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Branching Functions for Admissible Representations of Affine Lie Algebras and Super-Virasoro Algebras

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Abstract: We explicitly calculate the branching functions arising from the tensor product decompositions between level 2 and principal admissible representations over $\widehat{\mathfrak{sl}}_2$. In addition, investigating the characters of the minimal series representations of super-Virasoro algebras, we present the tensor product decompositions in terms of the minimal series representations of super-Virasoro algebras for the case of principal admissible weights.

Keywords: branching functions; admissible representations; characters; affine Lie algebras; super-Virasoro algebras

1. Introduction

One of the basic problems in representation theory is to find the decomposition of a tensor product between two irreducible representations. In fact, the study of tensor product decompositions plays an important role in quantum mechanics and in string theory [1,2], and it has attracted much attention from combinatorial representation theory [3]. In addition, recent studies reveal that tensor product decompositions are also closely related to the representation theory of Virasoro algebra and W-algebras [4–6].

In [6], the authors extensively study decompositions of tensor products between integrable representations over affine Lie algebras. They also investigate relationships among tensor products, branching functions and Virasoro algebra through integrable representations over affine Lie algebras.

In the present paper we shall follow the methodology appearing in [6]. However, we will focus on admissible representations of affine Lie algebras. Admissible representations are not generally integrable over affine Lie algebras, but integrable with respect to a subroot system of the root system attached to a given affine Lie algebra. Kac and Wakimoto showed that admissible representations satisfy several nice properties such as Weyl-Kac type character formula and modular invariance [5,7]. In their subsequent works, they also established connections between admissible representations of affine Lie algebras and the representation theory of W-algebras [4,8]. In addition, Kac and Wakimoto expressed in ([9], Theorem 3.1) the branching functions arising from the tensor product decompositions between principal admissible and integrable representations as the q-series involving the associated dominant integral weights and string functions.

One of the main results of this paper is the explicit calculations of the branching functions appearing in ([9], Theorem 3.1). We are particularly interested in the calculations of the branching functions obtained from certain tensor product decompositions of level 2 integrable and principal admissible representations over $\widehat{\mathfrak{sl}}_2$ (see Theorem 4). We shall see that these branching functions connect the representation theory of affine Lie algebras with the representation theory of super-Virasoro algebras.

We usually apply the theory of modular functions for calculations of string functions [10]. However, in the current work we shall not use the tools of modular functions for the calculations of the string functions appearing in ([9], Theorem 3.1). Instead, we shall use both the invariance properties of string functions under the action of affine Weyl group and the character formula whose summation is taken over maximal weights (see Theorem 5). It seems like that this approach provides a simpler way for computations of the string functions in our cases.

We would like to point out that in ([5], Corollary 3(c)) the authors expressed the branching functions in terms of theta functions. We shall show that our expressions for the branching functions appearing in Theorem 4 are actually same as those of ([5], Corollary 3(c)) through the investigations of the characters of the minimal series representations of super-Virasoro algebras. Comparing our calculations of the branching functions over $\widehat{\mathfrak{sl}}_2$ with the characters of the minimal series representations of super-Virasoro algebras, we also present the tensor product decompositions between level 2 integrable and principal admissible representations in terms of the minimal series representations of super-Virasoro algebras (see Theorem 6). This generalizes the decomposition formula appearing in ([6], Section 4.1(a)) to the case of principal admissible weights.

2. Preliminaries

Let $A = (a_{ij})_{1\le i,j\le n}$ be a symmetrizable generalized Cartan matrix and $\mathfrak{g}$ the Kac-Moody Lie algebra associated with A. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Fix the set of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ of $\mathfrak{h}$ and simple coroots $\Pi^\vee = \{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$ of $\mathfrak{h}^*$, respectively. Assume that Π and $\Pi^\vee$ satisfy the condition $\alpha_j(\alpha_i^\vee) = a_{ij}$. We denote by $(\,|\,)$ the non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. Write Δ, Δ_+ and Δ_- for the set of all roots, positive roots and negative roots of $\mathfrak{g}$, respectively. Put $\Delta^{re} = \{\alpha \in \Delta \mid (\alpha|\alpha) > 0\}$ and $\Delta^{im} = \{\alpha \in \Delta \mid (\alpha|\alpha) \le 0\}$. For each $i = 1, \cdots, n$, we define the fundamental reflection r_{α_i} of $\mathfrak{h}^*$ by

$$r_{\alpha_i}(\lambda) = \lambda - \lambda(\alpha_i^\vee)\,\alpha_i \ \ (\lambda \in \mathfrak{h}^*).$$

The subgroup W of $GL(\mathfrak{h}^*)$ generated by all fundamental reflections is called the *Weyl group* of $\mathfrak{g}$.

Among symmetrizable Kac-Moody Lie algebras, the most important Lie algebras are affine Lie algebras whose associated Cartan matrices are called *affine Cartan matrices*. It is known that every affine Cartan matrix is a positive semidefinite of corank 1. Each affine Cartan matrix is in one-to-one correspondence with the affine Dynkin diagram of type $X_n^{(r)}$, where $X = A, B, C, D, E, F$ or G and $r = 1, 2$ or 3. The number r is called the *tier number* (see [11,12] for details). Given an affine Cartan matrix $A = (a_{ij})_{0\le i,j\le l}$, two $(l+1)$-tuples $(a_i^\vee)_{0\le i\le l}$ and $(a_i)_{0\le i\le l}$ of positive integers are uniquely determined by the conditions

1. $(a_0^\vee, a_1^\vee, \cdots, a_l^\vee)\, A = \mathbf{O}$,
2. $A \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_l \end{pmatrix} = \mathbf{O}$,
3. $gcd(a_0^\vee, a_1^\vee, \cdots, a_l^\vee) = gcd(a_0, a_1, \cdots, a_l) = 1$,

where $\mathbf{O}$ is the zero vector. We call $(a_i)_{0\le i\le l}$ (resp. $(a_i^\vee)_{0\le i\le l}$) the *label* (resp. *colabel*) of the affine matrix A. The corresponding positive integer $h = \sum_{i=0}^{l} a_i$ (resp. $h^\vee = \sum_{i=0}^{l} a_i^\vee$) is called the *Coxeter number* (resp. *dual Coxeter number*). Notice that the element $K = \sum_{i=0}^{l} a_i^\vee \alpha_i^\vee$ satisfies $\alpha_i(K) = 0$ for $0 \le i \le l$, and we call this element the *central element*. Through the non-degenerate bilinear form $(\,|\,)$ defined on $\mathfrak{g}$, the central element K corresponds to $\delta = \sum_{i=0}^{l} a_i\alpha_i$ in $\mathfrak{h}^*$.

Suppose that $\mathfrak{g}$ is the affine Lie algebra associated to an affine Cartan matrix $A = (a_{ij})_{0\le i,j\le l}$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The Cartan subalgebra $\mathfrak{h}$ is $(l+2)$-dimensional, and we can decompose $\mathfrak{h}$ and $\mathfrak{h}^*$ as follows:

$$\mathfrak{h} = \overline{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

$$\mathfrak{h}^* = \overline{\mathfrak{h}}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0,$$

where $\overline{\mathfrak{h}} = \sum_{i=1}^{l} \mathbb{C}\alpha_i^\vee$ and $\overline{\mathfrak{h}}^* = \sum_{i=1}^{l} \mathbb{C}\alpha_i$.

The lattice $Q = \sum_{i=0}^{l} \mathbb{Z}\alpha_i$ and $Q^\vee = \sum_{i=0}^{l} \mathbb{Z}\alpha_i^\vee$ are called the *root lattice* and *coroot lattice*, respectively. Set

$$M = \begin{cases} \overline{Q}^\vee & \text{if } r = 1 \text{ or } A = A_{2l}^{(2)}, \\ \overline{Q} & \text{if } r \geq 2 \text{ and } A \neq A_{2l}^{(2)}. \end{cases}$$

For an element $\alpha \in Q$, we define $t_\alpha \in GL(\mathfrak{h}^*)$ by

$$t_\alpha(\lambda) = \lambda + (\lambda|\delta)\alpha - \left\{ \frac{|\alpha|^2}{2}(\lambda|\delta) + (\lambda|\alpha) \right\} \delta \ \ (\lambda \in \mathfrak{h}^*).$$

We call $t_\alpha \ (\alpha \in Q)$ the *translation operator*. It is known that the Weyl group W of the affine Lie algebra $\mathfrak{g}$ is also given by $\overline{W} \ltimes t_M$, where $\overline{W} = \langle r_{\alpha_i} | 1 \leq i \leq l \rangle$ and $t_M = \{ t_\alpha | \alpha \in M \}$.

For a non-twisted affine Lie algebra (i.e., $r = 1$), recall that

$$\Delta^{im} = \{ n\delta | n \in \mathbb{Z} - \{0\} \}$$

and

$$\Delta^{re} = \{ n\delta + \alpha | n \in \mathbb{Z}, \ \alpha \in \overline{\Delta} \},$$

where $\overline{\Delta}$ is the set of all roots of the finite-dimensional simple Lie algebra associated with the finite Cartan matrix $\overline{A} = (a_{ij})_{1 \leq i,j \leq l}$.

Set

$$\begin{aligned} P &= \{ \lambda \in \mathfrak{h}^* | \lambda(\alpha_i^\vee) \in \mathbb{Z} \text{ for } 0 \leq i \leq l \}, \\ P^m &= \{ \lambda \in P | \lambda(K) = m \}, \\ P_+ &= \{ \lambda \in \mathfrak{h}^* | \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \text{ for } 0 \leq i \leq l \}, \\ P_+^m &= P^m \cap P_+. \end{aligned}$$

An element in P (reps. P_+) is called an *integral weight* (resp. a *dominant integral weight*). Let ρ be the dominant integral weight defined by $\rho(\alpha_i^\vee) = 1$ for $0 \leq i \leq l$. The element ρ is called the *Weyl vector* of $\mathfrak{g}$. It is sometimes convenient to choose the Weyl vector satisfying the additional condition $\rho(d) = 0$, and we get $\rho = \overline{\rho} + h^\vee \Lambda_0$ in this case.

