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Hybrid Deduction–Refutation Systems

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Received: 27 August 2019; Accepted: 10 October 2019; Published: 21 October 2019

Abstract: Hybrid deduction–refutation systems are deductive systems intended to derive both valid and non-valid, i.e., semantically refutable, formulae of a given logical system, by employing together separate derivability operators for each of these and combining 'hybrid derivation rules' that involve both deduction and refutation. The goal of this paper is to develop a basic theory and 'meta-proof' theory of hybrid deduction–refutation systems. I then illustrate the concept on a hybrid derivation system of natural deduction for classical propositional logic, for which I show soundness and completeness for both deductions and refutations.

Keywords: deductive refutability; refutation systems; hybrid deduction–refutation rules, derivative hybrid rules, soundness, completeness, natural deduction, meta-proof theory

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1. Introduction

1.1. Semantic vs. Deductive Refutability

Consider a generic logical system L, comprising a formal logical language with a given semantics. The basic semantic notion is that of L-*validity*: an L-formula A is said to be L-*valid* in L, denoted $\models_{\mathrm{L}} A$, iff it is true in every L-model. Respectively, an L-formula A is said to be *refutable* in L (L-*refutable*), or L-*falsifiable*, denoted $\not\models_{\mathrm{L}} A$, iff there is an L-model falsifying A, i.e., if it is not L-valid. (Note that in the usual logical semantics "not L-valid" means the same as "L-falsifiable", but, in some non-classical logical systems, such as paraconsistent logic, the semantics may allow the same formula to be both true and false, hence it may turn out both valid and falsifiable. This leads, inter alia, to terminological complications. To avoid these, I will exclude from consideration such paraconsistent semantics here, but will briefly discuss that case in Remark 6.)

Now, consider a deductive system **D** for L, with a derivation relation $\vdash_{\mathbf{D}}$. Then, the basic deductive notion associated with **D** is *provability in* **D** of a given L-formula A, denoted as usual by $\vdash_{\mathbf{D}} A$. If **D** is sound and complete for L, then $\vdash_{\mathbf{D}} A$ corresponds precisely to validity, i.e., $\models_{\mathrm{L}} A$. In general, this may not be the case, but, still, provability is the intended syntactic counterpart of validity (and, more generally, of logical consequence).

Then, what is the precisely matching syntactic notion to semantic refutability? One can argue that it is *not* "non-provability" but is rather the notion of "*deductive refutability*", i.e., existence of a formal derivation in a suitable *derivation system for* L-*refutable formulae*. Following Łukasiewicz, a new symbol, $\dashv_{\mathrm{L}}$, can be introduced for that notion, where $\dashv_{\mathrm{L}} A$ means "*A is deductively refutable in* L", i.e., "*the non-validity of A in* L *is established deductively*", or "*the refutation of A in* L *is formally derived/derivable*".

Thus, the notion of *formal (deductive) refutation* arises. The idea goes far back in history, already to Aristotle, who essentially applied that idea to 'derive' some non-valid syllogisms from others, but it was not pursued further, until Łukasiewicz revived it in the early-mid 20th century and introduced the notion of *(deductive) refutation system*. For further details on the origins and history of refutation systems, see [1–3]. For a recent comprehensive overview on the research and literature on refutation systems, see [4].

It should, of course, be noted that the idea of formally proving unprovability in a given formal deductive system has been crucial for the development of Proof Theory since its very inception by Gentzen and others, and has been fundamental in Logic since Gödel's incompleteness theorems. Moreover, results in Structural Proof Theory (see [5]), such as normalisation results for systems of Natural Deduction (see [6,7]) make it possible to obtain precise mathematical proofs of unprovability in such system, which can be appropriately formalised. What the theory of refutation systems proposes further is to consider the concept of deductive refutability as a first-class citizen and treat it as an object of study in its own right.

1.2. Related Work and Main Contributions

The overall development of refutations systems so far has been driven by the ideas to employ and 'simulate' traditional deductive systems, rather than to interact with them. In particular, a commonly pursued goal has been to design *pure* refutations systems, involving *only* the relation of refutability, but not at all that of provability. Even in the cases of refutations systems involving both refutability and provability, the latter is typically used as an auxiliary, 'black box' operator, to enable the applications of some 'mixed' rules of refutation inference, such as Modus Tollens (see further).

An alternative philosophy, promoted in the present work, is *to treat both notions of provability and refutation on a par* and to seek to develop *combined* deductive systems where both notions not only coexist, but actually interact and cooperate with each other, for the sake of ultimately deriving the correct validity/non-validity status of the formula or logical consequence in question.

Related Work

The idea of combining deductions and refutations in common systems of derivations, here called *hybrid* (The use of the term "hybrid" in the context of this work is not related to deduction in the so-called "hybrid logic"). I hope that the use of that term here would not create terminological confusion in the literature) *deduction–refutation systems* (also, for short, *hybrid derivation systems*), can be traced back implicitly to some works of Łukasiewicz and Carnap. However, to my knowledge, it was first explicitly proposed in [3] but apparently not pursued further since then. Still, several similar or related ideas have been proposed and discussed (though not as follow-up works to [3]) in the meantime, including (chronologically):

- The idea of 'complementary systems' for sentential logic, suggested by Bonatti and Varzi in [8] is related in spirit, though technically different from the idea of hybrid refutation systems, as it considers the complementary systems, for deductions and for refutations, acting separately.
- Similarly, in [9], Skura studies 'symmetric inference systems', that is, pairs of essentially non-interacting inference systems, and shows how they can be used for characterizing maximal non-classical logic with certain properties. In particular, the method is applied there to paraconsistent logic.
- In [10], Wybraniec-Skardowska and Waldmajer explore the general theory of deductive systems employing the two dual consequence operators, the standard logical consequence, inferring validities, and the refutation consequence, inferring non-validities. Again, no interaction of these consequence operators is considered there.
- In [11], Caferra and Peltier, motivated by potential applications to automated reasoning, take a unifying perspective on deriving accepting or rejecting propositions from other, already accepted or rejected, propositions, thus considering separately each of the four consequence relations arising as combinations.
- In [12], Goré and Postniece combine derivations and refutations to obtain cut-free complete systems for bi-intuitionistic logic.

- In [13], Negri explores the duality of proofs and countermodels in labelled sequent calculi and develops a method for unifying proof search and countermodel construction for some modal and intuitionistic propositional logic over classes of Kripke frames with suitable frame conditions. In particular, for some of this logic, the method provides a decision procedure.
- In [14], Citkin considers essentially multiple-conclusion generalisations of hybrid inference rules studied here. Citkin discusses consequence relations and inference systems employing such rules and proposes a meta-logic for formalising propositional reasoning about such systems. Even though with different motivation and agenda, and with no technical results of the type pursued here, this work appears to be the closest in spirit to the idea of hybrid deduction–refutation systems studied in the present work.
- Likewise, in [15], Fiorentini and Ferrari explore the duality between unprovability and provability in forward proof-search for intuitionistic propositional logic and develop a refutation-complete sequent-based forward refutation calculus for it, following on their previous work [16].
- In [17], Rumfitt considers "reversals" of the rules of propositional Natural Deduction, to formalise derivations between "accepted" and "rejected" sentences. While the motivation is different from the one related to refutation systems, most (but not all!) resulting rules are essentially the same as the "hybrid refutation rules" obtained by contrapositive inversion of the rules of propositional Natural Deduction considered in Section 4. See Remark 7 on the distinction between the two types of rules.

Contributions and Structure of the Paper

The goal of this paper is to develop a basic proof theory and 'meta-proof' theory of hybrid deduction–refutation systems. After a brief description of refutation rules and systems in the preliminary Section 2, I present in Section 3 a basic theory of hybrid derivation rules and systems, including the notion of *inversion* of deduction and refutation rules and *canonical hybrid extensions* of deductive and refutation systems. In Section 4, I then illustrate these concepts on the natural deduction system for classical propositional logic $\mathbf{ND}^{\mathsf{PL}}$, for which I develop a 'standard hybrid extension' $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$, for which I prove soundness and completeness with respect to both deductions and refutations in Section 5. Then, Section 6 discusses the 'meta-theory' of hybrid derivation systems. The paper ends with some concluding remarks on potential applications and further work on hybrid derivation systems in Section 7.

