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Article

Measure of Noncompactness for Hybrid Langevin Fractional Differential Equations

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Abstract: In this research article, we introduce a new class of hybrid Langevin equation involving two distinct fractional order derivatives in the Caputo sense and Riemann–Liouville fractional integral. Supported by three-point boundary conditions, we discuss the existence of a solution to this boundary value problem. Because of the important role of the measure of noncompactness in fixed point theory, we use the technique of measure of noncompactness as an essential tool in order to get the existence result. The modern analysis technique is used by applying a generalized version of Darbo's fixed point theorem. A numerical example is presented to clarify our outcomes.

Keywords: hybrid Langevin fractional differential equation; measure of noncompactness; Darbo's fixed point theorem; boundary value problem

MSC: 26A33; 34A08; 34A12

1. Introduction

A hybrid system is a dynamic system that interacts with discrete and continuous dynamics. The appearance of novel multiplex engineering systems that have several types of process and abstract decision-making units present the image of different systems simultaneously exhibiting continuous and discrete time dynamics, discrete events, logic commands and jumps. In addition, the concept of hybrid systems is of great importance in embedded control systems [1].

Fractional differential equations engender in either systems of mathematical modeling or operations the phenomena in many diverse fields, such as engineering, physics, chemistry, the phenomena of blood flow, image processing, etc. [2–12]. For some recent developments regarding this, and in particular, fractional Langevin equations, see [13–23] and the references therein.

The Langevin equation is an ideal method to depict mathematical physics. This can help the scientists to represent processes like anomalous diffusion effectively in a descent manner. In the field of economy, the operations include price index fluctuations [24]. In critical dynamics theory, the generic formula to the Langevin equation for noise sources with correlations performs an important role [25]. Many generic Langevin equations have been applied to certain types of dynamical operations in media, such as Langevin equation in general [26,27]. The way in which nonlinear fractional Langevin equations were remodeled by Mainardi and Pironi was remarkable [28].

In the field of fractional differential equations, many of the mathematicians who are interested in fractional differential equations have discussed hybrid fractional differential equations (see [29–33]).

Recently, in [34], Sitho et al. studied and proved the existence of a solution supported via certain initial value problems for the following equation:

$$\begin{cases} D^{\alpha}\left[\frac{D^{\beta}u(t)-\sum_{i=1}^{m} I^{\omega_i} f_i(t,u(t))}{g(t,u(t))}\right] = h(t,u(t),I^{\gamma}u(t)), \ \ t\in[0,1], \\ u(0) = D^{\beta}u(0) = 0, \end{cases}$$

where D^{α} and D^{β} represent the Riemann–Liouville fractional derivatives, where $0<\alpha,\beta\leq 1$.

In [35], Jamil et al. proved the existence of a solution to the following system of hybrid fractional sequential integro-differential equations with two distinct orders of Caputo derivatives as follows:

$$\begin{cases} {}^{c}D^{\alpha}\left[\frac{{}^{c}D^{\beta}u(t)-\sum_{i=1}^{m} I_i^{\omega} f_i(t,u(t))}{g(t,u(t))}\right] = h(t,u(t),I^{\gamma}u(t)), \ \ t\in[0,1], \\ u(0) \ = \ {}^{c}D^{\beta}u(t) = 0, \ \ u(1) = \delta u(\eta), \ \ 0<\delta<1, \ \ 0<\eta<1. \end{cases}$$

According to recent contributions concerning hybrid fractional differential equations, we introduce our research point as the following boundary value problem to the nonlinear fractional hybrid Langevin differential equation:

$$\begin{cases} {}^{c}D^{\alpha}\left[{}^{c}D^{\beta}\left[\dfrac{u(t)}{f(t,u(\nu(t)))}\right]-\lambda u(t)\right] = g(t,u(\mu(t)),I^{\gamma}u(\mu(t))), \ \ t\in J=[0,1], \\ u(0)=0, \ {}^{c}D^{\beta}\left[\dfrac{u(t)}{f(t,u(\nu(t)))}\right]_{t=0} = 0, \ \ u(1)=\zeta u(\eta), \ \ 0<\eta<1, \ \zeta\in\mathbb{R}, \end{cases} \tag{1}$$

where both ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ are Caputo derivatives of orders $0<\alpha\leq 1$ and $1<\beta\leq 2$, respectively. Here, $\lambda\in\mathbb{R}\setminus\{0\}$, I^{γ} denotes the Riemann–Liouville fractional integral of order $0<\gamma<1$, $\zeta = \dfrac{f(1,u(\nu(1)))}{f(\eta,u(\nu(\eta)))}$, $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$, and $g\in C(J\times\mathbb{R}^2,\mathbb{R})$. Furthermore, μ and ν are two continuous functions from J into itself.

This demonstrates that the bridge between the hybrid and Langevin equation is considerable, and is a new form in the study of fractional differential equations, especially considering the use of the measure of noncompactness technique. This encourages us to advance the previous boundary value problem. Therefore, our study relies on what is known as Darbo's fixed point theorem, in general form, for the product of two operators through implementing the measure of noncompactness technique to a hybrid Langevin equation.

2. Basic Concepts and Relevant Lemmas

In this part, we provide some important points that we need as a basis for the coming sections.

Definition 1 ([12])**.** *The Caputo fractional derivative for a function Θ that has an absolutely continuous derivative up to the order $(k-1)$ is given as*

$${}^{c}D^{\alpha}_{0^+}\Theta(t) = \int_0^t \frac{(t-s)^{k-\alpha-1}\Theta^{(k)}(s)}{\Gamma(k-\alpha)}\,ds,$$

where $k-1<\alpha\leq k$, $k\in\mathbb{N}$.