Define the fundamental weights $\Lambda_i \in \mathfrak{h}^* \ (0 \leq i \leq l)$ by $\Lambda_i(\alpha_j^\vee) = \delta_{ij} \ (0 \leq j \leq l)$ and $\Lambda_i(d) = 0$. Similarly, we define the fundamental coweights $\Lambda_i^\vee \in \mathfrak{h} \ (0 \leq i \leq l)$ by $(\Lambda_i^\vee | \alpha_j) = \delta_{ij} \ (0 \leq j \leq l)$ and $(\Lambda_i^\vee | d) = 0$. Let $\overline{\Lambda}_i$ and $\overline{\Lambda}_i^\vee$ be the restrictions of Λ_i and $\Lambda_i^\vee$ to $\overline{\mathfrak{h}}^*$ and $\overline{\mathfrak{h}}$, respectively. Put $\overline{P} = \sum_{i=1}^{l} \mathbb{Z}\overline{\Lambda}_i$ and $\overline{P}^\vee = \sum_{i=1}^{l} \mathbb{Z}\overline{\Lambda}_i^\vee$, and let us introduce a lattice

$$\widetilde{M} = \begin{cases} \overline{P}^\vee & \text{if } r = 1 \text{ or } A = A_{2l}^{(2)}, \\ \overline{P} & \text{if } r \geq 2 \text{ and } A \neq A_{2l}^{(2)}. \end{cases}$$

Then, the group $\widetilde{W} = \overline{W} \ltimes t_{\widetilde{M}}$ is called the *extended affine Weyl group* of $\mathfrak{g}$.

3. Branching functions for admissible weights

Let $\mathfrak{g}$ be the Kac-Moody Lie algebra associated to a symmetrizable generalized Cartan matrix A, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. An element $\lambda \in \mathfrak{h}^*$ satisfying conditions

1. $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Q} - \mathbb{Z}_{\leq 0}$ for all $\alpha \in \Delta_+^{re} := \Delta^{re} \cap \Delta_+$,
2. $\mathbb{Q}$-span of $\{ \alpha \in \Delta_+^{re} | \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z} \} = \mathbb{Q}$-span of Δ_+^{re}

is called an *admissible weight*. When λ is an admissible weight, the corresponding irreducible highest weight $\mathfrak{g}$-module $L(\lambda)$ is called an *admissible* $\mathfrak{g}$*-module* or *admissible representation*. Write

$$\Delta_\lambda^{\vee re} = \left\{\alpha^\vee | \alpha \in \Delta^{re} \text{ and } \left\langle \lambda + \rho, \alpha^\vee \right\rangle \in \mathbb{Z}_{\geq 1}\right\}.$$

Then, it is easy to see that $\Delta_\lambda^{\vee re}$ forms a subroot system of the coroot system $\Delta^\vee$. We denote by $\Pi_\lambda^\vee$ a base of $\Delta_\lambda^{\vee\, re}$, and put $W_\lambda = \langle r_\alpha | \alpha \in \Pi_\lambda^\vee \rangle$.

An admissible weight λ is called a *principal admissible weight* if $\Pi_\lambda^\vee$ is isomorphic to $\Pi^\vee$. In general, the level of a principal admissible weight is a rational number. In fact, it is known from [7] that a rational number $m = \frac{v}{u}$ $(u \in \mathbb{Z}_{\geq 1},\ v \in \mathbb{Z},\ gcd(u, v) = 1)$ is the level of principal admissible weights if and only if it satisfies

1. $gcd\,(u, r^\vee) = 1$,
2. $u\,(m + h^\vee) \geq h^\vee$,

where $r^\vee$ is the tier number of the transposed generalized Cartan matrix A^t and $h^\vee$ denotes the dual Coxeter number of $\mathfrak{g}$.

Henceforth, we assume that $\mathfrak{g}$ is an affine Lie algebra with a simple coroot system $\Pi^\vee = \{\alpha_0^\vee, \cdots, \alpha_l^\vee\}$.

Given $u \in \mathbb{Z}_{\geq 1}$, put $\gamma_0 = (u-1)c + \alpha_0^\vee$ and $\gamma_i = \alpha_i^\vee$ $(1 \leq i \leq l)$. Define $S_{(u)} = \{\gamma_i \mid 0 \leq i \leq l\}$. Then, $S_{(u)}$ becomes a simple coroot system of $\Delta^\vee \cap \left(\sum_{i=0}^l \mathbb{Z}\gamma_i\right)$ if $gcd\,(u, r^\vee) = 1$ (see [13], Lemma 3.2.1). Moreover, the following theorems are known.

Theorem 1. *Let $m = \frac{v}{u}$ with $u \in \mathbb{Z}_{\geq 1}$, $v \in \mathbb{Z}$ and $gcd(u, v) = 1$. Assume that $y \in \widetilde{W}$ satisfies $y\left(S_{(u)}\right) \subset \Delta_+^\vee$. Write $P_{u,y}^m$ for the set of all principal admissible weights λ of level m with $\Pi_\lambda^\vee = y\left(S_{(u)}\right)$. Then, we have*

$$P_{u,y}^m = \left\{y\left(\lambda^0 - (u-1)\left(m + h^\vee\right)\Lambda_0 + \rho\right) - \rho \mid \lambda^0 \in P_+^{u(m+h^\vee)-h^\vee}\right\}.$$

Proof. See ([7], Theorem 2.1) or ([9], Proposition 1.5). □

Theorem 2. *Let $m = \frac{v}{u}$ with $u \in \mathbb{Z}_{\geq 1}$, $v \in \mathbb{Z}$ and $gcd(u, v) = 1$. Let P_+^m be the set of all principal admissible weights of level m (we use the same notation as the case of dominant integral weights). Then, $P_+^m = \bigcup_y P_{u,y}^m$, where y runs over $\left\{y \in \widetilde{W} | y\left(S_{(u)}\right) \subset \Delta_+^\vee\right\}$.*

Proof. See ([9], Proposition 1.5). □

Let us now review branching functions and their connections with the Virasoro algebra.

Recall the Virasoro algebra is an infinite dimensional Lie algebra $Vir = \left(\bigoplus_{n\in\mathbb{Z}} \mathbb{C}\ell_n\right) \oplus \mathbb{C}c$ with brackets

$$[\ell_m, c] = 0 \text{ for all } m \in \mathbb{Z}$$

and

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}\, c \text{ for all } m, n \in \mathbb{Z}.$$

Let $\bar{\mathfrak{g}}$ be a finite dimensional simple Lie algebra, and $\mathfrak{g} = \mathbb{C}\left[t, t^{-1}\right] \otimes \bar{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathbb{C}d$ the non-twisted affine Lie algebra over $\bar{\mathfrak{g}}$. Let V be the highest weight $\mathfrak{g}$-module of level m such that $m + h^\vee \neq 0$. Define the operators $L_n^{\mathfrak{g}}$ $(n \in \mathbb{Z})$ via

$$L^{\mathfrak{g}}(z) = \sum_{n\in\mathbb{Z}} L_n^{\mathfrak{g}} z^{-n-2} = \frac{1}{2\left(m + h^\vee\right)} \sum_{i=1}^{dim\bar{\mathfrak{g}}} : u^i(z)u_i(z) :, \qquad (1)$$

where $\{u^i\}$ and $\{u_i\}$ are bases of $\overline{\mathfrak{g}}$ satisfying $(u_i|u^j) = \delta_{ij}$. It is well-known that V becomes a *Vir*-module by letting

$$\ell_n \longmapsto L_n^{\mathfrak{g}} \ (n \in \mathbb{Z}) \text{ and } c \longmapsto \frac{m\dim\overline{\mathfrak{g}}}{m+h^\vee}. \tag{2}$$

The Virasoro action (2) satisfies the following properties:

$$\left[\ell_n, t^j \otimes X\right] = -jt^{j+n} \otimes X \ (X \in \overline{\mathfrak{g}},\ n \in \mathbb{Z} - \{0\}\,,\ j \in \mathbb{Z})\,, \tag{3}$$

$$\ell_0 = \frac{(\Lambda + 2\rho|\Lambda)}{2\,(m+h^\vee)}\mathrm{Id} - d. \tag{4}$$

Let $\overline{\mathfrak{p}}$ be a reductive subalgebra of $\overline{\mathfrak{g}}$. Then, $\overline{\mathfrak{p}}$ is decomposed as $\overline{\mathfrak{p}} = \overline{\mathfrak{p}}_0 \oplus \overline{\mathfrak{p}}_1 \oplus \cdots \oplus \overline{\mathfrak{p}}_s$, where $\overline{\mathfrak{p}}_0$ is the center of $\overline{\mathfrak{p}}$ and each $\overline{\mathfrak{p}}_i \ (i = 1, \cdots, s)$ is a simple Lie algebra. Assume that

$$\overline{\mathfrak{p}}_0 \oplus \left(\sum_{i=1}^{s} \overline{\mathfrak{h}}_i\right) \subset \overline{\mathfrak{h}}$$

and

$$\sum_{i=1}^{s} \overline{\mathfrak{p}}_{i+} \subset \overline{\mathfrak{g}}_+,$$

where $\overline{\mathfrak{h}}_i$ (resp. $\overline{\mathfrak{h}}$) is a Cartan subalgebra of $\overline{\mathfrak{p}}_i$ (resp. $\overline{\mathfrak{g}}$) and $\overline{\mathfrak{p}}_{i+}$ (resp. $\overline{\mathfrak{g}}_+$) is the sum of the positive root spaces of $\overline{\mathfrak{p}}_i$ (resp. $\overline{\mathfrak{g}}$). Consider the affinization $\mathfrak{p} = \left(\mathbb{C}[t, t^{-1}] \otimes \overline{\mathfrak{p}}\right) \oplus \mathbb{C}\dot{K} \oplus \mathbb{C}d$ of $\overline{\mathfrak{p}}$. Since V is the highest weight $\mathfrak{g}$-module, V is also the highest weight $\mathfrak{p}$-module. However, the action of the central element $\dot{K}$ on V is somewhat complicated. We refer to ([11], Chapter 12) for the details of the action of the central element $\dot{K}$. Let $\dot{m}$ be the level of V as a $\mathfrak{p}$-module, and write $(\ |\ )'$ for the standard bilinear form on $\overline{\mathfrak{p}}$. Set

$$L^{\mathfrak{p}} = \sum_{n\in\mathbb{Z}} L_n^{\mathfrak{p}} z^{-n-2} = \frac{1}{2\,(\dot{m} + \dot{h}^\vee)} \sum_{i=1}^{dim\overline{\mathfrak{p}}} :\dot{u}^i(z)\dot{u}_i(z):, \tag{5}$$

where $\{\dot{u}^i\}$ and $\{\dot{u}_i\}$ are bases of $\overline{\mathfrak{p}}$ satisfying $(\dot{u}^i|\dot{u}_j)' = \delta_{ij}$ and $\dot{h}^\vee$ is the dual Coxeter number of $\overline{\mathfrak{p}}$. Using (1) and (5), define

$$L^{\mathfrak{g};\mathfrak{p}}(z) = L^{\mathfrak{g}} - L^{\mathfrak{p}} = \sum_{n\in\mathbb{Z}} L_n^{\mathfrak{g};\mathfrak{p}} z^{-n-2}.$$