2. Preliminaries

I will assume that the reader is familiar with basic logical notation and terminology for proof systems for classical logic. If necessary, see, e.g., [5,7], or [18].

2.1. Refutation Rules and Systems: Basic Concepts

Let us consider and fix a logical system L, comprising a formal logical language with a given semantics, defining the notion of validity and, respectively, logical consequence. Here, I will introduce the basic concepts of (axiomatic) refutation systems, generally (but not fully) following notation and terminology from [3,19], to which the reader is referred for further details; see also [4] for a bibliographic overview on refutation systems.

A **pure rule of refutation inference** is a rule scheme of the type

$$\frac{\dashv \psi_1, \ldots, \dashv \psi_n}{\dashv \gamma},$$

where $\psi_1, \ldots, \psi_n, \gamma$ are propositional formulae. (This definition can be naturally extended to first-order languages. (Here, and further: the commas used to separate premises in the rules should not be considered as part of the formal syntax, but rather as typographical indication to separate these premisses. That way, derivations can be regarded as trees, as usual.) The intuitive meaning of that rule with respect to the logical system L is that, for any uniform substitution σ of formulae for the propositional variables occurring in the formulae $\psi_1, \ldots, \psi_n, \gamma$, if each of the formulae $\sigma(\psi_1), \ldots, \sigma(\psi_n)$, has been derived as non-valid (in L), then $\sigma(\gamma)$ is derived as non-valid (in L) too. In that sense, the rule is actually a *rule scheme*, and this will apply likewise for all propositional inference rules considered further in the paper.

A typical example of a pure rule of refutation inference is the **Disjunction rule**:

$$\frac{\dashv \varphi, \ \dashv \psi}{\dashv \varphi \vee \psi}.$$

Another important example is Łukasiewicz's **Reverse substitution rule scheme**:

$$\frac{\dashv \tau(\varphi)}{\dashv \varphi},$$

where τ is a uniform substitution. (Note that this is a substitutional scheme in two senses.)

Usually, pure refutation rules do not suffice to capture adequately semantic refutability in a refutation system, so we also consider a more general type of refutation rules, called **mixed refutation rules**, which are relativised to a given underlying deductive system **D** for the logical system L, as follows:

$$\frac{\vdash_{\mathbf{D}} \varphi_1, \ldots, \vdash_{\mathbf{D}} \varphi_m, \ \dashv \psi_1, \ldots, \dashv \psi_n}{\dashv \gamma},$$

where $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n, \gamma$ are (here, propositional) formulae. The intuitive meaning of that rule with respect to the logical system L is that, for any uniform substitution σ of formulae for the propositional variables occurring in the formulae $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n, \gamma$, if each of the formulae $\sigma(\varphi_1), \ldots, \sigma(\varphi_m)$ is derived by **D** (hence, assuming soundness of **D**, proved valid in L) and each $\sigma(\psi_1), \ldots, \sigma(\psi_n)$ has been derived as non-valid in L, then $\sigma(\gamma)$ is derived as non-valid in L too. A typical example is Łukasiewicz's rule **Reverse modus ponens** (aka, **Modus Tollens**):

$$\frac{\vdash_{\mathbf{D}} \varphi \to \psi, \dashv \psi}{\dashv \varphi}.$$

A **refutation system (associated with a given underlying deductive system D)** is a set $\mathcal{R}$ of (generally, mixed) refutation rules (where $\vdash$ is indexed with **D**). Refutation rules with no premises are called **structural refutation axioms**, and I will write them simply as $\dashv \theta$.

Remark 1. *Substitution closure of inference rules is a standard structurality condition in most non-classical propositional logic. However, in the case of mixed refutation rules, it has a rather non-trivial nature, as it makes an interesting connection with the notions of* unification and unifiability *of propositional formulae; see [20]. Indeed, a refutation axiom is sound under substitution instances precisely when it is* not *unifiable. In particular, this will make it necessary further to consider more general, not closed under substitutions' refutation axiom schemes, in addition to structural ones. Furthermore, unifiability of formulae is closely related to admissibility of rules. On the connections of these with refutation rules and systems, see [21].*

The issue of substitution closure of inferences will come up again later when hybrid inference rules are considered.

When defining refutation derivations in $\mathcal{R}$, one typically assumes that the necessary derivations in the underlying deductive system **D** are done separately, in advance or "on demand", whenever needed for the derivation of the target refutation, and as part of that derivation. In either case, the deductive system **D** is assumed to play only an auxiliary role for the functioning of the refutation system $\mathcal{R}$. Formally, a **refutation derivation in** $\mathcal{R}$, or just an $\mathcal{R}$**-derivation**, for a formula θ is a sequence $S_1, ..., S_t$, where S_t is $\dashv \theta$ and every S_i is either a refutation axiom, or is of the form $\vdash_{\mathbf{D}} \psi$

or is obtained from some already listed items in the sequence by applying a refutation rule from $\mathcal{R}$, by deriving the conclusion from suitable substitution instances of the premises. We now say that a formula θ is **refutable in** $\mathcal{R}$ (or, just $\mathcal{R}$**-refutable**) iff there is a refutation derivation for θ in $\mathcal{R}$.

Given a logical system L, we say that a refutation system $\mathcal{R}$ is:

- **refutation-sound**, or **Ł-sound, for** L, if *only* non-valid in L-formulae (more generally, logical consequences in L) are $\mathcal{R}$-refutable,
- **refutation-complete**, or **Ł-complete, for** L, if *all* non-valid in L-formulae (more generally, logical consequences in L) are $\mathcal{R}$-refutable.

2.2. Basic Refutation Systems for Classical Logic

The most common type of deductive systems are axiomatic (aka, Hilbert style) systems—respectively, the most common type of refutation systems are axiomatic refutation systems. Here is such a refutation system $\mathsf{Ref}^{\mathsf{CPC}}$, for any fixed sound and complete deductive system CPC for the Classical Propositional Logic PL, due to Łukasiewicz:

Refutation axiom: $\dashv \bot$.

Refutation rules:

Reverse Substitution **RS**:

$$\frac{\dashv \sigma(\varphi)}{\dashv \varphi}$$

for any uniform substitution σ.

Modus Tollens **MT**:

$$\frac{\vdash_{\mathsf{CPC}} \varphi \to \psi,\ \dashv \psi}{\dashv \varphi}.$$

Remark 2. *Note that a refutation system can be Ł-complete for more than one logic. Indeed,* $\mathsf{Ref}^{\mathsf{CPC}}$ *is Ł-complete not only for the* CPC*, but also for both maximal normal modal logic* $\mathbf{K} + \Box\bot$ *and* $\mathbf{K} + (p \leftrightarrow \Box p)$ *(see [3]), provided that the deductive system* CPC *in* $\vdash_{\mathsf{CPC}}$ *in* ***MT*** *is replaced by one for the respective modal logic.*

Besides those in axiomatic style, some refutation systems have also been constructed for sequent calculi and in natural deduction style. In [22], Tiomkin constructed a sequent-style refutation calculus for FOL without function symbols and with the only logical connectives being $\vee, \neg, \forall$, and sketched a proof of its Ł-completeness for the formulae refutable in finite models. Independently, Goranko developed in [3] an Ł-complete sequent refutation calculi for the full language of PL, also extended there to some important normal modal logic. In [2], Tamminga developed a system of natural deduction for deriving the non-theorems of PL, proved there to be Ł-sound and Ł-complete.

3. Hybrid Derivation Systems: Basic Theory

3.1. Hybrid Deduction–Refutation Rules and Systems

Again, let us consider and fix a logical system L, comprising a formal propositional logical language with a given semantics defining the notion of L-validity and, more generally, logical consequence in L.

