the Jacobi series coefficients and is of particular interest. We obtain a sufficient condition for a representation of a function by the fractional integral in terms of the Jacobi series coefficients. We consider several modifications of the Jacobi polynomials, which gives us the opportunity to study the invariant property of the Riemann–Liouville operator. In this direction, we have shown that the fractional integral operator acting in the weighted spaces of Lebesgue square integrable functions has a sequence of the included invariant subspaces.

Keywords: fractional derivative; fractional integral; Riemann–Liouville operator; Jacobi polynomials; Legendre polynomials; invariant subspace

MSC: 26A33; 47A15; 47A46; 12E10

Blessed memory of Isai I. Mikaelyan is devoted.

1. Introduction

First, in this paper, we aim to reformulate the well-known theorems on the Riemann–Liouville operator action in terms of the Jacobi series coefficients. Even though this type of problems was well studied by such mathematicians as Rubin B.S. [1–3], Vakulov B.G. [4], Samko S.G. [5,6], and Karapetyants N.K. [7,8] (the results in [1,2,7] are also presented in [9]) in several spaces and for various generalizations of the fractional integral operator, the method suggested in this work allows us to notice interesting properties of the fractional integral and fractional derivative operators. We suggest using properties of the Jacobi polynomials for studying the Riemann–Liouville operator, but we should make a remark that this idea was previously used in the following papers [10–15]. For instance, in the papers [11,12], the operational matrices of the Riemann–Liouville fractional integral and the Caputo fractional derivative for shifted Jacobi polynomials were considered; in the paper [10], the fractional derivative formula was obtained applicably to the general class of polynomials introduced by Srivastava; and, in the paper [13], a general formulation for the fractional-order Legendre functions was constructed to obtain the solution of the fractional order differential equations. Interesting in itself, the fractional calculus theory was applied in [16–18] to study the Jacobi polynomials. However, our main interest lies in a rather different field of studying: the mapping theorems for the Riemann–Liouville operator via the Jacobi polynomials. This approach gives us the advantage of getting results in terms of the Jacobi series coefficients, as well as the concrete achievements. The central point of our method of study is to use the basis property of the Jacobi polynomials system. In this way, we aim to obtain a sufficient condition of existence and uniqueness of the Abel equation solution with the right part belonging to the weighted space of Lebesgue p-th integrable functions. In addition, the usage of the weak topology gives us an opportunity to cover some cases in the mapping theorems that were not previously obtained. Besides, having filled some conditions gaps and formulated the unified result, we aim to systematize the mapping theorems established in the monograph [9].

Secondly, we notice that the question on existence of a non-trivial invariant subspace for an arbitrary linear operator acting in a Hilbert space is still relevant today. In 1935, J. von Neumann proved that an arbitrary non-zero compact operator acting in a Hilbert space has a non-trivial invariant subspace [19]. This approach obtained the further generalizations in the works [20,21], but the established results are based on the compact property of the operator. In the general case, the results in [22,23] are of particular interest. The overview of results in this direction can be found in [24–26]. Due to many difficulties in solving this problem in the general case, some scientists have paid attention to special cases and one of these cases is the Volterra integral operator acting in the space of Lebesgue square-integrable functions on a compact of the real axis. The invariant subspaces of this operator were carefully studied and described in the papers [27–29]. We make an attempt to study invariant subspaces of the Riemann–Liouville fractional integral operator acting in the weighted space of Lebesgue square-integrable functions on a compact of the real axis. In this regard, the following question is relevant: whether the Riemann–Liouville fractional integral has such an invariant subspace on which one would be selfadjoint.

This paper is organized as follows, In Section 2, the auxiliary formulas of fractional calculus are given as well as a brief remark on the Jacobi polynomials system basis property. In Section 3, the main results are presented, the mapping theorems established in the monograph [9] are systematized and reformulated in terms of the Jacoby series coefficients, and the invariant subspaces of the Riemann–Liouville operator are studied. The conclusions are given in Section 4.

2. Preliminaries

2.1. Some Fractional Calculus Formulas

Throughout this paper, we consider complex functions of a real variable; we use the following denotation for weighted complex Lebesgue spaces $L_p(I,\omega)$, $1 \le p < \infty$, where $I = (a,b)$ is an interval of the real axis and the weighted function ω is a real-valued function. In addition, we use the denotation $p' = p/(p-1)$. If $\omega = 1$, then we use the notation $L_p(I)$. Using the denotations of the paper [9], let us define the left-side and right-side fractional integrals and derivatives of real order, respectively,

$$\left(I^{\alpha}_{a+}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int\limits_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt,\ \left(I^{\alpha}_{b-}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int\limits_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\,dt,\ f \in L_1(I);$$

$$\left(D^{\alpha}_{a+}f\right)(x) = \frac{d^n}{dx^n}\left(I^{n-\alpha}_{a+}f\right)(x),\ f \in I^{\alpha}_{a+}(L_1);\ \left(D^{\alpha}_{b-}f\right)(x) = (-1)^n\frac{d^n}{dx^n}\left(I^{n-\alpha}_{a+}f\right)(x),\ f \in I^{\alpha}_{b-}(L_1),$$

$$\alpha \ge 0,\ n = [\alpha]+1,$$

where $I^{\alpha}_{a+}(L_1)$, $I^{\alpha}_{b-}(L_1)$ are the classes of functions, which can be represented by the fractional integrals (see [9] (p. 43)). Further, we use as a domain of definition of the fractional differential operators, mainly the set of polynomials on which these operators are well defined. We use the shorthand notation $L_2 := L_2(I)$ and denote by $(\cdot,\cdot)$ the inner product on the Hilbert space $L_2(I)$. Using Definition 1.5 of [9] (p. 4), we consider the space $H^{\lambda}_0(\bar I, r) := \{f : f(x)r(x) \in H^{\lambda}(\bar I),\ f(a)r(a) = f(b)r(b) = 0\}$ endowed with the norm:

$$\|f\|_{H^{\lambda}_0(\bar I, r)} = \max_{x\in I}|f(x)r(x)| + \sup_{\substack{x_1,x_2\in I\\ x_1\neq x_2}}\frac{|f(x_1)r(x_1) - f(x_2)r(x_2)|}{|x_1-x_2|^{\lambda}},\ r(x) = (x-a)^{\beta}(b-x)^{\gamma},\ \beta,\gamma \in \mathbb{R}.$$

Denote by $C, C_i,\ i \in \mathbb{N}$ positive real constants. We mean that the values of C can be different in various parts of formulas, but the values of $C_i,\ i \in \mathbb{N}$ are certain. We use the following special denotation:

$$\binom{\eta}{\mu} := \Gamma(\eta+1)/\Gamma(\eta-\mu+1),\ \eta,\mu \in \mathbb{R},\ \mu \neq -1,-2,\ldots .$$

Further, we need the following formulas for multiple integrals. Note that under the assumption $\varphi \in L_1(I)$, we have

$$\frac{1}{\Gamma(\alpha-m)} \underbrace{\int_a^x dx \int_a^x dx \ldots \int_a^x}_{\text{m+1 integrals}} \varphi(t)(x-t)^{\alpha-m-1}dt = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\varphi(t)dt;$$

$$\frac{1}{\Gamma(\alpha-m)} \underbrace{\int_x^b dx \int_x^b dx \ldots \int_x^b}_{\text{m+1 integrals}} \varphi(t)(t-x)^{\alpha-m-1}dt = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1}\varphi(t)dt,\ m = \begin{cases} [\alpha],\ \alpha \in \mathbb{R}^+ \setminus \mathbb{N}, \\ [\alpha]-1, \quad \alpha \in \mathbb{N}. \end{cases} \tag{1}$$

Suppose $f(x) \in AC^n(\bar{I}),\ n \in \mathbb{N}$; then, using the previous formulas, we have the representations:

$$f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f^{(n)}(t)dt + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k;$$

$$f(x) = \frac{(-1)^n}{(n-1)!} \int_x^b (t-x)^{n-1} f^{(n)}(t)dt + \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!}(b-x)^k.$$

Now, assume that $n = [\alpha]+1$ in the previous formulas; then, due to Theorem 2.5 of [9] (p. 46) and formulas of the fractional integral of a power function ((2.44) and (2.45) [9] (p. 40)), we have in the left-side case

$$(D^\alpha_{a+}f)(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)}(x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, \tag{2}$$

and, in the right-side case

$$(D^\alpha_{b-}f)(x) = \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(b)}{\Gamma(k+1-\alpha)}(b-x)^{k-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt. \tag{3}$$

2.2. Riemann–Liouville Operator via the Jacobi Polynomials

The orthonormal system of the Jacobi polynomials is denoted by

$$p_n^{(\beta,\gamma)}(x) = \delta_n(\beta,\gamma)\, y_n^{(\beta,\gamma)}(x),\ n \in \mathbb{N}_0,$$

where the normalized multiplier $\delta_n(\beta,\gamma)$ is defined by the formula

$$\delta_n(\beta,\gamma) = (-1)^n \frac{\sqrt{\beta+\gamma+2n+1}}{(b-a)^{n+(\beta+\gamma+1)/2}} \cdot \sqrt{\frac{\Gamma(\beta+\gamma+n+1)}{n!\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}},$$

$$\delta_0(\beta,\gamma) = \frac{1}{\sqrt{\Gamma(\beta+1)\Gamma(\gamma+1)}},\ \beta+\gamma+1=0,$$

and the orthogonal polynomials $y_n^{(\beta,\gamma)}$ are defined by the formula

$$y_n^{(\beta,\gamma)}(x) = (x-a)^{-\beta}(b-x)^{-\gamma}\frac{d^n}{dx^n}\left[(x-a)^{\beta+n}(b-x)^{\gamma+n}\right], \beta,\gamma > -1.$$

For convenience, we use the following functions:

$$\varphi_n^{(\beta,\gamma)}(x) = (x-a)^{n+\beta}(b-x)^{n+\gamma}.$$

If misunderstanding does not appear, we use the shorthand denotations in various parts of this work

$$p_n^{(\beta,\gamma)}(x) := p_n(x),\ y_n^{(\beta,\gamma)}(x) := y_n(x),\ \varphi_n^{(\beta,\gamma)}(x) := \varphi_n(x),\ \delta_n(\beta,\gamma) := \delta_n.$$

