

Article

The *Zahl-Anzahl* Distinction in Gottlob Frege: Arithmetic of Natural Numbers with *Anzahl* as a Primitive Term

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Abstract: The starting point is Peano's expression of the axiomatics of natural numbers in the framework of Leśniewski's elementary ontology. The author enriches elementary ontology with the so-called Frege's predication scheme and goes on to propose the formulations of this axiomatic, in which the original natural number (*N*) term is replaced by the term *Anzahl* (*A*). The functor of the successor (*S*) is defined in it.

Keywords: Peano's axiomatics of natural numbers; Leśniewski's elementary ontology; Frege's predication scheme; Frege's *Zahl-Anzahl* distinction

1. Introduction

The term *Anzahl* functions in German in various numerical phrases, more or less defined. It appears in such contexts as "Anzahl der Äpfel" (i.e., "number of apples") or "neun ist Anzahl der Planeten des Sonnensystems" (i.e., "nine is the number of planets of the Solar System"). In numerical phrases of the indefinite type, we use the combination of the number name (*Zahl*), which is an abstract object, and the general name, which is the product of a general name (here, the word apple/apples) with a name present only implicitly, which somehow characterizes the objects falling within the extension of the first name (usually a general name referring to the place they occupy). However, number phrases of the definite type (v. gr. "neun ist Anzahl den Planeten des Sonnensystem" or "Anzahl... ist gleich dem Anzahl...") are discussed in Gottlob Frege's works. Number phrases of the definite type are generally given to us through elementary expressions of the type "*a* is *Ax*", where *a* is a number and *x* a (definite) general name. There is a specific functor *A* (from "Anzahl") of the *n/n* category. In Frege, "Anzahl" refers to a concept determined by a single-argument predicate (*Begriffswort*). Given these remarks, we enrich the framework of elementary ontology with the so-called Frege's predication scheme, to propose the formulations of these axiomatics in which the original natural number (*N*) term is replaced by the term *Anzahl* (*A*), and we present an original depiction of the arithmetic of natural numbers with *Anzahl* as a primitive term.

2. Preliminaries

2.1. Elementary Ontology

The specific axiom of elementary ontology (OE) is:

$$A0 \quad x\epsilon y \leftrightarrow \Sigma z(z\epsilon x) \wedge \Pi z(u(z\epsilon x \wedge u\epsilon x \rightarrow z\epsilon u) \wedge \Pi z(z\epsilon x \rightarrow z\epsilon y)$$

The secondary rules, which are direct consequences of axiom A0 include:

$$R1 \quad x\epsilon y / x\epsilon x$$

$$R2 \quad x\epsilon y \wp y\epsilon z / x\epsilon z$$

$$R3 \quad x\epsilon y \wp y\epsilon z / y\epsilon x$$

The nominal constants of object and contradictory object are defined as follows:

$$DV \quad x\epsilon V \leftrightarrow x\epsilon x \quad x \text{ is an object;}$$

$$DL \quad x\epsilon \Lambda \leftrightarrow x\epsilon x \wp \sim x\epsilon x \quad x \text{ is a contradictory object.}$$

The functors of existence, singularity, being an object, weak inclusion, strong inclusion, extension identity, identity, and negation are also introduced by definition:

$$Dex \quad ex(x) \leftrightarrow \Sigma z(z\epsilon x) \quad x \text{ exists;}$$

$$Dsol \quad sol(x) \leftrightarrow \Pi zu(z\epsilon x \wedge u\epsilon x \rightarrow z\epsilon u) \quad \text{at most one object is } x;$$

$$Dob \quad ob(x) \leftrightarrow x\epsilon x \quad \text{is object } x;$$

$$D\delta \quad x \subset y \leftrightarrow \Pi z(z\epsilon x \rightarrow z\epsilon y) \quad x \text{ is included in } y \text{ (weak inclusion);}$$

$$D' \quad x \sqsubset y \leftrightarrow \Sigma z(z\epsilon x) \wedge \Pi z(z\epsilon x \rightarrow z\epsilon y) \quad x \text{ is included in } y \text{ (strong inclusion);}$$

$$D \quad x \square y \leftrightarrow \Pi z(z\epsilon x \leftrightarrow z\epsilon y) \quad x \text{ is extension identical with } y;$$

$$D= \quad x = y \leftrightarrow x\epsilon y \wedge y\epsilon x \quad x \text{ is identical with } y;$$

$$Dn \quad x\epsilon ny \leftrightarrow x\epsilon x \wedge \sim x\epsilon y \quad x \text{ is non } y.$$