Due to (3), it follows that

$$\left[L_n^{\mathfrak{g};\mathfrak{p}}, t^j \otimes X\right] = 0 \text{ for all } X \in \overline{\mathfrak{p}},\ n \in \mathbb{Z} - \{0\} \text{ and } j \in \mathbb{Z}. \tag{6}$$

Applying the operator product expansions, we can verify that $L^{\mathfrak{g};\mathfrak{p}}(z)$ is, in fact, a Virasoro field with the central charge $c^{\mathfrak{g};\mathfrak{p}} = \frac{m dim\overline{\mathfrak{g}}}{m+h^\vee} - \frac{\dot{m} dim\overline{\mathfrak{p}}}{\dot{m}+\dot{h}^\vee}$ (see [13,14] for the details). We call the Virasoro field $L^{\mathfrak{g};\mathfrak{p}}(z)$ the *coset Virasoro field*.

In the remaining part of this section, we assume that $V = L(\Lambda)$ for a dominant integral weight Λ of level m. Let $\dot{\overline{\mathfrak{h}}}$ be a Cartan subalgebra of $\overline{\mathfrak{p}}$, and $\mathfrak{p}_+$ the positive part of $\mathfrak{p}$. For $\nu \in \left(\dot{\overline{\mathfrak{h}}} \oplus \mathbb{C}\dot{K}\right)^*$, set

$$V_\nu^{\mathfrak{g};\mathfrak{p}} = \left\{v \in L(\Lambda)\,|Xv = 0 \ (\forall\, X \in \mathfrak{p}_+)\,,\ Hv = \nu(H)\,v \ \left(\forall\, H \in \dot{\overline{\mathfrak{h}}} \oplus \mathbb{C}\dot{K}\right)\right\}.$$

Due to (4) and (6), $V_\nu^{\mathfrak{g};\mathfrak{p}}$ is stable under the actions of $L_n^{\mathfrak{g};\mathfrak{p}} \ (n \in \mathbb{Z})$. So, $V_\nu^{\mathfrak{g};\mathfrak{p}}$ becomes a *Vir*-module. We call this module the *coset Virasoro module*. Notice that $L(\Lambda)$ is decomposed as a $Vir \oplus [\mathfrak{p}, \mathfrak{p}]$-module into

$$L(\Lambda) = \bigoplus_{\nu \in \dot{\overline{\mathfrak{h}}}^* \oplus \mathbb{C}\dot{\delta} \bmod \mathbb{C}\dot{\delta}} \left(V_\nu^{\mathfrak{g};\mathfrak{p}} \otimes \dot{L}(\nu)\right), \tag{7}$$

where $\dot{L}(\nu)$ is the irreducible $[\mathfrak{p},\mathfrak{p}]$-module with highest weight ν and $\dot{\delta}$ is identified with $\dot{K}$ via the non-degenerate bilinear form on $\mathfrak{p}$. From (7), we define a function

$$c_{\nu}^{\Lambda}(q) = Tr_{V_{\nu}^{\mathfrak{g};\mathfrak{p}}} q^{-d} = \sum_{j\in\mathbb{Z}_{\geq 0}} \mathrm{mult}_{\Lambda}\left(\nu - j\dot{\delta};\mathfrak{p}\right) q^{j} \left(q = e^{-\delta}\right), \tag{8}$$

where the multiplicity is defined as in ([6], Section 1.6). The function (8) is called the *string function*. Using the string function (8), the decomposition (7) yields the following formula for the character of $L(\Lambda)$:

$$chL(\Lambda) = \sum_{\nu\in\dot{\mathfrak{h}}^{*}\oplus\mathbb{C}\dot{\delta} \bmod \mathbb{C}\dot{\delta}} c_{\nu}^{\Lambda}(q) ch\dot{L}(\nu). \tag{9}$$

Let us now introduce the following numbers:

- $m_{\Lambda} = \frac{|\Lambda+\rho|^2}{2(m+h^{\vee})} - \frac{|\rho|^2}{2h^{\vee}}$,
- $\dot{m}_{\nu} = \frac{|\nu+\dot{\rho}|^2}{2(\dot{m}+\dot{h}^{\vee})} - \frac{|\dot{\rho}|^2}{2\dot{h}^{\vee}}$,

where $\dot{\rho}$ is the Weyl vector associated with $\mathfrak{p}$.

Then, we define the branching function as $b_{\nu}^{\Lambda}(\tau) = q^{m_{\Lambda}-\dot{m}_{\nu}} c_{\nu}^{\Lambda}(q)$ for $q = e^{2\pi i\tau}$. By the strange formula and (4), we see that the branching function also can be written as $b_{\nu}^{\Lambda}(\tau) = q^{-\frac{1}{24}c^{\mathfrak{g};\mathfrak{p}}} Tr_{V_{\nu}^{\mathfrak{g};\mathfrak{p}}} q^{\ell_0}$ (see [11] (Chapter 12) for the strange formula).

Recall that the normalized character $ch^{'}L(\Lambda)$ is defined as

$$ch^{'}L(\Lambda) = e^{-m_{\Lambda}\delta} chL(\Lambda).$$

Introducing the coordinate (τ, z, t) for $h = 2\pi i(-\tau d + z + tK) \in \mathfrak{h}$, we obtain that $ch^{'}_{L(\Lambda)}(\tau,z,t) = q^{m_{\Lambda}} ch_{L(\Lambda)}(\tau,z,t)$. So, the Formula (9) can be rewritten as

$$ch^{'}_{L(\Lambda)}(\tau,z,t) = \sum_{\nu\in\dot{\mathfrak{h}}^{*}\oplus\mathbb{C}\dot{\delta} \bmod \mathbb{C}\dot{\delta}} b_{\nu}^{\Lambda}(\tau)\, ch^{'}_{\dot{L}(\nu)}(\tau,z,t).$$

4. Tensor Product Decompositions

In this section, we fix an affine Lie algebra $\mathfrak{g} = \left(\mathbb{C}\left[t,t^{-1}\right]\otimes\bar{\mathfrak{g}}\right) \oplus \mathbb{C}K \oplus \mathbb{C}d$ over a finite dimensional simple Lie algebra $\bar{\mathfrak{g}}$. We also fix a Cartan subalgebra $\bar{\mathfrak{h}}$ of $\bar{\mathfrak{g}}$. For $\lambda, \mu \in \mathfrak{h}^{*}$, let $L(\lambda)$ and $L(\mu)$ be irreducible highest weight modules over $\mathfrak{g}$. We denote by π_{λ} and π_{μ} the representations of $\mathfrak{g}$ on $L(\lambda)$ and $L(\mu)$, respectively. Put $m = \lambda(K)$ and $m^{'} = \mu(K)$. Assume that $m + h^{\vee} \neq 0$, $m^{'} + h^{\vee} \neq 0$ and $m + m^{'} + h^{\vee} \neq 0$. It follows from (2) that the Virasoro algebra Vir acts on $L(\lambda)$ and $L(\mu)$. The corresponding Virasoro fields are

$$L^{\lambda}(z) = \frac{1}{2\left(m+h^{\vee}\right)} \sum_{i=1}^{dim\bar{\mathfrak{g}}} : \pi_{\lambda}\left(u_i(z)\right) \pi_{\lambda}\left(u^{i}(z)\right) :$$

and

$$L^{\mu}(z) = \frac{1}{2\left(m^{'}+h^{\vee}\right)} \sum_{i=1}^{dim\bar{\mathfrak{g}}} : \pi_{\mu}\left(u_i(z)\right) \pi_{\mu}\left(u^{i}(z)\right) : .$$

Notice that the Virasoro algebra Vir acts on $L(\lambda)\otimes L(\mu)$ via the tensor product action

$$L^{\lambda,\mu}(z) = L^{\lambda}(z)\otimes \mathrm{Id}_{L(\mu)} + \mathrm{Id}_{L(\lambda)}\otimes L^{\mu}(z)$$

with the central charge $\frac{m dim\bar{\mathfrak{g}}}{m+h^{\vee}} + \frac{m^{'} dim\bar{\mathfrak{g}}}{m^{'}+h^{\vee}}$.

On the other hand, we may consider the whole tensor product $L(\lambda) \otimes L(\mu)$ as the highest weight $\mathfrak{g}$-module. Applying (2) to the highest weight $\mathfrak{g}$-module $L(\lambda) \otimes L(\mu)$, we get the associated Virasoro field

$$L^{\lambda \otimes \mu}(z) = \frac{1}{2\left(m + m' + h^{\vee}\right)} \sum_{i=1}^{dim \overline{\mathfrak{g}}} \left(\pi_{\lambda} \otimes \pi_{\mu}\right)\left(u_i(z)\right)\left(\pi_{\lambda} \otimes \pi_{\mu}\right)\left(u^i(z)\right)$$

with the central charge $\frac{\left(m+m'\right) dim \overline{\mathfrak{g}}}{m+m'+h^{\vee}}$.

Using (3), we have

$$\begin{aligned} \left[L_m^{\lambda,\mu}, t^n \otimes X\right] &= -nt^{n+m} \otimes X \ \left(X \in \overline{\mathfrak{g}},\ m \in \mathbb{Z} - \{0\},\ n \in \mathbb{Z}\right), \\ \left[L_m^{\lambda \otimes \mu}, t^n \otimes X\right] &= -nt^{n+m} \otimes X \ \left(X \in \overline{\mathfrak{g}},\ m \in \mathbb{Z} - \{0\},\ n \in \mathbb{Z}\right). \end{aligned} \tag{10}$$

Set $\widetilde{L}(z) = L^{\lambda,\mu}(z) - L^{\lambda \otimes \mu}(z) = \sum_{n \in \mathbb{Z}} \widetilde{L}_n z^{-n-2}$. According to ([15], Proposition 10.3), the field $\widetilde{L}(z)$ yields the coset Virasoro field on $L(\lambda) \otimes L(\mu)$ with central charge $\frac{m dim \overline{\mathfrak{g}}}{m+h^{\vee}} + \frac{m' dim \overline{\mathfrak{g}}}{m'+h^{\vee}} - \frac{\left(m+m'\right) dim \overline{\mathfrak{g}}}{m+m'+h^{\vee}}$.