Definition 2 ([12])**.** *Let Θ be a continuous function on the interval $[0,\infty]$. The fractional integral of order $\delta>0$ is given by*

$$I^{\delta}_{0^+}\Theta(t) = \int_0^t \frac{(t-\eta)^{\delta-1}\Theta(\eta)}{\Gamma(\delta)}\,d\eta.$$

Lemma 1 ([12]). *Let $\gamma, \kappa \geq 0$, and $V \in L^1([0,1])$. Then,*

$$I_{0+}^{\gamma} I_{0_+}^{\kappa} V(t) = I_{0+}^{\gamma+\kappa} V(t) \quad and \quad {}^{c}D_{0+}^{\kappa} I_{0+}^{\kappa} V(t) = V(t)\ \forall t \in [0,1].$$

Lemma 2 ([12]). *Let $\lambda > 0$, where $m-1 < \lambda \leq m$, then*

$$I_{0+}^{\lambda}\ {}^{c}D_{0+}^{\lambda}\ \Phi(\eta) = \Phi(\eta) - \sum_{i=0}^{m-1} b_i \eta^i, \ \ b_i \in \mathbb{R}.$$

Let $\mathbf{B}(x,r)$ be a closed ball in Banach space $\mathbb{E}$ centered at x with radius r; in the case where $x = 0$, we denote $\mathbf{B}_r$ instead of $\mathbf{B}(0,r)$. Let X be a nonempty subset of $\mathbb{E}$, such that $\overline{X}$ and $ConvX$ are a closure and a convex closure of X, respectively. Throughout this work, the symbol $\mathfrak{M}_{\mathbb{E}}$ represents the family of the nonempty and bounded subsets of $\mathbb{E}$, while $\mathfrak{N}_{\mathbb{E}}$ denotes the subfamily of all relatively compact subsets of $\mathfrak{M}_{\mathbb{E}}$.

Definition 3 ([36]). *A mapping $\chi : \mathfrak{M}_{\mathbb{E}} \to [0,\infty)$ is called a noncompactness measure in $\mathbb{E}$ if all the conditions below hold:*

(ω_1) $\ker\chi = \{X \in \mathfrak{M}_{\mathbb{E}} : \chi(X) = 0\} \neq , \ \ \ker\chi \subset \mathfrak{N}_{\mathbb{E}}$,
(ω_2) $Y \subset X$, *then* $\chi(Y) \leq \chi(X)$,
(ω_3) $\chi(Y) = \chi(\overline{Y}) = \chi(ConvY)$,
(ω_4) $\chi(\lambda_1 Y + \lambda_2 X) \leq \lambda_1\chi(Y) + \lambda_2\chi(X)$, $\lambda_1 + \lambda_2 = 1$,
(ω_5) *If (Y_n) is a sequence of closed subsets of $\mathfrak{M}_{\mathbb{E}}$ with $Y_{n+1} \subset Y_n$ $(n \geq 1)$ and $\lim_{n\to\infty}\chi(Y_n) = 0$, then $\cap_{n=1}^{\infty} Y_n \neq \phi$.*

Definition 4 ([36]). *Let Y be a bounded nonempty subset of Banach space $\mathbb{E}(J)$. The function $f \in Y$ is said to be a modulus of continuous function, denoted by $\omega(f,\epsilon)$; if $\forall f \in Y$ and $\forall \epsilon > 0$, we have*

$$\omega(f,\epsilon) = \sup\{|f(t) - f(s)| : t,s \in J, \ |t-s| \leq \epsilon\}.$$

In addition,

$$\omega(Y,\epsilon) = \sup\{\omega(f,\epsilon) : f \in Y\},$$

and

$$\omega_0(Y) = \lim_{\epsilon\to 0}\omega(Y,\epsilon).$$

Definition 5 ([37]). *Let $\mathbb{E}(J)$ be Banach algebra. A noncompactness measure χ in $\mathbb{E}(J)$ satisfies condition (m) if the condition below holds:*

$$\chi(fg) \leq \|f\|\chi(g) + \|g\|\chi(f)$$

for all $f, g \in \mathfrak{M}_{C(J)}$.

Lemma 3 ([38]). *The condition (m) can be held by the noncompactness measure ω_0 on $\mathbb{E}(J)$.*

Consider the following class $\mathcal{S}$ of all functions $\psi : (0,\infty) \to (b,\infty)$ such that the following condition holds:

$$\forall\,(x_n) \subset (0,\infty)\,, \ \lim_{n\to\infty}\psi(x_n) = b \iff \lim_{n\to\infty} x_n = 0.$$

Now, we introduce Darbo's fixed point theorem, which we can rely on to illustrate the existence of at least one fixed point.

Theorem 1 ([39]). *Consider Banach space $\mathbb{E}(J)$ containing a bounded, closed, convex, and nonempty subset, for example, Ω. Let $T : \Omega \longrightarrow \Omega$ be a continuous mapping. Assume that there is $\theta \in [0,1)$ with χ as a noncompactness measure in $\mathbb{E}(J)$ satisfying the following:*

$$\chi(TY) \leq \theta\chi(Y), \;\; \phi \neq Y \subseteq \Omega.$$

Then, T has a fixed point in Ω.

The generalization of Darbo's fixed point theorem is very important for the forthcoming results.