We also adopt the definition of the of satisfying functor:

$$Dstsf \quad x\epsilon stsf(\phi) \leftrightarrow x\epsilon x \wedge \phi(x) \quad x \text{ is satisfying } \phi$$

The system of elementary ontology can be founded on the calculus of predicates without identity. Jerzy Słupecki's work [1] may serve as an introduction to elementary ontology.

2.2. Peano's Axiomatics

The system of arithmetic of natural numbers can be founded on the axiomatics given by Peano [2]. The formulation of these axiomatics in the framework of elementary ontology can be found in Leśniewski's works ([3], chapter 4, p. 129):

$$LA1 \quad 1\epsilon N$$

$$LA2 \quad a\epsilon N \rightarrow Sa\epsilon N$$

$$LA3 \quad a\epsilon N \rightarrow \sim Sa\epsilon 1$$

$$LA4 \quad a\epsilon N \wedge b\epsilon N \wedge Sa = Sb \rightarrow a = b$$

$$LA5 \quad 1\epsilon x \wedge \Pi b(b\epsilon N \wedge b\epsilon x \rightarrow Sb\epsilon x) \wedge a\epsilon N \rightarrow a\epsilon x$$

The last axiom is the axiom of mathematical induction. Axioms A1–A5, as depicted below in Section 4, are equivalent to the axioms given by Peano.

2.3. Elementary Ontology with Frege's Predication Scheme

Here, the system of elementary ontology is extended with the functor *sub* of the *n/n* category. The elementary expression " $x\epsilon sub(y)$ " is read here as: " x is subordinated to y ". The specific axioms of this system (\mathbf{OE}^{sub}) have the following shapes (see [4]):

$$SA1 \quad x\epsilon sub(y) \rightarrow sub(x) \subset sub(y)$$

$$SA2 \quad x\epsilon sub(y) \rightarrow \sim y \subset sub(x)$$

$$SA3 \quad x\epsilon sub(y) \rightarrow y\epsilon y$$

$$SA4 \quad sub(x) \square sub(y) \rightarrow y \square x$$

The functor of *concept* is introduced by definition:

$$DC \quad x\epsilon Cy \leftrightarrow x\epsilon x \wedge \Pi z(z\epsilon y \leftrightarrow z\epsilon sub(x)) \quad x \text{ is a concept } y$$

The key theses with functors *sub* and *C* include:

$$x\epsilon sub(Cy) \rightarrow x\epsilon y \quad \text{If } x \text{ is subordinated to the concept } y, \text{ then } x \text{ is } y,$$

and:

$$Cx\epsilon sub(Cy) \rightarrow x \subset y \quad \text{If the concept } x \text{ is subordinated to the concept } y, \text{ then } x \text{ is included in } y$$

3. Idea

The term “Anzahl” functions in German in various numerical phrases, more or less defined. The word *Anzahl* appears in such contexts as: *Anzahl der Äpfel* (number of apples), *Anzahl von Freuden* (number of friends), *fünf is Anzahl der Bäumen in meinem Garten* (five is the number of trees in my garden), or *neun ist Anzahl der Planeten des Sonnensystems* (nine is the number of planets of the Solar System).

3.1. An Indefinite Numerical Phrase with the Term Anzahl

In number phrases of the indefinite type we have to deal with the combination of the number name (*Zahl*) being an abstract object with the general name, which is the product of a general name (e.g., apple/apples) and a name present only implicitly, which somehow characterizes the objects falling within the extension of the first name (usually a general name referring to the place they occupy). For example, “five apples” occupy a certain place on the table.