For $\mu \in \overline{\mathfrak{h}}^* \oplus \mathbb{C}\delta$, we define

$$V_{\mu}^{\lambda,\mu} = \left\{ v \in L(\lambda) \otimes L(\mu) \,|\, Xv = 0 \ \left(\forall\, x \in \mathfrak{g}_+\right),\ Hv = \mu(H)\, v \ \left(\forall\, H \in \overline{\mathfrak{h}} \oplus \mathbb{C}K\right) \right\}.$$

It follows from (10) that the space $V_{\mu}^{\lambda,\mu}$ becomes a Vir-module via the coset Virasoro field $\widetilde{L}(z)$. Notice that $L(\lambda) \otimes L(\mu)$ is decomposed as a $Vir \oplus [\mathfrak{g}, \mathfrak{g}]$-module into

$$L(\lambda) \otimes L(\mu) = \sum_{\mu \in \overline{\mathfrak{h}}^* \oplus \mathbb{C}\delta \bmod \mathbb{C}\delta} V_{\mu}^{\lambda,\mu} \otimes L(\mu). \tag{11}$$

We obtain from (11) a string function

$$c_{\nu}^{\lambda \otimes \mu}(q) = Tr_{V_{\mu}^{\lambda,\mu}} q^{-d} = \sum_{j \in \mathbb{Z}_{\geq 0}} \mathrm{mult}_{\lambda \otimes \mu}\left(\mu - j\delta; \mathfrak{g}\right) q^j. \tag{12}$$

Using (11) and (12), we get

$$ch(\lambda)\, ch(\mu) = \sum_{\mu \in \overline{\mathfrak{h}}^* \oplus \mathbb{C}\delta \bmod \mathbb{C}\delta} c_{\nu}^{\lambda \otimes \mu}(q) ch L(\nu). \tag{13}$$

If we define the normalized branching function by

$$b_{\nu}^{\lambda \otimes \mu}(\tau) = q^{m_{\lambda} + m_{\mu} - m_{\nu}} c_{\nu}^{\lambda \otimes \mu}(q),$$

then the Formula (13) yields

$$ch'_{L(\lambda)}(\tau, z, t)\, ch'_{L(\mu)}(\tau, z, t) = \sum_{\nu} b_{\nu}^{\lambda \otimes \mu}(\tau)\, ch'_{L(\nu)}(\tau, z, t). \tag{14}$$

Let Λ be a dominant integral weight and μ a principal admissible weight of the affine Lie algebra $\mathfrak{g}$. Then, the branching function of the tensor product $L(\Lambda) \otimes L(\mu)$ can be expressed in terms of the string functions of $L(\Lambda)$ as follows.

Theorem 3. *Let $\mathfrak{g}$ be any affine Lie algebra and $m \in \mathbb{Z}_{\geq 0}$. Let $m' = \frac{v}{u}$ with $u \in \mathbb{Z}_{\geq 1}$, $v \in \mathbb{Z}$ and $gcd(u, v) = 1$. Assume that Λ and μ^0 are dominant integral weights of level m and $u\left(m' + h^{\vee}\right) - h^{\vee}$,* *respectively. Write* $\widetilde{c}^{\Lambda}_{\xi}(q)$ *for the modified string function* $q^{m_\Lambda - \frac{|\xi|^2}{2m}} c^{\Lambda}_{\xi,(\mathfrak{g};\mathfrak{g})}(q)$ *for* $\xi \in \overline{\mathfrak{h}}^* \oplus \mathbb{C}\delta$*, where* $c^{\Lambda}_{\xi,(\mathfrak{g};\mathfrak{g})}(q)$ *is the string function defined with respect to the pair* $(\mathfrak{g};\mathfrak{g})$ *(i.e.,* $\mathfrak{p} = \mathfrak{g}$ *in (8)). Then, for a principal admissible weight* $\mu = y\left(\mu^0 - (u-1)\left(m' + h^\vee\right)\Lambda_0 + \rho\right) - \rho \in P^{m'}_{u,y}$*, the following formula holds:*

$$ch'_{L(\Lambda)}(\tau,z,t)\, ch'_{L(\mu)}(\tau,z,t) = \sum_{\nu \in P^{m+m'}_{u,y} \text{ s.t. } \nu \equiv \Lambda+\mu \bmod Q} b^{\Lambda\otimes\mu}_{\nu}(\tau)\, ch'_{L(\nu)}(\tau,z,t),$$

where

$$b^{\Lambda\otimes\mu}_{\nu}(\tau) = \sum_{w\in W} \epsilon(w) q^{\frac{\left(m'+h^\vee\right)\left(m+m'+h^\vee\right)}{2m}\left|\frac{w\left(\nu^0+\rho\right)}{m+m'+h^\vee} - \frac{\mu^0+\rho}{m'+h^\vee}\right|^2} \widetilde{c}^{\Lambda}_{y(w(\nu^0+\rho)-(\mu^0+\rho)-(u-1)m\Lambda_0)}(q).$$

Proof. See ([9], Theorem 3.1). □

In the next section, we simply write $\widetilde{c}^{\Lambda}_{\lambda}$ for $\widetilde{c}^{\Lambda}_{\lambda}(q)$ if no confusion seems likely to arise, and will calculate explicitly the branching functions for some specific cases.

5. Explicit Calculations of Branching Functions

Let Λ_0 and Λ_1 be the fundamental weights of $\widehat{\mathfrak{sl}}_2$, and λ a principal admissible weight of $\widehat{\mathfrak{sl}}_2$. In this section, we explicitly calculate the branching functions arising from the tensor product decompositions of $(L(2\Lambda_0) \oplus L(2\Lambda_1)) \otimes L(\lambda)$ and $L(\rho) \otimes L(\lambda)$.

Let us write $\Pi = \{\alpha\}$ for the simple root system of $\mathfrak{sl}_2$. Then it is easy to check

$$h^\vee = 2,\ \rho = \Lambda_0 + \Lambda_1 = \overline{\rho} + h^\vee\Lambda_0 \text{ and } \Lambda_1 = \Lambda_0 + \frac{1}{2}\alpha \tag{15}$$

for $\widehat{\mathfrak{sl}_2}$. Let $m = \frac{v}{u}$ $(u \in 2\mathbb{Z}_{\geq 1},\ v \in 2\mathbb{Z}+1)$, and choose a principal admissible weight λ of level m satisfying $\lambda = \lambda^0 - (u-1)(m+2)\Lambda_0 \in P^m_{u,1}$ for $\lambda^0 \in P^{u(m+2)-2}_+$ (see Theorems 1 and 2).

Applying Theorem 3 to the tensor product representations $L(2\Lambda_0) \otimes L(\lambda)$ and $L(2\Lambda_1) \otimes L(\lambda)$, we obtain

$$\begin{aligned} & ch'_{L(2\Lambda_0)}(\tau,z,t)\, ch'_{L(\lambda)}(\tau,z,t) \\ = & \sum_{\nu \in P^{m+2}_{u,1} \text{ s.t. } \nu \equiv 2\Lambda_0+\lambda \bmod Q} b^{2\Lambda_0\otimes\lambda}_{\nu}(\tau)\, ch'_{L(\nu)}(\tau,z,t), \end{aligned} \tag{16}$$

where

$$b^{2\Lambda_0\otimes\lambda}_{\nu}(\tau) = \sum_{w\in W} \epsilon(w) q^{\frac{(m+2)(m+4)}{4}\left|\frac{w\left(\nu^0+\rho\right)}{m+4} - \frac{\lambda^0+\rho}{m+2}\right|^2} \widetilde{c}^{2\Lambda_0}_{w(\nu^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0}$$

and

$$\begin{aligned} & ch'_{L(2\Lambda_1)}(\tau,z,t)\, ch'_{L(\lambda)}(\tau,z,t) \\ = & \sum_{\widetilde{\nu} \in P^{m+2}_{u,1} \text{ s.t. } \widetilde{\nu} \equiv 2\Lambda_1+\lambda \bmod Q} b^{2\Lambda_1\otimes\lambda}_{\widetilde{\nu}}(\tau)\, ch'_{L(\widetilde{\nu})}(\tau,z,t), \end{aligned} \tag{17}$$

where

$$b^{2\Lambda_1\otimes\lambda}_{\widetilde{\nu}}(\tau) = \sum_{w\in W} \epsilon(w) q^{\frac{(m+2)(m+4)}{4}\left|\frac{w\left(\widetilde{\nu}^0+\rho\right)}{m+4} - \frac{\lambda^0+\rho}{m+2}\right|^2} \widetilde{c}^{2\Lambda_1}_{w(\widetilde{\nu}^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0}.$$

Similarly, if we apply Theorem 3 to the tensor product representation $L(\rho) \otimes L(\lambda)$ then we have

$$\begin{aligned} & ch^{'}_{L(\rho)}(\tau,z,t)\, ch^{'}_{L(\lambda)}(\tau,z,t) \\ = & \sum_{\nu \in P^{m+2}_{u,1} \text{ s.t. } \nu \equiv \rho+\lambda \bmod Q} b^{\rho\otimes\lambda}_{\nu}(\tau)\, ch^{'}_{L(\nu)}(\tau,z,t), \end{aligned} \tag{18}$$

where

$$b^{\rho\otimes\lambda}_{\nu}(\tau) = \sum_{w\in W} \epsilon(w) q^{\frac{(m+2)(m+4)}{4}\left|\frac{w\left(\nu^0+\rho\right)}{m+4} - \frac{\lambda^0+\rho}{m+2}\right|^2} \widetilde{c}^{\rho}_{w(\nu^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0}.$$

For $\lambda^0 \in P^{u(m+2)-2}_{+}$ and $\nu^0 \in P^{u(m+4)-2}_{+}$, let us write

$$\begin{aligned} \lambda^0 &= (u(m+2)-2-n)\,\Lambda_0 + n\Lambda_1, \\ \nu^0 &= \left(u(m+4)-2-n^{'}\right)\Lambda_0 + n^{'}\Lambda_1 \end{aligned} \tag{19}$$

for some $n \in \mathbb{Z}$ and $n^{'} \in \mathbb{Z}$. Then, we can rewrite λ and ν in (16) as

$$\lambda = \lambda^0 - (u-1)(m+2)\Lambda_0 = (m-n)\Lambda_0 + n\Lambda_1$$

and

$$\nu = \nu^0 - (u-1)(m+4)\Lambda_0 = \left(m-n^{'}+2\right)\Lambda_0 + n^{'}\Lambda_1.$$

Since $2\Lambda_0 - (\nu-\lambda) \in Q$ and $2\Lambda_1 - 2\Lambda_0 = \alpha$, we should have $n \equiv n^{'} \pmod 2$.

