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Observations on the Separable Quotient Problem for Banach Spaces

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Abstract: The longstanding Banach–Mazur separable quotient problem asks whether every infinite-dimensional Banach space has a quotient (Banach) space that is both infinite-dimensional and separable. Although it remains open in general, an affirmative answer is known in many special cases, including (1) reflexive Banach spaces, (2) weakly compactly generated (WCG) spaces, and (3) Banach spaces which are dual spaces. Obviously (1) is a special case of both (2) and (3), but neither (2) nor (3) is a special case of the other. A more general result proved here includes all three of these cases. More precisely, we call an infinite-dimensional Banach space X dual-like, if there is another Banach space E, a continuous linear operator T from the dual space E^* onto a dense subspace of X, such that the closure of the kernel of T (in the relative weak* topology) has infinite codimension in E^*. It is shown that every dual-like Banach space has an infinite-dimensional separable quotient.

Keywords: Banach space; separable space; quotient space; weakly compactly generated; dual space; separable quotient problem; Markushevich base; biorthogonal system

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We work in the category of Banach spaces, where the quotient by a closed (i.e., complete) subspace is always another Banach space. The Banach–Mazur separable quotient problem, which asks whether every infinite-dimensional Banach space has a quotient space which is both separable and infinite-dimensional, has remained unsolved for 85 years (the dual problem, finding a separable infinite-dimensional subspace in a given Banach space, is almost trivial). Reflexive Banach spaces constitute one case which is easily resolved. If R is reflexive and infinite-dimensional, then so is its dual R^*. Choose any infinite-dimensional separable subspace $S \subset R^*$. Then S is the annihilator M^0 of some subspace M of R, and $(R/M)^* \cong M^0 = S$ is separable, whence R/M is also separable.

For a comprehensive account of known results, we refer to [1–3]. These give an affirmative answer in a large number of special cases, of which we just mention one omnibus result now (Corollary 17, [3]): If a Banach space X or its dual X^* contains a subspace isomorphic to either c_0 or ℓ_1, then X has an infinite-dimensional separable quotient. This covers most known concrete examples of Banach spaces, in particular the classical function spaces, as each is reflexive or has a subspace isomorphic to either c_0 or ℓ_1. We will focus on two natural generalisations of reflexive spaces, namely weakly compactly generated (WCG) spaces and dual spaces, and examine what they have in common.

Weakly compactly generated (WCG) spaces were introduced to the world by Amir and Lindenstrauss [4]: a Banach space is *WCG* if it is generated by (i.e., is the closed linear span of) a weakly compact subset. This includes all reflexive Banach spaces, because the unit ball of a reflexive space is weakly compact. Amir and Lindenstrauss showed that WCG spaces admit many projections, in particular, every separable subspace of a WCG space is contained in a complemented separable subspace. (For a survey of this topic see (Sections 3 and 4, [5]); and for a modern viewpoint, using the concept of projectional skeleton, see [6] and the references therein.) Since every complemented subspace of a Banach space is isomorphic to a quotient space, it is immediate that every infinite-dimensional WCG space has an infinite-dimensional separable quotient.

One easier proof of this comes from appealing to Proposition 2 below. However the easiest proof is this argument from (Theorem 1, [7]). If X is WCG, choose an infinite-dimensional countable subset of X^*, and let S be the weak* closure of its linear span. Being weak* closed, S is the annihilator M^0 of some subspace M of X, and $(X/M)^* \cong M^0 = S$ is weak* separable. Set $Y = X/M$; then Y^* is weak* separable, which implies that every weakly compact set in Y is weakly metrisable, hence separable. But Y is obviously WCG, hence also separable.

Every reflexive space is also a dual space. Another old question in Banach space theory is whether every dual space has an infinite-dimensional reflexive quotient (equivalently, whether every bidual space contains an infinite-dimensional reflexive subspace). If this were true, it would easily imply that dual spaces have separable quotients. However, a counterexample for this question appeared in 2006 (Theorem 6.27, [8]). Nevertheless in 2008 Argyros, Dodos, and Kanellopoulos [9] succeeded in proving that if X is the Banach dual of any infinite-dimensional Banach space, then X has a separable infinite-dimensional quotient Banach space; this is a result of considerable depth.

It should be noted that neither of the properties *WCG* and *dual* implies the other. It is easy to show that that every separable Banach space is generated by a sequence which converges to zero (i.e., by a norm compact set); however some separable Banach spaces (e.g., c_0 and $L_1(0,1)$) are not isomorphic to dual spaces. Thus, not all WCG spaces are Banach duals. On the other hand, the Banach space ℓ_∞ is the dual of the separable space ℓ_1, which ensures that every weakly compact subset of ℓ_∞ is separable. Since ℓ_∞ is not separable, it cannot be WCG, despite being a dual space. Nevertheless, the following folklore result gives a relationship between WCG spaces and dual spaces, which partially motivates our work.