In such cases, we would like reader see carefully the denotations corresponding to a concrete paragraph. Specifically, in the case of the Jacobi polynomials, when $\beta = \gamma = 0$, we have the Legendre polynomials. If we consider the Hilbert space $L_2(I)$, then the Legendre orthonormal system has a basis property due to the general property of complete orthonormal systems in Hilbert spaces, but the question on the basis property of the Legendre system for an arbitrary $p \geq 1$, $p \neq 2$ had been still relevant until the first half of the last century. In the direction of solving this problem, the following works are known [30–33]. In particular, in the paper [31], Pollard H. proved that the Legendre system has a basis property in the case $4/3 < p < 4$ and, for the values of $p \in [1,4/3] \cup [4,\infty)$, the Legendre system does not have a basis property in $L_p(I)$ space. The cases $p = 4/3, p = 4$ were considered by Newman J. and Rudin W. in the paper [30], where it is proved that in these cases the Legendre system also does not have a basis property in $L_p(I)$ space. It is worth noting that the criterion of a basis property for the Jacobi polynomials was proved by Pollard H. in the work [33]. In that paper, Pollard H. formulated the theorem proposing that the Jacobi polynomials have a basic property in the space $L_p(I_0,\omega)$, $I_0 := (-1,1)$, $\beta,\gamma \geq -1/2$, $M(\beta,\gamma) < p < m(\beta,\gamma)$ and do not have a basis property, when $p < M(\beta,\gamma)$ or $p > m(\beta,\gamma)$, where

$$m(\beta,\gamma) = 4\min\left\{\frac{\beta+1}{2\beta+1},\frac{\gamma+1}{2\gamma+1}\right\},\ M(\beta,\gamma) = 4\max\left\{\frac{\beta+1}{2\beta+3},\frac{\gamma+1}{2\gamma+3}\right\}.$$

However, this result was subsequently improved by Muckenhoupt B. in the paper [34]. Note that the linear transform

$$l : [-1,1] \to [a,b],\ y = \frac{b-a}{2}x + \frac{b+a}{2}$$

shows us that all results of the orthonormal polynomials theory obtained for the segment $[-1,1]$ are true for the segment $[a,b] \subset \mathbb{R}$. We use the denotation $S_k f := \sum_{n=0}^{k} f_n p_n^{(\beta,\gamma)}$, $k \in \mathbb{N}_0$, where f_n are the Jacobi series coefficients of the function f.

Consider the orthonormal Jacobi polynomials

$$p_n^{(\beta,\gamma)}(x) = \delta_n y_n(x) = \delta_n (x-a)^{-\beta}(b-x)^{-\gamma}\varphi_n^{(n)}(x),\ \beta,\gamma > -1/2,\ n \in \mathbb{N}_0.$$

Further, we need some formulas. Using the Leibnitz formula, we get

$$y_n(x) = \sum_{i=0}^{n}(-1)^i C_n^i \binom{n+\beta}{n-i}(x-a)^i\binom{n+\gamma}{i}(b-x)^{n-i} = \sum_{i=0}^{n}(-1)^{n+i}C_n^i\binom{n+\beta}{i}(x-a)^{n-i}\binom{n+\gamma}{n-i}(b-x)^i. \quad (4)$$

Using again the Leibnitz formula, we obtain

$$y_n^{(k)}(x) = \sum_{i=0}^{n}(-1)^i C_n^i\binom{n+\beta}{n-i}\binom{n+\gamma}{i}\sum_{j=c}^{i}(-1)^{k+j}C_k^j\binom{i}{j}(x-a)^{i-j}\binom{n-i}{k-j}(b-x)^{n+j-i-k}, \quad (5)$$

where $c = \max\{0, k+i-n\}$, $k \le n$.

In accordance with Equation (5), we have

$$y_n^{(k)}(a) = (-1)^k (b-a)^{n-k} \sum_{i=0}^{n} C_n^i \binom{n+\beta}{n-i}\binom{n+\gamma}{i} C_k^i \binom{n-i}{k-i} i!,\ k \le n; \tag{6}$$

$$p_n^{(k)}(a) = \frac{(-1)^{n+k}\sqrt{\beta+\gamma+2n+1}}{(b-a)^{k+(\beta+\gamma+1)/2}} \cdot \sqrt{\frac{\Gamma(\beta+\gamma+n+1)}{n!\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}} \sum_{i=0}^{n} C_n^i \binom{n+\beta}{n-i}\binom{n+\gamma}{i} C_k^i \binom{n-i}{k-i} i!,\ k \le n.$$

In the same way, we get

$$y_n^{(k)}(x) = \sum_{i=0}^{n} (-1)^{n+i} C_n^i \binom{n+\beta}{i}\binom{n+\gamma}{n-i} \sum_{j=c}^{i} (-1)^i C_k^j \binom{n-i}{k-j} (x-a)^{n+j-i-k} \binom{i}{j} (b-x)^{i-j},\ k \le n. \tag{7}$$

Hence,

$$y_n^{(k)}(b) = (-1)^n (b-a)^{n-k} \sum_{i=0}^{n} C_n^i \binom{n+\beta}{i}\binom{n+\gamma}{n-i} C_k^i \binom{n-i}{k-i} i!,\ k \le n; \tag{8}$$

$$p_n^{(k)}(b) = \frac{\sqrt{\beta+\gamma+2n+1}}{n!(b-a)^{k+(\beta+\gamma+1)/2}} \cdot \sqrt{\frac{n!\Gamma(\beta+\gamma+n+1)}{\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}} \sum_{i=0}^{n} C_n^i \binom{n+\beta}{n-i}\binom{n+\gamma}{i} C_k^i \binom{n-i}{k-i} i!,\ k \le n.$$

Let $\mathfrak{C}_n^{(k)}(\beta,\gamma) := (-1)^{n+k} p_n^{(k)}(a)(b-a)^k$, then $p_n^{(k)}(b)(b-a)^k = \mathfrak{C}_n^{(k)}(\gamma,\beta)$. Using the Taylor series expansion for the Jacobi polynomials, we get

$$p_n^{(\beta,\gamma)}(x) = \sum_{k=0}^{n} (-1)^{n+k} (b-a)^{-k} \frac{\mathfrak{C}_n^{(k)}(\beta,\gamma)}{k!} (x-a)^k = \sum_{k=0}^{n} (-1)^k (b-a)^{-k} \frac{\mathfrak{C}_n^{(k)}(\gamma,\beta)}{k!} (b-x)^k.$$

Applying Formulas (2.44) and (2.45) of the fractional integral and derivative of a power function in [9] (p. 40), we obtain

$$(I_{a+}^{\alpha} p_n)(x) = \sum_{k=0}^{n} (-1)^{n+k} (b-a)^{-k} \frac{\mathfrak{C}_n^{(k)}(\beta,\gamma)}{\Gamma(k+1+\alpha)} (x-a)^{k+\alpha},$$

$$(I_{b-}^{\alpha} p_n)(x) = \sum_{k=0}^{n} (-1)^k (b-a)^{-k} \frac{\mathfrak{C}_n^{(k)}(\gamma,\beta)}{\Gamma(k+1+\alpha)} (b-x)^{k+\alpha},\ \alpha \in (-1,1),$$

where we use the formal denotation $I_{a+}^{-\alpha} := D_{a+}^{\alpha}$. Thus, using integration by parts, we get

$$\int_a^b p_m(x)(I_{a+}^{\alpha} p_n)(x)\omega(x)dx = \delta_m \int_a^b \varphi_m^{(m)}(x)(I_{a+}^{\alpha} p_n)(x)dx =$$

$$= -\delta_m \int_a^b \varphi_m^{(m-1)}(x)(I_{a+}^{\alpha} p_n)^{(1)}(x)dx = (-1)^m \delta_m \int_a^b \varphi_m(x)(I_{a+}^{\alpha} p_n)^{(m)}(x)dx =$$

$$= (-1)^m \delta_m \int_a^b \sum_{k=0}^{n} (-1)^{n+k} (b-a)^{-k} \frac{\mathfrak{C}_n^{(k)}(\beta,\gamma)}{\Gamma(k+\alpha-m+1)} (x-a)^{k+\alpha+\beta} (b-x)^{m+\gamma} dx =$$

$$= (-1)^n \hat{\delta}_m \sum_{k=0}^{n} (-1)^k \frac{\mathfrak{C}_n^{(k)}(\beta,\gamma) B(\alpha+\beta+k+1, \gamma+m+1)}{\Gamma(k+\alpha-m+1)},$$

where

$$\hat{\delta}_m = (b-a)^{\alpha+(\beta+\gamma+1)/2}\sqrt{\frac{(\beta+\gamma+2m+1)\Gamma(\beta+\gamma+m+1)}{m!\Gamma(\beta+m+1)\Gamma(\gamma+m+1)}}.$$

In the same way, we get

$$(p_m, I^\alpha_{b-}p_n)_{L_2(I,\omega)} = (-1)^m\hat{\delta}_m\sum_{k=0}^{n}(-1)^k\frac{\mathfrak{C}^{(k)}_n(\gamma,\beta)B(\alpha+\gamma+k+1,\beta+m+1)}{\Gamma(k+\alpha-m+1)}.$$

Using the denotation

$$A^{\alpha,\beta,\gamma}_{mn} := \hat{\delta}_m\sum_{k=0}^{n}(-1)^k\frac{\mathfrak{C}^{(k)}_n(\beta,\gamma)B(\alpha+\beta+k+1,\gamma+m+1)}{\Gamma(k+\alpha-m+1)},$$

we have

$$(p_m, I^\alpha_{a+}p_n)_{L_2(I,\omega)} = (-1)^n A^{\alpha,\beta,\gamma}_{mn},\;\; (p_m, I^\alpha_{b-}p_n)_{L_2(I,\omega)} = (-1)^m A^{\alpha,\gamma,\beta}_{mn}. \tag{9}$$

We claim the following formulas without any proof because of the absolute analogy with the proof corresponding to the fractional integral operators

$$(p_m, D^\alpha_{a+}p_n)_{L_2(I,\omega)} = (-1)^n A^{-\alpha,\beta,\gamma}_{mn},\;\; (p_m, D^\alpha_{b-}p_n)_{L_2(I,\omega)} = (-1)^m A^{-\alpha,\gamma,\beta}_{mn}. \tag{10}$$

Further, we use the following denotations:

$$A^{\alpha,\beta,\gamma}_{+} := \begin{pmatrix} A^{\alpha,\beta,\gamma}_{00} & -A^{\alpha,\beta,\gamma}_{01} & \ldots \\ A^{\alpha,\beta,\gamma}_{10} & -A^{\alpha,\beta,\gamma}_{11} & \ldots \\ . & & \\ . & & \\ . & & \ldots \end{pmatrix},\;\; A^{\alpha,\gamma,\beta}_{-} := \begin{pmatrix} A^{\alpha,\gamma,\beta}_{00} & A^{\alpha,\gamma,\beta}_{01} & \ldots \\ -A^{\alpha,\gamma,\beta}_{10} & -A^{\alpha,\gamma,\beta}_{11} & \ldots \\ . & & \\ . & & \\ . & & \ldots \end{pmatrix}, \alpha\in\mathbb{R}. \tag{11}$$

This allows us to consider the integro-differential operators in the matrix form of notation.