Theorem 2 ([39]). *Let $\mathcal{U}$ be a bounded, closed, convex, and nonempty subset of Banach space $\mathbb{E}(J)$ and let $\mathbf{T} : \mathcal{U} \longrightarrow \mathcal{U}$ be a continuous mapping. Assume there exists $\Phi \in \mathcal{S}$ and $\theta \in [0,1)$ such that for any nonempty subset $\mathcal{D}$ of $\mathcal{U}$ with $\chi(\mathbf{T}\mathcal{D}) > 0$,*

$$\Phi(\chi(\mathbf{T}\mathcal{D})) \leq (\Phi(\chi(\mathcal{D})))^{\theta},$$

where χ is a noncompactness measure in $\mathbb{E}(J)$. Then, $\mathbf{T}$ has a in $\mathcal{U}$.

Lemma 4. *Assume that $0 < \alpha \leq 1$, $1 < \beta \leq 2$, $0 < \gamma < 1$, and both functions f, g satisfy Equation (1). The unique solution of Equation (1) is given by the following formula:*

$$\begin{aligned} u(t) = f(t, u(\nu(t))) \Bigg\{ & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^{\gamma}u(\mu(s))\, ds + \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(s)\, ds \\ & + \frac{t}{1-\eta} \Bigg[\int_0^{\eta} \frac{(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^{\gamma}u(\mu(s)))\, ds \\ & - \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^{\gamma}u(\mu(s)))\, ds + \lambda \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} u(s)\, ds \\ & - \lambda \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} u(s)\, ds \Bigg] \Bigg\}. \end{aligned} \tag{2}$$

Proof. By using Lemma 2, we get

$${}^{c}D^{\beta} \left[\frac{u(t)}{f(t, u(\nu(t)))} \right] = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, u(\mu(s)), I^{\gamma}u(\mu(s)))\, ds + \lambda u(t) + c_2.$$

According to the first and second boundary conditions, we obviously see that $c_2 = 0$. Again, we apply Lemma 2 to obtain the following form:

$$\frac{u(t)}{f(t, u(\nu(t)))} = \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^{\gamma}u(\mu(s)))\, ds + \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(s)\, ds + c_1 t + c_0.$$

From the first boundary condition, we have $c_0 = 0$. The unique solution from Equation (2) is provided, as shown above, once we use third condition. Conversely, it is not difficult to see that when inserting Equation (2) into the left side of Equation (1), and employing Lemma 2, we get the right side. Clearly, Equation (2) satisfies the boundary value conditions in Equation (1). □

3. Main Results

Now, to give a clear view, consider $(\mathbb{E}, \|\cdot\|)$ to be the space of all continuous real-valued functions defined on the unit interval $J = [0,1]$. It is obvious that it is a Banach space equipped with the following norm:

$$\|x\| = \sup_{t \in J} |x(t)|, \qquad x \in \mathbb{E}.$$

Before introducing the main results, we investigate Equation (2) under the following assumptions:

(i) Both functions $\nu, \mu : [0,1] \longrightarrow [0,1]$ are continuous;
(ii) The function $f \in C([0,1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and the function $g \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$;
(iii) There exists a real number $p \in (0,1)$ such that

$$|f(t,x_2) - f(t,x_1)| \leq (|x_2 - x_1| + b)^p - b^p \ \forall t \in [0,1], \ \ x_1, x_2 \in \mathbb{R}, \ \ b \in \mathbb{R}^+;$$

(iv) There exist a continuous function $\omega \in L^1(J,(0,\infty))$, and a continuous nondecreasing function

$$\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$$

with $\psi(0) = 0$ such that

$$|g(t,x_1(t),x_2(t))| \leq \|\omega\|_{L^1}\psi(\|x\|), \ \forall \in [0,1], \ \ x_1, x_2 \in \mathbb{R};$$

(v) There is a positive real number r_0 conditionally:

$$\left[(r_0+b)^p - b^p + N\right]\left\{\frac{\psi(r_0)}{\Gamma(\alpha+\beta+1)}\left[1 + \frac{\eta^{\alpha+\beta}+1}{1-\eta}\right] + \frac{|\lambda| r_0}{\Gamma(\beta+1)}\left[1 + \frac{\eta^{\beta}+1}{1-\eta}\right]\right\} \leq r_0,$$

where $N = \sup\{|f(t,0)| : t \in [0,1]\}$.

The main result is given via the following theorem.

Theorem 3. *Consider that all of the assumptions* **(i)–(v)** *hold. Then, Equation* (1) *has a solution in Banach algebra* $\mathbb{E}(J)$ *if*

$$\mathcal{R} \leq 1$$

where

$$\mathcal{R} = \frac{\psi(r_0)}{\Gamma(\alpha+\beta+1)}\left[1 + \frac{\eta^{\alpha+\beta}+1}{1-\eta}\right] + \frac{|\lambda| r_0}{\Gamma(\beta+1)}\left[1 + \frac{\eta^{\beta}+1}{1-\eta}\right].$$

Proof. Consider the operator $T : \mathbb{E}(J) \to \mathbb{E}(J)$ as

$$(Tu)(t) = (Fu)(t)(Gu)(t), \tag{3}$$

where

$$\begin{aligned}
(Fu)(t) &= f(t, u(\nu(t))), \\
(Gu)(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^\gamma u(\mu(s)) \, ds + \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(s) \, ds \\
&\quad + \frac{t}{1-\eta}\left[G_\alpha(g) + \lambda G_0(u)\right],
\end{aligned}$$

where

$$\begin{aligned}
G_\alpha(g) &= \int_0^\eta \frac{(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^\gamma u(\mu(s))) \, ds \\
&\quad - \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^\gamma u(\mu(s))) \, ds,
\end{aligned}$$

and

$$G_0(u) = \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} u(s) \, ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} u(s) \, ds \, .$$

It is easy to see that

$$\|G_0(u)\| = \frac{1+\eta^{\beta}}{\Gamma(\beta+1)}\|u\| \tag{4}$$

and

$$\|G_\alpha(g)\| = \frac{1+\eta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|\omega\|_{L^1}\psi((1+\frac{1}{\Gamma(1+\gamma)})\|u\|). \tag{5}$$

The proof of this theorem depends on different parts.