For greater generality and for the purposes of Section 4, the basic notions of hybrid deduction–refutation systems will be given here in terms of *sequents of formulae*, readily reducible to single formulae. By a **(single-conclusion) sequent**, we mean an expression of the type $\Gamma \bowtie \theta$, where Γ is a list (treated as a set) of formulae in L, θ is a formula in L, and $\bowtie \in \{\vdash, \dashv\}$. Sequents of the type $\Gamma \vdash \theta$ will be called **deductions**, while those of the type $\Gamma \dashv \theta$ will be called **refutations**. (From a general perspective, both deductions and refutations in our sense are treated syntactically as logical deductions, but we need a more differentiating and unambiguous terminology here.)

Semantically, $\Gamma \vdash \theta$ is meant to claim that the logical consequence $\Gamma \models \theta$ is valid in L, whereas $\Gamma \dashv \theta$ is meant to claim that $\Gamma \models \theta$ is falsifiable, hence not valid in L, i.e. that $\Gamma \not\models \theta$ holds in L. Thus, we say that a sequent $\Gamma \vdash \theta$ is **sound** in L if $\Gamma \models \theta$ is a valid logical consequence in L. Respectively, a sequent $\Gamma \dashv \theta$ is **sound** in L if $\Gamma \models \theta$ is a non-valid logical consequence in L.

Remark 3. *Note an important semantic distinction between deduction and refutation sequents: $\Gamma \models \theta$ is typically monotone by inclusion with respect to Γ, meaning that, if $\Gamma \models \theta$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \models \theta$; on the other hand, by contraposition, $\Gamma \not\models \theta$ is typically* anti-monotone *with respect to Γ, i.e., if $\Gamma \not\models \theta$ and $\Gamma' \subseteq \Gamma$, then $\Gamma' \not\models \theta$ but generally not vice versa. This will have the practical consequence that all inferences that involve refutation sequents $\Gamma \dashv \theta$ are sensitive, and generally intolerant, to adding extra formulae to Γ. See also Remark 1.*

Now, we will extend the refutation rules to *hybrid deduction–refutation rules*, also by adding premises as contexts, which now becomes essential in view of the remark above. These rules fall in two complementary types, defined below, where $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n, \theta$ are (generally) schemes of formulae and $\Gamma, \Gamma_1, ..., \Gamma_m, \Delta, \Delta_1, ..., \Delta_n$ are sets of schemes of formulae in L. In the propositional case treated here, one can alternatively assume that all these are concrete formulae, but the rules employ uniform substitutions; see further. Because of the opposite monotonicity properties of the deduction and refutation sequents (see Remark 3), the hybrid rules generally have to employ sequents with different sets of premises.

A **hybrid deduction rule of inference** (based on a given deductive system **D**) is a rule of the type:

$$\textbf{HDR} \qquad \frac{\Gamma_1 \vdash \varphi_1, \ldots \Gamma_m \vdash \varphi_m,\ \Delta_1 \dashv \psi_1, \ldots, \Delta_n \dashv \psi_n}{\Gamma \vdash \theta}.$$

A **hybrid refutation rule of inference** (based on a given deductive system **D**) is a rule of the type:

$$\textbf{HRR} \qquad \frac{\Gamma_1 \vdash \varphi_1, \ldots \Gamma_m \vdash \varphi_m,\ \Delta_1 \dashv \psi_1, \ldots, \Delta_n \dashv \psi_n}{\Delta \dashv \theta}.$$

The two types of rules above will be called collectively **hybrid rules of inference**. Hybrid rules with no premises will be called respectively **deduction axioms** and **structural refutation axioms**, and we write them simply as sequents $\Gamma \vdash \theta$, respectively $\Delta \dashv \theta$.

These hybrid rules of inference will be regarded as *rule schemes under substitution*, like the earlier defined refutation rules, in the following sense. Every uniform substitution of formulae for the propositional variables occurring in the sequents in the rule creates an instance of the rule. For any such uniform substitution σ if all sequents resulting from applying σ to the premises of the rule, viz. $\sigma(\Gamma_1) \vdash \sigma(\varphi_1), \ldots, \sigma(\Gamma_m) \vdash \sigma(\varphi_m), \sigma(\Delta_1) \dashv \sigma(\psi_1), \ldots, \sigma(\Delta_n) \dashv \sigma(\psi_n)$ are derivable / have been derived, then the rule allows the derivation of the sequent resulting from applying σ to the conclusion, i.e., $\sigma(\Gamma) \vdash \sigma(\theta)$ in the case of **HDR**, resp. $\sigma(\Delta) \dashv \sigma(\theta)$ in the case of **HRR**. The respective semantic interpretation of the hybrid rules above in the case of propositional logical systems can be given as follows: for any uniform substitution σ, if each of the logical consequences $\sigma(\Gamma_1) \models \sigma(\varphi_1), \ldots, \sigma(\Gamma_m) \models \sigma(\varphi_m)$ is derived as valid and each of $\sigma(\Delta_1) \models \sigma(\psi_1), \ldots, \sigma(\Delta_n) \models \sigma(\psi_n)$ has been derived as non-valid, then $\sigma(\Gamma) \models \sigma(\theta)$ is derived as valid in the case of **HDR**, respectively $\sigma(\Delta) \models \sigma(\theta)$ derived as non-valid in the case of **HRR**, as defined above.

In addition (see Remark 1), we also need to allow more general, non-structural **refutation axiom schemes** of the type $\Gamma \dashv \theta$, where closure under substitution is not assumed, but syntactic constraints are imposed on Γ and θ. A simplest example is a scheme $p \dashv q$, where $p \neq q$. Clearly, allowing closure under substitution would produce unsound refutation sequents, such as $p \dashv p$. Of course, structural refutation axioms are special kinds of refutation axiom schemes, but it would be helpful to consider both types separately. Structural refutation axioms and refutation axiom schemes will be called collectively just **refutation axioms**. Note that, to make the general hybrid rules applicable, they must act in combination with some rules with no premises, i.e., deduction and refutation axioms, which provide an initial stock of derived sequents.

Remark 4. *Note that, unlike in standard refutation systems, in hybrid derivation systems, we will no longer assume that these rules act in the context of a separately pre-defined, purely deductive system* **D***, which provides the initial stock of derived sequents* $\Gamma \vdash \theta$ *only, but rather that they define the notions of deduction derivations and refutation derivations* on a par*, by a mutual induction defined as expected, which I combine in one notion of* hybrid derivation*, defined further.*

A hybrid inference rule is **sound** for a given logical system L if it respects the intuitive interpretation above, i.e., whenever applied to sound premises in L, it produces a sound conclusion in L.

Here are some examples of hybrid rules:

- All standard deduction rules (in particular, axioms) are particular cases of hybrid deduction rules. In particular, such are all rules of sequent calculi and systems of natural deduction.
- The refutation rules defined in Section 2.1 are particular cases of hybrid refutation rules.
- In addition, suitable meta-properties of the given logical system L can be used to extract and justify specific new hybrid inference rules for it. An important example is the **Deductive consistency rule**

$$(\mathbf{Cons}) \qquad \frac{\vdash \varphi}{\dashv \neg\varphi},$$

which is justified ('sound') whenever the underlying deductive relation $\vdash$ is sound (hence consistent) for L. More generic examples will be given further.

A **hybrid deduction–refutation system**, or (for shorter) a **hybrid derivation system**, is a set $\mathcal{H}$ of hybrid rules of inference for a given logical language. A **hybrid derivation in** $\mathcal{H}$, or just an $\mathcal{H}$**-derivation**, for a sequent $\Gamma \bowtie \theta$ is a sequence of sequents $S_1, ..., S_t$, where S_t is $\Gamma \bowtie \theta$ and every S_i is either a deduction axiom or a refutation axiom, or is obtained from some already listed sequents in the sequence by applying a hybrid rule of inference from $\mathcal{H}$. Then, we say that the sequent $\Gamma \bowtie \theta$ is **derivable in** $\mathcal{H}$. Furthermore, we say that the logical consequence $\Gamma \models \theta$ is **deduced/deducible in** $\mathcal{H}$ if $\Gamma \vdash \theta$ is derivable in $\mathcal{H}$, and that $\Gamma \models \theta$ is **refuted/refutable in** $\mathcal{H}$ if $\Gamma \dashv \theta$ is derivable in $\mathcal{H}$.