Proposition 1. *For a Banach space X, the following are equivalent:*

(i) *X is weakly compactly generated.*
(ii) *There is a Banach space Y, and a weak* to weak continuous linear injection $T : Y^* \to X$, with dense range.*
(iii) *There is a Banach space Y, and a weak* to weak continuous linear injection $T : X^* \to Y$.*
(iv) *There is a Banach space Y, and a weak* to weak continuous linear injection $T : X^* \to Y$, with dense range.*
(v) *There is a Banach space Y, and a weak* to weak continuous linear operator $T : Y^* \to X$, with dense range.*

Proof. (sketch)

$(i) \Rightarrow (iii)$: There are several possible choices for Y and T. The first historically, albeit with the most difficult proof, is that Y can be $c_0(\Gamma)$ for a suitably large set Γ (Proposition 2, [4]).

The simplest argument is perhaps the following, which appears in the proof of (Theorem 2.3, [10]). If K is a weakly compact generating subset of the Banach space X, consider the restriction operator $T : X^* \to C(K)$. This is clearly continuous from the topology of uniform convergence on weakly compact subsets of X (i.e., the Mackey* topology $\tau(X^*, X)$) to the norm topology on $C(K)$. It must therefore be continuous in the corresponding weak topologies. But the dual of X^* under $\tau(X^*, X)$ is just X (p. 62, Theorem 7, [11]), so T is weak* to weak continuous. Since K generates X, T is also injective.

Another particularly interesting possibility [12] is that Y can be a reflexive Banach space.

$(iii) \Rightarrow (iv)$: Simply replace Y by the closure of the range of T.

$(iv) \Rightarrow (ii)$: Note that the adjoint $T^* : Y^* \to X$ will be weak* to weak continuous, injective, and have dense range.

$(ii) \Rightarrow (v)$: This is obvious.

$(v) \Rightarrow (i)$: The unit ball of Y is weak* compact, so its image under T will be a weakly compact generating set. □

Note that if we replace *dense range* by *surjective* in condition (ii) above, it becomes a characterisation of reflexivity.

We now introduce a class of Banach spaces which, by virtue of the preceding result, includes all WCG spaces and all dual spaces, and show that all of its members have separable quotients.

Definition 1. *A Banach space X is said to be* dual-like *if there is another Banach space E and a continuous linear operator T from the dual space E^* onto a dense subspace of X, such that the kernel W of T is not too large, in the sense that its closure in the weak*-topology on E^* has infinite codimension in E^*.*

Remark 1. Clearly every dual Banach space is dual-like, as is every WCG space.

Remark 2. If X and E are Banach spaces and there exists a one-to-one continuous linear operator from E^* onto a dense subspace of X, then X is dual-like.

Before presenting our main result, we highlight the following beautiful result of Saxon and Wilansky [1]. Recall that a (closed linear) subspace A of a Banach space X is said to be *quasicomplemented* if there is another subspace B with $A \cap B = \{0\}$ and $A + B$ dense in X. A complemented subspace is clearly quasicomplemented; a proper quasicomplemented subspace is one which is not complemented.

Proposition 2. *For a Banach space X, the following are equivalent:*

(i) *X has an infinite-dimensional separable quotient Banach space.*
(ii) *X has a dense nonbarrelled subspace.*
(iii) *X has a separable infinite-dimensional quasicomplemented subspace.*
(iv) *X has a proper quasicomplemented subspace.*

Theorem 1. *Any infinite-dimensional dual-like Banach space has a quotient Banach space which is infinite-dimensional and separable.*

Proof. Let X be dual-like, then there exist a Banach space E and a continuous linear operator $T : E^* \to X$ such that $T(E^*)$ is dense in X and the weak*-closure of the kernel W of T has infinite codimension in E^*.

Firstly consider the case that T is surjective. Let $F = \{f \in E : w(f) = 0, \text{ for all } w \in W\}$ be the annihilator of W in E. Then let $V = \{v \in E^* : v(f) = 0 \text{ for all } f \in F\}$ be the annihilator in E^* of F. By the Bipolar Theorem (p. 35, Theorem 4 [11]), V is the weak*-closure of W, and by our assumption we have that V has infinite codimension in E^*. By the open mapping theorem $X \cong E^*/W$. Now E^*/W has E^*/V as a quotient space, and E^*/V is isomorphic to F^*. As an infinite-dimensional dual Banach space, by [9], F^* has an infinite-dimensional separable quotient Banach space, and therefore X does too.