Firstly, we show that $\forall\ u \in \mathbb{E}(J)$ implies that $(Tu) \in \mathbb{E}(J)$. In other words, $(Fu)(Gu) \in \mathbb{E}(J)$ for all $u \in \mathbb{E}(J)$. Indeed, from the assumptions **(i)** and **(ii)**, $\forall\ u \in \mathbb{E}(J)$, it yields that $(Fu) \in \mathbb{E}(J)$. It remains to prove $(Gu) \in \mathbb{E}(J)\ \forall\ u \in \mathbb{E}(J)$. Let $u \in \mathbb{E}(J)$ and $t_2, t_1 \in J$ be taken arbitrarily with $t_2 > t_1$. By using assumption **(iv)**, we get

$$\begin{aligned}
|Gu(t_2) - Gu(t_1)| = \Bigg| & \int_0^{t_1} \frac{(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^{\gamma}u(\mu(s))ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^{\gamma}u(\mu(s))ds \\
& + \lambda \int_0^{t_1} \frac{(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}}{\Gamma(\beta)} u(s)\ ds - \lambda \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} u(s) ds \\
& + \frac{t_2-t_1}{1-\eta}\left[G_\alpha(g) + \lambda G_0(u)\right] \Bigg| \\
& \le \|\omega\|_{L^1}\psi((1+\frac{1}{\Gamma(1+\gamma)})\|u\|)\left[\int_0^{t_1} \frac{(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds \right.\\
& \left. + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds\right] + |\lambda|\|u\|\left[\int_0^{t_1} \frac{(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}}{\Gamma(\beta)} ds \right.\\
& \left. + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} ds\right] + \frac{|t_2-t_1|}{1-\eta}\left[\|G_\beta(g)\| + |\lambda|\|G_0(u)\|\right] \\
& = \frac{\|\omega\|_{L^1}\psi((1+\frac{1}{\Gamma(1+\gamma)})\|u\|)}{\Gamma(\alpha+\beta+1)}|t_2^{\alpha+\beta} - t_1^{\alpha+\beta}| + \frac{|\lambda|\|u\|}{\Gamma(\beta+1)}|t_2^{\beta} - t_1^{\beta}| \\
& + \left[\frac{1+\eta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|\omega\|_{L^1}\psi((1+\frac{1}{\Gamma(1+\gamma)})\|u\|) + \frac{|\lambda|(1+\eta^{\beta})}{\Gamma(\beta+1)}\|u\|\right]\frac{|t_2-t_1|}{1-\eta},
\end{aligned}$$

which tends zero uniformly once $t_2 \to t_1$. It is clear that $Gu \in \mathbb{E}(J)$ for all $u\ \in \mathbb{E}(J)$. Moreover, for all $u \in \mathbb{E}(J)$ and $t \in J$, we are able to estimate the absolute value of the operator Tu as

$$
\begin{aligned}
|(Tu)(t)| &= |(Fu)(t)(Gu)(t)| \\
&\leq (|f(t,u(\nu(t))) - f(t,0)| + |f(t,0)|) \left\{ \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |g(s,u(\mu(s)), I^\gamma u(\mu(s))| \, ds \right. \\
&\left. + |\lambda| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |u(s)| \, ds + \frac{t}{1-\eta} \left[|G_\alpha(g)| + |\lambda||G_0(u)| \right] \right\} \\
&\leq \left[(\|u\|+b)^p - b^p + N \right] \left\{ \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \|\omega\|_{L^1} \psi((1+\frac{1}{\Gamma(1+\gamma)})\|u\|) \, ds \right. \\
&+ |\lambda| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \|u\| \, ds + \frac{t}{1-\eta} \left[\frac{1+\eta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \|\omega\|_{L^1} \psi((1+\frac{1}{\Gamma(1+\gamma)})\|u\|) \right. \\
&\left.\left. + \frac{|\lambda|(1+\eta^\beta)}{\Gamma(\beta+1)} \|u\| \right] \right\} \\
&\leq \left[(\|u\|+b)^p - b^p + N \right] \left\{ \frac{\|\omega\|_{L^1} \psi((1+\frac{1}{\Gamma(1+\gamma)})\|u\|)}{\Gamma(\alpha+\beta+1)} \left[1 + \frac{\eta^{\alpha+\beta}+1}{1-\eta} \right] \right. \\
&\left. + \frac{|\lambda|\|u\|}{\Gamma(\beta+1)} \left[1 + \frac{\eta^\beta+1}{1-\eta} \right] \right\}.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
\|Tu\| \leq \left[(\|u\|+b)^p - b^p + N \right] & \left\{ \frac{\|\omega\|_{L^1} \psi((1+\frac{1}{\Gamma(1+\gamma)})\|u\|)}{\Gamma(\alpha+\beta+1)} \left[1 + \frac{\eta^{\alpha+\beta}+1}{1-\eta} \right] \right. \\
& \left. + \frac{|\lambda|\|u\|}{\Gamma(\beta+1)} \left[1 + \frac{\eta^\beta+1}{1-\eta} \right] \right\}.
\end{aligned}
$$

Let $\mathcal{B}_{r_0}$ be a subset of $\mathbb{E}(J)$ given as

$$
\mathcal{B}_{r_0} = \{u(t) \in \mathbb{E}(J) : \|u\| \leq r_0 : \ t \in J\}, \tag{6}
$$

with a fixed radius r_0, which satisfies the inequality mentioned in assumption **v**. Furthermore, we see that the operator T defined in Equation (3) maps $\mathcal{B}_{r_0}$ into itself.