Throughout this paper, the results are formulated and proved for the left-side case. One may reformulate them for the right-side case with no difficulty.

3. Main Results

3.1. Mapping Theorems

The following lemma aims to establish more simplified and at the same time applicable form of the results proven in Theorem 3.10 of [9] (p. 78) and Theorem 3.12 of [9] (p. 81), and is devoted to the description of the operator I^α_{a+} action in the space $L_p(I,\omega)$. More precisely, these theorems describe the action $I^\alpha_{a+} : L_p(I,\omega) \to L_q(I,r)$ with rather inconveniently formulated conditions, from the point of view of operator theory, regarding to the weighted functions and indexes p, q. To justify this claim, we can easily see that there are some cases in the theorems conditions for which the bounded action $I^\alpha_{a+} : L_p(I,\omega) \to L_p(I,\omega)$, $\alpha\in(0,1)$, $\omega(x) = (x-a)^\beta(b-x)^\gamma$ does not follow easily from the theorems, for instance in the case $2 < p < 1/(1-\alpha)$, $\beta\in\mathbb{R}$, $0<\gamma\leq\alpha p-1$, the above mentioned bounded action of I^α_{a+} cannot be obtained by using the theorems and estimating, as we show below that the proof of this fact requires involving the weak topology methods.

Lemma 1. *Suppose* $\omega(x) = (x-a)^\beta(b-x)^\gamma$, $\beta,\gamma\in[-1/2,1/2]$, $M(\beta,\gamma) < p < m(\beta,\gamma)$; *then,*

$$\|I^\alpha_{a+}f\|_{L_p(I,\omega)} \leq C\|f\|_{L_p(I,\omega)},\; f\in L_p(I,\omega),\; \alpha\in(0,1). \tag{12}$$

Proof. By direct calculation, we can verify that β satisfies the inequality $2t^2 + t - 1 < 0$. We see that

$$2t^2 + t - 1 \leq 0;\ 2t^2 + 3t \leq 2t + 1;\ t \leq \frac{2t+1}{2t+3};\ t + 1 \leq 4\frac{t+1}{2t+3}.$$

Let us substitute β for t, we have

$$\beta + 1 \leq 4\frac{\beta+1}{2\beta+3} \leq M(\beta,\gamma) < p.$$

Hence, $\beta < p - 1$. We have absolutely analogous reasoning for γ i.e., $\gamma < p - 1$. Let us consider the various relations between p and α.

(i) $p < 1/\alpha$. If $\gamma > \alpha p - 1$, then, in accordance with Theorem 3.10 of [9] (p. 78), we get

$$\|I^{\alpha}_{a+}f\|_{L_q(I,r)} \leq C\|f\|_{L_p(I,\omega)},\ q = p/(1-\alpha p),\ r(x) = (x-a)^{\frac{\beta q}{p}}(b-x)^{\frac{\gamma q}{p}}. \tag{13}$$

Using the Hölder inequality, we obtain

$$\left(\int_a^b |I^{\alpha}_{a+}f|^p\,\omega\,dx\right)^{\frac{1}{p}} = \left(\int_a^b \left|\omega^{\frac{1}{p}} I^{\alpha}_{a+}f\right|^p dx\right)^{\frac{1}{p}} \leq C\left(\int_a^b |I^{\alpha}_{a+}f|^q\,\omega^{\frac{q}{p}}dx\right)^{\frac{1}{q}} = C\left(\int_a^b |I^{\alpha}_{a+}f|^q\,r dx\right)^{\frac{1}{q}}.$$

Combining this inequality with Equation (13), we obtain Equation (12). If $\gamma \leq \alpha p - 1$, then we have the following reasoning

$$\left(\int_a^b |I^{\alpha}_{a+}f|^p\,\omega\,dx\right)^{\frac{1}{p}} = \left(\int_a^b \left|(x-a)^{\frac{\beta}{p}} I^{\alpha}_{a+}f(x)\right|^p (b-x)^{\gamma}\,dx\right)^{\frac{1}{p}} =$$

$$= \left(\int_a^b \left|(x-a)^{\frac{\beta}{p}} I^{\alpha}_{a+}f(x)\right|^p (b-x)^{\frac{\gamma}{\xi}}(b-x)^{\frac{\gamma}{\xi'}}\,dx\right)^{\frac{1}{p}} = I_1,\ \xi = 1/(1-\alpha p).$$

Using the Hölder inequality, we get

$$I_1 = \left(\int_a^b \left|(x-a)^{\frac{\beta}{p}}(b-x)^{\frac{\gamma}{p\xi}} I^{\alpha}_{a+}f(x)\right|^p (b-x)^{\frac{\gamma}{\xi'}}\,dx\right)^{\frac{1}{p}} \leq$$

$$\leq \left(\int_a^b \left|(x-a)^{\frac{\beta}{p}}(b-x)^{\frac{\gamma}{p\xi}} I^{\alpha}_{a+}f(x)\right|^{p\xi} dx\right)^{\frac{1}{p\xi}} \times \left(\int_a^b (b-x)^{\gamma}\,dx\right)^{\frac{1}{p\xi'}} =$$

$$C\left(\int_a^b |I^{\alpha}_{a+}f(x)|^q (x-a)^{\frac{\beta q}{p}}(b-x)^{\gamma}\,dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b |I^{\alpha}_{a+}f(x)|^q (x-a)^{\frac{\beta q}{p}}(b-x)^{\nu}\,dx\right)^{\frac{1}{q}},\ -1 < \nu < \gamma.$$

Applying Theorem 3.10 of [9] (p. 78), we obtain

$$\left(\int_a^b |I^{\alpha}_{a+}f(x)|^q (x-a)^{\frac{\beta q}{p}}(b-x)^{\nu}\,dx\right)^{\frac{1}{q}} \leq C\|f\|_{L_p(I,\omega)}.$$

Hence, Equation (12) is fulfilled.

(ii) $1/\alpha < p$. We have several cases.

(a) $\gamma \leq 0$ or $\gamma > \alpha p - 1$. If $\gamma \leq 0$, then applying Theorem 3.8 of [9] (p. 74), we obtain

$$\|I_{a+}^{\alpha} f\|_{H_0^{\alpha-1/p}(I,r_1)} \leq C \left(\int_a^b |f(x)|^p (x-a)^{\beta} dx \right)^{\frac{1}{p}} \leq C \left(\int_a^b |f(x)|^p \,\omega(x) dx \right)^{\frac{1}{p}},$$

where $r_1(x) = (x-a)^{\frac{\beta}{p}}$. We have the following estimate:

$$\left(\int_a^b |I_{a+}^{\alpha} f|^p \,\omega(x) dx \right)^{\frac{1}{p}} = \left(\int_a^b \left| (x-a)^{\frac{\beta}{p}} I_{a+}^{\alpha} f \right|^p (b-x)^{\gamma} dx \right)^{\frac{1}{p'}} \leq$$

$$\leq \|I_{a+}^{\alpha} f\|_{H_0^{\alpha-1/p}(I,r_1)} \left(\int_a^b (b-x)^{\gamma} dx \right)^{\frac{1}{p}} = C \|I_{a+}^{\alpha} f\|_{H_0^{\alpha-1/p}(I,r_1)}.$$

Hence, Equation (12) is fulfilled. If $\gamma > \alpha p - 1$, then we get

$$\left(\int_a^b |I_{a+}^{\alpha} f(x)|^p \,\omega(x) dx \right)^{\frac{1}{p}} = \left(\int_a^b \left| (x-a)^{\frac{\beta}{p}} (b-x)^{\frac{\gamma}{p}} I_{a+}^{\alpha} f(x) \right|^p dx \right)^{\frac{1}{p}} \leq C \|I_{a+}^{\alpha} f(x)\|_{H_0^{\alpha-1/p}(I,r_1)},$$

where $r_1(x) = (x-a)^{\frac{\beta}{p}} (b-x)^{\frac{\gamma}{p}}$. Applying Theorem 3.12 [9] (p. 81), we obtain

$$\|I_{a+}^{\alpha} f\|_{H_0^{\alpha-1/p}(I,r_1)} \leq C \left(\int_a^b |f(x)|^p \,\omega(x) dx \right)^{\frac{1}{p}}.$$

Hence, Equation (12) is fulfilled.

(b) $p \leq 2$, $0 < \gamma \leq \alpha p - 1$. As a consequence of the condition $p \leq 2$, we get $\gamma - (\alpha p - 1) > -1$. We obtain the estimate

$$\left(\int_a^b |I_{a+}^{\alpha} f|^p \,\omega(x) dx \right)^{\frac{1}{p}} = \left(\int_a^b \left| (x-a)^{\frac{\beta}{p}} (b-x)^{\frac{\theta}{p}} I_{a+}^{\alpha} f(x) \right|^p (b-x)^{\gamma-\theta} dx \right)^{\frac{1}{p}} \leq$$

$$\leq \|I_{b-}^{\alpha} \varphi_m\|_{H_0^{\alpha-1/p}(I,r_1)} \left(\int_a^b (b-x)^{\gamma-\theta} dx \right)^{\frac{1}{p}},$$

where $\theta = (\alpha p - 1) + p\,\delta$, $\delta > 0$, $r_1(x) = (x-a)^{\frac{\beta}{p}} (b-x)^{\frac{\theta}{p}}$. Taking into account that $\gamma - (\alpha p - 1) > -1$, we get for sufficiently small $\delta > 0$

$$\left(\int_a^b (b-x)^{\gamma-\theta} dx \right)^{\frac{1}{p}} < \infty.$$

Applying Theorem 3.12 of [9] (p. 81), we obtain

$$\|I_{a+}^{\alpha} f\|_{H_0^{\alpha-1/p}(I,r_1)} \leq C \left(\int_a^b |f(x)|^p \,\omega(x) dx \right)^{\frac{1}{p}}.$$