In particular, $\mathcal{H}$ may contain all axioms and rules of a given traditional deduction system **D** (which can be an axiomatic system, a sequent calculus, a system of natural deduction, or a system of semantic tableaux). In such case, the derivations in $\mathcal{H}$ extend those in **D**, by enabling not only derivations of refutations based on **D**, but also possibly of some deductions not derivable in **D** (esp. in case **D** is incomplete).

Remark 5. *Note that hybrid derivation systems do not employ separate rules of uniform substitution, even to derived sequents* $\Gamma \vdash \theta$ *because of the non-preservation of refutations (that may have been used in the derivation) under such substitutions. (A similar remark is made in [14].) Still, uniform substitutions are used here for generating instances of the inference rules, as explained earlier..*

Some basic terminology will be needed in what follows. Recall (see footnote 1) the assumption that "not valid" and "non-valid" means "falsifiable" (but see also Remark 6). Given a logical system L, we say that a hybrid derivation system $\mathcal{H}$ is:

- **deductively sound for** L, or **D-sound for** L, if only logical consequences that are valid in L are $\mathcal{H}$-deducible.
- **refutationally sound for** L, or **R-sound for** L, if only logical consequences that are non-valid in L are $\mathcal{H}$-refutable.

- **Ł-sound for** L, if it is both D-sound and R-sound for L.
- **Ł-consistent**, if there is no Γ and θ such that both $\Gamma \vdash \theta$ and $\Gamma \dashv \theta$ are derivable in $\mathcal{H}$.
- **deductively complete for** L, or **D-complete for** L, if all logical consequences that are valid in L are $\mathcal{H}$-deducible.
- **refutationally complete for** L, or **R-complete for** L, if all logical consequences that are non-valid in L are $\mathcal{H}$-refutable.
- **Łukasiewicz-complete for** L, or **Ł-complete for** L, if it is both D-complete and R-complete for L.
- **Ł-saturated**, if for all Γ and θ, either $\Gamma \vdash \theta$ or $\Gamma \dashv \theta$ (possibly both) is derivable in $\mathcal{H}$.
- **Ł-adequate for** L, if it is both Ł-sound and Ł-complete for L.
- **Ł-balanced**, if it is both Ł-consistent and Ł-saturated.

Proposition 1. *Let* L *be a logical system and* $\mathcal{H}$ *a hybrid deduction–refutation system for* L*. Then:*

1. *If* $\mathcal{H}$ *is Ł-sound for* L*, then* $\mathcal{H}$ *is Ł-consistent.*
2. *If* $\mathcal{H}$ *is Ł-complete for* L*, then* $\mathcal{H}$ *is Ł-saturated.*
3. *If* $\mathcal{H}$ *is Ł-adequate for* L*, then* $\mathcal{H}$ *is Ł-balanced.*
4. *If* $\mathcal{H}$ *has a recursive set of rules and is Ł-adequate for* L*, then it provides a decision procedure for the valid logical consequences in* L*.*

Proof. Here, 1 and 2 are straightforward, since, for any Γ and θ, the logical consequence $\Gamma \models \theta$ is either valid or non-valid, but not both. Then, 3 follows immediately.

Likewise, 4 is immediate, as the recursiveness of $\mathcal{H}$ implies that all derived sequents in $\mathcal{H}$ can be recursively enumerated, the Ł-completeness of $\mathcal{H}$ means that, for every Γ and ϕ, either $\Gamma \vdash \phi$ or $\Gamma \dashv \phi$ (but not both) will eventually appear in that enumeration, and the Ł-soundness guarantees that, whatever the case is, it will correctly imply validity, resp. non-validity, of $\Gamma \models \phi$. □

Remark 6. *All notions defined above are meant to apply, in particular, to most general cases of derivation systems, which may possibly extend unsound, or even to paraconsistent deductive systems. (In a similar spirit, Citkin defines in [14] a more general motion of a logical system, as a pair consisting of a set of accepted and a set of rejected propositions, without assuming that these must be complementary, nor even disjoint.) However, in the case of paraconsistent semantics where a formula or logical consequence can be both valid and falsifiable, the term "non-valid" in the definitions of* ***R****-soundness and* ***R****-completeness should be replaced by "falsifiable", without assuming that the latter implies the former. Still, claims 1 and 3 in Proposition 1 will no longer hold for such semantics. (Thanks to the reviewer who pointed that out.) Still, note that, even if the deduction fragment of a hybrid derivation system may be D-unsound, or D-incomplete, for the given logical system, its refutation fragment may still be R-sound, or R-complete, and vice versa. An interesting example is the simple Ł-complete refutation system for Medvedev's logic of finite problems (for which no recursive axiomatization is known yet, but it has a co-r.e. set of validities) designed in [23], employing as the underlying deductive system the weaker Kreisel–Putnam's logic KP. Thus, the resulting hybrid system is D-incomplete but R-complete for Medvedev's logic.*

3.2. Inversion of Rules and Derivative Hybrid Rules

New hybrid rules can be defined in a uniform way as **derivative rules** from existing ones by using **inversion**: swapping one premise with the conclusion of the given rule and swapping $\vdash$ with $\dashv$ in both sequents. (The use of the term 'inversion' here is different from 'inversion principle' widely used in proof theory, see [5], but related to the term 'inversion' used in [2], when applied to single-premise rules. In addition, the idea of inverting inference rules was essentially used in the design and proof of completeness of the sequential refutation system for PL in [3].) For example, applying inversion to the rule Modus Ponens

$$\frac{\Gamma \vdash \phi,\ \Gamma \vdash \phi \to \psi}{\Gamma \vdash \psi}$$

produces the following derivative rules:

$$\frac{\Gamma \dashv \psi,\ \Gamma \vdash \phi \to \psi}{\Gamma \dashv \phi} \text{ and } \frac{\Gamma \vdash \phi, \Gamma \dashv \psi}{\Gamma \dashv \phi \to \psi}.$$

Likewise, the Disjunction rule:

$$\frac{\Gamma \dashv \varphi,\ \Gamma \dashv \psi}{\Gamma \dashv \varphi \vee \psi}$$

produces the following derivative rules:

$$\frac{\Gamma \vdash \varphi \vee \psi,\ \Gamma \dashv \psi}{\Gamma \vdash \varphi} \text{ and } \frac{\Gamma \dashv \varphi,\ \Gamma \vdash \varphi \vee \psi}{\Gamma \vdash \psi}.$$

The general definitions follow.

3.2.1. Inversion of Deduction Rules

The deduction rule

$$\frac{\Gamma_1 \vdash \varphi_1, \ldots, \Gamma_i \vdash \varphi_i, \ldots, \Gamma_m \vdash \varphi_m,\ \Delta_1 \dashv \psi_1, \ldots, \Delta_j \dashv \psi_j, \ldots, \Delta_n \dashv \psi_n}{\Gamma \vdash \theta}$$

produces each of the following derivative rules

$$\frac{\Gamma_1 \vdash \varphi_1, \ldots, \Gamma \dashv \theta, \ldots, \Gamma_m \vdash \varphi_m,\ \Delta_1 \dashv \psi_1, \ldots, \Delta_j \dashv \psi_j, \ldots, \Delta_n \dashv \psi_n}{\Gamma_i \dashv \varphi_i}$$

for each $i = 1, ..., m$, and

$$\frac{\Gamma_1 \vdash \varphi_1, \ldots, \Gamma_i \vdash \varphi_i, \ldots, \Gamma_m \vdash \varphi_m,\ \Delta_1 \dashv \psi_1, \ldots, \Gamma \dashv \theta, \ldots, \Delta_n \dashv \psi_n}{\Delta_j \vdash \psi_j}$$

for each $j = 1, ..., n$.