The second step depends on the continuity of the operator T on $\mathcal{B}_{r_0}$. To briefly demonstrate this, we show the continuity for both F and G on $\mathcal{B}_{r_0}$, separately. We claim that F is continuous on $\mathcal{B}_{r_0}$. Indeed, for all $\epsilon > 0$ and $u, w \in \mathcal{B}_{r_0}$, there exists $0 < \delta < (\epsilon + b^p)^{1/p} - b$ such that $\|u - w\| \leq \delta$, which implies that, with fixed $t \in J$, it yields

$$
\begin{aligned}
|Fu(t) - Fw(t)| &= |f(t,u(\nu(t))) - f(t,w(\nu(t)))| \\
&\leq (|u(\nu(t)) - w(\nu(t))| + b)^p - b^p \\
&\leq (\|u-w\| + b)^p - b^p \\
&\leq (\delta + b)^p - b^p < \epsilon.
\end{aligned} \tag{7}
$$

Thus, F satisfies the continuity condition on $\mathcal{B}_{r_0}$. Lebesgue dominant convergence is used in order to have continuity proof for the operator G on $\mathcal{B}_{r_0}$. Take a convergent sequence (u_n), and its limit u is in $\mathcal{B}_{r_0}$ with $\|u_n - u\| \to 0$ as $n \to 0$. Since $\mu : J \to J$ is continuous, we can say

$$
|u_n(\mu(t))| \leq r_0 \ \forall n \in \mathbb{N} \ \ \forall t \in J.
$$

Since g is continuous on $J \times [-r_0, r_0]$, it is uniformly continuous on $J \times [-r_0, r_0]$. Now, set

$$G_0 = \max_{(t,u) \in J \times [-r_0, r_0]} |(Gu)(t)| \tag{8}$$

$$\kappa_0 = \frac{G_0}{\Gamma(\alpha+\beta+1)}(1-\eta^{\alpha+\beta}) + \frac{|\lambda| r_0}{\Gamma(\beta+1)}(1-\eta^{\beta}). \tag{9}$$

By applying Lebesgue dominant convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} (Gu_n)(t) = \lim_{n\to\infty} &\left\{ \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u_n(\mu(s)), I^{\gamma} u_n(\mu(s)) ds - \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u_n(s) ds \right. \\
&+ \frac{t}{1-\eta} \left[G_\alpha(u_n) + \lambda G_0(u_n) \right] \\
&= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^{\gamma} u(\mu(s)) ds - \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(s)\, ds \\
&+ \frac{t}{1-\eta} \left[G_\alpha(u) + \lambda G_0(u) \right] \\
&= (Gu)(t).
\end{aligned}$$

Since both F and G are continuous operators, it yields that T is a continuous operator on $\mathcal{B}_{r_0}$.

The final step is based on estimating the limit of the modulus of continuity for the operator T. This helps us to estimate $\omega_0(F\Omega)$ and $\omega_0(G\Omega)$ for nonempty Ω as a subset of $\mathcal{B}_{r_0}$.

Since $\nu : J \to J$ is uniformly continuous, we have $\forall\, \epsilon > 0\ \exists\, \delta > 0$ with $(\delta < \epsilon)$ $\forall t_1, t_2 \in J$ with $|t_2 - t_1| < \delta$, which implies $|\nu(t_2) - \nu(t_1)| < \epsilon$. Choose $u \in \Omega$ and $t_1, t_2 \in J$ with $|t_2 - t_1| < \delta$. By using assumption **(v)**, we obtain

$$\begin{aligned}
|(Fu)(t_2) - (Fu)(t_1)| &= |f(t_2, u(\nu(t_2))) - f(t_1, u(\nu(t_1)))| \\
&\le |f(t_2, u(\nu(t_2))) - f(t_2, u(\nu(t_1)))| + |f(t_2, u(\nu(t_1))) - f(t_1, u(\nu(t_1)))| \\
&\le [(|u(\nu(t_2)) - u(\nu(t_1))| + b)^p - b^p] + \omega(f, \epsilon) \\
&\le [(\omega(\Omega, \epsilon) + b)^p - b^p] + \omega(f, \epsilon).
\end{aligned} \tag{10}$$

Considering

$$\omega(f, \epsilon) = \sup\{|f(t_2, u) - f(t_1, u)| : t_1, t_2 \in J, |t_2 - t_1| < \epsilon, u \in [-r_0, r_0]\},$$

Equation (10) can be written as

$$\omega(F\Omega, \epsilon) \le [(\omega(\Omega, \epsilon) + b)^p - b^p] + \omega(f, \epsilon). \tag{11}$$