Now we consider the case that T is not surjective. The conclusion follows immediately from (Corollary 3.4 [2]); let us repeat the short argument. Since the image $T(E^*)$ is a dense proper subspace, it must be an incomplete normed space. The open mapping theorem for continuous operators mapping a Banach space onto a barrelled locally convex space (p. 116, Theorem 7 and Corollary 1, [11]) then ensures that $T(E^*)$ is not barrelled. Proposition 2 now completes the proof. $\square$

Corollary 1. *Let X be an infinite-dimensional Banach space which is either reflexive, or weakly compactly generated (WCG), or a dual space. Then X has a quotient Banach space which is infinite-dimensional and separable.*

It is well known that Banach spaces with suitable biorthogonal systems, in particular Markushevich bases, admit separable quotients. We show that the idea of dual-like leads to this conclusion under much weaker hypotheses. For a comprehensive account of biorthogonal systems, we refer to [13]. Now we just recall the definitions we need.

Let X be a Banach space, and Γ a nonempty index set. A family $\{(x_i, f_i) : i \in \Gamma\} \subset X \times X^*$ is called a biorthogonal system if $f_i(x_j) = \delta_{ij}$, where δ denotes the Kronecker delta, for all $i, j \in \Gamma$.

A family $\{x_i : i \in \Gamma\} \subset X$ is called a *minimal system* if there exists a family $\{f_i : i \in \Gamma\} \subset X^*$ such that $\{(x_i, f_i) : i \in \Gamma)\}$ is a biorthogonal system (in $X \times X^*$). A family $\{x_i : i \in \Gamma\} \subset X$ is called *fundamental* if it generates X, i.e., the closure of its linear span is all of X. A family $\{f_i : i \in \Gamma\} \subset X^*$ is called *total* if it separates the points of X, equivalently if its linear span is weak* dense in X^*. A fundamental and total biorthogonal system $\{(x_i, f_i) : i \in \Gamma)\} \subset X \times X^*$ is called a *Markushevich basis* for X, or more simply an *M-basis* in X. If the context is clear, one sometimes uses the abbreviated notation $\{x_i : i \in \Gamma\} \subset X$ for an M-basis in X.

It is straightforward to verify that any Banach space with an M-basis has a separable infinite-dimensional quasicomplemented subspace (Prop. 5.73 [13]). Thus every Banach space with an M-basis has a separable quotient. We now use our main theorem to generalise this.

We will call an indexed family $\{(x_i, f_i) : i \in \Gamma\} \subset X \times X^*$ *pseudo-orthogonal* if there is an infinite subset $\Gamma_1 \subset \Gamma$ such that $f_j(x_i) = \delta_{ij}$, whenever $i \in \Gamma$ and $j \in \Gamma_1$.

Lemma 1. *A Banach space with a fundamental pseudo-orthogonal family* $\{(x_i, f_i) : i \in \Gamma\} \subset X \times X^*$ *is dual-like.*

Proof. Without loss of generality, we suppose that $\{x_i : i \in \Gamma\}$ is a bounded subset of X. The Banach space $\ell_1(\Gamma)$ is a dual space, the dual of $c_0(\Gamma)$. We denote its standard basis by $\{e_i : i \in \Gamma\}$. Then the linear operator $T : \ell_1(\Gamma) \to X$,

$$T((\lambda_i)_{i\in\Gamma}) = \sum_{i\in\Gamma} \lambda_i x_i$$

is well defined and has dense range.

If an element $(\lambda_i)_{i\in\Gamma} \in \ell_1(\Gamma)$ lies in the kernel of T, then $\sum_{i\in\Gamma} \lambda_i x_i = 0$. Let Γ_1 be defined as above. Then for any $j \in \Gamma_1$, we have

$$\lambda_j = \sum_{i\in\Gamma} \lambda_i \delta_{ji} = \sum_{i\in\Gamma} \lambda_i f_j(x_i) = f_j(\sum_{i\in\Gamma} \lambda_i x_i) = 0.$$

Thus the support of $(\lambda_i)_{i\in\Gamma}$ is contained in $\Gamma_0 = \Gamma \setminus \Gamma_1$, in other words $\ker T \subseteq \ell_1(\Gamma_0)$. But $\ell_1(\Gamma_0)$ is a weak* closed subspace of $\ell_1(\Gamma)$, and has infinite codimension. In particular, the weak* closure of $\ker T$ has infinite codimension. □

Corollary 2. *A Banach space with a fundamental pseudo-orthogonal family has an infinite-dimensional separable quotient.*

We remark that, unlike the case of an M-basis, the existence of a fundamental pseudo-orthogonal family does not trivially imply the existence of a separable quasicomplemented subspace.

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