In particular, a deduction rule with no premises, i.e., a deduction axiom $\Gamma \vdash \theta$, will be regarded—without essential effect—as the rule

$$\frac{\vdash \top}{\Gamma \vdash \theta}.$$

Thus, it has one derivative rule:

$$\frac{\Gamma \dashv \theta}{\dashv \top}.$$

3.2.2. Inversion of Refutation Rules

Likewise, the refutation rule

$$\frac{\Gamma_1 \vdash \varphi_1, \ldots, \Gamma_i \vdash \varphi_i, \ldots, \Gamma_m \vdash \varphi_m,\ \Delta_1 \dashv \psi_1, \ldots, \Delta_j \dashv \psi_j, \ldots, \Delta_n \dashv \psi_n}{\Delta \dashv \theta}$$

produces each of the following derivative rules

$$\frac{\Gamma_1 \vdash \varphi_1, \ldots, \Delta \vdash \theta, \ldots, \Gamma_m \vdash \varphi_m,\ \Delta_1 \dashv \psi_1, \ldots, \Delta_j \dashv \psi_j, \ldots, \Delta_n \dashv \psi_n}{\Gamma_i \dashv \varphi_i}$$

for each $i = 1, ..., m$, and

$$\frac{\Gamma_1 \vdash \varphi_1, \ldots, \Gamma_i \vdash \varphi_i, \ldots, \Gamma_m \vdash \varphi_m,\ \Delta_1 \dashv \psi_1, \ldots, \Delta \vdash \theta, \ldots, \Delta_n \dashv \psi_n}{\Delta_j \vdash \psi_j}$$

for each $j = 1, ..., n$.

A refutation rule with no premises, i.e., a structural refutation axiom $\Gamma \dashv \theta$, will be regarded—again without essential effect—as the rule

$$\frac{\dashv \bot}{\Gamma \dashv \theta}.$$

Respectively, it also has one derivative rule

$$\frac{\Gamma \vdash \theta}{\vdash \bot}.$$

3.2.3. Soundness of Derivative Rules

Proposition 2. *Let* L *be a logical system and let* R *be a hybrid inference rule in the language of* L*, which is sound for* L*. Then, every derivative rule of* R *is sound for* L *too.*

Proof. Suppose first that R is a hybrid deduction rule

$$\frac{\Gamma_1 \vdash \varphi_1, \ldots, \Gamma_i \vdash \varphi_i, \ldots, \Gamma_m \vdash \varphi_m,\ \Delta_1 \dashv \psi_1, \ldots, \Delta_j \dashv \psi_j, \ldots, \Delta_n \dashv \psi_n}{\Gamma \vdash \theta}.$$

Consider the derivative refutation rule

$$\frac{\Gamma_1 \vdash \varphi_1, \ldots, \Gamma \dashv \theta, \ldots, \Gamma_m \vdash \varphi_m,\ \Delta_1 \dashv \psi_1, \ldots, \Delta_j \dashv \psi_j, \ldots, \Delta_n \dashv \psi_n}{\Gamma_i \dashv \varphi_i}$$

for $i \in \{1, ..., m\}$. To prove its soundness, consider any uniform substitution σ and suppose that all premises obtained after applying σ are sound, i.e., each of the logical consequences $\sigma(\Gamma_k) \models \sigma(\varphi_k)$, for $k = 1, ..., i-1, i+1, ..., m$, is valid and each of $\sigma(\Delta_k) \models \sigma(\psi_k)$, for $k = 1, \ldots, n$, as well as $\sigma(\Delta) \models \sigma(\theta)$, is non-valid. Then, $\sigma(\Gamma_i) \models \sigma(\varphi_i)$ must be non-valid too; otherwise, the soundness of R would imply the validity of $\sigma(\Delta) \models \sigma(\theta)$.

The argument for the soundness of derivative deduction rules is similar.

The proof when R is a hybrid refutation rule is completely analogous.

□

3.3. Canonical Hybrid Extensions of Deductive Systems

Given any deductive system **D**, its **canonical hybrid extension** $\mathcal{H}(\mathbf{D})$ is obtained by adding to **D** the derivative rules of all deduction rules (incl. axioms) of **D**.

Note that the sequent refutation systems proposed for PL and FOL in [3,22] are essentially constructed as (subsystems of) the canonical hybrid extensions of respective standard sequent deduction systems for these logic.

Proposition 2 implies that, if **D** is D-sound for a given logical system L, then $\mathcal{H}(\mathbf{D})$ is Ł-sound for L. If **D** is also D-complete for L, then $\mathcal{H}(\mathbf{D})$ cannot add more derivable deduction sequents, so it is D-complete too. In this case, $\mathcal{H}(\mathbf{D})$ extends **D** conservatively with respect to deductions, but it generally does add derivable refutation sequents. However, even then $\mathcal{H}(\mathbf{D})$ may generally not be R-complete, hence not Ł-complete, either. In particular, it *cannot* be R-complete if L is not decidable. The question of when $\mathcal{H}(\mathbf{D})$ is Ł-complete is one of the main questions of the general theory of hybrid derivation systems.

Likewise, given any refutation system $\mathcal{R}$, its **canonical hybrid extension** $\mathcal{H}(\mathcal{R})$ is obtained by adding to $\mathcal{R}$ the derivative rules of all refutation rules (incl. axioms) of $\mathcal{R}$. Again, by Proposition 2, if $\mathcal{R}$ is R-sound for a logical system L, then $\mathcal{H}(\mathcal{R})$ is Ł-sound for L. The question of Ł-completeness of $\mathcal{H}(\mathcal{R})$ is, again, generally open.

As for Ł-soundness, using Proposition 2, a straightforward induction on derivations proves the following.

Corollary 1. *Let* $\boldsymbol{D}$ *be a sound deductive system for a given logical system* L. *Then,* $\mathcal{H}(\boldsymbol{D})$ *is Ł-sound for* L.

4. Hybrid Extensions of the System of Natural Deduction for PL

I will illustrate here the concept of canonical hybrid extension, applied to the system of Natural Deduction for the classical propositional logic PL.

4.1. *Hybrid Derivatives of the Rules for Natural Deduction for* PL

Let us fix a standard version $\mathbf{ND}^{\mathsf{PL}}$ of a sound and complete system of Natural Deduction (ND) for PL (see [6], or [7], or [18]).

Every pure inference rule of $\mathbf{ND}^{\mathsf{PL}}$ produces one or two derivative hybrid rules. Note that the derivatives of introduction rules for $\vdash$ typically become hybrid elimination rules for $\dashv$ and vice versa.

Note also that the open assumptions must be explicitly listed in the rules because of the anti-monotonicity of the refutations (see Remark 3). For that reason and for better readability, the rules are presented further as rules over sequents.

4.2. *Hybrid Derivatives of the Introduction Rules of* $\boldsymbol{ND}^{\mathsf{PL}}$

For the record, here are the derivative rules produced from the introduction rules of $\mathbf{ND}^{\mathsf{PL}}$, where the arrows $\Rightarrow$ below indicate the respective transformations of deduction rules to their derivative hybrid rules:

$$(\wedge\mathrm{I})\ \frac{\Gamma \vdash \phi,\ \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(\wedge\mathrm{HE}^{l})\ \frac{\Gamma \dashv \phi \wedge \psi,\ \Gamma \vdash \psi}{\Gamma \dashv \phi} \quad (\wedge\mathrm{HE}^{r})\ \frac{\Gamma \vdash \phi,\ \Gamma \dashv \phi \wedge \psi}{\Gamma \dashv \psi},$$

$$(\vee\mathrm{I}^{l})\ \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \qquad (\vee\mathrm{I}^{r})\ \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(\vee\mathrm{HE}^{l})\ \frac{\Gamma \dashv \phi \vee \psi}{\Gamma \dashv \phi} \qquad (\vee\mathrm{HE}^{r})\ \frac{\Gamma \dashv \phi \vee \psi}{\Gamma \dashv \psi},$$

$$(\rightarrow \mathrm{I})\ \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \qquad (\neg\mathrm{I})\ \frac{\Gamma, \phi \vdash \bot}{\Gamma \vdash \neg\phi}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(\rightarrow \mathrm{HE}^{1})\ \frac{\Gamma \dashv \phi \rightarrow \psi}{\Gamma, \phi \dashv \psi} \qquad (\neg\mathrm{HE}^{1})\ \frac{\Gamma \dashv \neg\phi}{\Gamma, \phi \dashv \bot}.$$