A phrase of this type, which is a nominal expression, can be rendered as follows:

$$a \circ x,$$

where a is a number (*Anzahl*) and x is the name of object of a given sort, which, in combination with a hidden name characterizing these objects, gives us our universe of discourse. Since a and x are names, $a \circ x$ is also a name, then \circ is a functor of the n/n category.

3.2. A Definite Numerical Phrase with the Term Anzahl

Numerical phrases of the definite type (“neun ist Anzahl den Planeten des Sonnensystem” or “Anzahl... ist gleich dem Anzahl...”) are discussed in Gottlob Frege’s works (see [5], p. 20 and [6], p. 88). In number phrases of the definite type, we use an elementary expression of the type:

$$a \varepsilon Ax$$

where a is a number, ε is a functor (of the s/n category), and x is a (definite) general name. A specific functor A (from *Anzahl*) of the n/n category appears here.

In Frege, “Anzahl” refers to a concept designated by a single-argument predicate (*Begriffswort*), but also a functor of the $n/(s/n)$ category. The elementary phrase with this functor could be expressed in the following way: $a \varepsilon A(P)$, where the (single-argument) predicate P is equivalent to the name x (from the phrase “ $a \varepsilon Ax$ ”). Our expression existing in the framework of the calculus of names is closer to natural language, in which the category of names is understood broadly (individual and general names), as opposed to only individual names, as in Frege’s language (*Namen=Eigennamen*).

We will continue to deal with definite number phrases with the term *Anzahl*. We shall return to indefinite phrases with this term in the final part of the present paper.

4. Arithmetic of Natural Numbers

As our starting point, we shall adopt the following five axioms given by Leśniewski, including 0 in natural numbers, which is in accordance with the contemporary formulations of Peano’s axiomatics:

- A1 $0 \varepsilon N$
- A2 $a \varepsilon N \rightarrow Sa \varepsilon N$
- A3 $a \varepsilon N \rightarrow \sim Sa \varepsilon 0$
- A4 $a \varepsilon N \wedge b \varepsilon N \wedge Sa = Sb \rightarrow a = b$
- A5 $0 \varepsilon x \wedge \Pi b (b \varepsilon N \wedge b \varepsilon x \rightarrow Sb \varepsilon x) \wedge a \varepsilon N \rightarrow a \varepsilon x$

Our initial system is OE^{sub} enriched with these axiomatics. The language of this system ($OE^{sub}[A1,A2,A3,A4,A5]$) is extended with number nominal variables (a,b,c), referring to natural numbers. Apart from the standard rule of detachment (MP):

$$MP \quad \alpha, \alpha \rightarrow \beta / \beta$$

we shall adopt the rule of substitution:

$$RS \quad \alpha / \alpha[x/t],$$

where t is a nominal variable/constant or (in particular) a number nominal variable/constant:

$$\alpha / \alpha[a/t],$$

where t is a number nominal variable/constant.

Having adopted the definition of the natural number:

$$DN \quad a \in N \leftrightarrow a \in a \wedge \Sigma x(a \in Ax),$$

we can eliminate constant N by means of functor A :

- B1 $0 \in A \wedge$
- B2 $a \in Ax \rightarrow \Sigma z(Sa \in Az)$
- B3 $a \in Ax \rightarrow \sim Sa \in A \wedge$
- B4 $a \in Ax \wedge b \in Ay \wedge Sa \in Sb \rightarrow a \in b$
- B5 $0 \in x \wedge \Pi by(b \in x \wedge b \in Ay \rightarrow Sb \in x) \wedge a \in Az \rightarrow a \in x$

Axioms A1–A5 are consequences of Axioms B1–B5:

- (A1) $0 \in N$ [B1, DN]
- (A2) $a \in N \rightarrow Sa \in N$ [DN, B2]
- (A3) $a \in N \rightarrow \sim Sa \in 0$ [DN, B1, R2, B3]
- (A4) $a \in N \wedge b \in N \wedge Sa = Sb \rightarrow a = b$ [DN, D=, B4]
- (A5) $0 \in x \wedge \Pi b(b \in N \wedge b \in x \rightarrow Sb \in x) \wedge a \in N \rightarrow a \in x$ [DN, B5]

Next, we shall eliminate from axiomatics the functor of the successor (S) by introducing it by definition. We shall leave it only in the last axiom, for the purpose of shortening.