Hence, Equation (12) is fulfilled.
(c) $p > 2$, $0 < \gamma \le \alpha p - 1$. In this case, we should consider various subcases.
(1) $p' > 1/\alpha$. If $\beta \ge 0$, then we note that $\varphi_m^{(m)}(x)(b-x)^{-\gamma} \in L_\infty(I)$. Hence,

$$\int_a^b |\varphi_m^{(m)}(x)|^{p'}(b-x)^{\gamma(1-p')}dx < \infty. \tag{14}$$

It is easily shown that

$$\left(\int_a^b \left|I_{b-}^\alpha \varphi_m^{(m)}\right|^{p'} \omega^{1-p'}(x)dx\right)^{\frac{1}{p'}} = \left(\int_a^b \left|(b-x)^{\frac{\gamma(1-p')}{p'}} I_{b-}^\alpha \varphi_m^{(m)}(x)\right|^{p'} (x-a)^{\beta(1-p')}dx\right)^{\frac{1}{p'}} \le$$

$$\le \|I_{b-}^\alpha \varphi_m^{(m)}(x)\|_{H_0^{\alpha-1/p'}(I,r_1)} \left(\int_a^b (x-a)^{\beta(1-p')}dx\right)^{\frac{1}{p'}},$$

where $r_1(x) = (b-x)^{\gamma(1-p')/p'}$. Solving the quadratic equality, we can verify that, under the assumptions $0 < \beta \le 1/2$, we have $4(\beta+1)/(2\beta+1) \le (\beta+1)/\beta$. Since it can easily be checked that $p' < m(\beta,\gamma) \le 4(\beta+1)/(2\beta+1)$, then $p' < (\beta+1)/\beta$ or $\beta(1-p') > -1$. Hence.

$$\left(\int_a^b \left|I_{b-}^\alpha \varphi_m^{(m)}\right|^{p'} \omega^{1-p'}(x)dx\right)^{\frac{1}{p'}} \le C\|I_{b-}^\alpha \varphi_m^{(m)}(x)\|_{H_0^{\alpha-1/p'}(I,r_1)}.$$

It is obvious that $\gamma(1-p') < p'-1$. Combining the relation in Equation (14) and Theorem 3.8 of [9] (p. 74), we obtain

$$\|I_{b-}^\alpha \varphi_m^{(m)}(x)\|_{H_0^{\alpha-1/p'}(I,r_1)} \le C\left(\int_a^b \left|\varphi_m^{(m)}(x)\right|^{p'}(b-x)^{\gamma(1-p')}dx\right)^{\frac{1}{p'}} < \infty.$$

Since $\beta(1-p') \le 0$, then

$$\left(\int_a^b \left|\varphi_m^{(m)}(x)\right|^{p'}(b-x)^{\gamma(1-p')}dx\right)^{\frac{1}{p'}} \le (b-a)^{\beta(p'-1)}\left(\int_a^b \left|\varphi_m^{(m)}(x)\right|^{p'}(x-a)^{\beta(1-p')}(b-x)^{\gamma(1-p')}dx\right)^{\frac{1}{p'}} =$$

$$= (b-a)^{\beta(p'-1)}\left(\int_a^b |p_m(x)|^{p'}(x-a)^\beta(b-x)^\gamma dx\right)^{\frac{1}{p'}}.$$

Taking into account the above considerations, we obtain

$$\left(\int_a^b \left|I_{b-}^\alpha \varphi_m^{(m)}\right|^{p'} \omega^{1-p'}(x)dx\right)^{\frac{1}{p'}} \le C\|p_m\|_{L_{p'}(I,\omega)},\ m \in \mathbb{N}_0. \tag{15}$$

Thus, we get $\omega^{-1} I_{b-}^\alpha \varphi_m^{(m)} \in L_{p'}(I,\omega)$. Using the Hölder inequality and the previous reasoning, we get

$$I_2 = \left|\int_a^b f(x)dx\int_x^b \varphi_m^{(m)}(t)(t-x)^{\alpha-1}dt\right| = \left|\int_a^b \left\{\omega^{-1}(x)\int_x^b \varphi_m^{(m)}(t)(t-x)^{\alpha-1}dt\right\} f(x)\,\omega(x)dx\right| \le$$

$$\leq \left\{ \int_a^b |f(x)|^p \omega(x) dx \right\}^{1/p} \left\{ \int_a^b \left| \omega^{-1}(x) \int_x^b \varphi_m^{(m)}(t)(t-x)^{\alpha-1} dt \right|^{p'} \omega(x) dx \right\}^{1/p'} \leq$$

$$\leq C \|f\|_{L_p(I,\omega)} \|p_m\|_{L_{p'}(I,\omega)} < \infty, \ f \in L_p(I,\omega), \ m \in \mathbb{N}_0.$$

Hence, in accordance with the consequence of the Fubini theorem, we get

$$(I_{a+}^{\alpha} f, p_m)_{L_2(I,\omega)} = \left(f, \omega^{-1} I_{b-}^{\alpha} \varphi_m^{(m)} \right)_{L_2(I,\omega)}, \ m \in \mathbb{N}_0. \tag{16}$$

Consider the functional

$$l_f(p_m) = (I_{a+}^{\alpha} f, p_m)_{L_2(I,\omega)} = \left(f, \omega^{-1} I_{b-}^{\alpha} \varphi_m^{(m)} \right)_{L_2(I,\omega)}.$$

Applying Equation (15), we obtain

$$|l_f(p_m)| \leq C \|f\|_{L_p(I,\omega)} \|p_m\|_{L_{p'}(I,\omega)}, \ m \in \mathbb{N}_0. \tag{17}$$

We see that the previous inequality is true for all linear combinations

$$|l_f(\mathcal{L}_m)| \leq C \|f\|_{L_p(I,\omega)} \|\mathcal{L}_m\|_{L_{p'}(I,\omega)}, \ \mathcal{L}_m := \sum_{n=0}^{m} c_n p_n, \ c_n = \text{const}, \ m \in \mathbb{N}_0. \tag{18}$$

Since it can easily be checked that $M(\beta,\gamma) < p' < m(\beta,\gamma)$, then, in accordance with the results of the paper [33], the system $\{p_m\}_0^{\infty}$ has a basis property in the space $L_{p'}(I,\, \omega)$. Using this fact, we pass to the limit in both sides of the inequality in Equation (18), thus we get

$$|l_f(g)| \leq C \|f\|_{L_p(I,\omega)} \|g\|_{L_{p'}(I,\omega)}, \ \forall g \in L_{p'(I,\omega)}. \tag{19}$$

In the terms of the given above denotation, we can write

$$|\, (I_{a+}^{\alpha} f, g)_{L_2(I,\omega)} \,| \leq C \|f\|_{L_p(I,\omega)} \|g\|_{L_{p'}(I,\omega)}, \ \forall g \in L_{p'(I,\omega)}.$$

In its turn, this inequality can be rewritten in the following form:

$$\left| \left(\frac{I_{a+}^{\alpha} f}{\|f\|_{L_p(I,\omega)}}, g \right)_{L_2(I,\omega)} \right| \leq C \|g\|_{L_{p'}(I,\omega)}, \ \forall g \in L_{p'(I,\omega)}.$$

Hence, the set

$$\mathcal{F} := \left\{ \frac{I_{a+}^{\alpha} f}{\|f\|_{L_p(I,\omega)}}, \ f \in L_p(I,\omega) \right\}$$

is weekly bounded. Therefore, in accordance with the well-known theorem, this set is bounded with respect to the norm $L_p(I,\omega)$. It implies that Equation (12) holds. If $\beta < 0$, then it is easy to show that $\beta(1-p') - \alpha p' + 1 > -1$. Under the assumptions $\beta(p'-1) \leq \alpha p' - 1$, we have

$$\left(\int_a^b |I_{b-}^{\alpha} \varphi_m|^{p'} \omega^{1-p'}(x) dx \right)^{\frac{1}{p'}} = \left(\int_a^b \left| (x-a)^{\frac{\theta}{p'}} (b-x)^{\frac{\gamma(1-p')}{p'}} I_{b-}^{\alpha} \varphi_m(x) \right|^{p'} (x-a)^{\beta(1-p')-\theta} dx \right)^{\frac{1}{p'}} \leq$$

$$\leq \|I^{\alpha}_{b-}\varphi_m(x)\|_{H_0^{\alpha-1/p'}(I,r_1)} \left(\int\limits_a^b (x-a)^{\beta(1-p')-\theta} dx\right)^{\frac{1}{p'}},$$

where $\theta = (\alpha p' - 1) + p'\delta$, $\delta > 0$, $r_1(x) = (x-a)^{\theta/p'}(b-x)^{\gamma(1-p')/p'}$. Hence, using the condition $\beta(1-p') - \theta > -1$, we get for sufficiently small $\delta > 0$

$$\left(\int\limits_a^b \left|I^{\alpha}_{b-}\varphi_m\right|^{p'} \omega^{1-p'}(x)dx\right)^{\frac{1}{p'}} \leq C\|I^{\alpha}_{b-}\varphi_m(x)\|_{H_0^{\alpha-1/p'}(I,r_1)}.$$

On the other hand, under the assumptions $\beta(1-p') > \alpha p' - 1$, we can evaluate directly

$$\left(\int\limits_a^b \left|I^{\alpha}_{b-}\varphi_m\right|^{p'} \omega^{1-p'}(x)dx\right)^{\frac{1}{p'}} = \left(\int\limits_a^b \left|(x-a)^{\frac{\beta(1-p')}{p'}}(b-x)^{\frac{\gamma(1-p')}{p'}} I^{\alpha}_{b-}\varphi_m(x)\right|^{p'} dx\right)^{\frac{1}{p'}} \leq$$

$$\leq C\|I^{\alpha}_{b-}\varphi_m(x)\|_{H_0^{\alpha-1/p'}(I,r_1)},$$

where $r_1(x) = (x-a)^{\frac{\beta(1-p')}{p'}}(b-x)^{\frac{\gamma(1-p')}{p'}}$. Applying Theorem 3.12 of [9] (p. 81), we obtain

$$\|I^{\alpha}_{b-}\varphi_m(x)\|_{H_0^{\alpha-1/p'}(I,r_1)} \leq C\left(\int\limits_a^b |\varphi_m(x)|^{p'} \omega^{1-p'}(x)dx\right)^{\frac{1}{p'}} = \left(\int\limits_a^b |p_m(x)|^{p'} \omega(x)dx\right)^{\frac{1}{p'}}.$$

Hence, the inequality in Equation (15) holds. Arguing as above, we obtain Equation (12).

(2) $p' < 1/\alpha$. Applying the reasoning used in (i), we obtain easily Equation (15). Further, we get Equation (12) in the way considered above.