Clearly, $f(t, u)$ is uniformly continuous on $J \times [-r_0, r_0]$, and $\omega(f, \epsilon) \longrightarrow 0$ once $\epsilon \longrightarrow 0$. Equation (11) becomes as we want:

$$\omega_0(F\Omega) \le (\omega_0(\Omega) + b)^p - b^p. \tag{12}$$

Since $\mu : J \to J$ is uniformly continuous, we have $\forall\, \epsilon > 0\ \exists\, \delta > 0$ with $(\delta = \delta(\epsilon))$ $\forall t_1, t_2 \in J$ with $|t_2 - t_1| < \delta$, which implies $|\mu(t_2) - \mu(t_1)| < \epsilon$. Taking into account Equations (6), (8) and (9); $\forall\, \epsilon > 0$, therefore, let

$$\delta = \min\left\{ \frac{1}{2}, \frac{(1-\eta)\epsilon}{\kappa_0}, \frac{\Gamma(\beta+1)\epsilon}{|\lambda| r_0}, \frac{\Gamma(\alpha+\beta+1)\epsilon}{8G_0} \right\}.$$

Arbitrarily choose $u \in \Omega$ and $t_1, t_2 \in J$ with $|t_2 - t_1| \leq \delta$, we have

$$\begin{aligned}
|Gu(t_2) - Gu(t_1)| = \Bigg| & \int_0^{t_1} \frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^{\gamma} u(\mu(s))\, ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^{\gamma} u(\mu(s))\, ds \\
& + \int_0^{t_1} \frac{(t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, u(\mu(s)), I^{\gamma} u(\mu(s))\, ds \\
& + \lambda \int_0^{t_1} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} u(s)\, ds + \lambda \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} u(s)\, ds \\
& + \lambda \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} u(s)\, ds + \frac{t_2 - t_1}{1-\eta} \left[G_\alpha(u) + \lambda G_0(u) \right] \Bigg| \\
& \leq \frac{G_0}{\Gamma(\alpha+\beta+1)} |t_2^{\alpha+\beta} - t_1^{\alpha+\beta}| + \frac{|\lambda| r_0}{\Gamma(\beta+1)} |t_2^{\beta} - t_1^{\beta}| \\
& + \frac{\kappa_0}{1-\eta} |t_2 - t_1|.
\end{aligned} \tag{13}$$

The factors $t_2^{\alpha+\beta} - t_1^{\alpha+\beta}$ and $t_2^{\beta} - t_1^{\beta}$ can be estimated as follows:

Case 1 : If $0 \leq t_1 < \delta$, $t_2 \leq 2\delta$. Then,
$t_2^{\beta} - t_1^{\beta} \leq t_2^{\beta} < (2\delta)^{\beta} \leq 2^{\beta}\delta \leq 4\delta$ and
$t_2^{\alpha+\beta} - t_1^{\alpha+\beta} \leq t_2^{\alpha+\beta} < (2\delta)^{\alpha+\beta} \leq 2^{\alpha+\beta}\delta \leq 8\delta$.

Case 2 : If $0 < t_1 < t_2 \leq \delta$. Then,
$t_2^{\beta} - t_1^{\beta} \leq t_2^{\beta} < \delta^{\beta} \leq \beta\delta < 4\delta$ and
$t_2^{\alpha+\beta} - t_1^{\alpha+\beta} \leq t_2^{\alpha+\beta} < \delta^{\alpha+\beta} \leq (\alpha+\beta)\delta < 8\delta$.

Case 3 : If $\delta \leq t_1 < t_2 \leq 1$. Then,
$t_2^{\beta} - t_1^{\beta} < \beta\delta < 4\delta$ and
$t_2^{\alpha+\beta} - t_1^{\alpha+\beta} < (\alpha+\beta)\delta < 8\delta$.

Accordingly, we obtain

$$|Gu(t_2) - Gu(t_1)| \leq \epsilon,$$

which implies that

$$\omega(Gu, \epsilon) \leq \epsilon.$$

Letting $\epsilon \to 0$, we have

$$\omega_0(G\Omega) = 0. \tag{14}$$

The estimation of $\omega_0(T\Omega)$ can be achieved by Lemma 3 and Equations (6), (12) and (14) as

$$\begin{aligned}
\omega_0(T\Omega) &= \omega_0(F\Omega . G\Omega) \\
& \|F\Omega\| \omega_0(G\Omega) + \|G\Omega\| \omega_0(F\Omega) \\
& \leq \|F(\mathcal{B}_{r_0})\| \omega_0(G\Omega) + \|G(\mathcal{B}_{r_0})\| \omega_0(F\Omega) \\
& \leq \left[(\omega_0(\Omega) + b)^p - b^p \right] \left[\frac{\psi(r_0)}{\Gamma(\alpha+\beta+1)} \left[1 + \frac{\eta^{\alpha+\beta}+1}{1-\eta} \right] + \frac{|\lambda| r_0}{\Gamma(\beta+1)} \left[1 + \frac{\eta^{\beta}+1}{1-\eta} \right] \right] \\
& = \left[(\omega_0(\Omega) + b)^p - b^p \right] \mathcal{R}.
\end{aligned}$$

By using assumption **(v)** and since $\mathcal{R} \leq 1$, we have

$$\omega_0(T\Omega) + b^p \leq (\omega_0(\Omega) + b)^p.$$

According to Theorem 1, the contractive condition has been satisfied with $\psi(x) = x + b$, where $\psi \in \mathcal{S}$. By using Theorem 2, we conclude that T has at least fixed point in $\mathcal{B}_{r_0}$. Hence, Equation (1) possesses at least one solution in $\mathcal{B}_{r_0}$. □