We shall adopt the axiomatics:

- C1 $a \in a$
- C2 $a \in Ax \rightarrow ex(nx)$
- C3 $\sim ex(x) \rightarrow 0 \in Ax$
- C4 $a \in Ax \wedge b \in Ax \rightarrow a \in b$
- C5 $0 \in x \wedge \Pi by(b \in x \wedge b \in Ay \rightarrow Sb \in x) \wedge a \in Az \rightarrow a \in x$ (=B5)

We shall introduce the functor of the successor by definition:

$$DS \quad a \in Sb \leftrightarrow a \in nb \wedge \Pi xz(b \in Ax \wedge z \in nx \leftrightarrow a \in A(x \cup z) \wedge z \in nx) \wedge \Pi xz(0 \in Ax \wedge z \in nx \rightarrow a \in Az) \wedge \Pi yz(a \in Az \wedge \sim ex(z) \rightarrow \sim b \in Ay) \wedge \Pi z(a \in Az \wedge z \in z \rightarrow b = 0)$$

- T1.1 $sol(Ax)$ [C4, Dsol]
- T1.2 $A \wedge = 0$ [OE, C3, T1.1, R3, D=]
- T1.3 $a \in Ax \wedge b \in Ax \rightarrow a = b$ [C4, D=]
- $Sa \in Sa \leftrightarrow Sa \in na \wedge \Pi xz(a \in Ax \wedge z \in nx \leftrightarrow Sa \in A(x \cup z) \wedge z \in nx) \wedge \Pi xz(0 \in Ax \wedge z \in nx \rightarrow Sa \in Az)$
- T1.4 $\wedge \Pi yz(Sa \in Az \wedge \sim ex(z) \rightarrow \sim a \in Ay) \wedge \Pi z(Sa \in Az \wedge z \in z \rightarrow a = 0)$ [DS]
- T1.5 $Sa \in A(x \cup z) \wedge z \in nx \rightarrow a \in Ax$ [R1, T1.4]
- T1.6 $Sa \in Ax \wedge x \in x \rightarrow a = 0$ [R1, T1.4]
- T1.7 $\sim Sa \in a$ [R1, T1.4, Dn]
- T2.1 $0 \in A \wedge$ (=B1) [OE, C3]
- T2.2 $a \in Ax \rightarrow \Sigma z(Sa \in A$ (=B2) [C2, C1, Dex, T1.4]
- T2.3 $a \in Ax \rightarrow \sim Sa \in A \wedge$ (=B3) [R1, T1.4, OE]
- T2.4 $a \in Ax \wedge b \in Ay \wedge Sa \in Sb \rightarrow a \in b$ (=B4) [OE, T1.4, C2, Dex, T1.5, T1.3]
- T2.5 $\phi(0) \wedge \Pi b(b \in N \wedge \phi(b) \rightarrow \phi(Sb)) \wedge a \in N \rightarrow \phi(a)$ [C1, Dstsf, DN, C5]

Of these, T2.5 is a typical formulation of an induction axiom.

5. Addition and Multiplication: Logical and Philosophical Analysis

Addition and multiplication as operations on numbers—with the numerical functor (*Anzahl*) as a primary functor—is, from a logical and philosophical point of view, an operation more complex than it usually appears.