Similarly, for $\widetilde{\nu}^0 = \left(u(m+4)-2-n^{''}\right)\Lambda_0 + n^{''}\Lambda_1 \in P^{u(m+4)-2}_{+}$, we obtain $\widetilde{\nu} = \left(m-n^{''}+2\right)\Lambda_0 + n^{''}\Lambda_1$ $\left(n^{''} \in \mathbb{Z}\right)$. From the condition $2\Lambda_1 - (\widetilde{\nu}-\lambda) \in \mathbb{Z}\alpha$, we have the same condition $n \equiv n^{''} \pmod 2$ as the case of ν. For this reason, we shall identify $\widetilde{\nu}$ with ν in the following Theorem 4. The same argument yields that the condition $\nu \equiv \rho + \lambda \bmod Q$ in (18) is equivalent to the condition $n^{'} \equiv n+1 \pmod 2$ in (19).

Theorem 4. *Let $m = \frac{v}{u}$ for $u \in 2\mathbb{Z}_{\geq 1}$ and $v \in 2\mathbb{Z}+1$, and let $p = u(m+4)$ and $p^{'} = u(m+2)$.*

1. *Suppose that*

$$\lambda^0 = (u(m+2)-2-n)\,\Lambda_0 + n\Lambda_1$$

and

$$\nu^0 = \left(u(m+2)-2-n^{'}\right)\Lambda_0 + n^{'}\Lambda_1$$

for some $n \in 4\mathbb{Z}$ and $n^{'} \in \mathbb{Z}$ satisfying $n \equiv n^{'} \pmod 2$. Then, the branching functions in (16) and (17) are explicitly given by

$$\begin{aligned} b^{2\Lambda_0\otimes\lambda}_{\nu}(\tau) &= \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp^{'}}\left(2pp^{'}j+\left(n^{'}+1\right)p^{'}-(n+1)p\right)^2} A \\ &\quad - \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp^{'}}\left(2pp^{'}j-\left(n^{'}+1\right)p^{'}-(n+1)p\right)^2} B \end{aligned} \tag{20}$$

and

$$b_{\nu}^{2\Lambda_1\otimes\lambda}(\tau) = \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j+\left(n'+1\right)p'-(n+1)p\right)^2} B - \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j-\left(n'+1\right)p'-(n+1)p\right)^2} A, \tag{21}$$

where $\begin{cases} A = \widetilde{c}^{2\Lambda_0}_{2\Lambda_0},\ B = \widetilde{c}^{2\Lambda_0}_{2\Lambda_1} & \text{if } n' \equiv 0 \pmod 4 \\ A = \widetilde{c}^{2\Lambda_0}_{2\Lambda_1},\ B = \widetilde{c}^{2\Lambda_0}_{2\Lambda_0} & \text{if } n' \equiv 2 \pmod 4. \end{cases}$

2. *Assume that*

$$\lambda^0 = (u(m+2)-2-n)\,\Lambda_0 + n\Lambda_1$$

and

$$\nu^0 = \left(u(m+2)-2-n'\right)\Lambda_0 + n'\Lambda_1$$

for some $n \in 4\mathbb{Z}$ *and* $n' \in 4\mathbb{Z}+1$. *Then, the branching function in (18) is explicitly given by*

$$b_{\nu}^{\rho\otimes\lambda}(\tau) = \sum_{j\in\mathbb{Z}} \left(q^{\frac{1}{8pp'}\left(2pp'j+\left(n'+1\right)p'-(n+1)p\right)^2} - q^{\frac{1}{8pp'}\left(2pp'j-\left(n'+1\right)p'-(n+1)p\right)^2} \right) \widetilde{c}^{\rho}_{\rho}. \tag{22}$$

Proof. We first prove (20) and (21).

Recall that the Weyl group W of $\widehat{\mathfrak{sl}_2}$ is given by $\{t_{j\alpha},\ t_{j\alpha}r_\alpha | j \in \mathbb{Z}\}$.

By (15) and (19), we have

$$\nu^0 + \rho = u(m+4)\Lambda_0 + \frac{n'+1}{2}\alpha$$

and

$$\lambda^0 + \rho = u(m+2)\Lambda_0 + \frac{n+1}{2}\alpha.$$

So, we get

$$t_{j\alpha}\left(\nu^0+\rho\right) - \left(\lambda^0+\rho\right) - 2(u-1)\Lambda_0 = 2\Lambda_0 + \left(u(m+4)j + \frac{n'-n}{2}\right)\alpha - \left(u(m+4)j^2 + \left(n'+1\right)j\right)\delta \tag{23}$$

and

$$t_{j\alpha}r_\alpha\left(\nu^0+\rho\right) - \left(\lambda^0+\rho\right) - 2(u-1)\Lambda_0 = 2\Lambda_0 + \left(u(m+4)j - \frac{n'+n+2}{2}\right)\alpha + \left(u(m+4)j^2 - \left(n'+1\right)j\right)\delta. \tag{24}$$

Notice from ([11], (12.7.9)) that we have

$$\widetilde{c}^{2\Lambda_0}_{w(\lambda')+2\gamma+a\delta} = \widetilde{c}^{2\Lambda_0}_{\lambda'} \tag{25}$$

for $\lambda' \in \mathfrak{h}^*$, $w \in \overline{W}$, $\gamma \in \mathbb{Z}\alpha$ and $a \in \mathbb{C}$. Since $\overline{W} = \{1, r_\alpha\}$, we see from (25) that

$$\widetilde{c}^{2\Lambda_0}_{\lambda+(2n+1)\alpha+a\delta} = \widetilde{c}^{2\Lambda_0}_{(\lambda+\alpha)+2n\alpha+a\delta} = \widetilde{c}^{2\Lambda_0}_{\lambda+\alpha}$$

and

$$\widetilde{c}^{2\Lambda_0}_{r_\alpha(\lambda)+(2n+1)\alpha+a\delta}=\widetilde{c}^{2\Lambda_0}_{r_\alpha(\lambda+\alpha)+(2n+2)\alpha+a\delta}=\widetilde{c}^{2\Lambda_0}_{\lambda+\alpha}.$$

Hence, in any case we obtain

$$\widetilde{c}^{2\Lambda_0}_{w(\lambda)+(2n+1)\alpha+a\delta}=\widetilde{c}^{2\Lambda_0}_{\lambda+\alpha} \tag{26}$$

for $w\in\overline{W}$. Since u is even, we have

$$u(m+4)j+\frac{n'-n}{2}\equiv\frac{n'-n}{2}\pmod 2$$

and

$$u(m+4)j-\frac{n'+n+2}{2}\equiv-\frac{n'+n+2}{2}\pmod 2.$$

Since $n\in 4\mathbb{Z}$ and $n\equiv n' \pmod 2$, we obtain $n'\equiv 0\pmod 4$ or $n'\equiv 2\pmod 4$. If $n'\equiv 0\pmod 4$, then $\frac{n'-n}{2}\equiv 0\pmod 2$ and $-\frac{n'+n+2}{2}\equiv 1\pmod 2$. Thus, by (23), (24), (25) and (26) we get

$$\widetilde{c}^{2\Lambda_0}_{t_{j\alpha}(\nu^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0}=\widetilde{c}^{2\Lambda_0}_{2\Lambda_0} \tag{27}$$

and

$$\widetilde{c}^{2\Lambda_0}_{t_{j\alpha}r_\alpha(\nu^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0}=\widetilde{c}^{2\Lambda_0}_{2\Lambda_0+\alpha}=\widetilde{c}^{2\Lambda_0}_{2\Lambda_1}. \tag{28}$$

Similarly, if $n'\equiv 2\pmod 4$, then $\frac{n'-n}{2}\equiv 1\pmod 2$ and $-\frac{n'+n+2}{2}\equiv 0\pmod 2$. So, in this case we have

$$\widetilde{c}^{2\Lambda_0}_{t_{j\alpha}(\lambda^0+\rho)-(\mu^0+\rho)-2(u-1)\Lambda_0}=\widetilde{c}^{2\Lambda_0}_{2\Lambda_0+\alpha}=\widetilde{c}^{2\Lambda_0}_{2\Lambda_1} \tag{29}$$

and

$$\widetilde{c}^{2\Lambda_0}_{t_{j\alpha}r_\alpha(\lambda^0+\rho)-(\mu^0+\rho)-2(u-1)\Lambda_0}=\widetilde{c}^{2\Lambda_0}_{2\Lambda_0}. \tag{30}$$

We now compute the exponent $\frac{(m+2)(m+4)}{4}\left|\frac{w(\nu^0+\rho)}{m+4}-\frac{\lambda^0+\rho}{m+2}\right|^2$ of q in (16) and (17).