4.3. *Hybrid Derivatives of the Elimination Rules of* $\boldsymbol{ND}^{\mathsf{PL}}$

Here are the derivative rules produced from the elimination rules of $\mathbf{ND}^{\mathsf{PL}}$, where, again, the arrows $\Rightarrow$ below indicate the respective transformations of deduction rules to their derivative hybrid rules:

$$(\wedge\mathrm{E}^{l})\ \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi} \qquad (\wedge\mathrm{E}^{r})\ \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(\wedge \mathrm{HI}^l)\ \frac{\Gamma \dashv \phi}{\Gamma \dashv \phi \wedge \psi} \qquad (\wedge \mathrm{HI}^r)\ \frac{\Gamma \dashv \psi}{\Gamma \dashv \phi \wedge \psi},$$

$$(\vee \mathrm{E})\ \frac{\Gamma, \phi \vdash \theta,\ \Gamma, \psi \vdash \theta}{\Gamma, \phi \vee \psi \vdash \theta}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(\vee \mathrm{HI}^l)\ \frac{\Gamma, \phi \vee \psi \dashv \theta,\ \Gamma, \psi \vdash \theta}{\Gamma, \phi \dashv \theta}, \qquad (\vee \mathrm{HI}^r)\ \frac{\Gamma, \phi \vdash \theta,\ \Gamma, \phi \vee \psi \dashv \theta}{\Gamma, \psi \dashv \theta},$$

$$(\rightarrow \mathrm{E})\ \frac{\Gamma \vdash \phi,\ \Gamma \vdash \phi \rightarrow \psi}{\Gamma \vdash \psi}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(\rightarrow \mathrm{HE}^2)\ \frac{\Gamma \dashv \psi,\ \Gamma \vdash \phi \rightarrow \psi}{\Gamma \dashv \phi}, \qquad (\rightarrow \mathrm{HI})\ \frac{\Gamma \vdash \phi, \Gamma \dashv \psi}{\Gamma \dashv \phi \rightarrow \psi},$$

$$(\neg \mathrm{E})\ \frac{\Gamma \vdash \phi,\ \Gamma \vdash \neg\phi}{\Gamma \vdash \bot}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(\neg \mathrm{HE}^2)\ \frac{\Gamma \dashv \bot,\ \Gamma \vdash \neg\phi}{\Gamma \dashv \phi}, \qquad (\neg \mathrm{HI})\ \frac{\Gamma \vdash \phi,\ \Gamma \dashv \bot}{\Gamma \dashv \neg\phi}.$$

4.4. Hybrid Derivatives of "Ex Falso" and "Reductio ad Absurdum"

The hybrid derivative of "Ex falso quodlibet" is produced as follows:

$$(\mathrm{EFQ})\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}$$

$$\Downarrow$$

$$(\mathrm{HEQF})\ \frac{\Gamma \dashv \phi}{\Gamma \dashv \bot}.$$

Respectively, here is the hybrid derivative of "Reductio ad absurdum":

$$(\mathrm{RAA})\ \frac{\Gamma, \neg\phi \vdash \bot}{\Gamma \vdash \phi}$$

$$\Downarrow$$

$$(\mathrm{HRAA})\ \frac{\Gamma \dashv \phi}{\Gamma, \neg\phi \dashv \bot}.$$

Note that this refutation rule is sound for PL, but not for the inuitionistic logic.

Remark 7. *Rumfitt considers in [17] (thanks to an anonymous reviewer for this reference) "reversals" of the rules of* $\boldsymbol{ND}^{\mathsf{PL}}$ *to formalise derivations between "signed sentences"* $+A$ *and* $-A$ *used "to abbreviate Smiley's amalgams of questions with answers 'Is it the case that A? Yes' and 'Is it the case that A? No' " (ibid.). While the motivation is different from the one coming from refutation inference rules, most (but not all) resulting rules are essentially the same as the hybrid derivative rules for **ND** obtained here. However, there is an essential distinction between the meanings of the two types of rules, e.g.,: whereas rejection of a sentence implies acceptance of its negation, and deductive refutation of the validity of a sentence does not imply deduction of the validity of its negation. That distinction is manifested e.g., by the rules* $+\neg I$ *and* $-\neg E$ *in [17] as compared to the hybrid derivative rules for* $\neg$ *obtained and employed here.*

4.5. Atomic Refutations and Monotonicity Rules

The canonical extension $\mathcal{H}(\mathbf{ND}^{\mathsf{PL}})$ constructed above is easily seen to be too weak for deriving refutations, as it does not contain any refutation axioms nor hybrid refutation rules that only have deduction sequents as premises; hence, it cannot enable derivation of any refutation sequents yet. In order to compensate for that, we also need to add the following atomic **refutation axiom scheme** $\mathsf{RefAx}^{\mathsf{PL}}$:

$$\Gamma \dashv \phi,$$

where ϕ is a literal or $\perp$, all formulae in Γ are literals, Γ does not contain a complementary pair of literals, and $\phi \notin \Gamma$. Note that $\mathsf{RefAx}^{\mathsf{PL}}$ is a non-structural refutation axiom scheme, i.e., not closed under uniform substitution.

In addition, the rules of $\mathcal{H}(\mathbf{ND}^{\mathsf{PL}})$ do not enable explicitly removing formulae from the left-hand side of a refutation sequent. To solve that deficiency and to streamline the hybrid derivation system, we also add the following two monotonicity rules:

- The rule $\mathsf{Mon}^{\vdash}$: **Monotonicity of** $\vdash$
$$\frac{\Gamma \vdash \phi,\ \Gamma \subseteq \Gamma'}{\Gamma' \vdash \phi},$$
(Usually this rule is implicitly assumed in any traditional system of natural deduction.)
- The rule $\mathsf{Mon}^{\dashv}$: **Anti-monotonicity of** $\dashv$
$$\frac{\Gamma \dashv \phi,\ \Gamma' \subseteq \Gamma}{\Gamma' \dashv \phi}.$$

Let us denote by $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$ the extension of $\mathcal{H}(\mathbf{ND}^{\mathsf{PL}})$ obtained by adding the rules $\mathsf{RefAx}^{\mathsf{PL}}$, $\mathsf{Mon}^{\vdash}$, and $\mathsf{Mon}^{\dashv}$. The system $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$ will be called the **standard hybrid extension of** $\mathbf{ND}^{\mathsf{PL}}$.

5. Some Results about the Standard Hybrid Extension of $\mathbf{ND}^{\mathsf{PL}}$

5.1. Soundness and Some Properties of $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$

Proposition 3.

1. *Every rule of $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$ is sound.*
2. *$\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$ is Ł-sound for* PL *and hence Ł-consistent.*
3. *If Γ is a satisfiable set of formulae, then $\Gamma \vdash \perp$ is not derivable in $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$.*

Proof. 1. The soundness of all derivative rules for PL follows from the D-soundness of **ND** for PL and Proposition 2. Proving the soundness of $\mathsf{RefAx}^{\mathsf{PL}}$, $\mathsf{Mon}^{\vdash}$, and $\mathsf{Mon}^{\dashv}$ for PL is quite routine, and I leave out the details.

2. Now, the Ł-soundness of $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$ for PL follows by a straightforward induction on hybrid derivations (Corollary 1). In particular, $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$ extends conservatively $\mathbf{ND}^{\mathsf{PL}}$ with respect to deduction sequents.

3. Follows immediately from 2.
□

Lemma 1.