5.1. Addition

The addition of numbers in the context of a numerical functor (*Anzahl*) establishes the number of objects of the same sort. For example, with five trees in the front garden and seven trees in the back garden, we reach the conclusion that we have 12 trees altogether. Using names y and z so that $y \sqsubset$ a tree in the front garden and $z \sqsubset$ a tree in the back garden, and taking into account the fact that the extensions of these names have no common elements ($\sim ex(y \cap z)$), we create in this case a common name $x \sqsubset$ a tree in the garden, so that $x \sqsubset y \cup z$ —in order to state the following: $12 \in Ax$ with $5 \in Ay$ and $7 \in Az$, where between x , y , and z there are the connections described before. The operation of addition in this context can be generally defined as follows:

$$D+ \quad a \in b + c \leftrightarrow a \in a \wedge \exists xyz(x \sqsubset y \cup z \wedge \sim ex(y \cap z) \wedge a \in Ax \wedge b \in Ay \wedge c \in Az)$$

5.2. Multiplication

The multiplication of numbers in the context of the numerical functor (*Anzahl*) is more complicated. Let us take a similar example. Let us assume that in the garden we have trees which are grouped in four groups, each comprising five trees. There are two groups in front of the house (on the left and on the right) and two groups behind the house (one on the left and one on the right, too). We begin our operation by distinguishing four groups, which we treat as distributive classes determined by four names: y , z , u , and v , which are, respectively a tree in the front garden on the left, a tree in the front garden on the right, a tree in the back garden on the left, and a tree in the back garden on the right.

We shall mark these classes, respectively, as: Cy , Cz , Cu , and Cv . This distinction is accompanied by an ascertainment that for a certain name x with the extension $x \sqsubset y \cup z \cup u \cup v$ ($x \sqsubset$ a tree in the garden), these names are separate in terms of extension— $\sim ex(y \cap z)$, $\sim ex(z \cap u)$, and $\sim ex(u \cap v)$. Next, we state that there are four such groups/classes of trees— $4 \in Aw$ $4 \in Aw$, where w is a shortening of the name “a group of trees in the garden”, and we finally state that each group of trees comprises five trees and each tree in the garden can be characterized as belonging strictly to one of these groups and as one of five trees belonging to this group. This last sentence can be shortly expressed as follows: each tree in the garden is characterized by a pair [one of the groups, one of the trees of a given bank of trees assigned to this group], to be eventually able to say that the number of trees in the garden is identical to the number of such pairs. Symbolically, it can be briefly expressed as follows:

$$D\odot \quad a \in b \odot c \leftrightarrow a \in a \wedge \exists xyz(a \in Ax \wedge b \in Ay \wedge c \in Az \wedge \Pi u(u \in y \leftrightarrow \Sigma v(u \in Cv \wedge v \subset x)) \wedge x \in y \times z)$$

where \times is a functor of Cartesian product of the n/n category.

6. The New Formulation of the Arithmetic of Natural Numbers

We shall replace axiom C3 with a more intuitive one:

$$C3^* \quad x \sqsubset y \rightarrow Ax \sqsubset Ay$$

The system of arithmetic of natural numbers in this formulation is: $OE^{sub}[C1, C2, C3^*, C4, C5]$. In the formulation of axiom C5, the functor of successor occurs, which we are introducing similarly, by means of the already given definition DS.

Here, the term C from OE^{sub} is interpreted in terms of class (see [7]):

$$DC \quad x \in Cy \leftrightarrow x \in x \wedge \Pi z(z \in y \leftrightarrow z \in sub(x)) \quad x \text{ is a distributive class (of objects which are) } y.$$

In addition to the rules of substitution and detachment (MP), we shall also adopt the rules of omission and introduction for the list operator in the form (see [8]):

$$\text{OL } x\epsilon [z_1, \dots, z_n] / x\epsilon z_1 \vee \dots \vee x\epsilon z_n \quad [x_1, \dots, x_n]\epsilon y / x_1\epsilon y \wedge \dots \wedge z_n\epsilon y \quad \text{if } y \text{ is not list}$$

$$\text{IL } x\epsilon z_1 \vee \dots \vee x\epsilon z_n / x\epsilon [z_1, \dots, z_n] \quad x_1\epsilon y \wedge \dots \wedge z_n\epsilon y / [x_1, \dots, x_n]\epsilon y$$

and the rule:

$$\text{RL } [x_1, \dots, x_n]\epsilon [z_1, \dots, z_n] / x_1\epsilon z_1 \wedge [x_2, \dots, x_n]\epsilon [z_2, \dots, z_n] \quad [x]\epsilon [z] / x\epsilon z$$

Thanks to this rule, we obtain the property for two-element lists, $[x, y] = [z, u] \rightarrow x = z \wedge y = u$, which is the equivalent of property for an orderly pair in the framework of the set theory.