Since $p=u(m+4)$ in assumption, we see that

$$t_{j\alpha}\left(\nu^0+\rho\right)=p\Lambda_0+\left(pj+\frac{n'+1}{2}\right)\alpha-\left(pj^2+\left(n'+1\right)j\right)\delta. \tag{31}$$

and

$$t_{j\alpha}r_\alpha\left(\nu^0+\rho\right)=p\Lambda_0+\left(pj-\frac{n'+1}{2}\right)\alpha-\left(pj^2-\left(n'+1\right)j\right)\delta. \tag{32}$$

It also follows from the assumption $p'=u(m+2)$ that

$$\begin{aligned}
&\frac{(m+2)(m+4)}{4}\left|\frac{w\left(\nu^0+\rho\right)}{m+4}-\frac{\lambda^0+\rho}{m+2}\right|^2\\
&=\frac{u^2(m+2)(m+4)}{4}\left|\frac{w\left(\nu^0+\rho\right)}{u(m+4)}-\frac{\lambda^0+\rho}{u(m+2)}\right|^2 \qquad (33)\\
&=\frac{pp'}{4}\left|\frac{w\left(\nu^0+\rho\right)}{p}-\frac{\lambda^0+\rho}{p'}\right|^2.
\end{aligned}$$

Notice from (31) and (32) that

$$\begin{aligned} & \frac{t_{j\alpha}\left(\nu^0+\rho\right)}{p} - \frac{\lambda^0+\rho}{p^{'}} \\ = & \left(\Lambda_0 + \left(j + \frac{n^{'}+1}{2p}\right)\alpha\right) - \frac{1}{p^{'}}\left(p^{'}\Lambda_0 + \frac{n+1}{2}\alpha\right) \bmod \mathbb{C}\delta \\ = & \left(j + \frac{n^{'}+1}{2p} - \frac{n+1}{2p^{'}}\right)\alpha \bmod \mathbb{C}\delta \end{aligned}$$

and

$$\begin{aligned} & \frac{t_{j\alpha}r_\alpha\left(\nu^0+\rho\right)}{p} - \frac{\lambda^0+\rho}{p^{'}} \\ = & \left(\Lambda_0 + \left(j - \frac{n^{'}+1}{2p}\right)\alpha\right) - \frac{1}{p^{'}}\left(p^{'}\Lambda_0 + \frac{n+1}{2}\alpha\right) \bmod \mathbb{C}\delta \\ = & \left(j - \frac{n^{'}+1}{2p} - \frac{n+1}{2p^{'}}\right)\alpha \bmod \mathbb{C}\delta. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \left|\frac{t_{j\alpha}\left(\nu^0+\rho\right)}{p} - \frac{\lambda^0+\rho}{p^{'}}\right|^2 &= \frac{1}{2\left(pp^{'}\right)^2}\left(2pp^{'}j + \left(n^{'}+1\right)p^{'} - (n+1)p\right)^2, \\ \left|\frac{t_{j\alpha}r_\alpha\left(\nu^0+\rho\right)}{p} - \frac{\lambda^0+\rho}{p^{'}}\right|^2 &= \frac{1}{2\left(pp^{'}\right)^2}\left(2pp^{'}j - \left(n^{'}+1\right)p^{'} - (n+1)p\right)^2. \end{aligned} \tag{34}$$

Hence, if $n^{'} \equiv 0 \pmod 4$, then we obtain from (27), (28), (33) and (34) that

$$\begin{aligned} b_\nu^{2\Lambda_0\otimes\lambda}(\tau) = & \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp^{'}}\left(2pp^{'}j+\left(n^{'}+1\right)p^{'}-(n+1)p\right)^2}\widetilde{c}_{2\Lambda_0}^{2\Lambda_0} \\ & -\sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp^{'}}\left(2pp^{'}j-\left(n^{'}+1\right)p^{'}-(n+1)p\right)^2}\widetilde{c}_{2\Lambda_1}^{2\Lambda_0}. \end{aligned}$$

If $n^{'} \equiv 2 \pmod 4$ then we also obtain from (29), (30), (33) and (34)

$$\begin{aligned} b_\nu^{2\Lambda_0\otimes\lambda}(\tau) = & \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp^{'}}\left(2pp^{'}j+\left(n^{'}+1\right)p^{'}-(n+1)p\right)^2}\widetilde{c}_{2\Lambda_1}^{2\Lambda_0} \\ & -\sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp^{'}}\left(2pp^{'}j-\left(n^{'}+1\right)p^{'}-(n+1)p\right)^2}\widetilde{c}_{2\Lambda_0}^{2\Lambda_0}. \end{aligned}$$

The Formula (20) now follows.

Applying the same argument as above to the case of $b_\nu^{2\Lambda_1\otimes\lambda}(\tau)$, we obtain

$$\begin{cases} \widetilde{c}^{2\Lambda_1}_{t_{j\alpha}(\nu^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0} = \widetilde{c}^{2\Lambda_1}_{2\Lambda_0} & \text{if } n^{'} \equiv 0 \pmod 4 \\ \widetilde{c}^{2\Lambda_1}_{t_{j\alpha}r_\alpha(\nu^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0} = \widetilde{c}^{2\Lambda_1}_{2\Lambda_1} & \text{if } n^{'} \equiv 0 \pmod 4. \end{cases} \tag{35}$$

and

$$\begin{cases} \widetilde{c}^{2\Lambda_1}_{t_{j\alpha}(\nu^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0} = \widetilde{c}^{2\Lambda_1}_{2\Lambda_1} & \text{if } n' \equiv 2 \pmod 4 \\ \widetilde{c}^{2\Lambda_1}_{t_{j\alpha}r_\alpha(\nu^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0} = \widetilde{c}^{2\Lambda_1}_{2\Lambda_0} & \text{if } n' \equiv 2 \pmod 4 . \end{cases} \tag{36}$$

Notice that we have $\widetilde{c}^{M\Lambda_0+N\Lambda_1}_{m\Lambda_0+n\Lambda_1} = \widetilde{c}^{N\Lambda_0+M\Lambda_1}_{n\Lambda_0+m\Lambda_1}$ due to the outer automorphism of $\widehat{\mathfrak{sl}}_2$.

Hence, we obtain that

$$\widetilde{c}^{2\Lambda_1}_{2\Lambda_0} = \widetilde{c}^{2\Lambda_0}_{2\Lambda_1} \text{ and } \widetilde{c}^{2\Lambda_1}_{2\Lambda_1} = \widetilde{c}^{2\Lambda_0}_{2\Lambda_0} \tag{37}$$

Therefore, if $n' \equiv 0 \pmod 4$ then we get from (35), (36), (33), (34) and (37) that

$$\begin{aligned} b^{2\Lambda_1\otimes\lambda}_{\nu}(\tau) &= \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j+\left(n'+1\right)p'-(n+1)p\right)^2} \widetilde{c}^{2\Lambda_0}_{2\Lambda_1} \\ &\quad - \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j-\left(n'+1\right)p'-(n+1)p\right)^2} \widetilde{c}^{2\Lambda_0}_{2\Lambda_0}. \end{aligned}$$

Similarly, if $n' \equiv 2 \pmod 4$ then we obtain that

$$\begin{aligned} b^{2\Lambda_1\otimes\lambda}_{\nu}(\tau) &= \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j+\left(n'+1\right)p'-(n+1)p\right)^2} \widetilde{c}^{2\Lambda_0}_{2\Lambda_0} \\ &\quad - \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j-\left(n'+1\right)p'-(n+1)p\right)^2} \widetilde{c}^{2\Lambda_0}_{2\Lambda_1}. \end{aligned}$$

The Formula (21) now follows.

Let us now prove (22).

The proof is exactly the same as those of (20) and (21) except for calculations of the string function $\widetilde{c}^{\rho}_{w(\nu^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0}$. Recall from the assumption that $n \in 4\mathbb{Z}$ and $n' \in 4\mathbb{Z}+1$. Then, by (23)–(25) we obtain

$$\widetilde{c}^{\rho}_{t_{j\alpha}(\nu^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0} = \widetilde{c}^{\rho}_{2\Lambda_0+\frac{n'-n}{2}\alpha} = \widetilde{c}^{\rho}_{2\Lambda_0+\frac{1}{2}\alpha} = \widetilde{c}^{\rho}_{\rho}$$

and

$$\widetilde{c}^{\rho}_{t_{j\alpha}r_\alpha(\nu^0+\rho)-(\lambda^0+\rho)-2(u-1)\Lambda_0} = \widetilde{c}^{\rho}_{2\Lambda_0-\frac{n'+n+2}{2}\alpha} = \widetilde{c}^{\rho}_{2\Lambda_0-2\alpha+\frac{1}{2}\alpha} = \widetilde{c}^{\rho}_{\rho}.$$

The result now follows. □

It is immediate from Theorem 4 that the branching function of $(L(2\Lambda_0) \oplus L(2\Lambda_1)) \otimes L(\lambda)$ for $\widehat{\mathfrak{sl}}_2$ is given by

$$\begin{aligned} & b^{2\Lambda_0\otimes\lambda}_{\nu}(\tau) + b^{2\Lambda_1\otimes\lambda}_{\nu}(\tau) \\ = & \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j+\left(n'+1\right)p'-(n+1)p\right)^2} \left(\widetilde{c}^{2\Lambda_0}_{2\Lambda_0} + \widetilde{c}^{2\Lambda_0}_{2\Lambda_1}\right) \\ & - \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j-\left(n'+1\right)p'-(n+1)p\right)^2} \left(\widetilde{c}^{2\Lambda_0}_{2\Lambda_0} + \widetilde{c}^{2\Lambda_0}_{2\Lambda_1}\right). \end{aligned}$$

In the following theorem, we explicitly calculate $\widetilde{c}^{2\Lambda_0}_{2\Lambda_0} + \widetilde{c}^{2\Lambda_0}_{2\Lambda_1}$ and $\widetilde{c}^{\rho}_{\rho}$ in terms of the Dedekind eta function.