(iii) $\alpha = 1/p$. We already know that, due to the condition $M(\beta,\gamma) < p < m(\beta,\gamma)$, we have $\beta,\gamma < p-1$. Let $p_1 = p - \varepsilon$, $\varepsilon > 0$, $\beta,\gamma < p_1 - 1$. If $\gamma \geq 0$, then we should use the following reasoning:

$$\left(\int\limits_a^b |I^{1/p}_{a+}f|^p \omega dx\right)^{\frac{1}{p}} = \left(\int\limits_a^b |\omega^{\frac{1}{p_1}} I^{1/p}_{a+}f|^p \omega^{1-\frac{p}{p_1}} dx\right)^{\frac{1}{p}} \leq C\left(\int\limits_a^b |I^{1/p}_{a+}f|^q \omega^{\frac{q}{p_1}} dx\right)^{\frac{1}{q}} \left(\int\limits_a^b \omega^{\left(1-\frac{p}{p_1}\right)\xi'} dx\right)^{\frac{1}{p\xi'}},$$

where $q = p\,\xi$, $\xi = p_1/p(1 - p_1 p^{-1})$. Thus, for sufficiently small ε, we obtain

$$\int\limits_a^b \omega^{\left(1-\frac{p}{p_1}\right)\xi'} dx < \infty.$$

Taking into account that $\gamma > p_1 p^{-1} - 1$ and applying Theorem 3.10 of [9] (p. 78), we get

$$\left(\int\limits_a^b |I^{1/p}_{a+}f(x)|^p \omega(x)dx\right)^{\frac{1}{p}} \leq C\left(\int\limits_a^b |I^{1/p}_{a+}f(x)|^q \omega^{\frac{q}{p_1}}(x)dx\right)^{\frac{1}{q}} \leq C\left(\int\limits_a^b |f(x)|^{p_1} \omega(x)dx\right)^{\frac{1}{p_1}}.$$

Thus, noticing that $\|f\|_{L_{p_1}(I,\omega)} \leq C\|f\|_{L_p(I,\omega)}$, we obtain Equation (12). If $\gamma < 0$, then we can choose ε so that $\gamma < p_1 p^{-1} - 1$. We have the following reasoning:

$$\left(\int\limits_a^b \left|I^{1/p}_{a+}f\right|^p \omega\, dx\right)^{\frac{1}{p}} = \left(\int\limits_a^b \left|(x-a)^{\frac{\beta}{p_1}} I^{1/p}_{a+}f(x)\right|^p (x-a)^{\beta\left(1-\frac{p}{p_1}\right)}(b-x)^{\gamma\left(\frac{1}{\xi}+\frac{1}{\xi'}\right)} dx\right)^{\frac{1}{p}} = I_1,$$

where $q = p\,\xi$, $\xi = p_1/p(1-p_1p^{-1})$. Using the Hölder inequality, we get

$$I_1 = \left(\int_a^b \left| (x-a)^{\frac{\beta}{p_1}} (b-x)^{\frac{\gamma}{p\xi}} I_{a+}^{1/p} f(x) \right|^p (x-a)^{\beta\left(1-\frac{p}{p_1}\right)} (b-x)^{\frac{\gamma}{\xi'}} \, dx \right)^{\frac{1}{p}} \leq$$

$$\leq \left(\int_a^b \left| (x-a)^{\frac{\beta}{p_1}} (b-x)^{\frac{\gamma}{p\xi}} I_{a+}^{1/p} f(x) \right|^{p\xi} dx \right)^{\frac{1}{p\xi}} \times \left(\int_a^b (x-a)^{\beta\left(1-\frac{p}{p_1}\right)\xi'} (b-x)^{\gamma} \, dx \right)^{\frac{1}{p\xi'}}.$$

We can choose ε so that we have $\beta\,(1-p/p_1)\,\xi' > -1$. Therefore,

$$I_1 \leq C \left(\int_a^b \left| (x-a)^{\frac{\beta}{p_1}} (b-x)^{\frac{\gamma}{p\xi}} I_{a+}^{1/p} f(x) \right|^{p\xi} dx \right)^{\frac{1}{p\xi}} = C \left(\int_a^b |I_{a+}^{\alpha} f(x)|^q \, (x-a)^{\frac{\beta q}{p_1}} (b-x)^{\gamma} \, dx \right)^{\frac{1}{q}} \leq$$

$$\leq C \left(\int_a^b |I_{a+}^{\alpha} f(x)|^q \, (x-a)^{\frac{\beta q}{p_1}} (b-x)^{\nu} \, dx \right)^{\frac{1}{q}}, \; -1 < \nu < \gamma.$$

Applying Theorem 3.10 of [9] (p. 78), we get

$$\left(\int_a^b |I_{a+}^{\alpha} f(x)|^q \, (x-a)^{\frac{\beta q}{p}} (b-x)^{\nu} \, dx \right)^{\frac{1}{q}} \leq C\|f\|_{L_{p_1}(I,\omega)}.$$

Taking into account that $\|f\|_{L_{p_1}(I,\omega)} \leq C\|f\|_{L_p(I,\omega)}$, we obtain Equation (12). □

The results of the monograph [9] (see Chapter 1) give us a description of the fractional integral mapping properties in the space $L_p(I,\omega)$, $1 < p < \infty$, $p \neq 1/\alpha$, where ω is some power function. Actually, the following question is still relevant: What happens in the case $p = 1/\alpha$? In the non-weighted case, the approach to this question is given in the paper [35]. In addition, it is found in a more convenient form in the monograph [9] (p. 92), where the following inequality is given:

$$\|I_{a+}^{1/p} f\|^* \leq C\|f\|_{L_p(I)},$$

where

$$\|f\|^* = \sup_{J \subset I} m_J f\,, \; m_J f = \frac{1}{|J|} \int_J |f(x) - f_J| dx, \; f_J = \int_J f(x) dx.$$

It is remarkable that there is no mention on the weighted case in the historical review of the monograph [9]. In contrast to the said mentioned approaches, we obtain a description of the fractional integral mapping properties in the space $L_p(I,\omega)$ in terms of the Jacobi series coefficients. This approach is principally different from ones used in [9], in particular it allows us to avoid problems confected with the case $p = 1/\alpha$. Further, in this section, we deal with the normalized Jacobi polynomials $p_n^{(\alpha,\beta)}$, $n \in \mathbb{N}_0$.

Theorem 1. *Suppose*

$$\psi \in L_p(I,\omega), \; \omega(x) = (x-a)^{\beta}(b-x)^{\gamma}, \; \beta,\gamma \in [-1/2, 1/2]\,, \; M(\beta,\gamma) < p < m(\beta,\gamma); \tag{20}$$

then,

$$I_{a+}^{\alpha} \psi = f, \; \alpha \in (0,1),$$

where

$$f_m = \sum_{n=0}^{\infty} (-1)^n \psi_n A_{mn}^{\alpha,\beta,\gamma}, \; m \in \mathbb{N}_0.$$

This theorem can be formulated in the matrix form

$$A_{+}^{\alpha,\beta,\gamma} \times \psi = f, \quad \sim \quad \begin{pmatrix} A_{00}^{\alpha,\beta,\gamma} & -A_{01}^{\alpha,\beta,\gamma} & \ldots \\ A_{10}^{\alpha,\beta,\gamma} & -A_{11}^{\alpha,\beta,\gamma} & \ldots \\ \cdot & & \\ \cdot & & \\ \cdot & & \ldots \end{pmatrix} \times \begin{pmatrix} \psi_0 \\ \psi_1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}.$$

Proof. Note that, according to the results of the paper [33], the system of the normalized Jacobi polynomials has a basis property in $L_p(I,\omega)$, $M(\beta,\gamma) < p < m(\beta,\gamma)$. Hence,

$$\sum_{n=0}^{l} \psi_n p_n \stackrel{L_p(I,\omega)}{\longrightarrow} \psi \in L_p(I,\omega), \; l \to \infty.$$

Using Lemma 1, we obtain

$$\sum_{n=0}^{l} \psi_n I_{a+}^{\alpha} p_n \stackrel{L_p(I,\omega)}{\longrightarrow} I_{a+}^{\alpha} \left(\sum_{n=0}^{\infty} \psi_n p_n \right) = I_{a+}^{\alpha} \psi, \; l \to \infty.$$

Hence,

$$\sum_{n=0}^{l} \psi_n \left(I_{a+}^{\alpha} p_n, p_m \right)_{L_p(I,\omega)} \longrightarrow \left(I_{a+}^{\alpha} \psi, p_m \right)_{L_p(I,\omega)}, \; l \to \infty.$$

Applying first formula of Equation (9), we obtain

$$f_m = \left(I_{a+}^{\alpha} \psi, p_m \right)_{L_p(I,\omega)} = \sum_{n=0}^{\infty} (-1)^n \psi_n A_{mn}^{\alpha,\beta,\gamma}.$$

Using the denotations in Equation (11), we obtain the matrix form for the statement of this theorem. □

The following result is formulated in terms of the Jacobi series coefficients and is devoted to the representation of a function by the fractional integral. Consider the Abel equation under most general assumptions relative to the right part

$$I_{a+}^{\alpha} \varphi = f, \; \alpha \in (0,1). \tag{21}$$

If the next conditions hold

$$I_{a+}^{1-\alpha} f \in AC(\bar{I}), \; (I_{a+}^{1-\alpha} f)(a) = 0, \tag{22}$$

then there exists a unique solution of Equation (21) in the class $L_1(I)$ (see Theorem 2.1 [9] (p. 31)). The sufficient conditions for existence and uniqueness of the Abel equation solution are established in the following theorem under the minimum assumptions relative to the right part of Equation (21). In comparison with the ordinary Abel equation, we avoid imposing conditions similar to Equation (22). Moreover, we refuse the assumption that the right part is a Lebesgue integrable function.