1. *If $\Gamma \dashv \phi$ is derivable in $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$, then $\Gamma, \neg\phi \dashv \phi$ is derivable in $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$.*
2. *If $\Gamma, \phi \dashv \psi$ is derivable in $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$, then $\Gamma \dashv \phi \to \psi$ is derivable in $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$.*

3. *If* $\Gamma \vdash \phi \to \psi$ *is derivable in* $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$ *and* $\Gamma, \phi \dashv \theta$ *is derivable in* $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$*, then* $\Gamma, \psi \dashv \theta$ *is derivable in* $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$*.*

   Consequently, if $\Gamma \vdash \phi \leftrightarrow \psi$ *is derivable in* $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$*, then* $\Gamma, \phi \dashv \theta$ *is derivable in* $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$ *iff* $\Gamma, \psi \dashv \theta$ *is derivable in* $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$*.*
4. *If* $\Gamma \vdash \phi \leftrightarrow \psi$ *is derivable in* $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$*, then* $\Gamma, \theta \dashv \phi$ *is derivable in* $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$ *iff* $\Gamma, \theta \dashv \psi$ *is derivable in* $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$*.*
5. $\Gamma, \psi_1, ..., \psi_k \dashv \theta$ *is derivable in* $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$ *iff* $\Gamma, \psi_1 \wedge ... \wedge \psi_k \dashv \theta$ *is derivable in* $\mathcal{H}^s(\boldsymbol{ND}^{\mathsf{PL}})$*.*

Proof.

1. Let $\Gamma \dashv \phi$ be derived in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$.

   Then, $\Gamma, \neg\phi \dashv \bot$ is derived in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$, by (HRAA).

   Hence, $\Gamma, \neg\phi \dashv \phi$ is derived in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$, by ($\neg$HE^2).
2. Suppose $\Gamma, \phi \dashv \psi$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$. Since $\Gamma, \phi \vdash \phi$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$, we derive $\Gamma, \phi \dashv \phi \to \psi$ by ($\to$ HI). Then, by the Anti-Monotonicity rule $\mathsf{Mon}^{\dashv}$, $\Gamma \dashv \phi \to \psi$ is derived in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$.
3. Let $\Gamma \vdash \phi \to \psi$ be derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$.

   Since $(\phi \to \psi) \to ((\psi \to \theta) \to (\phi \to \theta))$ is a classical tautology,
   $\Gamma \vdash (\phi \to \psi) \to ((\psi \to \theta) \to (\phi \to \theta))$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$.

   Hence, by Modus Ponens, $\Gamma \vdash (\psi \to \theta) \to (\phi \to \theta)$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$. (*)

   Now, suppose that $\Gamma, \phi \dashv \theta$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$.

   Then, by item 2, $\Gamma \dashv \phi \to \theta$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$.

   Therefore, $\Gamma \dashv \psi \to \theta$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$ by ($\to$ HE^2) applied to the latter and (*). Then, finally, $\Gamma, \psi \dashv \theta$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$, by ($\to$ HE^1).
4. Let $\Gamma \vdash \phi \leftrightarrow \psi$ be derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$.

   Suppose that $\Gamma, \theta \dashv \phi$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$.

   Then, $\Gamma, \dashv \theta \to \phi$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$, by claim 2. (**)

   Since $(\phi \leftrightarrow \psi) \to ((\theta \to \psi) \to (\theta \to \phi))$ is a classical tautology,
   $\Gamma \vdash (\phi \leftrightarrow \psi) \to ((\theta \to \psi) \to (\theta \to \phi))$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$.

   Therefore, $\Gamma \vdash (\theta \to \psi) \to (\theta \to \phi)$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$.

   Hence, $\Gamma, \dashv \theta \to \psi$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$, by ($\to$ HE^2) applied to the latter and (**).

   Then, finally, $\Gamma, \theta \dashv \psi$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$, by ($\to$ HE^1).
5. It suffices to prove the claim when $k = 2$ and then apply a straightforward induction.

   Suppose $\Gamma, \psi_1, \psi_2 \dashv \theta$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$.

   Then, $\Gamma \dashv (\psi_1 \to (\psi_2 \to \theta))$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$, by applying claim 2 twice.

   Since $(\psi_1 \to (\psi_2 \to \theta)) \leftrightarrow ((\psi_1 \wedge \psi_2) \to \theta)$ is a classical tautology, $\Gamma \dashv (\psi_1 \wedge \psi_2) \to \theta$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$, by claim 4.

   Then, finally, $\Gamma, \psi_1 \wedge \psi_2 \dashv \theta$ is derivable in $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$, by ($\to$ HE^1).

   The converse direction is similar.

□

Given a truth assignment $\delta : \mathsf{Prop} \to \{\mathsf{f}, \mathsf{t}\}$, for any propositional variable $p \in \mathsf{Prop}$, let us define $p^\delta := p$ if $\delta(p) = \mathsf{t}$, else $p^\delta := \neg p$.

Lemma 2. *Let Γ be a finite set of propositional formulae and let $\{p_1, ..., p_n\}$ contain all propositional variables occurring in formulae in Γ. Suppose δ is a truth assignment satisfying Γ and let $\Gamma^\delta = \Gamma \cup \{p_1^\delta, ..., p_n^\delta\}$. Then, $\Gamma^\delta \dashv \bot$ is derivable in $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$.*

Proof. By items 3 and 5 of Lemma 1, it suffices to prove the claim assuming that all formulae in Γ are transformed to equivalent ones in CNF and then replaced by the list of elementary disjunctions occurring as conjuncts in that CNF. Thus, without loss of generality, we can assume that $\Gamma = \{\gamma_1, ..., \gamma_k\}$, where all γ_i are elementary disjunctions.

Take the satisfying assignment δ. By definition, δ also satisfies all literals in $\{p_1^\delta, ..., p_n^\delta\}$. Furthermore, $p_1^\delta, ..., p_n^\delta \dashv \bot$ is an atomic refutation axiom, hence derivable in $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$.

Now, select from each γ_i in Γ a literal disjunct α_i that is satisfied by δ. Then, α_i must be in $\{p_1^\delta, ..., p_n^\delta\}$. Hence, $\{p_1^\delta, ..., p_n^\delta, \alpha_1, ..., \alpha_n\} = \{p_1^\delta, ..., p_n^\delta\}$.

Therefore, $p_1^\delta, ..., p_n^\delta, \alpha_1, ..., \alpha_n \dashv \bot$ is an atomic refutation axiom, hence derivable in $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$. (*)

In addition, $\vdash \alpha_i \to \gamma_i$ is derivable in $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$, for each $i = 1, ..., n$. Therefore, by applying repeatedly item 3 of Lemma 1, we can replace successively each α_i by γ_i in (*), thereby eventually proving the claim. □

By Anti-Monotonicity of $\dashv$, Lemma 2 immediately implies the following.

Corollary 2. *Let Γ be a finite satisfiable set of propositional formulae. Then, $\Gamma \dashv \bot$ is derivable in $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$.*

5.2. *Ł-Completeness and Ł-Adequacy of $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$*

Theorem 1. *The hybrid derivation system $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$ is Ł-complete for the classical propositional logic* PL.

Proof. Due to the deductive completeness of $\textbf{ND}^{\mathsf{PL}}$, of which $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$ is a deductively conservative extension, it suffices to prove the R-completeness of $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$, i.e., that the refutation of every non-valid in PL sequent is derivable there. Let $\Gamma \not\models \theta$. Then, there is a truth assignment δ satisfying Γ and falsifying θ. Therefore, δ satisfies $\Gamma \cup \{\neg\theta\}$. By Corollary 2, it follows that $\Gamma, \neg\theta \dashv \bot$ is derivable in $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$. Then, by Rule ($\neg$HE^2), $\Gamma, \neg\theta \dashv \theta$ is derivable in $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$. Finally, by the Anti-Monotonicity Rule $\mathsf{Mon}^{\dashv}$, we obtain that $\Gamma \dashv \theta$ is derivable in $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$. QED. □

Proposition 3 and Theorem 1 together imply the following.

Corollary 3. *The hybrid derivation system $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$ is Ł-adequate for* PL *and, therefore, it provides a syntactic decision procedure for* PL.