Now, let us define the functor of Cartesian product (compare [9], p. 176):

$$\text{D}\times [x, y] \epsilon z \times u \leftrightarrow x\epsilon z \wedge y\epsilon u$$

Now, we can—quite formally—adopt the definitions of the operations of addition (+) and multiplication (\odot) in accordance with their interpretation:

$$\text{D+ } a\epsilon b + c \leftrightarrow a\epsilon a \wedge \Sigma xyz(x\Box y \cup z \wedge \sim ex(y \cap z) \wedge a\epsilon Ax \wedge b\epsilon Ay \wedge c\epsilon Az)$$

$$\text{D}\odot a\epsilon b \odot c \leftrightarrow a\epsilon a \wedge \Sigma xyz(a\epsilon Ax \wedge b\epsilon Ay \wedge c\epsilon Az \wedge \Pi u(u\epsilon y \leftrightarrow \Sigma v(u\epsilon Cv \wedge v \subset x)) \wedge x\epsilon y \times z)$$

According to D+, addition is a symmetrical operation: $a\epsilon b + c \leftrightarrow c + b$ ($a + b = b + a$).

The operation of multiplication in the sense \odot , in accordance with the definition D \odot , is not symmetrical. We shall introduce multiplication as a symmetrical operation (\cdot) by definition:

$$\text{D}\cdot a\epsilon b \cdot c \leftrightarrow (a\epsilon b \odot c) \vee a\epsilon c \odot b$$

The theses which are consequences of definition D0 and axiom A3* include:

$$\begin{array}{ll} \text{T2.6 } 0\Box A\Lambda & [\text{D0,D }] \\ \text{T2.7 } \sim ex(x) \rightarrow 0\epsilon Ax & (=C3) [\text{Dex,OE,D}\delta,\text{D } ,C3^*,R1,T2.6,R2] \end{array}$$

7. The Term *Anzahl* in the Indefinite Sense

We shall now deal with number phrases with the term “Anzahl” in the indefinite sense, where, “Numbers (Anzahlen) are always numbers (Anzahlen) of something. They can be added (five apples and two apples make seven apples). However, it is impossible to multiply numbers (Anzahlen) by numbers (Anzahlen)” (see [10], p. 7).

Compounds such as five apples or two dogs fall, in accordance with the previous arrangements, under the scheme aBx . We shall introduce the functor \circ by definition:

$$\text{D}\circ x\epsilon a \circ y \leftrightarrow x\epsilon x \wedge \Sigma z(a\epsilon A(y \cap z))$$

The numbers (Anzahlen) in this sense behave like the so-called *denominate numbers* (like, for example: 5 m or 2 kg) present in the so-called *dimensional analysis*, which appears in physics. They can be added, provided that the same unit is preserved. Addition in such contexts can be defined as follows:

$$\text{D}\oplus x\epsilon(a \circ y) \oplus (b \circ y) \leftrightarrow x\epsilon(a + b) \circ y$$

Denominate numbers (*benannte Zahlen*) are the subject of analysis in one of Hermann von Helmholtz’s works (see [11], p. 12).

The numbers (Anzahlen) in the indefinite sense cannot be multiplied, as opposed to denominate numbers in dimensional analysis.

8. Conclusions

A new, original depiction of arithmetic of natural numbers with *Anzahl* as a primitive term has been presented. The basis of this depiction is the calculus of names, which is a certain extension of elementary ontology (OE^{sub}). The notion of a pair (list) was introduced by means of rules OL, IL, and

RL, the last of them playing a significant role. I have recently noticed that it is possible to define an ordered pair in the framework of the OE^{sub} system.

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