Theorem 2. *Suppose* $\omega(x) = (x-a)^{\beta}(b-x)^{\gamma}$, $\beta,\gamma \in [-1/2, 1/2]$, $M(\beta,\gamma) < p < m(\beta,\gamma)$, *the right part of Equation* (21) *such that*

$$\|D^{\alpha}_{a+}S_k f\|_{L_p(I,\omega)} \leq C,\ k \in \mathbb{N}_0,\ \left|\sum_{n=0}^{\infty}(-1)^n f_n A^{-\alpha,\beta,\gamma}_{mn}\right| \sim m^{-\lambda},\ m \to \infty,\ \lambda \in [0,\infty); \tag{23}$$

then there exists a unique solution of Equation (21) in $L_p(I,\omega)$, the solution belongs to $L_q(I,\omega)$, where: $q = p$, when $0 \leq \lambda \leq 1/2$; $q = \max\{p,t\}$, $t < (2s-1)/(s-\lambda)$, when $1/2 < \lambda < s$ $(s = 3/2 + \max\{\beta,\gamma\})$; q is arbitrary large, when $\lambda \geq s$. Moreover, the solution is represented by a convergent in $L_q(I,\omega)$ series

$$\psi(x) = \sum_{m=0}^{\infty} p_m(x) \sum_{n=0}^{\infty}(-1)^n f_n A^{-\alpha,\beta,\gamma}_{mn}. \tag{24}$$

Proof. Applying the first formula in Equation (10), we obtain the following relation

$$(D^{\alpha}_{a+}S_k f, p_m)_{L_2(I,\omega)} \longrightarrow \sum_{n=0}^{\infty}(-1)^n f_n A^{-\alpha,\beta,\gamma}_{mn},\ k \to \infty,\ m \in \mathbb{N}_0, \tag{25}$$

We can easily verify that $M(\beta,\gamma) < p' < m(\beta,\gamma)$. Hence, due to Theorem A in [33], the system $\{p_n\}_0^{\infty}$ has a basis property in the space $L_{p'}(I,\omega)$. Since the relation in Equation (25) holds and the sequence $\{D^{\alpha}_{a+}S_k f\}_0^{\infty}$ is bounded with respect to the norm $L_p(I,\omega)$, then, due to the well-known theorem, we have that the sequence $\{D^{\alpha}_{a+}S_k f\}_0^{\infty}$ converges weakly to some function $\psi \in L_p(I,\omega)$. Using Theorem 2.4 of [9] (p. 44) and the Dirichlet formula (see Theorem 1.1 [9] (p. 9)), we get

$$(S_k f, p_m)_{L_2(I,\omega)} = (I^{\alpha}_{a+}D^{\alpha}_{a+}S_k f, p_m)_{L_2(I,\omega)} = \left(D^{\alpha}_{a+}S_k f, \omega^{-1}I^{\alpha}_{b-}\varphi_m\right)_{L_2(I,\omega)}.$$

Let us show that $\omega^{-1}I^{\alpha}_{b-}\varphi_m \in L_{p'}(I,\omega)$. For this purpose, consider the functional

$$l_1(f) := \left(f, \omega^{-1}I^{\alpha}_{b-}\varphi_m\right)_{L_2(I,\omega)}.$$

Using the Hölder inequality, Lemma 1, we have

$$(I^{\alpha}_{a+}f, p_m)_{L_2(I,\omega)} \leq C\|f\|_{L_p(I,\omega)}\|p_m\|_{L_{p'}(I,\omega)} < \infty. \tag{26}$$

Hence, using the Dirichlet formula, we have

$$(I^{\alpha}_{a+}f, p_m)_{L_2(I,\omega)} = \left(f, \omega^{-1}I^{\alpha}_{b-}\varphi_m\right)_{L_2(I,\omega)}.$$

By virtue of this fact, we can rewrite the relation in Equation (26) in the following form

$$|l_1(f)| \leq C\|f\|_{L_p(I,\omega)},\ \forall f \in L_p(I,\omega).$$

Using the Riesz representation theorem, we obtain $\omega^{-1}I^{\alpha}_{b-}\varphi_m \in L_{p'}(I,\omega)$. Hence, we get

$$\left(D^{\alpha}_{a+}S_k f, \omega^{-1}I^{\alpha}_{b-}\varphi_m\right)_{L_2(I,\omega)} \to \left(\psi, \omega^{-1}I^{\alpha}_{b-}\varphi_m\right)_{L_2(I,\omega)}.$$

Using the Dirichlet formula, we obtain

$$\left(\psi, \omega^{-1}I^{\alpha}_{b-}\varphi_m\right)_{L_2(I,\omega)} = (I^{\alpha}_{a+}\psi, p_m)_{L_2(I,\omega)},\ m \in \mathbb{N}_0. \tag{27}$$

Hence,

$$(S_k f, p_m)_{L_2(I,\omega)} \longrightarrow (I^{\alpha}_{a+}\psi, p_m)_{L_2(I,\omega)},\ k \to \infty,\ m \in \mathbb{N}_0.$$

Taking into account that

$$(S_k f, p_m)_{L_2(I,\omega)} = \begin{cases} f_m, \; k \geq m, \\ \; 0, \; k < m \end{cases},$$

we obtain

$$(I^{\alpha}_{a+}\psi, p_m)_{L_2(I,\omega)} = f_m, \; m \in \mathbb{N}_0.$$

Using the uniqueness property of the Jacobi series expansion, we obtain $I^{\alpha}_{a+}\psi = f$ almost everywhere. Hence, there exists a solution to the Abel Equation (21). If we assume that there exists another solution $\phi \in L_p(I,\omega)$, then we get $I^{\alpha}_{a+}\psi = I^{\alpha}_{a+}\phi$ almost everywhere. Consider the function $\eta \in C_0^{\infty}(I)$. Using Theorem 2.4 of [9] (p. 44) and the Dirichlet formula, we have

$$(\psi - \phi, \eta)_{L_2(I)} = \left(\psi - \phi, I^{\alpha}_{b-} D^{\alpha}_{b-}\eta\right)_{L_2(I)} = \left(I^{\alpha}_{a+}[\psi - \phi], D^{\alpha}_{b-}\eta\right)_{L_2(I)} = 0.$$

Consider an interval $I' \subset I$. Note that $\psi, \phi \in L_p(I')$, $\forall I'$. Since $C_0^{\infty}(I') \subset C_0^{\infty}(I)$, here we assume that the functions belonging to $C_0^{\infty}(I')$ have the zero extension outside of I', then we obtain

$$(\psi - \phi, \eta)_{L_2(I')} = 0, \; \forall \eta \in C_0^{\infty}(I').$$

We claim that $\psi \neq \phi$. Hence, in accordance with the consequence of the Hahn–Banach theorem, there exists the element $\vartheta \in L_{p'}(I')$, such that

$$(\psi - \phi, \vartheta)_{L_2(I')} = \|\psi - \phi\|_{L_p(I')} > 0.$$

On the other hand, there exists the sequence $\{\eta_n\}_1^{\infty} \subset C_0^{\infty}(I')$, such that $\eta_n \stackrel{L_{p'}(I')}{\longrightarrow} \vartheta$. Hence,

$$0 = (\psi - \phi, \eta_n)_{L_2(I')} \to (\psi - \phi, \vartheta)_{L_2(I')}.$$

Thus, we come to the contradiction. Hence, $\psi = \phi$ almost everywhere on I', $\forall I' \subset I$. It implies that $\psi = \phi$ almost everywhere on I. The uniqueness has been proved. Now, let us proceed to the following part of the proof. Note that, $\psi \in L_p(I,\omega)$ was proved above, when $0 \leq \lambda < \infty$. Let us show that $\psi \in L_q(I,\omega)$, where $q < (2s-1)/(s-\lambda)$, $1/2 < \lambda < s$. In accordance with the reasoning given above, we have

$$(D^{\alpha}_{a+} S_k f, p_m)_{L_2(I,\omega)} \longrightarrow (\psi, p_m)_{L_2(I,\omega)}, \; m \in \mathbb{N}_0.$$

Combining this fact with Equation (25), we get

$$\psi_m = (\psi, p_m)_{L_2(I,\omega)} = \sum_{n=0}^{\infty} (-1)^n f_n A^{-\alpha,\beta,\gamma}_{mn}, \; m \in \mathbb{N}_0. \tag{28}$$

Using the theorem conditions, we have

$$|\psi_m| \sim m^{-\lambda}, \; m \to \infty.$$

Now, we need an adopted version of the Zigmund–Marczinkevich theorem (see [36]), which establishes the following. Let $\{\phi_n\}$ be an orthogonal system on the segment $\bar{I}$ and $\|\phi_n\|_{L_\infty(I)} \leq M_n$, $(n = 1, 2, ...)$, where M_n is a monotone increasing sequence of real numbers. If $q \geq 2$ and we have

$$\Omega_q(c) = \left(\sum_{n=1}^{\infty} |c_n|^q n^{q-2} M_n^{q-2}\right)^{1/q} < \infty, \tag{29}$$

then the series $\sum_{n=1}^{\infty} c_n \phi_n(x)$ converges in $L_q(I)$ to some function $f \in L_q(I)$ and $\|f\|_{L_q(I)} \leq C\Omega_q(c)$. We aim to apply this theorem to the case of the Jacobi system, however we need some auxiliary reasoning.

As the matter of fact, we deal with the weighted $L_p(I,\omega)$ spaces, but the Zigmund–Marczinkevich theorem in its pure form formulated in terms of the non-weighted case. Consider the following change of the variable $\int_a^x \omega(t)dt = \tau$. For the solution $\psi \in L_p(I,\omega)$, we have

$$\psi_n = \int_a^b \psi(x)p_n(x)\omega(x)dx = \int_0^B \tilde{\psi}(\tau)\phi_n(\tau)d\tau, \quad (30)$$

where $\tilde{\psi}(\tau) = \psi(\kappa(\tau))$, $\phi_n(\tau) = p_n(\kappa(\tau))$, $\kappa(\tau) = (b-a)^{-(\beta+\gamma+1)}B_\tau^{-1}(\beta+1,\gamma+1)$, $B = (b-a)^{\beta+\gamma+1}B(\beta+1,\gamma+1)$. Hence, if we note the estimate $|p_n(x)| \leq Cn^{a+1/2}$, $a = \max\{\beta,\gamma\}$, $x \in \bar{I}$ (see Theorem 7.3 [37] (p. 288)), then due to the change of the variable, we have $|\phi_n(\tau)| \leq V_n$, $\tau \in [0,B]$, $V_n = Cn^{a+1/2}$. In addition, it is clear that $(\phi_m,\phi_n)_{L_2(0,B)} = \delta_{mn}$, where δ_{mn} is the Kronecker symbol. Thus, $\{\phi_m\}_0^\infty$ is the orthonormal system on $[0,B]$ that satisfies the conditions of the Zigmund–Marczinkevich theorem. It can easily be checked that, due to the theorem conditions, the following series is convergent

$$\sum_{m=0}^{\infty} m^{q\,(s-\lambda)-2s} < \infty,\ 1/2 < \lambda < s,\ q < (2s-1)/(s-\lambda). \quad (31)$$

For the values $\lambda \geq s$, the series in Equation (31) converges for an arbitrary positive q. In accordance with given above, we have

$$\left\{\sum_{m=0}^{\infty} |\psi_m|^q m^{q-2} V_m^{q-2}\right\}^{1/q} \leq C\left\{\sum_{m=0}^{\infty} m^{q\,(s-\lambda)-2s}\right\}^{1/q} < \infty.$$

Thus, all conditions of the Zigmund–Marczinkevich theorem are fulfilled. Hence, we can conclude that there exists a function ν such that the next estimate holds

$$\|\nu\|_{L_q(0,B)} \leq C\left\{\sum_{m=0}^{\infty} |\nu_m|^q m^{q-2} M_m^{q-2}\right\}^{1/q} < \infty. \quad (32)$$

Since the system $\{p_m\}_0^\infty$ has a basis property in $L_p(I,\omega)$, then it is not hard to prove that the system $\{\phi_m\}_0^\infty$ has a basis property in $L_p(0,B)$. Since the functions ν and $\tilde{\psi}$ have the same Jacobi series coefficients, then $\nu = \tilde{\psi}$ almost everywhere on $(0,B)$. By virtue of the chosen change of the variable, we obtain $\|\psi\|_{L_q(I,\omega)} = \|\tilde{\psi}\|_{L_q(0,B)}$. Consequently, the solution ψ belongs to the space $L_q(I,\omega)$, $q < (2s-1)/(s-\lambda)$, when $1/2 < \lambda < s$ and the index q is arbitrary large, when $\lambda \geq s$. Taking into account Equation (30) and applying the Zigmund–Marczinkevich theorem, we have

$$\sum_{m=0}^{k} \phi_m\psi_m \xrightarrow{L_q(0,B)} \tilde{\psi},\ k \to \infty.$$

Using the inverse change of the variable and applying Equation (28), we obtain Equation (24). □

3.2. Non-Simple Property Problem

The questions related to existence of such an invariant subspace of the operator that the operator restriction to the subspace is selfadjont (the so-called non-simple property [38] (p. 275)) are still relevant for today. Thanks to the powerful tool provided by the Jacobi polynomials theory, we are able to approach a little close to solving this problem for the Riemann–Liouville operator.