The system $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$ and the ND-style refutation system developed in [2] are equivalent in terms of formal refutability, by virtue of the respective Ł-soundness and Ł-completeness results. Still, they are fairly different in style and it would be instructive to compare their proof-theoretic features, strengths and weaknesses, for the sake of possibly designing a better structured system of practical derivations based on $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$.

Remark 8. *Note that only some of the derived hybrid refutation rules were used in the proofs of Ł-soundness and Ł-completeness, hence the others must be derivable, or at least admissible, in the reduction of $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$ obtained by removing them. I leave the question of identifying a minimal Ł-complete subsystem of $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$ to future investigation. In particular, however, the rule* HRAA *is used in the proof of Lemma 1, hence that proof is not applicable to the system $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$ of Natural Deduction for the intuitionistic propositional logic* IPL. *Of course, it should not be applicable for* IPL*, e.g., because the refutation axiom $\dashv (p \vee \neg p)$ ought to be derivable there, while it is not in $\mathcal{H}^s(\textbf{ND}^{\mathsf{PL}})$.*

6. Towards a Meta-Proof Theory of Hybrid Derivation Systems

Adding the relation $\dashv$ for syntactic refutation and building systems of formal derivations that involve it together with the standard provability relation $\vdash$ can be regarded as first steps towards internalising the notion of hybrid derivation into the logical language and then developing a theory for that notion that mirrors the proof theory of $\vdash$. In particular, derivability and refutability can now be treated on a par, as two related primitive concepts rather than as complementary ones where refutability is to be represented syntactically by non-provability. (Note, however, that, for any complete logic or theory, $\vdash$ and $\dashv$ applied to sequents of sentences are readily inter-reducible as complementary relations.) Thus, a proof theory of hybrid derivation systems emerges, extending and combining both the traditional proof theory and the theory of refutation systems.

Furthermore, the basic logical concepts of soundness, completeness, consistency, and satisfiability that relate syntax and semantics of a given logical system can now be all expressed and treated purely syntactically in terms of $\vdash$ and $\dashv$. Thus, a "meta-proof theory" of hybrid derivation systems now emerges too, studying the meta-logic of these concepts respective to the given logical system L. Here, I will only set the stage for development of such meta-proof theory and will raise some generic questions, but I leave its systematic study to future work.

To begin with, let us add a new meta-symbol **F**, for "absurd", "falsum", or "contradiction", to the meta-language of hybrid derivation systems. Now, new hybrid derivation rules can be added to the thus extended framework, in order to reflect basic meta-properties of the given hybrid derivation system:

$\triangleright$ **Cons**, stating consistency:

$$\frac{\vdash \phi, \dashv \phi}{\mathbf{F}},$$

$\triangleright$ "Ex (meta-)falso quodlibet", **EFQ**:

$$\frac{\mathbf{F}}{\vdash \phi}, \quad \frac{\mathbf{F}}{\dashv \phi},$$

$\triangleright$ **Ł-Comp**: "Ł-completeness":

$$\frac{\begin{array}{c}[\vdash \phi]\\ \vdots \\ \mathbf{F}\end{array}}{\dashv \phi},$$

$\triangleright$ Ł-**RAA**: "Ł-Reductio ad absurdum"

$$\frac{\begin{array}{c}[\dashv \phi]\\ \vdots \\ \mathbf{F}\end{array}}{\vdash \phi}.$$

Deductive completeness and Ł-completeness can now be *internalised* and stated as additional hybrid rules:

$$(\mathbf{Ded})\ \frac{\begin{array}{cc}[\dashv \phi] & [\vdash \phi]\\ \vdots & \vdots \\ \vdash \psi & \vdash \psi\end{array}}{\vdash \psi} \qquad (\mathbf{Ref})\ \frac{\begin{array}{cc}[\dashv \phi] & [\vdash \phi]\\ \vdots & \vdots \\ \dashv \psi & \dashv \psi\end{array}}{\dashv \psi}.$$

Some natural questions arise:

1. Can any of these meta-rules strengthen the deductive power of a given (not complete) hybrid derivation system?
2. In particular, can any of these bring about deductive completeness or Ł-completeness, when it does not hold without them?

A next natural step would be to strengthen the meta-language even further, to a full-fledged logical meta-language, involving meta-variables and quantification over derivable and refutable formulae (or, sequents). Then, for instance, the semantic relationship between $\vdash$ and $\dashv$ can be postulated in the meta-language as $\Gamma \vdash \phi \;\; \Leftrightarrow \sim \Gamma \dashv \phi$ (where $\sim$ is the meta-negation). (Some initial steps into studying propositional meta-theory of acceptance and rejection of formulae (sequents with empty lists of premises) in a similar spirit can be found in [14].) I leave the general study of the meta-proof theory of hybrid derivation system to future work.

Remark 9. *It should be noted that what I call here 'meta-proof theory' has essentially been studied in great depth for theories of the arithmetic in the context of Gödel's incompleteness theorems and, more generally, in the context of* axiomatic theories of truth*; see [24]. However, the general meta-proof theory proposed here makes no assumptions about the expressiveness of the object logic regarding definability of truth predicates in it, or in general, and consequently it has a much wider scope.*

7. Conclusions

7.1. Some Applications of Hybrid Derivation Systems

Arguably, hybrid derivation systems have a number of potential applications, both conceptual and technical, including:

- Hybrid derivation systems put proofs and refutations on equal footing and thus enable their comparative study and of the development of meta-proof theory, where the interaction of the concepts of deduction and syntactic refutation for a given logic is the object of study.
- Hybrid derivation systems can yield purely deductive decision procedures, as indicated in Proposition 1 and illustrated for PL in Section 5.
- Hybrid derivation systems can capture important classes of non-valid formulae in recursively axiomatizable but undecidable logic, such as FOL. They can also provide complete refutation systems for logical theories with co-r.e. validity. Typically, this is logic defined over a class of finite models, such as FOL in the finite or Medvedev's logic of finite problems (see respectively [23,25] for R-complete refutation systems for these).
- Hybrid derivation systems can *possibly* provide more succinct proof systems. This hypothesis is yet to be tried and tested.

7.2. Current and Future Work

Due to space and time limitations, this paper leaves many open ends and related questions, some of which have already been mentioned so far. In addition, here are some topics of current and follow-up work:

- Develop and understand the general meta-proof theory of hybrid derivation systems.
- Design Ł-complete hybrid derivation systems for the intuitionistic propositional logic and for some important modal logic (extending such results from [3]) and for other non-classical logic.
- Extend/modify $\mathcal{H}^s(\mathbf{ND}^{\mathsf{PL}})$ to hybrid derivation systems for classical and intuitionistic FOL that are R-complete for the non-validities in the finite. Characterise the set of refutable non-validities in these systems.

- Relate more explicitly hybrid derivation systems with tableaux systems. As the latter are designed to check satisfiability, i.e., non-validity of the negated input, they are naturally related to refutations and, hence, to hybrid derivation systems.
- Analyze the relation of the present work with Negri's work on proofs and countermodels in [13,26] and explore the interaction of these two approaches to develop systems combining proofs, refutations, and counter-model constructions for various non-classical logic.
- Another potentially interesting direction (suggested by an anonymous referee) for related further research is to explore the relation between hybrid derivation systems and methods for proof certification [27].
- Last but not least: a challenge worth pursuing in this area would be to obtain new decidability results by designing Ł-adequate hybrid deductive systems for logic that is not yet known to be decidable, such as Medvedev's logic.

Funding: This work was partly supported by research grant 2015-04388 of the Swedish Research Council.

Acknowledgments: I thank Sara Negri and Tom Skura, as well as the anonymous referees for careful reading, helpful suggestions, and some important corrections. I also thank the participants in the Refutation Symposium in Poznań 2018 and in the CLLAM seminar at the Philosophy Department of Stockholm University for some useful comments on earlier versions of this work presented at these events.

Conflicts of Interest: The author declares no conflicts of interest.

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