4. Example

Consider the following hybrid Langevin fractional differential equation:

$$\begin{cases} {}^cD^{\frac{1}{2}}\left[{}^cD^{\frac{3}{2}}\left[\dfrac{u(t)}{\sqrt{u(\frac{e^{(t-1)}}{2})+16}}\right]-\frac{1}{100}u(t)\right]=\frac{t}{10}\left[\sin u(\sqrt{t})+I^{\frac{1}{3}}u(\sqrt{t})\right], \quad t\in J=[0,1] \\ u(0)=0, \quad {}^cD^{\frac{3}{2}}\left[\dfrac{u(t)}{\sqrt{u(\frac{e^{(t-1)}}{2})+16}}\right]_{t=0}=0, \quad u(1)=3u(\frac{1}{10}). \end{cases} \tag{15}$$

According to Equation (15), we see that
$\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $\lambda = \frac{1}{100}$, $\zeta = 3$, $\eta = \frac{1}{10}$, $\nu(t) = \frac{e^{(t-1)}}{2}$, $\mu(t) = \sqrt{t}$, $\omega(t) = \frac{t}{10}$,
$f(t,u) = \sqrt{|u|+16}$, $g(t,u,I^{\frac{1}{3}}u) = \frac{1}{10}\left[\sin u + I^{\frac{1}{3}}u\right]$, $b = 16$, and
$N = \sup_{t\in[0,1]} |f(t,0,0)| = 4.$

Both assumption **(i)** and **(ii)** hold. In assumption **(iii)**, we have $p = \frac{1}{2}$. Moreover, if we take $z(u) = \sqrt{|u|+16} - 4$, we see that $z(0) = 0$ and it is a concave function. Since $z(t)$ is concave, we conclude by using the subadditive property of the concave function, such that

$$\begin{aligned} |f(t,u_2) - f(t,u_1)| &= |z(u_2) - z(u_1)| \\ &\leq z(u_2 - u_1) = \sqrt{|u_2 - u_1| + 16} - 4 \end{aligned}$$

and

$$\begin{aligned} |g(t,u,I^{\frac{1}{3}}u)| &= |\frac{t}{10}\left[\sin u(t) + I^{\frac{1}{3}}u\right]| \\ &\leq \frac{1}{10}\left[|u(t)| + \int_0^t \frac{(t-s)^{\frac{-2}{3}}}{\Gamma(\frac{1}{3})}|u(s)|ds\right] \\ &\leq 0.21198465217\,\|u\| \quad \forall t \in J. \end{aligned}$$

This means that $\|\omega\|_{L^1} = \frac{1}{10}$ and $\psi(\|u\|) = 0.21198465217\,\|u\|$.
The last assumption **(v)** allows us to determine the range of r_0 which is obviously

$$0 < r_0 \leq 2.331413061.$$

Accordingly, Theorem 3 ensures that the numerical example (Equation (15)) has a solution in $\mathbb{E}(J)$ because of

$$\mathcal{R} = 0.544529299 < 1.$$

5. Conclusions

We introduce the proof of the existence of a solution to the hybrid Langevin fractional differential equation supported by certain boundary value conditions. The problem is given as a nonlinear equation, in which it implicitly relies on the unknown function, fractional derivative orders of both $\alpha \in (0,1)$ and $\beta \in (1,2)$. The generalized Darbo's fixed point theorem was applied in order to prove the existence of a solution for the given problem. In our paper, we consider the theorem mentioned for the product of the two operators associated with noncompactness measures. The outcomes are not only new in the formation shown, but we also used new assumptions in our results.

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References

1. Haddad, W.M.; Chellaboina, V.S.; Nersesov, S.G. *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity and Control*; Princeton University Press: Princeton, NJ, USA, 2006.
2. Keten, A.; Yavuz, M.; Baleanu, D. Nonlocal Cauchy Problem via a Fractional Operator Involving Power Kernel in Banach Spaces. *Fractal Fract.* **2019**, *3*, 1–8. [CrossRef]
3. Alqahtani, B.; Aydi, H.; Karapınar, E.; Rakocevi, V. A Solution for Volterra Fractional Integral Equations by Hybrid Contractions. *Mathematics* **2019**, *7*, 694. [CrossRef]
4. Abdeljawad, T.; Agarwal, R.P.; Karapınar, E.; Kumari, P.S. Solutions of the Nonlinear Integral Equation and Fractional Differential Equation Using the Technique of a Fixed Point with a Numerical Experiment in Extended *b*-Metric Space. *Symmetry* **2019**, *11*, 686. [CrossRef]
5. Karapınar, E.; Fulga, A.; Rashid, M.; Shahid, L.; Aydi, H. Large Contractions on Quasi-Metric Spaces with an Application to Nonlinear Fractional. *Mathematics* **2019**, *7*, 444. [CrossRef]
6. Karapınar, E.; Abdeljawad, T.; Jarad, F. Applying new fixed point theorems on fractional and ordinary differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 421. [CrossRef]
7. Javidi, M.; Ahmad, B. Dynamic analysis of time fractional order phytoplankton-toxic phytoplanktonzooplankton system. *Ecol. Model.* **2015**, *318*, 8–18. [CrossRef]
8. Fallahgoul, H.A.; Focardi, S.M.; Fabozzi, F.J. *Fractional Calculus and Fractional Processes with Applications to Financial Economics, Theory and Application*; Elsevier Academic Press: London, UK, 2017.
9. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
10. Sabatier, J.; Agrawal, R.P.; Machado, J.T. *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics And Engineering*; Springer: Dordrecht, The Netherlands, 2007.
11. Lakshmikantham, V.; Leela, S.; Devi, J.V. *Theory of Fractional Dynamic Systems*; Cambridge Academic Publichers: Cambridge, MA, USA, 2009.
12. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
13. Zhao, Y.; Sun, S.; Han, Z.; Li, Q. The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations. *Commun. Nonlinear Scri Numer. Simul.* **2011**, *16*, 2086–2097. [CrossRef]
14. Chen, W.; Zhao, Y. Solvability of boundary value problems of nonlinear fractional differential equations. *Adv. Differ Equ.* **2015**, *2015*, 36. [CrossRef]
15. Tariboon, J.; Ntouyas, S.K.; Sudsutad, W. Fractional integral problems for fractional differential equations via caputo derivative. *Adv. Differ. Equ.* **2014**, *2014*, 181. [CrossRef]
16. Salem, A.; Alnegga, M. Fractional Langevin equations with multi-point and non-local integral boundary conditions. *Cogent Math. Stat.* **2020**, *7*, 1758361. [CrossRef]
17. Salem, A.; Aghamdi, B. Multi-Strip and Multi-Point Boundary Conditions for Fractional Langevin Equation. *Fractal Fract.* **2020**, *4*, 18. [CrossRef]
18. Salem, A.; Alzahrani, F.; Alnegga, M. Coupled System of Non-linear Fractional Langevin Equations with Multi-point and Nonlocal Integral Boundary Conditions. *Math. Problem Eng.* **2020**, *2020*, 7345658.
19. Salem, A.; Alzahrani, F.; Alghamdi, B. Langevin equation involving two fractional orders with three-point boundary conditions. *Differ. Integral Equ.* **2020**, *33*, 163–180.
20. Salem, A.; Alzahrani, F.; Almaghamsi, L. Fractional Langevin equation with nonlocal integral boundary condition. *Mathematics* **2019**, *7*, 402. [CrossRef]
21. Salem, A.; Aghamdi, B. Multi-Point and Anti-Periodic Conditions for Generalized Langevin Equation with Two Fractional Orders. *Fractal Fract.* **2019**, *3*, 51. [CrossRef]