In this section, we deal with the so-called normalized ultraspherical polynomials $p_n^{(\beta,\beta)}(x)$ in the weighted space $L_p(I,\omega)$, $\omega(x) = [(x-a)(b-x)]^\beta$, $\beta \geq -1/2$, $1 \leq p < \infty$. In accordance with [32], the system of the normalized ultraspherical polynomials has a basis property in $L_p(I,\omega)$, if $1 - 1/(3+$

$2\beta) < p/2 < 1 + 1/(1+2\beta)$, $\lambda = \beta + 1/2$ and does not have a basis property, if $1/2 \leq p/2 < 1 - 1/(3+2\beta)$ or $p/2 > 1 + 1/(1+2\beta)$. Having noticed that $A^{\alpha,\beta,\beta}_{mn} = A^{\alpha,\beta,\beta}_{nm}$, $m,n \in \mathbb{N}_0$, using Equation (9), we obtain

$$\int_a^b \left(I^{\alpha}_{a+}p_n\right)(x)p_m(x)\omega(x)dx = (-1)^{n+m}\int_a^b p_n(x)\left(I^{\alpha}_{a+}p_m\right)(x)\omega(x)dx\,;$$

$$\int_a^b \left(I^{\alpha}_{b-}p_n\right)(x)p_m(x)\omega(x)dx = (-1)^{n+m}\int_a^b p_n(x)\left(I^{\alpha}_{b-}p_m\right)(x)\omega(x)dx, \quad m,n \in \mathbb{N}_0. \tag{33}$$

Taking into account these formulas, we conclude that the fractional integral operator is symmetric in the subspaces of $L_2(I,\omega)$ generated, respectively, by the even system $\{p_{2k}\}_0^{\infty}$ and the odd system $\{p_{2k+1}\}_0^{\infty}$ of the normalized ultraspherical polynomials. Let us denote by $L_2^{+}(I,\omega)$, $L_2^{-}(I,\omega)$ these subspaces, respectively. The following theorem gives us an alternative.

Theorem 3 (Alternative). *Suppose $\alpha \in (1/2, 3/2)$, $\omega(x) = (x-a)^{\beta}(b-x)^{\beta}$, $\alpha - 1/2 < \beta < 1$; then, we have the following alternative: either the fractional integral operator acting in $L_2(I,\omega)$ is non-simple or one has an infinite sequence of the included invariant subspaces having the non-zero intersection with both subspaces $L_2^{+}(I,\omega)$, $L_2^{-}(I,\omega)$.*

Proof. We provide the proof only for the left-side case, since the proof corresponding to the right case is analogous and can be obtained by simple repetition. Let us show that the operator $I^{\alpha}_{a+} : L_2(I,\omega) \to L_2(I,\omega)$ is compact. Using Theorem 3.12 of [9] (p. 81), we have the estimate

$$\|I^{\alpha}_{a+}f\|_{H_0^{\lambda}(\bar{I},r)} \leq C\|f\|_{L_2(I,\omega)}, \; \lambda = \alpha - 1/2, \tag{34}$$

where $r(x) = (x-a)^{\beta/2}(b-x)^{\beta/2}$, if $\beta > 2\alpha - 1$ and $r(x) = (x-a)^{\beta/2}(b-x)^{\alpha - 1/2 + \delta}$ for sufficiently small $\delta > 0$, if $\beta \leq 2\alpha - 1$. It can easily be checked that, in the case $(\beta > 2\alpha - 1)$, we have

$$\left(\int_a^b \left|(I^{\alpha}_{a+})f(x)\right|^2 \omega(x)dx\right)^{1/2} = \left(\int_a^b \left|r(x)(I^{\alpha}_{a+})f(x)\right|^2 dx\right)^{1/2} \leq C\|I^{\alpha}_{a+}f\|_{H_0^{\lambda}(\bar{I},r)},$$

and, in the case $(\beta \leq 2\alpha - 1)$, we have

$$\left(\int_a^b \left|(I^{\alpha}_{a+})f(x)\right|^2 \omega(x)dx\right)^{1/2} = \left(\int_a^b \left|r(x)(I^{\alpha}_{a+})f(x)\right|^2 (b-x)^{\beta - (2\alpha - 1 + 2\delta)}dx\right)^{1/2}.$$

Note that $\beta - (2\alpha - 1 + 2\delta) > -1$ for sufficiently small δ. Therefore,

$$\left(\int_a^b \left|(I^{\alpha}_{a+})f(x)r(x)\right|^2 (b-x)^{\beta - (2\alpha - 1 + 2\delta)}dx\right)^{1/2} \leq C\|I^{\alpha}_{a+}f\|_{H_0^{\lambda}(\bar{I},r)}.$$

Thus, using the estimate in Equation (34), we obtain

$$\|I^{\alpha}_{a+}f\|_{L_2(I,\omega)} \leq C\|f\|_{L_2(I,\omega)}. \tag{35}$$

Now, let us use the Kolmogorov criterion of compactness (see [39]), which claims that a set in the space $L_p(I,\omega)$, $1 \leq p < \infty$ is compact, if this set is bounded and equicontinuous with respect to the norm $L_p(I,\omega)$. Note that, by virtue of Equation (35), the set $I^{\alpha}_{a+}(\mathfrak{N})$ is bounded in $L_2(I,\omega)$, where $\mathfrak{N} := \left\{ f : \|f\|_{L_2(I,\omega)} \leq M,\ M > 0 \right\}$. Using (34), we get $\|I^{\alpha}_{a+}f\|_{H^{\lambda}_{0}(\bar{I},r)} \leq C_1$, $\forall f \in \mathfrak{N}$. Hence, in accordance with the definition, we have $|(I^{\alpha}_{a+}f)(x+t)r(x+t) - (I^{\alpha}_{a+}f)(x)r(x)| < C_1 t^{\lambda}$, $\forall f \in \mathfrak{N}$, $\forall x \in [a,b)$, where t is a sufficiently small positive number. Under the assumption that functions have a zero extension outside of $\bar{I}$, we have

$$\left\{ \int\limits_{a}^{b} \left|(I^{\alpha}_{a+}f)(x+t) - (I^{\alpha}_{a+}f)(x)\right|^2 \omega(x)dx \right\}^{\frac{1}{2}} \leq \left\{ \int\limits_{b-t}^{b} \left|(I^{\alpha}_{a+}f)(x)\right|^2 \omega(x)dx \right\}^{\frac{1}{2}} +$$

$$+ \left\{ \int\limits_{a}^{b-t} \left|(I^{\alpha}_{a+}f)(x+t) - (I^{\alpha}_{a+}f)(x)\right|^2 \omega(x)dx \right\}^{\frac{1}{2}} = \tilde{I} + I.$$

Assume that $f \in \mathfrak{N}$ and consider the case $(\beta \leq 2\alpha - 1)$. Note that due to Theorem 3.12 of [9] (p. 81), we obtain

$$\tilde{I} = \left\{ \int\limits_{b-t}^{b} \left|(I^{\alpha}_{a+}f)(x)r(x)\right|^2 (b-x)^{\beta-\mu}dx \right\}^{\frac{1}{2}} \leq C_1 \left\{ \int\limits_{b-t}^{b} (b-x)^{\beta-\mu}dx \right\}^{\frac{1}{2}} \leq Ct^{\beta-\mu+1},$$

where $\mu = 2\alpha - 1 + 2\delta$, $r(x) = (x-a)^{\beta/2}(b-x)^{\mu/2}$. Using the Minkowski inequality, we get

$$I \leq \left\{ \int\limits_{a}^{b-t} \left|(I^{\alpha}_{a+}f)(x+t)r(x+t) - (I^{\alpha}_{a+}f)(x)r(x)\right|^2 (b-x)^{\beta-\mu}dx \right\}^{\frac{1}{2}} +$$

$$+ \left\{ \int\limits_{a}^{b-t} \left|(I^{\alpha}_{a+}f)(x+t)[r(x+t) - r(x)]\right|^2 (b-x)^{\beta-\mu}dx \right\}^{\frac{1}{2}} = I_1 + I_2.$$

As before, applying Theorem 3.12 of [9] (p. 81), we get $I_1 \leq Ct^{\alpha-1/2}$. Using the inequality $(\tau+1)^{\nu} < \tau^{\nu} + 1$, $\tau > 1$, $0 < \nu < 1$, we obtain

$$\left|(x+t-a)^{\beta/2} - (x-a)^{\beta/2}\right| = t^{\beta/2} \left| \left(\frac{x-a}{t} + 1\right)^{\beta/2} - \left(\frac{x-a}{t}\right)^{\beta/2} \right| < t^{\beta/2},\ a+t < x < b.$$