Theorem 5. *$\widetilde{c}^{2\Lambda_0}_{2\Lambda_0} + \widetilde{c}^{2\Lambda_0}_{2\Lambda_1} = \frac{\eta(\tau)}{\eta\left(\frac{\tau}{2}\right)\eta(2\tau)}$ and $\widetilde{c}^{\rho}_{\rho} = \frac{\eta(2\tau)}{\eta(\tau)^2}$, where $\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$.*

Proof. It follows from the Weyl-Kac character formula that

$$chL\left(2\Lambda_0\right) = \frac{1}{e^{\rho}R}\sum_{w\in W}\epsilon(w)e^{w(2\Lambda_0+\rho)}, \tag{38}$$

where

$$W = \{t_{j\alpha},\ t_{j\alpha}r_{\alpha} | j \in \mathbb{Z}\}$$

and

$$R = \Pi_{n=1}^{\infty}\left(1-q^n\right)\left(1-e^{-\alpha}q^{n-1}\right)\left(1-e^{\alpha}q^n\right).$$

Calculating $w\left(2\Lambda_0+\rho\right)$ for $w \in W$, we obtain from (38)

$$chL\left(2\Lambda_0\right) = \frac{1}{e^{\rho}R}\left(\sum_{j\in\mathbb{Z}} e^{2\Lambda_0+\rho+4j\alpha-\frac{(4j)^2+4j}{4}\delta} - \sum_{j\in\mathbb{Z}} e^{2\Lambda_0+\rho+(-4j-1)\alpha-\frac{(-4j-1)^2+(-4j-1)}{4}\delta}\right). \tag{39}$$

Similarly, we can evaluate $chL\left(2\Lambda_1\right)$ as follows:

$$\frac{1}{e^{\rho}R}q^{-\frac{1}{2}}\left(\sum_{j\in\mathbb{Z}} e^{2\Lambda_0+\rho+(4j+1)\alpha-\frac{(4j+1)^2+(4j+1)}{4}\delta} - \sum_{j\in\mathbb{Z}} e^{2\Lambda_0+\rho+(-4j-2)\alpha-\frac{(-4j-2)^2+(-4j-2)}{4}\delta}\right). \tag{40}$$

Using (39), (40) and the Jacobi triple product identity, we have

$$\begin{aligned}
& chL\left(2\Lambda_0\right) - q^{\frac{1}{2}}chL\left(2\Lambda_1\right) \\
=\ & \frac{1}{e^{\rho}R}\sum_{j\in\mathbb{Z}}(-1)^j e^{2\Lambda_0+\rho+j\alpha-\frac{j^2+j}{4}\delta} \\
=\ & \frac{e^{2\Lambda_0}}{R}\sum_{j\in\mathbb{Z}}(-1)^j e^{j\alpha}q^{\frac{j^2+j}{4}} \\
=\ & \frac{e^{2\Lambda_0}}{R}\prod_{n=1}^{\infty}\left(1-q^{\frac{n}{2}}\right)\left(1-e^{\alpha}q^{\frac{n}{2}}\right)\left(1-e^{-\alpha}q^{\frac{n-1}{2}}\right) \\
=\ & e^{2\Lambda_0}\prod_{n=1}^{\infty}\left(1-q^{\frac{2n-1}{2}}\right)\left(1-e^{\alpha}q^{\frac{2n-1}{2}}\right)\left(1-e^{-\alpha}q^{\frac{2n-1}{2}}\right) \\
=\ & e^{2\Lambda_0}\prod_{n=1}^{\infty}\frac{1-q^{n-\frac{1}{2}}}{1-q^n}\prod_{n=1}^{\infty}\left(1-q^n\right)\left(1-e^{\alpha}q^{\frac{2n-1}{2}}\right)\left(1-e^{-\alpha}q^{\frac{2n-1}{2}}\right) \\
=\ & e^{2\Lambda_0}\prod_{n=1}^{\infty}\frac{1-q^{n-\frac{1}{2}}}{1-q^n}\sum_{j\in\mathbb{Z}}(-1)^j e^{j\alpha}q^{\frac{j^2}{2}}.
\end{aligned} \tag{41}$$

Recall from ([11], (12.7.1)) that

$$chL\left(2\Lambda_0\right) = \sum_{\lambda\in\max(2\Lambda_0)} c_{\lambda}^{2\Lambda_0}e^{\lambda}. \tag{42}$$

and

$$chL\left(2\Lambda_1\right) = \sum_{\lambda\in\max(2\Lambda_1)} c_{\lambda}^{2\Lambda_1}e^{\lambda}. \tag{43}$$

From (42) and (43), the coefficient of $e^{2\Lambda_0}$ in $chL\left(2\Lambda_0\right) - q^{\frac{1}{2}}chL\left(2\Lambda_1\right)$ should be equal to

$$c_{2\Lambda_0}^{2\Lambda_0} - q^{\frac{1}{2}}c_{2\Lambda_0}^{2\Lambda_1}. \tag{44}$$

Comparing (44) with the coefficient of $e^{2\Lambda_0}$ in (41), we obtain

$$c_{2\Lambda_0}^{2\Lambda_0} - q^{\frac{1}{2}} c_{2\Lambda_0}^{2\Lambda_1} = \prod_{n=1}^{\infty} \frac{1-q^{n-\frac{1}{2}}}{1-q^n}. \tag{45}$$

By substituting $x = q^{\frac{1}{2}}$, we obtain from (45)

$$c_{2\Lambda_0}^{2\Lambda_0} - x c_{2\Lambda_0}^{2\Lambda_1} = \prod_{n=1}^{\infty} \frac{1-x^{2n-1}}{1-x^{2n}}.$$

By letting $x \longmapsto -x$, we get

$$c_{2\Lambda_0}^{2\Lambda_0} + x c_{2\Lambda_0}^{2\Lambda_1} = \prod_{n=1}^{\infty} \frac{1+x^{2n-1}}{1-x^{2n}},$$

and this implies

$$c_{2\Lambda_0}^{2\Lambda_0} + q^{\frac{1}{2}} c_{2\Lambda_0}^{2\Lambda_1} = \prod_{n=1}^{\infty} \frac{1+q^{n-\frac{1}{2}}}{1-q^n}. \tag{46}$$

On the other hand, it is easy to check that $m_{2\Lambda_0} - \frac{|2\Lambda_0|^2}{4} = -\frac{1}{16}$ and $m_{2\Lambda_1} - \frac{|2\Lambda_0|^2}{4} = \frac{7}{16}$, and these yield that $\widetilde{c}_{2\Lambda_0}^{2\Lambda_0} = q^{-\frac{1}{16}} c_{2\Lambda_0}^{2\Lambda_0}$ and $\widetilde{c}_{2\Lambda_0}^{2\Lambda_1} = q^{\frac{7}{16}} c_{2\Lambda_0}^{2\Lambda_1}$. So, (46) gives rise to

$$q^{\frac{1}{16}} \left(\widetilde{c}_{2\Lambda_0}^{2\Lambda_0} + \widetilde{c}_{2\Lambda_0}^{2\Lambda_1} \right) = \prod_{n=1}^{\infty} \frac{1+q^{n-\frac{1}{2}}}{1-q^n}. \tag{47}$$

Thus,

$$\begin{aligned}
&\frac{\eta(\tau)}{\eta\left(\frac{\tau}{2}\right)\eta(2\tau)} \\
=\;& \frac{q^{-\frac{1}{16}}}{\prod_{n=1}^{\infty}\left(1-q^{\frac{n}{2}}\right)\prod_{n=1}^{\infty}(1+q^n)} \\
=\;& \frac{q^{-\frac{1}{16}}\prod_{n=1}^{\infty}\left(1+q^{\frac{n}{2}}\right)}{\prod_{n=1}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(1+q^n)} \\
=\;& \frac{q^{-\frac{1}{16}}\prod_{n=1}^{\infty}\left(1+q^{n-\frac{1}{2}}\right)}{\prod_{n=1}^{\infty}(1-q^n)} \\
=\;& \widetilde{c}_{2\Lambda_0}^{2\Lambda_0} + \widetilde{c}_{2\Lambda_0}^{2\Lambda_1} \;\; (\text{see } (47)).
\end{aligned}$$

Next, we compute $\widetilde{c}_{\rho}^{\rho}$.

Replacing all positive roots α by $k\alpha$ $(k \in \mathbb{Z}_{\geq 1})$, we obtain from the denominator identity that

$$e^{k\rho} \prod_{\alpha \in \Delta_+} \left(1 - e^{-k\alpha}\right)^{\text{mult}(\alpha)} = \sum_{w \in W} \epsilon(w) e^{w(k\rho)}.$$

Thus, it follows from the Jacobi triple identity that

$$\begin{aligned}
&chL\,(\rho)\\
&=\frac{1}{e^{\rho}R}\sum_{w\in W}\epsilon(w)e^{w(2\rho)}\\
&=\frac{1}{e^{\rho}R}e^{2\rho}\prod_{\alpha\in\Delta_+}\left(1-e^{-2\alpha}\right)^{\mathrm{mult}(\alpha)}\\
&=e^{\rho}\prod_{j=1}^{\infty}\frac{\left(1-q^{2j}\right)\left(1-e^{-2\alpha}q^{2(j-1)}\right)\left(1-e^{2\alpha}q^{2j}\right)}{\left(1-q^{j}\right)\left(1-e^{-\alpha}q^{(j-1)}\right)\left(1-e^{\alpha}q^{j}\right)}\ \left(q=e^{-\delta}\right)\\
&=e^{\rho}\prod_{j=1}^{\infty}\frac{\left(1+q^{j}\right)}{\left(1-q^{j}\right)}\prod_{j=1}^{\infty}\left(1-q^{j}\right)\left(1+e^{-\alpha}q^{(j-1)}\right)\left(1+e^{\alpha}q^{j}\right)\\
&=\prod_{j=1}^{\infty}\frac{\left(1+q^{j}\right)}{\left(1-q^{j}\right)}\sum_{j\in\mathbb{Z}}e^{\rho-j\alpha}q^{\frac{j^2-j}{2}}\\
&=\prod_{j=1}^{\infty}\frac{\left(1+q^{j}\right)}{\left(1-q^{j}\right)}\sum_{j\in\mathbb{Z}}e^{\rho-j\alpha-\frac{j^2-j}{2}\delta}.
\end{aligned}\tag{48}$$

On the other hand, we get from ([11], (12.7.1)) that

$$chL\,(\rho)=\sum_{\lambda\in\max(\rho)}c_{\lambda}^{\rho}e^{\lambda}.\tag{49}$$

Comparing the coefficients of e^{ρ} in (48) and (49), we have

$$c_{\rho}^{\rho}=\prod_{j=1}^{\infty}\frac{1+q^{j}}{1-q^{j}}.$$

Moreover, it is easy to check $m_{\rho}-\frac{|\rho|^2}{4}=0$ which implies $\widetilde{c}_{\rho}^{\rho}=c_{\rho}^{\rho}$.

The result now follows. □

6. Super-Virasoro algebras

In this section, we shall investigate relationships between our results on branching functions and the representation theory of super-Virasoro algebras. As by-products, we generalize the tensor product decomposition formulas ([6], (4.1.2a) and (4.1.2b)) to the case of principal admissible weights.