22. Fazli, H.; Nieto, J.J. Fractional Langevin equation with anti-periodic boundary conditions. *Chaos Solitons Fractals* **2018**, *114*, 332–337. [CrossRef]
23. Yukunthorn, W.; Ntouyas, S.K.; Tariboon, J. Nonlinear fractional Caputo Langevin equation with nonlocal Riemann Liouville fractional integral conditions. *Adv. Differ. Equ.* **2014**, *2014*, 315. [CrossRef]
24. West, B.J.; Picozzi, S. Fractional Langevin model of memory in financial time series. *Phys. Rev. E* **2002**, *65*, 037106. [CrossRef]
25. Hohenberg, P.C.; Halperin, B.I. Theory of dynamic critical phenomena. *Rev. Mod. Phys.* **1977**, *49*, 435–479. [CrossRef]
26. Wang, G.; Tokuyama, M. Nonequilibrium statistical description of anomalous diffusion. *Phys. A* **1999**, *265*, 341–351. [CrossRef]
27. Lutz, E. Fractional Langevin equation. *Phys. Rev. E* **2001**, *64*, 051106. [CrossRef] [PubMed]
28. Mainardi, F.; Pironi, P. The fractional Langevin equation: Brownian motion revisited. *Extr. Math.* **1996**, *11*, 140–154.
29. Hilal, K.; Kajouni, A. Boundary value problem for hybrid differential equations with fractional order. *Adv. Differ. Equ.* **2015**, *2015*, 183. [CrossRef]
30. Herzallah, M.A.E.; Baleanu, D. On fractional order hybrid differential equations. *Abstr. Appl. Anal.* **2014**, *2014*, 389386. [CrossRef]
31. Lu, H.; Sun, S.; Yang, D.; Teng, H. Theory of fractional hybrid differential equations with linear perturbations of second type, Bound. *Value Probl.* **2013**, *2013*, 23. [CrossRef]
32. Zhao, Y.; Sun, S.; Han, Z.; Li, Q. Theory of fractional hybrid differential equations. *Comput. Math. Appl.* **2011**, *62*, 1312–1324. [CrossRef]
33. Dhage, B.C.; Lakshmikantham, V. Basic results on hybrid differential equations. *Nonlinear Anal.* **2010**, *4*, 414–424. [CrossRef]
34. Sitho, S.; Ntoyus, S.K.; Tariboon, J. Existence results for hybrid fractional integro-differential equations. *Bound Value Probl.* **2015**, *2015*, 113. [CrossRef]
35. Jamil, M.; Khan, R.A.; Shan, K. Existence theory to a class of boundary value problems of hybrid fractional sequential integro-differential equations. *Bound Value Probl.* **2019**, *2019*, 77. [CrossRef]
36. Banas, J.; Goebel, K. *Measures of Noncompactness in Banach Spaces*; Lecture Notes in Pure and Applied Mathematics; Marcel Dekker: New York, NY, USA, 1980.
37. Banas, J.; Goebel, K. On a class of measures of noncompactness in babach algebras and their application to nonlinear integral equations. *J. Anal. Appl.* **2009**, *28*, 475–498.
38. Zidane, B.; Kaddour, G.; Mouffak, B.; Yong, Z. Boundary value problems for hybrid caputo fractional differential equations. *Mathematics* **2019**, *7*, 282.
39. Jleli, M.; Karapınar, E.; O'Regan, D.; Samet, B. Some generalizations of Darbo's theorem and applications to fractional integral equations. *Fixed Point Theory Appl.* **2016**, *2016*, 11. [CrossRef]