In the same way, using the inequality $(\tau-1)^{\nu} > \tau^{\nu} - 1$, $\tau > 1$, $0 < \nu < 1$, we get

$$\left|(b-x-t)^{\mu/2} - (b-x)^{\mu/2}\right| < t^{\mu/2},\ a < x < b-t.$$

Since $r(x)$ is a product of the functions that satisfy the Hölder condition, then it is not hard to prove that

$$|r(x+t) - r(x)| < C_2 t^{\beta/2},\ a+t < x < b-t.$$

Using the fact that $\|I^{\alpha}_{a+}f\|_{H^{\lambda}_{0}(\bar{I},r} \leq C_1$, we get

$$I_2^2 \leq C_1 \int\limits_{a}^{b-t} r^{-2}(x+t)\left|r(x+t) - r(x)\right|^2 (b-x)^{\beta-\mu}dx \leq$$

$$\leq C_1 \int\limits_{a+t}^{b-t} r^{-2}(x+t)\left|r(x+t) - r(x)\right|^2 (b-x)^{\beta-\mu}dx +$$

$$+C_1 \int\limits_a^{a+t} r^{-2}(x+t)\,|r(x+t)-r(x)|^2\,(b-x)^{\beta-\mu}dx = I_{21}+I_{22}.$$

Taking into account the above reasoning, we have

$$I_{21} \le Ct^{\beta} \int\limits_{a+t}^{b-t} r^{-2}(x+t)(b-x)^{\beta-\mu}dx \le Ct^{2\beta-\mu} \int\limits_{a+t}^{b-t} (x-a+t)^{-\beta}(b-t-x)^{-\mu}dx =$$

$$= Ct^{2\beta-\mu}\left\{(b-a)^{1-\beta-\mu}B(1-\beta,1-\mu) - \int\limits_{a-t}^{a+t}(x-a+t)^{-\beta}(b-t-x)^{-\mu}dx\right\}^{\frac{1}{2}} \le Ct^{2\beta-\mu};$$

$$I_{22} \le C \int\limits_a^{a+t} (x-a+t)^{-\beta}(b-t-x)^{-\mu}(b-x)^{\beta-\mu}dx \le C \int\limits_a^{a+t}(x-a+t)^{-\beta}dx \le Ct^{1-\beta}.$$

Hence, we conclude that $I_2 \le Ct^{\delta_1}$ and, as a consequence, we obtain $I \le Ct^{\delta_2}$, where δ_1, δ_2 are some positive numbers. To achieve the case $(\beta > 2\alpha - 1)$, we should just repeat the previous reasoning having replaced μ by β. The proof is omitted. Thus, in both cases considered above, we obtain

$$\forall \varepsilon > 0,\ \exists\, t := t(\varepsilon):\ \|(I^{\alpha}_{a+}f)(\cdot+t) - (I^{\alpha}_{a+}f)(\cdot)\|_{L_2(I,\omega)} < \varepsilon,\ \forall f \in \mathfrak{N}.$$

It implies that the conditions of the Kolmogorov criterion of compactness [39] are fulfilled. Hence, any bounded set with respect to the norm $L_2(I,\omega)$ has a compact image. Therefore, the operator $I^{\alpha}_{a+}: L_2(I,\omega) \to L_2(I,\omega)$ is compact. Now, applying the von Neumann theorem [40] (p. 204), we conclude that there exists a non-trivial invariant subspace of the operator I^{α}_{a+}, which we denote by $\mathfrak{M}$. On the other hand, using the basis property of the system $\{p_n\}_0^{\infty}$, we have $L_2(I,\omega) = L_2^{+}(I,\omega) \oplus L_2^{-}(I,\omega)$. It is quite sensible to assume that $\mathfrak{M} \cap L_2^{+}(I,\omega) \neq 0,\ \mathfrak{M} \cap L_2^{-}(I,\omega) \neq 0$. If we assume otherwise, then we have an invariant subspace on which the operator I^{α}_{a+}, by virtue of Equation (33), is selfadjoint and we get the first statement of the alternative. Continuing this line of reasoning, we see that under the assumption excluding the first statement of the alternative we come to conclusion that this process can be finished only in the case, when on some step we get a finite-dimensional invariant subspace. We claim that this cannot be! The proof is by reductio ad absurdum. Assume the converse, then we obtain a finite-dimensional restriction $\tilde{I}^{\alpha}_{a+}$ of the operator I^{α}_{a+}. Applying the reasoning of Theorem 2, we can easily prove that the point zero is not an eigenvalue of the operator I^{α}_{a+}, hence one is not an eigenvalue of the operator $\tilde{I}^{\alpha}_{a+}$. It implies that, in accordance with the fundamental theorem of algebra, the operator $\tilde{I}^{\alpha}_{a+}$ has at least one non-zero eigenvalue (since $\tilde{I}^{\alpha}_{a+}$ is finite-dimensional). It is clear that this eigenvalue is an eigenvalue of the operator I^{α}_{a+}. We can write

$$\exists \lambda \in \mathbb{C},\ \lambda \neq 0,\ f \in L_2(I,\omega),\ f \neq 0:\ I^{\alpha}_{a+}f = \lambda f \ \text{ a.e.} \tag{36}$$

Further, we use the method described in [41] (p. 14). Using the Cauchy–Schwarz inequality, we get

$$|f(x)|^2 \le |\lambda|^{-2}B(x)\int\limits_a^x |f(t)|^2\omega(t)dt \le |\lambda|^{-2}\|f\|^2_{L_2(I,\omega)}B(x), \tag{37}$$

where

$$B(x) = \Gamma^{-1}(\alpha)\int\limits_a^x (x-t)^{2\alpha-2}\omega^{-1}(t)dt.$$

Substituting $f(t)$ for $|\lambda|^{-2}\|f\|^2_{L_2(I,\omega)}B(x)$ in Equation (37), we get

$$|f(x)|^2 \le \|f\|^2_{L_2(I,\omega)} |\lambda|^{-4} B(x) \int_a^x B(t)\omega(t)dt.$$

Continuing this process, we obtain

$$|f(x)|^2 \le \|f\|^2_{L_2(I,\omega)} |\lambda|^{-2(n+1)} B(x) \underbrace{\int_a^x B(x_n)\omega(x_n)dx_n \int_a^{x_n} B(x_{n-1})\omega(x_{n-1})dx_{n-1} ... \int_a^{x_2} B(x_1)\omega(x_1)dx_1}_{\text{n integrals}},\ n \in \mathbb{N}.$$

Let

$$B_n(x) := \int_a^x B(x_n)\omega(x_n)dx_n \int_a^{x_n} B(x_{n-1})\omega(x_{n-1})dx_{n-1} ... \int_a^{x_2} B(x_1)\omega(x_1)dx_1,$$

thus

$$B_n(x) = \int_a^x B(t)B_{n-1}(t)\omega(t)dt,\ B_0(x) := 1,\ n \in \mathbb{N}. \tag{38}$$

Let us show that $B_n(x) = B_1^n(x)/n!$.

It is obviously true in the case $(n = 1)$. Assume that the relation $B_{n-1}(x) = B_1^{n-1}(x)/(n-1)!$ is fulfilled and let us deduce that $B_n(x) = B_1^n(x)/n!$.

Using Equation (38), we obtain

$$B_n(x) = \frac{1}{(n-1)!} \int_a^x B(t)B_1^{n-1}(t)\omega(t)dt = \frac{1}{(n-1)!} \int_a^x B_1^{n-1}(t)\frac{dB_1(t)}{dt}dt = \frac{B_1^n(x)}{n!}.$$

Hence,

$$|f(x)|^2 \le \frac{1}{n!} \|f\|^2_{L_2(I,\omega)} |\lambda|^{-2(n+1)} B(x) B_1^n(x),\ n \in \mathbb{N}.$$

Using the Dirichlet formula, we get

$$B_1(x) \le \frac{1}{\Gamma(\alpha)} \int_a^b \omega(y)dy \int_a^y (y-t)^{2\alpha-2}\omega^{-1}(t)dt = \frac{1}{\Gamma(\alpha)} \int_a^b \omega^{-1}(t)dt \int_t^b (y-t)^{2\alpha-2}\omega(y)dy =: J.$$

By virtue of the theorem conditions, we conclude that $J < \infty$. Hence, we have

$$|f(x)|^2 \le \frac{|\lambda|^{-2(n+1)} J^n}{n!} \|f\|^2_{L_2(I,\omega)} B(x),\ n \in \mathbb{N}.$$

Since $\left(|\lambda|^{-2(n+1)} J^n\right)/n! \to 0,\ n \to \infty$, then $f(x) = 0,\ x \in I$. We have obtained the contradiction with Equation (36), which allows us to conclude that there does not exist a finite dimensional invariant subspace. It implies that we have the sequence of the included invariant subspaces

$$\mathfrak{M}_1 \supset \mathfrak{M}_2 \supset ... \supset \mathfrak{M}_k \supset ...\ ,$$

$$\mathfrak{M}_k \cap L_2^+(I,\omega) \ne 0,\ \mathfrak{M}_k \cap L_2^-(I,\omega) \ne 0,\ k = 1,2,...\ .$$

□

4. Conclusions

In this paper, our first aim is to reformulate in terms of the Jacoby series coefficients the previously known theorems describing the Riemann–Liouville operator action in the weighted spaces of Lebesgue p-th power integrable functions, and our second aim is to approach a little bit closer to solving the problem of whether the Riemann–Liouville operator acting in the weighted space of Lebesgue square integrable functions is simple. The approach used in the paper is the following: to use the Jacobi polynomials special properties that allow us to apply novel methods of functional analysis and theory of functions of a real variable, which are rather different in comparison with the perviously applied methods for studying the Riemann–Liouville operator. Besides the main results of the paper, we stress that there is arranged some systematization of the previously known facts of the Riemann–Liouville operator action in the weighted spaces of Lebesgue p-th power integrable functions, when the weighted function is represented by some kind of a power function. It should be noted that the previously known description of the Riemann–Liouville operator action in the weighted spaces of Lebesgue p-th power integrable functions consists of some theorems in which the conditions imposed on the weight function have the gaps, i.e., some cases corresponding to the concrete weighted functions were not considered. Motivated by this, among the unification of the known results, we managed to fill the gaps of the conditions and formulated this result as a separate lemma. The following main results were obtained in terms of the Jacoby series coefficients: the theorem on the Riemann–Liouville operator direct action was proved, the existence and uniqueness theorem for Abel equation in the weighted spaces of Lebesgue p-th power integrable functions was proved and the solution formula was given, the alternative in accordance with which the Riemann–Liouville operator is either simple or one has the sequence of the included invariant subspaces was established. Note that these results give us such a view of the fractional calculus that has many advantages. For instance, we can reformulate Theorem 2 under more general assumptions relative to the integral operator on the left side of Equation (21), at the same time having preserved the main scheme of the reasonings. In this case, the most important problem may be, in what way we are able to calculate the coefficients given by Equation (10). Besides, the notorious case $p = 1/\alpha$, which was successfully achieved in this paper is also worth noticing. Thus, the obtained results make a prerequisite of researching in the direction of fractional calculus.

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