Let us first review the theta functions associated to an affine Lie algebra $\mathfrak{g}=\mathbb{C}\left[t,t^{-1}\right]\otimes\overline{\mathfrak{g}}\oplus\mathbb{C}K\oplus\mathbb{C}d$ and its Cartan subalgebra $\mathfrak{h}$.

For $\lambda\in P^{m}\ \ (m\in\mathbb{Z}_{\geq 0})$, the theta function θ_{λ} is defined as

$$\theta_{\lambda}=e^{-\frac{|\lambda|^2}{2m}\delta}\sum_{\alpha\in\overline{Q}}e^{t_{\alpha}\lambda},$$

where $\overline{Q}$ is the root lattice of $\overline{\mathfrak{g}}$. Using the coordinate (τ,z,t) for the Cartan subalgebra $\mathfrak{h}$, we get

$$\theta_{\lambda}\,(\tau,z,t)=e^{2\pi imt}\sum_{\gamma\in\overline{Q}+\frac{\overline{\lambda}}{m}}q^{\frac{m}{2}|\gamma|^2}e^{2\pi im(\gamma|z)},$$

where $\overline{\lambda}$ is the projection of λ onto $\overline{\mathfrak{h}}$.

In particular, if we take $\lambda = md + \frac{1}{2}n\alpha + rK \in P^m$ for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ then the corresponding theta function is

$$\theta_\lambda(\tau, z, t) = e^{2\pi imt} \sum_{k\in\mathbb{Z}+\frac{n}{m}} q^{mk^2} e^{2\pi im(k\alpha|z)}. \tag{50}$$

Evaluating (50) at $(\tau, 0, 0)$, we have

$$\theta(\tau, 0, 0) = \sum_{j\in\mathbb{Z}} q^{m\left(j+\frac{n}{2m}\right)^2} \quad \left(q = e^{2\pi i\tau}\right). \tag{51}$$

For convenience, we shall simply write $\theta_{n,m}$ for (51) in the remaining part of this section.

Next, we review the super-Virasoro algebras Vir_ϵ $\left(\epsilon = 0, \frac{1}{2}\right)$. (For $\epsilon = 0$ or $\frac{1}{2}$, Vir_ϵ is called the *Ramond* and *Neveu-Schwarz* superalgebra, respectively.)

The super-Virasoro algebra Vir_ϵ is the complex superalgebra with a basis $\{c, \ell_j, g_m | j \in \mathbb{Z}$ and $m \in \epsilon + \mathbb{Z}\}$, and it satisfies commutation relations

1. $[\ell_i, \ell_j] = (i-j)\ell_{i+j} + \frac{1}{12}\left(i^3 - i\right)\delta_{i+j,0}c$,
2. $[c, \ell_j] = 0$,
3. $[g_m, \ell_n] = \left(m - \frac{n}{2}\right) g_{m+n}$,
4. $[g_m, c] = 0$,
5. $\{g_m, g_n\} = 2\ell_{m+n} + \frac{1}{3}\left(m^2 - \frac{1}{4}\right)\delta_{m+n,0}c$,

where $\{\ ,\ \}$ denotes an anti-commutator bracket between two odd elements.

Recall that every minimal series irreducible module of Vir_ϵ corresponds to the pair of numbers $\left(z^{(p,p')}, h^{(p,p')}_{r,s;\epsilon}\right)$. Here, $z^{(p,p')}$ is the central charge equals $z^{(p,p')} = \frac{3}{2}\left(1 - \frac{2\left(p-p'\right)^2}{pp'}\right)$, and $h^{(p,p')}_{r,s;\epsilon}$ is the minimal eigenvalue of ℓ_0 equals $h^{(p,p')}_{r,s;\epsilon} = \frac{\left(pr-p's\right)^2 - \left(p-p'\right)^2}{8pp'} + \frac{1}{16}(1-2\epsilon)$ for $p, p', r, s \in \mathbb{Z}$, $2 \le p' < p$, $p - p' \in 2\mathbb{Z}$, $gcd\left(\frac{p-p'}{2}, p'\right) = 1$, $1 \le r \le p' - 1$, $1 \le s \le p - 1$ and $r - s \in 2\mathbb{Z}$ (we refer to ([16], Theorem 5.2) for the details).

Write $V_\epsilon\left(z^{(p,p')}, h^{(p,p')}_{r,s;\epsilon}\right)$ for the minimal series module over Vir_ϵ corresponding to $\left(z^{(p,p')}, h^{(p,p')}_{r,s;\epsilon}\right)$. According to [17,18], it follows that

$$chV_\epsilon\left(z^{(p,p')}, h^{(p,p')}_{r,s;\epsilon}\right) = q^{\frac{1}{24}z^{(p,p')}} \eta_\epsilon(\tau)\left(\theta_{\frac{pr-p's}{2}, \frac{pp'}{2}} - \theta_{\frac{pr+p's}{2}, \frac{pp'}{2}}\right),$$

where $\eta_\epsilon(\tau) = \begin{cases} \frac{\eta(2\tau)}{\eta(\tau)^2} & \text{if } \epsilon = 0 \\ \frac{\eta(\tau)}{\eta\left(\frac{\tau}{2}\right)\eta(2\tau)} & \text{if } \epsilon = \frac{1}{2}. \end{cases}$

By (51), we see that

$$\theta_{\frac{pr-p's}{2}, \frac{pp'}{2}} = \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j + pr - p's\right)^2},$$
$$\theta_{\frac{pr+p's}{2}, \frac{pp'}{2}} = \sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j + pr + p's\right)^2}.$$

So, the normalized character of $V_\epsilon\left(z^{\left(p,p'\right)},h_{r,s;\epsilon}^{\left(p,p'\right)}\right)$ is

$$\begin{aligned} & \chi_{r,s;\epsilon}^{\left(p,p'\right)}\left(\tau\right) \\ = \ & \eta_\epsilon\left(\tau\right)\left(\sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j+pr-p's\right)^2}-\sum_{j\in\mathbb{Z}} q^{\frac{1}{8pp'}\left(2pp'j+pr+p's\right)^2}\right), \end{aligned} \quad (52)$$

where $\chi_{r,s;\epsilon}^{\left(p,p'\right)}\left(\tau\right)=q^{-\frac{1}{24}z^{\left(p,p'\right)}}chV_\epsilon\left(z^{\left(p,p'\right)},h_{r,s;\epsilon}^{\left(p,p'\right)}\right)$.

Let $r=-(n+1)$ and $s=\left(n'+1\right)$ in (52). Then, by Theorem 4, Theorem 5 and (52), we obtain the following result.

Proposition 1. *Let* $m=\frac{v}{u}$ $(u\in 2\mathbb{Z}_{\geq 1},\ v\in 2\mathbb{Z}+1)$. *Suppose that* λ *is a principal admissible weight of* $\widehat{\mathfrak{sl}_2}$ *such that* $\lambda=\lambda^0-(u-1)(m+2)\Lambda_0\in P_{u,1}^m$ *for* $\lambda^0\in P_+^{u(m+2)-2}$. *Then, the branching function* $b_\nu^{2\Lambda_0\otimes\lambda}\left(\tau\right)+b_\nu^{2\Lambda_1\otimes\lambda}\left(\tau\right)$ *(resp.* $b_\nu^\rho\left(\tau\right)$*) of* $\left(L\left(2\Lambda_0\right)\oplus L\left(2\Lambda_1\right)\right)\otimes L\left(\lambda\right)$ *(resp.* $L\left(\rho\right)\otimes L\left(\lambda\right)$*) is the same as the normalized character* $\chi_{-(n+1),n'+1;\frac{1}{2}}^{\left(p,p'\right)}\left(\tau\right)$ *(resp.* $\chi_{-(n+1),n'+1;0}^{\left(p,p'\right)}\left(\tau\right)$*) of the Neveu-Schwarz (resp. Ramond) superalgebra.*

It follows from Section 4 that

$$\left(L\left(2\Lambda_0\right)\oplus L\left(2\Lambda_1\right)\right)\otimes L\left(\lambda\right)=\sum_\nu\left(V_\nu^{2\Lambda_0,\lambda}\oplus V_\nu^{2\Lambda_1,\lambda}\right)\otimes L\left(\nu\right) \quad (53)$$

and

$$L\left(\rho\right)\otimes L\left(\lambda\right)=\sum_{\nu'}V_{\nu'}^{\rho,\lambda}\otimes L\left(\nu'\right), \quad (54)$$

where ν and ν' are taken over $P_{u,1}^{m+2}$ such that $\nu\equiv 2\Lambda_0+\lambda \bmod Q$ and $\nu'\equiv\rho+\lambda \bmod Q$, respectively.

According to [17] the coset Virasoro action introduced in Section 4 can be extended to the action of super-Virasoro algebras, and (53) and (54) can be considered as decompositions of $V_\epsilon\oplus[\mathfrak{g},\mathfrak{g}]$-module. Thus, (14) and Proposition 1 imply that $V_\nu^{2\Lambda_0}\oplus V_\nu^{2\Lambda_1}$ (resp. V_ν^ρ) should be isomorphic to the minimal series module $V_{\frac{1}{2}}\left(z^{p,p'},h_{-(n+1),n'+1;\frac{1}{2}}^{p,p'}\right)$ (resp. $V_0\left(z^{p,p'},h_{-(n+1),n'+1;0}^{p,p'}\right)$) as $Vir_{\frac{1}{2}}$-modules (resp. Vir_0-modules). Hence, we obtain the following theorem.

Theorem 6. *Let* m *and* λ *be the same as Proposition 1. Then, we have*

$$\left(L\left(2\Lambda_0\right)\oplus L\left(2\Lambda_1\right)\right)\otimes L\left(\lambda\right)=\sum_\nu V_{\frac{1}{2}}\left(z^{p,p'},h_{-(n+1),n'+1;\frac{1}{2}}^{p,p'}\right)\otimes L\left(\nu\right)$$

and

$$L\left(\rho\right)\otimes L\left(\lambda\right)=\sum_{\nu'}V_0\left(z^{p,p'},h_{-(n+1),n'+1;0}^{p,p'}\right)\otimes L\left(\nu'\right),$$

where ν *and* ν' *are taken over* $P_{u,1}^{m+2}$ *such that* $\nu\equiv 2\Lambda_0+\lambda \bmod Q$ *and* $\nu'\equiv\rho+\lambda \bmod Q$*, respectively.*

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